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MONADS FOR FRAMED SHEAVES ON HIRZEBRUCH SURFACES

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ABSTRACT. We define monads for framed torsion-free sheaves on Hirzebruch surfaces and use them to construct moduli spaces for these objects. These moduli spaces are smooth algebraic varieties, and we show that they are fine by constructing a universal monad.

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1. Introduction

Moduli spaces of framed sheaves over projective surfaces have been the object of some interest over the last few years. In the case of the complex projective plane — generalizing the classical result in [6] — these moduli spaces are resolutions of singularities of the moduli space of ideal instantons on the four-sphere S^4 [22], and as such, they have been used to compute Nekrasov's partition function, i.e., the partition function of a (suitably twisted) $N = 2$ topological super Yang-Mills theory (see [24, 2] and also [9] for the more general case of toric surfaces).

More generally, they are at the basis of the so-called instanton counting [23]. Fine moduli spaces of framed sheaves were constructed by Huybrechts and Lehn [16, 17] by introducing a stability condition. Bruzzo and Markushevich [3] showed that framed sheaves on projective surfaces, even without a stability condition, give rise to fine moduli spaces.

Moduli spaces of framed bundles on the projective plane where considered by Donaldson [6, 7] (establishing their isomorphism with the moduli of instantons on S^4). This was extended by King to the case of the blowup of the projective plane at a point [19] and by Buchdahl [5] to the case of multiple blowups. The degenerate case (including torsion-free sheaves) was first considered, as already cited, by Nakajima [22] for the projective plane, and then by Nakajima and Yoshioka [23] for the blownup plane. The case of multiple blowups was studied by Henni [13].

In this paper we consider torsion-free sheaves on a Hirzebruch surface Σ_n, for $n > 0$, that are framed to the trivial bundle on a generic line ℓ_∞ in Σ_n. We construct a moduli space for such sheaves by using monads. This allows us to obtain a moduli space which is a smooth quasi-projective variety. These moduli spaces are also shown to be fine, and this implies that they are isomorphic to the moduli spaces constructed in [3] (and therefore embed as open subschemes into Huybrechts-Lehn's moduli spaces).

These results are the basis for further work where we give a detailed description of the moduli spaces when the topological invariants satisfy the lower bound $2c = na(1 - a)$. Moreover, considering the rank one case, they allow us to obtain a rather explicit construction of the Hilbert schemes of points of the total spaces of the line bundles $\mathcal{O}_{\mathbb{P}^1}(-n)$. In both cases the results are achieved by using explicit ADHM descriptions [1].

The monads we use generalize to torsion-free sheaves those introduced by Buchdahl [4] for the locally-free case (he was actually interested in μ-stable vector bundles on Hirzebruch surfaces with $c_1 = 0$, $c_2 = 2$). Indeed, we show in Corollary 4.6 that for any framed torsion-free sheaf $(\mathcal{E}, \theta)$ on Σ_n, the underlying sheaf $\mathcal{E}$ is isomorphic to the cohomology of a monad

$$\mathcal{U}_{\vec{k}} \xrightarrow{\alpha} \mathcal{V}_{\vec{k}} \xrightarrow{\beta} \mathcal{W}_{\vec{k}} \tag{1.1}$$

whose terms depend only on the Chern character $\mathrm{ch}(\mathcal{E}) = (r, aE, -c - \frac{1}{2}na^2)$ (here E is the exceptional curve in Σ_n, i.e., the unique irreducible curve in Σ_n squaring to $-n$, and we put $\vec{k} = (n, r, a, c)$). This provides a map

$$(\mathcal{E}, \theta) \longmapsto (\alpha, \beta) \in \mathrm{Hom}(\mathcal{U}_{\vec{k}}, \mathcal{V}_{\vec{k}}) \oplus \mathrm{Hom}(\mathcal{V}_{\vec{k}}, \mathcal{W}_{\vec{k}}). \tag{1.2}$$

We call $L_{\vec{k}}$ the image of (1.2) and we prove in Proposition 4.8 and in Lemma 4.9 that $L_{\vec{k}}$ is a smooth variety. We construct a principal $\mathrm{GL}(r, \mathbb{C})$-bundle $P_{\vec{k}}$ over $L_{\vec{k}}$ whose fibre over a point (α, β) is naturally identified with the space of framings for the cohomology of the complex (1.1). This implies that the map (1.2) lifts to a map

$$(\mathcal{E}, \theta) \longmapsto \theta \in P_{\vec{k}}.$$

The group $G_{\vec{k}}$ of isomorphisms of monads of the form (1.1) acts on $P_{\vec{k}}$, and the moduli space $\mathcal{M}^n(r, a, c)$ of framed sheaves on Σ_n with the given Chern character is set-theoretically defined as the quotient $P_{\vec{k}}/G_{\vec{k}}$. Theorem 5.1 proves that $\mathcal{M}^n(r, a, c)$ inherits from $P_{\vec{k}}$ a structure of smooth algebraic variety; moreover Lemma 4.7 states that two monads of the form (1.1) are isomorphic if and only it their cohomologies are isomorphic, and this ensures that there is a bijection between $\mathcal{M}^n(r, a, c)$ and set of isomorphism classes of framed sheaves on Σ_n (isomorphisms of framed sheaves are introduced in Definition 3.3). This enables us to show that the set of these classes is nonempty if and only if $2c \geq na(1-a)$ (one should note that this lower bound may correspond to moduli spaces of strictly positive dimension).

We prove the fineness of the moduli space by constructing a universal family $\left(\mathfrak{E}_{\vec{k}}, \Theta_{\vec{k}}\right)$ of framed sheaves on Σ_n parametrized by $\mathcal{M}^n(r, a, c)$ (for the precise notion of family see Definition 4.2).

If not otherwise specified, by "scheme" we mean a noetherian reduced scheme of finite type over $\mathbb{C}$. If X and S are schemes, $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X \times S}$-modules, and F is a morphism between two such sheaves, we shall denote by $\mathcal{F}_s$ (resp. F_s) the restriction of $\mathcal{F}$ (resp. F) to the fibre of $X \times S \longrightarrow S$ over the point $s \in S$.

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2. Monads

A monad M on a scheme T is a three-term complex of locally-free $\mathcal{O}_T$-modules, having nontrivial cohomology only in the middle term:

$$M: \qquad 0 \longrightarrow \mathcal{U} \xrightarrow{a} \mathcal{V} \xrightarrow{b} \mathcal{W} \longrightarrow 0\,. \tag{2.1}$$

The cohomology of the monad will be denoted by $\mathcal{E}(M)$. It is a coherent $\mathcal{O}_T$-module. A *morphism* (*isomorphism*) *of monads* is a morphism (isomorphism) of complexes.

The *display* of the monad (2.1) is the commutative diagram (with exact rows and columns)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{U} & \xrightarrow{a} & \mathcal{B} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{U} & \xrightarrow{a} & \mathcal{V} & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \nu} & & \\
 & & & & \mathcal{W} & = & \mathcal{W} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.2}$$

where $\mathcal{A} := \operatorname{coker} a$, $\mathcal{B} := \ker b$, $\mathcal{E} = \mathcal{E}(M)$, and all morphisms are naturally induced.

Let us suppose that T is a product $T = X \times S$, where X is a smooth connected projective variety and S a scheme; we denote by t_i, $i = 1, 2$ the canonical projections onto the first and second factor, respectively.

Lemma 2.1. *Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism between two coherent sheaves on T. If for all* closed *points $s \in S$ the restricted morphism φ_s is surjective (resp. injective), then φ is surjective (resp. injective).*

Proof. For all $\mathcal{O}_T$-modules $\mathcal{N}$ and all closed points $(x, s) \in T$ there are isomorphisms

$$\mathcal{N}_{(x,s)} \simeq (\mathcal{N}_s)_x \otimes_{\mathcal{O}_{(T_s)_x}} \mathcal{O}_{T,(x,s)}\,. \tag{2.3}$$

and this is a flat base change. Let $\mathcal{Q} = \operatorname{coker} \phi$, and (x, s) a closed point in T. By (2.3), since $\mathcal{Q}_s$ vanishes by hypothesis, we have $\mathcal{Q} = 0$ on the subset of closed points of T, and then $\mathcal{Q} = 0$ everywhere [8, Lemma 2.8b]. So ϕ is surjective.

Assume instead that ϕ_s is injective for all *closed* points $s \in S$, and put $\mathcal{K} = \ker \phi$. One has an exact sequence

$$0 \longrightarrow \mathcal{F}_s \xrightarrow{\phi_s} \mathcal{G}_s \longrightarrow \mathcal{Q}_s \longrightarrow 0\,.$$

For s closed one has $T_s \simeq X$. If $x \in X$ is a closed point, consider the previous sequence on the stalks at x, and apply the base change (2.3), getting

$$0 \longrightarrow \mathcal{F}_{(s,x)} \xrightarrow{f_{(s,x)}} \mathcal{G}_{(s,x)} \longrightarrow \mathcal{Q}_{(s,x)} \longrightarrow 0$$

It follows that $\mathcal{K}_{(x,s)} = 0$ for all closed points $(x,s) \in T$. Again by [8, Lemma 2.8b] one deducse $\mathcal{K} = 0$. This completes the proof. □

Lemma 2.2. *Let M be a monad over the product scheme $T = X \times S$ (see eq. (2.1)). The cohomology $\mathcal{E}(M)$ is flat over S if and only if the restricted morphisms a_s are injective for all closed points $s \in S$.*

Proof. By Lemma 2.1.4 in [18], what we need to prove is that if a_s is injective for all *closed* then it is so for all s. Since flatness over S is local over S, we can assume $S = \operatorname{Spec} R$. If s is not closed, it is the generic point of a closed irreducible subscheme $Z \subseteq S$. We know that $a_{|Z}$ is injective for all closed points $s \in Z$. By Lemma 2.1, $a_{|Z}$ is injective. Since $\{s\} \hookrightarrow Z$ is a flat morphism, we conclude that a_s is injective. □

Suppose now that $\dim X = 2$.

Proposition 2.3. *If $\mathcal{E}$ and $\mathcal{E}'$ are coherent sheaves on T, flat on S, and for all closed points s in S the restrictions $\mathcal{E}_s$ and $\mathcal{E}'_s$ are torsion-free, the sheaf $\mathcal{H}om(\mathcal{E},\mathcal{E}')$ is flat on S.*

The proof of this Proposition relies on a few intermediate results. Let A be a commutative ring, B and R *not necessarily flat* A-algebras, with R noetherian; let $R' := R \otimes_A B$.

Lemma 2.4. *Let M, N be R-modules, and suppose that M is finitely generated as a R-module, and flat as an A-module. Then*

$$\operatorname{Ext}^i_R(M,N) \otimes_R R' \simeq \operatorname{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R') \quad \textit{for } i \geq 0\,.$$

Proof. It is an easy modification of the proof of [20, Prop (3.E)]. □

We notice that, if $\mathcal{E}$ is flat on S, for every point $s \in S$ there are isomorphisms

$$\left[\mathcal{E}xt^i_{\mathcal{O}_T}(\mathcal{E},\mathcal{O}_T)\right]_s \simeq \mathcal{E}xt^i_{\mathcal{O}_{T_s}}(\mathcal{E}_s,\mathcal{O}_{T_s}) \qquad i \geq 0\,. \tag{2.4}$$

In particular, for all $s \in S$ one has $(\mathcal{E}^*)_s \simeq (\mathcal{E}_s)^*$. Since $\dim X = 2$ and X is smooth it follows that $(\mathcal{E}^*)_s$ is locally free on T_s.

Lemma 2.5. *The dual $\mathcal{E}^*$ of a coherent $\mathcal{O}_T$-module $\mathcal{E}$ flat on S is locally free.*

Proof. We claim that, for any point (closed or not) $s \in S$ one has $(\mathcal{E}^*)_s \simeq (\mathcal{E}_s)^*$. Indeed one has

$$(\mathcal{E}^*)_s = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{E}, \mathcal{O}_T) \otimes_{\mathcal{O}_T} \mathcal{O}_{T_s}$$
$$(\mathcal{E}_s)^* = \mathcal{H}om_{\mathcal{O}_{T_s}}(\mathcal{E} \otimes_{\mathcal{O}_T} \mathcal{O}_{T_s}, \mathcal{O}_{T_s})$$

Since $\mathcal{O}_{T_s} = \mathcal{O}_T \otimes_{\mathcal{O}_S} k(s)$ and $\mathcal{E}$ is flat on S, the claim follows from Lemma 2.4.

In particular, whenever s is *closed*, $\mathcal{E}_s^*$ is locally free since $\dim X = 2$. This implies that, for all closed points $x \in T_s$, the stalk $(\mathcal{E}_s)_x^*$ is a free $\mathcal{O}_{T_s,x}$-module. By equation (2.3) it follows that for all closed points $(x,s) \in T$ the stalk $\mathcal{E}^*_{(x,s)}$ is a free $\mathcal{O}_{T_{(s,x)}}$-module. By Exercise 6.2 in [8], $\mathcal{E}^*$ is locally free. □

Lemma 2.6. *Let $\mathcal{E}$ be a coherent $\mathcal{O}_T$-module such that the restrictions $\mathcal{E}_s$ are torsion-free for all closed $s \in S$. Then $\mathcal{E}$ is torsion-free.*

Proof. By hypothesis, for all $x \in T_s$, the stalk $(\mathcal{E}_s)_x$ is torsion-free. By equation (2.3), for all closed $(x,s) \in T$, the stalk $\mathcal{E}_{(x,s)}$ is torsion-free. Let $\mathcal{T}$ be torsion submodule of $\mathcal{E}$, so that one has an exact sequence $0 \to \mathcal{T} \to \mathcal{E} \to \mathcal{E}^{**}$. By localizing at a closed point $(x,s) \in T$, we obtain $\mathcal{T}_{(x,s)} = 0$. By Lemma 2.8 in [8], one has $\mathcal{T} = 0$. □

Proof of Proposition 2.3. One has the following natural morphism of sheaves on T

$$\mathcal{H}om(\mathcal{E}, \mathcal{E}') \xrightarrow{f} \mathcal{H}om(\mathcal{E}^{**}, (\mathcal{E}')^{**}) . \tag{2.5}$$

By Lemma 2.6 $\mathcal{E}$ and $\mathcal{E}'$ are torsion-free, hence the morphism f is injective. Since for all closed $s \in S$ the sheaves $\mathcal{E}_s$ and $\mathcal{E}'_s$ are torsion-free, by appling Lemma 2.4 one deduces that also the restricted morphisms f_s are injective for all closed $s \in S$. At the same time the sheaf $\mathcal{H}om(\mathcal{E}^{**}, (\mathcal{E}')^{**})$ is flat on S by its local freeness and the flatness of the projection t_2. From this we can prove that $\operatorname{coker} f$ is flat on S by a reasoning similar to the proof of Lemma 2.2. The thesis follows from [12, Prop. III 9.1A.(e)]. □

3. Statement of the main result

Let Σ_n be the n-th Hirzebruch surface, i.e., the projectivization of the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$. We denote by F the class in $\operatorname{Pic}(\Sigma_n)$ of the natural ruling $\Sigma_n \longrightarrow \mathbb{P}^1$ and by H the class of the "line at infinity" $\ell_\infty \simeq \mathbb{P}^1$. One has $\operatorname{Pic}(\Sigma_n) = \mathbb{Z}H \oplus \mathbb{Z}F$, and for any sheaf $\mathcal{E}$ of $\mathcal{O}_{\Sigma_n}$-modules we shall write

$$\mathcal{E}(p,q) := \mathcal{E} \otimes \mathcal{O}_{\Sigma_n}(pH + qF) \qquad p, q \in \mathbb{Z},$$

Lemma 3.1.

$$H^0(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \qquad \textit{if and only if} \qquad \begin{cases} p \geq 0 \\ np + q \geq 0\,; \end{cases}$$

$$H^1(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \qquad \textit{if and only if} \qquad \begin{cases} p \geq 0 \\ q \leq -2 \end{cases} \qquad \textit{or} \qquad \begin{cases} p \leq -2 \\ q \geq n\,; \end{cases}$$

$$H^2(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \qquad \textit{if and only if} \qquad \begin{cases} p \leq -2 \\ np + q \leq -(n+2)\,. \end{cases}$$

Proof. Similar to King's proof for the case $n = 1$ [19, pp. 22-23]. □

Among the several definitions of framed sheaves available in the literature, we shall adopt the following.

Definition 3.2. *A* framed sheaf *is a pair* $(\mathcal{E}, \theta)$*, where*

(1) $\mathcal{E}$ *is a torsion-free sheaf on* Σ_n *such that*

$$\mathcal{E}|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus r}, \tag{3.1}$$

with $r = \mathrm{rk}(\mathcal{E})$.

(2) θ *is a fixed isomorphism* $\theta : \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$*;*

Condition (3.1) implies $c_1(\mathcal{E}) \propto E$, where $E = H - nF$. The isomorphism θ is the so-called *framing at infinity*. By "sheaf trivial at infinity" we shall mean a sheaf satisfying condition (3.1) (without any assigned framing).

Definition 3.3. *An isomorphism* Λ *between two framed sheaves* $(\mathcal{E}, \theta)$ *and* $(\mathcal{E}', \theta')$ *is an isomorphism* $\Lambda : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{E}|_{\ell_\infty} & \xrightarrow{\theta} & \mathcal{O}_{\ell_\infty}^{\oplus r} \\ {\scriptstyle \Lambda_\infty}\downarrow & \nearrow{\scriptstyle \theta'} & \\ \mathcal{E}'|_{\ell_\infty} & & \end{array} \tag{3.2}$$

where $\Lambda_\infty := \Lambda|_{\ell_\infty}$.

Let $\mathcal{M}^n(r,a,c)$ be the set of isomorphism classes of framed sheaves on Σ_n having rank r, first Chern class aE, and second Chern class c: we shall prove that this set can be endowed with a structure of a smooth algebraic variety. We restrict ourselves to the case $n \geq 1$ and assume that the framed sheaves are normalized in such a way that $0 \leq a \leq r-1$.

In order to simplify the statement of Theorem 3.4 we introduce some notation.

- We denote by $\vec{k}$ a quadruple (n, r, a, c), and define $k_i \in \mathbb{Z}$, $i = 1, \ldots, 4$ as follows:

$$k_1 = c + \frac{1}{2} na(a-1) \qquad \text{and} \qquad \begin{cases} k_2 = k_1 + na \\ k_3 = k_1 + (n-1)a \\ k_4 = k_1 + r - a\,. \end{cases} \tag{3.3}$$

  The other definitions needed to state Theorem 3.4 makes sense only under the assumpition $k_1 \geq 0$. The Theorem itself will give a deeper meaning to this inequality.
- We introduce the locally-free sheaves

$$\begin{cases} \mathcal{U}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(0,-1)^{\oplus k_1} \\ \mathcal{V}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus k_2} \oplus \mathcal{O}_{\Sigma_n}^{\oplus k_4} \\ \mathcal{W}_{\vec{k}} := \mathcal{O}_{\Sigma_n}(1,0)^{\oplus k_3}. \end{cases} \tag{3.4}$$

  We shall write $\mathcal{U}_{\vec{k},\infty}$ in place of $\mathcal{U}_{\vec{k}}\big|_{\ell_\infty}$, etc.
- We introduce the vector space

$$\mathbb{V}_{\vec{k}} := \operatorname{Hom}\left(\mathcal{U}_{\vec{k}}, \mathcal{V}_{\vec{k}}\right) \oplus \operatorname{Hom}\left(\mathcal{V}_{\vec{k}}, \mathcal{W}_{\vec{k}}\right) ,$$

  whose elements will be denoted by (α, β).
- Let $\overline{L}_{\vec{k}}$ the affine subvariety of $\mathbb{V}_{\vec{k}}$ cut by the equation $\beta \circ \alpha = 0$. One has the associated complex

$$M(\alpha,\beta): \qquad \mathcal{U}_{\vec{k}} \xrightarrow{\alpha} \mathcal{V}_{\vec{k}} \xrightarrow{\beta} \mathcal{W}_{\vec{k}}\,.$$

- We define the quasi-affine variety $L_{\vec{k}}$ as the open subset of $\overline{L}_{\vec{k}}$ characterized by the following four conditions:
  (c1) the sheaf morphism α is a monomorphism;
  (c2) the sheaf morphism β is an epimorphism;
  (c3) the vector space morphisms $\alpha \otimes k(y)$ have maximal rank for all closed points $y \in \ell_\infty$;
  (c4) if we consider the display associated with the monad $M(\alpha,\beta)$ as in eq. (2.2), and we restrict it to ℓ_∞, after twisting by $\mathcal{O}_{\ell_\infty}(-1)$ and taking cohomology, we get a vector space morphism $\Phi := H^0\left(\nu|_{\ell_\infty}(-1)\right) : H^0\left(\mathcal{A}|_{\ell_\infty}(-1)\right) \longrightarrow H^0\left(\mathcal{W}_{\vec{k},\infty}(-1)\right)$; we require that

$$\det \Phi \neq 0\,.$$

(c5) the cohomology $\mathcal{E}_{\alpha,\beta}$ of the monad $M(\alpha,\beta)$ is torsion-free.

We shall prove in Lemma 4.9 that $L_{\vec{k}}$ is a smooth variety. We let $\mathcal{E}_{\alpha,\beta,\infty} = \mathcal{E}_{\alpha,\beta}|_{\ell_\infty}$.

- We consider the algebraic group

$$G_{\vec{k}} = \operatorname{Aut}\left(\mathcal{U}_{\vec{k}}\right) \times \operatorname{Aut}\left(\mathcal{V}_{\vec{k}}\right) \times \operatorname{Aut}\left(\mathcal{W}_{\vec{k}}\right).$$

This group acts naturally on $L_{\vec{k}}$ according to the following formulas:

$$\begin{cases} \alpha & \mapsto & \alpha' = \psi\alpha\phi^{-1} \\ \beta & \mapsto & \beta' = \chi\beta\psi^{-1} \end{cases} \qquad \bar{\psi} = (\phi,\psi,\chi) \in G_{\vec{k}}\,. \tag{3.5}$$

This action will be called $\rho_0 : L_{\vec{k}} \times G_{\vec{k}} \to L_{\vec{k}}$.

- We introduce (see subsection 4.2) a principal $\mathrm{GL}(r,\mathbb{C})$ bundle $P_{\vec{k}} \xrightarrow{\tau} L_{\vec{k}}$ whose fibre over (α,β) is naturally identified with the space of framings at infinity for $\mathcal{E}_{\alpha,\beta}$, namely, a point $\theta \in P_{\vec{k}}$ is an isomorphism $\theta : \mathcal{E}_{\alpha,\beta,\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r}$, where $(\alpha,\beta) = \tau(\theta)$.
- One can lift ρ_0 to an action ρ: $P_{\vec{k}} \times G_{\vec{k}} \to P_{\vec{k}}$ by letting

$$\rho(\theta,\bar{\psi}) = \theta \circ \Lambda_\infty\left(\alpha,\beta;\bar{\psi}\right)^{-1}, \tag{3.6}$$

where $(\alpha,\beta) = \tau(\theta)$, and after letting $(\alpha',\beta') = \bar{\psi}\cdot(\alpha,\beta)$, the isomorphism $\Lambda_\infty\left(\alpha,\beta;\bar{\psi}\right) : \mathcal{E}_{\alpha,\beta,\infty} \longrightarrow \mathcal{E}_{\alpha',\beta',\infty}$ is induced by $\bar{\psi} : M(\alpha,\beta) \longrightarrow M(\alpha',\beta')$. The identity

$$\Lambda_\infty(\alpha,\beta,\bar{\psi}'\cdot\bar{\psi}) = \Lambda_\infty(\alpha',\beta',\bar{\psi}') \circ \Lambda_\infty(\alpha,\beta,\bar{\psi})$$

ensures that ρ is indeed an action. We notice that the projection τ: $P_{\vec{k}} \to L_{\vec{k}}$ becomes a $G_{\vec{k}}$-equivariant morphism.

We have now all ingredients needed to state our main result.

Theorem 3.4. *The set $\mathcal{M}^n(r,a,c)$ is nonempty if and only if $c+\frac{1}{2}na(a-1) \geq 0$. If this is the case, it can be given a structure of smooth algebraic variety of dimension $2rc+(r-1)na^2$ by representing it as the quotient $\mathcal{M}^n(r,a,c) = P_{\vec{k}}/G_{\vec{k}}$. Moreover, $\mathcal{M}^n(r,a,c)$ turns out to be a fine moduli space of framed sheaves on Σ_n.*

Note that

$$\dim \mathcal{M}^n(r,a,c) = 2rc + (r-1)na^2 = 2r\Delta,$$

where $\Delta = c_2 - \frac{r-1}{2r}c_1^2$ is the discriminant of the sheaves parametrized by $\mathcal{M}^n(r,a,c)$.

4. Families of framed sheaves

In this section we explain how the varieties $L_{\vec{k}}$ and $P_{\vec{k}}$ arise and construct a canonical family $\left(\widetilde{\mathfrak{E}}_{\vec{k}}, \widetilde{\Theta}_{\vec{k}}\right)$ on the product $\Sigma_n \times P_{\vec{k}}$.

For any scheme S, let $T = \Sigma_n \times S$ and let t_i, $i = 1,2$ be the projections onto the first and the second factor, respectively. Analogously, we introduce the product scheme $T_\infty = \ell_\infty \times S$, toghether with the projections u_i, $i = 1,2$.

Definition 4.1. *Let $\vec{k} = (n, r, a, c) \in \mathbb{Z}^4$ with $n \geq 1$, $r \geq 1$ and $0 \leq a \leq r-1$. A coherent sheaf $\mathfrak{F}$ on T is fullfills condition $\vec{k}$ if and only if it is flat on S and for all closed points $s \in S$*

- *the restricted sheaf $\mathfrak{F}_s$ is torsion-free and trivial at infinity on $T_s \simeq \Sigma_n$;*
- *the Chern character of $\mathfrak{F}_s$ is $(r, aE, -c - \frac{1}{2}na^2)$.*

Note that by Lemma 2.6 $\mathfrak{F}$ is torsion-free.

Definition 4.2. *Given a vector $\vec{k}$ and a product scheme T as above, a family of framed sheaves on Σ_n is a pair $(\mathfrak{F}, \Theta)$, where:*

(1) *$\mathfrak{F}$ is a sheaf on T fullfilling condition $\vec{k}$;*
(2) *Θ is an isomorphism $\mathfrak{F}|_{T_\infty} \to \mathcal{O}_{T_\infty}^{\oplus r}$.*

For any sheaf $\mathfrak{G}$ of $\mathcal{O}_T$-modules we let

$$\mathfrak{G}(p,q) = \mathfrak{G} \otimes t_1^* \mathcal{O}_{\Sigma_n}(p,q) \qquad \text{for all } (p,q) \in \mathbb{Z}.$$

Proposition 4.3. *A sheaf $\mathfrak{F}$ on T that satisfies condition $\vec{k}$ is isomorphic to the cohomology of a canonically associated monad $M(\mathfrak{F})$ on T*

$$M(\mathfrak{F}): \qquad 0 \longrightarrow \mathfrak{U} \xrightarrow{A} \mathfrak{V} \xrightarrow{B} \mathfrak{W} \longrightarrow 0 \ , \tag{4.1}$$

where the locally-free sheaves $\mathfrak{U}$ and $\mathfrak{W}$ are

$$\begin{cases} \mathfrak{U} = \mathcal{O}_T(0,-1) \otimes t_2^* R^1 t_{2*} \left[\mathfrak{F}(-2, n-1)\right] \\ \mathfrak{W} = \mathcal{O}_T(1,0) \otimes t_2^* R^1 t_{2*} \left[\mathfrak{F}(-1,0)\right]. \end{cases}$$

The locally-free sheaf $\mathfrak{V}$ is defined as an extension

$$0 \longrightarrow \mathfrak{V}_- \xrightarrow{\mathfrak{c}} \mathfrak{V} \xrightarrow{\mathfrak{d}} \mathfrak{V}_+ \longrightarrow 0\,, \tag{4.2}$$

$$\textit{where} \qquad \begin{cases} \mathfrak{V}_+ := t_2^* R^1 t_{2*} \left[\mathfrak{F}(-2, n)\right] \\ \mathfrak{V}_- := \mathcal{O}_T(1,-1) \otimes t_2^* R^1 t_{2*} \left[\mathfrak{F}(-1,-1)\right] \end{cases}$$

and the morphisms $\mathfrak{c}$ *and* $\mathfrak{d}$ *are canonically determined by* $\mathfrak{F}$.

Moreover the association $\mathfrak{F} \mapsto M(\mathfrak{F})$ *is covariantly functorial.*

To prove this Proposition, one needs the following result.

Lemma 4.4. *Let* $\mathcal{E}$ *be a torsion-free sheaf on* Σ_n *trivial at infinity. One has*

$$\begin{aligned} H^0(\mathcal{E}(p,q)) = 0 \qquad &\text{for} \qquad np+q \leq -1\,, \\ H^2(\mathcal{E}(p,q)) = 0 \qquad &\text{for} \qquad np+q \geq -(n+1)\,. \end{aligned}$$

Proof. When $\mathcal{E}$ is locally free, the proof is essentially the same as in [19, p. 24]. Otherwise we get the thesis by using the injection $\mathcal{E} \rightarrowtail \mathcal{E}^{**}$. □

Proof of Proposition 4.3. Let us consider the product scheme $\Sigma_n \times \Sigma_n \times S$, together with the canonical projections p_{12}, p_{13} and p_{23}. Buchdahl [4] proved the existence of a three-term locally-free resolution $\mathcal{G}^\bullet \twoheadrightarrow \mathcal{O}_\Delta$ of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subseteq \Sigma_n \times \Sigma_n$. Given any sheaf $\mathfrak{F}$ on T fullfilling condition $\vec{k}$, we introduce the complex

$$\mathcal{C}^\bullet = \; \mathcal{C}^{-2} \longrightarrow \mathcal{C}^{-1} \longrightarrow \mathcal{C}^0 \; = (p_{12}^*\mathcal{G}^\bullet) \otimes [p_{23}^*(\mathfrak{F}(-1,0))] \;.$$

There are two spectral sequences both abutting to hyperdirect image $\mathbb{R}^\bullet p_{13*}(\mathcal{C}^\bullet)$. From the first spectral sequence one gets

$$\mathbb{R}^i p_{13*}(\mathcal{C}^\bullet) = \begin{cases} \mathfrak{F}(-1,0) & \text{if } i=0 \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.3}$$

By using Lemma 4.4, one can obtain from the second exact sequence a complex that, when twisted by $\mathcal{O}_T(1,0)$, yields $M(\mathfrak{F})$.

The functoriality follows from the construction. □

This proof implies that the sheaves $R^1 t_{2*}\left(\mathfrak{F}(p,q)\right)$ are locally free for

$$(p,q) \in \mathscr{I} = \{(-2,n-1),(-1,0),(-2,n),(-1,-1)\}\,.$$

4.1. **The variety $L_{\vec{k}}$.** As a straightforward consequence of Proposition 4.3 we get the following result.

Corollary 4.5. *Let the sheaf* $\mathfrak{F}$ *and the monad* $M(\mathfrak{F})$ *be as in Proposition 4.3. Assume that* S *is affine, and that the sheaves* $R^1 t_{2*}\left(\mathfrak{F}(p,q)\right)$ *are trivial for* $(p,q) \in \mathscr{I}$. *There are isomorphisms*

$$\mathfrak{U} \simeq t_1^*\mathcal{U}_{\vec{k}}\,; \qquad \mathfrak{V} \simeq t_1^*\mathcal{V}_{\vec{k}}\,; \qquad \mathfrak{W} \simeq t_1^*\mathcal{W}_{\vec{k}}\,,$$

the sheaves $\mathcal{U}_{\vec{k}}$, $\mathcal{V}_{\vec{k}}$ *and* $\mathcal{W}_{\vec{k}}$ *being defined as in eq.* (3.4).

Proof. A trivialization for $R^1t_{2*}\left(\mathfrak{F}(p,q)\right)$ amounts to choosing a closed point $s_0 \in S$ and an isomorphism

$$R^1t_{2*}\left(\mathfrak{F}(p,q)\right) \xrightarrow{\sim} \mathcal{O}_S \otimes \left[R^1t_{2*}\left(\mathfrak{F}(p,q)\right) \otimes k(s_0)\right].$$

Since $\mathfrak{F}_{s_0}$ is torsion-free and trivial at infinity, from Lemma 4.4 and from the Semicontinuity Theorem one obtains the isomorphism

$$R^1t_{2*}\left(\mathfrak{F}(p,q)\right) \otimes k(s_0) \simeq H^1\left(\mathfrak{F}_{s_0}(p,q)\right).$$

The dimensions of the vector spaces $H^1(\mathfrak{F}_{s_0}(p,q))$ can be computed by means of Riemann-Roch Theorem and Lemma 4.4:

$$h^1\left(\mathfrak{F}_{s_0}(p,q)\right) = \begin{cases} k_1 & \text{for } (p,q) = (-2, n-1) \\ k_2 & \text{for } (p,q) = (-1,-1) \\ k_3 & \text{for } (p,q) = (-1,0) \\ k_4 & \text{for } (p,q) = (-2,n) \end{cases}$$

where k_i, $i = 1, \ldots, 4$ are as in eq. (3.3). As a consequence the sheaves $R^1t_{2*}\left(\mathfrak{F}(-2,n-1)\right)$, $R^1t_{2*}\left(\mathfrak{F}(-1,-1)\right)$, $R^1t_{2*}\left(\mathfrak{F}(-1,0)\right)$ and $R^1t_{2*}\left(\mathfrak{F}(-2,n)\right)$ are free of ranks $k_1, \ldots, k_4$ respectively, so that

$$\mathfrak{U} \simeq \mathcal{O}_T(0,-1)^{\oplus k_1}; \qquad \mathfrak{W} \simeq \mathcal{O}_T(1,0)^{\oplus k_3}; \qquad \mathfrak{V}_- \simeq \mathcal{O}_T^{\oplus k_4}; \qquad \mathfrak{V}_+ \simeq \mathcal{O}_T(1,-1)^{\oplus k_2}.$$

The thesis follows for $\mathfrak{U}$ and $\mathfrak{W}$. By plugging $\mathfrak{V}_-$ and $\mathfrak{V}_+$ into the sequence eq. (4.2), the latter splits, since

$$\begin{aligned} \operatorname{Ext}^1\left(\mathcal{O}_T(1,-1), \mathcal{O}_T\right) &\simeq H^1\left(T, \mathcal{O}_T(-1,1)\right) \simeq \\ &\simeq H^0\left(S, R^1t_{2*}\mathcal{O}_T(-1,1)\right) \simeq \qquad (*) \\ &\simeq H^0\left(S, R^1t_{2*}\left[t_1^*\mathcal{O}_{\Sigma_n}(-1,1)\right]\right) \simeq \\ &\simeq H^0(S, \mathcal{O}_S) \otimes H^1\left(\Sigma_n, \mathcal{O}_{\Sigma_n}(-1,1)\right) = 0 \end{aligned}$$

(the isomorphism $(*)$ holds true as S is affine and the vanishing is a consequence of Lemma 4.4). This ends the proof. □

The following result is the absolute case of Proposition 4.3, obtained by letting $S = \operatorname{Spec}\mathbb{C}$, and follows easily from Corollary 4.5.

Corollary 4.6. *Any sheaf $\mathcal{E}$ on Σ_n that is torsion-free and trivial at infinity is isomorphic to the cohomology of a monad $M_{\vec{k}}(\mathcal{E})$, which is of the form $M(\alpha,\beta)$ for a suitable $(\alpha,\beta) \in \overline{L}_{\vec{k}}$. Notice that we do not require $k_1 \geq 0$* a priori. *As a consequence however, if $k_1 = c + \frac{1}{2}na(a-1) < 0$, the set $\mathcal{M}^n(r,a,c)$ is empty.*

We fix a $\vec{k}$ such that $k_1 \geq 0$.

The functoriality of $M_{\vec{k}}(-)$ implies that $M_{\vec{k}}(\mathcal{E}) \simeq M_{\vec{k}}(\mathcal{E}')$ whenever $\mathcal{E} \simeq \mathcal{E}'$. In particular, $M_{\vec{k}}(-)$ provides a map between the set of isomorphism classes of torsion-free sheaves on Σ_n that are trivial at infinity and the set of isomorphism classes of monads of the form $M(\alpha, \beta)$. The following two results establish the injectivity of this map, and enable us to characterize its image.

Lemma 4.7. *Let $(\alpha, \beta), (\alpha', \beta')$ be any two points in $\overline{L}_{\vec{k}}$ satisfying the conditions (c1) and (c2) introduced in section 3 (therefore $M = M(\alpha, \beta)$ and $M' = M(\alpha', \beta')$ are monads). Then*

$$M \simeq M' \qquad \text{if and only if} \qquad \mathcal{E}(M) \simeq \mathcal{E}(M').$$

Proof. The proof of [25, Lemma 4.1.3] holds true for non locally free sheaves as well. □

Proposition 4.8. *For any point $(\alpha, \beta) \in \overline{L}_{\vec{k}}$ satisfying conditions (c1) and (c2), the cohomology $\mathcal{E}$ of the monad $M(\alpha, \beta)$ is trivial at infinity if and only if the morphisms (α, β) satisfy conditions (c3) and (c4).*

Proof. Condition (c3) is equivalent to the local freeness of $\mathcal{E}|_{\ell_\infty}$. As for condition (c4), the display of $M(\alpha, \beta)$ produces the exact sequence

$$H^0\left(\mathcal{E}|_{\ell_\infty}(-1)\right) \rightarrowtail H^0\left(\mathcal{A}|_{\ell_\infty}(-1)\right) \xrightarrow{\Phi} H^0\left(\mathcal{W}_{\vec{k},\infty}(-1)\right) \twoheadrightarrow H^1\left(\mathcal{E}|_{\ell_\infty}(-1)\right).$$

Condition (c4) is equivalent to the vanishing of $H^i\left(\mathcal{E}|_{\ell_\infty}(-1)\right)$, $i = 0, 1$. The thesis follows easily. □

This result enables us to identify $L_{\vec{k}}$ with the subset of $\mathbb{W}_{\vec{k}}$ whose points correspond to cohomology sheaves $\mathcal{E}_{\alpha,\beta}$ that are torsion-free and trivial at infinity.

Lemma 4.9. *The variety $L_{\vec{k}}$ is smooth of dimension $\dim L_{\vec{k}} = \dim \mathbb{V}_{\vec{k}} - \dim \mathbb{W}_{\vec{k}}$, where $\mathbb{W}_{\vec{k}} = \operatorname{Hom}\left(\mathcal{U}_{\vec{k}}, \mathcal{W}_{\vec{k}}\right)$.*

Proof. The proof is essentially the same as in [25, Lemma 4.1.7]. We define the map

$$\begin{array}{rccc} \zeta: & \mathbb{V}_{\vec{k}} & \longrightarrow & \mathbb{W}_{\vec{k}} \\ & (\alpha, \beta) & \longmapsto & \beta\alpha. \end{array}$$

So $\overline{L}_{\vec{k}}$ is the set $\{\zeta = 0\}$. The differential $\mathrm{d}\zeta$ at the point $(\alpha_0, \beta_0) \in L_{\vec{k}}$ is the linear map

$$\begin{array}{rccc} (\mathrm{d}\zeta)|_{(\alpha_0,\beta_0)}: & \mathbb{V}_{\vec{k}} & \longrightarrow & \mathbb{W}_{\vec{k}} \\ & (\alpha, \beta) & \longmapsto & \beta_0\alpha + \beta\alpha_0. \end{array}$$

The rank of this map is equal to $d = \dim \mathbb{V}_{\vec{k}} - \dim L_{\vec{k}}$ on the (non empty) nonsingular locus of $L_{\vec{k}}$, and outside of this set is bounded above by d (see [12, pp. 31-33]).

It is not difficult to prove that for any point $(\alpha, \beta) \in L_{\vec{k}}$, there is an isomorphism

$$\operatorname{coker}(\mathrm{d}\zeta)|_{(\alpha,\beta)} \simeq H^2\left(\mathcal{E}^*_{\alpha,\beta} \otimes \mathcal{E}_{\alpha,\beta}\right). \tag{4.4}$$

Now, $\mathcal{E}^*_{\alpha,\beta} \otimes \mathcal{E}_{\alpha,\beta}$ is torsion-free and trivial at infinity, so that $H^2\left(\mathcal{E}^*_{\alpha,\beta} \otimes \mathcal{E}_{\alpha,\beta}\right) = 0$ by Lemma 4.4 and $\mathrm{d}\zeta$ has maximal rank everywhere on $L_{\vec{k}}$. Since the singular locus of $L_{\vec{k}}$ coincides with the set of points in which $\mathrm{d}\zeta$ fails to have maximal rank, $L_{\vec{k}}$ is smooth. □

4.2. **The variety $P_{\vec{k}}$.** Let us introduce the varieties $\Upsilon = \Sigma_n \times L_{\vec{k}}$ and $\Upsilon_\infty = \ell_\infty \times L_{\vec{k}}$, together with the canonical projections shown in the following diagram:

$$\begin{array}{ccc} \ell_\infty & \hookrightarrow & \Sigma_n \\ \uparrow{\scriptstyle \breve{\mathfrak{u}}_1} & & \uparrow{\scriptstyle \breve{\mathfrak{t}}_1} \\ \Upsilon_\infty & \hookrightarrow & \Upsilon \\ & {\scriptstyle \breve{\mathfrak{u}}_2}\searrow & \downarrow{\scriptstyle \breve{\mathfrak{t}}_2} \\ & & L_{\vec{k}}\,. \end{array} \tag{4.5}$$

On Υ we define the complex

$$\breve{\mathbb{M}}_{\vec{k}}: \qquad \breve{\mathfrak{t}}_1^*\mathcal{U}_{\vec{k}} \xrightarrow{\breve{\mathfrak{t}}_2^* f_A} \breve{\mathfrak{t}}_1^*\mathcal{V}_{\vec{k}} \xrightarrow{\breve{\mathfrak{t}}_2^* f_B} \breve{\mathfrak{t}}_1^*\mathcal{W}_{\vec{k}}$$

where f_A and f_B are the restrictions to $L_{\vec{k}}$ of the projections of $\mathbb{V}_{\vec{k}} = \operatorname{Hom}\left(\mathcal{U}_{\vec{k}}, \mathcal{V}_{\vec{k}}\right) \oplus \operatorname{Hom}\left(\mathcal{V}_{\vec{k}}, \mathcal{W}_{\vec{k}}\right)$ onto its direct summands. This complex is actually a monad by Lemma 2.1: let $\breve{\mathfrak{E}}_{\vec{k}}$ be its cohomology. This sheaf satisfies condition $\vec{k}$, and more precisely one has the following isomorphism for all points $(\alpha, \beta) \in L_{\vec{k}}$:

$$\left(\breve{\mathfrak{E}}_{\vec{k}}\right)_{(\alpha,\beta)} \simeq \mathcal{E}_{\alpha,\beta}\,.$$

The restriction $\breve{\mathbb{M}}_{\vec{k},\infty}$ of $\breve{\mathbb{M}}_{\vec{k}}$ to Υ_∞ is isomorphic to the monad

$$0 \longrightarrow \breve{\mathfrak{u}}_1^*\mathcal{U}_{\vec{k},\infty} \xrightarrow{\breve{\mathfrak{u}}_2^* g_B} \breve{\mathfrak{u}}_1^*\mathcal{V}_{\vec{k},\infty} \xrightarrow{\breve{\mathfrak{u}}_2^* g_B} \breve{\mathfrak{u}}_1^*\mathcal{W}_{\vec{k},\infty} \longrightarrow 0\,. \tag{4.6}$$

where $g_A = (\cdot|_{\ell_\infty}) \circ f_A : \mathbb{V}_{\vec{k}} \longrightarrow \operatorname{Hom}\left(\mathcal{U}_{\vec{k},\infty}, \mathcal{V}_{\vec{k},\infty}\right)$ and analogously for g_B. The cohomology of $\breve{\mathbb{M}}_{\vec{k},\infty}$ will be denoted by $\breve{\mathfrak{E}}_{\vec{k},\infty}$. It turns out that

$$\left(\breve{\mathfrak{E}}_{\vec{k},\infty}\right)_{(\alpha,\beta)} \simeq \mathcal{E}_{\alpha,\beta,\infty} \tag{4.7}$$

for all points $(\alpha, \beta) \in L_{\vec{k}}$.

Let $\mathfrak{N}_{\vec{k}}$ denote the direct image $\breve{\mathfrak{u}}_{2*}\breve{\mathfrak{E}}_{\vec{k},\infty}$. Since $\breve{\mathfrak{E}}_{\vec{k},\infty}$ is a trivial vector bundle on each fibre of $\breve{\mathfrak{u}}_2$, the sheaf $\mathfrak{N}_{\vec{k}}$ is locally free of rank r. Let $P_{\vec{k}}$ be its bundle of linear frames, which is a principal $\mathrm{GL}(r)$ bundle on $L_{\vec{k}}$, whose fibre at $(\alpha,\beta)\in L_{\vec{k}}$ may be identified with the bundle of linear frames of the vector bundle $\mathcal{E}_{\alpha,\beta,\infty}$. Moreover, if $\tau\colon P_{\vec{k}}\to L_{\vec{k}}$ is the projection, the vector bundle $\widetilde{\mathfrak{N}}_{\vec{k}}=\tau^*\mathfrak{N}_{\vec{k}}$ is trivial. Note that a framed sheaf $(\mathcal{E},\theta)$ determines a point of $P_{\vec{k}}$. Indeed the monad whose cohomology is $\mathcal{E}$ gives a point $(\alpha,\beta)\in L_{\vec{k}}$, and the framing θ gives a point of $P_{\vec{k}}$ in the fibre over (α,β).

4.3. **The family $\left(\widetilde{\mathfrak{E}}_{\vec{k}},\widetilde{\Theta}_{\vec{k}}\right)$.** The geometrical environment of this subsection is provided by the varieties $\Sigma_n\times P_{\vec{k}}$ and $\ell_\infty\times P_{\vec{k}}$, together with the canonical projections shown in the following diagram:

$$\begin{array}{ccc} \ell_\infty & \hookrightarrow & \Sigma_n \\ \uparrow{\scriptstyle \tilde{\mathfrak{u}}_1} & & \uparrow{\scriptstyle \tilde{\mathfrak{t}}_1} \\ \ell_\infty\times P_{\vec{k}} & \hookrightarrow & \Sigma_n\times P_{\vec{k}} \\ & \searrow{\scriptstyle \tilde{\mathfrak{u}}_2} & \downarrow{\scriptstyle \tilde{\mathfrak{t}}_2} \\ & & P_{\vec{k}}\,. \end{array}$$

Proposition 4.10. *Let $\widetilde{\mathfrak{E}}_{\vec{k}}:=(\mathrm{id}_{\Sigma_n}\times\tau)^*\,\breve{\mathfrak{E}}_{\vec{k}}$. This sheaf satisfies condition $\vec{k}$, and in particular for any point $\theta\in P_{\vec{k}}$ one has the natural isomorphism*

$$\left(\widetilde{\mathfrak{E}}_{\vec{k}}\right)_\theta\simeq\mathcal{E}_{\tau(\theta)}\,. \tag{4.8}$$

We shall call $\widetilde{\mathfrak{E}}_{\vec{k},\infty}$ the restriction of $\widetilde{\mathfrak{E}}_{\vec{k}}$ to $\ell_\infty\times P_{\vec{k}}$, so that $\widetilde{\mathfrak{E}}_{\vec{k},\infty}\simeq(\mathrm{id}_{\mathbb{P}^1}\times\tau)^*\,\breve{\mathfrak{E}}_{\vec{k},\infty}\simeq\tilde{\mathfrak{u}}_2^*\widetilde{\mathfrak{N}}_{\vec{k}}$.

Proof. The flatness of $\widetilde{\mathfrak{E}}_{\vec{k}}$ on $P_{\vec{k}}$ follows from the identification $\Sigma_n\times P_{\vec{k}}\simeq P_{\vec{k}}\times_{L_{\vec{k}}}\Upsilon$, while the isomorphism (4.8) comes from [12, Prop 9.3], as $\ell_\infty\times P_{\vec{k}}\simeq P_{\vec{k}}\times_{L_{\vec{k}}}\Upsilon_\infty$. The last statement is trivial. □

One can extend the action ρ of $G_{\vec{k}}$ on $P_{\vec{k}}$ to actions $\tilde{\rho}$ on $\Sigma_n\times P_{\vec{k}}$ and ρ_∞ on $\ell_\infty\times P_{\vec{k}}$ by setting

$$\begin{cases} \tilde{\rho}:=\mathrm{id}_{\Sigma_n}\times\rho \\ \rho_\infty:=\mathrm{id}_{\ell_\infty}\times\rho\,. \end{cases}$$

Proposition 4.11. *The sheaves* $\widetilde{\mathfrak{E}}_{\vec{k}}$ *and* $\widetilde{\mathfrak{E}}_{\vec{k},\infty}$ *are, respectively, isomorphic to the cohomologies of the monads*

$$\widetilde{\mathbb{M}}_{\vec{k}} := (\mathrm{id}_{\Sigma_n} \times \tau)^* \, \breve{\mathbb{M}}_{\vec{k}};$$
$$\widetilde{\mathbb{M}}_{\vec{k},\infty} := (\mathrm{id}_{\ell_\infty} \times \tau)^* \, \breve{\mathbb{M}}_{\vec{k},\infty}.$$

Both monads $\widetilde{\mathbb{M}}_{\vec{k}}$ *and* $\widetilde{\mathbb{M}}_{\vec{k},\infty}$ *are* $G_{\vec{k}}$*-equivariant.*

Corollary 4.12. *The sheaf* $\widetilde{\mathfrak{E}}_{\vec{k}}$ *admits a* $G_{\vec{k}}$*-linearization* Ψ *satisfying the isomorphism*

$$\left(\Psi|_{\ell_\infty \times P_{\vec{k}} \times G_{\vec{k}}}\right)^{-1}_{\left(\theta,\bar{\psi}\right)} \simeq \Lambda_\infty\left(\alpha, \beta; \bar{\psi}\right)$$

for any point $\left(\theta, \bar{\psi}\right) \in P_{\vec{k}} \times G_{\vec{k}}$*.*

Proof. The linearization Ψ is induced by the isomorphism

$$m_{12}^* \breve{\mathbb{M}}_{\vec{k}} \xrightarrow{m_3^*\left(\mathrm{id}_{G_{\vec{k}}}\right)} (\mathrm{id}_{\Sigma_n} \times \rho_0)^* \, \breve{\mathbb{M}}_{\vec{k}}\,,$$

where

$$G_{\vec{k}} \xleftarrow{m_3} \Upsilon \times G_{\vec{k}} \xrightarrow{m_{12}} \Upsilon$$

are the canonical projections. □

Since $P_{\vec{k}}$ is the bundle of linear frames of $\mathfrak{N}_{\vec{k}}$, there exists a canonical isomorphism $\widetilde{\mathfrak{N}}_{\vec{k}} \longrightarrow \mathcal{O}_{P_{\vec{k}}}^{\oplus r}$, which can be regarded as a framing $\widetilde{\Theta}_{\vec{k}}$ for the sheaf $\widetilde{\mathfrak{E}}_{\vec{k}}$. As a consequence of eq. (3.6), the morphism $\widetilde{\Theta}_{\vec{k}}$ is $G_{\vec{k}}$-equivariant, namely, the following diagram is commutative

$$\begin{array}{ccc} l_{12}^* \widetilde{\mathfrak{E}}_{\vec{k},\infty} & \xrightarrow{l_{12}^* \widetilde{\Theta}_{\vec{k}}} & \mathcal{O}^{\oplus r}_{\ell_\infty \times P_{\vec{k}} \times G_{\vec{k}}} \\ {\scriptstyle \Psi_\infty}\big\downarrow & \nearrow_{\rho_\infty^* \widetilde{\Theta}_{\vec{k}}} & \\ \rho_\infty^* \widetilde{\mathfrak{E}}_{\vec{k},\infty} & & \end{array} \tag{4.9}$$

where $\Psi_\infty = \Psi|_{\ell_\infty \times P_{\vec{k}} \times G_{\vec{k}}}$ and $l_{12} : \ell_\infty \times P_{\vec{k}} \times G_{\vec{k}} \longrightarrow \ell_\infty \times P_{\vec{k}}$ is the projection.

5. The moduli space $\mathcal{M}^n(r,a,c)$

In this section we shall give the moduli space $\mathcal{M}^n(r,a,c)$ a scheme structure, and prove the first part of the Main Theorem. The space $\mathcal{M}^n(r,a,c)$ can be set-theoretically identified with the quotient $P_{\vec{k}}/G_{\vec{k}}$. We denote by $\pi : P_{\vec{k}} \longrightarrow \mathcal{M}^n(r,a,c)$ the natural projection.

Theorem 5.1. *The orbit space $\mathcal{M}^n(r,a,c)$ is a smooth algebraic variety, and $P_{\vec{k}}$ is a locally trivial principal $G_{\vec{k}}$-bundle over it.*

In order to prove this Theorem, we need to investigate some properties of the $G_{\vec{k}}$-action on $P_{\vec{k}}$.

Lemma 5.2. *Let $\mathcal{E}$ and $\mathcal{E}'$ be sheaves on Σ_n that are torsion-free and trivial at infinity. There is an injection*

$$0 \longrightarrow \operatorname{Hom}(\mathcal{E},\mathcal{E}') \xrightarrow{R} \operatorname{Hom}(\mathcal{E}|_{\ell_\infty},\mathcal{E}'|_{\ell_\infty}) \simeq \operatorname{End}(\mathbb{C}^r)$$

where R is the restriction morphism.

Proof. If $\mathcal{E}$ and $\mathcal{E}'$ are locally free, one has $\operatorname{Hom}(\mathcal{E},\mathcal{E}') \simeq H^0(\mathcal{E}^*\otimes\mathcal{E}')$. The sheaf $\mathcal{E}^*\otimes\mathcal{E}'$ is locally free and trivial at infinity, so that the result follows by twisting the structure sequence of ℓ_∞ by it and taking cohomology (see Lemma 4.4). In the general case one has an injection

$$0 \longrightarrow \operatorname{Hom}(\mathcal{E},\mathcal{E}') \longrightarrow \operatorname{Hom}(\mathcal{E}^{**},\mathcal{E}'^{**})\,.$$

Since $\mathcal{E}^{**}$ and $\mathcal{E}'^{**}$ are locally free, this ends the proof. □

This result generalizes to the relative situation. Let S be a scheme, and let $\mathfrak{F}$ and $\mathfrak{F}'$ be two sheaves on $T = \Sigma_n \times S$ satisfying condition $\vec{k}$.

Corollary 5.3. *The restriction morphism*

$$\operatorname{Hom}(\mathfrak{F},\mathfrak{F}') \xrightarrow{R} \operatorname{Hom}(\mathfrak{F}|_{T_\infty},\mathfrak{F}'|_{T_\infty})$$

is injective.

Proof. Let $\mathscr{H} = \mathcal{H}om(\mathfrak{F},\mathfrak{F}')$. Since both $\mathfrak{F}$ and $\mathfrak{F}'$ are locally free along T_∞ one gets $\mathcal{T}or_i(\mathscr{H},\mathcal{O}_{T_\infty}) = 0$ for $i > 0$ (see for example [10, p. 700]). Thus, if we twist the structure sequence of the divisor T_∞ by $\mathscr{H}$, we get

$$0 \longrightarrow \mathscr{H}(-T_\infty) \longrightarrow \mathscr{H} \longrightarrow \mathscr{H}|_{T_\infty} \longrightarrow 0\,,$$

where $\mathscr{H}|_{T_\infty} \simeq \mathcal{H}om\,(\mathfrak{F}|_{T_\infty}, \mathfrak{F}'|_{T_\infty})$. It follows that

$$\ker R = H^0\left(\mathscr{H}(-T_\infty)\right) = H^0\left(t_{2*}\left(\mathscr{H}(-T_\infty)\right)\right).$$

By Propositions 4.3 and 2.3, the sheaf $\mathscr{H}(-T_\infty)$ is flat on S. Moreover Lemmas 2.4 and 5.2 yield the following vanishing result for all closed points $s \in S$:

$$H^0\left(\mathscr{H}(-T_\infty)_s\right) = H^0\left(\mathrm{Hom}\left(\mathfrak{F}_s, \mathfrak{F}'_s\right)(-\ell_\infty)\right) = 0\,.$$

The Semicontinuity Theorem entails the vanishing of the sheaf $t_{2*}\left(\mathscr{H}(-T_\infty)\right)$. This ends the proof. □

Corollary 5.4. *The action of $G_{\vec{k}}$ on $P_{\vec{k}}$ is free.*

Proof. Let $\left(\alpha, \beta; \bar{\psi}\right) \in L_{\vec{k}} \times G_{\vec{k}}$, and put $(\alpha', \beta') = \bar{\psi} \cdot (\alpha, \beta)$. It follows from Lemma 5.2 that a morphism $\Lambda \in \mathrm{Hom}\,(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'})$ is fully determined by its restriction Λ_∞ to ℓ_∞. It is not difficult to see [25, Lemma 4.1.3] that Λ is induced by a unique isomorphism $\bar{\psi} : M(\alpha, \beta) \longrightarrow M(\alpha', \beta')$ between the corresponding monads.

Whenever $\bar{\psi}$ lies in the stabilizer of a point $\theta \in P_{\vec{k}}$, one has $\Lambda_\infty\left(\alpha, \beta; \bar{\psi}\right) = \mathrm{id}_{\mathcal{E}_{\alpha,\beta,\infty}}$, where $(\alpha, \beta) = \tau(\theta)$. Since $\bar{\psi}$ is uniquely determined, this implies $\bar{\psi} = \mathrm{id}_{G_{\vec{k}}}$. □

Proposition 5.5. *The graph Γ of the action ρ is closed in $P_{\vec{k}} \times P_{\vec{k}}$.*

Proof. Let $x = \left(\theta_{\alpha,\beta}, \theta'_{\alpha',\beta'}\right)$ be a point in Γ; by $\theta_{\alpha,\beta}$ we mean that θ belongs to the fibre over (α, β). One has

$$\Lambda_\infty := \left(\theta'_{\alpha',\beta'}\right)^{-1} \circ \theta_{\alpha,\beta} \in \mathrm{Iso}\left(\mathcal{E}_{\alpha,\beta,\infty}, \mathcal{E}_{\alpha',\beta',\infty}\right)\,.$$

We define the vector space $\mathrm{Hom}_\Lambda\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right)$ as the fibre product

$$\begin{array}{ccc} \mathrm{Hom}_\Lambda\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right) & \xrightarrow{\quad i \quad} & \mathbb{C} \\ \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle \cdot\Lambda_\infty} \\ \mathrm{Hom}\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right) & \xrightarrow{R} & \mathrm{Hom}\left(\mathcal{E}_{\alpha,\beta,\infty}, \mathcal{E}_{\alpha',\beta',\infty}\right)\,, \end{array}$$

where R is restriction morphism to ℓ_∞, and $\cdot\Lambda_\infty$ is the multiplication by Λ_∞. Both morphisms i and j are injective, since R is injective by Lemma 5.2, while $\cdot\Lambda_\infty$ is injective by the invertibility of Λ_∞ [14, Lemma 1.2].

Thus, $\mathrm{Hom}_\Lambda\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right)$ is the subspace of homomorphisms between $\mathcal{E}_{\alpha,\beta}$ and $\mathcal{E}_{\alpha',\beta'}$ that at infinity reduce to multiples of Λ_∞. By Lemma [14, Lemma 1.1] one has the short exact sequence

$$0 \to \mathrm{Hom}_\Lambda\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right) \to \mathbb{C} \oplus \mathrm{Hom}\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right) \xrightarrow{(\cdot\Lambda_\infty, -(\cdot)|_{\ell_\infty})} \mathrm{Hom}\left(\mathcal{E}_{\alpha,\beta,\infty}, \mathcal{E}_{\alpha',\beta',\infty}\right) \to 0\,. \quad (5.1)$$

It is clear that

$$\Gamma = \left\{ \left(\theta_{\alpha,\beta}, \theta'_{\alpha',\beta'}\right) \in P_{\vec{k}} \times P_{\vec{k}} \,\middle|\, \dim \mathrm{Hom}_\Lambda\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right) = 1 \right\}\,. \quad (5.2)$$

Let us consider the following product varieties, along with the associated canonical projections:

$$\begin{array}{l} \mathfrak{X} := \Sigma_n \times P_{\vec{k}} \times P_{\vec{k}} \overset{q_{12}}{\underset{q_{13}}{\rightrightarrows}} \Sigma_n \times P_{\vec{k}} \\ \qquad \searrow^{q_{23}} \\ \qquad\qquad P_{\vec{k}} \times P_{\vec{k}}\,, \end{array}$$

$$\mathfrak{Y} := \ell_\infty \times P_{\vec{k}} \times P_{\vec{k}} \overset{p_{12}}{\underset{p_{13}}{\rightrightarrows}} \ell_\infty \times P_{\vec{k}}\,.$$

One can pull-back the family $\left(\widetilde{\mathfrak{E}}_{\vec{k}}, \widetilde{\Theta}_{\vec{k}}\right)$ to $\mathfrak{X}$ in two different ways, getting $\left(q_{1i}^*\widetilde{\mathfrak{E}}_{\vec{k}}, p_{1i}^*\widetilde{\Theta}_{\vec{k}}\right)$ with $i = 2, 3$. Out of these two pairs one defines

$$\mathscr{K} = \mathcal{H}om\left(q_{12}^*\widetilde{\mathfrak{E}}_{\vec{k}}, q_{13}^*\widetilde{\mathfrak{E}}_{\vec{k}}\right)$$

$$\Omega = \left(p_{13}^*\widetilde{\Theta}_{\vec{k}}\right)^{-1} \circ \left(p_{12}^*\widetilde{\Theta}_{\vec{k}}\right) \in \mathrm{Iso}\left(p_{12}^*\widetilde{\mathfrak{E}}_{\vec{k},\infty}, p_{13}^*\widetilde{\mathfrak{E}}_{\vec{k},\infty}\right)\,.$$

Since $q_{1i}^*\widetilde{\mathfrak{E}}_{\vec{k}}$ for $i = 2, 3$ are locally free along $\mathfrak{Y}$, there is an isomorphism

$$\mathscr{K}_\infty := \mathcal{H}om\left(p_{12}^*\widetilde{\mathfrak{E}}_{\vec{k},\infty}, p_{13}^*\widetilde{\mathfrak{E}}_{\vec{k},\infty}\right) \simeq \mathscr{K}|_{\mathfrak{Y}}\,.$$

We introduce the sheaf $\mathscr{K}_\Omega$ by means of the exact sequence

$$0 \longrightarrow \mathscr{K}_\Omega \longrightarrow \mathcal{O}_{\mathfrak{Y}} \oplus \mathscr{K} \xrightarrow{\left(\cdot\Omega, -(\cdot)|_{\mathfrak{Y}}\right)} \mathscr{K}_\infty \longrightarrow 0\,. \quad (5.3)$$

By Lemma 2.4, for any point $x = \left(\theta_{\alpha,\beta}, \theta'_{\alpha',\beta'}\right) \in P_{\vec{k}} \times P_{\vec{k}}$ the restriction of this sequence to the fibre of q_{23} over x is isomorphic to the sequence (5.1). In particular one gets the isomorphism

$$H^0\left(\mathscr{K}_{\Omega,x}\right) \simeq \mathrm{Hom}_\Lambda\left(\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}\right)\,. \quad (5.4)$$

Since, by Proposition 2.3, the sheaf $\mathscr{K}$ is flat on $P_{\vec{k}} \times P_{\vec{k}}$, the sheaf $\mathscr{K}_\Omega$ is flat on $P_{\vec{k}} \times P_{\vec{k}}$ as well [12, Prop 9.1A.(e)]. eq. (5.4) and the Semicontinuity Theorem ensure that Γ, as characterized in (5.2), is closed. $\square$

The smooth algebraic varieties $P_{\vec{k}}$ and $G_{\vec{k}}$ have unique compatible structures of complex manifolds $P_{\vec{k}}^{an}$ and $G_{\vec{k}}^{an}$. Note that Γ is closed in $P_{\vec{k}}^{an} \times P_{\vec{k}}^{an}$ as well.

Corollary 5.6. *The action of $G_{\vec{k}}^{an}$ on $P_{\vec{k}}^{an}$ is locally proper.*

Proof. Let K_{θ_0} be a compact neighbourhood of a point $\theta_0 \in P_{\vec{k}}^{an}$. We consider the morphism

$$\begin{array}{rccc} \gamma_{\theta_0}: & K_{\theta_0} \times G_{\vec{k}}^{an} & \longrightarrow & P_{\vec{k}}^{an} \times P_{\vec{k}}^{an} \\ & \left(\theta; \bar{\psi}\right) & \longmapsto & \left(\theta, \bar{\psi} \cdot \theta\right) . \end{array}$$

Since the action of $G_{\vec{k}}^{an}$ is free, γ_{θ_0} is injective, so that its image is

$$\operatorname{im} \gamma_{\theta_0} = \Gamma \cap \left(K_{\theta_0} \times P_{\vec{k}}^{an}\right) . \tag{5.5}$$

We have to prove that, for any compact subset $K \subset P_{\vec{k}}$, the counterimage $\left(\rho|_{K_{\theta_0} \times G_{\vec{k}}^{an}}\right)^{-1}(K)$ is compact. But it is easy to see that

$$\left(\rho|_{K_{\theta_0} \times G_{\vec{k}}^{an}}\right)^{-1}(K) = \gamma_{\theta_0}^{-1}\left(\Gamma \cap \left(K_{\theta_0} \times K\right)\right) .$$

As Γ is closed by Proposition 5.5, the thesis follows. □

We recall that an algebraic group G is said to be *special* if every locally isotrivial G-principal bundle is locally trivial [26] (a fibration is said to be isotrivial if it is trivial in the étale topology).

Lemma 5.7. *The group $G_{\vec{k}}$ is special.*

Proof. For any two positive integers p, q, let $H_{p,q}$ the subgroup of $\mathrm{GL}(p+q, \mathbb{C})$ whose elements are the matrices

$$\begin{pmatrix} \mathbf{1}_q & A \\ 0 & \mathbf{1}_p \end{pmatrix} , \quad \text{where } A \in \operatorname{Hom}\left(\mathbb{C}^p, \mathbb{C}^q\right) .$$

This group is isomorphic to the direct product of copies of the additive group $\mathbb{C}$, and therefore it is special [11, Prop 1]. We have

$$G_{\vec{k}} \simeq \mathrm{GL}(k_1, \mathbb{C}) \times \operatorname{Aut}\left(\mathcal{V}_{\vec{k}}\right) \times \mathrm{GL}(k_3, \mathbb{C}) ,$$

where $\operatorname{Aut}\left(\mathcal{V}_{\vec{k}}\right)$ can be embedded as a closed subgroup in $\mathrm{GL}(nk_2 + k_4, \mathbb{C})$. Moreover H_{k_4, nk_2} is a normal subgroup of $\operatorname{Aut}\left(\mathcal{V}_{\vec{k}}\right)$, and we get the short exact sequence of groups:

$$1 \longrightarrow H_{k_4,nk_2} \longrightarrow G_{\vec{k}} \longrightarrow \mathrm{GL}(k_1, \mathbb{C}) \times \mathrm{GL}(k_2, \mathbb{C}) \times \mathrm{GL}(k_4, \mathbb{C}) \times \mathrm{GL}(k_3, \mathbb{C}) \longrightarrow 1 .$$

Since the group $\mathrm{GL}(p, \mathbb{C})$ is special for any p [26, Thm 2], it turns out that $G_{\vec{k}}$ is special as well [26, Lemma 6]. □

We have now all ingredients to prove Theorem 5.1.

Proof of Theorem 5.1. Since $\mathcal{M}^n(r,a,c)$ is defined as a quotient set, the canonical projection π induces both the quotient topology, which makes $\mathcal{M}^n(r,a,c)$ into a noetherian topological space, and a canonical structure of locally ringed space (see for example [27]).

Let $\mathcal{M}^n(r,a,c)^{an} := P_{\vec{k}}^{an}/G_{\vec{k}}^{an}$ and let $\pi^{an} : P_{\vec{k}}^{an} \longrightarrow \mathcal{M}^n(r,a,c)^{an}$ be the projection. Since the action of $G_{\vec{k}}^{an}$ on $P_{\vec{k}}^{an}$ is free and locally proper, [15, Satz 24] implies that $\mathcal{M}^n(r,a,c)^{an}$ with its natural structure of locally ringed space is a complex manifold.

We have a commutative diagram of locally ringed spaces:

$$\begin{array}{ccc} P_{\vec{k}}^{an} & \longrightarrow & P_{\vec{k}} \\ \downarrow{\scriptstyle \pi^{an}} & & \downarrow{\scriptstyle \pi} \\ \mathcal{M}^n(r,a,c)^{an} & \longrightarrow & \mathcal{M}^n(r,a,c)\,. \end{array}$$

It follows plainly that $\mathcal{M}^n(r,a,c)$ is an algebraic variety, and is smooth since $\mathcal{M}^n(r,a,c)^{an}$ is (see [28, p. 109]).

Moreover, the morphism $P_{\vec{k}} \times G_{\vec{k}} \longrightarrow P_{\vec{k}} \times P_{\vec{k}}$ defined by the action ρ is a closed immersion, and $\mathcal{M}^n(r,a,c)$ is a geometric quotient of $P_{\vec{k}}$ modulo $G_{\vec{k}}$. It follows from [21, Prop 0.9] that $P_{\vec{k}}$ is a principal $G_{\vec{k}}$-bundle over $\mathcal{M}^n(r,a,c)$, in particular it is locally isotrivial. Lemma 5.7 says that $P_{\vec{k}}$ is actually locally trivial, and this completes the proof. □

We can now embark on proving the Main Theorem.

Proof of the Main Theorem, first part. From Corollary 4.6, if $k_1 < 0$ the set $\mathcal{M}^n(r,a,c)$ is empty. *Vice versa*, let $\vec{k} = (n,r,a,c)$ be such that $k_1 \geq 0$, and define the sheaf $\mathcal{E}_{\vec{k}}$ as follows:

$$\mathcal{E}_{\vec{k}} = \begin{cases} \mathcal{I}_{c\cdot x} & \text{if } r = 1 \quad (\Rightarrow a = 0) \\ \mathcal{I}_{c\cdot x} \oplus \mathcal{O}_{\Sigma_n}(aE) & \text{if } r = 2 \\ \mathcal{I}_{c\cdot x} \oplus \mathcal{O}_{\Sigma_n}(aE) \oplus \mathcal{O}_{\Sigma_n}^{\oplus(r-2)} & \text{if } r > 2 \end{cases}$$

where $\mathcal{I}_{c\cdot x}$ is the ideal sheaf of a 0-dimensional scheme of length c concentrated at a point $x \notin \ell_\infty$. The Chern character of $\mathcal{E}_{\vec{k}}$ is $(r, aE, -c - \frac{1}{2}na^2)$. It follows that $\mathcal{M}^n(r,a,c)$ is empty if and only if $k_1 < 0$.

By Theorem 5.1 $\mathcal{M}^n(r,a,c)$ is a smooth algebraic variety, and its dimension can be computed from the dimensions of $L_{\vec{k}}$, $\mathrm{GL}(r,\mathbb{C})$ and $G_{\vec{k}}$. □

6. The universal family

In this section we prove that the moduli space $\mathcal{M}^n(r,a,c)$ is fine by constructing a universal family of framed sheaves on Σ_n. Let us define the varieties $\Xi = \Sigma_n \times \mathcal{M}^n(r,a,c)$ and $\Xi_\infty = \ell_\infty \times \mathcal{M}^n(r,a,c)$, together with the canonical projections shown in the following diagram:

$$\begin{array}{ccc} \ell_\infty & \hookrightarrow & \Sigma_n \\ \uparrow{\scriptstyle \mathfrak{u}_1} & & \uparrow{\scriptstyle \mathfrak{t}_1} \\ \Xi_\infty & \hookrightarrow & \Xi \\ & {\scriptstyle \mathfrak{u}_2}\searrow & \downarrow{\scriptstyle \mathfrak{t}_2} \\ & & \mathcal{M}^n(r,a,c)\,. \end{array}$$

We define the projections

$$\begin{aligned} \boldsymbol{q} = \mathrm{id}_{\Sigma_n} \times \pi : \quad & \Sigma_n \times P_{\vec{k}} \longrightarrow \Xi\,; \\ \boldsymbol{p} = \mathrm{id}_{\ell_\infty} \times \pi : \quad & \ell_\infty \times P_{\vec{k}} \longrightarrow \Xi_\infty\,. \end{aligned}$$

where $\pi : P_{\vec{k}} \longrightarrow \mathcal{M}^n(r,a,c)$ is the quotient morphism. We define the sheaf

$$\mathfrak{E}_{\vec{k}} = \left(\boldsymbol{q}_* \widetilde{\mathfrak{E}}_{\vec{k}}\right)^G$$

where $()^G$ denotes taking invariants with respect to the action of $G_{\vec{k}}$ on $P_{\vec{k}}$.

Proposition 6.1. *$\mathfrak{E}_{\vec{k}}$ is a rank r coherent sheaf, satisfying condition $\vec{k}$. Actually, for any point $[\theta] \in \mathcal{M}^n(r,a,c)$ with $\tau(\theta) = (\alpha,\beta)$ one has the isomorphism*

$$\left(\mathfrak{E}_{\vec{k}}\right)_{[\theta]} \simeq \mathcal{E}_{\alpha,\beta}\,. \tag{6.1}$$

Furthermore, by considering the restriction at infinity $\mathfrak{E}_{\vec{k},\infty} := \mathfrak{E}_{\vec{k}}\big|_{\Xi_\infty}$ we get

$$\mathfrak{E}_{\vec{k},\infty} \simeq \left(\boldsymbol{p}_* \widetilde{\mathfrak{E}}_{\vec{k},\infty}\right)^G\,. \tag{6.2}$$

We need to prove a few preliminary results. First, we take the monad $\widetilde{\mathbb{M}}_{\vec{k}} = (\mathrm{id}_{\Sigma_n} \times \tau)^* \breve{\mathbb{M}}_{\vec{k}}$ as in Proposition 4.11, and we let

$$\widetilde{\mathbb{M}}_{\vec{k}} = \qquad 0 \longrightarrow \widetilde{\mathfrak{U}}_{\vec{k}} \xrightarrow{\widetilde{A}_{\vec{k}}} \widetilde{\mathfrak{V}}_{\vec{k}} \xrightarrow{\widetilde{B}_{\vec{k}}} \widetilde{\mathfrak{W}}_{\vec{k}} \longrightarrow 0\,.$$

Analogously, we let

$$\widetilde{\mathbb{M}}_{\vec{k},\infty} = \qquad 0 \longrightarrow \widetilde{\mathfrak{U}}_{\vec{k},\infty} \xrightarrow{\widetilde{A}_{\vec{k},\infty}} \widetilde{\mathfrak{V}}_{\vec{k}} \xrightarrow{\widetilde{B}_{\vec{k},\infty}} \widetilde{\mathfrak{W}}_{\vec{k},\infty} \longrightarrow 0\,.$$

We introduce the subsheaves

$$\mathfrak{U}_{\vec{k}} = \left(\boldsymbol{q}_*\widetilde{\mathfrak{U}}_{\vec{k}}\right)^G ; \qquad \mathfrak{V}_{\vec{k}} = \left(\boldsymbol{q}_*\widetilde{\mathfrak{V}}_{\vec{k}}\right)^G ; \qquad \mathfrak{W}_{\vec{k}} = \left(\boldsymbol{q}_*\widetilde{\mathfrak{W}}_{\vec{k}}\right)^G ;$$
$$\mathfrak{U}_{\vec{k},\infty} = \left(\boldsymbol{p}_*\widetilde{\mathfrak{U}}_{\vec{k},\infty}\right)^G ; \qquad \mathfrak{V}_{\vec{k},\infty} = \left(\boldsymbol{p}_*\widetilde{\mathfrak{V}}_{\vec{k},\infty}\right)^G ; \qquad \mathfrak{W}_{\vec{k},\infty} = \left(\boldsymbol{p}_*\widetilde{\mathfrak{W}}_{\vec{k},\infty}\right)^G ,$$

Lemma 6.2. *One has isomorphisms*

$$\mathfrak{U}_{\vec{k}} \simeq \mathfrak{t}_1^*\mathcal{U}_{\vec{k}} ; \qquad \mathfrak{V}_{\vec{k}} \simeq \mathfrak{t}_1^*\mathcal{V}_{\vec{k}} ; \qquad \mathfrak{W}_{\vec{k}} \simeq \mathfrak{t}_1^*\mathcal{W}_{\vec{k}} ;$$
$$\mathfrak{U}_{\vec{k},\infty} \simeq \mathfrak{u}_1^*\mathcal{U}_{\vec{k},\infty} ; \qquad \mathfrak{V}_{\vec{k},\infty} \simeq \mathfrak{u}_1^*\mathcal{V}_{\vec{k},\infty} ; \qquad \mathfrak{W}_{\vec{k},\infty} \simeq \mathfrak{u}_1^*\mathcal{W}_{\vec{k},\infty} .$$

Thus, the sheaves $\mathfrak{U}_{\vec{k}}$, $\mathfrak{V}_{\vec{k}}$ and $\mathfrak{W}_{\vec{k}}$ are locally free of rank k_1, $k_2 + k_4$ and k_3, respectively. Furthermore, there is an isomorphism $\boldsymbol{q}^*\mathfrak{U}_{\vec{k}} \simeq \widetilde{\mathfrak{U}}_{\vec{k}}$, and similarly for the other sheaves.

Proof. It is a straigthforward consequence of the projection formula. □

Remark 6.3. *Since the restriction morphism $\cdot|_{\ell_\infty \times P_{\vec{k}}}$ is $G_{\vec{k}}$-equivariant, the triples of sheaves $\left(\mathfrak{U}_{\vec{k}}, \mathfrak{V}_{\vec{k}}, \mathfrak{W}_{\vec{k}}\right)$ on Ξ and $\left(\mathfrak{U}_{\vec{k},\infty}, \mathfrak{V}_{\vec{k},\infty}, \mathfrak{W}_{\vec{k},\infty}\right)$ on Ξ_∞ are related by restriction at infinity, that is, $\mathfrak{U}_{\vec{k},\infty} \simeq \mathfrak{U}_{\vec{k}}\big|_{\Xi_\infty}$, $\mathfrak{V}_{\vec{k},\infty} \simeq \mathfrak{V}_{\vec{k}}\big|_{\Xi_\infty}$ and $\mathfrak{W}_{\vec{k},\infty} \simeq \mathfrak{W}_{\vec{k}}\big|_{\Xi_\infty}$.*

We introduce now the "universal" monad.

Proposition 6.4. *One has the following commutative diagram of monads on Ξ:*

$$\begin{array}{llcccccccc} \mathbb{M}_{\vec{k}} : & 0 & \longrightarrow & \mathfrak{U}_{\vec{k}} & \xrightarrow{A_{\vec{k}}} & \mathfrak{V}_{\vec{k}} & \xrightarrow{B_{\vec{k}}} & \mathfrak{W}_{\vec{k}} & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle \cdot|_{\Xi_\infty}} & & \downarrow{\scriptstyle \cdot|_{\Xi_\infty}} & & \downarrow{\scriptstyle \cdot|_{\Xi_\infty}} & & \\ \mathbb{M}_{\vec{k},\infty} : & 0 & \longrightarrow & \mathfrak{U}_{\vec{k},\infty} & \xrightarrow{A_{\vec{k},\infty}} & \mathfrak{V}_{\vec{k},\infty} & \xrightarrow{B_{\vec{k},\infty}} & \mathfrak{W}_{\vec{k},\infty} & \longrightarrow & 0 \end{array} \tag{6.3}$$

where

$$\begin{cases} A_{\vec{k}} := \left(\boldsymbol{q}_*\widetilde{A}_{\vec{k}}\right)\Big|_{\mathfrak{U}_{\vec{k}}} \\ B_{\vec{k}} := \left(\boldsymbol{q}_*\widetilde{B}_{\vec{k}}\right)\Big|_{\mathfrak{V}_{\vec{k}}} \end{cases} \quad \text{and} \quad \begin{cases} A_{\vec{k},\infty} := \left(\boldsymbol{p}_*\widetilde{A}_{\vec{k},\infty}\right)\Big|_{\mathfrak{U}_{\vec{k},\infty}} \\ B_{\vec{k},\infty} := \left(\boldsymbol{p}_*\widetilde{B}_{\vec{k},\infty}\right)\Big|_{\mathfrak{V}_{\vec{k},\infty}} \end{cases} .$$

The sheaves $\mathfrak{E}_{\vec{k}}$ and $\mathfrak{E}_{\vec{k},\infty}$ are isomorphic to the cohomologies of the monads $\mathbb{M}_{\vec{k}}$ and $\mathbb{M}_{\vec{k},\infty}$, respectively.

Proof. The morphisms of $\mathbb{M}_{\vec{k}}$ are well defined due to by the $G_{\vec{k}}$-equivariance of $\widetilde{A}_{\vec{k}}$ and $\widetilde{B}_{\vec{k}}$. The condition $B_{\vec{k}} \circ A_{\vec{k}} = 0$ follows from the functoriality of $\boldsymbol{q}_*$. The injectivity of $A_{\vec{k}}$ is apparent since $A_{\vec{k}}$ is the restriction of the injective morphism $\boldsymbol{q}_*\widetilde{A}_{\vec{k}}$. Lemma 6.2 implies $\boldsymbol{q}^*B_{\vec{k}} \simeq \widetilde{B}_{\vec{k}}$ and $\boldsymbol{q}^*\left(\operatorname{coker} B_{\vec{k}}\right) \simeq \operatorname{coker} \widetilde{B}_{\vec{k}} = 0$; the vanishing of $\operatorname{coker} B_{\vec{k}}$ follows from the faithful flatness of $\boldsymbol{q}$.

The proof for $\mathbb{M}_{\vec{k},\infty}$ is analogous, and the commutativity of the diagram is an easy consequence of Remark 6.3.

The last statement follows from Proposition 4.11. □

Proof of Proposition 6.1. The coherence of $\mathfrak{E}_{\vec{k}}$ follows from $\mathfrak{E}_{\vec{k}} \simeq \mathcal{E}\left(\mathbb{M}_{\vec{k}}\right)$, and its rank can be computed easily from the ranks of $\mathfrak{U}_{\vec{k}}$, $\mathfrak{V}_{\vec{k}}$ and $\mathfrak{W}_{\vec{k}}$ given by Lemma 6.2. From Lemma 6.2 and Proposition 6.4 we get isomorphisms $\boldsymbol{q}^*\mathbb{M}_{\vec{k}} \simeq \widetilde{\mathbb{M}}_{\vec{k}}$, $\boldsymbol{q}^*\mathfrak{E}_{\vec{k}} \simeq \widetilde{\mathfrak{E}}_{\vec{k}}$ and $\boldsymbol{q}^*A_{\vec{k}} \simeq \widetilde{A}_{\vec{k}}$. Note that $\left(A_{\vec{k}}\right)_{[\theta]} = \alpha$ for all points $[\theta] \in \mathcal{M}^n(r,a,c)$, with $\tau(\theta) = (\alpha,\beta)$. Now 2.2 implies the flatness of $\mathfrak{E}_{\vec{k}}$, and eq. (6.1) follows from eq. (4.8). This is enough to show that $\mathfrak{E}_{\vec{k}}$ satisfies condition $\vec{k}$.

Finally, eq. (6.2) is a consequence of the commutativity of the diagram (6.3). □

The morphism $\widetilde{\Theta}_{\vec{k}}$ (cf. subsection 4.3) provides a framing for the sheaf $\mathfrak{E}_{\vec{k}}$. Note that this morphism is $G_{\vec{k}}$-equivariant.

Definition 6.5. *We define the isomorphism $\Theta_{\vec{k}}$ as the restriction of $\boldsymbol{p}_*\widetilde{\Theta}_{\vec{k}}$ to the $G_{\vec{k}}$-invariant subsheaf $\mathfrak{E}_{\vec{k},\infty}$:*

$$\Theta_{\vec{k}} := \left(\boldsymbol{p}_*\widetilde{\Theta}_{\vec{k}}\right)\Big|_{\mathfrak{E}_{\vec{k},\infty}} : \mathfrak{E}_{\vec{k},\infty} \xrightarrow{\sim} \mathcal{O}_{\Xi_\infty}^{\oplus r} .$$

We shall call $\Theta_{\vec{k}}$ the universal framing.

We show how to canonically associate a scheme morphism $f_{[(\mathfrak{F},\Theta)]} : S \longrightarrow \mathcal{M}^n(r,a,c)$ with an isomorphism class $[(\mathfrak{F},\Theta)]$ of families of framed sheaves on $T = \Sigma_n \times S$. We begin by describing some properties of the monad $M(\mathfrak{F})$.

Lemma 6.6. *Let $\mathfrak{F}$ be a sheaf on T which satisfies condition $\vec{k}$, and let $M(\mathfrak{F})$ be the canonically associated monad.*

- *For any closed point $s \in S$ there is an isomorphism of complexes*
$$M(\mathfrak{F})_s \simeq M(\alpha(s),\beta(s)) , \tag{6.4}$$
where $(\alpha(s),\beta(s)) = (A_s,B_s) \in L_{\vec{k}}$.
- *The restriction $M(\mathfrak{F})_\infty$ of the monad $M(\mathfrak{F})$ to T_∞ is a monad, whose cohomology is isomorphic to $\mathfrak{F}|_{T_\infty}$. For any closed point $s \in S$ there is an isomorphism*

$$M(\mathfrak{F})_{\infty,s} \simeq \quad 0 \longrightarrow \mathcal{U}_{\vec{k},\infty} \xrightarrow{\alpha(s)|_{\ell_\infty}} \mathcal{V}_{\vec{k},\infty} \xrightarrow{\beta(s)|_{\ell_\infty}} \mathcal{W}_{\vec{k},\infty} \longrightarrow 0 . \tag{6.5}$$

Proof. The proof splits in two steps.

- Let $s \in S$ be any closed point. By Corollary 4.5, there is an isomorphism

$$M(\mathfrak{F})_s \simeq \mathcal{U}_{\vec{k}} \xrightarrow{A_s} \mathcal{V}_{\vec{k}} \xrightarrow{B_s} \mathcal{W}_{\vec{k}} .$$

It is enough to show that A_s is injective. This is a consequence of the application of [18, Lemma 2.1.4] to the short exact sequence.

$$0 \longrightarrow \mathfrak{U} \xrightarrow{A} \ker B \longrightarrow \mathfrak{F} \longrightarrow 0 .$$

Hence $M(\mathfrak{F})_s$ is a monad, whose cohomology is isomorphic to $\mathfrak{F}_s$; the latter sheaf is torsion-free and trivial at infinity. Proposition 4.8 implies $(A_s, B_s) \in L_{\vec{k}}$.

- eq. (6.5) follows from eq. (6.4), since the condition $(\alpha(s), \beta(s)) \in L_{\vec{k}}$ implies that $\alpha(s)|_{\ell_\infty}$ is injective. By Lemma 2.1 this condition ensures that A_∞ is injective.

□

Remark 6.7. *Suppose that $(\mathfrak{F}, \Theta)$ is a family of framed sheaves on T. For any closed point $s \in S$, let $\theta(s) = \Theta_s$ be the restricted framing. One has*

$$\theta(s) \in \operatorname{Iso}\left(\mathcal{E}_{\alpha(s),\beta(s),\infty}, \mathcal{O}_{\mathbb{P}^1}^{\oplus r}\right) .$$

We proceed with the construction of the morphism $f_{[(\mathfrak{F},\Theta)]} : S \longrightarrow \mathcal{M}^n(r, a, c)$ by defining it first on closed points. We choose an open affine cover $S = \bigcup_{a\in\mathscr{A}} S_a = \bigcup_{a\in\mathscr{A}} \operatorname{Spec} \mathscr{S}_a$ where the S_a's satisfy the conditions of Corollary 4.5. Moreover, if (A, B) is the pair of morphisms in the monad $M(\mathfrak{F})$, we introduce the following notation:

$$(A_a, B_a, \Theta_a) := \left(A|_{\Sigma_n\times S_a}, B|_{\Sigma_n\times S_a}, \Theta|_{\ell_\infty\times S_a}\right) .$$

Recall that $t_2 : T \longrightarrow S$ is the projection. Note that by applying the functor t_{2*} to the monad $M(\mathfrak{F})$ restricted to $\Sigma_n \times S_a$ we obtain a complex of trivial sheaves on S_a, so that

$$(t_{2*}A_a, t_{2*}B_a) \in \mathbb{V}_{\vec{k}} \otimes \mathscr{S}_a \simeq \mathscr{S}_a^{\oplus d} .$$

If we define $(\alpha_a(s), \beta_a(s)) = (t_{2*}A_a, t_{2*}B_a) \otimes_{\mathscr{S}_a} k(s)$ we obtain the same morphisms as in Lemma 6.6. We define a morphism $\bar{f}_a : S_a \longrightarrow L_{\vec{k}}$ by letting $\bar{f}_a(s) = (\alpha_a(s), \beta_a(s))$. We complete this to a scheme morphism by defining the rings homomorphism

$$\bar{f}_a^\sharp : \mathbb{C}[z_1, \ldots, z_d] \longrightarrow \mathscr{S}_a$$

which maps the polynomial g to $\tilde{g}\,(A_a, B_a)$, where $\tilde{g}$ is the natural extension of g to the ring $\mathscr{S}_a[z_1, \ldots, z_d]$.

In the same way, we define $\theta_a(s) = u_{2*}\Theta \otimes_{\mathscr{S}_a} k(s)$. This allows us to lift the morphisms $\bar{f}_a$ to morphisms $\tilde{f}_a : S_a \longrightarrow P_{\vec{k}}$ which we define on closed points as $\bar{f}_a(s) = \theta_a(s)$. Again, these extend to scheme morphisms. By composing these morphisms with the projection

$\pi : P_{\vec{k}} \longrightarrow \mathcal{M}^n(r,a,c)$ we obtain morphisms $f_a : S_a \longrightarrow \mathcal{M}^n(r,a,c)$, which glue to a morphism $f_{[(\mathfrak{F},\Theta)]} : S \longrightarrow \mathcal{M}^n(r,a,c)$, since on overlaps the different $\tilde{f}_a$'s differ by the action of $G_{\vec{k}}$.

6.1. **Fineness.** In this section we prove that $\mathcal{M}^n(r,a,c)$ represents a moduli functor, i.e., it is a fine moduli space. Let $\mathfrak{Sch}$ the category of noetherian reduced schemes of finite type over $\mathbb{C}$ and $\mathfrak{Sets}$ the category of sets.

Definition 6.8. *For any vector $\vec{k}$ such that $k_1 \geq 0$ we introduce the contravariant functor $\mathcal{M}od_{\vec{k}} : \mathfrak{Sch} \longrightarrow \mathfrak{Sets}$ by the following prescriptions:*

- *for any object $S \in \operatorname{Ob}(\mathfrak{Sch})$ we define the set $\mathcal{M}od_{\vec{k}}(S)$ as*

$$\mathcal{M}od_{\vec{k}}(S) := \frac{\left\{ \begin{array}{l} \textit{families } (\mathfrak{F},\Theta) \\ \textit{of framed sheaves} \\ \textit{on } \Sigma_n \times S \end{array} \right\}}{\{\textit{framing-preserving isomorphisms}\}} ;$$

- *for any morphism $\varphi : S' \longrightarrow S$ we define the set-theoretic map*

$$\begin{array}{rccc} \mathcal{M}od_{\vec{k}}(\varphi) : & \mathcal{M}^n(r,a,c)(S) & \longrightarrow & \mathcal{M}od_{\vec{k}}(S') \\ & [(\mathfrak{F},\Theta)] & \longmapsto & [(\varphi_\Sigma^*\mathfrak{F}, \varphi_\infty^*\Theta)] , \end{array}$$

  where $\varphi_\Sigma = \mathrm{id}_{\Sigma_n} \times \varphi$ and $\varphi_\infty = \mathrm{id}_{\ell_\infty} \times \varphi$.

Observe that $\mathcal{M}od_{\vec{k}}(\operatorname{Spec}\mathbb{C})$ is the set underlying $\mathcal{M}^n(r,a,c)$. The key property of this functor is the following.

Proposition 6.9. *The functor $\mathcal{M}od_{\vec{k}}(-)$ is represented by the scheme $\mathcal{M}^n(r,a,c)$, that is there is a natural isomorphism of functors*

$$\mathcal{M}od_{\vec{k}}(-) \simeq \operatorname{Hom}(-, \mathcal{M}^n(r,a,c)) .$$

This implies that $\mathcal{M}^n(r,a,c)$ is a fine moduli space of framed sheaves on Σ_n. The pair $\left(\mathfrak{E}_{\vec{k}}, \Theta_{\vec{k}}\right)$ on $\Sigma_n \times \mathcal{M}^n(r,a,c)$, is the universal family of framed sheaves on Σ_n.

We divide the proof of this Proposition in three Lemmas.

Lemma 6.10. *Let S be any scheme.*

- *We define the map $\boldsymbol{\mu}_S$ as*

$$\begin{array}{rccc} \boldsymbol{\mu}_S : & \mathcal{M}od_{\vec{k}}(S) & \longrightarrow & \operatorname{Hom}(S, \mathcal{M}^n(r,a,c)) \\ & [(\mathfrak{F},\Theta)] & \longmapsto & f_{[(\mathfrak{F},\Theta)]} . \end{array}$$

- *We define the map* $\boldsymbol{\nu}_S$ *as*

$$\begin{array}{rccl} \boldsymbol{\nu}_S: & \mathrm{Hom}\,(S,\mathcal{M}^n(r,a,c)) & \longrightarrow & \mathcal{M}od_{\vec{k}}(S) \\ & f & \longmapsto & \left[\left(f_\Sigma^*\mathfrak{E}_{\vec{k}}, f_\infty^*\Theta_{\vec{k}}\right)\right] . \end{array}$$

In this way we get natural tranformations $\boldsymbol{\mu}: \mathcal{M}od_{\vec{k}}(-) \longrightarrow \mathrm{Hom}\,(-,\mathcal{M}^n(r,a,c))$ *and* $\boldsymbol{\nu}: \mathrm{Hom}\,(-,\mathcal{M}^n(r,a,c)) \longrightarrow \mathcal{M}od_{\vec{k}}(-)$.

Proof. The naturality of $\boldsymbol{\nu}$ is straightforward since

$$\left[\left(\varphi_\Sigma^* f_\Sigma^*\mathfrak{E}_{\vec{k}}, \varphi_\infty^* f_\infty^*\Theta_{\vec{k}}\right)\right] = \left[\left((f\circ\varphi)_\Sigma^*\,\mathfrak{E}_{\vec{k}}, (f\circ\varphi)_\infty^*\,\Theta_{\vec{k}}\right)\right] ,$$

whenever a composition of morphisms $S' \xrightarrow{\varphi} S \xrightarrow{f} \mathcal{M}^n(r,a,c)$ is given.

To prove the naturality of $\boldsymbol{\mu}$ we need to show that, given any morphism $S' \xrightarrow{\varphi} S$ and any family $(\mathfrak{F},\Theta)$ on T, the following equality holds true:

$$f_{[(\mathfrak{F},\Theta)]}\circ\varphi = f_{\left[\left(\varphi_\Sigma^*\mathfrak{F},\varphi_\infty^*\Theta\right)\right]} . \tag{6.6}$$

To simplify the notation we let $f = f_{\left[(\mathfrak{F},\Theta)_{\vec{k}}\right]}$ and $f' = f_{\left[\left(\varphi_\Sigma^*\mathfrak{F},\varphi_\infty^*\Theta\right)_{\vec{k}}\right]}$. We can assume $S = \mathrm{Spec}\,\mathscr{S}$ and $S' = \mathrm{Spec}\,\mathscr{S}'$, so that φ is induced by a ring homomorphism $\mathscr{S} \xrightarrow{\varphi^\sharp} \mathscr{S}'$. If (A,B) and (A',B') are the morphisms in the monads $M(\mathfrak{F})$ and $M(\varphi^*\mathfrak{F})$, respectively, one has

$$(A',B') = \left(\varphi^\sharp\right)^{\oplus d}(A,B) ;$$

in particular, for any polynomial $g\in\mathbb{C}[z_1,\dots,z_d]$

$$\begin{gathered} \mathrm{ev}_{(A',B')}\,\tilde{g} = \left(\varphi^\sharp\circ\mathrm{ev}_{(A,B)}\right)\tilde{g} , \\ \text{where} \qquad g\mapsto\tilde{g}\in\mathscr{S}'[z_1,\dots,z_d] \end{gathered}$$

so that $\bar{f}\circ\varphi = \bar{f}' : S' \longrightarrow L_{\vec{k}}$. We can assume that $\mathrm{im}\,\bar{f}'$ is contained in an open affine subset $V = \mathrm{Spec}\,\mathscr{L}$ of $L_{\vec{k}}$ which trivializes $\mathfrak{N}_{\vec{k}}$, and we put $N_{\vec{k}} = \Gamma\left(V,\mathfrak{N}_{\vec{k}}\right)$ which turns out to be a free $\mathscr{L}$-module of rank r.

We put $N = \Gamma\left(T_\infty, \mathfrak{F}|_{T_\infty}\right)$ which is a free $\mathscr{S}$-module of rank r, and similarly we put $N' = \Gamma\left(T'_\infty, \varphi_\infty^*\left(\mathfrak{F}|_{T_\infty}\right)\right)$ which is a free $\mathscr{S}'$-module of the same rank; By the homomorphisms $f^\sharp$ and $\phi^\sharp$ one has $N \simeq N_{\vec{k}}\otimes_{\mathscr{L}}\mathscr{S}$ and $N' \simeq N\otimes_{\mathscr{S}}\mathscr{S}'$. Moreover one has $\varphi_\infty^*\Theta \simeq \Theta\otimes_{\mathscr{S}} 1_{\mathscr{S}'}$.

For any polynomial $h\in\mathrm{Sym}_{\mathscr{L}}(H^*)$, where $H\simeq\mathrm{Hom}_{\mathscr{L}}\left(N_{\vec{k}},\mathscr{L}^{\oplus r}\right)$, we get

$$\begin{aligned} \mathrm{ev}_{\varphi_\infty^*\Theta}\left(h\otimes_{\mathscr{L}} 1_{\mathscr{S}'}\right) &= \mathrm{ev}_{\left(\Theta\otimes_{\mathscr{S}}1_{\mathscr{S}'}\right)}\left[\left(h\otimes_{\mathscr{L}} 1_{\mathscr{S}}\right)\otimes_{\mathscr{S}} 1_{\mathscr{S}'}\right] = \\ &= \left[\mathrm{ev}_\Theta\left(h\otimes_{\mathscr{L}} 1_{\mathscr{S}}\right)\right]\otimes_{\mathscr{S}} 1_{\mathscr{S}'} . \end{aligned}$$

Hence

$$\tilde{f}\circ\varphi=\tilde{f}':S'\longrightarrow P_{\vec{k}}\,.$$

By applying the projection π to both sides of this equation, we obtain eq. (6.6). □

Lemma 6.11. *For any scheme S one has $\boldsymbol{\nu}_S\circ\boldsymbol{\mu}_S=\mathrm{id}_{\mathcal{M}od_{\vec{k}}(S)}$.*

Proof. We need to prove that

$$\left(f_\Sigma^*\mathfrak{E}_{\vec{k}},f_\infty^*\Theta_{\vec{k}}\right)\simeq(\mathfrak{F},\Theta)\ ,$$

for any family $(\mathfrak{F},\Theta)$ of framed sheaves on Σ_n parametrized by S, where $f=f_{[(\mathfrak{F},\Theta)]}$. It is enough to show that there is an isomorphism

$$f_\Sigma^*\mathbb{M}_{\vec{k}}\simeq M(\mathfrak{F}) \tag{6.7}$$

and that this isomorphism is compatible with the framings. By Lemma 6.2 there are isomorphisms $f_\Sigma^*\mathfrak{U}_{\vec{k}}\simeq t_1^*\mathcal{U}_{\vec{k}}$, $f_\Sigma^*\mathfrak{V}_{\vec{k}}\simeq t_1^*\mathcal{V}_{\vec{k}}$, $f_\Sigma^*\mathfrak{W}_{\vec{k}}\simeq t_1^*\mathcal{W}_{\vec{k}}$, toghether with $\left(\boldsymbol{q}^*A_{\vec{k}},\boldsymbol{q}^*B_{\vec{k}},\boldsymbol{p}^*\Theta_{\vec{k}}\right)\simeq\left(\widetilde{A}_{\vec{k}},\widetilde{B}_{\vec{k}},\widetilde{\Theta}_{\vec{k}}\right)$. When S is affine and satisfies the conditions of Corollary 4.5, we have in addition the isomorphisms $\mathfrak{U}\simeq t_1^*\mathcal{U}_{\vec{k}}$, $\mathfrak{V}\simeq t_1^*\mathcal{V}_{\vec{k}}$ and $\mathfrak{W}\simeq t_1^*\mathcal{W}_{\vec{k}}$, and we have

$$\begin{aligned}f_\Sigma^*\left(A_{\vec{k}},B_{\vec{k}}\right)&=\tilde{f}_\Sigma^*\boldsymbol{q}^*\left(\widetilde{A}_{\vec{k}},\widetilde{B}_{\vec{k}}\right)=\tilde{f}_\Sigma^*\left(\widetilde{A}_{\vec{k}},\widetilde{B}_{\vec{k}}\right)=f_\Sigma^*\mathfrak{t}_2^*\tau^*\left(\mathrm{id}_{L_{\vec{k}}}\right)=t_2^*\tilde{f}^*\tau^*\left(\mathrm{id}_{L_{\vec{k}}}\right)\\&=t_2^*\bar{f}^*\left(\mathrm{id}_{L_{\vec{k}}}\right)=(A,B)\,.\end{aligned}$$

where $\tilde{f}_\Sigma=\mathrm{id}_{\Sigma_n}\times\tilde{f}$. This proves eq. (6.7) locally, and similarly the compatibility of $\Theta_{\vec{k}}$ can be shown. By Corollary 5.3 we get the thesis. □

Lemma 6.12. *For any vector $\vec{k}$ such that $k_1\geq 0$ and for any scheme S one has*

$$\boldsymbol{\mu}_S\circ\boldsymbol{\nu}_S=\mathrm{id}_{\mathrm{Hom}(S,\mathcal{M}^n(r,a,c))}\ .$$

Proof. Let $g:S\longrightarrow\mathcal{M}^n(r,a,c)$ be any scheme morphism. We need to show that:

$$g=f_{\left[\left(g_\Sigma^*\mathfrak{E}_{\vec{k}},g_\infty^*\Theta_{\vec{k}}\right)\right]}\,;$$

for simplicity we set $f=f_{\left[\left(g_\Sigma^*\mathfrak{E}_{\vec{k}},g_\infty^*\Theta_{\vec{k}}\right)\right]}$. Let $M\left(g_\Sigma^*\mathfrak{E}_{\vec{k}}\right)$ be the monad

$$0\longrightarrow\mathfrak{U}\xrightarrow{\ A\ }\mathfrak{V}\xrightarrow{\ B\ }\mathfrak{W}\longrightarrow 0$$

canonically associated with $g_\Sigma^*\mathfrak{E}_{\vec{k}}$. We can work locally by assuming that $S=\operatorname{Spec}\mathscr{S}$ satisfies the hypotheses of Corollary 4.5 for the sheaves $\mathfrak{U}$, $\mathfrak{V}$, $\mathfrak{W}$ and that $\operatorname{im}g\subseteq W$, where $W \subseteq \mathcal{M}^n(r,a,c)$ is a trivializing open subset for the $G_{\vec{k}}$-principal bundle $P_{\vec{k}}$. Thus, there exists a local section $\sigma : W \longrightarrow P_{\vec{k}}$ lifting g to $P_{\vec{k}}$:

$$\begin{array}{ccc} & & P_{\vec{k}} \\ & \overset{\sigma\circ g =: \tilde{g}}{\nearrow} & \downarrow \pi \\ S & \xrightarrow[g]{} & \mathcal{M}^n(r,a,c)\,. \end{array} \tag{6.8}$$

Under our assumptions, the complex

$$g_\Sigma^* \mathbb{M}_{\vec{k}} : \qquad g_\Sigma^* \mathfrak{U}_{\vec{k}} \xrightarrow{g_\Sigma^* A_{\vec{k}}} g_\Sigma^* \mathfrak{V}_{\vec{k}} \xrightarrow{g_\Sigma^* B_{\vec{k}}} g_\Sigma^* \mathfrak{W}_{\vec{k}} \longrightarrow 0$$

is a monad. Indeed the morphism $g_\Sigma^* A_{\vec{k}}$ is injective, as it follows from diagram (6.8) and Lemma 2.1. This monad is isomorphic to $M\left(g_\Sigma^* \mathfrak{E}_{\vec{k}}\right)$: as a matter of fact, their cohomologies are isomorphic and [25, Lemma 4.1.3] applies. Because of this isomorphism, in view of Proposition 6.1 one has

$$\left(g_\Sigma^* \mathfrak{E}_{\vec{k}}\right)_s \simeq \mathcal{E}_{\alpha(s),\beta(s)},$$

where $g(s) = [\theta(s)]$ and $(\alpha(s), \beta(s)) = \tau(\theta(s))$ for any closed point $s \in S$. This ends the proof. □

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