Rate-independent damage in thermo-viscoelastic materials with inertia

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Abstract

We present a model for rate-independent, unidirectional, partial damage in visco-elastic materials with inertia and thermal effects. The damage process is modeled by means of an internal variable, governed by a rate-independent flow rule. The heat equation and the momentum balance for the displacements are coupled in a highly nonlinear way. Our assumptions on the corresponding energy functional also comprise the case of the Ambrosio-Tortorelli phase-field model (without passage to the brittle limit). We discuss a suitable weak formulation and prove an existence theorem obtained with the aid of a (partially) decoupled time-discrete scheme and variational convergence methods. We also carry out the asymptotic analysis for vanishing viscosity and inertia and obtain a fully rate-independent limit model for displacements and damage, which is independent of temperature.

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1 Introduction

Gradient damage models have been extensively studied in recent years, in particular in order to understand the behavior of brittle or quasi-brittle materials. In this paper we present a model for rate-independent, unidirectional, partial damage in visco-elastic materials with inertia and thermal effects. Thus we deal with a PDE system composed of the (damped) equation of elastodynamics, a rate-independent flow rule for the damage variable, and the heat equation, coupled in a highly nonlinear way. We prove an existence result basing on time-discretization and variational convergence methods, where the analytical difficulties arise from the interaction of rate-independent and rate-dependent phenomena. We study also the relationship of our model with a fully rate-independent system by time rescaling.

Following Frémond's approach [Fré02], damage is represented through an internal variable, in the context of generalized standard materials [HN75]. The damage process is unidirectional, meaning that no healing is allowed; we do not use the term "irreversibility" to avoid confusion with thermodynamical notions. In our model the evolution of this variable is rate-independent: this choice is due to the consideration that, to damage a certain portion of the material, one needs a quantity of energy that is independent of the velocity, see e.g. [KMR06]. Rate-independent damage has been widely explored

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over the last years, cf. e.g. [MR06, FG06, BMR09, TM10, GL09, Tho13, FKS12, KRZ13]. For different studies on rate-dependent damage we refer to e.g. [FN96, BS04, BSS05] in the isothermal case and [BB08, RR14a, RR14b, HK12] for temperature-dependent systems.

Energy can be dissipated not only by damage growth, but also by viscosity and heat, both phenomena having a rate-dependent nature. Rate-independent processes coupled with viscosity, inertia, and also temperature have first been analyzed in the two pioneering papers [Rou09, Rou10]. Under the assumption of small strains, the momentum equation is linearized and is formulated using Kelvin-Voigt rheology and inertia. The nonlinear heat equation is coupled with the momentum balance through a thermal expansion term: this reflects the fact that temperature changes produce additional stresses. Here, we extend Roubíček's ansatz for the temperature-dependent setting to a unidirectional process, thus dealing with a discontinuous rate-independent dissipation potential, cf. (1.2) below. Existence results for an Ambrosio-Tortorelli-type system with unidirectional damage, inertia, and damping were already provided in [LOS10] in the isothermal case.

The PDE system. More precisely, we address the analysis of the following PDE system:

$$\rho\ddot{u} - \mathrm{div}\,(\mathbb{D}(z,\theta)e(\dot{u}) + \mathbb{C}(z)e(u) - \theta\,\mathbb{B}) = f_{\mathrm{V}} \qquad \text{in } (0,T)\times\Omega\,, \tag{1.1a}$$

$$\partial\mathrm{R}_1(\dot{z}) + \mathrm{D}_z G(z,\nabla z) - \mathrm{div}\,(\mathrm{D}_\xi G(z,\nabla z)) + \tfrac{1}{2}\mathbb{C}'(z)e(u) : e(u) \ni 0 \qquad \text{in } (0,T)\times\Omega\,, \tag{1.1b}$$

$$\dot{\theta} - \mathrm{div}\,(\mathbb{K}(z,\theta)\nabla\theta) = \mathrm{R}_1(\dot{z}) + \mathbb{D}(z,\theta)e(\dot{u}) : e(\dot{u}) - \theta\,\mathbb{B} : e(\dot{u}) + H \qquad \text{in } (0,T)\times\Omega\,, \tag{1.1c}$$

where the unknowns are the displacement vector field u, the damage variable z, and the absolute temperature θ, all the three being functions of the time $t \in (0,T)$ and of the position x in the reference configuration of a material Ω, a bounded subset of $\mathbb{R}^d$, with $d \in \{2,3\}$. Here, $e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$ denotes the small strain tensor.

In (1.1a), the constant $\rho > 0$ is the mass density. Moreover, $\mathbb{D}(z,\theta)$ and $\mathbb{C}(z)$ are the viscous and the elastic stress tensors and are both bounded, symmetric, and positive definite, uniformly in z and θ. This reflects two hypotheses of the model, motivated by analytical reasons: first, we cannot renounce the presence of some damping in the momentum balance; second, we restrict ourselves to the case of partial damage, assuming that even in its most damaged state the material keeps some elastic properties. In order to account for the phenomenological effect that an increase of damage reduces the stored elastic energy, see e.g. [LD05], it is assumed that the elastic tensor $\mathbb{C}(z)$ depends monotonically on the internal variable z, cf. also [FN96, Fré02, MWH10].

According to the rate-independent and unidirectional nature of the damage process, R_1 is a 1-homogeneous dissipation potential of the form

$$\mathrm{R}_1(v) := \begin{cases} |v| & \text{if } v \le 0\,, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.2}$$

which enforces the internal variable z to be nonincreasing in time. Indeed, we assume that $z = 1$ marks the sound material and $z = 0$ the most damaged state.

The gradient term $G(z,\nabla z)$ is needed to regularize damage; in particular, this term also allows for a nonconvex dependence on z as in many phase-field models. Moreover, for suitable choices we retrieve the Modica-Mortola term appearing in the Ambrosio-Tortorelli functional, see Remark 2.3. The flow rule (1.1b) is given as a subdifferential inclusion, where ∂ denotes the subdifferential in the sense of convex analysis of R_1 while D_z and D_ξ stand for the Gâteaux derivatives of $G(\cdot,\xi)$ and $G(z,\cdot)$, respectively. This is a compact way to write a (semi)-stability condition of Kuhn-Tucker type.

The term $\theta\,\mathbb{B}$, where $\mathbb{B}$ is a fixed symmetric matrix, derives from thermodynamical considerations and is a coupling term between the momentum (1.1a) and the heat equation (1.1c). The information on the heat conductivity of the material is contained in the symmetric matrix $\mathbb{K}(z,\theta)$. We suppose that $\mathbb{K}(z,\cdot)$ satisfies subquadratic growth conditions uniformly in z, which are borrowed from [RR14b] and which are in the same spirit as in [FPR09]. These conditions are fundamental in the proof of some a priori estimates; see the discussion below (1.4) for appropriate examples from materials science.

All the aforementioned quantities are independent of time and space, whilst the external force f_V and the heat source H are functions of both. The system is complemented with the natural boundary conditions

$$(\mathbb{D}(z,\theta)e(\dot{u}) + \mathbb{C}(z)e(u) - \theta\,\mathbb{B})\,\nu = f_S \quad \text{on } (0,T)\times\partial_N\Omega\,, \tag{1.3a}$$

$$u = 0 \quad \text{on } (0,T)\times\partial_D\Omega\,, \tag{1.3b}$$

$$\mathrm{D}_\xi G(z,\nabla z)\,\nu = 0 \quad \text{on } (0,T)\times\partial\Omega\,, \tag{1.3c}$$

$$\mathbb{K}(z,\theta)\nabla\theta\cdot\nu = h \quad \text{on } (0,T)\times\partial\Omega\,, \tag{1.3d}$$

where $\partial_D\Omega$ and $\partial_N\Omega := \partial\Omega\backslash\partial_N\Omega$ are the Dirichlet and the Neumann part of the boundary, ν denotes the outer unit normal vector to $\partial\Omega$, and f_S and h are prescribed external data depending on time and space. As for the Dirichlet data, we restrict to homogeneous boundary conditions, see Remark 2.8 for a discussion on this choice. Moreover, Cauchy conditions are given on $u(0)$, $\dot{u}(0)$, $z(0)$, and $\theta(0)$. We refer to Section 2.1 for the precise assumptions on the domain and the given data.

The energetic formulation. Due to the rate-independent character of the flow rule (1.1b) and to the nonconvexity of the underlying energy, proving the existence of solutions to the PDE system (1.1) in its pointwise form seems to be out of reach. As customary in rate-independent processes, we will resort to a weak solvability concept, based on the notion of *energetic solution*, see [Mie05] and references therein. For fully rate-independent systems, governed (in the classical PDE-formulation) by the static momentum balance for u and the rate-independent flow rule for z, the energetic formulation consists of two properties:

- *global stability:* at each time t the configuration $(u(t), z(t))$ is a global minimizer of the sum of energy and dissipation;
- *energy-dissipation balance:* the sum of the energy at time t and of the dissipated energy in $[0,t]$ equals the initial energy plus the work of external loadings.

Over the last decade, this approach has been extensively applied to several mechanical problems and in particular to fracture, see e.g. [FL03, DMFT05, DML10], and damage, see e.g. [MR06, TM10, Tho13].

However, in a context where other rate-dependent phenomena are present, the global stability condition is too restrictive. Following [Rou09, Rou10] we will replace it with a *semistability* condition, where the sum of energy and dissipation is minimized with respect to the internal variable z only, while the displacement $u(t)$ is kept fixed, see also [RR11, BR11, Rou13b]. Accordingly, we will weakly formulate system (1.1) by means of

- semistability,
- the (dynamic) momentum equation in a weak sense,
- a suitable energy-dissipation balance,
- the heat equation in a weak sense.

Existence result. Theorem 2.7 states the existence of energetic solutions to the initial-boundary value problem for system (1.1). For the proof we rely on a well-established method for showing existence for rate-independent processes [Mie05], adjusted to the coupling with viscosity, inertia, and temperature in [Rou10]. Although we follow the approach of the latter paper, let us point out that the results therein do not account for some properties of our model, namely,

- the unidirectionality of damage, see (1.2),
- the dependence of the viscous tensor $\mathbb{D}(z,\theta)$ on damage and temperature.

These features are important for the modeling of volume-damage, as well as for the phase-field approximation of fracture and surface damage models, see also Remark 2.3, and cause some analytical difficulties.

As in many works on rate-independent systems, our existence proof is based on time-discretization and approximation by means of solutions to incremental problems. Differently from [Rou10], in our discrete scheme the approximate flow rule is decoupled from the other two equations, which may produce more efficient numerical simulations. Moreover, the assumption of a constant heat capacity allows us to avoid a so-called enthalpy transformation and, together with the subquadratic growth of the heat conductivity, to deduce a priori estimates and the positivity of the temperature by carefully adapting the methods developed in [FPR09, RR14b].

Some remarks on the thermal properties of system (1.1) **and its applicability.** For the thermodynamical derivation of the PDE system (1.1) one may follow the thermomechanical modeling by Frémond in [Fré02, Chapter 12] or Roubíček in [Rou10]. In particular, the free energy density associated with (1.1) is given by

$$F(e(u),z,\nabla z,\theta) := \tfrac{1}{2}\mathbb{C}(z)e(u):e(u) + G(z,\nabla z) + \varphi(\theta) - \theta\,\mathbb{B}:e(u)\,, \tag{1.4}$$

which leads to the entropy density S and the internal energy density U of the form

$$\begin{aligned} S(e(u),z,\nabla z,\theta) &= -\partial_\theta F = \mathbb{B}:e(u) - \varphi'(\theta)\,,\\ U(e(u),z,\nabla z,\theta) &= F + \theta\,S = \tfrac{1}{2}\mathbb{C}(z)e(u):e(u) + G(z,\nabla z) + \varphi(\theta) - \theta\,\varphi'(\theta)\,, \end{aligned}$$

where φ is a function such that $c_{\mathrm{V}}(\theta) := \partial_\theta U = -\theta\,\varphi''(\theta)$ is the specific heat capacity, and S and U satisfy a Gibbs' relation: $\partial_\theta U = \theta\,\partial_\theta S$. Starting from the entropy equation, which balances the changes of entropy with the heat flux and the heat sources given by the dissipation rate and the external sources H,

$$\theta\,\partial_\theta S\,\dot\theta + \operatorname{div} j = \mathrm{R}_1(\dot z) + (\mathbb{D}(z,\theta)e(\dot u) - \theta\,\mathbb{B}):e(\dot u) + H\,,$$

and then invoking Fourier's law $j = -\mathbb{K}(z,\theta)\nabla\theta$ as well as the above Gibbs' relation, the choice $\varphi(\theta) = \theta(1-\log\theta)$ indeed results in the heat equation (1.1c) with $c_{\mathrm{V}}(\theta) = \text{const.} = 1$.

In fact, the temperature dependence of the heat capacity can be described by the classical Debye model, see e.g. [Wed97, Sect. 4.2, p. 761]. In a first approximation it predicts a cubic growth of c_{V} with respect to temperature up to a certain, material-specific temperature, the so-called Debye temperature θ_{D}, whereas for $\theta \gg \theta_{\mathrm{D}}$ it can be approximated by $c_{\mathrm{V}} \equiv \text{const}$. Thus, the use of (1.1c) with $c_{\mathrm{V}}(\theta) = \text{const.}$ (normalized to $c_{\mathrm{V}}(\theta) = 1$ for shorter presentation) is justified if the temperature range of application is assumed to be above Debye temperature, i.e., $\theta \gg \theta_{\mathrm{D}}$. Indeed, our main existence Theorem 2.7, see also Proposition 3.2, contains an enhanced positivity estimate, which ensures that the temperature θ, as a component of an energetic solution (u,z,θ), always stays above a tunable threshold (to be tuned to θ_{D}), provided that the initial temperature and the heat sources H are suitably large, see (2.24).

In this context, let us here also allude to our hypothesis on the heat conductivity tensor $\mathbb{K}(z,\theta)$, which is assumed to have subquadratic growth in θ, see (2.11b). According to experimental findings, cf. [Eie64, Kle12], polymers such as e.g. polymethylmethacrylate (PMMA), exhibit such a subquadratic growth of the heat conductivity. In contrast, for metals the heat conductivity is ruled by the electron thermal conductivity. For this, the Wiedemann-Franz law states a linear dependence on the temperature, cf. [CR12, Chapter 17]. Moreover, let us mention that the analytical results in [FPR09] are obtained under the assumption of superquadratic growth, which is justified by the examples on nonlinear heat conduction given in [ZR02], that are related to radiation heat conduction or electron/ion heat conduction in a plasma. Thus, in conclusion, the thermal properties of our model rather comply with polymers than with metals.

Vanishing viscosity and inertia. Finally, we address the asymptotic analysis of (1.1) for vanishing viscosity and inertia in the momentum equation. Let us point out that our analysis is substantially different from the so-called vanishing-viscosity approach to fully rate-independent systems. There, the approximation by vanishing viscosity concerns the flow rule for the internal variable, and leads to *parameterized*/BV solutions, encoding information on the energetic behavior of the system at jumps, see e.g. [EM06, MRS09, MRS12, DMDS11] and [KMZ08, LT11, KRZ13] for applications to fracture and damage.

For isothermal, rate-independent processes with dynamics, the analysis for vanishing viscosity and inertia has been addressed, for the momentum equation only, in [Rou09, Rou13a]; more recently, in [DS13, Sca14] this was done also in the flow rule, leading to an energetic-type notion of solution. However, we have no knowledge of vanishing viscosity approaches to systems including thermal effects. We will develop this in Section 5, via a suitable time rescaling technique.

For this limit passage, it will be essential to assume an appropriate scaling of the tensor of heat conduction coefficients. This reflects the fact that in the slow-loading regime heat propagates at infinite speed. Thus in the limit we will obtain that the temperature is spatially constant and its evolution is fully decoupled from the one of the mechanical variables (u, z). Indeed, the latter are *local solutions* (according to the notion introduced in [Mie11, Rou13a]) to the system consisting of the (quasistatic) momentum balance and of the rate-independent flow rule, cf. Theorem 5.3.

Plan of the paper. The assumptions on the material quantities and the statement of the existence results for energetic solutions are given in Section 2. In Section 3 we present the properties of time-discrete solutions, hence in Section 4 we prove the main theorem by passing to the time-continuous limit by variational convergence techniques. Finally, Section 5 is devoted to the asymptotics for vanishing viscosity and inertia.

2 Setup and main result

Notation: Throughout this paper, for a given Banach space X we will denote by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X, and by $\mathrm{BV}([0,T];X)$, resp. $\mathrm{C}^0_{\mathrm{weak}}([0,T];X)$, the space of the bounded variation, resp. weakly continuous, functions with values in X. Notice that we shall consider any $v \in \mathrm{BV}([0,T];X)$ to be defined *at all* $t \in [0,T]$. We also mention that the symbols c, C, C' ... will be used to denote a positive constant depending on given data, and possibly varying from line to line. Furthermore in proofs, the symbols I_i, $i = 1, \ldots$, will be place-holders for several integral terms popping up in the various estimates. We warn the reader that we will not be self-consistent with the numbering so that, for instance, the symbol I_1 will occur in several proofs with different meanings.

2.1 Assumptions

We now specify the assumptions on the domain Ω, on the nonlinear functions featured in (1.1), on the initial data, and on the loading and source terms, under which our existence result, Theorem 2.7, holds. Let us mention in advance that, in order to simplify the exposition in Sections 2–4, and in view of the analysis for vanishing viscosity and inertia in Section 5, cf. (5.33), we will suppose that the matrix of thermal expansion coefficients is a given symmetric matrix $\mathbb{B} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. We instead allow the elasticity and viscosity tensors to depend on the state variables z and (z, θ), respectively, thus we need to impose suitable growth and coercivity conditions. We will also make growth assumptions for the matrix of heat conduction coefficients, which are suited for our analysis and which are in the line of [FPR09, RR14b]. These growth conditions will play a key role in the derivation of estimates for the temperature θ, in that it will allow us to cope with the quadratic right-hand side of (1.1c).

Assumptions on the domain. We assume that

$$\begin{aligned}&\Omega \subset \mathbb{R}^d,\ d \in \{2,3\},\ \text{is a bounded domain with Lipschitz-boundary } \partial\Omega \text{ such that}\\ &\partial_{\mathrm{D}}\Omega \subset \partial\Omega \text{ is nonempty and relatively open and } \partial_{\mathrm{N}}\Omega := \partial\Omega\backslash\partial_{\mathrm{D}}\Omega\,.\end{aligned} \tag{2.1}$$

Moreover, we will use the following notation for the state spaces for u and z:

$$\begin{aligned}H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d) &:= \{v \in H^1(\Omega;\mathbb{R}^d)\colon\ v = 0 \text{ on } \partial_{\mathrm{D}}\Omega \text{ in the trace sense}\}\,,\\ \mathcal{Z} &:= \{z \in W^{1,q}(\Omega)\colon z \in [0,1] \text{ a.e. in } \Omega\}\,,\end{aligned} \tag{2.2}$$

with fixed $q > 1$, cf. (2.9d). Analogous notation will be employed for the Sobolev spaces $W^{1,\gamma}_{\mathrm{D}}$, $\gamma \geq 1$.

Assumptions on the material tensors. We require that the tensors $\mathbb{B}\in\mathbb{R}^{d\times d}$, $\mathbb{C}\colon\mathbb{R}\to\mathbb{R}^{d\times d\times d\times d}$, and $\mathbb{D}\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}^{d\times d\times d\times d}$ fulfill

$$\mathbb{B}\in\mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ and set } C_{\mathbb{B}}:=|\mathbb{B}|\,, \tag{2.3a}$$
$$\mathbb{C}\in \mathrm{C}^{0,1}(\mathbb{R};\mathbb{R}^{d\times d\times d\times d}) \text{ and } \mathbb{D}\in \mathrm{C}^{0}(\mathbb{R}\times\mathbb{R};\mathbb{R}^{d\times d\times d\times d})\,, \tag{2.3b}$$
$$\mathbb{C}(z),\,\mathbb{D}(z,\theta)\in\mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}} \text{ and are positive definite for all } z\in\mathbb{R}\,,\ \theta\in\mathbb{R}\,, \tag{2.3c}$$
$$\exists\, C^1_{\mathbb{C}},\,C^2_{\mathbb{C}}>0\ \ \forall\, z\in\mathbb{R}\ \ \forall\, A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\colon\quad C^1_{\mathbb{C}}\,|A|^2\le \mathbb{C}(z)A:A\le C^2_{\mathbb{C}}\,|A|^2\,, \tag{2.3d}$$
$$\exists\, C^1_{\mathbb{D}},\,C^2_{\mathbb{D}}>0\ \ \forall\, z\in\mathbb{R}\ \ \forall\theta\in\mathbb{R}\ \ \forall\, A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\colon\quad C^1_{\mathbb{D}}\,|A|^2\le \mathbb{D}(z\,,\theta)A:A\le C^2_{\mathbb{D}}\,|A|^2\,. \tag{2.3e}$$

In the expressions above, $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ denotes the subset of symmetric matrices in $\mathbb{R}^{d\times d}$ and $\mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$ is the subset of symmetric tensors in $\mathbb{R}^{d\times d\times d\times d}$. In particular,

$$\mathbb{C}(z)_{ijkl}{=}\mathbb{C}(z)_{jikl}{=}\mathbb{C}(z)_{ijlk}{=}\mathbb{C}(z)_{klij} \ \text{ and }\ \mathbb{D}(z,\theta)_{ijkl}{=}\mathbb{D}(z,\theta)_{jikl}{=}\mathbb{D}(z,\theta)_{ijlk}{=}\mathbb{D}(z,\theta)_{klij}\,.$$

In addition to (2.3), we impose that $\mathbb{C}(\cdot)$ is monotonically nondecreasing, i.e.,

$$\forall\, A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\ \ \forall\, 0\le z_1\le z_2\le 1\colon\quad \mathbb{C}(z_1)A:A\le\mathbb{C}(z_2)A:A\,. \tag{2.4}$$

Remark 2.1 (Square root and square of fourth order tensors)**.** Given $A,B\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\mathbb{C},\mathbb{D}\in\mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}}$, recall that

$$\mathbb{D}A:B=\sum_{i,j,k,l=1}^{d}B_{ij}\mathbb{D}_{ijkl}A_{kl}\quad\text{and}\quad \mathbb{C}\,\mathbb{D}=\Big(\sum_{m,n}\mathbb{C}_{ijmn}\mathbb{D}_{mnlk}\Big)_{i,j,k,l=1}^{d}\,. \tag{2.5}$$

Exploiting the symmetry relations (2.3c) we also observe that

$$\begin{aligned}|\mathbb{D}A|^2&=\sum_{i,j}\Big(\sum_{k,l}\mathbb{D}_{ijkl}A_{kl}\Big)^2=\sum_{i,j}\sum_{k,l}\mathbb{D}_{ijkl}A_{kl}\sum_{m,n}\mathbb{D}_{ijmn}A_{mn}=\sum_{k,l,m,n}A_{kl}A_{mn}\sum_{i,j}\mathbb{D}_{klij}\mathbb{D}_{ijmn}\\&=\mathbb{D}^2A:A\,.\end{aligned} \tag{2.6}$$

In view of (2.3d)–(2.3e) we thus obtain the following bounds for the square of $\mathbb{C}(z)$, resp. $\mathbb{D}(z,\theta)$:

$$\exists\, C^1_{\mathbb{C}}\,,\,C^2_{\mathbb{C}}>0\ \ \forall\, z\in\mathbb{R}\ \ \forall\, A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\colon\quad (C^1_{\mathbb{C}})^2\,|A|^2\le\mathbb{C}(z)^2A:A\le(C^2_{\mathbb{C}})^2\,|A|^2\,, \tag{2.7a}$$
$$\exists\, C^1_{\mathbb{D}}\,,\,C^2_{\mathbb{D}}>0\ \ \forall\, z\in\mathbb{R}\ \ \forall\theta\in\mathbb{R}\ \ \forall\, A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\colon\quad (C^1_{\mathbb{D}})^2\,|A|^2\le\mathbb{D}(z\,,\theta)^2A:A\le(C^2_{\mathbb{D}})^2\,|A|^2\,. \tag{2.7b}$$

To find these relations for the constants, we may argue as follows: Being a linear mapping on $\mathbb{R}^{d\times d}$, we can fix a notation to rewrite any tensor $A\in\mathbb{R}^{d\times d}$ as a vector of d^2 components and $\mathbb{C}(z)$, $\mathbb{D}(z,\theta)$ as $\mathbb{R}^{d^2\times d^2}$-matrices, which are symmetric and positive definite. Exploiting the spectral decomposition of these two matrices we see that the constants $C^1_{\mathbb{C}},C^2_{\mathbb{C}}$, in (2.3d), resp. $C^1_{\mathbb{D}},C^2_{\mathbb{D}}$ in (2.3e), are bounds for the smallest, resp. largest, eigenvalues of the $\mathbb{R}^{d^2\times d^2}$-matrices corresponding to $\mathbb{C}(z)$, resp. $\mathbb{D}(z,\theta)$. Following our chosen notation, this transfers to the fourth order tensors, hence (2.7).

In a similar manner, exploiting the symmetry and (uniform) positive definiteness introduced in Assumption (2.3c) as well as the spectral decomposition of the corresponding $\mathbb{R}^{d^2\times d^2}$-matrix, we may conclude the existence of the *square root* of $\mathbb{D}(z,\theta)$, i.e., for all $z,\theta\in\mathbb{R}$ there is

$$\mathbb{U}(z,\theta)\in\mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym}} \text{ positive definite, s.t. } \mathbb{D}(z,\theta)=\mathbb{U}(z,\theta)^2\,. \tag{2.8a}$$

By symmetry, with calculations similar to those performed in (2.6) we thus have

$$\mathbb{D}(z,\theta)A:A=\mathbb{U}(z,\theta)A:\mathbb{U}(z,\theta)A\,. \tag{2.8b}$$

In addition, (2.3e) for $\mathbb{D}$ implies the following for $\mathbb{U}$:

$$\exists\, C^1_{\mathbb{D}}\,,\,C^2_{\mathbb{D}}>0\ \ \forall\, z\in\mathbb{R}\ \ \forall\theta\in\mathbb{R}\ \ \forall\, A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\colon\quad \sqrt{C^1_{\mathbb{D}}}|A|^2\le\mathbb{U}(z,\theta)A:A\le\sqrt{C^2_{\mathbb{D}}}|A|^2\,. \tag{2.8c}$$

The existence of a square root for the positive definite, symmetric fourth order tensor $\mathbb{D}(z,\theta)$ is found again by exploiting the spectral properties of the corresponding $\mathbb{R}^{d^2\times d^2}$-matrix. After diagonal transform, for this matrix the entries of its square root matrix are found by taking the square root of the eigenvalues. This also yields (2.8c), since, as already mentioned, the constants $C^1_{\mathbb{D}}, C^2_{\mathbb{D}}$ in (2.3e) are bounds for the smallest, resp. largest, eigenvalue of $\mathbb{D}(z,\theta)$.

Analogously, thanks to (2.3c), the square root $\mathbb{V}(z) \in \mathbb{R}^{d\times d\times d\times d}$ of the tensor $\mathbb{C}(z)$ is well-defined for all $z \in \mathbb{R}$ and it fulfills $\mathbb{C}(z)A : A = \mathbb{V}(z)A : \mathbb{V}(z)A$ and the analogue of (2.8c).

Assumptions on the damage regularization. We require that $G\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ fulfills

Indicator: For every $(z,\xi) \in \mathbb{R}\times\mathbb{R}^d$: $\quad G(z,\xi) < \infty \;\Rightarrow\; z \in [0,1]\,;$ (2.9a)

Continuity: G is continuous on its domain $\mathrm{dom}(G)$ and $G(0,0) = 0\,;$ (2.9b)

Convexity: For every $z \in \mathbb{R}$, $G(z,\cdot)$ is convex; (2.9c)

Growth: There exist constants $q > 1$ and $C^1_G, C^2_G > 0$ such that for every $(z,\xi) \in \mathrm{dom}(G)$

$$C^1_G(|\xi|^q - 1) \le G(z,\xi) \le C^2_G(|\xi|^q + 1)\,. \tag{2.9d}$$

Remark 2.2 (Properties of the regularizing term)**.** Since we are encompassing the feature that $z(\cdot,x)$ is decreasing for almost all $x \in \Omega$, starting from an initial datum $z_0 \in [0,1]$ a.e. in Ω, the z-component of any energetic solution to (1.1) will fulfill $z(t,x) \le 1$ a.e. in Ω. Therefore, we could weaken (2.9a) and just require that the domain of G is a subset of $[0,\infty)$.

Furthermore, we may require the second of (2.9b) without loss of generality, since adding a constant to G shall not affect our analysis.

Further observe that the above assumptions (2.9) ensure that the integral functional

$$\mathcal{G}\colon L^r(\Omega)\times L^q(\Omega;\mathbb{R}^d) \to \mathbb{R}\cup\{\infty\}\,,\quad \mathcal{G}(z,\xi) := \int_\Omega G(z,\xi)\,\mathrm{d}x \tag{2.10}$$

is lower semicontinuous with respect to strong convergence in $L^r(\Omega)$ for any $r \in [1,\infty)$ and weak convergence in $L^q(\Omega;\mathbb{R}^d)$, cf. e.g. [FL07, Theorem 7.5, p. 492]. In addition, $\mathcal{G}$ is continuous with respect to strong convergence in $\big(L^r(\Omega)\times L^q(\Omega;\mathbb{R}^d)\big)\cap\mathrm{dom}(G)$.

Remark 2.3 (Example: Phase-field approximation of fracture)**.** Starting from the work of Ambrosio and Tortorelli [AT90], gradient damage models have been extensively used in recent years to predict crack propagation in brittle or quasi-brittle materials, by means of phase-field approximation [BFM08]. In this approach, a sharp crack is regularized by defining an internal variable that interpolates continuously between sound and fractured material. In the mathematical literature, evolutionary problems for phase-field models were considered for instance in the fully quasistatic case [Gia05], in viscoelasticity as a gradient flow [BM14], and in dynamics [LOS10], always for isothermal systems. A thermodynamical model for regularized fracture with inertia was proposed and treated numerically e.g. in [MWH10]. The passage to the limit from phase-field to sharp crack, though successfully treated in the quasistatic [Gia05] and in the viscous case [BM14], is by now an open problem in dynamics and is outside the scope of this contribution.

In this context, typical examples for the regularizing term are functionals of Modica-Mortola type,

$$\mathcal{G}^q_{\mathrm{MM}}(z,\nabla z) = \int_\Omega G^q_{\mathrm{MM}}(z,\nabla z)\,\mathrm{d}x \qquad \text{with } G^q_{\mathrm{MM}}(z,\nabla z) := |\nabla z|^q + W(z) + I_{[0,1]}(z)\,,$$

where $q > 1$, W is a suitable potential, and $I_{[0,1]}(z) := 0$ if $z \in [0,1]$, $I_{[0,1]}(z) := +\infty$ otherwise. Such regularization agrees with the above assumptions up to an additive constant.

Notice that in Section 3, to construct discrete solutions, we will consider unilateral minimum problems of the type

$$\min_{z\in\mathcal{Z}}\left\{\int_\Omega \tfrac{1}{2}\mathbb{C}(z)e(u) : e(u)\,\mathrm{d}x + \int_\Omega G(z,\nabla z)\,\mathrm{d}x + \mathcal{R}_1(z-\bar{z})\right\}$$

for given $u \in H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)$ and a given $\bar{z} \in \mathcal{Z}$ defined in (2.2). Setting $\mathbb{C}(z) := (z^2+\delta)\, I$ with $\delta > 0$, and $G := G^2_{\mathrm{MM}}$ with $W(z) := \frac{1}{2}(1+z^2)$, the minimum problem is equivalent to

$$\min_{0\leq z\leq \bar{z}} \left\{ \int_\Omega (\tfrac{1}{2}(z^2+\delta)\,|e(u)|^2 \,\mathrm{d}x + \int_\Omega \tfrac{1}{2}(1-z)^2 \,\mathrm{d}x + \int_\Omega |\nabla z|^2 \,\mathrm{d}x \right\},$$

that is the classical minimization of the Ambrosio-Tortorelli functional, see [AT90, Gia05]. The generalization to $G = G^q_{\mathrm{MM}}$ with $q > 1$ was considered in [Iur13]. In this case one may want an effective dependence of the viscous tensor on z, choosing $\mathbb{D}(z,\theta) = \mathbb{C}(z)$ as in [LOS10].

Assumptions on the heat conductivity. On $\mathbb{K} \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{d\times d}$ we assume that

$$\mathbb{K} \in \mathrm{C}^0(\mathbb{R}\times\mathbb{R};\mathbb{R}^{d\times d}), \quad \mathbb{K}(z,\theta) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \text{ for all } z \in \mathbb{R},\ \theta \in \mathbb{R}, \tag{2.11a}$$

$$\exists\, \kappa \in (1,\kappa_d)\ \ \exists c_1,\, c_2 > 0\ \ \forall\, (z,\theta) \in \mathbb{R}\times\mathbb{R}\ \ \forall \xi \in \mathbb{R}^d: \quad \begin{cases} c_1(|\theta|^\kappa+1)|\xi|^2 \leq \mathbb{K}(z,\theta)\xi\cdot\xi, \\ |\mathbb{K}(z,\theta)| \leq c_2(|\theta|^\kappa+1), \end{cases} \tag{2.11b}$$

where $\kappa_d = 5/3$ for $d{=}3$ and $\kappa_d = 2$ for $d{=}2$.

The bound κ_d essentially comes into play in the derivation of the *Fifth a priori estimate* (cf. the proof of Proposition 3.4), and when passing from time-discrete to continuous in the heat equation, cf. Proposition 4.8. Essentially, it arises as a consequence of the enhanced integrability of the approximating temperature variables obtained by interpolation in (3.36k).

Assumptions on the initial data. We impose that

$$u_0 \in H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d), \quad \dot{u}_0 \in L^2(\Omega;\mathbb{R}^d), \quad z_0 \in \mathcal{Z}, \tag{2.12a}$$

$$\theta_0 \in L^1(\Omega), \quad \text{and } \theta_0 \geq \theta_* > 0 \ \text{ a.e. in } \Omega, \tag{2.12b}$$

where the state spaces $H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)$ and $\mathcal{Z}$ are defined in (2.2).

Assumptions on the loading and source terms. On the data f_{V}, f_{S}, H, and h we require that

$$f_{\mathrm{V}} \in H^1(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*), \qquad f_{\mathrm{S}} \in H^1(0,T;L^2(\partial_{\mathrm{N}}\Omega;\mathbb{R}^d)), \tag{2.13a}$$

$$\begin{aligned} &H \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)^*), \quad H \geq 0 \text{ a.e. in } (0,T)\times\Omega, \\ &h \in L^1(0,T;L^2(\partial\Omega)), \quad h \geq 0 \text{ a.e. in } (0,T)\times\partial\Omega, \end{aligned} \tag{2.13b}$$

For later convenience, we also introduce $f \colon [0,T] \to H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*$ defined by

$$\langle f(t), v\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} := \langle f_{\mathrm{V}}(t), v\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} + \int_{\partial_{\mathrm{N}}\Omega} f_{\mathrm{S}}\cdot v \,\mathrm{d}\mathcal{H}^{d-1}(x) \qquad \text{for all } v \in H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d). \tag{2.14}$$

It follows from (2.13a) that $f \in H^1(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*)$.

2.2 Weak formulation and main existence result

As already mentioned, following [Rou10], the *energetic* formulation of (the initial-boundary value problem associated with) system (1.1) consists of the variational formulation of the momentum and of the heat equations (1.1a) and (1.1c), with suitable test functions, and of a semistability condition joint with a *mechanical energy* balance, providing the weak formulation of the damage equation (1.1b). The latter relations feature the mechanical (quasistatic) energy associated with (1.1), i.e.,

$$\mathcal{E}(t,u,z) := \int_\Omega (\tfrac{1}{2}\mathbb{C}(z)e(u) : e(u) + G(z,\nabla z)) \,\mathrm{d}x - \langle f(t), u\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}, \tag{2.15}$$

as well as the rate-independent dissipation potential, given as the integrated version of (1.2)

$$\mathcal{R}_1(\dot z) := \int_\Omega \mathrm{R}_1(\dot z)\,\mathrm{d}x\,. \tag{2.16}$$

The choice of the test functions for the weak momentum equation reflects the regularity (2.17a) required for u, which in turn will derive from the standard energy estimates that can be performed on system (1.1). As we will see, such estimates only yield $\theta \in L^\infty(0,T;L^1(\Omega))$. In fact, the further regularity (2.17c) for θ shall result from a careful choice of test functions for the time-discrete version of (1.1c), and from refined interpolation arguments, drawn from [FPR09]. Finally, the $\mathrm{BV}([0,T];W^{2,d+\delta}(\Omega)^*)$-regularity for θ follows from a comparison argument. The choice of the test functions in (2.19d) is the natural one in view of (2.17).

Definition 2.4 (Energetic solution)**.** Given a quadruple of initial data $(u_0,\dot u_0,z_0,\theta_0)$ satisfying (2.12), we call a triple (u,z,θ) an *energetic solution* of the Cauchy problem for the PDE system (1.1) complemented with the boundary conditions (1.3) if

$$u \in H^1(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))\,, \tag{2.17a}$$

$$\begin{aligned} & z \in L^\infty(0,T;W^{1,q}(\Omega)) \cap L^\infty((0,T)\times\Omega) \cap \mathrm{BV}([0,T];L^1(\Omega))\,,\\ & z(t,x) \in [0,1] \text{ for a.a. } (t,x) \in (0,T)\times\Omega\,, \end{aligned} \tag{2.17b}$$

$$\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{2,d+\delta}(\Omega)^*)\,, \tag{2.17c}$$

such that the triple (u,z,θ) complies with the initial conditions

$$u(0)=u_0\,, \quad \dot u(0) = \dot u_0\,, \quad z(0)=z_0\,, \quad \theta(0)=\theta_0 \quad \text{a.e. in } \Omega\,, \tag{2.18}$$

and with the following properties:

- *unidirectionality*: for a.a. $x\in\Omega$, the function $z(\cdot,x)\colon [0,T]\to[0,1]$ is nonincreasing;
- *semistability*: for every $t\in[0,T]$

$$\forall\,\tilde z \in \mathcal{Z}: \quad \mathcal{E}(t,u(t),z(t)) \le \mathcal{E}(t,u(t),\tilde z) + \mathcal{R}_1(\tilde z - z(t))\,, \tag{2.19a}$$

  where $\mathcal{Z}$ is defined in (2.2);
- *weak formulation of the momentum equation*: for all $t\in[0,T]$

$$\begin{aligned} & \rho\int_\Omega \dot u(t)\cdot v(t)\,\mathrm{d}x - \rho\int_0^t\int_\Omega \dot u\cdot\dot v\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega (\mathbb{D}(z,\theta)e(\dot u) + \mathbb{C}(z)e(u) - \theta\,\mathbb{B}) : e(v)\,\mathrm{d}x\,\mathrm{d}s \\ & = \rho\int_\Omega \dot u_0\cdot v(0)\,\mathrm{d}x + \int_0^t \langle f, v\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}s \end{aligned} \tag{2.19b}$$

  for all test functions $v \in L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)) \cap W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^d))$;
- *mechanical energy equality*: for all $t\in[0,T]$

$$\begin{aligned} & \tfrac{\rho}{2}\int_\Omega |\dot u(t)|^2\,\mathrm{d}x + \mathcal{E}(t,u(t),z(t)) + \int_\Omega (z_0 - z(t))\,\mathrm{d}x + \int_0^t\int_\Omega (\mathbb{D}(z,\theta)e(\dot u) - \theta\,\mathbb{B}) : e(\dot u)\,\mathrm{d}x\,\mathrm{d}s \\ & = \tfrac{\rho}{2}\int_\Omega |\dot u_0|^2\,\mathrm{d}x + \mathcal{E}(0,u_0,z_0) + \int_0^t \partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s\,, \end{aligned} \tag{2.19c}$$

  where $\partial_t\mathcal{E}(t,u,z) = -\left\langle \dot f(t), u\right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}$;
- *weak formulation of the heat equation*: for all $t\in[0,T]$

$$\begin{aligned} & \langle\theta(t),\eta(t)\rangle_{W^{2,d+\delta}(\Omega)} - \int_0^t\int_\Omega \theta\,\dot\eta\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \mathbb{K}(\theta,z)\nabla\theta\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}s \\ & = \int_\Omega \theta_0\,\eta(0)\,\mathrm{d}x + \int_0^t\int_\Omega \eta\,|\dot z|\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega (\mathbb{D}(z,\theta)e(\dot u) : e(\dot u) - \theta\,\mathbb{B}) : e(\dot u)\eta\,\mathrm{d}x\,\mathrm{d}s \\ & \quad + \int_0^t\int_{\partial\Omega} h\eta\,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}s + \int_0^t\int_\Omega H\eta\,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{2.19d}$$

for all test functions $\eta \in H^1(0,T;L^2(\Omega)) \cap \mathrm{C}^0([0,T];W^{2,d+\delta}(\Omega))$, for some fixed $\delta > 0$. Here and in what follows, $|\dot z|$ denotes the total variation measure of z (i.e., the heat produced by the rate-independent dissipation), which is defined on every closed set of the form $A := [t_1,t_2]\times C \subset [0,T]\times\overline{\Omega}$ by

$$|\dot z|\,(A) := \int_C \mathrm{R}_1(z(t_2)-z(t_1))\,\mathrm{d}x\,,$$

and, for simplicity, we shall write $\int_0^t\int_\Omega \eta\,|\dot z|\,\mathrm{d}x\,\mathrm{d}s$ instead of $\iint_{(0,t)\times\Omega}\eta\,|\dot z|\,(\,\mathrm{d}s\,\mathrm{d}x)$.

Remark 2.5 (Total energy balance)**.** Testing the weak momentum balance (2.19b) by $\dot u$, cf. Remark 2.6, and the weak heat equation by $\eta \equiv 1$, and summing up, yields the total energy balance

$$\begin{aligned}\int_\Omega \tfrac{\rho}{2}|\dot u(t)|^2\,\mathrm{d}x + \mathcal{E}(t,u(t),z(t)) + \int_\Omega \theta(t)\,\mathrm{d}x = \int_\Omega \tfrac{\rho}{2}|\dot u_0|^2\,\mathrm{d}x + \mathcal{E}(0,u_0,z_0) + \int_\Omega \theta_0\,\mathrm{d}x \\ + \int_0^t \partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s + \int_0^t\int_\Omega H\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega} h\,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}s\,.\end{aligned} \tag{2.20}$$

Remark 2.6 (Improved regularity on $\ddot u$)**.** From the definition of energetic solution we can gain improved regularity for the time derivatives of the displacement. Indeed, let (u,z,θ) be as in (2.17) and such that the weak momentum equation (2.19b) holds. Then (1.1a) holds in the sense of distributions and

$$\rho\,\|\ddot u\|_{L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*)} = \sup_{\|v\|\le 1}\int_0^T\int_\Omega (\mathbb{D}(z,\theta)e(\dot u) + \mathbb{C}(z)e(u) - \theta\,\mathbb{B}) : e(v)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle f,v\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t\,,$$

where the supremum is taken over all functions such that $\|v\|_{L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))} \le 1$. The left-hand side of the previous equality is uniformly bounded thanks to (2.3), (2.14), and (2.17), thus we deduce that $\ddot u \in L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*)$. Since the spaces $H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d) \subset L^2(\Omega;\mathbb{R}^d) \subset H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*$ form a Gelfand triple, in view of e.g. [LM72, Chap. 1, Sec. 2.4, Prop. 2.2], we conclude that

$$\begin{aligned}&\int_{t_1}^{t_2}\langle \ddot u,\dot u\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t \\ &= \tfrac12\langle \dot u(t_2),\dot u(t_2)\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} - \tfrac12\langle \dot u(t_1),\dot u(t_1)\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} = \tfrac12\|\dot u(t_2)\|^2_{L^2(\Omega;\mathbb{R}^d)} - \tfrac12\|\dot u(t_1)\|^2_{L^2(\Omega;\mathbb{R}^d)}\end{aligned} \tag{2.21}$$

for every $t_1,t_2 \in [0,T]$. Hence, $\dot u$ can be used as a test function in (2.19b).

We are now in the position to state the main result of this paper. The last part of the assertion concerns the strict positivity of the absolute temperature θ. In particular, under (2.23) below we are able to specify, in terms of the given data, the constant which bounds θ from below.

Theorem 2.7 (Existence of energetic solutions)**.** *Under assumptions* (2.1)–(2.4)*,* (2.9)*, and* (2.11)*, and* (2.13) *on the data* f_{V}*,* f_{S}*,* H*, and* h*, for every quadruple* $(u_0,\dot u_0,z_0,\theta_0)$ *fulfilling* (2.12) *with* z_0 *satisfying* (2.19a)*, there exists an energetic solution* (u,z,θ) *to the Cauchy problem for system* (1.1)*.*

Moreover, there exists $\widetilde\theta > 0$ *such that*

$$\theta(t,x) \ge \widetilde\theta > 0 \qquad \text{for a.a. } (t,x)\in(0,T)\times\Omega\,. \tag{2.22}$$

Furthermore, if in addition

$$\exists\,H_* > 0\colon\; H(t,x) \ge H_* \text{ for a.a. } (t,x)\in(0,T)\times\Omega \;\text{ and }\; \theta_0(x) \ge \sqrt{H_*/\bar c}\; \text{ for a.a. } x\in\Omega\,, \tag{2.23}$$

where $\bar c := \frac{(C_{\mathbb{B}})^2}{2C^1_{\mathbb{D}}}$*, then*

$$\theta(t,x) \ge \max\left\{\widetilde\theta,\sqrt{H_*/\bar c}\right\} \quad \text{for a.a. } (t,x)\in(0,T)\times\Omega\,. \tag{2.24}$$

The proof of Theorem 2.7 will be developed in Sections 3 and 4 by time-discretization (see Propositions 4.1–4.2).

Remark 2.8 (Time-dependent Dirichlet loadings)**.** The existence of energetic solutions can be proven also when *time-dependent Dirichlet loadings* are considered for the displacement u instead of the homogeneous Dirichlet condition (1.3), in the case the viscous tensor $\mathbb{D}$ is *independent* of z and θ. This restriction is due to technical reasons, related to the derivation of suitable estimates for the approximate solutions to (1.1).

An alternative damage model, that still features a (z,θ)-dependence of $\mathbb{D}$, is discussed in [LRTT14b], where a time-dependent loading for u can be encompassed in the analysis, albeit under suitable stronger conditions.

Remark 2.9 (Failure of "entropic" solutions)**.** As already mentioned, the regularity for the temperature $\theta \in L^2(0,T;H^1(\Omega)) \cap \mathrm{BV}([0,T];W^{2,d+\delta}(\Omega)^*)$ results from careful estimates on the heat equation (1.1c), tailored on the quadratic character of its right-hand side and drawn from [FPR09]. There, the analysis of the *full system* for phase transitions proposed by Frémond [Fré02], featuring a heat equation with an L^1 right-hand side, was carried out.

The techniques from [FPR09] have been recently extended in [RR14b] to analyze a model for *rate-dependent* damage in thermo-viscoelasticity. Namely, in place of the 1-homogeneous dissipation potential R_1 from (1.2), the flow rule for the damage parameter in [RR14b] features the quadratic dissipation $\mathrm{R}_2(\dot z) = \frac12 |\dot z|^2$ if $\dot z \leq 0$, and $\mathrm{R}_2(\dot z) = \infty$ else. Consequently, the heat equation in [RR14b] is of the type

$$\dot\theta - \mathrm{div}\,(\mathbb{K}(z,\theta)\nabla\theta) = |\dot z|^2 + \mathbb{D}(z)e(\dot u) : e(\dot u) - \theta\,\mathbb{B} : e(\dot u) + H \qquad \text{in } (0,T)\times\Omega\,. \tag{2.25}$$

In [RR14b], under a weaker growth condition on $\mathbb{K}$ than the present (2.11), it was possible to prove an existence result for a weaker formulation of (2.25), consisting of an entropy inequality and of a total energy inequality. The resulting notion of "entropic" solution, originally proposed in [FPR09], indeed reflects the strict positivity of the temperature, and the fact that the entropy increases along solutions. Without going into details, let us mention that this entropy inequality is (formally) obtained by testing (2.25) by $\varphi\,\theta^{-1}$, with φ a smooth test function, and integrating in time. This procedure is fully justified because θ can be shown to be bounded away from zero by a positive constant, hence $\varphi(t)\,\theta^{-1}(t) \in L^\infty(\Omega)$ for almost all $t \in (0,T)$, and the integrals $\int_0^T \int_\Omega |\dot z|^2 \varphi\,\theta^{-1}\,\mathrm{d}x\,\mathrm{d}t$ and $\int_0^T \int_\Omega \mathbb{D}(z)e(\dot u) : e(\dot u)\varphi\,\theta^{-1}\,\mathrm{d}x\,\mathrm{d}t$ resulting from the first and second terms on the right-hand side of (2.25) are well-defined.

In the present *rate-independent* context, proving an existence result for the entropic formulation of (1.1c) seems to be out of reach. Indeed, in such formulation the term $\int_0^T \int_\Omega |\dot z|^2 \varphi\,\theta^{-1}\,\mathrm{d}x\,\mathrm{d}t$ would have to be replaced by $\int_{(0,T)\times\Omega} \varphi\,\theta^{-1} |\dot z|(\,\mathrm{d}x\,\mathrm{d}t)$, with $|\dot z|$ the total variation measure of z, cf. (2.19d), but the above integral is not well defined since $\varphi\,\theta^{-1}$ is not a continuous function.

3 Time-discretization

3.1 The time-discrete scheme

Given a partition

$$0 = t_n^0 < \cdots < t_n^n = T \qquad \text{with} \qquad t_n^k - t_n^{k-1} = \tfrac{T}{n} =: \tau_n\,,$$

we construct a family of discrete solutions $(u_n^k, z_n^k, \theta_n^k)_{k=1,\ldots,n}$ by solving recursively the time-discretization scheme (3.4) below, where the data f, H, and h are approximated by *local means* as follows

$$f_n^k := \tfrac{1}{\tau_n} \int_{t_n^{k-1}}^{t_n^k} f(s)\,\mathrm{d}s\,, \qquad H_n^k := \tfrac{1}{\tau_n} \int_{t_n^{k-1}}^{t_n^k} H(s)\,\mathrm{d}s\,, \qquad h_n^k := \tfrac{1}{\tau_n} \int_{t_n^{k-1}}^{t_n^k} h(s)\,\mathrm{d}s\,, \tag{3.1}$$

and the above integrals need to be understood in the Bochner sense.

Let us mention in advance that we have to add the regularizing term $-\tau_n\,\mathrm{div}\,(|e(u_n^k)|^{\gamma-2}e(u_n^k))$ in the discrete momentum equation, with $\gamma > 4$. As it will become evident in the proof of Proposition 3.2, the reason for this is that we need to compensate the quadratic term in $e(u_n^k)$ on the right-hand side of the discrete heat equation (3.4c). Because of this regularization, it will be necessary to further approximate the initial datum u_0 from (2.12a) by a sequence (cf. [Bur98, p. 56, Corollary 2])

$$(u_n^0)_n \subset W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d) \quad \text{such that } u_n^0 \to u_0 \quad \text{in } H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d) \text{ as } n \to \infty\,, \tag{3.2}$$

where $W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d) = \{v \in W^{1,\gamma}(\Omega;\mathbb{R}^d)\colon v = 0 \text{ on } \partial_{\mathrm{D}}\Omega \text{ in the trace sense}\}$.

For the weak formulation of the discrete heat equation, we also need to introduce the function space appropriate for θ, dependent on a given $\bar z \in L^\infty(\Omega)$

$$X_{\bar z} := \Big\{\vartheta \in H^1(\Omega)\colon \int_\Omega \mathbb{K}(\bar z,\vartheta)\nabla\vartheta\cdot\nabla v \,\mathrm{d}x \text{ is well defined for all } v \in H^1(\Omega)\Big\}\,. \tag{3.3}$$

We consider the following weakly-coupled discretization scheme (in fact, only the momentum and the heat equation are coupled, while the discrete equation for z is decoupled from them):

Problem 3.1. *Starting from*

$$u_n^0\,, \quad z_n^0 := z_0\,, \quad \theta_n^0 := \theta_0\,,$$

and setting $u_n^{-1} := u_n^0 - \tau_n \dot u_0$, *find* $(u_n^k, z_n^k, \theta_n^k)_{k=1}^n \subset W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d) \times W^{1,q}(\Omega) \times X_{z_n^k}$ *such that the following hold:*

- *Minimality of* z_n^k*:*

$$z_n^k \in \operatorname{argmin}\big\{\mathcal{R}_1(z - z_n^{k-1}) + \mathcal{E}(t_n^k, u_n^{k-1}, z)\colon z \in \mathcal{Z}\big\}\;; \tag{3.4a}$$

- *Time-discrete weak formulation of the coupled momentum balance and the heat equation: Find* $u_n^k \in W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)$ *and* $\theta_n^k \in X_{z_n^k}$ *such that*

$$\begin{aligned}
&\rho\int_\Omega \tfrac{u_n^k - 2u_n^{k-1} + u_n^{k-2}}{\tau_n^2}\cdot v\,\mathrm{d}x \\
&\quad + \int_\Omega \Big(\mathbb{D}(z_n^{k-1},\theta_n^{k-1})\, e\Big(\tfrac{u_n^k - u_n^{k-1}}{\tau_n}\Big) + \mathbb{C}(z_n^k)e(u_n^k) - \theta_n^k\,\mathbb{B} + \tau_n|e(u_n^k)|^{\gamma-2}e(u_n^k)\Big) : e(v)\,\mathrm{d}x \\
&= \big\langle f_n^k, v\big\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \qquad\qquad \textit{for all } v \in W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)\,,
\end{aligned} \tag{3.4b}$$

$$\begin{aligned}
&\int_\Omega \tfrac{\theta_n^k - \theta_n^{k-1}}{\tau_n}\eta\,\mathrm{d}x + \int_\Omega \mathbb{K}(z_n^k,\theta_n^k)\nabla\theta_n^k\cdot\nabla\eta\,\mathrm{d}x \\
&= \int_\Omega \tfrac{z_n^{k-1} - z_n^k}{\tau_n}\,\eta\,\mathrm{d}x + \int_\Omega \Big(\mathbb{D}(z_n^{k-1},\theta_n^{k-1})\, e\Big(\tfrac{u_n^k - u_n^{k-1}}{\tau_n}\Big) - \theta_n^k\,\mathbb{B}\Big) : e\Big(\tfrac{u_n^k - u_n^{k-1}}{\tau_n}\Big)\,\eta\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} h_n^k\eta\,\mathrm{d}\mathcal{H}^{d-1}(x) + \big\langle H_n^k, \eta\big\rangle_{H^1(\Omega)} \qquad\qquad \textit{for all } \eta \in H^1(\Omega)\,.
\end{aligned} \tag{3.4c}$$

The above time-discrete problem has been carefully designed in such a way as to be weakly-coupled in that, for each $k \in \{1,\dots,n\}$, it can be solved successively starting from (3.4a) and then solving the system (3.4b)–(3.4c). See [RR14b, Remark 4.3] for similar ideas.

Our existence result for Problem 3.1 reads:

Proposition 3.2. *Let the assumptions of Theorem 2.7 hold true. Then there exists a solution*

$$(u_n^k, z_n^k, \theta_n^k)_{k=1}^n \subset W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d) \times W^{1,q}(\Omega) \times H^1(\Omega)$$

to Problem 3.1.

Moreover, every solution satisfies the following properties: There exists $\widetilde\theta > 0$ *such that*

$$\theta_n^k \geq \widetilde\theta > 0 \qquad \textit{for all } k = 1,\dots,n\,, \quad \textit{for all } n \in \mathbb{N}\,. \tag{3.5}$$

Furthermore, if in addition (2.23) holds, then

$$\theta_n^k \geq \max\Big\{\widetilde\theta, \sqrt{H_*/\bar c}\Big\} > 0 \qquad \textit{for all } k = 1,\dots,n\,, \quad \textit{for all } n \in \mathbb{N}\,, \tag{3.6}$$

with H_* *and* $\bar c$ *from (2.23).*

While the existence of solutions for (3.4a) follows from the direct method of the calculus of variations in a straightforward manner, the existence proof for system (3.4b)–(3.4c) is more involved, due to the quasilinear character of the discrete heat equation. This is due to the fact that the viscous dissipation $\mathbb{D}(z_n^{k-1},\theta_n^{k-1})e\big(\frac{u_n^k-u_n^{k-1}}{\tau_n}\big):e\big(\frac{u_n^k-u_n^{k-1}}{\tau_n}\big)$ as well as the thermal stresses $\theta_n^{k-1}\,\mathbb{B}:e\big(\frac{u_n^k-u_n^{k-1}}{\tau_n}\big)$ only happen to be of L^1-regularity as a consequence of (3.4b). Observe in particular that $\mathbb{C}(z_n^k),\mathbb{D}(z_n^{k-1},\theta_n^{k-1})\in\big(L^\infty(\Omega)\cap W^{1,q}(\Omega)\big)^{d\times d\times d\times d}$, and we do not impose the assumption $q>d$, which would guarantee the continuity of the coefficients. As it is demonstrated by the counterexample in [NŠ76], in absence of continuous coefficients, it is not ensured that the solution of (3.4b) enjoys elliptic regularity. Because of this expected lack of additional regularity the existence of solutions for the coupled system (3.4b)–(3.4c) will be verified by means of an approximation procedure, in which the L^1 right-hand side in (3.4c) is replaced by a sequence of truncations. For this we proceed along the lines of [RR14b] where the analysis of a time-discrete system analogous to (3.4a)–(3.4c) was carried out. The existence of solutions to the approximate discrete system in turn follows from an existence result for a wide class of elliptic equations, in the framework of the Leray-Schauder theory of pseudo-monotone operators. We will then conclude the existence of solutions to (3.4b)–(3.4c) by passing to the limit with the truncation parameter. In such a step, we shall exploit the strict positivity of the approximate discrete temperatures, cf. (3.20) below. This property and the convergence of the approximate discrete temperatures clearly imply the strict positivity (3.5). Nonetheless, we shall prove it by arguing directly on the non-truncated discrete heat equation (3.4c), in order to make the structure of our argument more transparent and highlight that (3.5) holds for *all* discrete solutions to Problem 3.1, cf. (3.11). We will proceed similarly for the enhanced property (3.6) which, unlike (3.11), in fact provides a tunable threshold from below to the discrete temperatures.

In the forthcoming proof, we will use that for any *convex* (differentiable) function $\psi:\mathbb{R}\to(-\infty,+\infty]$

$$\psi(x)-\psi(y)\le\psi'(x)(x-y)\quad\text{for all } x,y\in\mathrm{dom}(\psi)\,. \tag{3.7}$$

Proof. **Existence of a minimizer to** (3.4a)**:** We first verify the coercivity of the functional $z\mapsto\mathcal{E}(t_n^k,u_n^{k-1},z)+\mathcal{R}_1(z-z_n^{k-1})\colon W^{1,q}(\Omega)\to\mathbb{R}\cup\{\infty\}$, where $\mathcal{R}_1$ is the dissipation potential (2.16). Indeed, by the positivity of $\mathcal{R}_1(\cdot)$ and assumption (2.9d) on the density G we have

$$\mathcal{E}(t_n^k,u_n^{k-1},z)+\mathcal{R}_1(z-z_n^{k-1})\ge\int_\Omega G(z,\nabla z)\,\mathrm{d}x-C\ge C_G^1\|z\|_{W^{1,q}(\Omega)}^q-C_G^1\mathcal{L}^d(\Omega)-C\,,$$

where we also used that $G(z(x),\nabla z(x))<\infty$ implies $z(x)\in[0,1]$, cf. (2.9a). By the convexity and the continuity assumptions (2.9b)–(2.9c) on G and by the properties of $\mathcal{R}_1$ we conclude that the functional

$$\mathcal{E}(t_n^k,u_n^{k-1},\cdot)+\mathcal{R}_1((\cdot)-z_n^{k-1})\colon W^{1,q}(\Omega)\to\mathbb{R}\cup\{\infty\}$$

is weakly sequentially lower semicontinuous. Since $\mathcal{Z}=\{z\in W^{1,q}(\Omega)\colon z\in[0,1]\text{ a.e. in }\Omega\}$, see (2.2), is a closed subset of a reflexive Banach space, the direct method of the calculus of variations ensures the existence of a minimizer $z_n^k\in\mathcal{Z}$.

Positivity of the discrete temperature, ad (3.5)**:** We prove that every solution satisfies (3.5), arguing by induction. Clearly, for $k=0$ the strict positivity (3.5) holds with $\widetilde{\theta}=\theta_*$, thanks to (2.12b). At the step k, following the lines of [RR14b, proof of Lemma 4.4] we develop a comparison argument drawn from [FPR09]. In this context, we will use the following estimate

$$\mathbb{D}(\bar z,\bar\theta)\bar e:\bar e-\bar\theta\,\mathbb{B}:\bar e\ge C_{\mathbb{D}}^1|\bar e|^2-|\bar e|C_{\mathbb{B}}|\bar\theta|\ge\tfrac{C_{\mathbb{D}}^1}{2}|\bar e|^2-\tfrac{(C_{\mathbb{B}})^2}{2C_{\mathbb{D}}^1}|\bar\theta|^2\,. \tag{3.8}$$

Exploiting (3.8) and also using the positivity (2.13b) of the data H and h and of θ_n^{k-1}, we deduce from (3.4c) that θ_n^k fulfills

$$\int_\Omega\theta_n^k\eta\,\mathrm{d}x+\tau_n\int_\Omega\mathbb{K}(z_n^k,\theta_n^k)\nabla\theta_n^k\cdot\nabla\eta\,\mathrm{d}x\ge\int_\Omega\theta_n^{k-1}\eta\,\mathrm{d}x-\tau_n\bar c\int_\Omega\big(\theta_n^k\big)^2\eta\,\mathrm{d}x \tag{3.9}$$

for all $\eta \in L^\infty(\Omega)$ with $\eta \geq 0$ a.e. in Ω, with the constant $\bar{c} = \frac{(C_{\mathbb{B}})^2}{2C_{\mathbb{D}}^1}$ independent of k. Hence, we compare θ_n^k with the solution $v_k \in \mathbb{R}$ of the finite difference equation

$$v_k = v_{k-1} - \tau_n \bar{c}\, v_k^2, \quad k = 1, \ldots, n, \qquad \text{with } v_0 := \theta_* > 0\,. \tag{3.10}$$

By testing the difference of (3.9) and (3.10) by a suitable function in $L^\infty(\Omega)$ in the very same way as in [RR14b, proof of Lemma 4.4], to which we refer for all details, it is possible to show for all $k = 1, \ldots, n$ that $\theta_n^k(x) \geq v_k$ for almost all $x \in \Omega$, and that

$$v_k \geq \widetilde{\theta} := \left(\bar{c}T + \frac{1}{\theta_*}\right)^{-1}. \tag{3.11}$$

Therefore, (3.5) ensues.

Refined positivity estimate for the discrete temperature, ad (3.6)**:** Under the additional strict positivity (2.23) of H, arguing as in the above lines we infer that θ_n^k fulfills

$$\int_\Omega \theta_n^k \eta \, \mathrm{d}x + \tau_n \int_\Omega \mathbb{K}(z_n^k, \theta_n^k) \nabla \theta_n^k \cdot \nabla \eta \, \mathrm{d}x \geq \int_\Omega \theta_n^{k-1} \eta \, \mathrm{d}x + \int_\Omega \tau_n \big(H_* - \bar{c}\,\big(\theta_n^k\big)^2\big) \eta \, \mathrm{d}x \tag{3.12}$$

for all $\eta \in L^\infty(\Omega)$ with $\eta \geq 0$ a.e. in Ω, with $\bar{c} > 0$ the same constant as in (3.9). Hence, we compare θ_n^k with the solution $\widetilde{v}_k \in \mathbb{R}$

$$\widetilde{v}_k = \widetilde{v}_{k-1} + \tau_n (H_* - \bar{c}\, \widetilde{v}_k^2)\,, \quad k = 1, \ldots, n\,, \quad \text{with } \ \widetilde{v}_0 := \max\Big\{\theta_*, \sqrt{H_*/\bar{c}}\Big\} > 0\,, \tag{3.13}$$

The very same arguments from [RR14b, proof of Lemma 4.4] allow us to show for all $k = 0, \ldots, n$ that $\theta_n^k(x) \geq \widetilde{v}_k$ for almost all $x \in \Omega$. Since $\widetilde{v}_k > \widetilde{v}_{k-1} - \tau_n \bar{c}\, \widetilde{v}_k^2$, and $\widetilde{v}_0 \geq v_0 = \theta_*$, a comparison with the solution v_k of the finite-difference equation (3.10) and induction over k yield that $\widetilde{v}_k \geq v_k$. Hence $\tilde{v}_k \geq \widetilde{\theta} > 0$. We now aim to prove that

$$\widetilde{v}_k \geq \sqrt{H_*/\bar{c}} \qquad \text{for all } k = 1, \ldots, n\,. \tag{3.14}$$

We proceed by contradiction and suppose that $H_* > \bar{c}\, \widetilde{v}_{\bar{k}}^2$ for a certain $\bar{k} \in \{1, \ldots, n\}$. Then, we read from (3.13) that $\widetilde{v}_{\bar{k}} > \widetilde{v}_{\bar{k}-1}$. Since $\widetilde{v}_{\bar{k}-1} > 0$, we then conclude that $H_* > \bar{c}\, \widetilde{v}_{\bar{k}}^2 > \bar{c}\, \widetilde{v}_{\bar{k}-1}^2$. Proceeding by induction, we thus conclude that $H_* > \bar{c}\, \widetilde{v}_0^2$, which is a contradiction to (3.13). Therefore, (3.14) ensues.

Existence of an approximate solution to system (3.4b)–(3.4c)**:** As in [RR14b, proof of Lemma 4.4], we approximate (3.4b)–(3.4c) by a suitable truncation of the heat conductivity matrix $\mathbb{K}$, in such a way as to reduce to an elliptic operator with *bounded* coefficients in the discrete heat equation. In a similar manner we treat the L^1 right-hand sides in order to improve their integrability. Accordingly, we truncate all occurrences of θ_n^k in the respective terms of system (3.4b)–(3.4c). We show that the approximate system thus obtained admits solutions by resorting to an existence result from the theory of elliptic systems featuring pseudo-monotone operators drawn from [Rou05]. Hence, we pass to the limit with the truncation parameter and conclude the existence of solutions to (3.4b)–(3.4c).

Let z_n^k be a solution of (3.4a). In what follows, we shall denote by $\overline{\mathbb{K}} = \overline{\mathbb{K}}(x, \theta)$ the function $\mathbb{K}(z_n^k(x), \theta)$. Let $M > 0$. We introduce the truncation operator

$$\mathcal{T}_M(\theta) := \begin{cases} -M & \text{if } \theta < -M, \\ \theta & \text{if } |\theta| \leq M, \\ M & \text{if } \theta > M, \end{cases} \tag{3.15}$$

and we set

$$\overline{\mathbb{K}}_M : \Omega \times \mathbb{R} \to \mathbb{R}^{d\times d}\,, \qquad \overline{\mathbb{K}}_M(x, \theta) := \overline{\mathbb{K}}(x, \mathcal{T}_M(\theta)). \tag{3.16}$$

Since $\mathbb{K} \in \mathrm{C}^0(\mathbb{R} \times \mathbb{R}; \mathbb{R}^{d\times d})$ and $0 \leq z_n^k(x) \leq 1$ for almost all $x \in \Omega$, it is immediate to check that there exists a positive constant C_M such that $|\overline{\mathbb{K}}_M(x, \theta)| \leq C_M$ for almost all $x \in \Omega$ and $\theta \in \mathbb{R}$. The truncated version of system (3.4b)–(3.4c) thus reads: find $(u,\theta) \in W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d) \times H^1(\Omega)$ such that

$$
\begin{aligned}
&\rho \int_\Omega \frac{u-2u_n^{k-1}+u_n^{k-2}}{\tau_n^2}\cdot v \,\mathrm{d}x \\
&\quad + \int_\Omega \Big(\mathbb{D}(z_n^{k-1},\theta_n^{k-1})\, e\Big(\frac{u-u_n^{k-1}}{\tau_n}\Big) + \mathbb{C}(z_n^k)e(u) - \mathcal{T}_M(\theta)\,\mathbb{B} + \tau_n |e(u)|^{\gamma-2}e(u)\Big) : e(v)\,\mathrm{d}x \\
&= \left\langle f_n^k, v\right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \qquad\qquad \text{for all } v \in W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)\,,
\end{aligned}
\tag{3.17a}
$$

$$
\begin{aligned}
&\int_\Omega \frac{\theta-\theta_n^{k-1}}{\tau_n}\eta\,\mathrm{d}x + \int_\Omega \overline{\mathbb{K}}_M(x,\theta)\nabla\theta\cdot\nabla\eta\,\mathrm{d}x \\
&= \int_\Omega \frac{z_n^{k-1}-z_n^k}{\tau_n}\,\eta\,\mathrm{d}x + \int_\Omega \Big(\mathbb{D}(z_n^{k-1},\theta_n^{k-1})\, e\Big(\frac{u-u_n^{k-1}}{\tau_n}\Big) - \mathcal{T}_M(\theta)\,\mathbb{B}\Big) : e\Big(\frac{u-u_n^{k-1}}{\tau_n}\Big)\,\eta\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} h_n^k \eta \,\mathrm{d}\mathcal{H}^{d-1}(x) + \left\langle H_n^k, \eta\right\rangle_{H^1(\Omega)} \qquad\qquad \text{for all } \eta \in H^1(\Omega)\,.
\end{aligned}
\tag{3.17b}
$$

Observe that system (3.17) rewrites as

$$
\begin{aligned}
&\rho \int_\Omega u\cdot v\,\mathrm{d}x \\
&\quad + \tau_n \int_\Omega \big(\mathbb{D}(z_n^{k-1},\theta_n^{k-1})e(u) + \tau_n\mathbb{C}(z_n^k)e(u) - \tau_n \mathcal{T}_M(\theta)\,\mathbb{B} + \tau_n^2|e(u)|^{\gamma-2}e(u)\big) : e(v)\,\mathrm{d}x \\
&= \rho\int_\Omega (2u_n^{k-1}-u_n^{k-2})\cdot v\,\mathrm{d}x + \tau_n\int_\Omega \mathbb{D}(z_n^{k-1},\theta_n^{k-1})e(u_n^{k-1}) : e(v)\,\mathrm{d}x + \tau_n^2\left\langle f_n^k, v\right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \\
&\qquad\qquad \text{for all } v \in W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)\,,
\end{aligned}
\tag{3.18a}
$$

$$
\begin{aligned}
&\int_\Omega \theta\,\eta\,\mathrm{d}x + \tau_n\int_\Omega \overline{\mathbb{K}}_M(x,\theta)\nabla\theta\cdot\nabla\eta\,\mathrm{d}x - \frac{1}{\tau_n}\int_\Omega \mathbb{D}(z_n^{k-1},\theta_n^{k-1})e(u):e(u)\eta\,\mathrm{d}x \\
&\quad + \int_\Omega \mathcal{T}_M(\theta)\,\mathbb{B} : e(u)\eta\,\mathrm{d}x + \frac{2}{\tau_n}\int_\Omega \mathbb{D}(z_n^{k-1},\theta_n^{k-1})e(u):e(u_n^{k-1})\eta\,\mathrm{d}x - \int_\Omega \mathcal{T}_M(\theta)\,\mathbb{B} : e(u_n^{k-1})\eta\,\mathrm{d}x \\
&= \int_\Omega \theta_n^{k-1}\eta\,\mathrm{d}x + \frac{1}{\tau_n}\int_\Omega \mathbb{D}(z_n^{k-1},\theta_n^{k-1})e(u_n^{k-1}):e(u_n^{k-1})\eta\,\mathrm{d}x \\
&\quad + \int_\Omega (z_n^{k-1}-z_n^k)\eta\,\mathrm{d}x + \tau_n\int_{\partial\Omega} h_n^k\eta\,\mathrm{d}\mathcal{H}^{d-1}(x) + \tau_n\left\langle H_n^k,\eta\right\rangle_{H^1(\Omega)} \qquad \text{for all } \eta\in H^1(\Omega)\,,
\end{aligned}
\tag{3.18b}
$$

which in turn can be recast in the form

$$
\mathcal{A}_{k,M}(u,\theta) = B_{k-1}\,. \tag{3.19}
$$

Here, $\mathcal{A}_{k,M} : W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)\times H^1(\Omega) \to W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*\times H^1(\Omega)^*$ is the elliptic operator, acting on the unknown (u,θ), defined by the left-hand sides of (3.18a) and (3.18b), while B_{k-1} is the vector defined by the right-hand side terms in system (3.18). It can be verified that $\mathcal{A}_{k,M}$ is a pseudo-monotone operator in the sense of [Rou05, Chapter II, Definition 2.1]. Furthermore, $\mathcal{A}_{k,M}$ is coercive on $W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)\times H^1(\Omega)$. This can be checked directly on system (3.18), testing (3.18a) by u and (3.18b) by θ and adding the resulting equations: it is then sufficient to deduce from these calculations an estimate for $\|u\|_{W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)}$ and $\|\theta\|_{H^1(\Omega)}$. We refer to [RR14b, proof of Lemma 4.4] for all the detailed calculations, which highlight the role of the regularizing term $-\tau_n \operatorname{div}(|e(u)|^{\gamma-2}e(u))$ added to the discrete momentum equation. It helps control the quadratic terms in $e(u)$ on the right-hand side of (3.17b), thus obtaining the bound for $\|\theta\|_{H^1(\Omega)}$, also exploiting that the operator with coefficients $\overline{\mathbb{K}}_M$ is uniformly elliptic thanks to (2.11b). Since $\mathcal{A}_{k,M}$ is pseudo-monotone and coercive, we are in the position to apply [Rou05, Chapter II, Theorem 2.6] to system (3.18), for every $M\in\mathbb{N}$ thus deducing the existence of a solution (u,θ) which shall be hereafter denoted as $(u^k_{n,M},\theta^k_{n,M})$.

Passage to the limit as $M\to\infty$: We now consider a family $(u^k_{n,M},\theta^k_{n,M})_M$ of solutions to the truncated system (3.17): we shall derive some a priori estimates on $(u^k_{n,M},\theta^k_{n,M})_M$ which will allow us to extract a (not relabeled) subsequence converging as $M\to\infty$ to a solution of system (3.4b)–(3.4c). For the ensuing calculations, it is crucial to observe that

$$\exists\,\widetilde{\theta} \quad \text{such that} \quad \theta^k_{n,M} \geq \widetilde{\theta} > 0 \quad \text{for all } M > 0\,. \tag{3.20}$$

This follows from the very same arguments as for (3.5): indeed, notice that $\widetilde{\theta}$ does not depend on M.

Hence, let us first test (3.17a) by $(u^k_{n,M}-u^{k-1}_n)/\tau_n$, (3.17b) by 1, and add the resulting relations. Taking into account the cancelation of the coupling terms between (3.17a) and (3.17b), by convexity, cf. (3.7), we obtain

$$\begin{aligned}
&\tfrac{\rho}{2\tau_n^3}\int_\Omega |u^k_{n,M}-u^{k-1}_n|^2\,\mathrm{d}x + \tfrac{1}{2\tau_n}\int_\Omega \mathbb{C}(z^k_n)e(u^k_{n,M}):e(u^k_{n,M})\,\mathrm{d}x + \tfrac{1}{\gamma}\int_\Omega |e(u^k_{n,M})|^\gamma\,\mathrm{d}x + \tfrac{1}{\tau_n}\int_\Omega \theta^k_{n,M}\,\mathrm{d}x\\
&\leq \tfrac{\rho}{2\tau_n^3}\int_\Omega |u^{k-1}_n-u^{k-2}_n|^2\,\mathrm{d}x + \tfrac{1}{2\tau_n}\int_\Omega \mathbb{C}(z^k_n)e(u^{k-1}_n):e(u^{k-1}_n)\,\mathrm{d}x + \tfrac{1}{\gamma}\int_\Omega |e(u^{k-1}_n)|^\gamma\,\mathrm{d}x + \tfrac{1}{\tau_n}\int_\Omega \theta^{k-1}_n\,\mathrm{d}x\\
&\quad + \left\langle f^k_n, \tfrac{u^k_{n,M}-u^{k-1}_n}{\tau_n}\right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} + \int_\Omega \left(\tfrac{z^{k-1}_n-z^k_n}{\tau_n}+H^k_n\right)\mathrm{d}x + \int_{\partial\Omega} h^k_n\,\mathrm{d}\mathcal{H}^{d-1}(x) \leq C\,,
\end{aligned}$$

where the constant C is uniform with respect to the truncation parameter M (but depends on k and n). Therefore, also on account of (3.20) we infer that

$$\|u^k_{n,M}\|_{W^{1,\gamma}(\Omega;\mathbb{R}^d)} + \|\theta^k_{n,M}\|_{L^1(\Omega)} \leq C\,. \tag{3.21}$$

As a straightforward consequence of (3.21), if we define

$$\mathcal{S}_M = \{x\in\Omega\,:\,\theta^k_{n,M}\leq M\}\,,$$

using Markov's inequality, it is not difficult to infer from (3.21) that

$$|\Omega\backslash\mathcal{S}_M| \to 0 \quad \text{as } M\to\infty\,. \tag{3.22}$$

Secondly, we test (3.17b) by $\mathcal{T}_M(\theta^k_{n,M})$. Using that

$$\theta\,\mathcal{T}_M(\theta) \geq |\mathcal{T}_M(\theta)|^2 \quad \text{and} \quad \overline{\mathbb{K}}_M(x,\theta)\nabla\theta\cdot\nabla\mathcal{T}_M(\theta) = \overline{\mathbb{K}}(x,\mathcal{T}_M(\theta))\nabla\mathcal{T}_M(\theta)\cdot\nabla\mathcal{T}_M(\theta),$$

we obtain

$$\begin{aligned}
&\tfrac{1}{2\tau_n}\int_\Omega |\mathcal{T}_M(\theta^k_{n,M})|^2\,\mathrm{d}x + \int_\Omega \overline{\mathbb{K}}(x,\mathcal{T}_M(\theta^k_{n,M}))\nabla\mathcal{T}_M(\theta^k_{n,M})\cdot\nabla\mathcal{T}_M(\theta^k_{n,M})\,\mathrm{d}x\\
&\leq \tfrac{1}{2\tau_n}\int_\Omega |\theta^{k-1}_n|^2\,\mathrm{d}x + I_1+I_2+I_3+I_4\,,
\end{aligned} \tag{3.23}$$

where, taking into account (2.3e) and the previously obtained (3.21), we have

$$\begin{aligned}
I_1 &:= \left|\int_\Omega \mathbb{D}(z^{k-1}_n,\theta^{k-1}_n)e\Big(\tfrac{u^k_{n,M}-u^{k-1}_n}{\tau_n}\Big):e\Big(\tfrac{u^k_{n,M}-u^{k-1}_n}{\tau_n}\Big)\mathcal{T}_M(\theta^k_{n,M})\,\mathrm{d}x\right|\\
&\leq C\left\|e\Big(\tfrac{u^k_{n,M}-u^{k-1}_n}{\tau_n}\Big)\right\|^4_{L^4(\Omega;\mathbb{R}^{d\times d})} + \tfrac{1}{8\tau_n}\int_\Omega |\mathcal{T}_M(\theta^k_{n,M})|^2\,\mathrm{d}x\,,\\
I_2 &:= \left|\int_\Omega \mathcal{T}_M(\theta^k_{n,M})\mathbb{B}:e\Big(\tfrac{u^k_{n,M}-u^{k-1}_n}{\tau_n}\Big)\mathcal{T}_M(\theta^k_{n,M})\,\mathrm{d}x\right|\\
&\leq C\left\|e\Big(\tfrac{u^k_{n,M}-u^{k-1}_n}{\tau_n}\Big)\right\|_{L^4(\Omega;\mathbb{R}^{d\times d})}\|\mathcal{T}_M(\theta^k_{n,M})\|_{L^2(\Omega)}\|\mathcal{T}_M(\theta^k_{n,M})\|_{L^4(\Omega)}\\
&\leq C\|\mathcal{T}_M(\theta^k_{n,M})\|^2_{L^4(\Omega)} \leq \tfrac{c_1}{4}\int_\Omega |\nabla\mathcal{T}_M(\theta^k_{n,M})|^2\,\mathrm{d}x + C\|\mathcal{T}_M(\theta^k_{n,M})\|^2_{L^1(\Omega)}\,,\\
I_3 &:= \left|\int_\Omega \tfrac{z^k_n-z^{k-1}_n}{\tau_n}\mathcal{T}_M(\theta^k_{n,M})\,\mathrm{d}x\right| \leq C + \tfrac{1}{8\tau_n}\int_\Omega |\mathcal{T}_M(\theta^k_{n,M})|^2\,\mathrm{d}x\,,
\end{aligned}$$

$$I_4 := \left| \left\langle H_n^k, \mathcal{T}_M(\theta_{n,M}^k) \right\rangle_{H^1(\Omega)} + \int_{\partial\Omega} h_n^k \, \mathcal{T}_M(\theta_{n,M}^k) \, \mathrm{d}\mathcal{H}^{d-1}(x) \right|$$
$$\leq \tfrac{1}{16\tau_n} \int_\Omega |\mathcal{T}_M(\theta_{n,M}^k)|^2 \, \mathrm{d}x + \tfrac{c_1}{2} \int_\Omega |\nabla \mathcal{T}_M(\theta_{n,M}^k)|^2 \, \mathrm{d}x + C \,.$$

where in the estimate for I_2 we have used the previously obtained bound (3.21), the Gagliardo-Nirenberg inequality $\|v\|_{L^4(\Omega)} \leq C\|v\|_{H^1(\Omega)}^{\sigma} \|v\|_{L^1(\Omega)}^{1-\sigma}$ for $\sigma = 9/10$, and the Young inequality. As by (2.11b) it is $\overline{\mathbb{K}}_M \xi \cdot \xi \geq c_1 |\xi|^2$, combining the above estimates with (3.23) and taking into account (3.21), we conclude that

$$\|\mathcal{T}_M(\theta_{n,M}^k)\|_{L^2(\Omega)} + \int_\Omega \overline{\mathbb{K}}(x, \mathcal{T}_M(\theta_{n,M}^k)) \nabla \mathcal{T}_M(\theta_{n,M}^k) \cdot \nabla \mathcal{T}_M(\theta_{n,M}^k) \, \mathrm{d}x \leq C \,. \tag{3.24}$$

Now, the coercivity (2.11b) implies

$$\int_\Omega \overline{\mathbb{K}}(x, \mathcal{T}_M(\theta_{n,M}^k)) \nabla \mathcal{T}_M(\theta_{n,M}^k) \cdot \nabla \mathcal{T}_M(\theta_{n,M}^k) \, \mathrm{d}x$$
$$\geq c_1 \int_\Omega |\mathcal{T}_M(\theta_{n,M}^k)|^\kappa |\nabla \mathcal{T}_M(\theta_{n,M}^k)|^2 \, \mathrm{d}x = c \int_\Omega |\nabla (\mathcal{T}_M(\theta_{n,M}^k))^{(\kappa+2)/2}|^2 \, \mathrm{d}x \,.$$

From this, recalling the continuous embedding $H^1 \subset L^6$ we infer

$$\|\mathcal{T}_M(\theta_{n,M}^k)\|_{H^1(\Omega)} + \|\mathcal{T}_M(\theta_{n,M}^k)\|_{L^{3\kappa+6}(\Omega)} \leq C \,. \tag{3.25}$$

Thirdly, we test (3.17b) by $\theta_{n,M}^k$. Relying on estimate (3.25) to bound the second term on the right-hand side of (3.17b) and mimicking the above calculations, we obtain

$$\|\theta_{n,M}^k\|_{H^1(\Omega)} + \|\theta_{n,M}^k\|_{L^{3\kappa+6}(\mathcal{S}_M)} \leq C \,. \tag{3.26}$$

With estimates (3.21), (3.25), and (3.26), combined with well-known compactness arguments, we find a pair (u, θ) such that, along a not relabeled subsequence, $(u_{n,M}^k, \theta_{n,M}^k) \rightharpoonup (u, \theta)$ in $W_{\mathrm{D}}^{1,\gamma}(\Omega; \mathbb{R}^d) \times H^1(\Omega)$. The argument for passing to the limit as $M \to \infty$ in (3.17), also based on (3.22), is completely analogous to the one developed in the proof of [RR14b, Lemma 4.4], therefore we refer to the latter paper for all details. This concludes the existence proof for system (3.4b)–(3.4c). □

3.2 Time-discrete version of the energetic formulation

We now define the approximate solutions to the energetic formulation of the initial-boundary value problem for system (1.1) by suitably interpolating the discrete solutions $(u_n^k, z_n^k, \theta_n^k)_{k=1}^n$ from Proposition 3.2. Namely, for $t \in (t_n^{k-1}, t_n^k]$, $k = 1, \ldots, n$, we set

$$\overline{u}_n(t) := u_n^k \,, \qquad \overline{\theta}_n(t) := \theta_n^k \,, \qquad \overline{z}_n(t) := z_n^k \,, \tag{3.27a}$$
$$\underline{u}_n(t) := u_n^{k-1} \,, \quad \underline{\theta}_n(t) := \theta_n^{k-1} \,, \quad \underline{z}_n(t) := z_n^{k-1} \,, \tag{3.27b}$$

and we also consider the piecewise linear interpolants, defined by

$$u_n(t) := \tfrac{t - t_n^{k-1}}{\tau_n} u_n^k + \tfrac{t_n^k - t}{\tau_n} u_n^{k-1} \,, \;\; z_n(t) := \tfrac{t - t_n^{k-1}}{\tau_n} z_n^k + \tfrac{t_n^k - t}{\tau_n} z_n^{k-1} \,, \;\; \theta_n(t) := \tfrac{t - t_n^{k-1}}{\tau_n} \theta_n^k + \tfrac{t_n^k - t}{\tau_n} \theta_n^{k-1} . \tag{3.27c}$$

In what follows, we shall understand the time derivative of the piecewise linear interpolant u_n to be defined at the nodes of the partition by

$$\dot{u}_n(t_n^k) := \tfrac{u_n^k - u_n^{k-1}}{\tau_n} \,, \qquad \text{for } k = 1, \ldots, n \,. \tag{3.27d}$$

We also introduce the piecewise constant and linear interpolants of the discrete data $(f_n^k, H_n^k, h_n^k)_{k=1}^n$ in (3.1) by setting for $t \in (t_n^{k-1}, t_n^k]$

$$\overline{f}_n(t) := f_n^k \,, \qquad \overline{H}_n(t) := H_n^k \,, \qquad \overline{h}_n(t) := h_n^k \,,$$

and $f_n(t) := \frac{t-t_n^{k-1}}{\tau_n} f_n^k + \frac{t_n^k - t}{\tau_n} f_n^{k-1}$ with time derivative $\dot{f}_n(t) := \frac{f_n^k - f_n^{k-1}}{\tau_n}$. It follows from (2.13) that, as $n\to\infty$,

$$
\begin{aligned}
&\overline{f}_n \to f \quad \text{in } L^p(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*) \text{ for all } 1 \le p < \infty\,, \quad \overline{f}_n \overset{*}{\rightharpoonup} f \quad \text{in } L^\infty(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*)\,, && (3.28a)\\
&f_n \rightharpoonup f \quad \text{in } H^1(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*)\,, && (3.28b)\\
&\overline{H}_n \to H \quad \text{in } L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)^*)\,, \qquad \overline{h}_n \to h \quad \text{in } L^1(0,T;L^2(\partial\Omega))\,. && (3.28c)
\end{aligned}
$$

Finally, we consider the piecewise constant interpolants associated with the partition, i.e.,

$$
\overline{\tau}_n(t) := t_n^k \quad \text{and} \quad \underline{\tau}_n(t) := t_n^{k-1} \quad \text{for } t \in (t_n^{k-1}, t_n^k]\,.
$$

In Proposition 3.3 we show that the approximate solutions introduced above indeed fulfill the discrete version of the energetic formulation from Definition 2.4. In order to check the discrete momentum equation (3.31b) and (3.31e), we shall make use of the following *discrete by-part integration* formula, for every $(r_k)_{k=1}^n \subset X$ and $(s_k)_{k=1}^n \subset X^*$, with X a given Banach space:

$$
\sum_{k=1}^n \langle s_k, r_k - r_{k-1}\rangle_X = \langle s_n, r_n\rangle_X - \langle s_0, r_0\rangle_X - \sum_{k=1}^n \langle s_k - s_{k-1}, r_{k-1}\rangle_X\,. \tag{3.29}
$$

In the discrete mechanical energy inequality (3.31c) below, the mechanical energy $\mathcal{E}$ will be replaced by

$$
\mathcal{E}_n(t,u,z) := \int_\Omega \Big(\tfrac12 \mathbb{C}(z)e(u) : e(u) + \tfrac{\tau_n}{\gamma}|e(u)|^\gamma\Big)\,\mathrm{d}x + \mathcal{G}(z,\nabla z) - \langle \overline{f}_n(t), u\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \quad \text{with } \tau_n = \tfrac{T}{n}\,. \tag{3.30}
$$

Proposition 3.3 (Time-discrete version of the energetic formulation (2.19) & total energy inequality)**.** *Let the assumptions of Theorem 2.7 hold true. Then the interpolants of the time-discrete solutions* $(\overline{u}_n, \underline{u}_n, u_n, \overline{z}_n, \underline{z}_n, z_n, \overline{\theta}_n, \underline{\theta}_n, \theta_n)$ *obtained via Problem 3.1 and* (3.27) *satisfy the following properties:*

- *unidirectionality: for a.a.* $x \in \Omega$, *the functions* $\overline{z}_n(\cdot,x)\colon [0,T] \to [0,1]$ *are nonincreasing;*
- *discrete semistability: for all* $t \in [0,T]$

$$
\forall\, \tilde{z} \in \mathcal{Z}: \quad \mathcal{E}_n(t,\underline{u}_n(t),\overline{z}_n(t)) \le \mathcal{E}_n(t,\underline{u}_n(t),\tilde{z}) + \mathcal{R}_1(\tilde{z} - \overline{z}_n(t))\,; \tag{3.31a}
$$

- *discrete formulation of the momentum equation: for all* $t \in [0,T]$ *and for every* $(n+1)$*-tuple* $(v_n^k)_{k=0,\dots,n} \subset W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)$, *setting* $\overline{v}_n(s) := v_n^k$ *and* $v_n(s) := \frac{s-t_n^{k-1}}{\tau_n}v_n^k + \frac{t_n^k-s}{\tau_n}v_n^{k-1}$ *for* $s \in (t_n^{k-1}, t_n^k]$,

$$
\begin{aligned}
&\rho\int_\Omega (\dot{u}_n(t)\cdot\overline{v}_n(t) - \dot{u}_0\cdot v_n(0))\,\mathrm{d}x - \rho\int_0^{\overline{\tau}_n(t)}\int_\Omega \dot{u}_n(s-\tau_n)\cdot\dot{v}_n(s)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + \int_0^{\overline{\tau}_n(t)}\int_\Omega \big(\mathbb{D}(\underline{z}_n,\underline{\theta}_n)e(\dot{u}_n) + \mathbb{C}(\overline{z}_n)e(\overline{u}_n) - \overline{\theta}_n\,\mathbb{B} + \tau_n|e(\overline{u}_n)|^{\gamma-2}e(\overline{u}_n)\big) : e(\overline{v}_n)\,\mathrm{d}x\,\mathrm{d}s \qquad (3.31b)\\
&= \int_0^{\overline{\tau}_n(t)} \langle \overline{f}_n, \overline{v}_n\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\,,
\end{aligned}
$$

  where we have extended u_n *to* $(-\tau_n, 0]$ *by setting* $u_n(t) := u_n^0 + t\dot{u}_0$;
- *discrete mechanical energy inequality: for all* $t \in [0,T]$

$$
\begin{aligned}
&\tfrac{\rho}{2}\int_\Omega |\dot{u}_n(t)|^2\,\mathrm{d}x + \mathcal{E}_n(t,\overline{u}_n(t),\overline{z}_n(t)) + \int_\Omega (z_0 - \overline{z}_n(t))\,\mathrm{d}x\\
&\quad + \int_0^{\overline{\tau}_n(t)}\int_\Omega \big(\mathbb{D}(\underline{z}_n,\underline{\theta}_n)e(\dot{u}_n) - \overline{\theta}_n\,\mathbb{B}\big) : e(\dot{u}_n)\,\mathrm{d}x\,\mathrm{d}s \qquad (3.31c)\\
&\le \tfrac{\rho}{2}\int_\Omega |\dot{u}_0|^2\,\mathrm{d}x + \mathcal{E}_n(0,u_n^0,z_0) - \int_0^{\overline{\tau}_n(t)} \big\langle \dot{f}_n, \underline{u}_n\big\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\,;
\end{aligned}
$$

- *discrete total energy inequality: for all* $t \in [0,T]$

$$
\begin{aligned}
&\frac{\varrho}{2}\int_\Omega |\dot{u}_n(t)|^2 \,\mathrm{d}x + \mathcal{E}_n(t, \overline{u}_n(t), \overline{z}_n(t)) + \int_\Omega \overline{\theta}_n(t)\,\mathrm{d}x \\
&\le \frac{\varrho}{2}\int_\Omega |\dot{u}_0|^2 \,\mathrm{d}x + \mathcal{E}_n(0, u_n^0, z_0) + \int_\Omega \theta_0 \,\mathrm{d}x \\
&\quad - \int_0^{\overline{\tau}_n(t)} \left\langle \dot{f}_n, \underline{u}_n \right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \mathrm{d}s + \int_0^{\overline{\tau}_n(t)} \left[\int_{\partial\Omega} \overline{h}_n \,\mathrm{d}\mathcal{H}^{d-1}(x) + \int_\Omega \overline{H}_n \,\mathrm{d}x \right] \mathrm{d}s\,;
\end{aligned}
\tag{3.31d}
$$

- *discrete formulation of the heat equation: for all* $t \in [0,T]$ *and for every* $(n+1)$*-tuple* $(\eta_n^k)_{k=0}^n \subset H^1(\Omega)$*, setting* $\overline{\eta}_n(s) := \eta_n^k$ *and* $\eta_n(s) := \frac{s-t_n^{k-1}}{\tau_n}\eta_n^k + \frac{t_n^k - s}{\tau_n}\eta_n^{k-1}$ *for* $s \in (t_n^{k-1}, t_n^k]$*,*

$$
\begin{aligned}
&\int_\Omega \overline{\theta}_n(t)\overline{\eta}_n(t)\,\mathrm{d}x - \int_\Omega \theta_0 \eta_n(0)\,\mathrm{d}x - \int_0^{\overline{\tau}_n(t)}\int_\Omega \underline{\theta}_n(s)\dot{\eta}_n(s)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^{\overline{\tau}_n(t)}\int_\Omega \left(\mathbb{K}(\overline{z}_n, \overline{\theta}_n)\nabla\overline{\theta}_n\right)\cdot\nabla\overline{\eta}_n \,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^{\overline{\tau}_n(t)}\int_\Omega \overline{\eta}_n\,|\dot{z}_n|\,\mathrm{d}x\,\mathrm{d}s \int_0^{\overline{\tau}_n(t)}\int_\Omega \left(\mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) - \overline{\theta}_n\,\mathbb{B}\right) : e(\dot{u}_n)\,\overline{\eta}_n \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^{\overline{\tau}_n(t)} \left[\int_{\partial\Omega} \overline{h}_n\,\eta_n \,\mathrm{d}\mathcal{H}^{d-1}(x) + \left\langle \overline{H}_n, \eta_n \right\rangle_{H^1(\Omega)} \right] \mathrm{d}s\,.
\end{aligned}
\tag{3.31e}
$$

Proof. The discrete momentum and heat equations (3.31b) and (3.31e) follow from testing (3.4b) and (3.4c) by the discrete test functions $(v_n^k)_{k=0}^n \subset W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)$ and $(\eta_n^k)_{k=0}^n \subset H^1(\Omega)$, respectively, and applying the discrete by-part integration formula (3.29). From the discrete minimum problem (3.4a) we infer

$$
\mathcal{E}(t_n^k, u_n^{k-1}, z_n^k) \le \mathcal{E}(t_n^k, u_n^{k-1}, \tilde{z}) + \int_\Omega (z_n^{k-1} - \tilde{z})\,\mathrm{d}x - \int_\Omega (z_n^{k-1} - z_n^k)\,\mathrm{d}x \le \mathcal{E}(t_n^k, u_n^{k-1}, \tilde{z}) + \int_\Omega (z_n^k - \tilde{z})\,\mathrm{d}x
$$

for all $\tilde{z} \in \mathcal{Z}$ with $\tilde{z} \le z_n^{k-1}$. By (3.4a) and the definition of the dissipation $\mathcal{R}_1$ we have $z_n^k \le z_n^{k-1}$, whence the unidirectionality and the discrete semistability (3.31a) hold.

To deduce the mechanical energy inequality (3.31c) we choose z_n^{k-1} as a competitor in (3.4a) and get

$$
\begin{aligned}
&\int_\Omega (z_n^{k-1} - z_n^k)\,\mathrm{d}x + \int_\Omega \left(\tfrac{1}{2}\mathbb{C}(z_n^k)e(u_n^{k-1}) : e(u_n^{k-1}) + G(z_n^k, \nabla z_n^k)\right)\,\mathrm{d}x \\
&\le \int_\Omega \left(\tfrac{1}{2}\mathbb{C}(z_n^{k-1})e(u_n^{k-1}) : e(u_n^{k-1}) + G(z_n^{k-1}, \nabla z_n^{k-1})\right)\,\mathrm{d}x\,.
\end{aligned}
\tag{3.32}
$$

Moreover, we test (3.4b) by $v = u_n^k - u_n^{k-1}$. To this aim, we observe that by convexity (3.7)

$$
\rho\int_\Omega \tfrac{u_n^k - 2u_n^{k-1} + u_n^{k-2}}{\tau_n^2}\cdot(u_n^k - u_n^{k-1})\,\mathrm{d}x \ge \rho\int_\Omega \left(\tfrac{1}{2}\tfrac{\left|u_n^k - u_n^{k-1}\right|^2}{\tau_n^2} - \tfrac{1}{2}\tfrac{\left|u_n^{k-1} - u_n^{k-2}\right|^2}{\tau_n^2} \right)\mathrm{d}x\,, \tag{3.33a}
$$

$$
\int_\Omega \mathbb{C}(z_n^k)e(u_n^k) : (e(u_n^k) - e(u_n^{k-1}))\,\mathrm{d}x \ge \int_\Omega \tfrac{1}{2}\Big(\mathbb{C}(z_n^k)e(u_n^k) : e(u_n^k) - \mathbb{C}(z_n^k)e(u_n^{k-1}) : e(u_n^{k-1})\Big)\,\mathrm{d}x\,, \tag{3.33b}
$$

$$
\int_\Omega \tau_n |e(u_n^k)|^{\gamma-2}e(u_n^k) : (e(u_n^k) - e(u_n^{k-1}))\,\mathrm{d}x \ge \int_\Omega \left(\tfrac{\tau_n}{\gamma}|e(u_n^k)|^\gamma - \tfrac{\tau_n}{\gamma}|e(u_n^{k-1})|^\gamma\right)\mathrm{d}x\,. \tag{3.33c}
$$

Further, let $t \in (0,T]$ be fixed, and let $1 \le j \le n$ fulfill $t \in (t_n^{j-1}, t_n^j]$. We sum (3.33)–(3.33c) over the index $k = 1, \ldots, j$. Applying the by-part integration formula (3.29) we conclude that

$$
\begin{aligned}
&\sum_{k=1}^{j} \left\langle f_n^k, u_n^k - u_n^{k-1} \right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} = \int_0^{\overline{\tau}_n(t)} \left\langle \overline{f}_n, \dot{u}_n \right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \mathrm{d}s \\
&= \left\langle \overline{f}_n(t), \overline{u}_n(t) \right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} - \left\langle f(0), u_0 \right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} - \int_0^{\overline{\tau}_n(t)} \left\langle \dot{f}_n, \underline{u}_n \right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \mathrm{d}s\,.
\end{aligned}
\tag{3.34}
$$

All in all we infer

$$
\begin{aligned}
&\tfrac{\varrho}{2}\int_\Omega |\dot u_n(t)|^2\,\mathrm{d}x + \int_0^{\overline{\tau}_n(t)}\int_\Omega (\mathbb{D}(\underline{z}_n,\underline{\theta}_n)\, e(\dot u_n) - \underline{\theta}_n\,\mathbb{B}) : e(\dot u_n)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_\Omega \tfrac12 \mathbb{C}(\overline{z}_n(t)) e(\overline{u}_n(t)) : e(\overline{u}_n(t))\,\mathrm{d}x + \int_\Omega \tfrac{\tau_n}{\gamma}|e(\overline{u}_n(t))|^\gamma\,\mathrm{d}x - \left\langle \overline{f}_n(t), \overline{u}_n(t)\right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \\
&\le \tfrac{\varrho}{2}\int_\Omega |\dot u_0|^2\,\mathrm{d}x + \int_\Omega \tfrac{\tau_n}{\gamma}|e(u_0)|^\gamma\,\mathrm{d}x - \langle f(0), u_0\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} - \int_0^{\overline{\tau}_n(t)} \left\langle \dot f_n, \underline{u}_n\right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}s \\
&\quad + \sum_{k=1}^{j}\int_\Omega \tfrac12 \mathbb{C}(z_n^k) e(u_n^{k-1}) : e(u_n^{k-1})\,\mathrm{d}x\,.
\end{aligned}
$$

We add the above inequality to (3.32), summed over $k=1,\ldots,j$. Observing the cancelation of the term $\sum_{k=1}^{j}\int_\Omega \frac12\mathbb{C}(z_n^k)e(u_n^{k-1}) : e(u_n^{k-1})\,\mathrm{d}x$, we conclude (3.31c).

Finally, the discrete total energy inequality ensues from adding the discrete mechanical energy inequality (3.31c) with the discrete heat equation (3.4c), tested for $\eta=\tau_n$ and added up over $k=1,\ldots,j$. We observe the cancelation of some terms, and readily conclude (3.31d). □

3.3 A priori estimates

The following result collects a series of a priori estimates on the approximate solutions, uniform with respect to $n\in\mathbb{N}$. Let us mention in advance that, in its proof we will start from the discrete total energy inequality (3.31d) and derive estimates (3.36a), (3.36b), (3.36d), (3.36h), for $\overline{u}_n$, $\dot u_n$, $\overline{z}_n$, as well as estimate (3.36i) below for $\|\overline{\theta}_n\|_{L^\infty(0,T;L^1(\Omega))}$. The next crucial step will be to obtain a bound for the $L^2(0,T;H^1(\Omega))$-norm of $\overline{\theta}_n$. For this, we will make use of a technique developed in [FPR09], cf. also [RR14b]. Namely, we will test the discrete heat equation (3.4c) by $(\theta_n^k)^{\alpha-1}$, with $\alpha\in(0,1)$. Exploiting the concavity of the function $F(\theta)=\theta^\alpha/\alpha$, we will deduce that

$$
\int_0^T\!\!\int_\Omega \mathbb{K}(\overline{z}_n,\overline{\theta}_n)\nabla\big(\overline{\theta}_n^{\alpha/2}\big)\cdot\nabla\big(\overline{\theta}_n^{\alpha/2}\big)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \tfrac{\theta_0^\alpha}{\alpha}\,\mathrm{d}x \le \int_\Omega \tfrac{\overline{\theta}_n^\alpha(T)}{\alpha}\,\mathrm{d}x + C\int_0^T\!\!\int_\Omega \overline{\theta}_n^{\alpha+1}(t)\,\mathrm{d}x\,\mathrm{d}t\,,
$$

where the positive and quadratic terms on the right-hand side of (3.4c) have been confined to the *left-hand side* and thus can be neglected. Hence, relying on the growth (2.11b) of $\mathbb{K}$, we will end up with an estimate for $\overline{\theta}_n^{\alpha/2}$ in $L^2(0,T;H^1(\Omega))$, from which we will ultimately infer the desired bound (3.36j), whence (3.36k) by interpolation. We will be then in the position to exploit the mechanical energy inequality in order to recover the *dissipative* estimate (3.36c). Estimate (3.36l) will finally ensue from a comparison in (3.4c).

In the following proof we will also use the concave counterpart to inequality (3.7), namely that for any *concave* (differentiable) function $\psi:\mathbb{R}\to(-\infty,+\infty]$

$$
\psi(x)-\psi(y)\le\psi'(y)(x-y)\quad\text{for all } x,y\in\mathrm{dom}(\psi)\,. \tag{3.35}
$$

Proposition 3.4 (A priori estimates). *Let the assumptions of Theorem 2.7 hold true and consider a sequence* $(\overline{u}_n,\underline{u}_n,u_n,\overline{z}_n,\underline{z}_n,\overline{\theta}_n,\underline{\theta}_n,\theta_n)_n$ *complying with Proposition 3.3. Then there exists a constant* $C>0$ *such that the following estimates hold uniformly with respect to* $n\in\mathbb{N}$*:*

$$
\|\overline{u}_n\|_{L^\infty(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))}\le C\,, \tag{3.36a}
$$

$$
\tau_n^{1/\gamma}\|\overline{u}_n\|_{L^\infty(0,T;W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d))}\le C\,, \tag{3.36b}
$$

$$
\|u_n\|_{H^1(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))}\le C\,, \tag{3.36c}
$$

$$
\|\dot u_n\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))}\le C\,, \tag{3.36d}
$$

$$
\|\dot u_n\|_{\mathrm{BV}([0,T];W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*)}\le C\,, \tag{3.36e}
$$

$$
\mathcal{R}_1(\underline{z}_n(T)-z_0)\le\mathcal{R}_1(\overline{z}_n(T)-z_0)\le C\,, \tag{3.36f}
$$

$$\|\overline{z}_n\|_{L^\infty((0,T)\times\Omega)} \le 1 \quad \textit{and} \quad \|\underline{z}_n\|_{L^\infty((0,T)\times\Omega)} \le 1\,, \tag{3.36g}$$

$$\|\overline{z}_n\|_{L^\infty(0,T;W^{1,q}(\Omega))} \le C \quad \textit{and} \quad \|\underline{z}_n\|_{L^\infty(0,T;W^{1,q}(\Omega))} \le C\,, \tag{3.36h}$$

$$\big\|\overline{\theta}_n\big\|_{L^\infty(0,T;L^1(\Omega))} \le C\,, \tag{3.36i}$$

$$\big\|\overline{\theta}_n\big\|_{L^2(0,T;H^1(\Omega))} \le C\,, \tag{3.36j}$$

$$\big\|\overline{\theta}_n\big\|_{L^p((0,T)\times\Omega)} \le C \quad \textit{for any } p \in \begin{cases} [1,8/3] & \textit{if } d{=}3\,, \\ [1,3] & \textit{if } d{=}2\,, \end{cases} \tag{3.36k}$$

$$\big\|\overline{\theta}_n\big\|_{\mathrm{BV}([0,T];W^{2,d+\delta}(\Omega)^*)} \le C \quad \textit{for every } \delta > 0\,, \tag{3.36l}$$

where $\mathcal{R}_1$ *is from* (2.16).

Proof. Estimate (3.36f) follows from (2.9a), (2.12a), the definition of $\mathcal{R}_1$, and the monotonicity of $\overline{z}_n$ and $\underline{z}_n$. We divide the proof of the other estimates in subsequent steps.

First a priori estimates, ad (3.36a)**,** (3.36b)**,** (3.36d)**,** (3.36g)**,** (3.36h)**,** (3.36i)**:** We start from the discrete total energy inequality (3.31d). For its left-hand side, we observe that the first and the third term are nonnegative. For the second one, we use that, in view of (2.3d), (2.9d), and (2.13a), we have

$$\begin{aligned}\mathcal{E}_n(t,\overline{u}_n(t),\overline{z}_n(t)) &\ge C^1_{\mathbb{C}} \int_\Omega |e(\overline{u}_n(t))|^2 \,\mathrm{d}x + C^1_G \int_\Omega |\nabla \overline{z}_n(t)|^q \,\mathrm{d}x + \tfrac{\tau_n}{\gamma} \int_\Omega |e(\overline{u}_n(t))|^\gamma \,\mathrm{d}x \\ &\quad - \big\|\overline{f}_n\big\|_{L^\infty(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*)} \|\overline{u}_n(t)\|_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} - C \\ &\ge C\left(\|\overline{u}_n(t)\|^2_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} + \tau_n \|\overline{u}_n(t)\|^\gamma_{W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)} + \|\overline{z}_n(t)\|^q_{W^{1,q}(\Omega)}\right) - C\,,\end{aligned} \tag{3.37}$$

for almost all $t \in (0,T)$, where we have also used Poincaré's and Korn's inequalities. Concerning the right-hand side of (3.31d), we use that $|\partial_t \mathcal{E}_n(t,\underline{u}_n(t),\underline{z}_n(t))| \le \|\dot{f}_n\|_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*} \|\underline{u}_n(t)\|_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}$ for almost all $t \in (0,T)$. The remaining terms on the right-hand side are bounded, uniformly with respect to $n \in \mathbb{N}$, in view of the properties of the initial and given data (2.12) and (3.2), and of (3.28c). All in all, from (3.31d) we deduce

$$C\|\overline{u}_n(t)\|^2_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \le C + \tfrac{1}{2} \int_0^{\overline{\tau}_n(t)} \|\underline{u}_n(s)\|^2_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}s + \tfrac{1}{2} \int_0^{\overline{\tau}_n(t)} \big\|\dot{f}_n\big\|^2_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*} \,\mathrm{d}s\,.$$

Also in view of the bounds on $\dot{f}_n$ by (3.28b), estimate (3.36a) then follows from the Gronwall Lemma. As a by-product, we conclude that

$$\int_0^{\overline{\tau}_n(t)} |\partial_t \mathcal{E}_n(s,\underline{u}_n(s),\underline{z}_n(s))| \,\mathrm{d}s \le C \int_0^{\overline{\tau}_n(t)} \big\|\dot{f}_n(s)\big\|_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*} \,\mathrm{d}s \le C\,. \tag{3.38}$$

Inserting this into (3.31d) we also infer estimates (3.36d), (3.36i), and that $|\mathcal{E}_n(t,\overline{u}_n(t),\overline{z}_n(t))| \le C$ for a constant independent of $n \in \mathbb{N}$ and $t \in (0,T)$. This implies (3.36b) and the first estimate in (3.36h) via (3.37). Then the second estimate in (3.36h) immediately follows from the very definition of the interpolants (3.27). Moreover, (3.36g) is a direct consequence of the boundedness of the energy, which implies $\overline{z}_n, \underline{z}_n \in [0,1]$ a.e. in Ω, for a.e. $t \in (0,T)$.

Second a priori estimate: We fix $\alpha \in (0,1)$. Exploiting that $\theta^k_n \ge \widetilde{\theta} > 0$, we may test (3.4c) by $(\theta^k_n)^{\alpha-1}$, thus obtaining

$$\begin{aligned}&\tfrac{4(1-\alpha)}{\alpha^2} \int_\Omega \mathbb{K}(z^k_n,\theta^k_n) \nabla(\theta^k_n)^{\alpha/2} \cdot \nabla(\theta^k_n)^{\alpha/2} \,\mathrm{d}x + \int_\Omega \mathbb{D}(z^k_n) e\big(\tfrac{u^k_n - u^{k-1}_n}{\tau}\big) : e\big(\tfrac{u^k_n - u^{k-1}_n}{\tau}\big) (\theta^k_n)^{\alpha-1} \,\mathrm{d}x \\ &\quad + \int_\Omega \tfrac{z^{k-1}_n - z^k_n}{\tau} (\theta^k_n)^{\alpha-1} \,\mathrm{d}x + \big\langle H^k_n, (\theta^k_n)^{\alpha-1} \big\rangle_{H^1(\Omega)} + \int_{\partial\Omega} h^k_n (\theta^k_n)^{\alpha-1} \,\mathrm{d}\mathcal{H}^{d-1} \\ &= \int_\Omega \tfrac{\theta^k_n - \theta^{k-1}_n}{\tau} (\theta^k_n)^{\alpha-1} \,\mathrm{d}x + \int_\Omega \theta^k_n \mathbb{B} : e\big(\tfrac{u^k_n - u^{k-1}_n}{\tau}\big) (\theta^k_n)^{\alpha-1} \,\mathrm{d}x \doteq I_1 + I_2\,,\end{aligned} \tag{3.39}$$

where we used that

$$\mathbb{K}(z^k_n,\theta^k_n)\nabla\theta^k_n \cdot \nabla(\theta^k_n)^{\alpha-1} = (\alpha-1)(\theta^k_n)^{\alpha-2} \mathbb{K}(z^k_n,\theta^k_n)\nabla\theta^k_n \cdot \nabla\theta^k_n = \tfrac{4(\alpha-1)}{\alpha^2} \mathbb{K}(z^k_n,\theta^k_n) \nabla(\theta^k_n)^{\alpha/2} \cdot \nabla(\theta^k_n)^{\alpha/2}$$

and moved the term $\int_\Omega \mathbb{K}(z_n^k,\theta_n^k)\nabla\theta_n^k\nabla(\theta_n^k)^{\alpha-1}\,\mathrm{d}x$ to the opposite side. It follows from (3.35) with $\psi(x) := \frac{x^\alpha}{\alpha}$ that

$$I_1 \le \int_\Omega \psi(\theta_n^k)\,\mathrm{d}x - \int_\Omega \psi(\theta_n^{k-1})\,\mathrm{d}x\,,$$

whereas we estimate I_2 by

$$I_2 \le \frac{C_{\mathbb{D}}^1}{2}\int_\Omega \Big|e\big(\tfrac{u_n^k-u_n^{k-1}}{\tau}\big)\Big|^2 (\theta_n^k)^{\alpha-1}\,\mathrm{d}x + C\int_\Omega |\theta_n^k|^2(\theta_n^k)^{\alpha-1}\,\mathrm{d}x\,,$$

where $C_{\mathbb{D}}^1$ from (2.3e) is such that $\int_\Omega \mathbb{D}(z_n^k)e\big(\frac{u_n^k-u_n^{k-1}}{\tau}\big) : e\big(\frac{u_n^k-u_n^{k-1}}{\tau}\big)(\theta_n^k)^{\alpha-1}\,\mathrm{d}x$ on the left-hand side of (3.39) is bounded from below by $C_{\mathbb{D}}^1\int_\Omega \Big|e\big(\frac{u_n^k-u_n^{k-1}}{\tau}\big)\Big|^2 (\theta_n^k)^{\alpha-1}\,\mathrm{d}x$. Thus the latter term absorbs I_2. Taking into account that the second, the third and the fourth integrals on the left-hand side of (3.39) are nonnegative also thanks to (2.13b) and summing up over the index k, we end up with

$$\frac{4(1-\alpha)}{\alpha^2}\int_0^{\overline{\tau}_n(t)}\!\!\int_\Omega \mathbb{K}\big(\overline{z}_n,\overline{\theta}_n\big)\nabla\big(\overline{\theta}_n^{\alpha/2}\big)\cdot\nabla\big(\overline{\theta}_n^{\alpha/2}\big)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \frac{\theta_0^\alpha}{\alpha}\,\mathrm{d}x \le \int_\Omega \frac{\overline{\theta}_n(t)^\alpha}{\alpha}\,\mathrm{d}x + C\int_0^{\overline{\tau}_n(t)}\!\!\int_\Omega \overline{\theta}_n(t)^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}s\,. \tag{3.40}$$

Since $\alpha\in(0,1)$,

$$\int_\Omega \frac{\overline{\theta}_n(t)^\alpha}{\alpha}\,\mathrm{d}x \le \frac{1}{\alpha}\int_\Omega \overline{\theta}_n(t)\,\mathrm{d}x + C \le C\,,$$

the latter estimate by (3.36i). In order to clarify the estimate for the second term on the right-hand side of (3.40), we now use the placeholder $w_n := (\overline{\theta}_n)^{\alpha/2}$, so that $(\overline{\theta}_n)^{\alpha+1} = (w_n)^{2(\alpha+1)/\alpha}$. Hence, neglecting the (positive) second term on the left-hand side of (3.40), we infer

$$\int_0^{\overline{\tau}_n(t)}\!\!\int_\Omega \mathbb{K}\big(\overline{z}_n,\overline{\theta}_n\big)\nabla w_n\cdot\nabla w_n\,\mathrm{d}x\,\mathrm{d}s \le C + C\int_0^{\overline{\tau}_n(t)}\!\!\int_\Omega |w_n|^q\,\mathrm{d}x\,\mathrm{d}s \qquad \text{with } q = \frac{2(\alpha+1)}{\alpha}\,. \tag{3.41}$$

We now proceed exactly in the same way as in [FPR09], cf. also [RR14b]. Namely, the Gagliardo-Nirenberg inequality for $d{=}3$ (for $d{=}2$ even better estimates hold true) yields

$$\|w_n\|_{L^q(\Omega)} \le C\|\nabla w_n\|_{L^2(\Omega;\mathbb{R}^d)}^\sigma \|w_n\|_{L^r(\Omega)}^{1-\sigma} + C'\|w_n\|_{L^r(\Omega)} \tag{3.42}$$

for suitable constants C and C', and for $1\le r\le q$ and σ satisfying $1/q = \sigma/6 + (1-\sigma)/r$. Hence $\sigma = 6(q-r)/q(6-r)$. Observe that $\sigma\in(0,1)$ if $q = 2(\alpha+1)/\alpha < 6$. This ultimately yields the restriction that $\alpha\in(1/2,1)$. Hence we transfer the Gagliardo-Nirenberg estimate into (3.41) and use Young's inequality in the estimate of the term

$$\int_0^{\overline{\tau}_n(t)} \|\nabla w_n\|_{L^2(\Omega;\mathbb{R}^d)}^{q\sigma}\|w_n\|_{L^r(\Omega)}^{q(1-\sigma)}\,\mathrm{d}s \le \frac{c_1}{2}\int_0^{\overline{\tau}_n(t)} \|\nabla w_n\|_{L^2(\Omega;\mathbb{R}^d)}^2\,\mathrm{d}s + C\int_0^{\overline{\tau}_n(t)} \|w_n\|_{L^r(\Omega)}^{2q(1-\theta)/(2-q\theta)}\,\mathrm{d}s\,,$$

where c_1 bounds $\mathbb{K}$ from below, cf. (2.11b). Therefore, the term $\frac{c_1}{2}\int_0^{\overline{\tau}_n(t)}\|\nabla w_n\|_{L^2(\Omega;\mathbb{R}^d)}^2\,\mathrm{d}s$ may be absorbed into the corresponding one on the left-hand side of (3.41). All in all, we conclude

$$c\int_0^{\overline{\tau}_n(t)}\!\!\int_\Omega \mathbb{K}\big(\overline{z}_n,\overline{\theta}_n\big)\nabla w_n\cdot\nabla w_n\,\mathrm{d}x\,\mathrm{d}s \le C + C\int_0^{\overline{\tau}_n(t)} \|w_n\|_{L^r(\Omega)}^{2q(1-\sigma)/(2-q\sigma)}\,\mathrm{d}s + C'\int_0^{\overline{\tau}_n(t)} \|w_n\|_{L^r(\Omega)}^q\,\mathrm{d}s\,. \tag{3.43}$$

Now, choosing $r\le 2/\alpha$, we have for almost all $t\in(0,T)$ that

$$\|w_n(t)\|_{L^r(\Omega)} = \left(\int_\Omega \big(\overline{\theta}_n(t)\big)^{r\alpha/2}\,\mathrm{d}x\right)^{1/r} \le \left(\int_\Omega \overline{\theta}_n(t)\,\mathrm{d}x\right)^{1/r} + C \le C\,, \tag{3.44}$$

for a constant independent of t, where again we have used estimate (3.36i). In the end, from (3.43)–(3.44) we infer

$$\int_0^{\overline{\tau}_n(t)}\int_\Omega \mathbb{K}\big(\overline{z}_n,\overline{\theta}_n\big)\nabla w_n\cdot\nabla w_n\,\mathrm{d}x\,\mathrm{d}s \le C\,. \tag{3.45}$$

Third a priori estimate, ad (3.36j) **and** (3.36k)**:** It follows from (3.45) and (2.11b) that

$$\begin{aligned} C &\geq \int_0^{\overline{\tau}_n(t)}\int_\Omega \mathbb{K}(\overline{z}_n,\overline{\theta}_n)\nabla\big(\overline{\theta}_n^{\alpha/2}\big)\cdot\nabla\big(\overline{\theta}_n^{\alpha/2}\big)\,\mathrm{d}x\,\mathrm{d}s \geq c_1\int_0^{\overline{\tau}_n(t)}\int_\Omega \big(\overline{\theta}_n\big)^{\kappa}|\nabla\big(\overline{\theta}_n^{\alpha/2}\big)|^2\,\mathrm{d}x\,\mathrm{d}s \\ &= C\int_0^{\overline{\tau}_n(t)}\int_\Omega \big|\big(\overline{\theta}_n\big)^{\kappa+\alpha-2}\big|\big|\nabla\overline{\theta}_n\big|^2\,\mathrm{d}x\,\mathrm{d}s = C\int_0^{\overline{\tau}_n(t)}\int_\Omega |\nabla\big(\overline{\theta}_n^{(\kappa+\alpha)/2}\big)|^2\,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{3.46}$$

with $\alpha\in(1/2,1)$ arbitrary. We now chose α such that $\kappa+\alpha-2\geq 0$ (this is possible, as $\kappa>1$). Therefore, (3.46) and the fact that $\overline{\theta}_n(t)\geq\widetilde{\theta}>0$ for almost all $t\in(0,T)$ yield that

$$\int_0^{\overline{\tau}_n(t)}\int_\Omega \big|\nabla\overline{\theta}_n\big|^2\,\mathrm{d}x\,\mathrm{d}s\leq C\,.$$

Then, (3.36j) follows in view of the previously obtained (3.36i) via Poincaré's inequality. Estimate (3.36k) ensues by interpolation between $L^2(0,T;H^1(\Omega))$ and $L^\infty(0,T;L^1(\Omega))$, relying on (3.36j) and (3.36i) and exploiting the Gagliardo-Nirenberg inequality. For later convenience, let us also point out that, as a by-product of the above calculations, we have for all $\alpha\in(1/2,1)$

$$\big\|(\overline{\theta}_n)^{(\kappa+\alpha)/2}\big\|_{L^2(0,T;H^1(\Omega))}\leq C\,. \tag{3.47}$$

Fourth a priori estimate, ad (3.36c) **and** (3.36e)**:** From the discrete mechanical energy inequality (3.31c) we infer

$$C_{\mathbb{D}}^1\int_0^{\overline{\tau}_n(t)}\int_\Omega |e(\dot{u}_n)|^2\,\mathrm{d}x\,\mathrm{d}s\leq C+\int_0^{\overline{\tau}_n(t)}\int_\Omega \overline{\theta}_n\,\mathbb{B}:e(\dot{u}_n)\,\mathrm{d}x\,\mathrm{d}s \tag{3.48}$$

where we have used (3.37), (3.38), and the fact that the terms $\int_\Omega|\dot{u}_0|^2\,\mathrm{d}x$ and $\mathcal{E}(0,u_n^0,z_0)$ are bounded, uniformly with respect to $n\in\mathbb{N}$, in view of (2.12a) and (3.2). Exploiting the previously obtained estimate (3.36j) we find

$$\begin{aligned} \int_0^{\overline{\tau}_n(t)}\int_\Omega \overline{\theta}_n\,\mathbb{B}:e(\dot{u}_n)\,\mathrm{d}x\,\mathrm{d}t &\leq \tfrac{C_{\mathbb{D}}^1}{2}\int_0^{\overline{\tau}_n(t)}\int_\Omega |e(\dot{u}_n)|^2\,\mathrm{d}x\,\mathrm{d}t + C\int_0^{\overline{\tau}_n(t)}\int_\Omega |\overline{\theta}_n|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\leq \tfrac{C_{\mathbb{D}}^1}{2}\int_0^{\overline{\tau}_n(t)}\int_\Omega |e(\dot{u}_n)|^2\,\mathrm{d}x\,\mathrm{d}t + C\,. \end{aligned}$$

Inserting this into (3.48) we conclude (3.36c) via Korn's inequality, again exploiting the definition of the interpolants (3.27). Finally, estimate (3.36e) ensues from a comparison argument in (3.4b), taking into account the previously proved (3.36b), (3.36c), (3.36j), as well as (3.28a).

Fifth a priori estimate, ad (3.36l)**:** Let κ be as in (2.11). In (3.4c) we use a test function $\eta\in W^{2,d+\delta}(\Omega)$ in order to have bounds on $\|\eta\|_{L^\infty(\Omega)}$ and $\|\nabla\eta\|_{L^\infty(\Omega;\mathbb{R}^d)}$, thus we find

$$\left|\int_\Omega \tfrac{\theta_n^k-\theta_n^{k-1}}{\tau_n}\eta\,\mathrm{d}x\right|\leq\left|\int_\Omega \mathbb{K}(z_n^k,\theta_n^k)\nabla\theta_n^k\cdot\nabla\eta\,\mathrm{d}x\right|+\left|\Big\langle \mathrm{RHS}_n^k,\eta\Big\rangle_{W^{2,d+\delta}(\Omega)}\right|, \tag{3.49}$$

where the terms on the right-hand side of (3.4c) are summarized in RHS_n^k. It follows from assumptions (2.3) and (2.13b) that

$$\begin{aligned} &\left|\Big\langle \mathrm{RHS}_n^k,\eta\Big\rangle_{W^{2,d+\delta}(\Omega)}\right| \\ &\leq C\left(\left\|e\left(\tfrac{u_n^k-u_n^{k-1}}{\tau_n}\right)\right\|_{L^2(\Omega;\mathbb{R}^{d\times d})}^2+\|\theta_n^k\|_{L^2(\Omega)}^2+\left\|\tfrac{z_n^k-z_n^{k-1}}{\tau_n}\right\|_{L^1(\Omega)}+\|h_n^k\|_{L^2(\partial\Omega)}+\|H_n^k\|_{L^1(\Omega)}\right)\|\eta\|_{L^\infty(\Omega)} \\ &\doteq \Lambda_n^k\|\eta\|_{L^\infty(\Omega)}\,. \end{aligned} \tag{3.50}$$

Furthermore, with (2.11) we find for every $\alpha \in (1/2, 1)$

$$
\begin{aligned}
&\left|\int_\Omega \mathbb{K}(z_n^k, \theta_n^k)\nabla\theta_n^k \cdot \nabla\eta \,\mathrm{d}x\right| \\
&\leq \|\nabla\eta\|_{L^\infty(\Omega;\mathbb{R}^d)} c_2 \|((\theta_n^k)^\kappa + 1)\nabla\theta_n^k\|_{L^1(\Omega;\mathbb{R}^d)} \qquad (3.51)\\
&\leq \|\nabla\eta\|_{L^\infty(\Omega;\mathbb{R}^d)} c_2 \left(\|(\theta_n^k)^{(\kappa-\alpha+2)/2}\|_{L^2(\Omega)} \|(\theta_n^k)^{(\kappa+\alpha-2)/2}\nabla\theta_n^k\|_{L^2(\Omega;\mathbb{R}^d)} + \mathcal{L}^d(\Omega)^{1/2}\|\nabla\theta_n^k\|_{L^2(\Omega;\mathbb{R}^d)}\right).
\end{aligned}
$$

Inserting (3.50) and (3.51) into (3.49) and summing over the index $k = 1, \ldots, n$, we find for every *time-dependent* function $\eta \in \mathrm{C}^0([0,T]; W^{2,d+\delta}(\Omega))$ that

$$
\begin{aligned}
&\left|\int_0^{\overline{\tau}_n(t)} \int_\Omega \dot{\theta}_n \,\eta \,\mathrm{d}x \,\mathrm{d}s\right| \\
&\leq C\|\nabla\eta\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^d)} \left(\|\overline{\theta}_n\|^{(\kappa-\alpha+2)/2}_{L^{\kappa-\alpha+2}((0,T)\times\Omega)} \|\left(\overline{\theta}_n\right)^{(\kappa+\alpha)/2}\|_{L^2(0,T;H^1(\Omega))} + \|\nabla\overline{\theta}_n\|_{L^2((0,T)\times\Omega;\mathbb{R}^d)}\right) \\
&\quad + \|\eta\|_{L^\infty((0,T)\times\Omega)} \int_0^{\overline{\tau}_n(t)} \overline{\Lambda}_n \,\mathrm{d}s\,, \qquad (3.52)
\end{aligned}
$$

where $\overline{\Lambda}_n$ denotes the piecewise constant interpolant of the values $(\Lambda_n^k)_k$. Note that the estimate on $\|(\theta_n^k)^{(\kappa+\alpha-2)/2}\nabla\theta_n^k\|_{L^2(\Omega;\mathbb{R}^d)}$ ensues from (3.46) and with (3.47) we find $\|\left(\overline{\theta}_n\right)^{(\kappa+\alpha)/2}\|_{L^2(0,T;H^1(\Omega))} \leq C$. Now, observe that

$$
\|\overline{\theta}_n\|^{(\kappa-\alpha+2)/2}_{L^{\kappa-\alpha+2}((0,T)\times\Omega)} \leq C
$$

thanks to (3.36k) if $p = \kappa - \alpha + 2$ satisfies the constraints in (3.36k). Since α can be chosen arbitrarily close to 1, this holds since, by (2.11b), $\kappa \in (1, \kappa_d]$ with $\kappa_d = 5/3$ if d=3 and $\kappa_d = 2$ if d=2. Finally, it follows from (3.28c), (3.36c), (3.36f), and (3.36j) that $\int_0^T \overline{\Lambda}_n \,\mathrm{d}t \leq C$. Ultimately, from (3.52) we conclude (3.36l). □

4 Passage from time-discrete to continuous

Based on the a priori bounds deduced in Proposition 3.4, exploiting a version of Helly's selection principle, we are now in a position to extract a suitably convergent subsequence of solutions of the time-discrete problems. Moreover, we will verify that the limit is an energetic solution of the time-continuous problem as stated in Definition 2.4.

Proposition 4.1 (Convergence of the time-discrete solutions). *Let the assumptions of Theorem 2.7 be satisfied. Then, there exists a triple* $(u, z, \theta)\colon [0,T]\times\Omega \to \mathbb{R}^d\times\mathbb{R}\times[0,\infty)$ *of regularity* (2.17) *such that for a.a.* $x \in \Omega$ *the function* $t \mapsto z(t,x) \in [0,1]$ *is nonincreasing,* (2.22) *holds, as well as* (2.24) *under the assumption* (2.23), *and there exists a subsequence of the time-discrete solutions* $(\overline{u}_n, \underline{u}_n, u_n, \overline{z}_n, \underline{z}_n, \overline{\theta}_n, \underline{\theta}_n)_n$ *from* (3.27) *such that*

$$
\begin{aligned}
&\overline{u}_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T; H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))\,, && (4.1a)\\
&u_n \rightharpoonup u \quad \text{in } H^1(0,T; H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))\,, && (4.1b)\\
&\dot{u}_n \overset{*}{\rightharpoonup} \dot{u} \quad \text{in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^d))\,, && (4.1c)\\
&\dot{u}_n(t) \rightharpoonup \dot{u}(t) \text{ in } L^2(\Omega;\mathbb{R}^d) \text{ for all } t \in [0,T]\,, && (4.1d)\\
&\overline{z}_n\,,\, \underline{z}_n \overset{*}{\rightharpoonup} z \quad \text{in } L^\infty(0,T; W^{1,q}(\Omega)) \cap L^\infty((0,T)\times\Omega)\,, && (4.1e)\\
&\overline{z}_n(t) \rightharpoonup z(t) \text{ in } W^{1,q}(\Omega) \text{ for all } t \in [0,T]\,, && (4.1f)\\
&\overline{z}_n(t) \to z(t) \text{ in } L^r(\Omega) \text{ for all } r \in [1,\infty) \text{ and for all } t \in [0,T]\,, && (4.1g)\\
&\underline{z}_n(t) \rightharpoonup z(t) \text{ in } W^{1,q}(\Omega) \text{ for all } t \in [0,T]\backslash J\,, && (4.1h)\\
&\underline{z}_n(t) \to z(t) \text{ in } L^r(\Omega) \text{ for all } r \in [1,\infty) \text{ and for all } t \in [0,T]\backslash J\,, && (4.1i)
\end{aligned}
$$

$$\overline{\theta}_n\,,\,\underline{\theta}_n \rightharpoonup \theta \qquad \text{in } L^2(0,T;H^1(\Omega))\,, \tag{4.1j}$$

$$\overline{\theta}_n\,,\,\underline{\theta}_n\,,\,\theta_n \to \theta \qquad \text{in } L^2(0,T;Y) \quad \text{for all } Y \text{ such that } H^1(\Omega) \Subset Y \subset W^{2,d+\delta}(\Omega)^*\,, \tag{4.1k}$$

$$\overline{\theta}_n\,,\,\underline{\theta}_n\,,\,\theta_n \to \theta \qquad \text{in } L^p((0,T)\times\Omega) \quad \text{for all } p \in \begin{cases} [1,8/3) & \text{if } d{=}3\,, \\ [1,3) & \text{if } d{=}2\,, \end{cases} \tag{4.1l}$$

$$\theta_n(t) \rightharpoonup \theta(t) \ \text{ in } W^{2,d+\delta}(\Omega)^* \text{ for all } t \in [0,T]\,, \tag{4.1m}$$

The set $J \subset [0,T]$ *appearing in* (4.1h)–(4.1i) *denotes the jump set of* $z \in \mathrm{BV}([0,T];L^1(\Omega))$*. Finally,*

$$|\dot{z}_n| \to |\dot{z}| \quad \text{in the sense of measures on } (0,T)\times\Omega\,. \tag{4.1n}$$

Proof. **Convergence of the displacements:** The convergences (4.1a), (4.1b), and (4.1c) follow by compactness from (3.36a), (3.36c), and (3.36d). As $u_n(t) - \overline{u}_n(t) = (t - t_n^k)\dot{u}_n(t)$ and $u_n(t) - \underline{u}_n(t) = (t - t_n^{k-1})\dot{u}_n(t)$, we immediately deduce from (4.1b) that the sequences u_n, $\overline{u}_n$, and $\underline{u}_n$ have the same limit in $L^\infty(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))$. Furthermore, due to estimate (3.36e), by compactness, there exists a further subsequence such that $\dot{u}_n \rightharpoonup \dot{u}$ in $\mathrm{BV}([0,T];W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*)$ as well as $\dot{u}_n(t) \rightharpoonup \dot{u}(t)$ in $W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*$ for all $t \in [0,T]$. Thanks to (3.36d), arguing by contradiction and using that $L^2(\Omega;\mathbb{R}^d)$ is dense in $W^{1,\gamma}_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*$, we may also conclude that $\dot{u}_n(t) \rightharpoonup \dot{u}(t)$ in $L^2(\Omega;\mathbb{R}^d)$ for all $t \in [0,T]$, i.e. (4.1d).

Convergence of the damage variables: From estimates (3.36f) on the $\mathcal{R}_1$-total variation of $(\overline{z}_n)_n$ (by monotonicity of $\overline{z}_n$), combined with (3.36h), a generalized version of Helly's selection principle, cf. e.g. [MT04, Theorem 6.1], allows us to extract a subsequence such that $\overline{z}_n(t) \rightharpoonup z(t)$ and $\underline{z}_n(t) \rightharpoonup \underline{z}(t)$ weakly in $W^{1,q}(\Omega)$ for all $t \in [0,T]$, and $z, \underline{z} \in L^\infty(0,T;W^{1,q}(\Omega))$. Moreover, the limit functions z and $\underline{z}$ inherit the monotonicity in time from $\overline{z}_n$ and $\underline{z}_n$, hence $z, \underline{z} \in \mathrm{BV}([0,T];L^1(\Omega))$, and their jump sets J and $\underline{J}$ are at most countable. Let $t \in [0,T]\backslash(J \cup \underline{J})$ fixed. Then, by (3.27), for every $n \in \mathbb{N}$ we have $\overline{z}_n(t - \tau_n) = \underline{z}_n(t)$ and therefore as $n \to \infty$ we get $z(t) = \underline{z}(t)$. Let now $t \in J \cup \underline{J}$ and let $(t_j^-)_j, (t_j^+)_j \subset [0,T]\backslash(J \cup \underline{J})$ be such that $t_j^- \nearrow t$ and $t_j^+ \searrow t$. Since z and $\underline{z}$ coincide on $[0,T]\backslash(J \cup \underline{J})$, we deduce that the left and the right limit satisfy $z^-(t) = \lim_j z(t_j^-) = \lim_j \underline{z}(t_j^-) = \underline{z}^-(t)$ and $z^+(t) = \lim_j z(t_j^+) = \lim_j \underline{z}(t_j^+) = \underline{z}^+(t)$. Therefore $J = \underline{J}$ and the convergences (4.1e), (4.1f), (4.1h) hold. From this, using (3.36g) we conclude that (4.1g) and (4.1i) hold true as well. In this line, we conclude by observing that (4.1n) follows from the fact that $\int_\Omega (z_n(0) - z_n(T))\,\mathrm{d}x$, i.e. the total variation of $\dot{z}_n$ on $[0,T]\times\overline{\Omega}$, converges to the total variation $\int_\Omega (z(0) - z(T))\,\mathrm{d}x$ of $\dot{z}$, also relying on the argument from [Rou10, Proposition 4.3, proof of (4.80)].

Convergence of the temperature variables: Due to estimate (3.36j) we have $\overline{\theta}_n \rightharpoonup \theta$ in $L^2(0,T;H^1(\Omega))$. Exploiting the definition of the interpolants (3.27), similarly to the arguments for the damage variables, we conclude that also $\underline{\theta}_n \rightharpoonup \theta$ in $L^2(0,T;H^1(\Omega))$, thus (4.1j) is proven. From this, convergences (4.1k) and (4.1l) for $(\overline{\theta}_n, \underline{\theta}_n)_n$ follow by a generalized Aubin-Lions Lemma, cf. [Rou05, Corollary 7.9, p. 196], making use of the estimates (3.36j), (3.36k), and (3.36l). Taking into account that $|\theta_n(t,x)| \le \max\{|\overline{\theta}_n(t,x)|, |\underline{\theta}_n(t,x)|\}$ for almost all $(t,x) \in (0,T)\times\Omega$, (a generalized version of) the Lebesgue Theorem yields convergence (4.1l) for $(\theta_n)_n$ as well. All in all, we conclude the weak convergence (4.1j), as well as (4.1k), for $(\theta_n)_n$. Convergence (4.1m) is a consequence of [MT04, Theorem 6.1]. The positivity properties (2.22) and (2.24) (under the additional (2.23)) then follow from their discrete analogues (3.5) and (3.6), respectively, combined with (3.36k). □

The fact that the limit triple (u, z, θ) is an energetic solution of the limit problem will be verified in Sections 4.1–4.3 right below. For this, in Section 4.1, we first pass from time-discrete to continuous in the weak momentum balance (3.31b) using suitably chosen time-discrete test functions and deduce a time-continuous limit *inequality* for the mechanical energy balance (3.30) by lower semicontinuity arguments. Secondly, in Section 4.2 we pass to the limit in the semistability inequality (3.31a) using mutual recovery sequences. As a further step in Section 4.3 it has to be verified that the limit triple (u, z, θ) indeed satisfies the mechanical energy balance as an *equality* by deducing the reverse inequality from the momentum balance and the semistability so far obtained. This result allows us to conclude the convergence of the viscous dissipation terms, which, in turn, is crucial for the limit passage in the heat equation (3.31e).

Altogether, these steps amount to the following

Proposition 4.2 (Energetic solution of the limit problem)**.** *Let the assumptions of Theorem 2.7 be satisfied and let* (u,z,θ) *be a triple of regularity* (2.17) *obtained as a limit, in the sense of convergences* (4.1)*, of a sequence of solutions to Problem 3.1. Then,* (u,z,θ) *is an energetic solution of the time-continuous problem* (1.1)*, supplemented with the boundary conditions* (1.3)*, in the sense of Definition 2.4.*

Proof. The statement of the theorem follows directly by combining Propositions 4.3, 4.5, 4.6, and 4.8. □

4.1 Limit passage in the momentum balance and the energy inequalities

Based on the convergence properties (4.1) we now pass from time-discrete to time-continuous in the weak momentum balance. By lower semicontinuity we will then carry out the limit passage in the mechanical as well as in the total energy inequality and obtain their analogues for the limit problem. Let us start with some considerations on the limit passage in the time-discrete momentum balance (3.31b). Here, one technicality arises from the regularization term of γ-growth ($\gamma > 4$) which was needed in order to handle the right-hand side of the heat equation. Observe that $|e|^{\gamma-2}e \in L^{\gamma'}(\Omega,\mathbb{R}^{d\times d})$ with $\gamma' = \gamma/(\gamma-1) < 2$ if $\gamma > 2$. Hence, test functions from the space $H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)$, which are used in the time-continuous momentum balance (2.19b), are not in duality with this regularization term. To compensate this lack of integrability for test functions one may argue by density, i.e., by [Bur98, p. 56, Corollary 2] and [Rou05, p. 189, Lemma 7.2], for any $v \in L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^d))$ and $\varepsilon > 0$ there is

$$\begin{aligned} &v^\star \in L^2(0,T;\mathrm{C}^1(\overline{\Omega};\mathbb{R}^d)) \cap L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^d)):\\ &\|v-v^\star\|_{L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))\cap H^1(0,T;L^2(\Omega;\mathbb{R}^d))} \le \varepsilon \ \text{ and } \ v^\star = v \text{ on } \partial_{\mathrm{D}}\Omega \text{ in the trace sense.} \end{aligned} \tag{4.2}$$

Furthermore, observe that (3.31b) only admits "discrete" test functions, that is why we will have to discretize each $v^\star$ with the enhanced spatial regularity (4.2). Namely, we construct the discrete test functions by evaluating $v^\star$ at the nodes of the partition, i.e., we set $(v^\star)^k_n := v(t^k_n)$ for all $k = 0,\dots,n$, and then define the piecewise constant and linear interpolants $\overline{v}^\star_n$ and $v^\star_n$. In view of (4.2), it can be checked that

$$\begin{aligned} &\overline{v}^\star_n \to v^\star \text{ in } L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)) \ \text{ and } \ v^\star_n \to v^\star \text{ in } H^1(0,T;L^2(\Omega;\mathbb{R}^d)),\\ &\tau_n^{1/\gamma}\|e(\overline{v}^\star_n)\|_{L^\gamma(0,T;L^\gamma(\Omega;\mathbb{R}^{d\times d}))} \to 0\,. \end{aligned} \tag{4.3a}$$

Observe that (4.3a) implies

$$v^\star_n(t) \to v^\star \text{ in } L^2(\Omega;\mathbb{R}^d) \text{ for all } t \in [0,T]\,. \tag{4.3b}$$

Using such sequences $(\overline{v}^\star_n, v^\star_n)_n$ of interpolants of smooth, dense test functions, we can now carry out the limit passage in the momentum balance. While most of the terms can be treated straight forward by exploiting the convergence properties (4.1), the quadratic terms arising from the stored elastic energy and the viscous dissipation, which involve state-dependent coefficients, i.e. $\mathbb{D}(\underline{z}_n,\underline{\theta}_n)$ and $\mathbb{C}(\overline{z}_n)$, need special attention. For these terms the limit will be deduced by exploiting the L^∞-bounds (2.3) on $\mathbb{C}$ and $\mathbb{D}$ and the dominated convergence theorem.

Proposition 4.3 (Limit passage in the weak momentum balance)**.** *Let the assumptions of Theorem 2.7 be satisfied. Then, a limit triple* (u,z,θ) *extracted as in Proposition 4.1 solves the time-continuous momentum balance* (2.19b) *at every* $t \in [0,T]$*. In particular,* $\dot u \in H^1(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)) \subset \mathrm{C}^0_{\mathrm{weak}}([0,T];L^2(\Omega;\mathbb{R}^d))$*.*

Proof. Given $v \in L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^d))$, let $v^\star$ be as in (4.2). We pass to the limit in (3.31b). By the convergence properties of the given data (3.28a) and for the smooth test functions

(4.3), together with the convergence results (4.1d), (4.1b) and (4.1j) we immediately find

$$
\begin{aligned}
&\rho\int_\Omega \big(\dot u_n(t)\cdot v_n^\star(t) - \dot u_0\cdot v_n^\star(0)\big)\,\mathrm{d}x - \int_0^{\overline{\tau}_n(t)}\!\!\Big(\int_\Omega \big(\rho \dot u_n(s{-}\tau_n)\cdot \dot v_n^\star - \overline{\theta}_n\,\mathbb{B}:e(\overline{v}_n^\star)\big)\,\mathrm{d}x - \big\langle \overline{f}_n, \overline{v}_n^\star\big\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}\Big)\,\mathrm{d}s\\
&\longrightarrow \rho\int_\Omega \big(\dot u(t)\cdot v^\star(t) - \dot u_0\cdot v^\star(0)\big)\,\mathrm{d}x - \int_0^t\!\Big(\int_\Omega \big(\rho \dot u\cdot \dot v^\star - \theta\,\mathbb{B}:e(v^\star)\big)\,\mathrm{d}x - \langle f, v^\star\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}\Big)\,\mathrm{d}s\,. \qquad (4.4)
\end{aligned}
$$

Moreover, the convergence of the term involving the γ-Laplacian follows from the estimate

$$
\left|\int_0^t\int_\Omega \tau_n|e(\overline{u}_n)|^{\gamma-2}e(\overline{u}_n):e(\overline{v}_n^\star)\,\mathrm{d}x\,\mathrm{d}s\right| \le \tau_n^{\frac{\gamma-1}{\gamma}}\,\|e(\overline{u}_n)\|^{\gamma-1}_{L^\gamma((0,T)\times\Omega;\mathbb{R}^{d\times d})}\tau_n^{\frac{1}{\gamma}}\|e(\overline{v}_n^\star)\|_{L^\gamma((0,T)\times\Omega;\mathbb{R}^{d\times d})} \to 0\,,
$$

due to the uniform bound (3.36b) and the convergence of $(v_n^\star)_n$ by (4.3).

Finally, in order to handle the quadratic terms with state-dependent coefficients, we prove that

$$
\big(\mathbb{D}(\underline{z}_n,\underline{\theta}_n) + \mathbb{C}(\overline{z}_n)\big)e(\overline{v}_n^\star) \to \big(\mathbb{D}(z,\theta)+\mathbb{C}(z)\big)e(v^\star) \text{ strongly in } L^2((0,T)\times\Omega;\mathbb{R}^{d\times d})\,. \qquad (4.5)
$$

Then, the convergence of the quadratic terms with state-dependent coefficients follows from weak-strong convergence, using that both $e(\dot u_n)\rightharpoonup e(\dot u)$ and $e(u_n)\rightharpoonup e(u)$ weakly in $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ by (4.1b). Now, to verify (4.5) we are going to apply the dominated convergence theorem. For this, we observe that for a.e. $t\in(0,T)$ we have $|\big(\mathbb{D}(\underline{z}_n(t),\underline{\theta}_n(t)) + \mathbb{C}(\overline{z}_n(t))\big):e(\overline{v}_n^\star(t))| \to |\big(\mathbb{D}(z(t),\theta(t))+\mathbb{C}(z(t))\big):e(v(t))|$ pointwise a.e. in Ω, by assumption (2.3b) and since by convergence results (4.1i) and (4.1k) we can resort to a subsequence $(\underline{z}_n(t),\overline{z}_n(t),\underline{\theta}_n)_n$ that converges pointwise a.e. in Ω for a.e. $t\in(0,T)$. Moreover, by assumption (2.3), more precisely by its consequence (2.7), we find an integrable, convergent majorant, i.e.,

$$
\big|\big(\mathbb{D}(\underline{z}_n,\underline{\theta}_n) + \mathbb{C}(\overline{z}_n)\big)e(\overline{v}_n^\star)\big| \le (C_{\mathbb{D}}^2 + C_{\mathbb{C}}^2)|e(\overline{v}_n^\star)| \;\to\; (C_{\mathbb{D}}^2 + C_{\mathbb{C}}^2)|e(v^\star)|
$$

pointwise a.e. in $(0,T)\times\Omega$ and with respect to the strong $L^2((0,T)\times\Omega))$-topology by (4.3). Hence, Pratt's version of the dominated convergence theorem yields (4.5). This concludes the limit passage in the momentum balance for smooth test function as in (4.2). By density this result carries over to all test functions $v\in L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))\cap H^1(0,T;L^2(\Omega;\mathbb{R}^d))$. As by (4.1d) we have $\dot u(t)\in L^2(\Omega;\mathbb{R}^d)$ for every $t\in[0,T]$, we immediately deduce that (2.19b) holds true at all $t\in[0,T]$.

The last assertion follows from Remark 2.6. □

Lemma 4.4 (Energy inequalities by lower semicontinuity)**.** *Let the assumptions of Theorem 2.7 be satisfied and let (u,z,θ) be a limit triple given by Proposition 4.1. Then for every $t\in[0,T]$ we have*

$$
\begin{aligned}
&\tfrac{\rho}{2}\int_\Omega |\dot u(t)|^2\,\mathrm{d}x + \mathcal{E}(t,u(t),z(t)) + \int_\Omega (z(t){-}z_0)\,\mathrm{d}x + \int_0^t\int_\Omega (\mathbb{D}(z,\theta)e(\dot u){-}\theta\,\mathbb{B}):e(\dot u)\,\mathrm{d}x\,\mathrm{d}s\\
&\le \tfrac{\rho}{2}\int_\Omega |\dot u_0|^2\,\mathrm{d}x + \mathcal{E}(0,u_0,z_0) - \int_0^t \Big\langle \dot f, v\Big\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}\,\mathrm{d}s\,.
\end{aligned}
\qquad (4.6)
$$

Proof. It is enough to pass to the limit in (3.31c) taking into account (3.28b), (4.1d), (4.1i), and (4.1k). □

4.2 Limit passage in the semistability inequality

In order to carry out the passage from time-discrete to continuous in the semistability inequality we follow the well-established method of circumventing a direct passage to the limit on the left- and on the right-hand side of the semistability inequality (3.31a). Instead, it is enough to prove a limsup inequality for the difference, cf. also [MR06, MRS08], using a so-called mutual recovery sequence. This procedure, which allows one to take advantage of some cancelations in the regularizing terms for the internal variable $\mathcal{G}(z,\nabla z)$, has been already employed in [MR06, TM10, Tho13] in problems concerned with (fully) rate-independent, partial, isotropic and unidirectional damage, featuring a $W^{1,q}(\Omega)$-gradient regularization, with $q{>}d$ in [MR06], any $q{>}1$ in [TM10] as in the present context, and $q{=}1$ in [Tho13]. In what follows, we verify that the recovery sequence constructed in [TM10], where $\mathcal{G}(z,\nabla z)=|\nabla z|^q$, is also suited in our setting of semistability with a general gradient term.

More precisely, let us fix $t\in[0,T]$ in the energy functionals $\mathcal{E}_n$ from (3.30), and a sequence $(v_n,\zeta_n)_n\subset H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)\times\mathcal{Z}$ such that

$$\begin{aligned} &v_n\rightharpoonup v \quad \text{weakly in } H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)\,,\quad \zeta_n\rightharpoonup\zeta \quad \text{weakly in } W^{1,q}(\Omega)\,,\\ &\mathcal{E}_n(t,v_n,\zeta_n)\le\mathcal{E}_n(t,v_n,\hat\zeta)+\mathcal{R}_1(\hat\zeta-\zeta_n) \quad \text{for all } \hat\zeta\in\mathcal{Z}\,, \end{aligned} \tag{4.7}$$

i.e., ζ_n is semistable for $\mathcal{E}_n(t,v_n,\cdot)$. Given $\tilde\zeta\in\mathcal{Z}$ let the recovery sequence $(\tilde\zeta_n)_n\subset\mathcal{Z}$ be defined by

$$\tilde\zeta_n:=\min\big\{\zeta_n,\max\{(\tilde\zeta-\delta_n,0)\}\big\}=\begin{cases}(\tilde\zeta-\delta_n) & \text{on } A_n=\big\{0\le(\tilde\zeta-\delta_n)\le\zeta_n\big\}\,,\\ \zeta_n & \text{on } B_n=\big\{\tilde\zeta-\delta_n>\zeta_n\big\}\,,\\ 0 & \text{on } C_n=\big\{\tilde\zeta-\delta_n<0\big\}\,,\end{cases} \tag{4.8}$$
$$\text{where } \delta_n:=\|\zeta_n-\zeta\|^{1/q}_{L^q(\Omega)}\,.$$

The sequence $(\tilde\zeta_n)_n$ was introduced in [TM10] where it was shown that

$$\tilde\zeta_n\rightharpoonup\tilde\zeta \;\text{ in } W^{1,q}(\Omega)\;\text{ for } q\in(1,\infty) \text{ from (2.9d) fixed.} \tag{4.9}$$

Note however that strong convergence in $W^{1,q}(\Omega)$ cannot be expected, since $\zeta_n\rightharpoonup\zeta$ weakly in $W^{1,q}(\Omega)$, only. This makes it impossible to show directly that $\mathcal{G}(\tilde\zeta_n,\nabla\tilde\zeta_n)\to\mathcal{G}(\tilde\zeta,\nabla\tilde\zeta)$, since this would require the strong convergence of the gradients. Nevertheless the following result holds.

Theorem 4.5. *Let the assumptions of Theorem 2.7 be satisfied. Let $t\in[0,T]$ be fixed and consider a sequence $(v_n,\zeta_n)_n\subset H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)\times\mathcal{Z}$ such that (4.7) holds. Given $\tilde\zeta\in\mathcal{Z}$, let $(\tilde\zeta_n)_n\subset\mathcal{Z}$ as in (4.8). Then*

$$0\le\limsup_{n\to\infty}\Big(\mathcal{E}_n(t,v_n,\tilde\zeta_n)-\mathcal{E}_n(t,v_n,\zeta_n)+\mathcal{R}_1(\tilde\zeta_n-\zeta_n)\Big)\le\mathcal{E}(t,v,\tilde\zeta)-\mathcal{E}(t,v,\zeta)+\mathcal{R}_1(\tilde\zeta-\zeta)\,. \tag{4.10}$$

Therefore the limit ζ is semistable for $\mathcal{E}(t,v,\cdot)$.

Proof. First of all note that, if $\tilde\zeta\in\mathcal{Z}$ does not satisfy $0\le\tilde\zeta\le\zeta$, then (4.10) trivially holds, since in this case $\mathcal{R}_1(\tilde\zeta-\zeta)=+\infty$.

Assume now $0\le\tilde\zeta\le\zeta$ for a.e. $x\in\Omega$. Let us estimate the left-hand side of (4.10) as follows:

$$\begin{aligned}&\limsup_{n\to\infty}\Big(\mathcal{E}_n(t,v_n,\tilde\zeta_n)-\mathcal{E}_n(t,v_n,\zeta_n)+\mathcal{R}_1(\tilde\zeta_n-\zeta_n)\Big)\\ &\le\limsup_{n\to\infty}\int_\Omega(\mathbb{C}(\tilde\zeta_n)-\mathbb{C}(\zeta_n))e(v_n):e(v_n)\,\mathrm{d}x+\limsup_{n\to\infty}\big(\mathcal{G}(\tilde\zeta_n,\nabla\tilde\zeta_n)-\mathcal{G}(\zeta_n,\nabla\zeta_n)\big)+\limsup_{n\to\infty}\mathcal{R}_1(\tilde\zeta_n-\zeta_n)\end{aligned} \tag{4.11}$$

and then treat each of the terms on the right-hand side of (4.11) separately. Since $\zeta_n\rightharpoonup\zeta$ in $W^{1,q}(\Omega)$, we may choose a (not relabeled) subsequence that converges pointwise a.e. in Ω.

Estimation of $\limsup_{n\to\infty}\big(\mathcal{G}(\tilde\zeta_n,\nabla\tilde\zeta_n)-\mathcal{G}(\zeta_n,\nabla\zeta_n)\big)$**:** Note that $G(\tilde\zeta_n,\nabla\tilde\zeta_n)=G(\zeta_n,\nabla\zeta_n)$ on B_n. If $\|\zeta_n-\zeta\|_{L^q(\Omega)}>0$, by Markov's inequality

$$\mathcal{L}^d(B_n)\le\mathcal{L}^d([\delta_n\le|\zeta_n-\zeta|])\le\tfrac{1}{\delta_n}\int_\Omega|\zeta_n-\zeta|\,\mathrm{d}x\le\tfrac{1}{\delta_n}\|\zeta_n-\zeta\|_{L^q(\Omega)}\to0\,, \tag{4.12}$$

with δ_n from (4.8), while for $\|\zeta_n-\zeta\|_{L^q(\Omega)}=0$ it is indeed $\mathcal{L}^d(B_n)=0$, thus

$$\mathcal{L}^d(A_n\cup C_n)\to\mathcal{L}^d(\Omega)\,. \tag{4.13}$$

In what follows, $\mathcal{X}_D$ will denote the characteristic function of a set D. By (2.9b), (2.9d) and (4.8), we deduce

$$\begin{aligned}&\limsup_{n\to\infty}\big(\mathcal{G}(\tilde\zeta_n,\nabla\tilde\zeta_n)-\mathcal{G}(\zeta_n,\nabla\zeta_n)\big)\\ &=\limsup_{n\to\infty}\int_{A_n}G((\tilde\zeta-\delta_n),\nabla\tilde\zeta)\,\mathrm{d}x+\int_{C_n}G(0,0)\,\mathrm{d}x-\int_{A_n\cup C_n}G(\zeta_n,\nabla\zeta_n)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\le \limsup_{n\to\infty} \Big(\int_\Omega G(\mathcal{X}_{A_n}(\tilde\zeta - \delta_n), \mathcal{X}_{A_n}\nabla\tilde\zeta)\,\mathrm{d}x + \int_\Omega G(0, \mathcal{X}_{C_n}\nabla\tilde\zeta)\,\mathrm{d}x - \int_\Omega G(\mathcal{X}_{A_n\cup C_n}\zeta_n, \mathcal{X}_{A_n\cup C_n}\nabla\zeta_n)\,\mathrm{d}x \Big) \\
&= \limsup_{n\to\infty} \Big(\int_\Omega G(\mathcal{X}_{A_n\cup C_n}(\tilde\zeta_n), \mathcal{X}_{A_n\cup C_n}\nabla\tilde\zeta)\,\mathrm{d}x - \int_\Omega G(\mathcal{X}_{A_n\cup C_n}\zeta_n, \mathcal{X}_{A_n\cup C_n}\nabla\zeta_n)\,\mathrm{d}x \Big) \\
&\le \mathcal{G}(\tilde\zeta, \nabla\tilde\zeta) - \liminf_{n\to\infty} \mathcal{G}(\mathcal{X}_{A_n\cup C_n}\zeta_n, \mathcal{X}_{A_n\cup C_n}\nabla\zeta_n) &&(4.14a)\\
&\le \mathcal{G}(\tilde\zeta, \nabla\tilde\zeta) - \mathcal{G}(\zeta, \nabla\zeta)\,, &&(4.14b)
\end{aligned}
$$

where in the second integral term in the third line we have used the obvious identity $\mathcal{X}_{C_n} 0 = 0$. To obtain (4.14a) we have used the dominated convergence theorem, while in order to prove (4.14b) we employed the lower semicontinuity of $\mathcal{G} : L^q(\Omega) \times L^q(\Omega;\mathbb{R}^d) \to \mathbb{R}\cup\{\infty\}$, since, by (4.9) and (4.13), we have $\mathcal{X}_{A_n\cup C_n}\zeta_n \to \zeta$ strongly in $L^q(\Omega)$ and $\mathcal{X}_{A_n\cup C_n}\nabla\zeta_n \rightharpoonup \nabla\zeta$ weakly in $L^q(\Omega;\mathbb{R}^d)$.

Estimation of the remaining terms in (4.11)**:** Since construction (4.8) ensures $\mathcal{R}_1(\tilde\zeta_n - \zeta_n) < \infty$ for every $n \in \mathbb{N}$, as well as $\tilde\zeta_n \to \tilde\zeta$ in $L^q(\Omega)$, due to $\zeta_n \to \zeta$ in $L^q(\Omega)$, we immediately conclude that $\mathcal{R}_1(\tilde\zeta_n - \zeta_n) \to \mathcal{R}_1(\tilde\zeta - \zeta)$.

In order to estimate the difference of the quadratic terms in the mechanical energy we note that $(\mathbb{C}(\tilde\zeta_n) - \mathbb{C}(\zeta_n))e(v_n) : e(v_n) \le 0$ due to $\tilde\zeta_n \le \zeta_n$ by construction and by the monotonicity assumption (2.4). We introduce the abbreviation $\mathbb{C}_n := (\mathbb{C}(\zeta_n) - \mathbb{C}(\tilde\zeta_n) + \delta_n \mathrm{Id}) \in \mathbb{R}^{d\times d\times d\times d}$ and note that for every $n \in \mathbb{N}$ the tensor $\mathbb{C}_n$ is symmetric and positively definite with $\mathrm{Id} \in \mathbb{R}^{d\times d\times d\times d}$ denoting the identity tensor. Since both $\zeta_n \to \zeta$ and $\tilde\zeta_n \to \tilde\zeta$ in $L^q(\Omega)$, as well as $\delta_n \to 0$, the Lipschitz-continuity of $\mathbb{C}$, cf. (2.3b), implies that $\mathbb{C}_n \to (\mathbb{C}(\zeta) - \mathbb{C}(\tilde\zeta))$ in $L^q(\Omega;\mathbb{R}^{d\times d\times d\times d})$.

Furthermore, we define the functional $\mathcal{C} : L^q(\Omega;\mathbb{R}^{d\times d\times d\times d}) \times L^2(\Omega;\mathbb{R}^{d\times d}) \to \mathbb{R}\cup\{\infty\}$,

$$
\mathcal{C}(\xi, e) := \int_\Omega C(\xi, e)\,\mathrm{d}x \quad \text{with} \quad C(\xi, e) := \begin{cases} \xi e : e & \text{if } \xi \in \mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym},C^1_{\mathbb{C}}}, \\ \infty & \text{otherwise}, \end{cases} \tag{4.15}
$$

where $\mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym},C^1_{\mathbb{C}}} := \{\xi \in \mathbb{R}^{d\times d\times d\times d},\ \xi \text{ symmetric and } \forall A \in \mathbb{R}^{d\times d}\colon C^1_{\mathbb{C}}|A|^2 \le \xi A : A\}$ with the constant $C^1_{\mathbb{C}} > 0$ from (2.3). Note that $C : \mathbb{R}^{d\times d\times d\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}\cup\{\infty\}$ is lower semicontinuous, since the set $\mathbb{R}^{d\times d\times d\times d}_{\mathrm{sym},C^1_{\mathbb{C}}}$ is closed and convex. Moreover, for all $\xi \in \mathbb{R}^{d\times d\times d\times d}$ fixed, $C(\xi,\cdot)$ is convex. Hence, the functional $\mathcal{C}$ is lower semicontinuous with respect to the strong convergence in $L^q(\Omega;\mathbb{R}^{d\times d\times d\times d})$ and the weak convergence in $L^2(\Omega;\mathbb{R}^{d\times d})$, as can be concluded e.g. by [FL07, Theorem 7.5, p. 492]. Thus, the first term on the right-hand side of (4.11) can be rewritten and estimated using (3.36c) and the lower semicontinuity of $\mathcal{C}$ as follows,

$$
\begin{aligned}
\limsup_{n\to\infty} \int_\Omega (\mathbb{C}(\tilde\zeta_n) - \mathbb{C}(\zeta_n))e(v_n) : e(v_n)\,\mathrm{d}x &= \limsup_{n\to\infty} \Big(\delta_n \|e(v_n)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} - \mathcal{C}(\mathbb{C}_n, e(v_n)) \Big) \\
&\le \lim_{n\to\infty} \delta_n C - \liminf_{n\to\infty} \mathcal{C}(\mathbb{C}_n, e(v_n)) \\
&\le -\mathcal{C}((\mathbb{C}(\zeta) - \mathbb{C}(\tilde\zeta)), e(v)) = \int_\Omega (\mathbb{C}(\tilde\zeta) - \mathbb{C}(\zeta))e(v) : e(v)\,\mathrm{d}x\,.
\end{aligned}
$$

Combining the above established estimates for the three terms on the right-hand side of (4.11) shows that condition (4.10) is satisfied. □

4.3 Energy equalities and limit passage in the heat equation

We now show that the limit triple (u, z, θ) satisfies the mechanical energy equality (2.19c). The inequality $(\le)$ has been proven in Lemma 4.4. The opposite inequality is found by approximation with Riemann sums, as common in existence proofs of rate-independent and rate-dependent evolutions, see e.g. [DMFT05].

Proposition 4.6 (Mechanical energy equality)**.** *Let the assumptions of Theorem 2.7 be satisfied, let (u, z, θ) be a triple given by Proposition 4.1, and let $t \in [0, T]$. Then* (2.19c) *holds.*

Proof. We fix a sequence of subdivisions $(s_n^k)_{0\le k\le k_n}$ of the interval $[0,t]$, with $0 = s_n^0 < s_n^1 < \cdots < s_n^{k_n-1} < s_n^{k_n} = t$, $\lim_n \max_k (s_n^k - s_n^{k-1}) = 0$, and

$$\left| \sum_{k=1}^{k_n} \int_{s_n^{k-1}}^{s_n^k} \int_\Omega \left[\mathbb{C}(z(s_n^k)) - \mathbb{C}(z(s)) \right] e(u(s)) : e(\dot u(s)) \,\mathrm{d}x \,\mathrm{d}s \right| \to 0 \,. \tag{4.16}$$

The existence of such a sequence is guaranteed by [Hah14], see also [Rou10, Proposition 4.3, Step 7]. Taking $z(s_n^k)$ as test function in the time-continuous semistability inequality (2.19a) at time s_n^{k-1} we get

$$\begin{aligned}
\mathcal{E}(s_n^{k-1}, u(s_n^{k-1}), z(s_n^{k-1})) \le\ & \mathcal{E}(s_n^{k-1}, u(s_n^{k-1}), z(s_n^k)) + \int_\Omega (z(s_n^{k-1}) - z(s_n^k)) \,\mathrm{d}x \\
&= \mathcal{E}(s_n^k, u(s_n^k), z(s_n^k)) + \int_\Omega (z(s_n^{k-1}) - z(s_n^k)) \,\mathrm{d}x - \int_{s_n^{k-1}}^{s_n^k} \partial_t \mathcal{E}(s, u(s), z(s)) \,\mathrm{d}s \\
&\quad + \int_{s_n^{k-1}}^{s_n^k} \langle f(s), \dot u(s) \rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}s - \int_{s_n^{k-1}}^{s_n^k} \int_\Omega \mathbb{C}(z(s_n^k)) e(u(s)) : e(\dot u(s)) \,\mathrm{d}x \,\mathrm{d}s \,.
\end{aligned}$$

Next we sum up the previous inequality over $k = 1, \ldots, k_n$ and we pass to the limit in n in the last term thanks to (4.16), obtaining

$$\begin{aligned}
\mathcal{E}(0, u_0, z_0) \le \mathcal{E}(t, u(t), z(t)) &+ \int_\Omega (z_0 - z(t)) \,\mathrm{d}x - \int_0^t \partial_t \mathcal{E}(s, u(s), z(s)) \,\mathrm{d}s \\
&+ \int_0^t \langle f(s), \dot u(s) \rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}s - \int_0^t \int_\Omega \mathbb{C}(z(s)) e(u(s)) : e(\dot u(s)) \,\mathrm{d}x \,\mathrm{d}s \,.
\end{aligned} \tag{4.17}$$

Further, thanks to Remark 2.6 we can test (2.19b) by $\dot u$ and get

$$\begin{aligned}
&\tfrac{\varrho}{2} \|\dot u(t)\|^2_{L^2(\Omega;\mathbb{R}^d)} + \int_0^t \int_\Omega (\mathbb{D}(z,\theta) e(\dot u) + \mathbb{C}(z) e(u) - \theta\, \mathbb{B}) : e(\dot u) \,\mathrm{d}x \,\mathrm{d}s \\
&= \tfrac{\varrho}{2} \|\dot u_0\|^2_{L^2(\Omega;\mathbb{R}^d)} + \int_0^t \langle f, \dot u \rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}s \,,
\end{aligned} \tag{4.18}$$

where we applied the by-part integration formula (2.21), as allowed by [Rou05, Lemma 7.3]. Summing up (4.18) with (4.17) we obtain

$$\begin{aligned}
\mathcal{E}(0, u_0, z_0) \le \mathcal{E}(t, u(t), z(t)) &+ \tfrac{\varrho}{2} \int_\Omega |\dot u(t)|^2 \,\mathrm{d}x + \int_\Omega (z_0 - z(t)) \,\mathrm{d}x - \int_0^t \partial_t \mathcal{E}(s, u(s), z(s)) \,\mathrm{d}s \\
&- \tfrac{\varrho}{2} \int_\Omega |\dot u_0|^2 \,\mathrm{d}x + \int_0^t \int_\Omega (\mathbb{D}(z(s), \theta(s)) e(\dot u(s)) - \theta(s)\, \mathbb{B}) : e(\dot u(s)) \,\mathrm{d}x \,\mathrm{d}s \,.
\end{aligned}$$

Combining this estimate with the reverse inequality (4.6) concludes the proof of (2.19c). □

Lemma 4.7 (Stronger convergences)**.** *Let the assumptions of Theorem 2.7 be satisfied and let* (u, z, θ) *be a triple given by Proposition 4.1. Then*

$$\lim_{n\to\infty} \int_0^T \int_\Omega \mathbb{D}(\underline{z}_n, \underline{\theta}_n) e(\dot u_n) : e(\dot u_n) \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega \mathbb{D}(z, \theta) e(\dot u) : e(\dot u) \,\mathrm{d}x \,\mathrm{d}t \,. \tag{4.19}$$

In particular, recalling that $\mathbb{U}(\cdot,\cdot)$ *denotes the square root tensor of* $\mathbb{D}(\cdot,\cdot)$*, see* (2.8a)*, we conclude that*

$$\mathbb{U}(\underline{z}_n, \underline{\theta}_n) e(\dot u_n) \to \mathbb{U}(z, \theta) e(\dot u) \qquad \text{strongly } in\ L^2((0,T) \times \Omega; \mathbb{R}^{d\times d}) \,. \tag{4.20}$$

Proof. By lower semicontinuity, taking into account the convergences already proven in Proposition 4.1, together with both the discrete mechanical energy inequality (3.31c) and the mechanical energy equality (2.19c), the following chain of inequalities holds

$$\begin{aligned}
&\int_0^T \int_\Omega \mathbb{D}(z, \theta) e(\dot u) : e(\dot u) \,\mathrm{d}x \,\mathrm{d}t + \int_\Omega (z_0 - z(T)) \,\mathrm{d}x \\
&\le \liminf_n \left(\int_0^T \int_\Omega \mathbb{D}(\underline{z}_n, \underline{\theta}_n) e(\dot u_n) : e(\dot u_n) \,\mathrm{d}x \,\mathrm{d}t + \int_\Omega (z_n(0) - z_n(T)) \,\mathrm{d}x \right)
\end{aligned}$$

$$
\begin{aligned}
&\le \limsup_n \left(\int_0^T \int_\Omega \mathbb{D}(\underline{z}_n, \underline{\theta}_n) e(\dot u_n) : e(\dot u_n) \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega (z_n(0) - z_n(T))\,\mathrm{d}x \right) \\
&\le \limsup_n \Bigg(-\mathcal{E}_n(T, u_n(T), z_n(T)) + \mathcal{E}_n(0, u_0, z_0) - \tfrac{\varrho}{2}\int_\Omega |\dot u_n(T)|^2 \,\mathrm{d}x + \tfrac{\varrho}{2}\int_\Omega |\dot u_0|^2 \,\mathrm{d}x \\
&\qquad\qquad + \int_0^T \int_\Omega \overline{\theta}_n \mathbb{B} : e(\dot u_n)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \partial_t \mathcal{E}_n(s, \underline{u}_n, \underline{z}_n)\,\mathrm{d}s \Bigg) \\
&\le -\mathcal{E}(T, u(T), z(T))) + \mathcal{E}(0, u_0, z_0) - \tfrac{\varrho}{2}\int_\Omega |\dot u(T)|^2 \,\mathrm{d}x + \tfrac{\varrho}{2}\int_\Omega |\dot u_0|^2 \,\mathrm{d}x \\
&\quad + \int_0^T \int_\Omega \theta \mathbb{B} : e(\dot u)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \partial_t \mathcal{E}(s, u, z)\,\mathrm{d}s \\
&= \int_0^T \int_\Omega \mathbb{D}(z, \theta) e(\dot u) : e(\dot u)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega (z_0 - z(T))\,\mathrm{d}x\,.
\end{aligned}
$$

Hence all inequalities above are actually equalities and we deduce that (4.19) holds.

To conclude (4.20), we observe that on the one hand

$$
\mathbb{U}(\underline{z}_n, \underline{\theta}_n) e(\dot u_n) \rightharpoonup \mathbb{U}(z, \theta) e(\dot u) \quad \text{in } L^2(0,T; L^2(\Omega; \mathbb{R}^{d\times d}))\,. \tag{4.21}
$$

This follows from the fact that $\mathbb{U}(\underline{z}_n, \underline{\theta}_n)A \to \mathbb{U}(z,\theta)A$ for every $A \in L^2(0,T; L^2(\Omega; \mathbb{R}^{d\times d}))$, which in turn can be verified relying on the boundedness (2.8c) of $\mathbb{U}(\cdot,\cdot)$, and arguing in the very same way as in the proof of Proposition 4.3, cf. (4.5). Since $e(\dot u_n) \to e(\dot u)$, (4.21) ensues. On the other hand, in view of (2.8b), equation (4.19) reads that

$$
\lim_{n\to\infty} \|\mathbb{U}(\underline{z}_n, \underline{\theta}_n) e(\dot u_n)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} = \|\mathbb{U}(z,\theta) e(\dot u)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))}\,,
$$

whence (4.20). □

Proposition 4.8 (Limit passage in the weak form of the heat equation)**.** *Let the assumptions of Theorem 2.7 be satisfied, Let (u,z,θ) be a triple given by Proposition 4.1, and let $t \in [0,T]$. Then the weak formulation of the heat equation* (2.19d) *holds.*

Proof. Let us fix $\eta \in H^1(0,T; L^2(\Omega)) \cap \mathrm{C}^0([0,T]; W^{2,d+\delta}(\Omega))$, define $\eta_n^k := \eta(t_n^k)$ for all $k = 0, \ldots, n$, and let η_n, $\overline{\eta}_n$ be the piecewise linear and constant interpolations of the values (η_n^k). It can be checked that

$$
\begin{aligned}
&\overline{\eta}_n \to \eta \quad \text{in } L^p(0,T; W^{2,d+\delta}(\Omega)) \text{ for all } 1 \le p < \infty\,, \qquad \overline{\eta}_n \overset{*}{\rightharpoonup} \eta \quad \text{in } L^\infty(0,T; W^{2,d+\delta}(\Omega))\,, \\
&\eta_n \to \eta \quad \text{in } H^1(0,T; L^2(\Omega)) \cap \mathrm{C}^0(0,T; W^{2,d+\delta}(\Omega))\,.
\end{aligned} \tag{4.22}
$$

We now pass to the limit in the discrete heat equation (3.31e) tested by η_n. The first three integral terms on the left-hand side of (3.31e) can be dealt with combining convergences (4.1k)–(4.1m) with (4.22). In order to pass to the limit in the fourth one, we argue along the lines of [RR14b, proof of Theorem 2.8] and derive a finer estimate for $(\mathbb{K}(\overline{z}_n, \overline{\theta}_n)\nabla\overline{\theta}_n)_n$. Indeed, thanks to (2.11b) we have

$$
|\mathbb{K}(\overline{z}_n, \overline{\theta}_n)\nabla\overline{\theta}_n| \le c_2(|\overline{\theta}_n|^{(\kappa-\alpha+2)/2} |\overline{\theta}_n|^{(\kappa+\alpha-2)/2} |\nabla\overline{\theta}_n| + |\nabla\overline{\theta}_n|) \qquad \text{a.e. in } (0,T)\times\Omega\,,
$$

with α as in (3.46). From this particular estimate we also gather that $|\overline{\theta}_n|^{(\kappa+\alpha-2)/2}|\nabla\overline{\theta}_n|$ is bounded in $L^2((0,T)\times\Omega)$. Since $(\overline{\theta}_n)_n$ is bounded in $L^{8/3}((0,T)\times\Omega)$ if d=3 (and in $L^3((0,T)\times\Omega)$ if d=2), choosing $\alpha \in (1/2, 1)$ such that $\kappa - \alpha < 2/3$ (which can be done, since $\kappa < 5/3$), we conclude that $|\overline{\theta}_n|^{(\kappa-\alpha+2)/2}$ is bounded in $L^{2+\delta}((0,T)\times\Omega)$ for some $\delta > 0$. All in all, we have that $\mathbb{K}(\overline{z}_n, \overline{\theta}_n)\nabla\overline{\theta}_n$ is bounded in $L^{1+\delta}((0,T)\times\Omega; \mathbb{R}^d)$ for some $\delta > 0$. With the very same arguments as in [RR14b, proof of Theorem 2.8], we show that

$$
\mathbb{K}(\overline{z}_n, \overline{\theta}_n)\nabla\overline{\theta}_n \rightharpoonup \mathbb{K}(z,\theta)\nabla\theta \quad \text{in } L^{1+\delta}((0,T)\times\Omega; \mathbb{R}^d)\,,
$$

which, combined with convergences (4.22) for $\overline{\eta}_n$, is enough to pass to the limit in the last term on the left-hand side of (3.31e).

Combining (4.1b) with (4.1l) and (4.22) yields $\int_0^{\overline{\tau}_n(t)}\int_\Omega \overline{\theta}_n \mathbb{B} : e(\dot{u}_n)\, \overline{\eta}_n \,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int_\Omega \theta\, \mathbb{B} : e(\dot{u})\, \eta \,\mathrm{d}x\,\mathrm{d}s$ as $n\to\infty$, while the passage to the limit in the term

$$\int_0^{\overline{\tau}_n(t)}\int_\Omega \mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) : e(\dot{u}_n)\, \overline{\eta}_n \,\mathrm{d}x\,\mathrm{d}s = \int_0^{\overline{\tau}_n(t)}\int_\Omega \mathbb{U}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) : \mathbb{U}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n)\, \overline{\eta}_n \,\mathrm{d}x\,\mathrm{d}s\,,$$

cf. (2.8b), results from (4.20) combined with (4.22). Convergence (4.1n) allows us to deal with the second term on the right-hand side of (3.31e), and we handle the last two terms via (3.28c) and (4.22), again. This concludes the proof of the weak heat equation and of the main existence result Theorem 2.7. □

5 Asymptotic behavior in the slow loading regime: the vanishing viscosity and inertia limit

In this section we address the limiting behavior of system (1.1) as the rate of the external load and of the heat sources becomes slower and slower. Accordingly, we will rescale time by a factor $\varepsilon > 0$. For analytical reasons we restrict to the case of a Dirichlet problem in the displacement, namely within this section we shall suppose that

$$\partial_{\mathrm{D}}\Omega = \partial\Omega\,. \tag{5.1}$$

Like in the previous sections, we assume that the Dirichlet datum is homogeneous, cf. (1.3b).

As $\varepsilon \downarrow 0$ we will *simultaneously* pass to

1. a rate-independent system for the limit displacement and damage variables (u, z), which does not display any temperature dependence and which formally reads

$$-\operatorname{div} \mathbb{C}(z)e(u) = f_{\mathrm{V}} \qquad \text{in } (0,T)\times\Omega\,, \tag{5.2a}$$

$$\partial\mathrm{R}_1(\dot{z}) + \mathrm{D}_z G(z, \nabla z) - \operatorname{div}(\mathrm{D}_\xi G(z, \nabla z)) + \tfrac{1}{2}\mathbb{C}'(z)e(u) : e(u) \ni 0 \quad \text{in } (0,T)\times\Omega \tag{5.2b}$$

   and will be weakly formulated through the concept of *local solution* to a rate-independent system;
2. a limit temperature $\theta = \Theta$, which is constant in space, but still time-dependent. The limit passage in the heat equation amounts to the trivial limit $0 = 0$, once more emphasizing that the limit system does not depend on temperature any more. A rescaling of the heat equation at level ε, however, reveals that Θ evolves in time according to an ODE in the sense of measures and the evolution is driven by the rate-independent dissipation and a measure originating from the viscous dissipation, cf. Remark 5.4.

Indeed, for the limit system we expect that, if a change of heat is caused at some spot in the material, then the heat must be conducted all over the material with infinite speed, so that the temperature is kept constant in space. This justifies a scaling of the tensor of heat conduction coefficients for the systems at level ε. More precisely, throughout this section we will suppose that

$$\mathbb{K}_\varepsilon(z,\theta) := \tfrac{1}{\varepsilon^2}\mathbb{K}(z,\theta) \quad \text{with } \mathbb{K} \text{ satisfying (2.11)}\,. \tag{5.3}$$

5.1 Time rescaling

Let us now set up the vanishing viscosity analysis following [Rou09], where this analysis was carried out for *isothermal* rate-independent processes in viscous solids, see also [DS13] in the context of perfect plasticity and [Rou13a, Sca14] for delamination, still in the isothermal case. We consider a family $(f_{\mathrm{V},\varepsilon}, H_\varepsilon, h_\varepsilon)_\varepsilon$ of data for system (1.1) and we rescale $f_{\mathrm{V},\varepsilon}$, H_ε, h_ε by the factor $\varepsilon > 0$, hence we introduce

$$f^\varepsilon(t) := f_{\mathrm{V},\varepsilon}(\varepsilon t) \qquad H^\varepsilon(t) := H_\varepsilon(\varepsilon t)\,, \qquad h^\varepsilon(t) := h_\varepsilon(\varepsilon t) \qquad \text{for } t \in [0, \tfrac{T}{\varepsilon}]\,. \tag{5.4}$$

Theorem 2.7 guarantees that for every $\varepsilon > 0$ there exists an energetic solution $(u^\varepsilon, z^\varepsilon, \theta^\varepsilon)$, defined on $[0, \frac{T}{\varepsilon}]$, to (the Cauchy problem for) system (1.1) supplemented with the data f^ε, H^ε, h^ε, and with the matrix of heat conduction coefficients $\mathbb{K}_\varepsilon$ from (5.3). For later convenience, let us recall that such solutions arise as limits of the time-discrete solutions to Problem 3.1. We now perform a rescaling of the solutions in such a way as to have them defined on the interval $[0, T]$. Namely, we set

$$u_\varepsilon(t) := u^\varepsilon(\tfrac{t}{\varepsilon}), \qquad z_\varepsilon(t) := z^\varepsilon(\tfrac{t}{\varepsilon}), \qquad \theta_\varepsilon(t) := \theta^\varepsilon(\tfrac{t}{\varepsilon}) \qquad \text{for } t \in [0, T]. \tag{5.5}$$

It is not difficult to check that, after transforming the time scale, the triple $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)$ (formally) solves the following system in $(0, T) \times \Omega$:

$$\varepsilon^2 \rho \ddot{u}_\varepsilon - \operatorname{div}\big(\varepsilon\, \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) + \mathbb{C}(z_\varepsilon) e(u_\varepsilon) - \theta_\varepsilon \mathbb{B}\big) = f_\varepsilon, \tag{5.6a}$$

$$\partial \mathrm{R}_1(\dot{z}_\varepsilon) + \mathrm{D}_z G(z_\varepsilon, \nabla z_\varepsilon) - \operatorname{div}\left(\mathrm{D}_\xi G(z_\varepsilon, \nabla z_\varepsilon)\right) + \tfrac{1}{2} \mathbb{C}'(z_\varepsilon) e(u_\varepsilon) : e(u_\varepsilon) \ni 0, \tag{5.6b}$$

$$\varepsilon \dot{\theta}_\varepsilon - \tfrac{1}{\varepsilon^2} \operatorname{div}\left(\mathbb{K}(z_\varepsilon, \theta_\varepsilon) \nabla \theta_\varepsilon\right) = \varepsilon \mathrm{R}_1(\dot{z}_\varepsilon) + \varepsilon^2 \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon) - \varepsilon \theta_\varepsilon \mathbb{B} : e(\dot{u}_\varepsilon) + H_\varepsilon, \tag{5.6c}$$

with the original data $f_\varepsilon := f_{\mathrm{V},\varepsilon}$, H_ε, and h_ε, and complemented with the boundary conditions (1.3). Since in the following we will be interested in the limit of (5.6) as $\varepsilon \downarrow 0$, for notational simplicity we shall henceforth set $\rho = 1$ in (5.6a).

Energetic solutions for the rescaled system. For later reference in the limit passage procedure as $\varepsilon \downarrow 0$, we recall the defining properties of energetic solutions. Given a quadruple of initial data $(u_\varepsilon^0, \dot{u}_\varepsilon^0, z_\varepsilon^0, \theta_\varepsilon^0)$ satisfying (2.12), a triple $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)$ is an energetic solution of the Cauchy problem for the PDE system (5.6) if it has the regularity (2.17), it complies with the initial conditions

$$u_\varepsilon(0) = u_\varepsilon^0, \quad \dot{u}_\varepsilon(0) = \dot{u}_\varepsilon^0, \quad z_\varepsilon(0) = z_\varepsilon^0, \quad \theta_\varepsilon(0) = \theta_\varepsilon^0 \quad \text{a.e. in } \Omega, \tag{5.7}$$

and fulfills

- *semistability and unidirectionality*: for a.a. $x \in \Omega$, $z_\varepsilon(\cdot, x) \colon [0, T] \to [0, 1]$ is nonincreasing and for all $t \in [0, T]$

$$\forall\, \tilde{z} \in \mathcal{Z},\ \tilde{z} \le z_\varepsilon(t): \quad \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) \le \mathcal{E}_\varepsilon(t, u_\varepsilon(t), \tilde{z}) + \mathcal{R}_1(z_\varepsilon(t) - \tilde{z}), \tag{5.8}$$

  with the mechanical energy

$$\mathcal{E}_\varepsilon(t, u, z) := \int_\Omega \left(\tfrac{1}{2} \mathbb{C}(z) e(u) : e(u) + G(z, \nabla z)\right) \mathrm{d}x - \langle f_\varepsilon(t), u \rangle_{H_{\mathrm{D}}^1(\Omega; \mathbb{R}^d)}; \tag{5.9}$$

- *weak formulation of the momentum equation*: for all test functions $v \in L^2(0, T; H_{\mathrm{D}}^1(\Omega; \mathbb{R}^d)) \cap W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^d))$

$$\begin{aligned} &\varepsilon^2 \int_\Omega \dot{u}_\varepsilon(t) \cdot v(t)\, \mathrm{d}x - \varepsilon^2 \int_0^t \int_\Omega \dot{u}_\varepsilon \cdot \dot{v}\, \mathrm{d}x\, \mathrm{d}t + \int_0^t \int_\Omega \left(\varepsilon\, \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) + \mathbb{C}(z_\varepsilon) e(u_\varepsilon) - \theta_\varepsilon \mathbb{B}\right) : e(v)\, \mathrm{d}x\, \mathrm{d}s \\ &= \varepsilon^2 \int_\Omega \dot{u}_\varepsilon^0 \cdot v(0)\, \mathrm{d}x + \int_0^t \langle f_\varepsilon, v \rangle_{H_{\mathrm{D}}^1(\Omega; \mathbb{R}^d)}\, \mathrm{d}s; \end{aligned} \tag{5.10}$$

- *mechanical energy equality*: for all $t \in [0, T]$

$$\begin{aligned} &\tfrac{\varepsilon^2}{2} \int_\Omega |\dot{u}_\varepsilon(t)|^2\, \mathrm{d}x + \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \int_\Omega (z_\varepsilon^0 - z_\varepsilon(t))\, \mathrm{d}x + \int_0^t \int_\Omega (\varepsilon\, \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) - \theta_\varepsilon \mathbb{B}) : e(\dot{u}_\varepsilon)\, \mathrm{d}x\, \mathrm{d}s \\ &= \tfrac{\varepsilon^2}{2} \int_\Omega |\dot{u}_\varepsilon^0|^2\, \mathrm{d}x + \mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) + \int_0^t \partial_t \mathcal{E}_\varepsilon(s, u(s), z(s))\, \mathrm{d}s; \end{aligned} \tag{5.11}$$

- *weak formulation of the heat equation*: for all $t \in [0,T]$

$$\begin{aligned}
&\varepsilon \langle \theta_\varepsilon(t), \eta(t)\rangle_{W^{2,d+\delta}} - \varepsilon \int_0^t \int_\Omega \theta_\varepsilon \dot{\eta} \,\mathrm{d}x\,\mathrm{d}s + \tfrac{1}{\varepsilon^2} \int_0^t \int_\Omega \mathbb{K}(\theta_\varepsilon, z_\varepsilon) \nabla\theta_\varepsilon \cdot \nabla\eta \,\mathrm{d}x\,\mathrm{d}s \\
&= \varepsilon \int_\Omega \theta_\varepsilon^0 \eta(0)\,\mathrm{d}x + \int_0^t \int_\Omega \left(\varepsilon^2 \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon) - \varepsilon\theta_\varepsilon \mathbb{B} : e(\dot{u}_\varepsilon)\right) \eta \,\mathrm{d}x\,\mathrm{d}s \qquad (5.12)\\
&\quad + \varepsilon \int_0^t \int_\Omega \eta \,|\dot{z}_\varepsilon| \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\partial\Omega} h_\varepsilon \eta \,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}s + \int_0^t \int_\Omega H_\varepsilon \eta \,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

for all test functions $\eta \in H^1(0,T;L^2(\Omega)) \cap \mathrm{C}^0(0,T;W^{2,d+\delta}(\Omega))$ (recall that $|\dot{z}_\varepsilon|$ denotes the total variation measure of z_ε).

Remark 5.1. Let us also observe that testing (5.12) by $\frac{1}{\varepsilon}$ and summing up with (5.11) leads to the rescaled total energy equality

$$\begin{aligned}
&\tfrac{\varepsilon^2}{2} \int_\Omega |\dot{u}_\varepsilon(t)|^2 \,\mathrm{d}x + \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \int_\Omega \theta_\varepsilon(t)\,\mathrm{d}x \\
&= \tfrac{\varepsilon^2}{2} \int_\Omega |\dot{u}_\varepsilon^0|^2 \,\mathrm{d}x + \mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) + \int_\Omega \theta_\varepsilon^0 \,\mathrm{d}x \qquad (5.13)\\
&\quad + \int_0^t \partial_t \mathcal{E}_\varepsilon(s, u_\varepsilon(s), z_\varepsilon(s))\,\mathrm{d}s + \tfrac{1}{\varepsilon} \int_0^t \int_{\partial\Omega} h_\varepsilon \,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}s + \tfrac{1}{\varepsilon} \int_0^t \int_\Omega H_\varepsilon \,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}$$

5.2 A priori estimates uniform with respect to ε

As done in the proof of Theorem 2.7, we shall derive the basic a priori estimates on the rescaled solutions $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)_\varepsilon$ from the total energy equality (5.13). Therefore, it is clear that we shall have to assume that the families of data $(H_\varepsilon)_\varepsilon$ and $(h_\varepsilon)_\varepsilon$ converge to zero in the sense that there exists $C > 0$ such that for all $\varepsilon > 0$

$$\int_0^t \int_\Omega H_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \leq C\varepsilon\,, \qquad \int_0^t \int_{\partial\Omega} h_\varepsilon \,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}s \leq C\varepsilon\,. \qquad (5.14)$$

Furthermore, we shall suppose that there exists f such that

$$f_\varepsilon \to f \qquad \text{in } H^1(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*)\,. \qquad (5.15)$$

We are now in the position to derive a priori bounds on the rescaled solutions $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)_\varepsilon$, uniform with respect to $\varepsilon > 0$. These estimates are the time-continuous counterpart of the *First–Third a priori estimates* in the proof of Proposition 3.4. Actually, the calculations underlying the *Second* and *Third* estimates (based on testing the heat equation by $\theta_\varepsilon^{\alpha-1}$), can be performed only formally when arguing on the energetic formulation of system (5.6). Indeed, the choice of the test function $\eta = \theta_\varepsilon^{\alpha-1}$ for the weak heat equation (5.12) is not admissible, since $\theta_\varepsilon^{\alpha-1} \notin \mathrm{C}^0([0,T];W^{2,d+\delta}(\Omega))$.

That is why Proposition 5.2 below will be stated not for *all* energetic solutions to the rescaled system (5.6), but just for those arising from the discrete solutions to (5.6) constructed in Section 3.1. Nonetheless, to avoid overburdening the exposition, from now on we shall overlook this point; in the next calculations we will stay on the time-continuous level to highlight how the a priori bounds are affected by ε. Therefore, some of the ensuing calculations will only be formal, but they can be rigorously justified by arguing on the time-discrete level and using "approximable solutions".

More precisely, we shall call "approximable solution" to the rescaled system (5.6) any triple obtained in the time-discrete to continuous limit, for which convergences (4.1) of Proposition 4.1 hold. In fact, in Section 4 we have shown that any approximable solution is an energetic solution. Now, it can be checked that some of the a priori estimates on the discrete solutions in Proposition 3.4 (i.e. those corresponding to (5.17) below), are uniform with respect to τ *and* ε as well. Therefore, Proposition 4.1 ensures that they are inherited by the "approximable" solutions in the limit $\tau \downarrow 0$, still uniformly with respect to ε.

Proposition 5.2 (A priori estimates). *Assume* (2.1)–(2.9), (5.3) *and let* $(H_\varepsilon)_\varepsilon \subset L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)^*)$, $(h_\varepsilon)_\varepsilon \subset L^1(0,T;L^2(\partial\Omega))$ *fulfill* (5.14), *and let* $(f_\varepsilon)_\varepsilon \subset H^1(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)^*)$ *comply with* (5.15). *In addition to* (2.12), *let the family of initial data* $(u^0_\varepsilon, \dot u^0_\varepsilon, z^0_\varepsilon, \theta^0_\varepsilon)_\varepsilon$ *fulfill*

$$|\mathcal{E}_\varepsilon(0,u^0_\varepsilon,z^0_\varepsilon)| + \varepsilon\|\dot u^0_\varepsilon\|_{L^2(\Omega;\mathbb{R}^d)} + \|\theta^0_\varepsilon\|_{L^1(\Omega)} \le C \tag{5.16}$$

for a constant C *independent of* ε. *Let* $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)_\varepsilon$ *be a family of approximable solutions to system* (5.6). *Then, there exists a constant* $C>0$ *such that the following estimates hold for all* $\varepsilon>0$*:*

$$\|u_\varepsilon\|_{L^\infty(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))} \le C\,, \tag{5.17a}$$

$$\varepsilon\|\dot u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \le C\,, \tag{5.17b}$$

$$\mathcal{R}_1(z_\varepsilon(T) - z^0_\varepsilon) \le C\,, \tag{5.17c}$$

$$\|z_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \le 1\,, \tag{5.17d}$$

$$\|z_\varepsilon\|_{L^\infty(0,T;W^{1,q}(\Omega))} \le C\,, \tag{5.17e}$$

$$\|\theta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C\,, \tag{5.17f}$$

$$\|\nabla\theta_\varepsilon\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} \le C\varepsilon\,, \tag{5.17g}$$

$$\|\theta_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C\,, \tag{5.17h}$$

$$\|\theta_\varepsilon\|_{L^p((0,T)\times\Omega)} \le C \quad \textit{for any } p \in \begin{cases} [1,8/3] & \textit{if } d{=}3\,, \\ [1,3] & \textit{if } d{=}2\,, \end{cases} \tag{5.17i}$$

with $\mathcal{R}_1$ *from* (1.2).

Sketch of the proof. **First a priori estimate: ad** (5.17a)**,** (5.17b)**,** (5.17c)**,** (5.17d) (5.17e)**,** (5.17f)**:** Estimate (5.17d) is obvious. Estimate (5.17c) follows from the definition of $\mathcal{R}_1$, (2.9a), and (2.12a), and the fact that the functions $z_\varepsilon(\cdot,x)$ are nonincreasing. We start from the total energy equality (5.13). Also thanks to (5.15), the energies $\mathcal{E}_\varepsilon$ enjoy the coercivity property (3.37) with constants independent of ε. Therefore, relying on the uniform bound (5.15) for $\dot f_\varepsilon$, and using that $\theta_\varepsilon > 0$ a.e. in $(0,T)\times\Omega$ for every $\varepsilon>0$, one can repeat the very same calculations as in the first step of the proof of Proposition 3.4, and conclude that the left-hand side of (5.13) is uniformly bounded from above and from below, whence (5.17a), (5.17b), (5.17e), (5.17f).

Second and third a priori estimates: ad (5.17g)**,** (5.17h)**, and** (5.17i)**:** We (formally) test (5.12) by $\theta_\varepsilon^{\alpha-1}$, integrate in time, and arrive at the (formally written) analogue of (3.39), viz.

$$\begin{aligned} &\tfrac{c}{\varepsilon^2}\int_0^t\int_\Omega \mathbb{K}(z_\varepsilon,\theta_\varepsilon)\nabla(\theta_\varepsilon^{\alpha/2})\cdot\nabla(\theta_\varepsilon^{\alpha/2})\,\mathrm{d}x\,\mathrm{d}s + \varepsilon^2\int_0^t\int_\Omega \mathbb{D}(z_\varepsilon,\theta_\varepsilon)e(\dot u_\varepsilon):e(\dot u_\varepsilon)\theta_\varepsilon^{\alpha-1}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \varepsilon\int_0^t\int_\Omega \theta_\varepsilon^{\alpha-1}|\dot z_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\partial\Omega} h_\varepsilon\theta_\varepsilon^{\alpha-1}\,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}s + \int_0^t\int_\Omega H_\varepsilon\theta_\varepsilon^{\alpha-1}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad = \varepsilon\int_0^t\int_\Omega \dot\theta_\varepsilon\theta_\varepsilon^{\alpha-1}\,\mathrm{d}x\,\mathrm{d}s + \varepsilon\int_0^t\int_\Omega \theta_\varepsilon\,\mathbb{B}:e(\dot u_\varepsilon)\theta_\varepsilon^{\alpha-1}\,\mathrm{d}x\,\mathrm{d}s \doteq I_1 + I_2\,. \end{aligned} \tag{5.18}$$

As in the proof of Proposition 3.4, we estimate

$$I_1 = \varepsilon\int_\Omega \tfrac{(\theta_\varepsilon^\alpha(t))^\alpha}{\alpha}\,\mathrm{d}x - \varepsilon\int_\Omega \tfrac{(\theta_\varepsilon^0)^\alpha}{\alpha}\,\mathrm{d}x\,, \tag{5.19}$$

whereas we estimate $I_2 = \iint \varepsilon\theta_\varepsilon\,\mathbb{B}:e(\dot u_\varepsilon)\theta_\varepsilon^{\alpha-1}$ by

$$I_2 \le \varepsilon^2\tfrac{C^1_{\mathbb{D}}}{2}\int_0^t\int_\Omega |e(\dot u_\varepsilon)|^2\theta_\varepsilon^{\alpha-1}\,\mathrm{d}x\,\mathrm{d}s + C\int_0^t\int_\Omega |\theta_\varepsilon|^2\theta_\varepsilon^{\alpha-1}\,\mathrm{d}x\,\mathrm{d}s\,, \tag{5.20}$$

where the constant C subsumes the norm $|\mathbb{B}|$ as well. Combining (5.18)–(5.20) and then arguing exactly in the same way as in the proof of Proposition 3.4, we end up with the analogue of (3.40), i.e.,

$$\tfrac{1}{\varepsilon^2}\int_0^t\!\!\int_\Omega \mathbb{K}(z_\varepsilon,\theta_\varepsilon)\nabla(\theta_\varepsilon^{\alpha/2})\cdot\nabla(\theta_\varepsilon^{\alpha/2})\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \tfrac{\varepsilon}{\alpha}(\theta_\varepsilon^0)^\alpha\,\mathrm{d}x \le \int_\Omega \tfrac{\varepsilon}{\alpha}(\theta_\varepsilon(t))^\alpha\,\mathrm{d}x + C\int_0^t\!\!\int_\Omega \theta_\varepsilon^{\alpha+1}(s)\,\mathrm{d}x\,\mathrm{d}s\,, \tag{5.21}$$

whence $\frac{1}{\varepsilon^2}\int_0^T\int_\Omega \mathbb{K}(z_\varepsilon,\theta_\varepsilon)\nabla(\theta_\varepsilon^{\alpha/2})\cdot\nabla(\theta_\varepsilon^{\alpha/2})\,\mathrm{d}x\,\mathrm{d}t \le C$. From this, with the same arguments as in the third step of the proof of Proposition 3.4, we infer that

$$\int_0^T\int_\Omega |\nabla\theta_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \le C\varepsilon^2,$$

i.e. (5.17g). Then, (5.17h) follows from (5.17g) and (5.17f), via the Poincaré inequality. Finally, (5.17i) ensues by interpolation, as in the proof of Proposition 3.4. □

Observe that in the proof of Proposition 5.2 we have not been able to repeat the calculations in the *Fourth–Fifth* estimates, cf. the proof of Proposition 3.4. In particular, from the mechanical energy equality (5.11) we have not been able to deduce an estimate for $\varepsilon^{1/2}e(\dot u_\varepsilon)$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$, since we cannot bound the term $\int_0^t\int_\Omega \theta_\varepsilon : e(\dot u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s$ on the right-hand side of (5.11). Therefore, in the proof of our convergence result for vanishing viscosity and inertia, Theorem 5.3 below, we shall have to resort to careful arguments in order to handle the terms containing $e(\dot u_\varepsilon)$, in the passage to the limit in the momentum equation and mechanical energy equality, cf. (5.31)–(5.34). In particular, differently from Proposition 3.4, for a vanishing sequence $(\varepsilon_n)_n$ the convergences

$$\begin{aligned} &\varepsilon_n e(\dot u_{\varepsilon_n}) \to 0 \quad \textit{strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d})) \quad \text{and} \quad \int_0^t\int_\Omega \theta_{\varepsilon_n} : e(\dot u_{\varepsilon_n})\,\mathrm{d}x\,\mathrm{d}s \to 0\,,\\ &\theta_\varepsilon \to \Theta \quad \textit{strongly in } L^2(0,T)\times\Omega) \end{aligned} \tag{5.22}$$

will now be extracted from the weak heat equation (5.12), using integration by parts and the information that Θ is constant in space. It is in this connection that we need to further assume homogeneous Dirichlet boundary conditions for the displacement on the whole boundary $\partial\Omega$, cf. (5.1).

5.3 Convergence to local solutions of the rate-independent limit system

Let us mention in advance that in Theorem 5.3 we will prove that, up to a subsequence, the functions $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)$ converge to a limit triple (u, z, Θ) such that Θ is spatially constant. As we will see, the pair (u, z) fulfills the (pointwise-in-time) *static* momentum balance (i.e. without viscosity and inertia), a semistability condition with respect to the energy $\mathcal{E}$ arising from $\mathcal{E}_\varepsilon$ (5.9) in the limit $\varepsilon\downarrow 0$, and an energy inequality, where the viscous, the inertial, and the thermal expansion contributions are no longer present. This inequality holds on $[0,t]$ *for every* $t\in[0,T]$ in the general case, and on $[s,t]$ for all $t\in[0,T]$ and almost every $s\in(0,t)$, under a further condition on the gradient term in the energy $\mathcal{E}$, i.e. that $q>d$. Indeed, the three properties (momentum balance, semistability, energy inequality) constitute the notion of *local solution* [Mie11, Rou13a, RTP13] to the rate-independent system driven by $\mathcal{R}_1$ and $\mathcal{E}$. Observe that, in fact, the spatially constant Θ does not appear in these relations, because it contributes with a zero term to the momentum balance.

Moreover, testing the weak heat equation (5.12) with functions η being constant in space – the property of the limit temperature Θ by (5.17g) – and taking into account the bounds (5.14), (5.16), (5.17f), and convergence (5.22), results in the limit relation $0 = 0$, which displays that the temporal evolution of Θ is irrelevant in the rate-independent limit model.

Theorem 5.3. *Assume* (2.1)–(2.4), (2.9), (2.13), *and, in addition, let* (5.1), (5.3), (5.14), *and* (5.15) *be satisfied. Let the initial data* $(u_\varepsilon^0, \dot u_\varepsilon^0, z_\varepsilon^0, \theta_\varepsilon^0)_\varepsilon$ *fulfill* (2.12), (5.16),

$$\varepsilon\dot u_\varepsilon^0 \to 0 \quad \textit{in } L^2(\Omega;\mathbb{R}^d)\,, \tag{5.23}$$

and suppose that there exist $u_0\in H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)$ *and* $z_0\in\mathcal{Z}$ *such that*

$$u_\varepsilon^0 \rightharpoonup u_0 \textit{ in } H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d), \qquad z_\varepsilon^0 \rightharpoonup z_0 \textit{ in } \mathcal{Z}, \qquad \mathcal{E}_\varepsilon(0,u_\varepsilon^0,z_\varepsilon^0)\to\mathcal{E}(0,u_0,z_0) \qquad \textit{as } \varepsilon\downarrow 0\,, \tag{5.24}$$

with $\mathcal{E}_\varepsilon$ *as in* (5.9).

Then, the functions $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)_\varepsilon$ *converge (up to subsequences) to a triple* (u, z, Θ) *such that*

$$
\begin{aligned}
&u \in L^\infty(0,T; H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)), \qquad z \in L^\infty(0,T; W^{1,q}(\Omega)) \cap L^\infty((0,T)\times\Omega) \cap \mathrm{BV}([0,T]; L^1(\Omega)),\\
&\Theta \textit{ is constant in space and } \Theta \in L^p(0,T) \quad \textit{for any } p \in \begin{cases} [1, 8/3] & \textit{if } d{=}3, \\ [1,3] & \textit{if } d{=}2. \end{cases}
\end{aligned}
\tag{5.25}
$$

The pair (u, z) *fulfills the unidirectionality as well as*

1. *the semistability condition* (2.19a) *for all* $t \in [0,T]$, *with the mechanical energy* $\mathcal{E}$ *defined as in* (5.9) *with* f_ε *replaced by the weak limit* f *of the sequence* $(f_\varepsilon)_\varepsilon$, *see* (5.15)*;*
2. *the weak momentum balance for all* $t \in [0,T]$

$$
\int_\Omega \mathbb{C}(z(t))e(u(t)) : e(v)\,\mathrm{d}x = \langle f(t), v\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \qquad \textit{for all } v \in H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d); \tag{5.26}
$$

3. *the mechanical energy inequality for all* $t \in [0,T]$

$$
\mathcal{E}(t, u(t), z(t)) + \int_\Omega (z(0){-}z(t))\,\mathrm{d}x \le \mathcal{E}(0, u(0), z(0)) + \int_0^t \partial_t \mathcal{E}(r, u(r), z(r))\,\mathrm{d}r. \tag{5.27}
$$

If in addition the function G *fulfills the growth condition* (2.9d) *with* $q > d$, *then* (u, z) *also fulfill*

$$
\mathcal{E}(t, u(t), z(t)) + \int_\Omega (z(s){-}z(t))\,\mathrm{d}x \le \mathcal{E}(s, u(s), z(s)) + \int_s^t \partial_t \mathcal{E}(r, u(r), z(r))\,\mathrm{d}r \tag{5.28}
$$

for all $t \in [0,T]$ *and for almost all* $s \in (0,t)$.

Proof. **Step 0, compactness:** It follows from Proposition 5.2 that for every vanishing sequence $(\varepsilon_n)_n$ there exist a (not relabeled) subsequence and a triple (u, z, Θ) as in (5.25) such that the following convergences hold

$$
\begin{aligned}
u_{\varepsilon_n} &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T; H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)), && (5.29a)\\
\varepsilon_n u_{\varepsilon_n} &\overset{*}{\rightharpoonup} 0 && \text{in } W^{1,\infty}(0,T; L^2(\Omega;\mathbb{R}^d)), && (5.29b)\\
z_{\varepsilon_n} &\overset{*}{\rightharpoonup} z && \text{in } L^\infty(0,T; W^{1,q}(\Omega)) \cap L^\infty((0,T)\times\Omega), && (5.29c)\\
z_{\varepsilon_n}(t) &\rightharpoonup z(t) && \text{in } W^{1,q}(\Omega) \quad \text{for all } t \in [0,T] && (5.29d)\\
z_{\varepsilon_n}(t) &\to z(t) && \text{in } L^r(\Omega) \quad \text{for all } 1 \le r < \infty \text{ and for all } t \in [0,T], && (5.29e)\\
\theta_{\varepsilon_n} &\rightharpoonup \Theta && \text{in } L^2(0,T; H^1(\Omega)) \cap L^p((0,T)\times\Omega) \text{ for all } p \text{ as in (5.17i)}. && (5.29f)
\end{aligned}
$$

Indeed, (5.29a) ensues from (5.17a), and it gives, in particular, that $\varepsilon_n u_{\varepsilon_n} \to 0$ in $L^\infty(0,T; H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))$. Then, convergence (5.29b) directly follows from estimate (5.17b). Convergences (5.29c)–(5.29e) ensue from the very same compactness arguments as in the proof of Proposition 4.1, also using the Helly Theorem. Furthermore, (5.29f) follows from estimates (5.17h)–(5.17i) by weak compactness. Observe that in view of (5.17g) we have that

$$
\nabla\theta_{\varepsilon_n} \to 0 \quad \text{in } L^2(0,T; L^2(\Omega;\mathbb{R}^d)). \tag{5.30}
$$

Therefore, we conclude that $\nabla\Theta = 0$ a.e. in $(0,T)\times\Omega$. Since Θ is spatially constant, hereafter we will write it as a function of the sole variable t.

We now prove the enhanced convergence

$$
\theta_{\varepsilon_n} \to \Theta \text{ in } L^2(0,T; L^2(\Omega)). \tag{5.31}
$$

In fact, we use the Poincaré inequality

$$
\|\theta_{\varepsilon_n} - \Theta\|_{L^2(0,T;L^2(\Omega))} \le \|\nabla(\theta_{\varepsilon_n} - \Theta)\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^d))} + C(\Omega,T)\left|\int_0^T\int_\Omega (\theta_{\varepsilon_n} - \Theta)\,\mathrm{d}x\,\mathrm{d}s\right| \longrightarrow 0,
$$

where the gradient term tends to 0 by (5.30), and the convergence of the second term follows from (5.29f).

Finally, let us show that

$$\varepsilon_n \, e(\dot u_{\varepsilon_n}) \to 0 \quad \textit{strongly} \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))\,. \tag{5.32}$$

Preliminarily, observe that, since the limit function Θ is constant in space, we have by integration by parts

$$\int_0^t \int_\Omega \Theta\,\mathbb{B} : e(\dot u_{\varepsilon_n})\,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_{\partial\Omega} \Theta\,\mathbb{B}\,\nu\cdot \dot u_{\varepsilon_n}\,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}s - \int_0^t\int_\Omega \mathrm{div}\,(\Theta\,\mathbb{B})\cdot \dot u_{\varepsilon_n}\,\mathrm{d}x\,\mathrm{d}s = 0\,, \tag{5.33}$$

where we used $\partial_{\mathrm{D}}\Omega = \partial\Omega$, hence $\dot u_{\varepsilon_n} \in L^2(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))$ implies that $\dot u_{\varepsilon_n} = 0$ a.e. in $(0,T)\times\partial\Omega$. Using (5.33) in the weak heat equation (5.12) tested by 1 and applying Young's inequality, we find

$$\begin{aligned} \varepsilon_n \left(\int_\Omega (\theta_{\varepsilon_n}(t) - \theta^0_{\varepsilon_n})\,\mathrm{d}x \right) &\geq \int_0^t \int_\Omega \left[\varepsilon_n^2 \mathbb{D}(z_{\varepsilon_n},\theta_{\varepsilon_n}) e(\dot u_{\varepsilon_n}) : e(\dot u_{\varepsilon_n}) - \varepsilon_n (\theta_{\varepsilon_n} - \Theta\,\mathbb{B}) : e(\dot u_{\varepsilon_n}) \right] \mathrm{d}x\,\mathrm{d}s \\ &\geq \int_0^t\int_\Omega \varepsilon_n^2 \tfrac{C_{\mathbb{D}}}{2} |e(\dot u_{\varepsilon_n})|^2\,\mathrm{d}x\,\mathrm{d}s - C\|\theta_{\varepsilon_n} - \Theta\|^2_{L^2(0,T;L^2(\Omega))} \end{aligned} \tag{5.34}$$

with $C = |\mathbb{B}|/2$. From this, taking into account that $(\theta^0_{\varepsilon_n})_n$ is bounded in $L^1(\Omega)$ by (5.16), estimate (5.17f) for $(\theta_{\varepsilon_n})_n$, and convergence (5.31), we conclude that $\lim_{\varepsilon_n\downarrow 0} \varepsilon_n \|e(\dot u_{\varepsilon_n})\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))} = 0$, whence (5.32).

In fact, by Korn's inequality we conclude that

$$\varepsilon_n u_{\varepsilon_n} \to 0 \quad \text{in } H^1(0,T;H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d))\,. \tag{5.35}$$

Step 1, passage to the limit in the momentum balance (5.10)**:** Convergence (5.35), joint with the boundedness (2.3e) of the tensor $\mathbb{D}$, ensures that the first and the second summands on the left-hand side of (5.10) tend to zero. Arguing as in the proof of Proposition 4.3, we show that for every test function v in (5.10), $\mathbb{C}(z_{\varepsilon_n})e(v) \to \mathbb{C}(z)e(v)$ in $L^2((0,T)\times\Omega;\mathbb{R}^{d\times d})$. We combine this with (5.29a) and, also using (5.29f), we pass to the limit in the third term on the left-hand side of (5.10), recalling that the fourth summand converges to zero similarly to (5.33). As for the right-hand side, by (5.16) we have

$$\varepsilon_n^2 \dot u^0_{\varepsilon_n} \to 0 \qquad \text{in } L^2(\Omega;\mathbb{R}^d), \tag{5.36}$$

hence the first term converges to zero. The second one tends to zero for almost all $t\in(0,T)$ by (5.29b), which in particular gives

$$\varepsilon_n^2 \dot u_{\varepsilon_n} \to 0 \qquad \text{in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)). \tag{5.37}$$

For the third one, we use (5.15). We thus conclude that (5.26) holds at almost all $t\in(0,T)$.

In order to check it at *every* $t\in[0,T]$, we observe that for every $t\in[0,T]$ from the bounded sequence $(u_{\varepsilon_n}(t))_n$ (along which convergences (5.29) hold) we can extract a subsequence, possibly depending on t, weakly converging to some $\bar u(t)$ in $H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)$. Relying on convergence (5.29e) for $(z_{\varepsilon_n}(t))_n$ and on (5.15) for $(f_{\varepsilon_n}(t))$, with the same arguments as above we conclude that $\int_\Omega \mathbb{C}(z(t))e(\bar u(t)) : e(v)\,\mathrm{d}x = \langle f(t), v\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}$ for all $v\in H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)$. Since this equation has a unique solution, we conclude that $\bar u(t) = u(t)$ for almost all $t\in(0,T)$, and that the *whole* sequence $u_{\varepsilon_n}(t)$ weakly converges to $\bar u(t)$ for every $t\in[0,T]$. In this way u extends to a function defined on $[0,T]$, such that

$$u_{\varepsilon_n}(t) \rightharpoonup u(t) \quad \text{in } H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d) \quad \text{for all } t\in[0,T]\,, \tag{5.38}$$

solving (5.26) at all $t\in[0,T]$.

Step 2, enhanced convergences for $(u_{\varepsilon_n})_n$: As a by-product of this limit passage, we also extract convergences (5.40) and (5.41) below for $(u_{\varepsilon_n})_n$, which we will then use in the passage to the limit in the semistability and in the mechanical energy inequality. Indeed, we test (5.10) by u_{ε_n}, thus obtaining

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^t \int_\Omega (\mathbb{C}(z_{\varepsilon_n})e(u_{\varepsilon_n}) - \theta_{\varepsilon_n}\mathbb{B}) : e(u_{\varepsilon_n}) \,\mathrm{d}x\,\mathrm{d}s \\
&\le \limsup_{n\to\infty} \varepsilon_n^2 \int_0^t \int_\Omega |\dot{u}_{\varepsilon_n}|^2 \,\mathrm{d}x\,\mathrm{d}t - \liminf_{n\to\infty} \int_0^t \int_\Omega \varepsilon_n \mathbb{D}(z_{\varepsilon_n}, \theta_{\varepsilon_n}) e(\dot{u}_{\varepsilon_n}) : e(u_{\varepsilon_n}) \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \limsup_{n\to\infty} \varepsilon_n^2 \int_\Omega \dot{u}^0_{\varepsilon_n} \cdot u^0_{\varepsilon_n} \,\mathrm{d}x - \liminf_{n\to\infty} \varepsilon_n^2 \int_\Omega \dot{u}_{\varepsilon_n}(t) \cdot u_{\varepsilon_n}(t) \,\mathrm{d}x + \limsup_{n\to\infty} \int_0^t \langle f_{\varepsilon_n}, u_{\varepsilon_n} \rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}s \\
&= 0 + 0 + 0 + 0 + \int_0^t \langle f, u \rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \,\mathrm{d}s = \int_0^t \int_\Omega \mathbb{C}(z) e(u) : e(u) \,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

where the first term in the right-hand side converges to zero thanks to (5.35), the second one by the boundedness of $\mathbb{D}$, (5.29a), and (5.35), the third one by (5.36) combined with the boundedness of $(u^0_{\varepsilon_n})_n$, the fourth one by (5.29a) and (5.37). The fifth term passes to the limit by (5.15) and (5.29a). The last identity follows from (5.26). Remark that the second term in the left-hand side converges to zero by (5.29a) and (5.29f), as done for (5.33). From the above chain of inequalities we thus conclude

$$\limsup_{n\to\infty} \int_0^t \int_\Omega \mathbb{V}(z_{\varepsilon_n}) e(u_{\varepsilon_n}) : \mathbb{V}(z_{\varepsilon_n}) e(u_{\varepsilon_n}) \,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \int_\Omega \mathbb{V}(z) e(u) : \mathbb{V}(z) e(u) \,\mathrm{d}x\,\mathrm{d}s,$$

where $\mathbb{V}(\cdot)$ is the square root of the tensor $\mathbb{C}(\cdot)$, cf. Remark 2.1. On the other hand, the very same arguments as in the proof of Proposition 4.3 (cf. also Lemma 4.7) yield that

$$\mathbb{V}(z_{\varepsilon_n}) e(u_{\varepsilon_n}) \text{ \textit{strongly} converges to } \mathbb{V}(z) e(u) \text{ in } L^2(0,T; L^2(\Omega;\mathbb{R}^{d\times d})) \,. \tag{5.39}$$

For later convenience, we observe that, in particular, this yields

$$\int_\Omega \mathbb{C}(z_{\varepsilon_n}(t)) e(u_{\varepsilon_n}(t)) : e(u_{\varepsilon_n}(t)) \,\mathrm{d}x \to \int_\Omega \mathbb{C}(z(t)) e(u(t)) : e(u(t)) \,\mathrm{d}x \qquad \text{for a.a. } t \in (0,T) \,. \tag{5.40}$$

Furthermore, we deduce

$$u_{\varepsilon_n} \to u \quad \text{in } L^2(0,T; H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)) \,. \tag{5.41}$$

Indeed, from (5.39) we have that $\mathbb{V}(z_{\varepsilon_n}) e(u_{\varepsilon_n}) \to \mathbb{V}(z) e(u)$ a.e. in $(0,T) \times \Omega$. It follows from the arguments of Remark 2.1 that for every z the inverse tensor $\mathbb{V}(z)^{-1}$ is well-defined, and that the mapping $z \mapsto \mathbb{V}(z)^{-1}$ is continuous with values in $\mathbb{R}^{d\times d\times d\times d}$. Therefore, we infer that $e(u_{\varepsilon_n}) = \mathbb{V}(z_{\varepsilon_n})^{-1} \mathbb{V}(z_{\varepsilon_n}) e(u_{\varepsilon_n}) \to \mathbb{V}(z)^{-1} \mathbb{V}(z) e(u) = e(u)$ a.e. in $(0,T) \times \Omega$. Property (2.3d) ensures that $|e(u_{\varepsilon_n})|^2 \le 1/C^1_{\mathbb{C}} \mathbb{V}(z_{\varepsilon_n}) e(u_{\varepsilon_n}) : \mathbb{V}(z_{\varepsilon_n}) e(u_{\varepsilon_n})$ a.e. in $(0,T) \times \Omega$. From this, and (5.39), we deduce that the sequence $(e(u_{\varepsilon_n}))_n$ is uniformly integrable with values in $L^2((0,T) \times \Omega; \mathbb{R}^{d\times d})$. Hence $e(u_{\varepsilon_n})$ strongly converges to $e(u)$ in the latter space and, by Korn's inequality, we ultimately infer (5.41).

Step 3, passage to the limit in the semistability condition: In view of the pointwise convergences (5.29d)–(5.29e) for z_{ε_n} and $u_{\varepsilon_n}(t) \to u(t)$ in $H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)$ (by (5.41)) for all $t \in [0,T]$, we may apply the mutual recovery sequence construction from Theorem 4.5 in order to pass to the limit as $\varepsilon_n \downarrow 0$ in the semistability (5.8). Also taking into account convergence (5.15) for $(f_{\varepsilon_n})_n$, we conclude that (u,z) comply with the semistability condition (2.19a) for every $t \in [0,T]$.

Step 4, passage to the limit in the mechanical energy inequality on $(0,t)$: By lower semicontinuity it follows from convergences (5.15), (5.38), (5.29d), and (5.29c) that

$$\liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}(t, u_{\varepsilon_n}(t), z_{\varepsilon_n}(t)) \ge \mathcal{E}(t, u(t), z(t)) \qquad \text{for all } t \in [0,T] \,. \tag{5.42}$$

Furthermore, combining (5.15) with (5.29a) we infer that

$$\partial_t \mathcal{E}_{\varepsilon_n}(t, u_{\varepsilon_n}, z_{\varepsilon_n}) = -\left\langle \dot{f}_{\varepsilon_n}(t), u_{\varepsilon_n} \right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} \to -\left\langle \dot{f}(t), u \right\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)} = \partial_t \mathcal{E}(t,u,z) \quad \text{in } L^2(0,T) \,. \tag{5.43}$$

We are now in the position to pass to the limit in the mechanical energy inequality (5.11). The first and the third terms on the left-hand side of (5.11) are positive. For the second one we use (5.42) and the fourth one converges to $\int_\Omega (z(0) - z(t))\,\mathrm{d}x$ by (5.29e). As for the right-hand side, we observe that the first term converges to zero by (5.23). The second term passes to the limit by the convergence (5.24) for the initial energies, and the fourth one by (5.43). As for the third one, we again argue as in (5.33)

$$\begin{aligned}\int_0^t\int_\Omega \theta_{\varepsilon_n}\,\mathbb{B} : e(\dot u_{\varepsilon_n})\,\mathrm{d}x\,\mathrm{d}s &= \int_0^t\int_{\partial\Omega} \theta_{\varepsilon_n}\,\mathbb{B}\,\nu\cdot \dot u_{\varepsilon_n}\,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}s - \int_0^t\int_\Omega \mathrm{div}\,(\theta_{\varepsilon_n}\,\mathbb{B})\cdot \dot u_{\varepsilon_n}\,\mathrm{d}x\,\mathrm{d}s \\ &= 0 - \int_0^t\int_\Omega \mathrm{div}\,(\theta_{\varepsilon_n}\,\mathbb{B})\cdot \dot u_{\varepsilon_n}\,\mathrm{d}x\,\mathrm{d}s\,,\end{aligned} \tag{5.44}$$

where we have used that $\dot u_{\varepsilon_n}$ complies with homogeneous Dirichlet conditions on $\partial_{\mathrm{D}}\Omega = \partial\Omega$, and then observe that

$$\|\,\mathrm{div}\,(\theta_{\varepsilon_n}\,\mathbb{B})\cdot \dot u_{\varepsilon_n}\|_{L^1((0,T)\times\Omega)} = \|\varepsilon_n^{-1}\,\mathrm{div}\,(\theta_{\varepsilon_n}\,\mathbb{B})\cdot \varepsilon_n\dot u_{\varepsilon_n}\|_{L^1((0,T)\times\Omega)} \le C\|\varepsilon_n\dot u_{\varepsilon_n}\|_{L^2((0,T)\times\Omega)} \to 0\,, \tag{5.45}$$

due to estimate (5.17g) and (5.35).

Step 5, case $q > d$, enhanced convergence for (z_{ε_n}) and energy convergence: We now prove that

$$\lim_{n\to\infty}\int_\Omega G(z_{\varepsilon_n}(t), \nabla z_{\varepsilon_n}(t))\,\mathrm{d}x = \int_\Omega G(z(t),\nabla z(t))\,\mathrm{d}x \qquad \text{for a.a. } t\in(0,T)\,, \tag{5.46}$$

which, combined with (5.15), (5.40) and (5.41) will yield the pointwise convergence of the energies

$$\lim_{n\to\infty} \mathcal{E}_{\varepsilon_n}(t, u_{\varepsilon_n}(t), z_{\varepsilon_n}(t)) = \mathcal{E}(t,u(t),z(t)) \qquad \text{for a.a. } t\in(0,T)\,. \tag{5.47}$$

We obtain (5.46) testing semistability (5.8) by a suitable recovery sequence $(\tilde z_{\varepsilon_n})_n$ for $\tilde z = z(t)$; in the following lines, to avoid overburdening notation we will drop t when writing $z_{\varepsilon_n}(t)$, $z(t)$, $u_{\varepsilon_n}(t)$, and $u(t)$. Following [MR06, Lemma 3.9], where the recovery sequence right below has been introduced to deduce energy convergence, we set

$$\tilde z_{\varepsilon_n} := \max\{0, z - \|z_{\varepsilon_n} - z\|_{L^\infty(\Omega)}\}\,. \tag{5.48}$$

Now, for $q > d$ the convergence $z_{\varepsilon_n} \rightharpoonup z$ in $W^{1,q}(\Omega)$, see (5.29d), implies $z_{\varepsilon_n} \to z$ in $L^\infty(\Omega)$. Thus, it can be checked that

$$\tilde z_{\varepsilon_n} \to z \textit{ strongly} \text{ in } W^{1,q}(\Omega)\,. \tag{5.49}$$

Since $\tilde z_{\varepsilon_n} \le z_{\varepsilon_n}$, we can choose it as a test function in (5.8). The term $-\langle f_{\varepsilon_n}(t), u_{\varepsilon_n}\rangle_{H^1_{\mathrm{D}}(\Omega;\mathbb{R}^d)}$ on both sides of the inequality cancels out and we deduce

$$\begin{aligned}&\limsup_{n\to\infty}\left(\int_\Omega (\tfrac12\mathbb{C}(z_{\varepsilon_n})e(u_{\varepsilon_n}) : e(u_{\varepsilon_n}) + G(z_{\varepsilon_n},\nabla z_{\varepsilon_n}))\,\mathrm{d}x\right) \\ &= \limsup_{n\to\infty}\left(\int_\Omega \tfrac12\mathbb{C}(\tilde z_n)e(u_{\varepsilon_n}) : e(u_{\varepsilon_n})\,\mathrm{d}x + \int_\Omega G(\tilde z_{\varepsilon_n},\nabla\tilde z_{\varepsilon_n})\,\mathrm{d}x\right) \le I_1 + I_2\,,\end{aligned} \tag{5.50}$$

where

$$I_1 := \lim_{n\to\infty}\int_\Omega \tfrac12\mathbb{C}(\tilde z_n)e(u_{\varepsilon_n}) : e(u_{\varepsilon_n})\,\mathrm{d}x \le \int_\Omega \tfrac12\mathbb{C}(z)e(u) : e(u)\,\mathrm{d}x\,, \tag{5.51}$$

combining (5.49) with (5.41) via the Lebesgue Theorem. It follows from (5.49), condition (2.9d) on the growth of G from above, and again the Lebesgue Theorem that

$$I_2 := \lim_{n\to\infty}\int_\Omega G(\tilde z_{\varepsilon_n},\nabla\tilde z_{\varepsilon_n})\,\mathrm{d}x = \int_\Omega G(z,\nabla z)\,\mathrm{d}x\,. \tag{5.52}$$

Taking into account the previously proven (5.40), from (5.50)–(5.52) we ultimately infer

$$\limsup_{n\to\infty}\int_\Omega G(z_{\varepsilon_n},\nabla z_{\varepsilon_n}))\,\mathrm{d}x \le \int_\Omega G(z,\nabla z))\,\mathrm{d}x\,,$$

whence (5.46).

Step 6, case $q > d$, passage to the limit in the mechanical energy inequality on (s,t): We now pass to the limit in (5.11) written on an interval $[s,t] \subset [0,T]$, for every $t \in [0,T]$ and almost all $s \in (0,t)$. Clearly, it is sufficient to discuss the limit passage on the right-hand side of (5.11), evaluated at s. The first summand tends to zero for almost all s, thanks to (5.35), which in particular ensures $\varepsilon_n \dot{u}_{\varepsilon_n}(s) \to 0$ in $L^2(\Omega;\mathbb{R}^d)$ for almost all $s \in (0,T)$. The second term passes to the limit by (5.47), while the third and the fourth ones can be dealt with by (5.44)–(5.45) and (5.43), respectively. □

Remark 5.4 (Temporal evolution of Θ)**.** Recall that the passage to the limit in the weak heat equation (5.12) leads to the trivial relation $0 = 0$. Thus, in order to gain insight into the time evolution of Θ, we now discuss the limit passage in the heat equation (5.12) rescaled by the factor $1/\varepsilon$ and tested by $\eta \in H^1(0,T)$, constant in space:

$$\begin{aligned}
&\eta(t)\int_\Omega \theta_\varepsilon(t)\,\mathrm{d}x - \int_0^t\int_\Omega \theta_\varepsilon\,\dot{\eta}\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_\Omega \theta_\varepsilon^0\,\eta(0)\,\mathrm{d}x + \int_0^t\int_\Omega (\varepsilon\,\mathbb{D}(z_\varepsilon,\theta_\varepsilon)e(\dot{u}_\varepsilon) - \theta_\varepsilon\,\mathbb{B}) : e(\dot{u}_\varepsilon)\eta\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega \eta\,|\dot{z}_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s + \tfrac{1}{\varepsilon}\int_0^t\int_{\partial\Omega} h_\varepsilon\,\eta\,\mathrm{d}\mathcal{H}^{d-1}(x)\,\mathrm{d}s + \tfrac{1}{\varepsilon}\int_0^t\int_\Omega H_\varepsilon\,\eta\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned} \tag{5.53}$$

For this, we consult the mechanical energy balance (5.11) and deduce by a comparison argument that

$$\varepsilon\int_0^T\int_\Omega \mathbb{D}(z_\varepsilon,\theta_\varepsilon)e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \le C\,, \text{ hence also } \varepsilon\int_0^T \eta\int_\Omega \mathbb{D}(z_\varepsilon,\theta_\varepsilon)e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \le C\|\eta\|_\infty \tag{5.54}$$

for every $\eta \in H^1(0,T)$, taking into account (5.15), (5.16) as well as (5.22). This allows us to conclude that there exists a Radon measure μ

$$\|\mathbb{D}(z_\varepsilon,\theta_\varepsilon)e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon)\|_{L^1(\Omega)} \to \mu \quad \text{in the sense of Radon measures in } [0,T]\,. \tag{5.55}$$

A comparison argument in (5.53) leads to

$$\left|\int_0^t\int_\Omega \eta\,\theta_\varepsilon\,\mathbb{B} : e(\dot{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s\right| \le C\|\eta\|_\infty\,, \tag{5.56}$$

also in view of the bounds (5.14), (5.17i) and (5.17c). Since η is constant in space, integration by parts and an argument along the lines of Step 4 of the proof of above Theorem 5.3 yield that indeed $\int_0^t\int_\Omega \eta\,\theta_\varepsilon\,\mathbb{B} : e(\dot{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s \to 0$. Moreover, the third convergence in (5.22) implies that $\theta_\varepsilon(t) \to \Theta(t)$ in $L^2(\Omega)$ for a.e. $t \in (0,T)$. Under the additional assumption that the positive heat sources satisfy $\frac{1}{\varepsilon}(\|H_\varepsilon\|_{L^1(\Omega)} + \|h_\varepsilon\|_{L^1(\partial\Omega)}) \rightharpoonup \widetilde{H}$ in $L^1(0,T)$ for some limit function $\widetilde{H} \in L^1(0,T)$, we find for Θ an ODE featuring measures, which arises as the limit of the rescaled right-hand side of (5.12), to be satisfied for a.a. $t \in (0,T)$:

$$\eta(t)\int_\Omega \Theta(t)\,\mathrm{d}x - \int_0^t\int_\Omega \Theta\,\dot{\eta}\,\mathrm{d}x\,\mathrm{d}s - \int_\Omega \Theta(0)\,\eta(0)\,\mathrm{d}x = \int_0^t \eta\,\mathrm{d}\mu(s) + \int_0^t\int_\Omega \eta\,|\dot{z}|\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \widetilde{H}\,\eta\,\mathrm{d}s\,.$$

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