

REPRESENTATIONS OF THE HEISENBERG ALGEBRA AND MODULI SPACES OF FRAMED SHEAVES

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ABSTRACT. Let $\mathcal{M}_{X,D}(r, c, n)$ be the moduli spaces of torsion free sheaves on a complex smooth connected projective surface X , framed along a smooth connected genus zero curve D . This paper gives a ‘geometrical’ construction of an action of a Heisenberg algebra on the homology of $\mathcal{M}_{X,D}(r, c, n)$ (more precisely, on the direct sum $\bigoplus_n \mathbb{H}_*(\mathcal{M}_{X,D}(r, c, n))$) using correspondences. This result generalizes Nakajima’s construction for the Hilbert schemes of points.

1. INTRODUCTION

Let X be a complex smooth connected quasi-projective surface and $X^{[n]}$ the Hilbert scheme of length n zero-dimensional subschemes of X . In [17] Nakajima builds a representation of the Heisenberg-Clifford super-algebra on the direct sum of the homology groups $\bigoplus_{n \geq 0} \mathbb{H}_*(X^{[n]})$, showing that the generating function of the Poincaré polynomials, found by Göttsche and Soergel in [11], coincides with the character formula of the representation. Nakajima’s result gives a supporting evidence to the ‘S-duality conjecture’ proposed by Vafa and Witten in [21].

The Hilbert scheme $X^{[n]}$ can be viewed as the moduli space of rank one sheaves over X with trivial determinant and second Chern class equals to n . Let X be a smooth projective surface; in [3] Baranovsky generalizes Nakajima’s result to the moduli spaces $\mathcal{M}_X(r, L, n)$ of Gieseker-stable torsion free sheaves on X of rank r , determinant L and second Chern class n .

Let D be a big and nef curve in X . In [5] Bruzzo and Markushevich built a fine moduli space $\mathcal{M}_{X,D}(r, c, n)$ for torsion free sheaves on X with invariants (r, c, n) , framed along the divisor D . The Hilbert scheme of points $(X \setminus D)^{[n]}$ can be viewed as the rank one case of this moduli space.

Moreover moduli spaces of framed sheaves are studied because they provide a desingularization of the moduli spaces of ideal instantons, so their equivariant cohomology under suitable toric actions is relevant to the computation of partition functions in topological quantum field theory (e.g. see [19], [4], [18] and [7]).

The aim of this work is to generalize Nakajima’s construction to these moduli spaces: we explicitly construct *Nakajima’s operators* on the direct sum of the homology groups

$\bigoplus_n H_*(\mathcal{M}_{X,D}(r, c, n))$ and show that they satisfy the Heisenberg commutation relations and hence we get a representation of the Heisenberg algebra generated by *Nakajima's operators* on this space.

Using this construction one can expect to generalize to higher rank case Carlsson and Okunkov result ([8]): they use Nakajima's operators for Hilbert schemes $X^{[n]}$ to define a class of vertex operators, whose trace gives the Nekrasov's partition function in the case we have a toric action on X . This could be relevant to provide a mathematical motivation for the conjecture about a correspondence between the Nekrasov's partition function for supersymmetric Yang-Mills topological field theories and the conformal blocks of a 2-dimensional conformal field theory (e.g. see [2]).

The article is structured in the following way: in section 2 we introduce framed sheaves over a smooth connected projective surface and we outline how one can construct moduli spaces of these; in section 3 we construct the *Nakajima operators*, which will lead us to the representation we are seeking; in section 4 we state and prove a generalization of Nakajima's theorem for the case of moduli spaces of framed sheaves, postponing to the last two sections a proof of the more technical propositions.

Technical remark. In this paper we shall assume that the framing divisor D is a smooth connected genus zero curve and this forces X to be a rational surface; we assume this hypothesis because we need the moduli spaces of framed sheaves to be smooth varieties and for the technical lemma 25. A priori, one can replace this assumption by '*let X and D be such that the moduli spaces $\mathcal{M}_{X,D}(r, c, n)$ are smooth*' and one can construct the operators in the same way, but we do not know how to prove the Heisenberg commutator relations in this general case. Anyway the main explicit examples of framed sheaves and their moduli (e.g. see chapter 2 in [17], [1], [20]) satisfy our hypothesis.

Conventions and notation. All schemes we are dealing with are of finite type over \mathbb{C} , by 'variety' we mean a reduced separated scheme; a 'sheaf' is always coherent.

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2. FRAMED SHEAVES AND THEIR MODULI

In this section we introduce the notion of framed sheaf and we give a construction of moduli spaces of these objects.

Generalities. Let X be a smooth connected projective surface. Fix a smooth connected curve D in X .

Definition 1. A *framed sheaf of rank r* on X with framing divisor D is a pair $\mathcal{E} = (E, \phi^E)$ where E is a torsion free sheaf of rank r on X , ϕ^E is a morphism $\phi^E : E \rightarrow O_D^{\oplus r}$ such that $\phi^E|_D$ is an isomorphism. If (E, ϕ^E) and (F, ϕ^F) are two framed sheaves of the same rank r , we define a *morphism of framed sheaves* as a morphism $\alpha : E \rightarrow F$ such that for some $\lambda \in \mathbb{C}$ we have a commutative diagram

$$(1) \quad \begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ \downarrow \phi^E & & \downarrow \phi^F \\ O_D^{\oplus r} & \xrightarrow{\cdot \lambda} & O_D^{\oplus r} \end{array}$$

Clearly the set of morphisms of framed sheaves is a linear subspace of the space of morphisms between the corresponding underlying sheaves.

Now let (E, ϕ^E) be a framed sheaf of rank r on X . Because of the framing, the stalks at the points on the divisor D are free of rank r , so E is locally free in a neighborhood of D , hence in this neighborhood the sheaves E and $E^{\vee\vee}$ are isomorphic. Thus we have a natural framing for $E^{\vee\vee}$. Moreover the support of A is disjoint from D . In the following we denote by $\mathcal{E}^{\vee\vee}$ the framed sheaf $(E^{\vee\vee}, \phi^E)$. Note that the inclusion of E in $E^{\vee\vee}$ induces a morphism between the corresponding framed sheaves.

Construction of the moduli space. We now come to the construction of the moduli space of framed sheaves on a surface.

Definition 2. Let S be a scheme. A *family of rank r framed sheaves parametrized by S* is a pair (G, ϕ^G) where G is a torsion free sheaf of rank r on $X \times S$ flat over S , ϕ^G is a morphism $\phi^G : G \rightarrow p_X^* O_D^{\oplus r}$ such that $\phi^G|_D : G|_{D \times S} \rightarrow p_X^* O_D^{\oplus r}$ is an isomorphism.

Remark that a family of rank r framed sheaves over X parametrized by S with framing divisor D is in particular a rank r framed sheaf over the product $X \times S$ with framing divisor $D \times S$.

We say that two families \mathcal{G} and \mathcal{G}' of rank r framed sheaves over X parametrized by S are isomorphic if they are isomorphic as rank r framed sheaves over $X \times S$.

Now fix invariants $r, n \in \mathbb{Z}$ with $r > 0$ and $c \in \text{NS}(X)$; we can define the functor

$$\underline{\mathcal{M}}_{X,D}(r, c, n) : (\text{Schemes})^\circ \longrightarrow (\text{Sets})$$

that associates to every scheme S the set of isomorphism classes of families (G, ϕ^G) of rank r framed sheaves parametrized by S such that the fibres G_s have rank r and Chern classes c and n , while to every morphism of schemes $f : T \rightarrow S$ associates the pull-back f^* that sends families parametrized by S in families parametrized by T .

Remark 3. A necessary condition for this functor to be nonempty is $\int_D c = 0$, indeed in general $\int_D c_1(E) = c_1(E|_D)$ and if E is trivial along D , then $c_1(E|_D) = c_1(O_D^{\oplus r}) = 0$.

In [5] Bruzzo and Markushevich give (in a more general setting) a proof of the representability of the functor $\underline{\mathcal{M}}_{X,D}(r, c, n)$; using their result we have the following:

Theorem 4. *Let X be a smooth connected projective surface and D a smooth connected curve in X which is a big and nef divisor. Then there exists a quasi-projective separated scheme $\mathcal{M}_{X,D}(r, c, n)$ that represents the functor $\underline{\mathcal{M}}_{X,D}(r, c, n)$.*

Note that $\mathcal{M}_{X,D}(r, c, n)$ is a *fine* moduli space for framed sheaves on X with framing divisor D and topological invariants r, c and n . The adjective ‘fine’ means the existence of a *universal* framed sheaf $\bar{\mathcal{E}} = (\bar{E}, \phi^{\bar{E}})$, i.e. a family of framed sheaves parametrized by $\mathcal{M}_{X,D}(r, c, n)$ with the following *universal* property: for any family (G, ϕ^G) of framed sheaves parametrized by S there exists a unique morphism $g : S \rightarrow \mathcal{M}_{X,D}(r, c, n)$ and an isomorphism $\alpha : G \rightarrow (\text{id} \times g)^*(\bar{E})$ such that $\phi^G|_{D \times S} = ((\text{id} \times g)^* \phi^{\bar{E}} \circ \alpha)|_{D \times S}$.

From now on we suppose that D is a smooth connected curve in X that satisfies the following conditions:

- (a) D is a big and nef divisor,
- (b) $D \cong \mathbb{C}P^1$.

Remark 5. Since D is isomorphic to the complex projective line and it has positive self intersection, X is a rational connected surface, hence it is a rational surface. Moreover the arithmetic genera of D and X are zero and $H_i(X, \mathbb{C}) = 0$ for $i = 1, 3$.

From these conditions follows the lemma:

Lemma 6. *Let (E, ϕ^E) and (F, ϕ^F) be framed sheaves on X . Then*

$$\text{Hom}(E, F) \cong \text{End}(\mathbb{C}^r).$$

Proof. Let f be a non trivial morphism in $\text{Hom}(E, F)$; since $E|_D$ and $F|_D$ are trivial, $f|_D$ is a constant matrix, i.e. is an element of $\text{End}(\mathbb{C}^r)$.

Consider the complete linear system $|D|$ and let Y be its base locus: we can define a morphism $\pi : X \setminus Y \rightarrow |D|$ that associates to each point x of $X \setminus Y$ the hyperplane in $|D|$ consisting of those sections vanishing at x .

Let V be the open set of divisor $C \in |D|$ such that C is a smooth curve of genus zero. Let U be the set consisting on curves $C \in V$ such that $E|_C$ and $F|_C$ are the trivial bundle. By lemma 2.3.1 in [14] and semicontinuity theorem U is open. We have that $f|_C$ is a constant matrix for all $C \in U$. So we get that f is a constant matrix in $\pi^{-1}(U)$,

that is a dense open subset of X . Thus f is constant in the whole X and the thesis is proved. \square

Now we want to study the local properties of the moduli space, in particular we want a description of its tangent space. We begin with a vanishing lemma:

Lemma 7. *Let (E, ϕ^E) be a framed sheaf on X with framing divisor D . Then*

- (a) $\text{Hom}(E, E(-D)) = 0$,
- (b) $\text{Ext}^2(E, E(-D)) = 0$.

Proof. **(a)** Let f be a non trivial morphism in $\text{Hom}(E, E(-D))$; since $E|_D$ is trivial we have that

$$\text{Hom}(E|_D, E(-D)|_D) \cong H^0(D, \mathcal{H}om(O_D^{\oplus r}, O_D^{\oplus r} \otimes O_X(-D))) \cong H^0(D, O_D(-a))^{\oplus r}$$

where $a = D.D$. Since D is a big and nef divisor, we get that a is positive, hence $H^0(D, O_D(-a)) = 0$ and therefore f is zero along D .

As before, consider the morphism $\pi : X \setminus Y \rightarrow |D|$. Let T be the set of divisor $C \in |D|$ such that $E|_C$ is a semistable sheaf on C . Note that $T \neq \emptyset$ because D belongs to T and T is open in $|D|$ because semistability is an open property. Since $p(E|_C, m) > p(E(-D)|_C, m)$ for all $C \in |D|$, by proposition 1.2.7 in [13] we get that $\text{Hom}(E|_C, E(-D)|_C) = 0$ for all $C \in T$ and therefore $f|_C = 0$. So we can use the same argument of lemma 6 and therefore f is zero in the whole X .

(b) Using Serre's duality we have

$$\text{Ext}^2(E, E(-D)) \cong \text{Hom}(E, E \otimes O_X(D) \otimes \omega_X)^\vee.$$

We have $\omega_D \cong (\omega_X \otimes O_X(D))|_D$ and $\deg \omega_D < 0$ by adjunction formula, so any element f in $\text{Hom}(E, E \otimes O_X(D) \otimes \omega_X)$ is zero when restricted to D ; in the same way, for all $C \in T$ we get $\deg O_X(K_X + D)|_C < 0$ and using the same argument as before, the thesis follows. \square

Thanks to this lemma and to theorem 4.3 in [5], we get

Theorem 8. *Let $\mathcal{E} = (E, \phi^E)$ be a framed sheaf with invariants r, c and n . Then the Zariski tangent space at the corresponding point $[\mathcal{E}]$ of $\mathcal{M}_{X,D}(r, c, n)$ is*

$$T_{[\mathcal{E}]} \mathcal{M}_{X,D}(r, c, n) \cong \text{Ext}^1(E, E(-D)).$$

Moreover $\mathcal{M}_{X,D}(r, c, n)$ is a smooth quasi-projective variety of dimension

$$\dim_{\mathbb{C}} \mathcal{M}_{X,D}(r, c, n) = 2rn - (r-1) \int_X c^2.$$

From now on we put $b := -(r-1) \int_X c^2$.

Remark 9. Note that there does not exist a general theorem about the nonemptiness of these moduli spaces. There are few known cases of nonempty moduli spaces of framed sheaves. The simplest case is given by $r = 1$, $c = 0$ and $n > 0$: the moduli space $\mathcal{M}_{X,D}(1, 0, n)$ is isomorphic to the Hilbert scheme of points $X_0^{[n]}$, where $X_0 = X \setminus D$. This space is a smooth quasi-projective variety of dimension $2n$. Another case is given by $X = \mathbb{P}^2$ and D a line in \mathbb{P}^2 (see for example [17], chapter II); in this case one shows that $\mathcal{M}_{\mathbb{P}^2,D}(r, c, n)$ is nonempty if and only if $c = 0$ and $n > 0$. A recently known case is one on the Hirzebruch surfaces: let $X = \mathbb{F}_p$ be the p -th Hirzebruch surface and D be the ‘line at infinity’, let $c = aE$, where $a \in \mathbb{Z}$, $0 \leq a < r - 1$, and E the only curve in \mathbb{F}_p with negative self-intersection, in [20] Rava proves that the moduli space $\mathcal{M}_{\mathbb{F}_p,D}(r, c, n)$ is nonempty if and only if the number $n + \frac{r-1}{2r}pa^2$ is an integer and $n + \frac{1}{2}pa(a-1) \geq 0$. Moreover he proves that when equality holds, $\mathcal{M}_{\mathbb{F}_1,D}(r, c, n) = \text{Grass}(a, r)$ and $\mathcal{M}_{\mathbb{F}_p,D}(r, c, n) = T(\text{Grass}(a, r))^{\oplus(p-1)}$ for $p > 1$, where $\text{Grass}(a, r)$ is the Grassmannian of a -planes inside \mathbb{C}^r .

In the following we assume that X, D, r, c, n are such that the moduli space $\mathcal{M}_{X,D}(r, c, n)$ is nonempty.

Tangent bundle of $\mathcal{M}_{X,D}(r, c, n)$. First of all we want to define the *Kodaira-Spencer map* for framed sheaves in terms of cocycles. Let S be a scheme.

Let p_S and p_X be the projections from $Y = X \times S$ to S and X respectively. Consider a family (G, ϕ^G) of framed sheaves parametrized by S . Let G^\bullet be a finite locally free resolution $G^\bullet \rightarrow G$.

Following chapter 10 in [13], we can define the *Atiyah class* $A(G^\bullet)$ in terms of connections: let $p_1, p_2 : Y \times Y \rightarrow Y$ be the projections to the two factors. Let \mathcal{I} be the ideal sheaf of the diagonal $\Delta \subset Y \times Y$ and let $\mathcal{O}_{2\Delta}$ denote the structure sheaf of the first infinitesimal neighbourhood of Δ .

Choose an open affine covering $\mathcal{U} = \{U_i | i \in I\}$ such that the restriction of the sequence

$$0 \longrightarrow G^q \otimes \Omega_Y \longrightarrow (p_1)_*(p_2^*G^q \otimes \mathcal{O}_{2\Delta}) \longrightarrow G^q \longrightarrow 0$$

to U_i splits for all q and i . Thus there are local connections $\nabla_i^q : G^q|_{U_i} \rightarrow G^q \otimes \Omega_Y|_{U_i}$. Note that the difference of two local connections is an \mathcal{O}_Y -linear map. Define cochains $\alpha' \in C^1(\mathcal{H}om^0(G^\bullet, G^\bullet \otimes \Omega_Y), \mathcal{U})$ and $\alpha'' \in C^0(\mathcal{H}om^1(G^\bullet, G^\bullet \otimes \Omega_Y), \mathcal{U})$ as follows:

$$\begin{aligned} \alpha'_{i_0 i_1}{}^q &= \nabla_{i_0}^q|_{U_{i_0 i_1}} - \nabla_{i_1}^q|_{U_{i_0 i_1}}, \\ \alpha''_i{}^q &= d_{G^\bullet} \circ \nabla_i^q - \nabla_i^{q+1} \circ d_{G^\bullet}, \end{aligned}$$

where d_{G^\bullet} is the differential of the complex G^\bullet . The element $\alpha = \alpha' + \alpha''$ is a cocycle in the total complex associated to the double complex $C^\bullet(\mathcal{H}om^\bullet(G^\bullet, G^\bullet \otimes \Omega_Y), \mathcal{U})$. The cohomology class of α in $\mathbb{E}xt^1(G^\bullet, G^\bullet \otimes \Omega_Y)$ is the *Atiyah class* $A(G^\bullet)$ of the sheaf G .

Now, we want to care about the morphism ϕ^G . The *connecting* morphism $\varepsilon : G^0 \rightarrow G$ induces a cohomology class $[\varepsilon \otimes \text{id}_{\Omega_Y}]$ in $\mathbb{E}xt^0(G^\bullet \otimes \Omega_Y, G \otimes \Omega_Y)$ and therefore $[\bar{\alpha}] := A(G^\bullet) \otimes [\varepsilon \otimes \text{id}_{\Omega_Y}]$ is a cohomology class in $\mathbb{E}xt^1(G^\bullet, G \otimes \Omega_Y)$, where \otimes is the Yoneda product for Ext-groups of complexes of sheaves.

Since $\mathcal{H}om^0(G^\bullet, G \otimes \Omega_Y) = \mathcal{H}om(G^0, G \otimes \Omega_Y)$ and $\mathcal{H}om^n(G^\bullet, G \otimes \Omega_Y) = 0$ for $n > 0$, we have that $\bar{\alpha}$ is an element in

$$C^1(\mathcal{H}om(G^0, G \otimes \Omega_Y)) \subset C^1(\mathcal{H}om^0(G^\bullet, G \otimes \Omega_Y \xrightarrow{\phi^G \otimes \text{id}} p_X^* O_D^{\oplus r} \otimes \Omega_Y)).$$

We define morphisms $\nabla_i : G^0|_{U_i} \rightarrow (G \otimes \Omega_Y)|_{U_i}$ as compositions

$$G^0|_{U_i} \xrightarrow{\nabla_i^0} (G^0 \otimes \Omega_Y)|_{U_i} \xrightarrow{\varepsilon \otimes \text{id}} (G \otimes \Omega_Y)|_{U_i}$$

and morphisms $\tilde{\phi}_i : G^0|_{U_i} \rightarrow (p_X^* O_D^{\oplus r} \otimes \Omega_Y)|_{U_i}$ as compositions

$$G^0|_{U_i} \xrightarrow{\nabla_i} (G \otimes \Omega_Y)|_{U_i} \xrightarrow{\phi^G \otimes \text{id}} (p_X^* O_D^{\oplus r} \otimes \Omega_Y)|_{U_i}$$

hence we obtain a cochain $\tilde{\phi} \in C^0(\mathcal{H}om^1(G^\bullet, G \otimes \Omega_Y \xrightarrow{\phi^G \otimes \text{id}} p_X^* O_D^{\oplus r} \otimes \Omega_Y))$.

Since $\tilde{\phi}_{i_0}|_{U_{i_0 i_1}} - \tilde{\phi}_{i_1}|_{U_{i_0 i_1}} = ((\phi^G \circ \varepsilon) \otimes \text{id}_{\Omega_Y})|_{U_{i_0 i_1}} \circ \alpha'_{i_0 i_1}$, we have that $A(G, \phi^G) := \bar{\alpha} + \tilde{\phi}$ defines an element of

$$\mathbb{E}xt^1(G^\bullet, G \otimes \Omega_Y \xrightarrow{\phi^G \otimes \text{id}} p_X^* O_D^{\oplus r} \otimes \Omega_Y) = \mathbb{E}xt^1(G, G \otimes \Omega_Y \xrightarrow{\phi^G \otimes \text{id}} p_X^* O_D^{\oplus r} \otimes \Omega_Y).$$

If we consider the induced section of $A(G, \phi^G)$ under the global-to-local map

$$\mathbb{E}xt^1(G, G \otimes \Omega_Y \xrightarrow{\phi^G \otimes \text{id}} p_X^* O_D^{\oplus r} \otimes \Omega_Y) \rightarrow H^0(S, \mathcal{E}xt_{p_S}^1(G, G \otimes \Omega_Y \xrightarrow{\phi^G \otimes \text{id}} p_X^* O_D^{\oplus r} \otimes \Omega_Y))$$

where $\mathcal{E}xt_{p_S}^1$ is the first right derived functor of $(p_S)_* \circ \mathcal{H}om$, then the direct sum decomposition $\Omega_Y = p_S^* \Omega_S \oplus p_X^* \Omega_X$ leads to an analogous decomposition $A(G, \phi^G) = A(G, \phi^G)' + A(G, \phi^G)''$. The *Kodaira-Spencer map* associated to (G, ϕ^G) is

$$KS : \Omega_S^\vee \xrightarrow{A(G, \phi^G)'} \Omega_S^\vee \otimes \mathcal{E}xt_{p_S}^1(G, G \otimes p_S^* \Omega_S \xrightarrow{\phi^G \otimes \text{id}} p_X^* O_D^{\oplus r} \otimes p_S^* \Omega_S) \rightarrow \mathcal{E}xt_{p_S}^1(G, G \xrightarrow{\phi^G} p_X^* O_D^{\oplus r})$$

Remark 10. It is easy to prove that the complex $G \xrightarrow{\phi^G} p_X^* O_D^{\oplus r}$ is quasi-isomorphic to $G \otimes p_X^* O_X(-D)$, hence $\mathcal{E}xt_{p_S}^1(G, G \xrightarrow{\phi^G} p_X^* O_D^{\oplus r}) \cong \mathcal{E}xt_{p_S}^1(G, G \otimes p_X^* O_X(-D))$.

Now we are able to give the following characterization of the tangent bundle of $\mathcal{M}_{X,D}(r, c, n)$.

Theorem 11. *Let $(\bar{E}, \phi^{\bar{E}})$ be the universal family of $\mathcal{M}_{X,D}(r, c, n)$ and let p be the projection from $\mathcal{M}_{X,D}(r, c, n) \times X$ to $\mathcal{M}_{X,D}(r, c, n)$. The Kodaira-Spencer map for $\mathcal{M}_{X,D}(r, c, n)$ induces a canonical isomorphism*

$$\mathbb{T}_{\mathcal{M}_{X,D}(r, c, n)} \cong \mathcal{E}xt_p^1(\bar{E}, \bar{E} \otimes p_X^* O(-D)).$$

Proof. To prove this theorem, we can use a similar argument of theorem 10.2.1 in [13]: by lemma 7 we get that the sheaf $\mathcal{E}xt_p^1(\bar{E}, \bar{E} \otimes p_X^* O(-D))$ commutes with base change and $\mathcal{E}xt_p^1(\bar{E}, \bar{E} \otimes p_X^* O(-D))$ is locally free.

Moreover, by example 10.1.9 in [13] we get that the Kodaira-Spencer map is an isomorphism on the fibres. \square

Donaldson-Uhlenbeck partial compactification. Let $\mathcal{M}_{X,D}^{reg}(r, c, n)$ be the open subset in $\mathcal{M}_{X,D}(r, c, n)$ consisting of framed sheaves (F, ϕ^F) with F locally free.

Now remember that each framed sheaf $\mathcal{E} = (E, \phi^E)$ has a natural inclusion in $\mathcal{E}^{\vee\vee} = (E^{\vee\vee}, \phi^E)$ and the quotient $E^{\vee\vee}/E$ is a zero-dimensional sheaf whose support is disjoint from D . In this way to any framed sheaf \mathcal{E} with invariants r, c, n we can associate the pair $(\mathcal{E}^{\vee\vee}, \sum_i a_i[x_i])$ where x_i are the points of the support of the quotient and a_i is the length of the quotient at x_i ; remark that if $c_2(E^{\vee\vee}) = s$ then $\text{length}(E^{\vee\vee}/E) = n - s$, hence $\sum_i a_i x_i$ is an element of the $(n - s)$ -th symmetric product $S^{n-s} X$ and $\mathcal{E}^{\vee\vee}$ is an element of $\mathcal{M}_{X,D}^{reg}(r, c, s)$.

This association gives a map

$$(2) \quad \pi_r : \mathcal{M}_{X,D}(r, c, n) \longrightarrow \widehat{\mathcal{M}_{X,D}(r, c, n)} := \prod_{s=0}^n \mathcal{M}_{X,D}^{reg}(r, c, s) \times S^{n-s} X;$$

in [6] the right hand side space is given a structure of scheme (similar to the Donaldson-Uhlenbeck partial compactification of instantons) such that the map π_r is a projective morphism.

Now we are going to study the fibers of this map, that will be useful later on: let $\mathbf{p} = ((G, \phi^G), \sum_j m_j[x_j])$ be a point in $\mathcal{M}_{X,D}^{reg}(r, c, s) \times S^{n-s} X$, the fiber $\pi_r^{-1}(\mathbf{p})$ parametrizes framed sheaves (E, ϕ^E) such that $E^{\vee\vee} \cong G$, the framing ϕ^E is just the composition of the injection in the double dual with ϕ^G (and hence is fixed if we fix \mathcal{G} and E) and the quotient $A = G/E$ is supported at x_j , with length over x_j equal to m_j . We can decompose $A = \bigoplus A_j$ as sum of skyscraper sheaves supported at x_j of length m_j ; each of this A_j may be seen as an Artin quotient of the trivial sheaf $O_X^{\oplus r}$ supported only at x_j and of length m_j , hence the possible choice of A_j are parametrized by the scheme $\text{Quot}_{x_j}(r, m_j)$; these schemes have been studied in the appendix of [3], we recall the following result:

Proposition 12. *Let X be a smooth projective surface, $x \in X$ a point and r, d a pair of positive integers. Then the scheme $\text{Quot}_x(r, d)$ parametrizing Artin quotients of the trivial sheaf $O_X^{\oplus r}$ supported at x of length d is irreducible of dimension $rd - 1$.*

We get as a corollary the following:

Corollary 13. *Let $\mathfrak{p} = (\mathcal{G}, \sum_{j=1}^l m_j[x_j])$ be a point of $\mathcal{M}_{X,D}^{reg}(r, c, s) \times S^{n-s}X$. Then $\pi_r^{-1}(\mathfrak{p})$ is isomorphic to the scheme $\prod_j \text{Quot}_{x_j}(r, m_j)$ as an irreducible closed subscheme of $\mathcal{M}_{X,D}(r, c, n)$ of dimension $\sum_j (rm_j - 1) = r(n - s) - l$.*

3. NAKAJIMA OPERATORS ON MODULI SPACES OF FRAMED SHEAVES

From now on we fix the rank r and the first Chern class c and we consider the moduli spaces $\mathcal{M}_{X,D}(r, c, n)$ all together by varying the second Chern class n . To emphasize this fact in the following we denote by $\mathcal{M}_{X,D}^{r,c}(n)$ the moduli space $\mathcal{M}_{X,D}(r, c, n)$ as if we consider n a variable and r, c constants.

Let $H_\bullet(\mathcal{M}_{X,D}^{r,c}(n))$ be the homology group of $\mathcal{M}_{X,D}^{r,c}(n)$ with complex coefficients. Denote by $\mathbb{H}(X)$ the direct sum $\bigoplus_n H_\bullet(\mathcal{M}_{X,D}^{r,c}(n))$ of all the homology groups of all the moduli spaces $\mathcal{M}_{X,D}^{r,c}(n)$.

In this section for any $i \in \mathbb{Z}$ and $\alpha \in H_\bullet(X)$ we define operators

$$P_\alpha[i] : \mathbb{H}(X) \rightarrow \mathbb{H}(X)$$

that we shall call *Nakajima operators* and in the following section we prove that they satisfy the Heisenberg commutator relations: in this way we define an action of a Heisenberg algebra associated to $H_\bullet(X)$ on the space $\mathbb{H}(X)$ in a geometric way.

First for any $i, n \in \mathbb{Z}$ and $\alpha \in H_\bullet(X)$ we are going to construct operators

$$P_\alpha^n[i] : H_\bullet(\mathcal{M}_{X,D}^{r,c}(n)) \longrightarrow H_\bullet(\mathcal{M}_{X,D}^{r,c}(n - i))$$

as *correspondence* for a suitable subvariety of $\mathcal{M}_{X,D}^{r,c}(n - i) \times \mathcal{M}_{X,D}^{r,c}(n)$. Moreover, for technical reasons due to the non-projectivity of the moduli spaces $\mathcal{M}_{X,D}^{r,c}(n)$ we have to use the Borel-Moore homology groups of our moduli spaces (for technical details about Borel-Moore homology we refer to Appendix B in [9]).

Now we recall briefly the construction of correspondences. Let M_1, M_2 be two oriented manifolds of dimension d_1, d_2 respectively and $Z \subseteq M_1 \times M_2$ an oriented submanifold such that the projection $p_2 : Z \rightarrow M_2$ is a proper map. We denote by $H_j^{lf}(M_a)$ the Borel-Moore homology group of M_a for $a = 1, 2$.

We can define operators

$$\begin{aligned} \tilde{\gamma}_Z : H_j^{lf}(M_1) &\longrightarrow H_{j+\dim(Z)-d_1}^{lf}(M_2) \\ \eta &\longmapsto p_{2*}(p_1^*\eta \cap [Z]) \end{aligned}$$

and for any $\alpha \in H_\bullet^{lf}(Z)$

$$\begin{aligned} \tilde{\gamma}_\alpha : H_j^{lf}(M_1) &\longrightarrow H_{j+\deg(\alpha)-d_1}^{lf}(M_2) \\ \eta &\longmapsto p_{2*}(p_1^*\eta \cap \alpha) \end{aligned}$$

where the \cap -product is the one on Borel-Moore homology.

Moreover, using the fact that the singular homology group of M_a is the direct limit of the Borel-Moore homology group of M_a with respect to the compact subsets of M_a (for $a = 1, 2$), one defines operators between the singular homology groups (that we call *correspondence*)

$$\gamma_Z : \mathbf{H}_j(M_2) \longrightarrow \mathbf{H}_{j+\dim(Z)-d_2}(M_1)$$

and for any $\alpha \in \mathbf{H}_\bullet^{lf}(Z)$

$$\gamma_\alpha : \mathbf{H}_j(M_2) \longrightarrow \mathbf{H}_{j+\deg(\alpha)-d_2}(M_1).$$

Now we want to study the composition of two such homomorphism: if M_1, M_2, M_3 are three oriented manifolds and $Z \subseteq M_1 \times M_2, W \subseteq M_2 \times M_3$ two oriented submanifolds such that the projections $Z \rightarrow M_2, W \rightarrow M_3$ are proper maps, we have the composition of the correspondences $\gamma_Z \circ \gamma_W : \mathbf{H}_\bullet(M_3) \rightarrow \mathbf{H}_\bullet(M_1)$: these turn out to be equal to the correspondence defined by the subvariety $A \subseteq M_1 \times M_3$,

$$A = p_{13}(p_{12}^{-1}(Z) \cap p_{23}^{-1}(W)),$$

where p_{ab} are the projections $M_1 \times M_2 \times M_3 \rightarrow M_a \times M_b$.

If we have $\alpha \in \mathbf{H}_\bullet(Z)$ and $\beta \in \mathbf{H}_\bullet(W)$, the composition of the correspondences γ_β and γ_α is given by the correspondence induced by the class $\epsilon \in \mathbf{H}_\bullet(A)$ defined by

$$\epsilon = p_{13*}(p_{12}^*\alpha \cap p_{23}^*\beta).$$

Now consider the moduli spaces $\mathcal{M}_{X,D}^{r,c}(n)$; let $n, i \in \mathbb{Z}, i > 0$ and consider the closed subvariety $P^n[i] \subseteq \mathcal{M}_{X,D}^{r,c}(n) \times \mathcal{M}_{X,D}^{r,c}(n+i) \times X$ defined by

$$P^n[i] = \left\{ (\mathcal{E}_1, \mathcal{E}_2, x) \in \mathcal{M}_{X,D}^{r,c}(n) \times \mathcal{M}_{X,D}^{r,c}(n+i) \times X \left| \begin{array}{l} \bullet \mathcal{E}_1^{\vee\vee} \cong \mathcal{E}_2^{\vee\vee}, \\ \bullet \mathcal{E}_2 \subseteq \mathcal{E}_1, \\ \bullet \text{supp}(E_1/E_2) = \{x\}. \end{array} \right. \right\}$$

We want to define *correspondences* using the fundamental class of $P^n[i]$. Through direct computations one can prove that the dimension of the closed subvariety $P^n[i]$ is $2rn + b + r + 1$.

Note that if $\pi : \mathcal{M}_{X,D}^{r,c}(n) \times \mathcal{M}_{X,D}^{r,c}(n+i) \times X \rightarrow \mathcal{M}_{X,D}^{r,c}(n) \times \mathcal{M}_{X,D}^{r,c}(n+i)$ is the projection, then the restriction $\pi|_{P^n[i]}$ is a proper injective map and therefore $\pi(P^n[i])$ is a closed subvariety.

Thus for any $\alpha \in \mathbf{H}_\bullet(X)$ we have a Borel-Moore homology class $P_\alpha^n[i] \in \mathbf{H}_\bullet^{lf}(\pi(P^n[i]))$ defined by

$$P_\alpha^n[i] = \pi_*(\tau^*\alpha \cap [P^n[i]])$$

where $\tau : \mathcal{M}_{X,D}^{r,c}(n) \times \mathcal{M}_{X,D}^{r,c}(n+i) \times X \rightarrow X$ is the projection on the surface. (With the previous notation: $P_\alpha^n[i] = \tilde{\gamma}_{P^n[i]}(\alpha)$).

Since the restriction to $\pi(P^n[i])$ of the projection from $\mathcal{M}_{X,D}^{r,c}(n) \times \mathcal{M}_{X,D}^{r,c}(n+i)$ to $\mathcal{M}_{X,D}^{r,c}(n+i)$ is a proper map, the class $P_\alpha^n[i]$ induces a correspondence operator

$$P_\alpha^n[i] : H_j(\mathcal{M}_{X,D}^{r,c}(n+i)) \longrightarrow H_{j+\deg(\alpha)-2ri-2}(\mathcal{M}_{X,D}^{r,c}(n))$$

(With the previous notation this would be $\gamma_{P_\alpha^n[i]}$).

In a similar way we can define for $i \in \mathbb{Z}, i > 0$ closed subvarieties

$$P^n[-i] \subseteq \mathcal{M}_{X,D}^{r,c}(n) \times \mathcal{M}_{X,D}^{r,c}(n-i)$$

as before exchanging the roles of \mathcal{E}_1 and \mathcal{E}_2 ; these subvarieties have dimension $2rn + b - ri + 1$. We can repeat the previous construction and for any $\beta \in H_\bullet(X)$ we get the operator

$$P_\beta^n[-i] : H_j(\mathcal{M}_{X,D}^{r,c}(n-i)) \longrightarrow H_{j+\deg(\beta)+2ri-2}(\mathcal{M}_{X,D}^{r,c}(n)).$$

For $i = 0$ we set $P_\alpha^n[0]$ equal to the identity map for any $n \in \mathbb{Z}$ and $\alpha \in H_\bullet(X)$.

Now we can extend these operators to $\mathbb{H}(X)$: for any $i \in \mathbb{Z}$ and $\alpha \in H_\bullet(X)$ define operators

$$P_\alpha[i] : \mathbb{H}_\bullet(X) \longrightarrow \mathbb{H}_\bullet(X)$$

acting on every summand $H_\bullet(\mathcal{M}_{X,D}^{r,c}(n))$ with $P_\alpha^n[i]$.

4. MAIN THEOREM

Now we want to study the commutation relations between the operators $P_\alpha[i]$ we constructed. In particular we want to prove the following result:

Theorem 14. *For any $i, j \in \mathbb{Z}, \alpha, \beta \in H_\bullet(X)$ the following holds:*

$$[P_\alpha[i], P_\beta[j]] = (-1)^{ri-1} ri \delta_{i+j,0} \langle \alpha, \beta \rangle \text{id}$$

This provides a representation of the Heisenberg algebra generated by $P_\alpha[i]/(-1)^{ri-1}$ and $P_\beta[j]$ on $\mathbb{H}(X)$.

The first step to get this result is the following:

Proposition 15. *There exist constants $c_{r,i,n}$ such that for any $v \in H_\bullet(\mathcal{M}_{X,D}^{r,c}(n))$ we have:*

$$[P_\alpha[i], P_\beta[j]]v = c_{r,i,n} \delta_{i+j,0} \langle \alpha, \beta \rangle v$$

A proof of this proposition may be given directly by investigating the cycles $P_\alpha^{n-j}[i]P_\beta^n[j]$ and $P_\beta^{n-i}[j]P_\alpha^n[i]$ in $\mathcal{M}_{X,D}^{r,c}(n-i-j) \times \mathcal{M}_{X,D}^{r,c}(n)$, and observing that they act in the same way whenever $i \neq j$, while when $i = j$ they differ by a constant times the intersection product of the cycles; in [17] one can find the details of this proof for the case when $r = 1$, i.e. the Hilbert scheme of points, and this directly generalize to our case.

Lemma 16. *The constants $c_{r,i,n}$ of the previous proposition are independent of n . (And so, from now on we shall denote them simply by $c_{r,i}$)*

Proof. Let $i, k \in \mathbb{Z}$ be such that $k \neq \pm i$, $\alpha, \beta, \gamma \in \mathbf{H}_\bullet(X)$ and $v \in \mathbf{H}_\bullet(\mathcal{M}_{X,D}^{r,c}(n))$. Since $P_\gamma[k]v \in \mathbf{H}_\bullet(\mathcal{M}_{X,D}^{r,c}(n-k))$, we get

$$\begin{aligned} [P_\gamma[k], [P_\alpha[i], P_\beta[-i]]]v &= P_\gamma[k][P_\alpha[i], P_\beta[-i]]v - [P_\alpha[i], P_\beta[-i]]P_\gamma[k]v = \\ &= \langle \alpha, \beta \rangle (c_{r,i,n} - c_{r,i,n-k})v \end{aligned}$$

But $k \neq \pm i$, hence $[P_\gamma[k], [P_\alpha[i], P_\beta[j]]] = 0$, and we can choose $\alpha, \beta, \gamma, k, v$ so that $\langle \alpha, \beta \rangle \neq 0$ and $P_\gamma[k]v \neq 0$.

So $c_{r,i,n} = c_{r,i,n+k}$; for $i \neq 0$, we can take $k = 1$ and have $c_{r,i,n} = c_{r,i,n+1}$ that implies the thesis; for $i = 1$ taking $k = 2, 3$ we have that $c_{r,1,n} = c_{r,1,n+2} = c_{r,1,n+3}$, and again we have the thesis. \square

5. CALCULATION OF CONSTANTS

In this section we compute explicitly the constants $c_{r,i}$: following Grojnowski and Nakajima, we find an isomorphism of the algebra of symmetric functions with some subspaces of $\mathbb{H}_\bullet(X)$, such that the Newton polynomials correspond to the Nakajima operators $P_{[C]}[i]$ for some smooth connected curves $C \subseteq X$, $C \neq D$. In the following section we recall some facts about the algebra of symmetric functions (for more informations see chapter one in [16]).

Notations: We use two different conventions for μ partition of a natural number k : if we write $\mu = (\mu_1, \dots, \mu_t)$ we mean that $\sum_a \mu_a = k$ and assume that $\mu_a \geq \mu_{a+1} > 0$; moreover we call *length* $|\mu|$ of the partition μ the natural number t . If we write $\mu = (1^{m_1}, \dots, k^{m_k})$ we mean that $m_l = \#\{a \mid \mu_a = l\}$.

Symmetric Functions. We call *algebra of symmetric polynomial in N variables* the subspace Λ_N of $\mathbb{C}[x_1, \dots, x_N]$ stable by the action of the N -th group of permutations σ_N . We have that Λ_N is a graded ring:

$$\Lambda_N = \bigoplus_n \Lambda_N^n$$

where Λ_N^n is the ring of homogeneous symmetric polynomials in N variables of degree n (together with the zero polynomial).

For any $M > N$ we have morphisms $\rho_{MN} : \Lambda_M \rightarrow \Lambda_N$ mapping the variables x_{N+1}, \dots, x_M to zero. Moreover the morphisms ρ_{MN} preserve the grading, hence we can define $\rho_{MN}^n : \Lambda_M^n \rightarrow \Lambda_N^n$; this allows us to define

$$\Lambda^n := \varprojlim_N \Lambda_N^n$$

and the *algebra of symmetric functions in infinitely many variables* as

$$\Lambda := \bigoplus_n \Lambda^n.$$

Now we want to define a basis for Λ , to do this we start by defining a basis in Λ_N . Let $\mu = (\mu_1, \dots, \mu_t)$ be a partition with $t \leq N$, we define the polynomial

$$m_\mu(x_1, \dots, x_N) = \sum_{\tau \in \sigma_N(\mu)} x_1^{\tau_1} \cdots x_N^{\tau_N}$$

where we set $\mu_j = 0$ for $j = t + 1, \dots, N$. The polynomial m_μ is symmetric, moreover the set of all m_μ for all the partitions μ with $|\mu| \leq N$ is a basis of Λ_N . If we consider the set of all m_μ for all the partitions μ with $|\mu| \leq N$ and $\sum_i \mu_i = n$, then this set is a basis of Λ_N^n .

Since for $M > N \geq t$ we have $\rho_{MN}^n(m_\mu(x_1, \dots, x_M)) = m_\mu(x_1, \dots, x_N)$, by using the definition of inverse limit we can define the *monomial symmetric functions* m_μ . By varying μ partition of n , these polynomials form a basis for Λ^n .

Now we want to define particular families of symmetric functions. Let $n \in \mathbb{N}, n \geq 1$, we define the *elementary symmetric function* e_n as

$$e_n = m_{(1^n)} = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n}$$

and we put $e_0 = 1$. For $\mu = (\mu_1, \dots, \mu_t)$ partition, we set $e_\mu = e_{\mu_1} e_{\mu_2} \cdots e_{\mu_t}$; one can prove that the set $\{e_\mu\}_\mu$ is a basis of Λ .

The generating function for the sequence $\{e_n\}_{n \in \mathbb{N}}$ is

$$E(z) = \sum_{n=0}^{\infty} e_n z^n = \prod_{i=1}^{\infty} (1 + x_i z).$$

Next, the n -th *complete symmetric function* is $h_n = \sum_{|\nu|=n} m_\nu$. Since we have the formal power series

$$\frac{1}{1 - x_i z} = 1 + x_i z + x_i^2 z^2 + \dots$$

the sequence $\{h_n\}_{n \in \mathbb{N}}$ has generating function

$$H(z) = \sum_{n=0}^{\infty} h_n z^n = \prod_{i=1}^{\infty} \frac{1}{1 - x_i z}.$$

In particular we have $E(z)H(-z) = 1$.

Finally, the n -th *Newton polynomial* p_n is $p_n = m_{(n)} = \sum_{i=1}^{\infty} x_i^n$. Its generating function is

$$P(z) = \sum_{n=1}^{\infty} p_n z^{n-1} = \sum_{n,i=1}^{\infty} x_i^n z^{n-1} = \sum_{i=1}^{\infty} \frac{d}{dz} \log \left(\frac{1}{1 - x_i z} \right) = \frac{d}{dz} \log H(z).$$

Hence we have the formulas:

$$(3) \quad \begin{aligned} H(z) &= \exp\left(\int P(z)\right) = \exp\left(\sum_{n=0}^{\infty} \frac{p_n z^n}{n}\right), \\ E(z) &= H(-z)^{-1} = \exp\left(\sum_{n=0}^{\infty} \frac{p_n z^n}{(-1)^{n-1} n}\right). \end{aligned}$$

To conclude this section, we state the following lemma:

Lemma 17. *For any $i \in \mathbb{N}$ we have*

$$p_i m_\mu = \sum_{\nu} a_{\mu\nu} m_\nu$$

where the summation is over partitions ν of $|\mu| + i$ which are obtained as follows: add i to a term in μ , say μ_j (possibly zero), and then arrange it in descending order. The coefficient $a_{\mu\nu}$ is $\#\{l \mid \exists k \text{ s.t. } \nu_l = \mu_k + i\}$.

Computation of $c_{r,i}$. We need to introduce some subvarieties of our moduli spaces: let C be a smooth connected curve in X , $C \neq D$, $s, k \in \mathbb{N}$; set

$$L_C^k(s)^\circ = \left\{ \mathcal{F} = (F, \phi^F) \in \mathcal{M}_{X,D}^{r,c}(s+k) \left| \begin{array}{l} \bullet \mathcal{F} \subset \mathcal{H} \text{ for some } \mathcal{H} \in \mathcal{M}_{X,D}^{r,c}(s), \\ \bullet H/F \text{ is an Artin sheaf of length } k, \\ \bullet \text{supp}(H/F) \subseteq C. \end{array} \right. \right\}$$

and $L_C^k(s)$ its closure in $\mathcal{M}_{X,D}^{r,c}(s+k)$; for μ partition of k , set

$$L_C^\mu(s)^\circ = \left\{ \mathcal{F} = (F, \phi^F) \in \mathcal{M}_{X,D}^{r,c}(s+k) \left| \begin{array}{l} \bullet \mathcal{F} \subset \mathcal{H} \text{ for some } \mathcal{H} \in \mathcal{M}_{X,D}^{r,c}(s), \\ \bullet \text{supp}(H/F) = \{x_1, \dots, x_t\} \subseteq C, \\ \bullet \text{length}_{x_a}(H/F) = \mu_a \text{ for } a = 1, 2, \dots, t. \end{array} \right. \right\}$$

and $L_C^\mu(s)$ its closure in $\mathcal{M}_{X,D}^{r,c}(s+k)$.

Now let $\mathcal{G} = (G, \phi^G) \in \mathcal{M}_{X,D}^{reg}(r, c, s)$ and set

$$L_C^k(\mathcal{G})^\circ = \left\{ \mathcal{F} = (F, \phi^F) \in \mathcal{M}_{X,D}^{r,c}(s+k) \left| \begin{array}{l} \bullet \mathcal{F}^{\vee\vee} = \mathcal{G}, \\ \bullet \text{supp}(G/F) \subseteq C. \end{array} \right. \right\}$$

and $L_C^k(\mathcal{G})$ its closure in $\mathcal{M}_{X,D}^{r,c}(s+k)$; for μ partition of k set

$$L_C^\mu(\mathcal{G})^\circ = \left\{ \mathcal{F} = (F, \phi^F) \in \mathcal{M}_{X,D}^{r,c}(s+k) \left| \begin{array}{l} \bullet \mathcal{F}^{\vee\vee} = \mathcal{G}, \\ \bullet \text{supp}(G/F) = \{x_1, \dots, x_t\} \subseteq C, \\ \bullet \text{length}_{x_a}(G/F) = \mu_a \text{ for } a = 1, 2, \dots, t. \end{array} \right. \right\}$$

and $L_C^\mu(\mathcal{G})$ its closure.

The first result we have is similar to that of proposition 4.5 in [3]:

Proposition 18. *The irreducible components of $L_C^k(s)$ are the closed subvarieties $L_C^\mu(s)$ for μ a partition of k , each of which has dimension $2rs + rk$.*

For any $\mathcal{G} \in \mathcal{M}_{X,D}^{reg}(r, c, s)$, the irreducible components of $L_C^k(\mathcal{G})$ are the closed subvarieties $L_C^\mu(\mathcal{G})$, for μ a partition of k , each of which has dimension rk .

Now we remark the following: let $\alpha = [C] \in H_2(X)$ and consider the corresponding operator

$$P_{[C]}[-i] : H_j(\mathcal{M}_{X,D}^{r,c}(n)) \rightarrow H_{j+2ri}(\mathcal{M}_{X,D}^{r,c}(n+i))$$

we have that the subspace of $\mathbb{H}(X)$ of *middle degrees*, i.e. $\bigoplus_n H_{2rn}(\mathcal{M}_{X,D}^{r,c}(n))$, is preserved by the action of $P_{[C]}[-i]$.

Moreover, since $P_{[C]}[-i]$ ‘changes the sheaves only at a point of C adding i to the length of a zero-dimensional quotient’, if μ is a partition of k we have that $P_{[C]}[-i][L_C^\mu(\mathcal{G})] \in H_{2r(k+i)}(L_C^{k+i}(\mathcal{G}))$; the next proposition calculates the coefficients of this element with respect to the basis of $H_{2r(k+i)}(L_C^{k+i}(\mathcal{G}))$ consisting of the classes of the irreducible components of $L_C^{k+i}(\mathcal{G})$.

Proposition 19. *For any $s, k \in \mathbb{N}$, μ a partition of k , $C \subseteq X$ smooth connected curve, $C \neq D$, the following holds:*

$$P_{[C]}[-i][L_C^\mu(s)] = \sum_{\nu \text{ partition of } |\mu|+i} a_{\mu\nu} [L_C^\nu(s)]$$

where the constants $a_{\mu\nu}$ are the same as in lemma 17.

For any $\mathcal{G} \in \mathcal{M}_{X,D}(r, c, s)$ we get

$$P_{[C]}[-i][L_C^\mu(\mathcal{G})] = \sum_{\nu \text{ partition of } |\mu|+i} [L_C^\nu(\mathcal{G})]$$

with the same constants $a_{\mu\nu}$.

The proof of this proposition is given in section 6.

Let $[vac]$ be the class of the point \mathcal{G} and $[VAC]$ the fundamental class of $\mathcal{M}_{X,D}^{r,c}(s)$. Let $L(\mathcal{G})$ (resp. $L(s)$) be the subspaces of $\mathbb{H}(X)$ generated by the classes $[L_C^\mu(\mathcal{G})]$ and $[vac]$ (resp. $[L_C^\mu(s)]$ and $[VAC]$). We define a \mathbb{C} -linear isomorphism from $L(\mathcal{G})$ (resp. $L(s)$) onto the algebra Λ of symmetric functions in infinitely many variables, sending $[vac]$ (resp. $[VAC]$) to $1 \in \Lambda$ and $[L_C^\mu(\mathcal{G})]$ (resp. $[L_C^\mu(s)]$) to the monomial symmetric function m_μ .

By proposition 19, the Nakajima operator $P_{[C]}[-i]$ corresponds under this isomorphism to multiplication by the i -th Newton function $p_i \in \Lambda$.

By this isomorphism, the elementary symmetric function e_n corresponds to $[L_C^{(1^n)}(\mathcal{G})]$ (resp. $[L_C^{(1^n)}(s)]$) and the equation 3 gives the following equalities:

$$(4) \quad \sum_{n=0}^{\infty} z^n [L_C^{(1^n)}(s)] = \exp\left(\sum_{i=1}^{\infty} \frac{z^i P_{[C]}[-i]}{(-1)^{i-1} i}\right) [VAC],$$

$$(5) \quad \sum_{n=0}^{\infty} z^n [L_C^{(1^n)}(\mathcal{G})] = \exp\left(\sum_{i=1}^{\infty} \frac{z^i P_{[C]}[-i]}{(-1)^{i-1} i}\right) [vac].$$

Now we need the following proposition, whose proof will be given in section 7:

Proposition 20. *Let L be a very ample divisor in X . Let C and C' be two smooth connected curves different from D , such that $C \cap C' \cap D = \emptyset$. Assume that C and C' meets trasversally and are in $|L|$. Then the following holds:*

$$\sum_{n=0}^{\infty} z^{2n} \langle [L_C^{(1^n)}(\mathcal{G})], [L_{C'}^{(1^n)}(s)] \rangle = (1 - (-1)^r z^2)^{r \langle [C], [C'] \rangle}.$$

We can finally conclude the proof of theorem 14, via the same computations of the proof of theorem 4.1 in [3]:

Proposition 21. *The constants $c_{r,i}$ of proposition 15 satisfy*

$$c_{r,i} = (-1)^{r i - 1} r i.$$

Proof. For any C smooth connected curve in X , $C \neq D$, set

$$C_+(z) = \sum_{i=1}^{\infty} \frac{P_{[C]}[i] z^i}{(-1)^{i-1} i}, \quad C_-(z) = \sum_{i=1}^{\infty} \frac{P_{[C]}[-i] z^i}{(-1)^{i-1} i}.$$

Since $P_{[C]}[i]$ is the adjoint to $P_{[C]}[-i]$ with respect to the intersection product on $\mathbb{H}(X)$, $C_+(z)$ is the adjoint to $C_-(z)$ (with respect to the intersection product extended by linearity to power series).

Now, by proposition 20 and formulas 4 and 5 we have the equalities

$$\begin{aligned} (1 - (-1)^r z^2)^{\langle [C], [C'] \rangle} &= \sum_{n=0}^{\infty} z^{2n} \langle [L_C^{(1^n)}(\mathcal{G})], [L_{C'}^{(1^n)}(s)] \rangle = \\ &= \langle \sum_{n=0}^{\infty} z^n [L_C^{(1^n)}(\mathcal{G})], \sum_{n=0}^{\infty} z^n [L_{C'}^{(1^n)}(s)] \rangle = \\ &= \langle \exp(C_-(z)) [vac], \exp(C'_-(z)) [VAC] \rangle = \\ &= \langle \exp(C'_+(z)) \exp(C_-(z)) [vac], [VAC] \rangle = \\ &= \langle \exp(C_-(z)) [\exp(-\text{ad} C_-(z)) (\exp(C'_+(z)))] [vac], [VAC] \rangle \end{aligned}$$

where the last equality follows from the general equality between operators $\exp(-A)B \exp(A) = \exp(-\text{ad}A)(B)$ (we put $A = C_-(z)$ and $B = \exp(C'_+(z))$).

An explicit computation shows that

$$[C_-(z), \exp(C'_+(z))] = -\Phi(z) \exp(C'_+(z))$$

with $\Phi(z) = \sum_{n=0}^{\infty} \frac{c_{r,n}}{n^2} \langle [C], [C'] \rangle z^{2n}$; from this and the fact that $\exp(-\text{ad}A)(B) = 1 - \text{ad}A(B) + \dots$ one gets that

$$\exp(-\text{ad}C_-(z))(\exp(Cd_+(z))) = \exp(\Phi(z)) \exp(Cd_+(z)),$$

and so

$$\begin{aligned} (1 - (-1)^r z^2)^{\langle [C], [C'] \rangle} &= \exp(\Phi(z)) \langle \exp(C_-(z)) \exp(C'_+(z)) [vac], [VAC] \rangle = \\ &= \exp(\Phi(z)) \langle \exp(C_-(z)) [vac], [VAC] \rangle = \\ &= \exp(\Phi(z)). \end{aligned}$$

where the second of this equalities follows from the fact that since $i > 0$, $P_{[C']}[i][vac] = 0$, while the last follows from the fact that every $P_{[C]}[-i]$ maps $[vac]$ in the orthogonal complement of $[VAC]$.

So finally we get

$$r \langle [C], [C'] \rangle \log(1 - (-1)^r z^2) = \sum_{n=1}^{\infty} \frac{c_{r,n}}{n^2} \langle [C], [C'] \rangle z^{2n}$$

from which, developing in power series the left hand side, we get $c_{r,n} = (-1)^{rn-1} rn$, that is the thesis. \square

6. PROOF OF PROPOSITION 19

Now we want to give a proof of proposition 19; we explain how to do with $L_C^\mu(\mathcal{G})$, the case of $L_C^\mu(s)$ is analogous.

First of all we need the following lemma:

Lemma 22. *Let λ be a partition of k . For \mathcal{F} generic in $L_C^\lambda(\mathcal{G})$ (i.e. in a open dense subset), around each x_j point of the support of G/F exist coordinates (η_j, ζ_j) such that*

$$\left(\frac{G}{F} \right)_{x_j} \cong \frac{\mathbb{C}[\zeta_j]}{(\zeta_j)^{\lambda_j}}.$$

Proof. It follows from the first paragraph of section 5 in [3]. \square

As we have already seen, $P_{[C]}[-i](L_C^\mu(\mathcal{G}))$ is an element of maximal degree in $H_\bullet(L_C^{k+i}(\mathcal{G}))$. Since $L_C^\nu(\mathcal{G})$ are the irreducible components of $L_C^{k+i}(\mathcal{G})$ for ν partition of $k+i$, $P_{[C]}[-i](L_C^\mu(\mathcal{G}))$ decomposes as a sum $\sum_\nu b_{\mu\nu} [L_C^\nu(\mathcal{G})]$ for some coefficients $b_{\mu\nu}$.

Put $n = k + s$. To compute the coefficients $b_{\mu\nu}$, consider the intersection

$$P_{[C]}^n[-i] \cap p_n^{-1}(L_C^\mu(\mathcal{G})) \cap p_{n+i}^{-1}(L_C^\nu(\mathcal{G}))$$

where the projection morphisms are:

$$\begin{array}{ccc} & \mathcal{M}_{X,D}^{r,c}(n+i) \times \mathcal{M}_{X,D}^{r,c}(n) & \\ \swarrow p_{n+i} & & \searrow p_n \\ \mathcal{M}_{X,D}^{r,c}(n+i) & & \mathcal{M}_{X,D}^{r,c}(n) \end{array}$$

We prove the proposition:

Proposition 23. *Let $\mathcal{F}_2 \in L_C^\mu(\mathcal{G}), \mathcal{F}_1 \in L_C^\nu(\mathcal{G})$ be such that $(\mathcal{F}_1, \mathcal{F}_2) \in P_{[C]}^n[-i]$ is a smooth point and such that they are generic in the sense of lemma 22. Then the intersection of the tangent spaces*

$$W = T_{(\mathcal{F}_1, \mathcal{F}_2)} p_n^{-1}(L_C^\mu(\mathcal{G})) \cap T_{(\mathcal{F}_1, \mathcal{F}_2)} P_{[C]}^n[-i]$$

is isomorphic via p_{n+i}^ to the tangent space $T_{\mathcal{F}_1} L_C^\nu(\mathcal{G})$.*

This result implies that the coefficients $b_{\mu\nu}$ may be computed by counting the number of points in a generic fiber of the morphism

$$p_{n+i} |_{P_{[C]}^n[-i] \cap p_n^{-1}(L_C^\mu(\mathcal{G})) \cap p_{n+i}^{-1}(L_C^\nu(\mathcal{G}))}.$$

Fix a generic (in the sense of lemma 22) $\mathcal{E}_2 \in L_C^\nu(\mathcal{G})$ such that the quotient sheaf $A_2 = G/E_2$ is of the form stated in lemma 22. For a generic framed sheaf $\mathcal{E}_1 \in L_C^\mu(\mathcal{G})$ such that $(\mathcal{E}_1, \mathcal{E}_2) \in P_{[C]}^n[-i]$ the quotient $A_1 = G/E_1$ satisfy the equality $(A_1)_{x_i} = \frac{\mathbb{C}[\zeta_i]}{(\zeta_i)^{\mu_i}}$, with $\mu_i = \nu_i$ for all i except for exactly one i_0 , for which $\mu_{i_0} = \nu_{i_0} - i$. So the number of generic $\mathcal{E}_1 \in L_C^\mu(\mathcal{G})$ such that $(\mathcal{E}_1, \mathcal{E}_2) \in P_{[C]}^n[-i]$ is exactly $a_{\mu\nu}$, thus $b_{\mu\nu} = a_{\mu\nu}$.

To prove proposition 23 we need to give a description of the tangent spaces to $L_C^\mu(\mathcal{G})$ and $P_{[C]}^n[-i]$.

Tangent to $L_C^\lambda(\mathcal{G})$. Let λ be a partition of k . Fix $\mathcal{F} = (F, \phi^F) \in L_C^\lambda(\mathcal{G})$. Remark that since $\mathcal{F}^{\vee\vee} = \mathcal{G}$, the framing ϕ^F is just the composition of the inclusion $F \hookrightarrow \mathcal{G}$ with the framing ϕ^G of \mathcal{G} . So the deformations of \mathcal{F} in $L_C^\lambda(\mathcal{G})$ (that in general would be deformations of F and ϕ^F) are uniquely determined by deformations of the sheaf F with respect to the functor $\mathcal{F}(L_C^\lambda(\mathcal{G}))$, where \mathcal{F} is the ‘framing-forgetting’ morphism of functors $\mathcal{M}_{X,D}^{r,c}(n) \rightarrow \mathcal{S}$, where \mathcal{S} is the contravariant functor from the category of schemes to the category of sets that associates to every scheme S the set consisting of isomorphism classes of flat families of torsion free sheaves over X parametrized by S .

The deformations of the sheaf F have been studied in [3], here we recall that study.

First note that the functor $\mathcal{F}(L_C^\lambda(\mathcal{G}))$ is a subfunctor of the Quot-scheme functor $\underline{Quot}_X(G, |\lambda|)$. It is well known that deformations at a point F in $\underline{Quot}_X(G, |\lambda|)$ are parametrized by $\text{Hom}(F, A)$, where $A = G/F$, that can be seen as a subspace of $\text{Ext}^1(F, F)$ via the connection morphism that comes applying the left-exact functor $\text{Hom}(F, \bullet)$ to the short exact sequence $0 \rightarrow F \rightarrow G \rightarrow A \rightarrow 0$.

Now, for $\mathcal{F} = (F, \phi^F) \in L_C^\lambda(\mathcal{G})$ we have the decomposition $A = \bigoplus_j A_j$ with A_j a skyscraper sheaf, supported at $x_j \in C$ of length λ_j : this gives a decomposition $\text{Hom}(F, A) = \bigoplus_j \text{Hom}(F, A_j)$. If we assume that \mathcal{F} is generic, around each x_j we have a neighborhood in which we have an inclusion $G(-\lambda_j C) \hookrightarrow F$ such that the diagram

$$(6) \quad \begin{array}{ccc} G(-\lambda_j C) & \hookrightarrow & F \\ \parallel & & \downarrow \\ G(-\lambda_j C) & \hookrightarrow & G \end{array}$$

commutes; it turns out that deformations of F with respect to $\mathcal{F}(L_C^\lambda(\mathcal{G}))$ are the one that preserves these diagrams. Now with deformation theory and diagram chasing one can prove

Proposition 24. *The set of deformations of a sheaf F with respect to the functor $\mathcal{F}(L_C^\lambda(\mathcal{G}))$ that preserve the diagram 6 are parametrized by*

$$\bigoplus_{j=1}^t \text{Hom} \left(\frac{F}{G(-\lambda_j C)}, \left(\frac{G}{F} \right)_{x_j} \right).$$

So the tangent space of $L_C^\lambda(\mathcal{G})$ at a generic framed sheaf \mathcal{F} is

$$T_{\mathcal{F}}(L_C^\lambda(\mathcal{G})) = \bigoplus_{j=1}^t \text{Hom} \left(\frac{F}{G(-\lambda_j C)}, \left(\frac{G}{F} \right)_{x_j} \right).$$

Remark that this is a subspace of $\text{Ext}^1(F, F(-D))$ via

$$\bigoplus_{j=1}^t \text{Hom} \left(\frac{F}{G(-\lambda_j C)}, \left(\frac{G}{F} \right)_{x_j} \right) \hookrightarrow \text{Hom}(E, A) = \text{Hom}(E, A(-D)) \hookrightarrow \text{Ext}^1(F, F(-D)).$$

Tangent to $P_{[C]}^n[-i]$. We recall that $P_{[C]}^n[-i]$ is the subset of $\mathcal{M}_{X,D}^{r,c}(n+i) \times \mathcal{M}_{X,D}^{r,c}(n)$ consisting of pairs $(\mathcal{F}_1, \mathcal{F}_2)$ such that

- (a) $\mathcal{F}_1^{\vee\vee} \cong \mathcal{F}_2^{\vee\vee}$;
- (b) $F_1 \subseteq F_2$;
- (c) $\text{supp}(F_2/F_1) = \{x\}$ with $x \in C$ and $\text{length}(F_2/F_1) = i$.

Condition (a) implies that the framing ϕ^{F_1} of F_1 is uniquely determined by the inclusion morphism and by the framing on F_2 . So the deformations of $(\mathcal{F}_1, \mathcal{F}_2)$ are one to one with triples (F'_1, F'_2, ϕ') where F'_a is a deformation of F_a for $a = 1, 2$ and ϕ' is a deformation of the framing ϕ^{F_2} . Moreover, note both pairs (F'_1, ϕ') and (F'_2, ϕ') defines deformations of \mathcal{F}_1 and \mathcal{F}_2 , hence define elements of $\text{Ext}^1(F_1, F_1(-D))$ and $\text{Ext}^1(F_2, F_2(-D))$ respectively.

The pair (F'_1, F'_2) is a deformation of (F_1, F_2) with respect to the functor $\mathcal{F}(\overline{P_{[C]}^n[-i]})$. This kind of deformations are studied by Baranovsky in [3], here we recall his study.

Conditions (b) and (c) are equivalent to having in a neighborhood of x the commutative diagram

$$\begin{array}{ccc} F_2(-iC) & \hookrightarrow & F_1 \\ \parallel & & \downarrow \\ F_2(-iC) & \hookrightarrow & F_2 \end{array}$$

where the lower arrow is the multiplication by i times a generator of the curve C . So a deformation of (F_1, F_2) with respect to $\mathcal{F}(\overline{P_{[C]}^n[-i]})$ is a pair of sheaves (F'_1, F'_2) over $X \times \text{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$, flat over $\text{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$, with fibers over the closed point isomorphic respectively to F_1 and F_2 satisfying the following conditions:

- (i) there is an inclusion $F'_1 \hookrightarrow F'_2$;
- (ii) in a neighborhood of x we have an inclusion $F'_2(-iC) \hookrightarrow F'_1$;
- (iii) in this same neighborhood we have a commutative diagram

$$(7) \quad \begin{array}{ccc} F'_2(-iC) & \hookrightarrow & F'_1 \\ \parallel & & \downarrow \\ F'_2(-iC) & \hookrightarrow & F'_2 \end{array}$$

where the lower arrow is the multiplication by i times a generator of the curve C .

Recall that for $a = 1, 2$, the sheaves F'_a fit in an exact sequence

$$0 \rightarrow F_a \rightarrow F'_a \rightarrow F_a \rightarrow 0$$

that corresponds to an element $v_a \in \text{Ext}^1(F_a, F_a)$. By diagram chasing, we get that condition (i) is equivalent to the fact that the images of v_1 and v_2 in $\text{Ext}^1(F_1, F_2)$ coincide.

Condition (ii) may be treated in the same way: first of all notice that $v_2 \in \text{Ext}^1(F_2, F_2)$ defines an element $\tilde{v}_2 \in \text{Ext}^1(F_2(-iC), F_2(-iC))$. One can prove that condition (ii) is equivalent to the fact that the images of v_1 and \tilde{v}_2 in $\text{Ext}^1(F_2(-iC), F_1)$ coincide.

Finally we come to condition (iii). Assume that (F'_1, F'_2) satisfies conditions (i) and (ii) and label the morphisms:

$$\tilde{\rho} : F'_2(-iC) \rightarrow F'_1, \quad \sigma : F'_2(-iC) \rightarrow F'_2$$

and ρ the composition of $\tilde{\rho}$ with the inclusion $F'_1 \rightarrow F'_2$. This allows us to define a morphism $\tau : F_2(-iC) \rightarrow F_1$ that keeps track of the commutativity of the diagram 7, i.e. such that τ vanish if and only if condition (iii) holds.

These three facts completely describe the deformations of pairs of sheaves (F_1, F_2) . So the tangent of $P_{[C]}^n[-i]$ at point (F_1, F_2) is parametrized by pairs

$$(v_1, v_2) \in \text{Ext}^1(F_1, F_1) \oplus \text{Ext}^1(F_2, F_2)$$

such that the corresponding extensions satisfy conditions (i), (ii) and (iii) and both $v_a \in \text{Ext}^1(F_a, F_a(-D))$ for $a = 1, 2$.

Proof of proposition 23. Let $w \in W$. By previous descriptions $w = (v_1, v_2) \in \text{Ext}^1(F_1, F_1(-D)) \oplus \text{Ext}^1(F_2, F_2(-D))$ is such that the corresponding extensions satisfy conditions (i), (ii) and (iii) and moreover $v_2 \in \bigoplus_{j=1}^t \text{Hom}\left(\frac{F_2}{G(-\mu_j C)}, (A_2)_{x_j}\right)$, where $A_2 = G/F_2$. We want to prove that $v_1 \in T_{\mathcal{F}_1} \mathcal{L}'_C(\mathcal{G})$ and that $v_1 = 0$ only if $v_2 = 0$.

Since $\mathcal{F}_1^{\vee\vee} \cong \mathcal{F}_2^{\vee\vee} \cong \mathcal{G}$, we have that v_1 is an element of $\text{Hom}(F_1, A_1)$, where A_1 is the quotient sheaf G/F_1 . We need to prove that actually $v_1 \in \bigoplus_{j=1}^t \text{Hom}\left(\frac{F_1}{G(-\nu_j C)}, (A_1)_{x_j}\right)$.

Using the inclusion morphism from F_a to its double dual G , for $a = 1, 2$, and the inclusion morphism $F_1 \rightarrow F_2$, we get the commutative diagram

$$(8) \quad \begin{array}{ccc} \text{Hom}(F_1, A_1) & \longrightarrow & \text{Ext}^1(F_1, F_1) \\ \downarrow & & \downarrow \\ \text{Hom}(F_1, A_2) & \longrightarrow & \text{Ext}^1(F_1, F_2) \\ \uparrow & & \uparrow \\ \text{Hom}(F_2, A_2) & \longrightarrow & \text{Ext}^1(F_2, F_2) \end{array}$$

Lemma 25. *The horizontal arrows of the previous diagram are injective morphisms.*

Proof. Consider the first row: it comes from the exact sequence

$$0 \rightarrow \text{Hom}(F_1, F_1) \rightarrow \text{Hom}(F_1, G) \rightarrow \text{Hom}(F_1, A_1) \rightarrow \text{Ext}^1(F_1, F_1)$$

It suffices to show that the first arrow is an isomorphism, but it is trivial because by lemma 6 we have $\text{Hom}(F_1, F_1) \cong \text{End}(\mathbb{C}^r) \cong \text{Hom}(F_1, G)$. By applying the same argument to the other arrows, we get the thesis. \square

By condition (ii) the images of v_1 and v_2 in $\text{Ext}^1(F_1, F_2)$ coincide. Since the diagram 8 commutes and the second horizontal arrow is injective, the images of v_1 and v_2 actually coincide in $\text{Hom}(F_1, A_2)$. Recall that the sheaves A_1 and A_2 are supported at points and differ only at x , so we can decompose A_1 and A_2 in the following way:

$$A_1 = \bigoplus_j A_{x_j} \oplus B_1 \quad A_2 = \bigoplus_j A_{x_j} \oplus B_2$$

where B_1, B_2 are sheaves supported in x , and $\text{length}(B_1) - \text{length}(B_2) = i$ (remark that B_2 may be zero). So we can decompose the morphisms v_1, v_2 in

$$v_1 = \bigoplus_j v_{1,j} \oplus u_1 \quad v_2 = \bigoplus_j v_{2,j} \oplus u_2$$

where $v_{a,j} : (F_a)_{x_j} \rightarrow A_{x_j}$ and $u_a : (F_a)_x \rightarrow B_a$ for $a = 1, 2$. Since the images of v_1, v_2 in $\text{Hom}(F_1, A_2)$ coincide and F_1 differs from F_2 only at x , we have that $v_{1,j} = v_{2,j}$. Moreover at each point x_j the morphism $v_{2,j}$ factors through $F_2/G(-\mu_j C)$, so $v_{1,j}$ factors through $F_1/G(-\nu_j C)$; so it remains to show only that u_1 factors through $F_1/G(-\nu_{j_0} C)$ knowing that u_2 factors through $F_2/G(-\mu_{j_0} C)$, and $\nu_{j_0} = \mu_{j_0} + i$. Recall that in a neighborhood of x we have an inclusion morphism $F_2(-iC) \hookrightarrow F_1$: this induces a diagram similar to 8:

$$\begin{array}{ccc} \text{Hom}(F_2(-iC), A_2(-iC)) & \longrightarrow & \text{Ext}^1(F_2(-iC), F_2(-iC)) \\ \downarrow & & \downarrow \\ \text{Hom}(F_2(-iC), A_1) & \longrightarrow & \text{Ext}^1(F_2(-iC), F_1) \\ \uparrow & & \uparrow \\ \text{Hom}(F_1, A_1) & \longrightarrow & \text{Ext}^1(F_1, F_1) \end{array}$$

with injective horizontal arrows. Condition (ii) assures that the images of \tilde{v}_2, v_1 in $\text{Hom}(F_2(-iC), A_1)$ coincide, hence it follows that v_1 factors through $F_1/G(-\nu_{j_0} C)$ with $\nu_{j_0} = \mu_{j_0} + i$.

Now assume that $v_1 = 0$, we want to show that this implies that $v_2 = 0$. From the commutativity of diagram 7 we get the commutativity of the diagram

$$\begin{array}{ccc} F_2(-iC) & \longrightarrow & F_1 \\ \downarrow \tilde{v}_2 & & \downarrow v_1 \\ A_2(-iC) & \longrightarrow & A_1 \end{array}$$

The upper horizontal arrow in this diagram is injective by hypothesis and this implies that the lower one is injective too. Now from the fact that $v_1 = 0$ follows that $\tilde{v}_2 = 0$, hence $v_2 = 0$ and this concludes the proof of proposition 19.

7. PROOF OF PROPOSITION 20

In this section we provide a proof of proposition 20; to achieve this, we compute the intersection number of the fundamental classes of the cycles $L_{C'}^{(1^n)}(s)$ and $L_C^{(1^n)}(\mathcal{G})$ in $\mathcal{M}_{X,D}^{r,c}(n+s)$.

First note that any framed sheaf \mathcal{F} in $L_C^{(1^n)}(\mathcal{G})$ corresponds to the kernel of a composition

$$G \rightarrow G|_C \rightarrow A \rightarrow 0$$

where A is an Artin sheaf of length n supported on C . Thus $L_C^{(1^n)}(\mathcal{G})$ is isomorphic to $Quot_C^n(G|_C)$, that is the Quot scheme that parametrizes the length n Artin quotient sheaves of $G|_C$.

Let $\chi : G|_C \rightarrow A$ be an element in $Quot_C^n(G|_C)$. Denote by F_1 the kernel of χ (note that it is a locally free sheaf). By *Serre's duality* we have

$$\text{Ext}^1(F_1, A) = \text{Hom}(A \otimes F_1^\vee, \omega_C) = H^1(C, A \otimes F_1^\vee)$$

and $H^1(C, A \otimes F_1^\vee) = 0$ because A is a zero-dimensional sheaf, hence by proposition 2.2.8 of [13] we get that $Quot_C^n(G|_C)$ is smooth at the point A . Thus $L_C^{(1^n)}(\mathcal{G})$ is smooth.

Moreover there exists an open subset $L_{C'}^{(1^n)}(s)^\bullet$ in $L_{C'}^{(1^n)}(s)$ isomorphic to a fiber bundle over $\mathcal{M}_{X,D}^{reg}(r, c, s)$ with fibers $L_{C'}^{(1^n)}(\mathcal{H})$ for $\mathcal{H} \in \mathcal{M}_{X,D}^{reg}(r, c, s)$.

Finally we observe that $L_C^{(1^n)}(\mathcal{G})$ does not intersect $L_{C'}^{(1^n)}(s) \setminus L_{C'}^{(1^n)}(s)^\bullet$ because for any $\mathcal{F} \in L_C^{(1^n)}(\mathcal{G})$ the length of the Artin sheaf $F^{\vee\vee}/F$ is equal to n while, on the other hand, if $\mathcal{F} \in L_{C'}^{(1^n)}(s) \setminus L_{C'}^{(1^n)}(s)^\bullet$, then $\text{length}(F^{\vee\vee}/F) \geq n+1$.

Now we compute the intersection in the case where $X = \mathbb{CP}^2$ and C, C' are two distinct lines intersecting at $x \notin D$; at the end of the section we explain how the general case follows from this.

$X = \mathbb{CP}^2$, C, C' **distinct lines**. Let $\mathcal{F} = (F, \phi^F) \in L_C^{(1^n)}(\mathcal{G}) \cap L_{C'}^{(1^n)}(s)$; then $\mathcal{F}^{\vee\vee} \cong \mathcal{G}$, the quotient G/F is of length n and supported on both C and C' , hence it is supported only at x ; moreover $(G/F)_x$ is a vector space of dimension n , and it is a quotient of G_x , which is an r -dimensional vector space. So the intersection $L_C^{(1^n)}(\mathcal{G}) \cap L_{C'}^{(1^n)}(s)$ is parametrized by the n -dimensional quotients of G_x , i.e. by the Grassmann variety $Gr(G_x, n)$ (in particular we observe that if $n > r$ the intersection is empty).

Now $L_C^{(1^n)}(\mathcal{G})$ and $L_{C'}^{(1^n)}(s)$ have complementary dimension in $\mathcal{M}_{X,D}^{r,c}(s+n)$, but their set-theoretic intersection has positive dimension; however we are dealing with this situation:

$$\begin{array}{ccc} Gr(G_x, n) & \xrightarrow{j} & L_{C'}^{(1^n)}(s) \\ g \downarrow & \searrow h & \downarrow f \\ L_C^{(1^n)}(\mathcal{G}) & \xrightarrow{i} & \mathcal{M}_{X,D}^{r,c}(n+s) \end{array}$$

where i is a regular imbedding of codimension $2rs + rn + b$ and j is a regular imbedding of codimension $2rs + n^2 + b$ by theorem 17.12.1 in [12]. Thus we can apply the *excess intersection formula* (see chapter 6 in [10]) to get

$$(9) \quad i_*[L_C^{(1^n)}(\mathcal{G})] \cdot f_*[L_{C'}^{(1^n)}(s)] = h_*(c_{n(r-n)}(V) \cap [Gr(G_x, n)])$$

where

$$V = \frac{g^*N_{L_C^{(1^n)}(\mathcal{G})}(\mathcal{M}_{X,D}^{r,c}(n+s))}{N_{Gr(G_x, n)}(L_{C'}^{(1^n)}(s))}$$

is a locally free sheaf of rank $n(r-n)$. It is called *excess bundle*.

The vector bundle V arises also from the following exact sequence

$$0 \longrightarrow T_{Gr(G_x, n)} \longrightarrow g^*T_{L_C^{(1^n)}(\mathcal{G})} \oplus j^*T_{L_{C'}^{(1^n)}(s)} \longrightarrow h^*T_{\mathcal{M}_{X,D}^{r,c}(n+s)} \longrightarrow V \longrightarrow 0$$

Using this sequence, we obtain the following relation among the full Chern classes of the tangent bundles:

$$(10) \quad c(V) = \frac{c(h^*T_{\mathcal{M}_{X,D}^{r,c}(n+s)})c(T_{Gr(G_x, n)})}{c(g^*T_{L_C^{(1^n)}(\mathcal{G})})c(j^*T_{L_{C'}^{(1^n)}(s)})}$$

Remark 26. For any pair of flat family of sheaves F_1 and F_2 on $\mathcal{M}_{X,D}^{r,c}(n+s) \times \mathbb{C}P^2$, we denote by $\mathcal{E}xt_p^i(F_1, F_2)$ the i th right derived functor of $p_* \circ \mathcal{H}om$, where p is the projection from $\mathcal{M}_{X,D}^{r,c}(n+s) \times \mathbb{C}P^2$ to $\mathcal{M}_{X,D}^{r,c}(n+s)$. We denote by q the projection from $\mathcal{M}_{X,D}^{r,c}(n+s) \times \mathbb{C}P^2$ to the other factor. If for all $x \in \mathcal{M}_{X,D}^{r,c}(n+s)$ the global Ext group $\mathcal{E}xt^i(p^{-1}(x); F_1, F_2)$ on the fiber $p^{-1}(x)$ is of constant dimension, then one can prove that $\mathcal{E}xt_p^i(F_1, F_2)$ is a vector bundle on $\mathcal{M}_{X,D}^{r,c}(n+s)$ which has the global Ext group above as the fiber over point x (see [15]). Similar remarks apply to any closed subspace of $\mathcal{M}_{X,D}^{r,c}(n+s)$.

First we prove the following technical lemma:

Lemma 27. *Let (F, ϕ^F) be a framed sheaf on X . Then $\mathcal{E}xt^j(F^{\vee\vee}, F(-D)) = 0$ for $j = 0, 2$.*

Proof. Consider the short exact sequence $0 \rightarrow F \rightarrow F^{\vee\vee} \rightarrow A \rightarrow 0$ and apply the functor $\text{Hom}(F^{\vee\vee}, \cdot)$, we get

$$\begin{aligned} 0 &\rightarrow \text{Hom}(F^{\vee\vee}, F(-D)) \rightarrow \text{Hom}(F^{\vee\vee}, F^{\vee\vee}(-D)) \rightarrow \text{Hom}(F^{\vee\vee}, A(-D)) \\ &\rightarrow \text{Ext}^1(F^{\vee\vee}, F(-D)) \rightarrow \text{Ext}^1(F^{\vee\vee}, F^{\vee\vee}(-D)) \rightarrow \text{Ext}^1(F^{\vee\vee}, A(-D)) \\ &\rightarrow \text{Ext}^2(F^{\vee\vee}, F(-D)) \rightarrow \text{Ext}^2(F^{\vee\vee}, F^{\vee\vee}(-D)) \rightarrow \text{Ext}^2(F^{\vee\vee}, A(-D)) \rightarrow 0 \end{aligned}$$

Since A is zero-dimensional, we have $\text{Ext}^i(F^{\vee\vee}, A(-D)) = 0$ for $i = 1, 2$, hence $\text{Ext}^2(F^{\vee\vee}, F(-D)) \cong \text{Ext}^2(F^{\vee\vee}, F^{\vee\vee}(-D))$. By lemma 7 we get $\text{Ext}^j(F^{\vee\vee}, F^{\vee\vee}(-D)) = 0$ for $j = 0, 2$, hence we get the thesis. \square

Let $\bar{\mathcal{E}} = (\bar{E}, \phi^{\bar{E}})$ be the universal family on $\mathcal{M}_{X,D}^{r,c}(n+s) \times \mathbb{C}\mathbb{P}^2$. Recall that the tangent bundle to $\mathcal{M}_{X,D}^{r,c}(n+s)$ is $\mathcal{E}xt_p^1(\bar{E}, \bar{E} \otimes q^*O_{\mathbb{C}\mathbb{P}^2}(-D))$.

We have the following result for the sheaf $h^*\mathcal{E}xt_p^1(\bar{E}, \bar{E} \otimes q^*O_{\mathbb{C}\mathbb{P}^2}(-D))$.

Lemma 28. *Let Q be the universal quotient bundle on the Grassmann variety $Gr(G_x, n)$. Then the full Chern class of $h^*\mathcal{E}xt_p^1(\bar{E}, \bar{E} \otimes q^*O_{\mathbb{C}\mathbb{P}^2}(-D))$ is equal to $(c(Q)c(Q^\vee))^r$, where $c(Q)$ (resp. $c(Q^\vee)$) is the full Chern class of Q (resp. of its dual).*

Proof. Let $\tilde{\mathcal{E}} = (h \times \text{id}_{\mathbb{C}\mathbb{P}^2})^*\bar{\mathcal{E}}$; then by universal property we get

$$\tilde{\mathcal{E}}|_{\{A\} \times \mathbb{C}\mathbb{P}^2} \cong \mathcal{F}$$

where F is the kernel of the composition morphism $G \rightarrow G_x \rightarrow A$ for $A \in Gr(G_x, n)$ and the framed sheaf \mathcal{F} is F with the framing induced by inclusion to G .

We denote by I_x the ideal sheaf of x and by \mathbb{C}_x the quotient sheaf O_x/I_x . Recall that the inclusion morphism from a torsion free sheaf on X in its double dual sheaf induces a short exact sequence of sheaves on X ; using the universal sheaf $\tilde{\mathcal{E}}$ one can give a ‘universal version’ of this exact sequence and obtain an exact sequence of sheaves on $Gr(G_x, n) \times \mathbb{C}\mathbb{P}^2$:

$$(11) \quad 0 \longrightarrow \tilde{E} \longrightarrow q^*G \longrightarrow B \longrightarrow 0$$

where $B = p^*Q \otimes q^*\mathbb{C}_x$ (we denote by p and q the projection morphisms from $Gr(G_x, n) \times \mathbb{C}\mathbb{P}^2$ to the first factor and the second factor respectively).

We want to compute the full Chern class of the sheaf

$$h^*\mathcal{E}xt_p^1(\bar{E}, \bar{E} \otimes q^*O_{\mathbb{C}\mathbb{P}^2}(-D)) \cong \mathcal{E}xt_p^1(\tilde{E}, \tilde{E} \otimes q^*O_{\mathbb{C}\mathbb{P}^2}(-D))$$

Applying the functor $\mathcal{H}om_p(\cdot, \tilde{E} \otimes q^*O_{\mathbb{C}\mathbb{P}^2}(-D))$ to the exact sequence 11, we obtain a long exact sequence of sheaves on the Grassmann variety:

$$\begin{aligned} 0 &\longrightarrow \mathcal{E}xt_p^1(B, \tilde{E} \otimes q^*O_{\mathbb{C}\mathbb{P}^2}(-D)) \longrightarrow \mathcal{E}xt_p^1(q^*G, \tilde{E} \otimes q^*O_{\mathbb{C}\mathbb{P}^2}(-D)) \longrightarrow \\ &\longrightarrow \mathcal{E}xt_p^1(\tilde{E}, \tilde{E} \otimes q^*O_{\mathbb{C}\mathbb{P}^2}(-D)) \longrightarrow \mathcal{E}xt_p^2(B, \tilde{E} \otimes q^*O_{\mathbb{C}\mathbb{P}^2}(-D)) \longrightarrow 0 \end{aligned}$$

Indeed, the sheaf $\mathcal{H}om_p(\tilde{E}, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D))$ is zero since its fiber over $A \in Gr(G_x, n)$ is $\text{Hom}(F, F(-D))$, that vanishes by lemma 7. The sheaf $\mathcal{E}xt_p^2(q^*G, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D))$ is zero because its fiber $\text{Ext}^2(G, F(-D))$ over any point A of the Grassmann variety, that it is zero by lemma 27.

Thus the full Chern class of $\mathcal{E}xt_p^1(\tilde{E}, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D))$ is equal to:

$$\frac{c(\mathcal{E}xt_p^2(B, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D)))c(\mathcal{E}xt_p^1(q^*G, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D)))}{c(\mathcal{E}xt_p^1(B, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D)))}$$

Now we compute the three terms in the formula above.

To compute $c(\mathcal{E}xt_p^1(q^*G, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D)))$, first we tensor the exact sequence 11 by the invertible sheaf $q^*O_{\mathbb{C}P^2}(-D)$ and then apply the functor $\mathcal{H}om_p(q^*G, \cdot)$. We get

$$\begin{aligned} 0 &\longrightarrow \mathcal{H}om_p(q^*G, q^*G \otimes q^*O_{\mathbb{C}P^2}(-D)) \longrightarrow \mathcal{H}om_p(q^*G, B \otimes q^*O_{\mathbb{C}P^2}(-D)) \longrightarrow \\ &\longrightarrow \mathcal{E}xt_p^1(q^*G, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D)) \longrightarrow \mathcal{E}xt_p^1(q^*G, q^*G \otimes q^*O_{\mathbb{C}P^2}(-D)) \longrightarrow 0 \end{aligned}$$

The fiber over $A \in Gr(G_x, n)$ of the sheaf $\mathcal{H}om_p(q^*G, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D))$ is equal to $\text{Hom}(G, F(-D))$, that vanishes by lemma 27. Thus the sheaf $\mathcal{H}om_p(q^*G, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D))$ is zero.

The sheaf $\mathcal{E}xt_p^1(q^*G, B \otimes q^*O_{\mathbb{C}P^2}(-D))$ is zero, because its fiber over $A \in Gr(G_x, n)$ is $H^1(\mathbb{C}P^2, G^\vee \otimes A(-D)) = 0$. Moreover, the sheaf $\mathcal{E}xt_p^1(q^*G, q^*G \otimes q^*O_{\mathbb{C}P^2}(-D))$ is a trivial vector bundle, because its fiber over all the points of $Gr(G_x, n)$ is $\text{Ext}^1(G, G(-D))$.

Thus

$$c(\mathcal{E}xt_p^1(q^*G, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D))) = c(\mathcal{H}om_p(q^*G, B \otimes q^*O_{\mathbb{C}P^2}(-D)))$$

We have

$$\mathcal{H}om_p(q^*G, B \otimes q^*O_{\mathbb{C}P^2}(-D)) \cong \mathcal{H}om_p(q^*G, q^*(\mathbb{C}_x \otimes O_{\mathbb{C}P^2}(-D))) \otimes Q$$

because Q is a locally free sheaf.

Moreover the stalk of $\mathcal{H}om(q^*G, q^*(\mathbb{C}_x \otimes O_{\mathbb{C}P^2}(-D)))$ is equal to $\text{Hom}(G_x, \mathbb{C}_x) \cong \mathbb{C}^r$ over the points $(A, x) \in Gr(G_x, n) \times \mathbb{C}P^2$ and it is equal to zero over the points (A, y) with $y \neq x$. Thus the sheaf $\mathcal{H}om_p(q^*G, q^*(\mathbb{C}_x \otimes O_{\mathbb{C}P^2}(-D)))$ is the constant sheaf \mathbb{C}^r on $Gr(G_x, n)$.

Thus

$$c(\mathcal{H}om_p(q^*G, B \otimes q^*O_{\mathbb{C}P^2}(-D))) = c(\mathbb{C}^r \otimes Q) = c(Q)^r$$

Following Baranovsky's computations in the proof of lemma 6.1 in [3], we get

$$c(\mathcal{E}xt_p^1(B, \tilde{E} \otimes q^*O_{\mathbb{C}P^2}(-D))) = c(\mathcal{H}om_p(B, B \otimes q^*O_{\mathbb{C}P^2}(-D))) = c(Q^\vee \otimes Q)$$

and

$$c(\mathcal{E}xt_p^2(B, \tilde{E} \otimes q^*O_{\mathbb{CP}^2}(-D))) = c(Q^\vee \otimes Q)c(Q^\vee)^r$$

Summing up the results of our computation, we get

$$c(\mathcal{E}xt_p^1(\tilde{E}, \tilde{E} \otimes q^*O_{\mathbb{CP}^2}(-D))) = \frac{c(Q^\vee \otimes Q)c(Q^\vee)^r c(Q)^r}{c(Q^\vee \otimes Q)} = (c(Q^\vee)c(Q))^r$$

□

By this lemma we get

$$c(h^*T_{\mathcal{M}_{X,D}^{r,c}(n+s)}) = c(h^*\mathcal{E}xt_p^1(E, E \otimes q^*O_{\mathbb{CP}^2}(-D))) = (c(Q^\vee)c(Q))^r$$

A similar approach can be used with $T_{L_C^{(1^n)}(\mathcal{G})}$ and $T_{L_{C'}^{(1^n)}(s)}$ (as sheaves on $Gr(G_x, n) \times C$ and on $Gr(G_x, n) \times C'$). We obtain the following result:

Lemma 29. *Let Q be the universal quotient bundle on the Grassmann variety $Gr(G_x, n)$. Then*

$$c\left(g^*T_{L_C^{(1^n)}(\mathcal{G})}\right) = c\left(j^*T_{L_{C'}^{(1^n)}(s)}\right) = c(Q)^r$$

Proof. Note that we can consider $L_C^{(1^n)}(\mathcal{G})$ (resp. $L_{C'}^{(1^n)}(s)$) as a variety that parametrizes Artin quotient of $G|_C$ on C of length n (resp. Artin quotient of $G'|_{C'}$ on C' of length n , where $\mathcal{G}' \in \mathcal{M}_{X,D}^{reg}(r, c, s)$). From this point of view, we can use the same result of Baranovsky (see lemma 6.2 in [3]) and get the thesis. □

Let S be the universal subbundle on the Grassmann variety $Gr(G_x, n)$. Recall that $T_{Gr(G_x, n)} \cong S^\vee \otimes Q$. Now using formula 10, we get

$$c(V) = \frac{(c(Q^\vee)c(Q))^r c(S^\vee \otimes Q)}{c(Q)^{2r}} = \frac{c(Q^\vee)^r c(S^\vee \otimes Q)}{c(Q)^r}$$

Since $c(Q)^r = c(S^\vee \otimes Q)c(Q^\vee \otimes Q)$ and $c(Q^\vee)^r = c(S \otimes Q^\vee)c(Q \otimes Q^\vee)$, we obtain

$$c(V) = \frac{(c(Q^\vee)c(S^\vee \otimes Q))}{c(Q)^r} = \frac{c(S^\vee \otimes Q)c(S \otimes Q^\vee)c(Q^\vee \otimes Q)}{c(S^\vee \otimes Q)c(Q^\vee \otimes Q)} = c(S \otimes Q^\vee)$$

The vector bundle $S \otimes Q^\vee$ is the cotangent bundle on the Grassmann variety, hence we have

$$(c_{n(r-n)}(V) \cap [Gr(G_x, n)]) = (-1)^{n(r-n)} \int_{Gr(G_x, n)} c_{n(r-n)}(T_{Gr(G_x, n)})$$

and therefore

$$i_*[L_C^{(1^n)}(\mathcal{G})] \cdot f_*[L_{C'}^{(1^n)}(s)] = (-1)^{n(r-n)} \chi(Gr(G_x, n)) = (-1)^{(r-1)n} \binom{r}{n}$$

where we denote by $\chi(Gr(G_x, n))$ the *topological Euler characteristic* of $Gr(G_x, n)$.

General case. Recall that by hypothesis, $C \cap C' \cap D = \emptyset$. Now C and C' intersects in q points x_1, x_2, \dots, x_q . Put $a_i = (-1)^{(r-1)i} \binom{r}{i}$.

The intersection between $L_C^{(1^n)}(\mathcal{G})$ and $L_{C'}^{(1^n)}(s)$ is, set-theoretically, the set of framed sheaves \mathcal{F} such that $\mathcal{F}^{\vee\vee} \cong \mathcal{G}$ and the length of the quotient sheaf $A := G/F$ is n . Note that A is a quotient of $\bigoplus_{x \in C \cap C'} G_x$.

Consider a subdivision $\nu = (\nu_1, \nu_2, \dots, \nu_q)$ of n , i.e. an ordered q -pla of non-negative integers ν_1, \dots, ν_q such that $\sum_{i=1}^q \nu_i = n$. Let $x = \sum_i \nu_i [x_i]$ be a cycle with $x_i \in C \cap C'$.

Let W_ν be the subset of $L_C^{(1^n)}(\mathcal{G}) \cap L_{C'}^{(1^n)}(s)$ formed by all framed sheaves \mathcal{F} that are kernels of morphisms $G \rightarrow A$, where A is an Artin sheaf supported at the cycle x . Thus W_ν is isomorphic to a products of a Grassmanians $\prod_{x_i \in C \cap C'} Gr(G_{x_i}, \nu_i)$, hence it is a irreducible closed subset of $L_C^{(1^n)}(\mathcal{G}) \cap L_{C'}^{(1^n)}(s)$ (note that W_ν does not depend from the fixed cycle x but only from the subdivision).

Moreover, by definition of W_ν follows

$$L_C^{(1^n)}(\mathcal{G}) \cap L_{C'}^{(1^n)}(s) = \bigcup_{\nu \text{ subdivision}} W_\nu$$

and therefore the subsets W_ν are the irreducible components of $L_C^{(1^n)}(\mathcal{G}) \cap L_{C'}^{(1^n)}(s)$. To each W_ν we can associate a excess bundle V as before.

Since all sheaves split into direct products over the individual Grassmanians, we get that V is direct sum of the excess bundles associates to each individual Grassmanian. Thus the top Chern class of V is equal to the product $a_{\nu_1} \cdots a_{\nu_q}$.

By lemma 7.1 (a) and by definition 6.1.2 in [10], to get the intersection number between $L_C^{(1^n)}(\mathcal{G})$ and $L_{C'}^{(1^n)}(s)$ we need to sum up the products $a_{\nu_1} \cdots a_{\nu_q}$ over all subdivisions $\nu = (\nu_1, \nu_2, \dots, \nu_q)$ of n . By a combinatorial argument and using the binomial formula, we get

$$\sum_{n=0}^{\infty} z^{2n} \langle [L_C^{(1^n)}(\mathcal{G})], [L_{C'}^{(1^n)}(s)] \rangle = (1 + (-1)^{r-1} z^2)^{rq}.$$

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