Semiclassical states for the nonlinear Klein-Gordon-Maxwell system

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Abstract

In the following we consider the Klein-Gordon-Maxwell system with some positive potentials in $\mathbb{R}^3$. We establish the existence of single spikes concentrating around critical points of the potentials.
Also necessary conditions for the concentration are given.

1 Introduction and main results

Let us consider the nonlinear Klein-Gordon equation

$$\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + m_0^2\psi = \lambda|\psi|^{p-1}\psi \tag{1.1}$$

where $\psi = \psi(x,t)$ is a complex function defined on $\mathbb{R}^3 \times \mathbb{R}$, $p \in (1,5)$, $\lambda > 0$ is a parameter and $m_0 > 0$ is the mass of the particle whose states are described, at a given moment, by the wave function ψ.
In recent years many papers have been devoted to find standing waves of (1.1), i.e. solutions of the form

$$\psi(x,t) = u(x)e^{-i\omega t}, \qquad u(x),\, \omega \in \mathbb{R}.$$

In this case (1.1) becomes a semilinear equation.
In this paper we want to study the interaction of ψ with its own electromagnetic field $(\mathbf{E},\mathbf{H})$ which is described, as usual, by the gauge potential $(\phi,\mathbf{A})$

$$\phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, \qquad \mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3.$$

Indeed $(\phi,\mathbf{A})$ is linked with $(\mathbf{E},\mathbf{H})$ through the Maxwell equations

$$\mathbf{E} := -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{H} := \nabla \times \mathbf{A}.$$

Following the ideas of [3], in order to study standing waves interacting with a purely electrostatic field (this means $\mathbf{A} = 0$ and $\phi = \phi(x)$) one can reduce to the study of the following system of equations

$$\begin{cases} -\Delta v + m_0^2 v - (\phi-\omega)^2 v = \lambda|v|^{p-1}v, & x \in \mathbb{R}^3 \\ \Delta\phi = (\phi-\omega)v^2. & x \in \mathbb{R}^3 \end{cases} \tag{1.2}$$

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For such a system existence and non-existence results have been established (see [3], [4], [5] and references therein).
Let us consider the following generalized version of the system (1.2):

$$\begin{cases} -\Delta v + a(x) m_0^2 v - b(x)(\phi - \omega)^2 v = \lambda |v|^{p-1} v, & x \in \mathbb{R}^3 \\ \Delta \phi = b(x)(\phi - \omega) v^2, & x \in \mathbb{R}^3 \end{cases} \tag{1.3}$$

where $a, b \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that

(a1) $\inf_{\mathbb{R}^3} a(x) > 0$, a and its derivatives are bounded;

(b1) $b(x) \geq 0$, $b \not\equiv 0$, b and its derivatives are bounded.

As we will see in Section 3 the problem (1.3) can be reduced into a single equation. In fact the equation

$$\Delta \phi = b(x)(\phi - \omega) v^2 \tag{1.4}$$

has a unique solution $\phi_v \in D^{1,2}(\mathbb{R}^3)$. If we substitute ϕ_v into the first equation of (1.3) we have to study the equivalent problem

$$-\Delta v + a(x) m_0^2 v - b(x)(\phi_v(x) - \omega)^2 v = \lambda |v|^{p-1} v, \qquad x \in \mathbb{R}^3. \tag{1.5}$$

Let us define $\epsilon^2 := \frac{1}{m_0^2}$. Then the equation (1.5) becomes

$$-\Delta v + \frac{a(x)}{\epsilon^2} v - b(x)(\phi_v(x) - \omega)^2 v = \lambda |v|^{p-1} v, \qquad x \in \mathbb{R}^3. \tag{1.6}$$

If we consider the problem (1.6) with a mass m_0 at a macroscopic scale, the value of ϵ will be then very small. We will interested not only in the existence of solutions for (1.6) but also in their behavior as $\epsilon \to 0$.
To the best of our knowledge no papers consider the concentration phenomena for such a problem.
We first state the main results of the paper.

Theorem 1.1 *Let (a1)-(b1) hold. Furthermore we assume*

(a2) *$a(x)$ has a non-degenerate local minimum or maximum at $x_0 \in \mathbb{R}^3$, namely $\nabla a(x_0) = 0$ and $D^2 a(x_0)$ is positive- or negative-definite.*

Then for $\epsilon > 0$ small, the equation (1.6) has a solution v_ϵ which concentrates at x_0.

If we assume

(a2') $a(x)$ has a degenerate local minimum or maximum at $x_0 \in \mathbb{R}^3$, namely there exists a positive integer $m > 0$ such that $D^k a(x_0) = 0$ for all $k < m$ and $D^m a(x_0)$ is positive- or negative-definite and such that

$$D^m a(x_0)[x] = \sum_{i=1}^{3} a_i x_i^m$$

where $a_i = \dfrac{\partial^m a(x_0)}{\partial x_i^m}$, $i = 1, 2, 3$;

(b2) $b(x_0) = 0$ and $b(x)$ has a (possibly) degenerate local minimum or maximum at $x_0 \in \mathbb{R}^3$, namely there exists a positive integer $n > 0$ such that $D^k b(x_0) = 0$ for all $k < n$ and $D^n b(x_0)$ is positive- or negative-definite and such that

$$D^n b(x_0)[x] = \sum_{i=1}^{3} b_i x_i^n$$

where $b_i = \dfrac{\partial^n b(x_0)}{\partial x_i^n}$, $i = 1, 2, 3$;

then the following result holds.

Theorem 1.2 *Let (a1)-(a2')-(b1)-(b2) hold. Let* $s = \min\{m, n+2\}$ *and we assume* $s < +\infty$. *If* $s = m < n+2$ *or* $s = n+2 < m$, *then the problem (1.6) has a solution* v_ϵ *which concentrates at* $x_0 \in \mathbb{R}^3$. *Moreover, if* $s = n+2 = m$ *the same holds provided*

$$C_{2,i} a_i - \widetilde{C}_{2,i} b_i \neq 0, \tag{1.7}$$

where $C_{2,i}, \widetilde{C}_{2,i}$ *are positive constants explicitly known.*

Remark 1.2.1 We note that if x_0 is a local minimum (resp. local maximum) for a, or for b, then $a_i, b_i > 0$ (resp. $a_i, b_i < 0$). Hence the assumption (1.7) is automatically satisfied if x_0 is a local maximum for $a(x)$ and a local minimum for $b(x)$ and vice versa.

Remark 1.2.2 The results of Theorem 1.1, 1.2 are not so different to those of [7] for the Schrödinger-Poisson problem. We remark that in Theorem 1.2 we require only that $C_{2,i} a_i - \widetilde{C}_{2,i} b_i \neq 0$ in the case $s = n+2 = m$. Nothing is assumed in other cases unlike in [7]. This can be done because of the particular form of the problem in the case in which we couple the Klein-Gordon equation with the Maxwell equations. Moreover the real difficulty is that, in this case, we do not have an expression for the solution of the second equation of problem (1.6). Indeed in [7] the second equation is a Poisson equation and it is known the expression of such a solution.

The paper is organized as follows. Section 2 is devoted to some notations and preliminaries. In Section 3 we study the existence and the uniqueness of the solution of $\Delta\phi = b(x)(\omega + \phi)$ and we derive some properties of it. In Section 4 we study the variational setting of the problem and in Section 5 we prove Theorem 1.1 and 1.2. In a final section (see Section 6) a necessary condition to the concentration phenomenon is discussed.

2 Preliminaries

Hereafter we use the following notation:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$(u, v) = \int_{\mathbb{R}^3} [\nabla u \nabla v + uv] dx; \qquad \|u\|^2 = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + u^2\right] dx.$$

- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

- $L^q(\mathbb{R}^3)$, $1 \leq q \leq +\infty$, denotes a Lebesgue space, the norm in L^q is denoted by $|u|_q$.
- $\bar{S}$ is the best constant in the Sobolev embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, namely

$$\bar{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_{D^{1,2}}}{|u|_6}.$$

- S_q is the best Sobolev constant for the embedding of $H^1(\mathbb{R}^3)$ in $L^q(\mathbb{R}^3)$, $q \in (2,6)$, that is

$$S_q = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|}{|u|_q}.$$

- C, C', C_i are various positive constants.

We consider a problem which will be useful in the sequel:

$$\begin{cases} -\Delta u + u = |u|^{p-1}u, & x \in \mathbb{R}^3 \\ u(x) \to 0 \quad \text{as } |x| \to +\infty. \end{cases} \tag{2.1}$$

It is well known that (2.1) has a unique positive radial solution U in $H^1(\mathbb{R}^3)$. This solution satisfies the following decay property (see [6]):

$$\lim_{r\to+\infty} U(r)re^r = C > 0, \qquad \lim_{r\to+\infty} \frac{U'(r)}{U(r)} = -1, \qquad r = |x|.$$

for some constant C.
The function U is also a critical point of the C^2 functional $I_0 : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\, dx, \tag{2.2}$$

Since (2.1) is translation invariant, it follows that any $z_\xi(x) := U(x-\xi)$, $\xi \in \mathbb{R}^3$, is also a solution of (2.1). In other words I_0 has a non-compact critical manifold given by

$$\mathcal{Z} = \{z_\xi(x) : \xi \in \mathbb{R}^3\}.$$

We know that $\mathcal{Z}$ is non-degenerate (see [2, Chapter 4, Section 4.2]).
Moreover, there exists a constant $C > 0$ such that

$$\|z_\xi\| \leq C; \qquad \left\|\frac{\partial z_\xi}{\partial x_j}\right\| \leq C, \quad \text{for all } \xi \in \mathbb{R}^3. \tag{2.3}$$

We also recall the following properties of the solution U.

Lemma 2.1 *Define the operator* $Q : H^1(\mathbb{R}^3) \to \mathbb{R}$ *as*

$$Q[\nu] := I_0''(U)[\nu,\nu] = \int_{\mathbb{R}^3} \left[|\nabla \nu|^2 + \nu^2 - pU^{p-1}\nu^2 \right] dx.$$

We denote $U_k = \frac{\partial U}{\partial x_k}$. *Then there hold:*

1. $Q[U] = (1-p)\|U\|^2 < 0$.
2. $Q[\frac{\partial U}{\partial x_j}] = 0,\ j = 1,2,3$.
3. $Q[\nu] \geq C\|\nu\|^2$ *for all* $\nu \perp U, \nu \perp \frac{\partial U}{\partial x_j}$, $j = 1,2,3$.

For a proof see for instance [2, Lemma 8.6].

3 Study of the equation (1.4)

Lemma 3.1 *For any* $v \in H^1(\mathbb{R}^3)$ *there exists a unique* $\phi_v \in D^{1,2}(\mathbb{R}^3)$ *of the equation (1.4) such that* $0 \leq \phi_v \leq \omega$.

Proof Let $v \in H^1(\mathbb{R}^3)$ and we define the following bilinear form

$$L : (w_1, w_2) \in D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \longmapsto \int_{\mathbb{R}^3} \left[\nabla w_1 \nabla w_2 + b(x) v^2 w_1 w_2 \right] dx \in \mathbb{R}.$$

It is easy to see that L is well defined. Moreover, since $b(x) \geq 0$, $L(w_1, w_1) \geq \|w_1\|^2_{D^{1,2}}$. Furthermore since $b(x)$ is bounded, by using the Hölder inequality, we have

$$\begin{aligned} L(w_1, w_2) &\leq \|w_1\|_{D^{1,2}} \|w_2\|_{D^{1,2}} + C(b) |v^2|_{\frac{3}{2}} \cdot |w_1 w_2|_3 \\ &\leq \|w_1\|_{D^{1,2}} \|w_2\|_{D^{1,2}} + C(b) |v|_3^2 |w_1|_6 |w_2|_6 \\ &\leq (1 + \widetilde{C}(b) \cdot |v|_3^2) \|w_1\|_{D^{1,2}} \|w_2\|_{D^{1,2}} \end{aligned}$$

where $\widetilde{C}(b) = \bar{S}^{-2} \cdot C(b)$.
Therefore L defines an inner product, equivalent to the standard inner product in $D^{1,2}(\mathbb{R}^3)$. Moreover $H^1(\mathbb{R}^3) \subset L^{12/5}(\mathbb{R}^3)$ and then

$$\left| \int_{\mathbb{R}^3} \omega b(x) v^2 w_1 \, dx \right| \leq C(b, \omega) |v|^2_{12/5} \|w_1\|_{D^{1,2}}.$$

Therefore the linear map

$$w_1 \in D^{1,2}(\mathbb{R}^3) \longmapsto \int_{\mathbb{R}^3} \omega b(x) v^2 w_1 \, dx \in \mathbb{R}$$

is continuous. Hence, by the Lax-Milgram theorem, there exists a unique $\phi_v \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} [\nabla \phi_v \nabla w_1 + b(x) v^2 \phi_v w_1] dx = \int_{\mathbb{R}^3} \omega b(x) v^2 w_1 dx, \qquad \forall\ w_1 \in D^{1,2}(\mathbb{R}^3)$$

namely ϕ_v is the unique solution of $-\Delta\phi + b(x)v^2\phi = \omega b(x)v^2$.
Arguing by contradiction, we assume that there exists an open subset $\Omega \subset \mathbb{R}^3$ such that

$$\phi_v > \omega. \tag{3.1}$$

Then, since ϕ_v solves (1.4), we have

$$-\Delta(\phi_v - \omega) + b(x)v^2(\phi_v - \omega) = -\Delta\phi_v + b(x)v^2\phi_v - \omega b(x)v^2 = 0.$$

So $\varphi = \phi_v - \omega$ satisfies

$$-\Delta\varphi + b(x)v^2\varphi = 0 \text{ in } \Omega, \;\; \varphi = 0 \text{ on } \partial\Omega.$$

Then $\varphi = 0$ contradicting (3.1).
An analogous argument shows that $\phi \geq 0$ (by using also the positivity of the function $b(x)$). □

Remark 3.1.1 Let $v \in H^1(\mathbb{R}^3)$. More in general one can consider, for all $h \in (D^{1,2}(\mathbb{R}^3))'$, the equation

$$-\Delta\phi + b(x)v^2\phi = h. \tag{3.2}$$

As done in Lemma 3.1 one can prove the existence and the uniqueness of the solution in $D^{1,2}(\mathbb{R}^3)$ of (3.2).

Now, let $v \in H^1(\mathbb{R}^3)$ and $h, k \in (D^{1,2}(\mathbb{R}^3))'$. We denote by ϕ^h the unique solution in $D^{1,2}(\mathbb{R}^3)$ of

$$-\Delta\phi^h + b(x)v^2\phi^h = h, \tag{3.3}$$

and by ϕ^k the unique solution in $D^{1,2}(\mathbb{R}^3)$ of

$$-\Delta\phi^k + b(x)v^2\phi^k = k. \tag{3.4}$$

Then

$$\int_{\mathbb{R}^3} h\phi^k \, dx = \int_{\mathbb{R}^3} k\phi^h \, dx. \tag{3.5}$$

Indeed, if we multiply by ϕ^k the equation (3.3) and by ϕ^h the equation (3.4) and we integrate on $\mathbb{R}^3$, we find

$$\int_{\mathbb{R}^3} h\phi^k \, dx = \int_{\mathbb{R}^3} \left[\nabla\phi^h \nabla\phi^k + b(x)v^2\phi^h\phi^k\right] \, dx = \int_{\mathbb{R}^3} k\phi^h \, dx.$$

Now we define the map

$$\Phi : v \in H^1(\mathbb{R}^3) \longmapsto \phi_v \in D^{1,2}(\mathbb{R}^3)$$

where ϕ_v is the unique solution of (1.4). From Lemma 3.1 it follows that Φ is well defined. As done in [4] one can prove that Φ is of class C^1 and, for every $u, v \in H^1(\mathbb{R}^3)$

$$(\Phi'[v])[u] = 2(\phi_1 - \phi_2) \tag{3.6}$$

where ϕ_1, ϕ_2 are respectively the solutions in $D^{1,2}(\mathbb{R}^3)$ of

$$-\Delta\phi_1 + b(x)v^2\phi_1 = \omega b(x)uv \tag{3.7}$$

$$-\Delta\phi_2 + b(x)v^2\phi_2 = b(x)\phi_v uv. \tag{3.8}$$

We remark that, since $\omega b(x)uv$, $b(x)\phi_v uv \in L^{6/5}(\mathbb{R}^3) \subset (D^{1,2}(\mathbb{R}^3))'$, from Remark 3.1.1 it follows the existence and uniqueness of ϕ_1 and ϕ_2.

Remark 3.1.2 Now, let $\beta > 0$ and we set $v_\beta(x) = \beta v(\beta x)$. Then $\phi := \Phi[v](\beta x)$ is the solution of

$$-\frac{1}{\beta^2}\Delta\phi + b(\beta x)v^2(\beta x)\phi = \omega b(\beta x)v^2(\beta x)$$

namely ϕ solves

$$-\Delta\phi + b(\beta x)v_\beta^2(x)\phi = \omega b(\beta x)v_\beta^2(x). \tag{3.9}$$

We set such a ϕ as ϕ_v^β.

We also note that if the function $b(x)$ is constant then one can infer that $\Phi[v](\beta x) = \Phi[v_\beta](x)$.

4 Variational setting

Let $x_0 \in \mathbb{R}^3$ be a critical point of $a(x)$, namely $\nabla a(x_0) = 0$. Without any loss of generality, we assume $x_0 = 0$ and $a(0) = 1$.

By suitably scaling, (1.6) can be reduced to the following equation:

$$-\Delta u + a(\epsilon x)u - \epsilon^2 b(\epsilon x)(\phi_v(\epsilon x) - \omega)^2 u = |u|^{p-1}u \tag{4.1}$$

where $\epsilon = \lambda^{3-p}$ and $u(x) = \epsilon v(\epsilon x)$.

From Remark 3.1.2 it follows that $\phi_v(\epsilon x) = \Phi[v](\epsilon x) = \phi_u^\epsilon$ where ϕ_u^ϵ is the solution of

$$-\Delta\phi_u^\epsilon + b(\epsilon x)u^2\phi_u^\epsilon = \omega b(\epsilon x)u^2. \tag{4.2}$$

Hence (4.1) becomes

$$-\Delta u + a(\epsilon x)u - \epsilon^2 b(\epsilon x)(\phi_u^\epsilon - \omega)^2 u = |u|^{p-1}u, \qquad x \in \mathbb{R}^3. \tag{4.3}$$

From (3.6) it follows that

$$(\phi_u^\epsilon)'[v] = 2(\phi_1^\epsilon - \phi_2^\epsilon) \tag{4.4}$$

where ϕ_1^ϵ, ϕ_2^ϵ are, respectively, the solutions of

$$-\Delta\phi_1^\epsilon + b(\epsilon x)u^2\phi_1^\epsilon = \omega b(\epsilon x)uv \tag{4.5}$$

$$-\Delta\phi_2^\epsilon + b(\epsilon x)u^2\phi_2^\epsilon = b(\epsilon x)\phi_u^\epsilon uv. \tag{4.6}$$

The solutions of (4.3) are the critical points of the functional $I_\epsilon : H^1(\mathbb{R}^3) \to \mathbb{R}$, $I_\epsilon \in C^2$

$$\begin{aligned} I_\epsilon(u) \quad = \quad & \frac{1}{2}\int_{\mathbb{R}^3} \left[|\nabla u|^2 + a(\epsilon x)u^2\right] dx - \frac{\epsilon^2}{2}\int_{\mathbb{R}^3} \omega^2 b(\epsilon x)u^2\, dx \\ & + \frac{\epsilon^2}{2}\int_{\mathbb{R}^3} \omega b(\epsilon x)\phi_u^\epsilon u^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}. \end{aligned} \tag{4.7}$$

Indeed, by using (3.6)

$$\begin{aligned} I'_\epsilon(u)[v] &= \int_{\mathbb{R}^3} \left[\nabla u \nabla v + a(\epsilon x)uv - \epsilon^2\omega^2 b(\epsilon x)uv - |u|^{p-1}uv\right] dx \\ &\quad + \epsilon^2 \int_{\mathbb{R}^3} \omega b(\epsilon x)\phi_u^\epsilon uv\, dx + \frac{\epsilon^2}{2}\int_{\mathbb{R}^3} \omega b(\epsilon x)(\phi_u^\epsilon)'[v]u^2\, dx \\ &= \int_{\mathbb{R}^3} \left[\nabla u \nabla v + a(\epsilon x)uv - \epsilon^2\omega^2 b(\epsilon x)uv - |u|^{p-1}uv\right] dx \\ &\quad + \epsilon^2 \int_{\mathbb{R}^3} \omega b(\epsilon x)\phi_u^\epsilon uv\, dx + \epsilon^2 \underbrace{\int_{\mathbb{R}^3} \omega b(\epsilon x)u^2\phi_1^\epsilon\, dx}_{(I)} - \epsilon^2 \underbrace{\int_{\mathbb{R}^3} \omega b(\epsilon x)u^2\phi_2^\epsilon\, dx}_{(II)} \end{aligned}$$

By using (3.5) with $h = \omega b(\epsilon x)u^2$ and $k = \omega b(\epsilon x)uv$ then, since $\phi^h \equiv \phi_u^\epsilon$ we find

$$(I) = \int_{\mathbb{R}^3} h\phi^k\, dx = \int_{\mathbb{R}^3} \omega b(\epsilon x)\phi_u^\epsilon uv\, dx.$$

Moreover let h as before and $k = b(\epsilon x)\phi_u^\epsilon uv$ then from (3.5) it follows

$$(II) = \int_{\mathbb{R}^3} h\phi^k\, dx = \int_{\mathbb{R}^3} b(\epsilon x)(\phi_u^\epsilon)^2 uv\, dx.$$

Hence

$$\begin{aligned} I'_\epsilon(u)[v] &= \int_{\mathbb{R}^3} \left[\nabla u \nabla v + a(\epsilon x)uv - \epsilon^2\omega^2 b(\epsilon x)uv - |u|^{p-1}uv\right] dx \\ &\quad + \epsilon^2 \int_{\mathbb{R}^3} \omega b(\epsilon x)\phi_u^\epsilon uv\, dx + \epsilon^2 \int_{\mathbb{R}^3} \omega b(\epsilon x)uv\phi_u^\epsilon\, dx - \epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)(\phi_u^\epsilon)^2 uv\, dx. \end{aligned}$$

At the end we find

$$\begin{aligned} I'_\epsilon(u)[v] &= \int_{\mathbb{R}^3} \left[\nabla u \nabla v + a(\epsilon x)uv - \epsilon^2\omega^2 b(\epsilon x)uv - |u|^{p-1}uv\right] dx \\ &\quad + \epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)(2\omega - \phi_u^\epsilon)\phi_u^\epsilon uv\, dx. \end{aligned} \tag{4.8}$$

In a similar way one can compute the second derivative of the functional I_ϵ finding

$$\begin{aligned} I''_\epsilon(u)[v,w] &= \int_{\mathbb{R}^3} \left[\nabla v \nabla w + a(\epsilon x)vw - \epsilon^2\omega^2 b(\epsilon x)vw - p|u|^{p-1}vw\right] dx \qquad (4.9) \\ &\quad + \epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)(2\omega - \phi_u^\epsilon)\phi_u^\epsilon vw\, dx + 4\epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)(\omega - \phi_u^\epsilon)(\bar\phi_1^\epsilon - \bar\phi_2^\epsilon)uv\, dx \end{aligned}$$

where $\bar\phi_1^\epsilon$ and $\bar\phi_2^\epsilon$ are, respectively, the solutions of

$$-\Delta\bar\phi_1^\epsilon + b(\epsilon x)u^2\bar\phi_1^\epsilon = \omega b(\epsilon x)uw, \qquad -\Delta\bar\phi_2^\epsilon + b(\epsilon x)u^2\bar\phi_2^\epsilon = \phi_u^\epsilon b(\epsilon x)uw$$

In the next lemma some estimates that are useful later are done.

Lemma 4.1 *Let be $u \in H^1(\mathbb{R}^3)$ and ϕ the solution in $D^{1,2}(\mathbb{R}^3)$ of*

$$-\Delta\phi + b(\epsilon x)z_\xi^2\phi = \omega b(\epsilon x)z_\xi u. \tag{4.10}$$

Then if (b1) holds then

$$\|\phi\|_{D^{1,2}} \leq C_{b,\omega} \|z_\xi\| \cdot \|u\|. \tag{4.11}$$

Instead, if (b1)-(b2) hold and $|\xi| \leq \bar{\xi}$*, then*

$$\|\phi\|_{D^{1,2}} \leq C_{\omega,\bar{\xi}} \cdot \epsilon^n \cdot \|u\|. \tag{4.12}$$

Proof Since ϕ solves (4.10) we have

$$\begin{aligned} \|\phi\|^2_{D^{1,2}} &= \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx \leq \int_{\mathbb{R}^3} \left[|\nabla \phi|^2 + b(\epsilon x) z_\xi^2 \phi^2 \right] \, dx \\ &= \int_{\mathbb{R}^3} \omega b(\epsilon x) z_\xi u \phi \, dx. \end{aligned}$$

If (b1) holds then

$$\|\phi\|^2_{D^{1,2}} \leq \omega \cdot C_b |\phi|_6 \cdot |z_\xi u|_{6/5} \leq C_{b,\omega} \|\phi\|_{D^{1,2}} \|z_\xi\| \cdot \|u\|$$

and (4.11) follows. If (b1)-(b2) hold then

$$b(\epsilon x) \leq C \cdot \epsilon^n |x|^n.$$

Hence

$$\|\phi\|^2_{D^{1,2}} \leq \omega |\phi|_6 \left(\int_{\mathbb{R}^3} b(\epsilon x)^{6/5} z_\xi^{6/5} u^{6/5} \, dx \right)^{5/6}.$$

Now, $z_\xi \in L^6(\mathbb{R}^3)$, hence $z_\xi^{6/5} \in L^5(\mathbb{R}^3)$. Moreover $u \in L^{3/2}(\mathbb{R}^3)$, hence $u^{6/5} \in L^{5/4}(\mathbb{R}^3)$. By using Hölder inequality we find

$$\begin{aligned} \|\phi\|^2_{D^{1,2}} &\leq C_\omega \|\phi\|_{D^{1,2}} \left(\int_{\mathbb{R}^3} b(\epsilon x)^6 z_\xi^6 \, dx \right)^{1/6} |u|_{3/2} \\ &\leq C_\omega \cdot \epsilon^n \|\phi\|_{D^{1,2}} \left(\int_{\mathbb{R}^3} |x + \xi|^{6n} U^6(x) \, dx \right)^{1/6} \|u\| \\ &\leq C_{\omega,\bar{\xi}} \cdot \epsilon^n \|\phi\|_{D^{1,2}} \|u\| \end{aligned}$$

since $\int_{\mathbb{R}^3} |x+\xi|^{6n} U^6(x) \, dx \leq C_{\bar{\xi}}$ provided $|\xi| \leq \bar{\xi}$. Then (4.12) follows. □

Remark 4.1.1 We note that (4.11) holds also if we substitute z_ξ with a function $v \in H^1(\mathbb{R}^3)$. Instead (4.12) holds only in that case since we use the exponential decay of the function U solution of (2.1) to obtain the result.

5 Proof of the main results

The proofs of our main results use a Lyapunov-Schmidt reduction.
Fix $\bar{\xi} \in \mathbb{R}^3$ and let us define

$$\mathcal{Z}_{\bar{\xi}} := \left\{ z_\xi \in \mathcal{Z}_\xi \ : \ |\xi| \leq \bar{\xi} \right\}.$$

For every $z_\xi \in \mathcal{Z}_{\bar\xi}$, we define $W = W_{z_\xi,\epsilon} = \left(T_{z_\xi}\mathcal{Z}_{\bar\xi}\right)^\perp$ and $P : H^1(\mathbb{R}^3) \to W$ the orthogonal projection onto W. Our approach is to find a pair $z_\xi \in \mathcal{Z}_{\bar\xi}$, $w \in W$, $\|w\| = O(\epsilon^s)$ ($s = 2$ in the assumptions of Theorem 1.1 and $s = \min\{m, n+2\}$ in the assumptions of Theorem 1.2), such that $I'_\epsilon(z_\xi + w) = 0$. Equivalently:

$$\begin{cases} \text{a) } PI'_\epsilon(z_\xi + w) = 0, \\ \\ \text{b) } (\mathcal{I} - P)I'_\epsilon(z_\xi + w) = 0. \end{cases} \tag{5.1}$$

The first equation above is called *auxiliary equation*, and the second one receives the name of *bifurcation equation*.

5.1 The auxiliary equation

Our intention now is to find a solution $w \in W$ of the auxiliary equation for any $z_\xi \in \mathcal{Z}_{\bar\xi}$. We begin with some estimates:

Proposition 5.1 *Let be (a1)-(a2)-(b1) hold.*
Then there exists $C = C(b, \omega, \bar\xi) > 0$ such that for all $\epsilon > 0$ we have

$$\|I'_\epsilon(z_\xi)\| \le C\epsilon^2. \tag{5.2}$$

If (a1)-(a2')-(b1)-(b2) hold then there exists $C = C(\omega, \bar\xi) > 0$ such that for all $\epsilon > 0$ small we have

$$\|I'_\epsilon(z_\xi)\| \le C\epsilon^s. \tag{5.3}$$

with $s = \min\{m, n+2\}$.

Proof Taking into account that z_ξ is a solution of (2.1) we have

$$\begin{aligned} |I'_\epsilon(z_\xi)[v]| &\le \int_{\mathbb{R}^3} |a(\epsilon x) - 1||z_\xi||v|\,dx + \epsilon^2\omega^2 \int_{\mathbb{R}^3} b(\epsilon x) z_\xi |v|\,dx \\ &\quad + \epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)|2\omega - \phi^\epsilon_{z_\xi}| \cdot |\phi^\epsilon_{z_\xi}| z_\xi |v|\,dx \\ &\le \left(\int_{\mathbb{R}^3} |a(\epsilon x) - 1|^2 z_\xi^2\,dx\right)^{1/2} |v|_2 + \epsilon^2\omega^2 \left(\int_{\mathbb{R}^3} b(\epsilon x)^2 z_\xi^2\,dx\right)^{1/2} |v|_2 \\ &\quad + \epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)|2\omega - \phi^\epsilon_{z_\xi}| \cdot |\phi^\epsilon_{z_\xi}| z_\xi |v|\,dx \\ &\le \underbrace{\left(\int_{\mathbb{R}^3} |a(\epsilon x) - 1|^2 z_\xi^2\,dx\right)^{1/2}}_{(I)} \|v\| + C_\omega \epsilon^2 \underbrace{\left(\int_{\mathbb{R}^3} b(\epsilon x)^2 z_\xi^2\,dx\right)^{1/2}}_{(II)} \|v\| \\ &\quad \underbrace{+\epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)|2\omega - \phi^\epsilon_{z_\xi}| \cdot |\phi^\epsilon_{z_\xi}| z_\xi |v|\,dx}_{(III)} \end{aligned}$$

If (a1)-(a2) hold then

$$|a(\epsilon x) - 1| \le C\epsilon^2|x|^2. \tag{5.4}$$

Therefore

$$(I) \le C\epsilon^2 \left(\int_{\mathbb{R}^3} |x + \xi|^4 U^2(x)\,dx\right)^{1/2} \le C_{\bar\xi}\epsilon^2 \tag{5.5}$$

provided $|\xi| \le \bar{\xi}$. If (a1)-(a2') hold then

$$|a(\epsilon x) - 1| \le C\epsilon^m |x|^m. \tag{5.6}$$

Therefore

$$(I) \le C\epsilon^m \left(\int_{\mathbb{R}^3} |x + \xi|^{2m} U^2(x)\, dx \right)^{1/2} \le C_{\bar{\xi}} \epsilon^m \tag{5.7}$$

provided $|\xi| \le \bar{\xi}$.
Moreover, if (b1) hold (by using (4.11) and (2.3)), we have

$$(II) \le C_b |z_\xi|_2 \le C_b \tag{5.8}$$

and

$$\begin{aligned} (III) &\le C_{b,\omega} \int_{\mathbb{R}^3} |\phi^\epsilon_{z_\xi}| z_\xi |v|\, dx + C_b \int_{\mathbb{R}^3} |\phi^\epsilon_{z_\xi}|^2 z_\xi |v|\, dx \\ &\le C_{b,\omega} |\phi^\epsilon_{z_\xi}|_6 |z_\xi v|_{6/5} + C_b |(\phi^\epsilon_{z_\xi})^2|_3 |z_\xi v|_{3/2} \\ &\le C_{b,\omega} \|\phi^\epsilon_{z_\xi}\|_{D^{1,2}} \|z_\xi\| \cdot \|v\| + C_b \|\phi^\epsilon_{z_\xi}\|^2_{D^{1,2}} \|z_\xi\| \cdot \|v\| \\ &\le C_{b,\omega} \cdot \|z_\xi\|^3 \|v\| + C_b \cdot \|z_\xi\|^5 \cdot \|v\| \\ &\le C_{b,\omega} \|v\|. \end{aligned} \tag{5.9}$$

If (b1)-(b2) hold then

$$b(\epsilon x) \le C\epsilon^n |x|^n.$$

Hence

$$(II) \le \epsilon^n \cdot \left(\int_{\mathbb{R}^3} |x + \xi|^{2n} U^2(x)\, dx \right)^{1/2} \le C_{\bar{\xi}} \epsilon^n \tag{5.10}$$

provided $|\xi| \le \bar{\xi}$ and

$$\begin{aligned} (III) &\le C_\omega \int_{\mathbb{R}^3} b(\epsilon x) |\phi^\epsilon_{z_\xi}| z_\xi |v|\, dx + \int_{\mathbb{R}^3} b(\epsilon x) |\phi^\epsilon_{z_\xi}|^2 z_\xi |v|\, dx \\ &\le C_\omega |\phi^\epsilon_{z_\xi}|_6 \cdot \left(\int_{\mathbb{R}^3} b(\epsilon x)^{6/5} |z_\xi v|^{6/5}\, dx \right)^{5/6} + |\phi^\epsilon_{z_\xi}|^2_6 \cdot \left(\int_{\mathbb{R}^3} b(\epsilon x)^{3/2} |z_\xi v|^{3/2}\, dx \right)^{2/3} \\ &\le C_\omega \|\phi^\epsilon_{z_\xi}\|_{D^{1,2}} \cdot \left(\int_{\mathbb{R}^3} b(\epsilon x)^6 z_\xi^6\, dx \right)^{1/6} \|v\| + \|\phi^\epsilon_{z_\xi}\|^2_{D^{1,2}} \cdot \left(\int_{\mathbb{R}^3} b(\epsilon x)^2 |z_\xi v|^2\, dx \right)^{1/2} \|v\| \\ &\le C_\omega \|\phi^\epsilon_{z_\xi}\|_{D^{1,2}} \cdot \epsilon^n \left(\int_{\mathbb{R}^3} |x + \xi|^{6n} U^6(x)\, dx \right)^{1/6} \|v\| + \\ &\quad + \|\phi^\epsilon_{z_\xi}\|^2_{D^{1,2}} \cdot \epsilon^n \left(\int_{\mathbb{R}^3} |x + \xi|^{2n} U^2(x)\, dx \right)^{1/2} \|v\| \\ &\le C_{\omega,\bar{\xi}} \cdot \epsilon^{2n} \|z_\xi\| \cdot \|v\| + C_{\bar{\xi}} \cdot \epsilon^{3n} \|z_\xi\|^2 \|v\|. \end{aligned}$$

So, by using (2.3) we have, for ϵ small,

$$(III) \le C_{\omega,\bar{\xi}} \epsilon^{2n} \|v\|. \tag{5.11}$$

Putting together (5.5)-(5.8)-(5.9) we obtain (5.2). Instead, putting together (5.7)-(5.10)-(5.11) we obtain, for ϵ small, (5.3). □

Now we are concerned with the invertibility of $I''_\epsilon(z_\xi)$ on $W = (T_{z_\xi}(\mathcal{Z}_{\bar\xi}))^\perp$. First we observe that $T_{z_\xi}\mathcal{Z}_{\bar\xi}$ is spanned by the functions

$$\dot{z}_i := -\frac{\partial U}{\partial x_i}(x-\xi), \quad i = 1,2,3. \tag{5.12}$$

Recall that P denotes the orthogonal projection onto W; we decompose: $W = A \oplus A^\perp$ where A is the space spanned by Pz_ξ.
A simple computation shows that

$$Pz_\xi \equiv z_\xi. \tag{5.13}$$

The following result holds:

Proposition 5.2 *Let be (a1)-(a2)-(b1) hold. For ϵ small and any $|\xi| \le \bar\xi$, $PI''_\epsilon(z_\xi) : W \to W$ is invertible and $\|[PI''_\epsilon(z_\xi)]^{-1}\| \le \bar{C}$.*

The above result follows directly from the following lemma (see [2]):

Lemma 5.3 *Let be (a1)-(a2)-(b1) hold. For all $\epsilon > 0$ sufficiently small there exist two positive constants C_1, C_2 such that*

(i) $I''_\epsilon(z_\xi)[u,u] \le -C_1\|u\|^2$, for all $u \in A$;

(ii) $I''_\epsilon(z_\xi)[u,u] \ge C_2\|u\|^2$, for all $u \in A^\perp$.

Proof Let be $u \in A$. Then by (5.13), $u = \alpha P z_\xi = \alpha z_\xi$, $\alpha \in \mathbb{R}$.
To prove (i) we just need to show

$$I''_\epsilon(z_\xi)[z_\xi, z_\xi] \le -C_1\|z_\xi\|^2.$$

From (4.9) it follows

$$\begin{aligned} I''_\epsilon(z_\xi)[z_\xi, z_\xi] &= \int_{\mathbb{R}^3} \left[|\nabla z_\xi|^2 + a(\epsilon x) z_\xi^2 - \epsilon^2\omega^2 b(\epsilon x) z_\xi^2 - p|z_\xi|^{p+1}\right] dx \\ &\quad + \epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)(2\omega - \phi^\epsilon_{z_\xi})\phi^\epsilon_{z_\xi} z_\xi^2\, dx + 4\epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)(\omega - \phi^\epsilon_{z_\xi})(\widetilde{\phi}^\epsilon_1 - \widetilde{\phi}^\epsilon_2) z_\xi^2\, dx \end{aligned}$$

where $\widetilde{\phi}^\epsilon_1$ solves $-\Delta\widetilde{\phi}^\epsilon_1 + b(\epsilon x) z_\xi^2 \widetilde{\phi}^\epsilon_1 = \omega b(\epsilon x) z_\xi^2$ and so, by uniqueness, $\widetilde{\phi}^\epsilon_1 \equiv \phi^\epsilon_{z_\xi}$ and $\widetilde{\phi}^\epsilon_2$ solves $-\Delta\widetilde{\phi}^\epsilon_2 + b(\epsilon x) z_\xi^2 \widetilde{\phi}^\epsilon_2 = \phi^\epsilon_{z_\xi} b(\epsilon x) z_\xi^2$. Hence

$$\begin{aligned} I''_\epsilon(z_\xi)[z_\xi, z_\xi] &= I''_0(z_\xi)[z_\xi, z_\xi] + \underbrace{\int_{\mathbb{R}^3} (a(\epsilon x) - 1) z_\xi^2\, dx}_{(A)} - \underbrace{\epsilon^2\omega^2 \int_{\mathbb{R}^3} b(\epsilon x) z_\xi^2\, dx}_{(B)} \\ &\quad + \underbrace{\epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)(2\omega - \phi^\epsilon_{z_\xi})\phi^\epsilon_{z_\xi} z_\xi^2\, dx}_{(C)} + 4\underbrace{\epsilon^2 \int_{\mathbb{R}^3} b(\epsilon x)(\omega - \phi^\epsilon_{z_\xi})(\phi^\epsilon_{z_\xi} - \widetilde{\phi}^\epsilon_2) z_\xi^2\, dx}_{(D)}. \end{aligned}$$

Reasoning as in the proof of Proposition 5.1 one can prove that $(A) = o(1)$, $(B) = o(1)$, $(C) = o(1)$, $(D) = o(1)$ as $\epsilon \to 0$ provided $|\xi| \le \bar\xi$. Therefore

$$I''_\epsilon(z_\xi)[z_\xi, z_\xi] = I''_0(z_\xi)[z_\xi, z_\xi] + o(1).$$

By using Lemma 2.1 we have, for ϵ small

$$I''_\epsilon(z_\xi)[z_\xi, z_\xi] \le (1-p)\|z_\xi\|^2 < -C_1 < 0.$$

So $I''_\epsilon(z_\xi)$ is negative definite on A. We now prove that $I''_\epsilon(z_\xi)$ is positive definite on $A^\perp$.
First we note that from (5.12) and (5.13) we have $A^\perp = \operatorname{span}\{z_\xi, \dot{z}_i\}^\perp$, so it suffices to prove (b) just for $u \perp \operatorname{span}\{z_\xi, \dot{z}_i\}$.
From (4.9) it follows

$$\begin{aligned} I''_\epsilon(z_\xi)[u,u] &= \underbrace{\int_{\mathbb{R}^3}\left[|\nabla u|^2 + a(\epsilon x)u^2 - p|z_\xi|^{p-1}u^2\right]dx}_{(A')} - \underbrace{\epsilon^2\omega^2\int_{\mathbb{R}^3} b(\epsilon x)u^2\,dx}_{(B')} \\ &\quad + \underbrace{\epsilon^2\int_{\mathbb{R}^3} b(\epsilon x)(2\omega - \phi^\epsilon_{z_\xi})\phi^\epsilon_{z_\xi}u^2\,dx}_{(C')} + \underbrace{4\,\epsilon^2\int_{\mathbb{R}^3} b(\epsilon x)(\omega - \phi^\epsilon_{z_\xi})(\hat{\phi}^\epsilon_1 - \hat{\phi}^\epsilon_2)z_\xi^2\,dx}_{(D')} \end{aligned}$$

where $\hat{\phi}^\epsilon_1$ solves $-\Delta\hat{\phi}^\epsilon_1 + b(\epsilon x)z_\xi^2\hat{\phi}^\epsilon_1 = \omega b(\epsilon x)z_\xi u$ and $\hat{\phi}^\epsilon_2$ solves $-\Delta\hat{\phi}^\epsilon_2 + b(\epsilon x)z_\xi^2\hat{\phi}^\epsilon_2 = \phi^\epsilon_{z_\xi}b(\epsilon x)z_\xi u$.
Since b and z_ξ are bounded we have $(B') = o(1)$, $(C') = o(1)$, $(D') = o(1)$ as $\epsilon \to 0$. Moreover, one can prove that there exists a positive constant C_2 such that

$$\int_{\mathbb{R}^3}\left[|\nabla u|^2 + a(\epsilon x)u^2 - pz_\xi^{p-1}u^2\right]dx \ge C_2\|u\|^2$$

(see [2, Lemma 8.9], where the last point of Lemma 2.1 was used too) and we conclude. □

With this estimates in hand we can now solve the auxiliary equation. Consider $z_\xi \in \mathcal{Z}_{\bar{\xi}}$ fixed, and define

$$B_\epsilon = \left\{u \in W : \|u\| \le 2\bar{C}\|I'_\epsilon(z_\xi)\|\right\},$$

where $\bar{C}$ is the positive constant given by Proposition 5.2. So, the solutions of the auxiliary equations are fixed points of the map $S_\epsilon : W \to W$

$$S_\epsilon(w) = w - [PI''_\epsilon(z_\xi)]^{-1}[PI'_\epsilon(z_\xi + w)].$$

It is easy to check that $\|S_\epsilon(0)\| \le \bar{C}\|I'_\epsilon(z_\xi)\|$. We now compute the derivative of S_ϵ:

$$S'_\epsilon(w)[v] = v - [PI''_\epsilon(z_\xi)]^{-1}PI''_\epsilon(z_\xi + w)[v] = [PI''_\epsilon(z_\xi)]^{-1}\left(PI''_\epsilon(z_\xi) - PI''_\epsilon(z_\xi + w)\right)[v].$$

Now observe that I''_ϵ is uniformly continuous in bounded sets, so

$$\|PI''_\epsilon(z_\xi + w) - PI''_\epsilon(z_\xi)\| \to 0 \qquad (\epsilon \to 0)$$

uniformly in $z_\xi \in \mathcal{Z}_{\bar{\xi}}$ and $w \in B_\epsilon$ (recall Proposition 5.1).
This implies that $\|S'_\epsilon(w)\| = o(1)$ for any $w \in B_\epsilon$. Therefore, S_ϵ is a contraction and, by using the mean value theorem, $S_\epsilon(B_\epsilon) \subset B_\epsilon$. We make use of the

Banach contraction theorem to find a unique fixed point $w = w_{\epsilon,z_\xi} \in B_\epsilon$ of S_ϵ. Moreover it follows that

$$\|w_{\epsilon,z_\xi}\| \leq 2\bar{C}\|I'_\epsilon(z_\xi)\|. \tag{5.14}$$

Hence, by (5.2), we find

$$\|w_{\epsilon,z_\xi}\| \leq C_{b,\omega,\bar{\xi}}\epsilon^2 \tag{5.15}$$

and by (5.3)

$$\|w_{\epsilon,z_\xi}\| \leq C_{\omega,\bar{\xi}} \cdot \epsilon^s, \tag{5.16}$$

with $s = \min\{m, n+2\}$.

5.2 The reduced functional

In this section we will find a solution for the bifurcation equation among the set of solutions of the auxiliary equation which is:

$$\bar{\mathcal{Z}}_{\bar{\xi}} = \left\{ z_\xi + w_{\epsilon,z_\xi} : z_\xi \in \mathcal{Z}_{\bar{\xi}},\ w_{\epsilon,z_\xi} \text{ solves } (5.1)(\text{a}), \text{ and satisfies } (5.15) \text{ or } (5.16) \right\}.$$

We remark that w_{ϵ,z_ξ} satisfies (5.15) if the assumptions of Theorem 1.1 hold and satisfies (5.16) if the assumptions of Theorem 1.2 hold.
By the Implicit Function Theorem it is easy to check that $\bar{\mathcal{Z}}_{\bar{\xi}}$ is a C^1 manifold. Moreover, it is well-known (see [2], for example) that $\bar{\mathcal{Z}}_{\bar{\xi}}$ is a natural constraint for I_ϵ for ϵ small. In other words, critical points of $I_\epsilon|_{\bar{\mathcal{Z}}_{\bar{\xi}}}$ are solutions of the bifurcation equation (5.1) (b), and hence solutions of (4.3).
So, let us define the reduced functional as the restriction of the functional I_ϵ to the natural constraint $\bar{\mathcal{Z}}_{\bar{\xi}}$, namely $\Gamma_\epsilon : B(0,\bar{\xi}) \to \mathbb{R}$, $\Gamma_\epsilon(\xi) = I_\epsilon(z_\xi + w(\epsilon, z_\xi))$, and we look for critical points of Γ_ϵ. Using the information on $\|w(\epsilon, z_\xi)\|$, we will be able to find an expansion of $\Gamma_\epsilon(\xi)$.
First of all, since I''_ϵ maps bounded sets onto bounded sets, we have

$$\Gamma_\epsilon(\xi) = I_\epsilon(z_\xi) + I'_\epsilon(z_\xi)[w(\epsilon, z_\xi)] + O(\|w(\epsilon, z_\xi)\|^2). \tag{5.17}$$

5.2.1 Expansion of Γ_ϵ in the assumptions of Theorem 1.1

If (a1)-(a2)-(b1) hold then $w(\epsilon, z_\xi)$ is a solution of the auxiliary equation which satisfies (5.15). Hence from (5.17)

$$\Gamma_\epsilon(\xi) = I_\epsilon(z_\xi) + O(\epsilon^4). \tag{5.18}$$

So we have to find an expansion of $I_\epsilon(z_\xi)$.

Lemma 5.4 *For any $|\xi| \leq \bar{\xi}$ and ϵ sufficiently small we have*

$$I_\epsilon(z_\xi) = C_0 + \epsilon^2 F_1(\xi) + o(\epsilon^2), \tag{5.19}$$

where

$$F_1(\xi) = C_1\langle D^2 a(0)\xi, \xi\rangle + C_2$$

and

$$C_0 = I_0(U), \qquad C_1 = \frac{1}{4}\int_{\mathbb{R}^3} U^2(x)\,dx$$

$$C_2 = \frac{1}{4}\int_{\mathbb{R}^3} \langle D^2 a(0)x, x\rangle U^2(x)\,dx - \frac{\omega^2 b(0)}{2}\int_{\mathbb{R}^3} U^2(x)\,dx + \frac{\omega b(0)}{2}\int_{\mathbb{R}^3} \phi_U^{b(0)} U^2(x)\,dx$$

with $\phi_U^{b(0)}$ *the solution of* $-\Delta\phi_U^{b(0)} + b(0)U^2\phi_U^{b(0)} = \omega b(0)U^2$.

Proof A simple computation shows that

$$\begin{aligned} I_\epsilon(z_\xi) &= I_0(U) + \underbrace{\frac{1}{2}\int_{\mathbb{R}^3} [a(\epsilon x) - 1]\, z_\xi^2\,dx}_{(A)} - \underbrace{\frac{\epsilon^2\omega^2}{2}\int_{\mathbb{R}^3} b(\epsilon x) z_\xi^2\,dx}_{(B)} \\ &\quad + \underbrace{\frac{\epsilon^2\omega}{2}\int_{\mathbb{R}^3} b(\epsilon x)\phi_{z_\xi}^\epsilon z_\xi^2\,dx}_{(C)}. \end{aligned}$$

We compute separately the various terms. Since (a1)-(a2) hold and $a(0) = 1$, we have

$$\begin{aligned} (A) &= \frac{\epsilon^2}{4}\int_{\mathbb{R}^3} \langle D^2 a(0)x, x\rangle U^2(x-\xi)\,dx + o(\epsilon^2) \\ &= \frac{\epsilon^2}{4}\int_{\mathbb{R}^3} \langle D^2 a(0)(x+\xi), (x+\xi)\rangle U^2(x)\,dx + o(\epsilon^2) \\ &= \epsilon^2\left[C_1 \cdot \langle D^2 a(0)\xi, \xi\rangle + \widetilde{C_2}\right] + o(\epsilon^2) \end{aligned} \tag{5.20}$$

where we have set

$$C_1 = \frac{1}{4}\int_{\mathbb{R}^3} U^2(x)\,dx, \qquad \widetilde{C}_2 = \frac{1}{4}\int_{\mathbb{R}^3} \langle D^2 a(0)x, x\rangle U^2(x)\,dx.$$

Moreover, since (b1) holds we find

$$\begin{aligned} (B) &= \frac{\epsilon^2\omega^2}{2}\int_{\mathbb{R}^3} [b(0) + \epsilon \cdot \langle \nabla b(\eta_1), x\rangle]\, U^2(x-\xi)\,dx \\ &= \epsilon^2 \frac{\omega^2 b(0)}{2}\int_{\mathbb{R}^3} U^2(x)\,dx + \frac{\epsilon^3\omega^2}{2}\int_{\mathbb{R}^3} \langle \nabla b(\eta_1), x+\xi\rangle U^2(x)\,dx \end{aligned}$$

with $\eta_1 \in B(0, \epsilon|x|)$. Since ∇b is bounded we have

$$\left|\int_{\mathbb{R}^3} \langle \nabla b(\eta_1), x+\xi\rangle U^2(x)\,dx\right| \le C\int_{\mathbb{R}^3} |x+\xi| U^2(x)\,dx \le C_{\bar{\xi}}$$

provided $|\xi| \le \bar{\xi}$. Hence

$$\frac{\epsilon^3\omega^2}{2}\int_{\mathbb{R}^3} \langle \nabla b(\eta_1), x+\xi\rangle U^2(x)\,dx = o(\epsilon^2).$$

At the end we find

$$(B) = \epsilon^2 C_3 + o(\epsilon^2) \tag{5.21}$$

where we have set

$$C_3 = \frac{\omega^2}{2} b(0)\int_{\mathbb{R}^3} U^2(x)\,dx.$$

It remains to compute (C).

$$
\begin{aligned}
(C) &= \frac{\epsilon^2\omega}{2}\int_{\mathbb{R}^3}[b(0)+\epsilon\cdot\langle\nabla b(\eta_2),x\rangle]\,\phi^{\epsilon}_{z_\xi}(x)U^2(x-\xi)\,dx\\
&= \frac{\epsilon^2\omega}{2}b(0)\int_{\mathbb{R}^3}\phi^{\epsilon}_{z_\xi}(x)U^2(x-\xi)\,dx+\frac{\epsilon^3\omega}{2}\int_{\mathbb{R}^3}\langle\nabla b(\eta_2),x\rangle\phi^{\epsilon}_{z_\xi}(x)U^2(x-\xi)\,dx\\
&= \frac{\epsilon^2\omega}{2}b(0)\int_{\mathbb{R}^3}\phi^{\epsilon}_{z_\xi}(x+\xi)U^2(x)\,dx+\frac{\epsilon^3\omega}{2}\int_{\mathbb{R}^3}\langle\nabla b(\eta_2),x+\xi\rangle\phi^{\epsilon}_{z_\xi}(x+\xi)U^2(x)\,dx
\end{aligned}
$$

with $\eta_2\in B(0,\epsilon|x|)$.
We observe that $\phi^{\epsilon}_{z_\xi}(x+\xi)$ solves

$$-\Delta\phi^{\epsilon}_{z_\xi}(x+\xi)+b(\epsilon(x+\xi))U^2(x)\phi^{\epsilon}_{z_\xi}(x+\xi)=\omega b(\epsilon(x+\xi))U^2(x).$$

We set $\phi^{\epsilon,\xi}_U:=\phi^{\epsilon}_{z_\xi}(x+\xi)$. So $\phi^{\epsilon,\xi}_U$ is the solution of

$$-\Delta\phi^{\epsilon,\xi}_U+b(\epsilon(x+\xi))U^2(x)\phi^{\epsilon,\xi}_U=\omega b(\epsilon(x+\xi))U^2. \tag{5.22}$$

Now, since ∇b is bounded, by using (4.11)

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3}\langle\nabla b(\eta_2),x+\xi\rangle\phi^{\epsilon,\xi}_U U^2(x)\,dx\right| &\le C\int_{\mathbb{R}^3}|x+\xi|\cdot\phi^{\epsilon,\xi}_U U^2(x)\,dx\\
&\le C|\phi^{\epsilon,\xi}_U|_6\cdot\left(\int_{\mathbb{R}^3}|x+\xi|^{6/5}U^{12/5}(x)\,dx\right)^{5/6}\\
&\le C_{\bar\xi}\|\phi^{\epsilon,\xi}_U\|_{D^{1,2}}\le C_{b,\omega,\bar\xi,U}
\end{aligned}
$$

provided $|\xi|\le\bar\xi$. Hence

$$\frac{\epsilon^3\omega}{2}\int_{\mathbb{R}^3}\langle\nabla b(\eta_2),x+\xi\rangle\phi^{\epsilon}_{z_\xi}(x+\xi)U^2(x)\,dx=o(\epsilon^2).$$

Let $\phi^{b(0)}_U$ the solution of

$$-\Delta\phi^{b(0)}_U+b(0)U^2\phi^{b(0)}_U=\omega b(0)U^2 \tag{5.23}$$

and we set $H:=\phi^{\epsilon,\xi}_U-\phi^{b(0)}_U$.

We remark that $\phi^{b(0)}_U$ exists and it is unique.

Claim:

$$\int_{\mathbb{R}^3}HU^2(x)\,dx=o(1).$$

If the claim holds then

$$
\begin{aligned}
(C) &= \frac{\epsilon^2\omega}{2}b(0)\int_{\mathbb{R}^3}\phi^{\epsilon,\xi}_U U^2(x)\,dx+o(\epsilon^2)=\frac{\epsilon^2\omega}{2}b(0)\int_{\mathbb{R}^3}\phi^{b(0)}_U U^2(x)\,dx+o(\epsilon^2)\\
&= \epsilon^2C_4+o(\epsilon^2) \qquad (5.24)
\end{aligned}
$$

where

$$C_4=\frac{\omega}{2}b(0)\int_{\mathbb{R}^3}\phi^{b(0)}_U U^2(x)\,dx.$$

Now we have to prove the claim.
Since $\phi_U^{\epsilon,\xi}$ solves (5.22) and $\phi_U^{b(0)}$ solves (5.23) then H solves

$$-\Delta H + b(0)U^2 H = \omega(b(\epsilon(x+\xi)) - b(0))U^2 - (b(\epsilon(x+\xi)) - b(0))U^2\phi_U^{\epsilon,\xi}. \quad (5.25)$$

Now multiplying (5.23) by H and integrating on $\mathbb{R}^3$ we find

$$b(0)\omega \int_{\mathbb{R}^3} HU^2(x)\,dx = \int_{\mathbb{R}^3} \left(\nabla\phi_U^{b(0)}\nabla H + b(0)U^2\phi_U^{b(0)}H\right)\,dx \quad (5.26)$$

and since H is the solution of (5.25), if we multiply by $\phi_U^{b(0)}$ the equation (5.25) and we integrate over $\mathbb{R}^3$, we find

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\nabla\phi_U^{b(0)}\nabla H + b(0)U^2\phi_U^{b(0)}H\right)\,dx = \\
&= \omega\int_{\mathbb{R}^3}(b(\epsilon(x+\xi)) - b(0))U^2\phi_U^{b(0)}\,dx - \int_{\mathbb{R}^3}(b(\epsilon(x+\xi)) - b(0))U^2\phi_U^{\epsilon,\xi}\phi_U^{b(0)}\,dx \\
&= +\epsilon\cdot\omega\underbrace{\int_{\mathbb{R}^3}\langle\nabla b(\eta_3), x+\xi\rangle U^2\phi_U^{b(0)}\,dx}_{(I)} - \epsilon\cdot\underbrace{\int_{\mathbb{R}^3}\langle\nabla b(\eta_4), x+\xi\rangle U^2\phi_U^{\epsilon,\xi}\phi_U^{b(0)}\,dx}_{(II)}
\end{aligned}$$

with $\eta_3, \eta_4 \in B(0, \epsilon|x|)$. Since ∇b is bounded (by using also (4.11))

$$\begin{aligned}
|(I)| &\leq C\int_{\mathbb{R}^3}|x+\xi|\cdot\phi_U^{b(0)}U^2\,dx \leq C\|\phi_U^{b(0)}\|_{D^{1,2}}\left(\int_{\mathbb{R}^3}|x+\xi|^{6/5}U^{12/5}\,dx\right)^{5/6} \\
&\leq C_{\omega,U,b(0),\bar{\xi}}
\end{aligned}$$

provided $|\xi| \leq \bar{\xi}$. Moreover

$$\begin{aligned}
|(II)| &\leq C\int_{\mathbb{R}^3}|x+\xi|\cdot\phi_U^{b(0)}\phi_U^{\epsilon,\xi}U^2\,dx \leq C\|\phi_U^{b(0)}\|_{D^{1,2}}\|\phi_U^{\epsilon,\xi}\|_{D^{1,2}}\left(\int_{\mathbb{R}^3}|x+\xi|^{3/2}U^3\,dx\right)^{2/3} \\
&\leq C_{\omega,U,b(0),b,\bar{\xi}}.
\end{aligned}$$

Hence

$$\epsilon\cdot\omega\int_{\mathbb{R}^3}\langle\nabla b(\eta_3), x+\xi\rangle U^2\phi_U^{b(0)}\,dx = \epsilon\cdot\int_{\mathbb{R}^3}\langle\nabla b(\eta_4), x+\xi\rangle U^2\phi_U^{\epsilon,\xi}\phi_U^{b(0)}\,dx = o(1).$$

At the end, recalling (5.26), the claim holds.
Putting together (5.20)-(5.21)-(5.24) and setting $C_0 = I_0(U)$ and $C_2 = \widetilde{C}_2 - C_3 + C_4$ (5.19) follows. □

From (5.18) and from (5.19) the reduced functional Γ_ϵ has the following expansion

$$\Gamma_\epsilon(\xi) = C_0 + \epsilon^2 F_1(\xi) + o(\epsilon^2) \quad (5.27)$$

with $F_1(\xi)$ defined in Lemma 5.19.

5.2.2 Expansion of Γ_ϵ in the assumptions of Theorem 1.2

If (a1)-(a2')-(b1)-(b2) hold then $w(\epsilon, z_\xi)$ is a solution of the auxiliary equation which satisfies (5.16). Hence from (5.17)

$$\Gamma_\epsilon(\xi) = I_\epsilon(z_\xi) + O(\epsilon^s) \tag{5.28}$$

where $s = \min\{m, n+2\}$. So we have to find an expansion of $I_\epsilon(z_\xi)$.

Lemma 5.5 *For any $|\xi| \le \bar{\xi}$ and ϵ sufficiently small we have*

$$I_\epsilon(z_\xi) = C_0 + \epsilon^s F_2(\xi) + o(\epsilon^s), \qquad s = \min\{m, n+2\} \tag{5.29}$$

where $C_0 = I_0(U)$,

$$F_2(\xi) = \begin{cases} f(\xi) & \text{if } s = m < n+2 \\ -g(\xi) & \text{if } s = n+2 < m \\ f(\xi) - g(\xi) & \text{if } s = m = n+2 \end{cases}$$

with

$$f(\xi) = \sum_{i=1}^{3} \sum_{\alpha=0}^{m} C_{\alpha,i} a_i \cdot \xi_i^{m-\alpha}, \qquad C_{\alpha,i} = \frac{1}{2m!} \int_{\mathbb{R}^3} x_i^\alpha U^2(x)\, dx$$

and

$$g(\xi) = \sum_{i=1}^{3} \sum_{\beta=0}^{n} \widetilde{C}_{\beta,i} b_i \cdot \xi_i^{n-\beta}, \qquad \widetilde{C}_{\beta,i} = \frac{\omega^2}{2n!} \int_{\mathbb{R}^3} x_i^\beta U^2(x)\, dx$$

Proof As in the proof of Lemma 5.19 we have to evaluate separately (A)-(B)-(C) taking account that now (a1)-(a2')-(b1)-(b2) hold. Here $s = \min\{m, n+2\} < \infty$. If $m = +\infty$ then

$$a(\epsilon x) - 1 = o(\epsilon^s)|x|^s.$$

Therefore

$$(A) = o(\epsilon^s) \int_{\mathbb{R}^3} |x|^s U^2(x-\xi)\, dx = o(\epsilon^s) \int_{\mathbb{R}^3} |x+\xi|^s U^2(x)\, dx = o(\epsilon^s) \tag{5.30}$$

provided $|\xi| \le \bar{\xi}$.
If $m < +\infty$ then

$$a(\epsilon x) - 1 = \frac{\epsilon^m}{m!} D^m a(0)[x] + o(\epsilon^m)|x|^m = \frac{\epsilon^m}{m!} \sum_{i=1}^{3} a_i x_i^m + o(\epsilon^m)|x|^m$$

and hence (since $|\xi| \le \bar{\xi}$)

$$\begin{aligned} (A) &= \frac{\epsilon^m}{2m!} \sum_{i=1}^{3} \int_{\mathbb{R}^3} a_i x_i^m U^2(x-\xi)\, dx + o(\epsilon^m) \int_{\mathbb{R}^3} |x|^m U^2(x-\xi)\, dx \\ &= \frac{\epsilon^m}{2m!} \sum_{i=1}^{3} \int_{\mathbb{R}^3} a_i (x_i+\xi_i)^m U^2(x)\, dx + o(\epsilon^m) \int_{\mathbb{R}^3} |x+\xi|^m U^2(x)\, dx \\ &= \frac{\epsilon^m}{2m!} \sum_{i=1}^{3} \sum_{\alpha=0}^{m} a_i \xi_i^{m-\alpha} \int_{\mathbb{R}^3} x_i^\alpha U^2(x)\, dx + o(\epsilon^m) \\ &= \epsilon^m f(\xi) + o(\epsilon^m) \end{aligned} \tag{5.31}$$

where

$$f(\xi) = \sum_{i=1}^{3} \sum_{\alpha=0}^{m} C_{\alpha,i} a_i \xi_i^{m-\alpha}, \qquad C_{\alpha,i} = \frac{1}{2m!} \int_{\mathbb{R}^3} x_i^{\alpha} U^2(x)\, dx.$$

If $n = +\infty$ then

$$b(\epsilon x) = o(\epsilon^{s-2})|x|^{s-2}.$$

Hence

$$(B) = \frac{\omega^2}{2} o(\epsilon^s) \int_{\mathbb{R}^3} |x|^{s-2} U^2(x-\xi)\, dx = \frac{\omega^2}{2} o(\epsilon^s) \int_{\mathbb{R}^3} |x+\xi|^{s-2} U^2(x)\, dx = o(\epsilon^s). \tag{5.32}$$

and

$$\begin{aligned} (C) &= \frac{\omega}{2} o(\epsilon^s) \int_{\mathbb{R}^3} |x|^{s-2} \phi^{\epsilon}_{z_\xi}(x) U^2(x-\xi)\, dx = \frac{\omega}{2} o(\epsilon^s) \int_{\mathbb{R}^3} |x+\xi|^{s-2} \phi^{\epsilon}_{z_\xi}(x+\xi) U^2(x)\, dx \\ &= \frac{\omega}{2} o(\epsilon^s) \int_{\mathbb{R}^3} |x+\xi|^{s-2} \phi_U^{\epsilon,\xi}(x) U^2(x)\, dx = o(\epsilon^s) \end{aligned} \tag{5.33}$$

provided $|\xi| \le \bar{\xi}$, where we set $\phi_U^{\epsilon,\xi} := \phi^{\epsilon}_{z_\xi}(x+\xi)$ as done in the proof of Lemma 5.19.

If $n < +\infty$ then

$$b(\epsilon x) = \frac{\epsilon^n}{n!} D^n b(0)[x] + o(\epsilon^n)|x|^n = \frac{\epsilon^n}{n!} \sum_{i=1}^{3} b_i x_i^n + o(\epsilon^n)|x|^n \tag{5.34}$$

and hence (since $|\xi| \le \bar{\xi}$)

$$\begin{aligned} (B) &= \omega^2 \frac{\epsilon^{n+2}}{2n!} \sum_{i=1}^{3} \int_{\mathbb{R}^3} b_i x_i^n U^2(x-\xi)\, dx + o(\epsilon^{n+2}) \int_{\mathbb{R}^3} |x|^n U^2(x-\xi)\, dx \\ &= \omega^2 \frac{\epsilon^{n+2}}{2n!} \sum_{i=1}^{3} \int_{\mathbb{R}^3} b_i (x_i+\xi_i)^n U^2(x)\, dx + o(\epsilon^{n+2}) \int_{\mathbb{R}^3} |x+\xi|^n U^2(x)\, dx \\ &= \omega^2 \frac{\epsilon^{n+2}}{2n!} \sum_{i=1}^{3} \sum_{\beta=0}^{n} b_i \xi_i^{n-\beta} \int_{\mathbb{R}^3} x_i^{\beta} U^2(x)\, dx + o(\epsilon^{n+2}) \\ &= \epsilon^{n+2} g(\xi) + o(\epsilon^{n+2}) \end{aligned} \tag{5.35}$$

where

$$g(\xi) = \sum_{i=1}^{3} \sum_{\beta=0}^{m} \widetilde{C}_{\beta,i} b_i \xi_i^{n-\beta}, \qquad \widetilde{C}_{\beta,i} = \frac{\omega^2}{2n!} \int_{\mathbb{R}^3} x_i^{\beta} U^2(x)\, dx.$$

Claim: $(C) = o(\epsilon^{n+2})$, namely

$$\underbrace{\int_{\mathbb{R}^3} b(\epsilon x) \phi^{\epsilon}_{z_\xi}(x) U^2(x-\xi)\, dx}_{(C')} = o(\epsilon^n).$$

Indeed:

$$\begin{aligned}(C') &= \int_{\mathbb{R}^3} b(\epsilon(x+\xi))\phi_U^{\epsilon,\xi} U^2(x)\,dx \\ &= \epsilon^n \sum_{i=1}^3 b_i \int_{\mathbb{R}^3} (x_i+\xi_i)^n \cdot \phi_U^{\epsilon,\xi} U^2(x)\,dx + o(\epsilon^n) \int_{\mathbb{R}^3} |x+\xi|^n \cdot \phi_U^{\epsilon,\xi} U^2(x)\,dx \\ &= \epsilon^n \sum_{i=1}^3 b_i \int_{\mathbb{R}^3} (x_i+\xi_i)^n \cdot \phi_U^{\epsilon,\xi} U^2(x)\,dx + o(\epsilon^n)\end{aligned}$$

since

$$\left| \int_{\mathbb{R}^3} |x+\xi|^n \cdot \phi_U^{\epsilon,\xi} U^2(x)\,dx \right| \leq C_{\bar{\xi}}$$

provided $|\xi| \leq \bar{\xi}$. Let now $\bar{\phi}$ the solution in $D^{1,2}(\mathbb{R}^3)$ of

$$-\Delta\bar{\phi} + b(\epsilon(x+\xi))U^2\bar{\phi} = \omega \sum_{i=1}^3 b_i(x_i+\xi_i)^n U^2 \tag{5.36}$$

and we set $H := \phi_U^{\epsilon,\xi} - \epsilon^n\bar{\phi}$. The function H solves

$$-\Delta H + b(\epsilon(x+\xi))U^2 H = \omega \left(b(\epsilon(x+\xi)) - \epsilon^n \sum_{i=1}^3 b_i(x_i+\xi_i)^n \right) U^2. \tag{5.37}$$

If we multiply (5.36) by H and we integrate over $\mathbb{R}^3$ we have

$$\int_{\mathbb{R}^3} \left[\nabla\bar{\phi}\nabla H + b(\epsilon(x+\xi))U^2\bar{\phi}H \right] dx = \omega \sum_{i=1}^3 b_i \int_{\mathbb{R}^3} (x_i+\xi_i)^n U^2 H\,dx.$$

Instead, if we multiply (5.37) by $\bar{\phi}$ and we integrate over $\mathbb{R}^3$ we find (by using (5.34))

$$\begin{aligned}\int_{\mathbb{R}^3} \left[\nabla\bar{\phi}\nabla H + b(\epsilon(x+\xi))U^2\bar{\phi}H \right] dx &= \omega \int_{\mathbb{R}^3} \left(b(\epsilon(x+\xi)) - \epsilon^n \sum_{i=1}^3 b_i(x_i+\xi_i)^n \right) U^2\bar{\phi}\,dx \\ &= o(\epsilon^n)\omega \int_{\mathbb{R}^3} |x+\xi|^n U^2\bar{\phi}\,dx = o(\epsilon^n)\end{aligned}$$

provided $|\xi| \leq \bar{\xi}$. Hence

$$\omega \sum_{i=1}^3 b_i \int_{\mathbb{R}^3} (x_i+\xi_i)^n U^2 H\,dx = o(\epsilon^n),$$

namely

$$\sum_{i=1}^3 b_i \int_{\mathbb{R}^3} (x_i+\xi_i)^n U^2 \phi_U^{\epsilon,\xi}\,dx = \epsilon^n \sum_{i=1}^3 b_i \int_{\mathbb{R}^3} (x_i+\xi_i)^n U^2\bar{\phi}\,dx + o(\epsilon^n).$$

Hence

$$(C') = \epsilon^{2n} \sum_{i=1}^3 b_i \int_{\mathbb{R}^3} (x_i+\xi_i)^n U^2\bar{\phi}\,dx + o(\epsilon^n)$$

and since for $|\xi| \leq \bar{\xi}$

$$\left|\sum_{i=1}^{3} b_i \int_{\mathbb{R}^3} (x_i + \xi_i)^n U^2 \bar{\phi}\, dx\right| \leq \left|\sum_{i=1}^{3} b_i \int_{\mathbb{R}^3} |x + \xi|^n U^2 \bar{\phi}\, dx\right| \leq C_{\bar{\xi}}$$

we find

$$(C') = o(\epsilon^n)$$

and the claim holds.
Putting together (A)-(B)-(C) we find (5.29) □

Remark 5.5.1 Repeating the same arguments of Lemma 3.1 one can prove, since $|\xi| \leq \bar{\xi}$, the existence and uniqueness of the solution of (5.36).

From (5.28) and from (5.29) the reduced functional Γ_ϵ has the following expansion

$$\Gamma_\epsilon(\xi) = C_0 + \epsilon^2 F_2(\xi) + o(\epsilon^2) \tag{5.38}$$

with $F_2(\xi)$ defined in Lemma 5.5.

5.3 Proofs of Theorem 1.1 and of Theorem 1.2

The following Lemma is nothing but Lemma 4.3 of [1] and provides a sufficient condition to have a non-degenerate minimum (or maximum) for the reduced functional.

Lemma 5.6 *Assume that the reduced functional Γ_ϵ has the following expansion*

$$\Gamma_\epsilon(\xi) = C_0 + \epsilon^\beta F(\xi) + o(\epsilon^\beta), \qquad |\xi| \leq \bar{\xi} \tag{5.39}$$

and that $\xi = 0$ is a non-degenerate minimum (or maximum) for F. Then Γ_ϵ has a minimum (or maximum) in some ξ_ϵ such that $\xi_\epsilon \to 0$ as $\epsilon \to 0$.

We are now ready to prove our main results.

Proof [Theorem 1.1, Theorem 1.2]
From the previous section it follows easily that $\xi = 0$ is a critical point for F_1 and F_2. Moreover, $\xi = 0$ is a local minimum (or maximum) for F_1 and F_2. In fact

$$D^2 F_1(0) = C_1 \cdot D^2 a(0)$$

and since $x_0 = 0$ is a minimum (or maximum) non degenerate for the function $a(x)$ then $D^2 F_1(0)$ is positive- (or negative-) definite.
Instead, if $s = m < n + 2$ then

$$D^2 F_2(0) = \begin{pmatrix} C_{2,1} a_1 & 0 & 0 \\ 0 & C_{2,2} a_2 & 0 \\ 0 & 0 & C_{2,3} a_3 \end{pmatrix}$$

and since $C_{2,i} > 0$ and $a_i > 0$ (or $a_i < 0$) then $D^2F_2(0)$ is positive- (or negative-) definite.
If $s = n + 2 < m$ then

$$D^2F_2(0) = \begin{pmatrix} -\widetilde{C}_{2,1}b_1 & 0 & 0 \\ 0 & -\widetilde{C}_{2,2}b_2 & 0 \\ 0 & 0 & -\widetilde{C}_{2,3}b_3 \end{pmatrix}$$

and, as before, we conclude that $D^2F_2(0)$ is positive- (or negative-) definite.
Finally, if $s = m = n + 2$ then

$$D^2F_2(0) = \begin{pmatrix} C_{2,1}a_1 - \widetilde{C}_{2,1}b_1 & 0 & 0 \\ 0 & C_{2,2}a_2 - \widetilde{C}_{2,2}b_2 & 0 \\ 0 & 0 & C_{2,3}a_3 - \widetilde{C}_{2,3}b_3 \end{pmatrix}$$

and, by the assumption (1.7), $D^2F_2(0)$ is positive- (or negative-) definite.
So, from Lemma 5.6, with $F = F_1$ and $\beta = 2$ for Theorem 1.1 and $F = F_2$ and $\beta = s$ for Theorem 1.2, it follows that Γ_ϵ has a minimum (or maximum) at ξ_ϵ such that $\xi_\epsilon \to 0$ as $\epsilon \to 0$.
Hence $u_\epsilon = z_{\xi_\epsilon} + w(\epsilon, \xi_\epsilon)$ is a critical point of I_ϵ, hence a solution of (4.3).
Recalling the change of variable we have

$$v_\epsilon(x) = \frac{1}{\epsilon} U\left(\frac{x}{\epsilon} - \xi_\epsilon\right) + \frac{1}{\epsilon} w(\epsilon, \xi_\epsilon) \sim \frac{1}{\epsilon} U\left(\frac{x}{\epsilon} - \xi_\epsilon\right)$$

since $\epsilon^{-1}w(\epsilon, \xi_\epsilon)$ is small for ϵ small (this means that $\|\epsilon^{-1}w(\epsilon, \xi_\epsilon\| \to 0$ as $\epsilon \to 0$ and this follows from (5.15) or (5.16)).
Since $U(\frac{x}{\epsilon} - \xi_\epsilon)$ has an exponential decay for $|\frac{x}{\epsilon} - \xi_\epsilon|$ large (and this means for ϵ small) then $\epsilon^{-1}U(\frac{x}{\epsilon} - \xi_\epsilon)$ decays exponentially for ϵ small. Hence v_ϵ is a solution of (1.6) which concentrates near $x_0 = 0$. □

Remark 5.6.1 If $n = 2$ then we can assume less on the function $b(x)$.
Indeed

$$b(\epsilon x) = \frac{\epsilon^2}{2}\langle D^2b(0)x, x\rangle + o(\epsilon^2)|x|^2$$

and so, in Lemma 5.5, the term (B) becomes (since $|\xi| \le \bar{\xi}$)

$$\begin{aligned} (B) &= \frac{\epsilon^4\omega^2}{4}\int_{\mathbb{R}^3}\langle D^2b(0)(x+\xi), x+\xi\rangle U^2(x)\,dx + o(\epsilon^4)\int_{\mathbb{R}^3}|x+\xi|^2U^2(x)\,dx \\ &= \frac{\epsilon^4\omega^2}{4}\langle D^2b(0)\xi,\xi\rangle\int_{\mathbb{R}^3}U^2(x)\,dx + o(\epsilon^4) \\ &= \epsilon^4\left[\bar{C}_1\langle D^2b(0)\xi,\xi\rangle + \bar{C}_2\right] + o(\epsilon^4) \end{aligned}$$

where

$$\bar{C}_1 := \frac{\omega^2}{4}\int_{\mathbb{R}}U^2(x)\,dx; \qquad \bar{C}_2 := \frac{\omega^2}{4}\int_{\mathbb{R}^3}\langle D^2b(0)x, x\rangle U^2(x)\,dx.$$

Moreover, as done in Lemma 5.5, one can prove that $(C) = o(\epsilon^4)$. Hence the function $g(\xi)$ becomes

$$g(\xi) = \left[\bar{C}_1\langle D^2b(0)\xi,\xi\rangle + \bar{C}_2\right]$$

and the Theorem 1.2 follows assuming only that b has a local minimum or maximum (non-degenerate) at $x_0 = 0$.

6 Necessary Condition

In this section we show that concentration necessarily occurs at stationary points of the function a.
Let $z = z_\xi \in \mathcal{Z}_{\bar{\xi}}$ and $w = w(\epsilon, \xi)$ the corresponding solution of the auxiliary equation.
Setting

$$T(\epsilon, u) = I'_\epsilon(u),$$

we obtain $T(0, z) = I'_0(z) = 0$ for all $z \in \mathcal{Z}_{\bar{\xi}}$, and we can consider $z \in \mathcal{Z}_{\bar{\xi}}$ as a bifurcation parameter. We say that $z \in \mathcal{Z}_{\bar{\xi}}$ is a bifurcation point for T if there exists $(\epsilon_n, u_n) \in \mathbb{R} \times H^1(\mathbb{R}^3)$ such that $T(\epsilon_n, u_n) = 0$, $\epsilon_n \neq 0$, $\epsilon_n \to 0$ and $u_n \to z$. It is well known (see [1, Section 7]) that a necessary condition for z to be a bifurcation point for T is that for all $\eta \in \mathbb{R}$ exists $v \in H^1(\mathbb{R}^3)$ such that

$$D_\epsilon I'_0(z)[h][\eta] = I''_0(z)[h, v], \qquad \forall\ h \in H^1(\mathbb{R}^3). \tag{6.1}$$

Theorem 6.1 *Let be (a1)-(a2)-(b1) hold and $p \in (1, 5)$. Suppose that v_ϵ is a solution of (1.6) such that the limit profile of v_ϵ is $\epsilon^{-1}U_\alpha(\frac{x-x_0}{\epsilon})$ with $\alpha^2 = a(x_0)$. Then $\nabla a(x_0) = 0$.*

Proof As before, assume $x_0 = 0$ and $\alpha^2 = a(x_0) = 1$. First we make the change $u_\epsilon(x) = \epsilon v(\epsilon x)$ and so the hypothesis on the limit profile of v_ϵ implies that $u_\epsilon(x) \to U(x)$. Define $z = U(x)$ then z is a bifurcation parameter and so we can use (6.1). By a direct calculation we find for all $h \in H^1(\mathbb{R}^3)$

$$D_\epsilon I'_0(z)[h][\eta] = \eta \int_{\mathbb{R}^3} \langle \nabla a(0), x \rangle z h \, dx;$$

$$I''_0(z)[h, v] = \int_{\mathbb{R}^3} [\nabla h \nabla v + hv - p|z|^{p-1} hv] \, dx.$$

Let $h \in T_z \mathcal{Z}_{\bar{\xi}}$, since such a h is a solution of the linearized equation in z, one has that

$$I''_0(z)[h, v] = \int_{\mathbb{R}^3} [\nabla h \nabla v + hv - p|z|^{p-1} hv] \, dx = 0.$$

So, from (6.1) and (5.12) we deduce that

$$-\int_{\mathbb{R}^3} \langle \nabla a(0), x \rangle U(x) \frac{\partial U(x)}{\partial x_k} dx = 0 \qquad \forall\ k = 1, 2, 3.$$

Therefore, we conclude that $\nabla a(0) = 0$. □

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