On wave operators for Schrödinger operators with threshold singularities in three dimensions

K. Yajima*

Abstract

We show that wave operators for three dimensional Schrödinger operators $H = -\Delta + V$ with threshold singularities are bounded in $L^1(\mathbb{R}^3)$ if and only if zero energy resonances are absent from H and the existence of zero energy eigenfunctions does not destroy the L^1-boundedness of wave operators for H with the regular threshold behavior. We also show in this case that they are bounded in $L^p(\mathbb{R}^3)$ for all $1 \leq p \leq \infty$ if all zero energy eigenfunctions $\phi(x)$ have vanishing first three moments: $\int_{\mathbb{R}^3} x^\alpha V(x)\phi(x)dx = 0$, $|\alpha| = 0, 1, 2$.

1 Introduction

Let $H_0 = -\Delta$ be the free Schrödinger operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^m)$ with domain $D(H_0) = \{u \in \mathcal{H} \colon \partial^\alpha u \in \mathcal{H},\ |\alpha| \leq 2\}$ and $H = -\Delta + V$, V being the multiplication with real measurable function $V(x)$ such that $|V(x)| \leq C\langle x\rangle^{-\delta}$ for some $\delta > 2$, $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$. Then, it is well known that wave operators $W_\pm$ defined by the strong limits in $\mathcal{H} = L^2(\mathbb{R}^m)$:

$$W_\pm = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0} \tag{1.1}$$

exist, they are unitary from $\mathcal{H}$ to the absolutely continuous spectral subspace $\mathcal{H}_{ac}(H)$ of $\mathcal{H}$ for H and enjoy the intertwining property:

$$f(H)P_{ac}(H) = W_\pm f(H_0)W_\pm^* \tag{1.2}$$

for any Borel functions on $\mathbb{R}^1$, where $P_{ac}(H)$ is the orthogonal projection onto $\mathcal{H}_{ac}(H)$. The intertwining property reduces the mapping properties

*SISSA-International School for Advanced Studies, Via Bonomea, 34136 Trieste, Italy. On leave from Department of Mathematics, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan. Supported by JSPS research No.16K05242

of $f(H)P_{ac}(H)$ to those of $f(H_0)$ provided that corresponding properties of $W_\pm$ are already established. For this reason the L^p-boundedness of $W_\pm$ has attracted various authors' interest and many important results are obtained (see [11] and references therein).

We recall here the results for three dimensions, restricting ourselves to the case $m = 3$ in what follows. We omit $\mathbb{R}^3$ from $L^p(\mathbb{R}^3)$ and etc. We write $L^2_\sigma = L^2(\mathbb{R}^3, \langle x\rangle^{2\sigma} dx)$ for $\sigma \in \mathbb{R}$ and

$$\langle f, g\rangle = \int_{\mathbb{R}^3} f(x)g(x)dx$$

whenever the right hand side makes sense. Define for $1/2 < \sigma < \delta - 1/2$ that

$$\mathcal{N} = \{u \in L^2_{-\sigma} \colon u + (-\Delta)^{-1}Vu = 0\} \tag{1.3}$$

and

$$\mathcal{E} = \{u \in L^2 \colon u + (-\Delta)^{-1}Vu = 0\}. \tag{1.4}$$

The space $\mathcal{N}$ is independent of σ; all $u \in \mathcal{N}$ satisfy

$$-\Delta u + Vu = 0;$$

$\mathcal{E} \subset \mathcal{N}$ is the zero energy eigenspace of H; $u \in \mathcal{N}$ belongs to $\mathcal{E}$ if and only if $\langle V, u\rangle = 0$ and $\dim \mathcal{N}/\mathcal{E} \leq 1$. Functions $\phi \in \mathcal{N} \setminus \mathcal{E}$ are called resonances.

(a) If $\mathcal{N} = \{0\}$ then, $W_\pm$ are bounded for all $1 \leq p \leq \infty$ ([9]).

(b) If $\mathcal{N} \neq \{0\}$, then $W_\pm$ are in general bounded in L^p for $1 < p < 3$. They are bounded in L^p for all $1 < p < \infty$ if and only if all $\phi \in \mathcal{N}$ satisfy $\langle V, x^\alpha\phi\rangle = 0$ for $|\alpha| \leq 1$ ([11]).

In this paper, being inspired by the approached employed by Goldberg and Green ([3]) for proving the L^p-boundedness of wave operators including $p = 1$ for higher dimensional Schrödinger operators with threshold singularities, we prove the following theorem for the end point cases $p = 1$ and $p = \infty$ for three dimensions which are missing from the results mentioned in (b).

Theorem 1.1. *Suppose that $|V(x)| \leq C\langle x\rangle^{-7-\varepsilon}$. Then, $W_\pm$ are bounded in L^1 if and only if $\mathcal{N} = \mathcal{E}$. They are bounded also in L^∞ if all $\phi \in \mathcal{N}$ satisfy $\langle V, x^\alpha\phi\rangle$ for $|\alpha| \leq 2$.*

Incidentally, the method of the proof of the theorem may be used for the proof of the "if" part of results in (b) which is different from the one given in [11]. We present it here, however, only for $1 < p < 3$ for avoiding too much repetition. We think that the assumption on V of the theorem is unnecessarily too strong, however, we do not pursue better conditions here.

We refer readers more about the L^p boundedness of wave operators to [11] and the literature therein and jump into the proof of the theorem immediately. We recall that $\phi \in \mathcal{N}$ satisfies

$$\phi(x) + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{V(y)\phi(y)}{|x-y|}dy = 0 \tag{1.5}$$

and how fast $\phi \in \mathcal{E}$ decays as $|x| \to \infty$ depends on how many first moments of $V\phi$ vanish: For $k = 0, 1, \ldots,$

$$\langle V, x^\alpha \phi\rangle = 0 \text{ for } |\alpha| \leq k \ \Rightarrow\ |\phi(x)| \leq C\langle x\rangle^{-(2+k)}, \quad x \in \mathbb{R}^3, \tag{1.6}$$

whereas for resonances

$$\phi(x) = c|x|^{-1} + O(|x|^{-2}), \quad c = -\langle V, \phi\rangle/4\pi \neq 0. \tag{1.7}$$

We shall often use Schur's lemma that the integral operator

$$Ku(x) = \int_Y K(x,y)d\nu(y) \tag{1.8}$$

is bounded from $L^p(Y, d\nu)$ to $L^p(X, d\mu)$ for all $1 \leq p \leq \infty$ if $K(x,y)$ satisfies

$$\sup_y \int_X |K(x,y)|d\mu(x) < \infty, \quad \sup_x \int_Y |K(x,y)|d\nu(y) < \infty. \tag{1.9}$$

We say that the integral kernel $K(x,y)$ is *admissible* if it satisfies (1.9). We often identify the integral operator K defined by (1.8) with its integral kernel $K(x,y)$ and say that $K(x,y)$ is an L^p-bounded kernel if K is bounded in L^p. We write $\chi(F)$ for the characteristic function of the set F and $a \leq_{|\cdot|} b$ means $|a| \leq |b|$.

Acknowledgement This work has been completed while the author is on leave from Gakushuin University and is visiting SISSA in Trieste. He would like to express his sincere gratitude to both institutions for making such a visit possible and for the warm hospitality in SISSA, to Gianfausto Dell'Antonio and Alessandro Michelangeli in particular.

2 Reduction to the low energy analysis

We prove the theorem only for W_- and write $W_- = W$ in the sequel. The conjugation $\mathcal{C}u(x) = \overline{u(x)}$ changes the direction of time and results for W_+

follows automatically from the ones for W_-. We write for $\lambda \in \mathbb{C}^+$, $\mathbb{C}^+$ being the upper half plane,

$$G_0(\lambda) = (H_0 - \lambda^2)^{-1}, \quad G(\lambda) = (H - \lambda^2)^{-1}.$$

The limiting absorption principle and the absence of positive eigenvalues imply that, for $\sigma > 1$, boundary values of $G_0(\lambda)$ for $\lambda \in \mathbb{R}$ and $G(\lambda)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ exist in $\mathbf{B}(L^2_\sigma, L^2_{-\sigma})$ and W can be represented via the boundary values in the following form (e.g. [1, 7]):

$$Wu = u - \frac{1}{\pi i} \int_0^\infty G(\lambda)V(G_0(\lambda) - G_0(-\lambda))u\lambda d\lambda \tag{2.1}$$

We decompose W into the high and the low energy parts

$$W = W_> + W_< \equiv W\Psi(H_0) + W\Phi(H_0), \tag{2.2}$$

by using the cut off functions $\Phi \in C_0^\infty(\mathbb{R})$ and $\Psi \in C^\infty(\mathbb{R})$ such that

$$\Phi(\lambda^2) + \Psi(\lambda^2) \equiv 1, \quad \Phi(\lambda^2) = 1 \text{ near } \lambda = 0 \text{ and } \Phi(\lambda^2) = 0 \text{ for } |\lambda| > \lambda_0$$

for a small constant $\lambda_0 > 0$. We have proven in the previous paper [10] that, under the assumption of this paper, $W_>$ is bounded in $L^p(\mathbb{R}^3)$ for all $1 \leq p \leq \infty$ and we have nothing to add in this paper for $W_>$. Thus, in what follows, we shall be devoted to studying the low energy part:

$$W_< = \Phi(H_0) - \frac{1}{\pi i} \int_0^\infty G(\lambda)V(G_0(\lambda) - G_0(-\lambda))\lambda\Phi(H_0)d\lambda. \tag{2.3}$$

Evidently $\Phi(H_0) \in \mathbf{B}(L^p(\mathbb{R}^3))$ for all $1 \leq p \leq \infty$ and we have only to study the operator Z defined by the integral of (2.3), which we rewrite as

$$Zu = -\frac{1}{\pi i} \int_0^\infty G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}(G_0(\lambda) - G_0(-\lambda))\lambda F(\lambda)ud\lambda \tag{2.4}$$

by using the resolvent identity $G(\lambda)V = G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}$ for $\lambda > 0$ and by defining $F(\lambda) = \Phi(\lambda^2)$.

2.1 Low energy behavior of $(1 + G_0(\lambda)V)^{-1}$.

Following [5], we say that H is of exceptional type of *the first kind* if $\mathcal{E} = \{0\}$, *the second* if $\mathcal{E} = \mathcal{N}$ and *the third kind* if $\{0\} \subsetneq \mathcal{E} \subsetneq \mathcal{N}$. The orthogonal projection in $\mathcal{H}$ onto the eigenspace $\mathcal{E}$ will be denoted by P. We let $D_0, D_1, \ldots$ be the integral operators defined by

$$D_j u(x) = \frac{1}{4\pi j!} \int_{\mathbb{R}^3} |x - y|^{j-1} u(y)dy, \quad j = 0, 1, \ldots.$$

so that we have a formal Taylor expansion

$$G_0(\lambda)u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i\lambda|x-y|}}{|x-y|}u(y)dy = \sum_{j=1}^{\infty}(i\lambda)^j D_j u.$$

If H is of exceptional type of the third kind, $-(Vu, u)$ is an inner product of $\mathcal{N}$ and there exists a unique $\psi \in \mathcal{N}$ such that

$$-(V\psi, u) = 0, \quad \forall u \in \mathcal{E}, \quad -(V\psi, \psi) = 1 \text{ and } (1, V\psi) > 0.$$

We define

$$\varphi = \psi + PVD_2V\psi \in \mathcal{N} \tag{2.5}$$

and call it *the canonical resonance* ([5]). If H is of exceptional type of the first kind, then $\dim \mathcal{N} = 1$ and there is a unique $\varphi \in \mathcal{N}$ such that $-(V\varphi, \varphi) = 1$ and $(1, V\varphi) > 0$ and we call this the canonical resonance. We have the following result (see e.g. [10]).

Proposition 2.1. *Let V satisfy $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > 3$. Suppose that H is of exceptional type of the third kind and let φ be the canonical resonance and $a = 4\pi i|\langle V, \varphi\rangle|^{-2}$. Then:*

$$(I + G_0(\lambda)V)^{-1} = \frac{PV}{\lambda^2} - \frac{PVD_3VPV}{\lambda} - \frac{a}{\lambda}|\varphi\rangle\langle\varphi|V + E(\lambda), \tag{2.6}$$

where $E(\lambda)$ is the operator valued function which, when substituted for $(1 + G_0(\lambda)V)^{-1}$ in (2.4), produces the operator which is bounded in L^p for all $1 \le p \le \infty$. If H is of exceptional type of the first or the second kind, (2.6) still holds with $P = 0$ or $\varphi = 0$ respectively.

3 L^1-unboundedness with resonances

If zero energy resonances are present, then Proposition 2.1 shows that their contribution to the wave operator W is given via the canonical resonance φ by

$$Z_r u = -\frac{ia}{\pi}\int_0^{\infty} G_0(\lambda)V\varphi\rangle\langle V\varphi|(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)d\lambda, \tag{3.1}$$

where $a = 4\pi i|\langle V, \varphi\rangle|^{-2} \neq 0$. We show that Z_r is bounded in L^p for $1 < p < 3$ but not for $p = 1$.

Lemma 3.1. *Suppose that* $K(x,y) \le_{|\cdot|} C\langle x\rangle^{-1}\langle y\rangle^{-1}\langle |x|-|y|\rangle^{-2-\varepsilon}$ *for some* $\varepsilon \ge 0$. *Then,*

$$Ku(x) = \int_{\mathbb{R}^3} K(x,y)u(y)dy$$

is bounded in L^p *for any* $1 \le p \le \infty$ *if* $\varepsilon > 0$ *and, for* $1 < p < \infty$ *if* $\varepsilon = 0$.

Proof. For $\varepsilon > 0$, the lemma follows from Schur's lemma. When $\varepsilon = 0$, we have

$$Ku(x) \le_{|\cdot|} C \int_0^\infty \frac{r^{2-\frac{2}{p}} r^{\frac{2}{p}} |M_u(r)| dr}{\langle x\rangle\langle r\rangle\langle |x|-r\rangle^2}, \quad M_u(r) = \int_{\mathbb{S}^2} u(r\omega)d\omega.$$

Since the right side is rotationally invariant, we have

$$\begin{aligned}\|Ku\|_p^p &\le C\int_0^\infty \left(\int_0^\infty \frac{\rho^{\frac{2}{p}} r^{2-\frac{2}{p}} (r^{\frac{2}{p}}|M_u(r)|)dr}{\langle\rho\rangle\langle r\rangle\langle\rho-r\rangle^2}\right)^p d\rho \\ &\le C\int_0^\infty \left(\int_0^\infty \frac{\langle\rho\rangle^{\frac{2}{p}-1}\langle r\rangle^{1-\frac{2}{p}} (r^{\frac{2}{p}}|M_u(r)|)dr}{\langle\rho-r\rangle^2}\right)^p d\rho\end{aligned}$$

We may estimate $\langle\rho\rangle^{\frac{2}{p}-1}\langle r\rangle^{1-\frac{2}{p}}$ by $C\langle\rho-r\rangle^{\frac{2}{p}-1}$ if $p \le 2$ and $C\langle\rho-r\rangle^{1-\frac{2}{p}}$ if $p \ge 2$. It follows that unless $p = 1$ or $p = \infty$ we have with $\gamma > 1$ that

$$\|Ku\|_p \le \left(\int_0^\infty \left(\int_0^\infty \frac{r^{\frac{2}{p}}|M_u(r)|dr}{\langle\rho-r\rangle^\gamma}\right)^p d\rho\right)^{1/p}$$

and Young's and Hölder's inequalities imply

$$\|Ku\|_p \le C\left(\int_0^\infty r^2|M_u(r)|^p dr\right)^{1/p} \le C\|u\|_p.$$

This completes the proof of lemma. □

Proposition 3.2. *Let* Z_r *be the operator defined by* (3.1). *Then,* Z_r *is bounded in* L^p *for* $1 < p < 3$ *but not for* $p = 1$.

Proof. It is known that Z_r is bounded in L^p for $1 < p < 3$ ([11]). Nevertheless, we give the proof for $1 < p < 3$ which is different from the one given in [11]. The integral kernel of Z_r is given by

$$Z_r(x,y) = \sum_{\pm} \frac{\mp ia}{\pi}\int_0^\infty \int e^{i\lambda(|x-z|\pm|w-y|)} \frac{(V\phi)(z)(V\phi)(w)F(\lambda)}{16\pi^2|x-z||w-y|} dwdzd\lambda. \tag{3.2}$$

Since $F \in C_0^\infty([0,\infty))$, we immediately see that, with a constant $C>0$,

$$Z_r(x,y) \le_{|\cdot|} C \int \frac{|(V\phi)(z)(V\phi)(w)|}{|x-z||w-y|} dwdz \le \frac{C}{\langle x\rangle\langle y\rangle} \tag{3.3}$$

and $\chi(||x|-|y|| \le 1) Z_r(x,y)$ and $\chi(||x|^2-|y|^2| \le 1) Z_r(x,y)$ are admissible kernels. Indeed, we have

$$\sup_y \int_{||x|-|y||\le 1} \frac{dx}{\langle x\rangle\langle y\rangle} = \sup_x \int_{||x|-|y||\le 1} \frac{dy}{\langle x\rangle\langle y\rangle} = C < \infty,$$

$\{(x,y)\colon ||x|^2-|y|^2| \le 1, ||x|-|y|| > 1\} \subset \{(x,y)\colon |x|<1, |y|<1\}$ and $C\langle x\rangle^{-1}\langle y\rangle^{-1}$ is obviously admissible on $\{(x,y)\colon |x|<1, |y|<1\}$. Thus *we may and do ignore the parts of $Z_r(x,y)$ where $||x|-|y||<1$ or $||x|^2-|y|^2|<1$ in the proof*. We decompose the exponential functions $e^{i\lambda|x-z|}$ and $e^{i\lambda|w-y|}$ in the form

$$e^{i\lambda|x-z|} = e^{i\lambda|x|} + e^{i\lambda|x|} r(\lambda,x,z), \quad r(\lambda,x,z) = e^{i\lambda(|x-z|-|x|)} - 1, \tag{3.4}$$
$$e^{i\lambda|w-y|} = e^{i\lambda|y|} + e^{i\lambda|y|} r(\lambda,y,w), \quad r(\lambda,y,w) = e^{i\lambda(|w-y|-|y|)} - 1, \tag{3.5}$$

and write $Z_r(x,y)$ as a sum of four kernels:

$$Z_r(x,y) = \frac{-ia}{\pi} \sum_{j=1}^{4} \iint_{\mathbb{R}^6} \frac{(V\phi)(z)(V\phi)(w)}{16\pi^2|x-z||w-y|} F_j(x,z,w,z) dwdz$$

where F_j, $j=1,2,3,4$ are respectively given by

$$F_1 = F_1(x,z,w,y) = \sum \pm \int_0^\infty e^{i\lambda(|x|\pm|y|)} r(\lambda,x,z) r(\pm\lambda,y,w) F(\lambda) d\lambda,$$
$$F_2 = F_2(x,w,y) = \sum \pm \int_0^\infty e^{i\lambda(|x|\pm|y|)} r(\pm\lambda,y,w) F(\lambda) d\lambda,$$
$$F_3 = F_3(x,z,y) = \sum \pm \int_0^\infty e^{i\lambda(|x|\pm|y|)} r(\lambda,x,z) F(\lambda) d\lambda,$$
$$F_4 = F_4(x,y) = \sum \pm \int_0^\infty e^{i\lambda(|x|\pm|y|)} F(\lambda) d\lambda.$$

Here and hereafter the sum $\sum$ is taken for $+$ and $-$. We estimate $F_1,\dots,F_4$ using integration by parts. We use the following properties of $r(t,x,y)$ for $k=1,2,3$:

$$r(0,x,y) = 0, \quad \partial_\lambda(r(\pm\lambda,x,y))|_{\lambda=0} = \pm i(|x-y|-|x|), \tag{3.6}$$
$$\partial_\lambda^k r(\lambda,x,y) \le |y|^k. \tag{3.7}$$

(1) We first show that $Z_1(x,y)$ is an admissible kernel. We apply integration by parts three times to F_1. Then, (3.6) and (3.7) imply

$$\begin{aligned} F_1(x,z,w,y) &= \frac{\mp i}{(|x|\pm|y|)^3}\partial_\lambda^2\{r(\lambda,x,z)r(\pm\lambda,y,w)F(\lambda)\}|_{\lambda=0} \\ &+ \frac{\mp i}{(|x|\pm|y|)^3}\int_0^\infty e^{i\lambda(|x|\pm|y|)}\partial_\lambda^3\{r(\lambda,x,z)r(\pm\lambda,y,w)F(\lambda)\}d\lambda \\ &\le_{|\cdot|} \sum C\frac{(1+|z|+|w|)^3}{(|x|\pm|y|)^3} \le C\frac{(1+|z|+|w|)^3}{(|x|-|y|)^3}. \end{aligned}$$

It follows that

$$Z_1(x,y) \le_{|\cdot|} C\int_{\mathbb{R}^6}\frac{(1+|z|+|w|)^3|(V\phi)(z)(V\phi)(w)|}{(|x|-|y|)^3|x-z||w-y|}dwdz \le \frac{C}{(|x|-|y|)^3\langle x\rangle\langle y\rangle}$$

and $Z_1(x,y)$ is admissible by virtue of Lemma 3.1 (recall that we are ignoring (x,y) with $||x|-|y||<1$ or $||x|^2-|y|^2|<1$).
(2) We apply integration parts twice to F_2 and write it in the form

$$\sum\frac{-i(|w-y|-|y|)}{(|x|\pm|y|)^2} + \sum\mp\int_0^\infty e^{i\lambda(|x|\pm|y|)}\frac{\partial_\lambda^2\{r(\pm\lambda,y,w)F(\lambda)\}}{(|x|\pm|y|)^2}d\lambda.$$

After another integration by parts we see that the integral terms are bounded by $C(1+|w|)^3(|x|\pm|y|)^{-3}$ and, when inserted into $Z_2(x,y)$, they produce admissible kernels bounded by $C\langle x\rangle^{-1}\langle y\rangle^{-1}(|x|\pm|y|)^{-3}$. Thus, modulo the admissible kernel

$$Z_2(x,y) \equiv \sum\frac{a}{\pi}\phi(x)\int_{\mathbb{R}^3}\frac{(|w-y|-|y|)(V\phi)(w)}{4\pi(|x|\pm|y|)^2\cdot|w-y|}dw. \tag{3.8}$$

Note that this is bounded in modulus by $C\langle x\rangle^{-1}\langle y\rangle^{-1}(|x|-|y|)^{-2}$ and Z_2 is bounded in L^p for any $1<p<\infty$ by virtue of Lemma 3.1.
(3) For F_3, we apply integration by twice as in (2):

$$F_3 = \sum\mp\frac{i(|z-x|-|x|)}{(|x|\pm|y|)^2} + \sum_\pm\mp\int_0^\infty e^{i\lambda(|x|\pm|y|)}\frac{\partial_\lambda^2\{r(\lambda,x,z)F(\lambda)\}}{(|x|\pm|y|)^2}d\lambda.$$

By applying integration by parts once more as in (2) we see that the integral terms are bounded by $C(1+|z|)^3(|x|\pm|y|)^{-3}$ and their sum produces the kernel bounded by $C(|x|-|y|)^{-3}\langle x\rangle^{-1}\langle y\rangle^{-1}$ when inserted into $Z_3(x,y)$, which is admissible. Thus modulo the admissible kernel

$$Z_3(x,y) \equiv \sum\frac{\mp a}{\pi}\int_{\mathbb{R}^6}\frac{(|z-x|-|x|)(V\phi)(z)(V\phi)(w)}{16\pi^2(|x|\pm|y|)^2|x-z||w-y|}dzdw$$

$$\leq_{|\cdot|} \frac{4a|x||y|\phi(y)}{\pi(|x|+|y|)^2(|x|-|y|)^2}\int_{\mathbb{R}^3}\frac{|z||(V\phi)(z)|}{4\pi|x-z|}dz \leq_{|\cdot|} \frac{C}{(|x|+|y|)^2(|x|-|y|)^2},$$

where we used (1.5) and (1.6) for $k=0$ in the second and the third step respectively. Thus, $Z_3(x,y)$ is admissible.
(4) Again an integration by parts shows that

$$F_4(x,y) = \sum \frac{\pm i}{(|x|\pm|y|)} + \sum \frac{\pm i}{(|x|\pm|y|)}\int_0^\infty e^{i\lambda(|x|\pm|y|)}F'(\lambda)d\lambda$$

Here $F' \in C_0^\infty((0,\infty))$ and the integral terms are bounded by $C\langle|x|\pm|y|\rangle^{-N}$ for any N. It follows that the sum of the integral terms produces an admissible kernel $C\langle|x|-|y|\rangle^N\langle x\rangle^{-1}\langle y\rangle^{-1}$ and, by virtue of (1.5), modulo the admissible kernel

$$Z_4(x,y) \equiv \sum \frac{\pm a\phi(x)\phi(y)}{\pi(|x|\pm|y|)}.$$

(5) We prove that Z_r is unbounded in L^1. The combination of (1) to (4) implies that modulo admissible kernel $Z_r(x,y)$ is equal to

$$\begin{aligned} Z_{red}(x,y) = \sum \frac{a}{\pi}\phi(x)\left(\frac{1}{(|x|+|y|)^2}+\frac{1}{(|x|-|y|)^2}\right)(-c+|y|\phi(y)) \\ + \frac{a}{\pi}\left(\frac{\phi(x)\phi(y)}{|x|+|y|}-\frac{\phi(x)\phi(y)}{|x|-|y|}\right) \equiv R_1(x,y)+R_2(x,y) \quad (3.9)\end{aligned}$$

where the constant $c \neq 0$ is defined in (1.7). We prove that the operator Z_{red} defined by $Z_{red}(x,y)$ is unbounded in L^1 by contradiction. So assume that Z_{red} is bounded in L^1. Take $u \in C_0^\infty(\mathbb{R}^3)$ such that $u(x)\geq 0$, $u(x)=0$ for $|x|\geq 1$ and $\int_{\mathbb{R}^3} u(x)dx = 1$ and, define

$$u_n(x) = n^3u(nx), \quad f_n(x) = \int_{\mathbb{R}^3} Z_{red}(x,y)u_n(y)dy, \quad n = 1,2,\ldots.$$

We have $\|u_n\|_1 = 1$, $n = 1,2,\ldots$. For any $2 \leq |x|$, we evidently have

$$\lim_{n\to\infty} f_n(x) = -\frac{2ac}{\pi}\frac{\phi(x)}{|x|^2}.$$

It follows by Fatou's lemma that

$$\begin{aligned} \frac{2|ac|}{\pi}\int_{2<|x|}\frac{|\phi(x)|}{|x|^2}dx \leq \liminf_{n\to\infty}\int_{2<|x|}|f_n(x)|dx \\ \leq \liminf_{n\to\infty}\|Z_{red}u_n\|_{L^1} \leq \|Z_{red}\|_{\mathbf{B}(L^1)} < \infty. \quad (3.10)\end{aligned}$$

By virtue of (1.7), $|\phi(x)| \geq C|x|^{-1}$ for a constant $C > 0$ for sufficiently large $|x|$ and (3.10) cannot happen. Thus, Z_{red} cannot be bounded in L^1.
(6) We finally prove that Z_{red} is bounded in L^p for $1 < p < 3$. We have shown in (2) that $R_1(x,y)$ of the right of (3.9) which is equal to (3.8) is L^p-bounded kernel for $1 < p < \infty$ and it suffices to show that

$$R_2(x,y) = -\frac{2a}{\pi}\frac{\phi(x)\phi(y)|y|}{|x|^2 - |y|^2}$$

is an L^p-bounded kernel on $\{(x,y)\colon |x|^2 - |y|^2 \geq 1\}$ for $1 < p < 3$. Since $\phi(x)|x| \in L^\infty$, it suffices to consider

$$Tu(x) = \int_{||x|^2-|y|^2|\geq 1} \frac{|x|^{-1}}{|x|^2 - |y|^2} u(y)dy$$

Since $Tu(x)$ is spherically symmetric, we have by using polar coordinates and by changing variables

$$\|Tu\|_p^p \leq 2^{-1-p}\pi \int_0^\infty \rho^{\frac{1}{2}-\frac{p}{2}} \left(\int_{|\rho - r|\geq 1} \frac{r^{\frac{1}{2}}}{\rho - r} |M_u(\sqrt{r})| dr \right)^p d\rho,$$

where $M_u(r) = \int_{\mathbb{S}^2} u(r\omega)d\omega$. Here $-1 < \frac{1}{2} - \frac{p}{2} < p - 1$ if $1 < p < 3$ and $\rho^{\frac{1}{2}-\frac{p}{2}}$ is an $(A)_p$ weight on $\mathbb{R}$. It follows by the weighted inequality (see e.g. Theorem 9.4.6 of [4]) that

$$\|Tu\|_p^p \leq C \int_0^\infty r^{\frac{1}{2}} |M_u(\sqrt{r})|^p dr \leq C \int_0^\infty r^2 |M_u(r)|^p dr \leq C\|u\|_p.$$

This completes the proof. □

Remark 3.3. If $\phi \in \mathcal{E}$, then $|\phi(x)| \leq C\langle x\rangle^{-2}$ and

$$|R_1(x,y)| + |R_2(x,y)| \leq C\langle x\rangle^{-2}\langle y\rangle^{-1}(|x| - |y|)^{-2}.$$

Hence Lemma 3.1 implies both R_1 and R_2 are L^p-bounded kernels for all $1 < p < \infty$. They are bounded also in L^1 because

$$\sup_y \int_{||x|-|y||\geq 1} \frac{dx}{\langle x\rangle^2 \langle y\rangle (|x| - |y|)} \leq 4\pi \sup_y \int_0^\infty \frac{dr}{\langle y\rangle\langle r - |y|\rangle^2} < \infty.$$

4 Contribution of zero-energy eigenfunctions

The following proposition together with Proposition 2.1 and Proposition 3.2 proves Theorem 1.1.

Proposition 4.1. *Let H be of exceptional type of the second kind. Then, $W_\pm$ is bounded in L^p for $1 \le p < 3$ in general. If $\langle V, x^\alpha \phi\rangle = 0$, $|\alpha| \le 2$, for all $\phi \in \mathcal{N}$, then $W_\pm$ is bounded in L^p for all $1 \le p \le \infty$.*

If H is of exceptional type of the second kind, then

$$S(\lambda) = \frac{PV}{\lambda^2} - \frac{PVD_3VPV}{\lambda}, \tag{4.1}$$

where D_3 is the operator of rank five with the kernel $-i|x-y|^2/4\pi$ and all $\phi \in \mathcal{E}$ satisfy $|\phi(x)| \le C\langle x\rangle^{-2}$ for a constant $C > 0$. We take the real orthonormal basis $\{\phi_1, \dots, \phi_d\}$ of $\mathcal{E}$ and write $Z_s u = Z_{s0}u + Z_{s1}u$, where with $a_{jk} = i\pi^{-1}\langle \phi_j | VD_3V | \phi_k\rangle \in \mathbb{R}$,

$$Z_{s0}u = \sum_{j,k=1}^{d} a_{jk} \int_0^\infty G_0(\lambda)V\phi_j\rangle\langle V\phi_k|(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)d\lambda \tag{4.2}$$

and, with extra singularity λ^{-1},

$$Z_{s1}u = \sum_{j=1}^{d} \frac{i}{\pi} \int_0^\infty G_0(\lambda)V\phi_j\rangle\langle V\phi_j|(G_0(\lambda) - G_0(-\lambda))u\rangle F(\lambda)\frac{d\lambda}{\lambda}. \tag{4.3}$$

Lemma 4.2. (1) *The operator Z_{s0} is bounded in L^p for any $1 \le p < \infty$.*
(2) *If all $\phi \in \mathcal{E}$ satisfy $\int_{\mathbb{R}^3} x_j V(x)\phi(x)dx = 0$ for all $j = 1, 2, 3$ then, Z_{s0} is bounded in L^p for all $1 \le p \le \infty$.*

Proof. Z_{s0} is the sum of $Z_{s0,ij}$, $1 \le i, j \le d$, whose kernels have the same structure as $Z_r(x, y)$ of (3.1) with only change of the constant a by $i\pi a_{ij}$ and of the canonical resonance φ by eigenfunctions ϕ_i and $\phi_j \in \mathcal{E}$, which satisfy all properties of φ which are necessary for proving Proposition 3.2. Then, if we proceed as in the proof of Proposition 3.2, all integral kernels which appear there are admissible except $R_{1,ij}(x, y)$ and $R_{2,ij}(x, y)$ which correspond to $R_1(x, y)$ and $R_2(x, y)$ of (3.9). If $\phi_i, \phi_j \in \mathcal{E}$, however, as remarked in Remark 3.3, they are estimated as

$$R_{1,ij}(x, y) = \sum_\pm C_{ij} \frac{\phi_i(x)|y|\phi_j(y)}{(|x| \pm |y|)^2} \le_{|\cdot|} C \sum_\pm \frac{\langle x\rangle^{-2}\langle y\rangle^{-1}}{(|x| \pm |y|)^2} \tag{4.4}$$

and likewise for $R_{2,ij}(x,y)$. Thus, they are L^p-bounded for $1 \le p < \infty$. If $\phi_i, \phi_j \in \mathcal{E}$ further satisfy the condition of (2), the right of (4.4) is bounded by $\langle x\rangle^{-3}\langle y\rangle^{-2}\langle |x|-|y|\rangle^{-2}$ and they are admissible kernels (recall that we are ignoring the part $\{(x,y)\colon ||x|-|y|| \le 1\}$). □

Lemma 4.3. (1) *The operator Z_{s1} is bounded in L^p for $1 \le p < 3$.*
(2) *If all $\phi \in \mathcal{E}$ satisfy $\langle V, x^\alpha \phi\rangle = 0$ for $|\alpha| \le 2$. Then Z_{s1} is bounded in L^p for all $1 \le p \le \infty$.*

Proof. Define for $j = 1, \dots, d$ that

$$Z_{s1,j}u = \frac{i}{\pi}\int_0^\infty G_0(\lambda)|V\phi_j\rangle\langle V\phi_j|(G_0(\lambda)-G_0(-\lambda))u\rangle F(\lambda)\lambda^{-1}d\lambda, \qquad (4.5)$$

so that $Z_{s1}u = \sum_{j=1}^d Z_{s1,j}u$. We prove the lemma only for $Z_{s1,1}$, which we denote by Z for short, as the proof for others is similar. As in the proof of Proposition 3.2, we decompose Z as

$$Zu = Z_1u + Z_2u + Z_3u + Z_4u$$

by splitting $e^{i\lambda|x-y|}$ and etc. as $e^{i\lambda|x-y|} = e^{i\lambda|x|} + e^{i\lambda|x|}r(\lambda,x,y)$ and etc. Thus the integral kernels of $Z_1, \dots, Z_4$ are given respectively by

$$Z_1 = \sum_{\pm}\frac{\pm i}{\pi}\int_0^\infty\int e^{i\lambda(|x|\pm|y|)}\frac{r(\lambda,x,z)r(\pm\lambda,w,y)V\phi(z)V\phi(w)F(\lambda)}{16\pi^2|x-z||w-y|}dwdz\frac{d\lambda}{\lambda},$$
$$Z_2 = \sum_{\pm}\frac{\mp i}{\pi}\int_0^\infty\int e^{i\lambda(|x|\pm|y|)}\frac{r(\pm\lambda,w,y)\phi(x)(V\phi)(w)F(\lambda)}{4\pi|w-y|}dw\frac{d\lambda}{\lambda},$$
$$Z_3 = \sum_{\pm}\frac{\mp i}{\pi}\int_0^\infty\int e^{i\lambda(|x|\pm|y|)}\frac{r(\lambda,x,z)(V\phi)(z)\phi(y)F(\lambda)}{4\pi|x-z|}dz\frac{d\lambda}{\lambda},$$
$$Z_4u = \frac{i}{\pi}\int_0^\infty\sum_{\pm}\pm e^{i\lambda(|x|\pm|y|)}\phi(x)\phi(y)F(\lambda)\frac{d\lambda}{\lambda},$$

where we used (1.5) for simplifying Z_2, Z_3 and Z_4 and put the sign $\sum$ inside the integral as it diverges separately. We write

$$r(\lambda,x,y) = i\lambda(|x-y|-|x|)r_1(\lambda,x,y), \quad r_1 = \int_0^1 e^{i\lambda(|x-y|-|x|)\theta}d\theta \qquad (4.6)$$

and etc. We have for $k = 0, 1, \dots,$

$$r_1^{(k)}(\lambda,x,y) \le_{|\cdot|} |y|^k/(k+1). \qquad (4.7)$$

We estimate $Z_1 u, \dots, Z_4 u$ individually in the following four lemmas. We obviously have $Z_j(x,y) \leq_{|\cdot|} \langle x \rangle^{-1} \langle y \rangle^{-1}$ for $j = 1,2,3$ and this holds also for $j=4$ as $|\phi(y)(e^{i\lambda|y|} - e^{-i\lambda|y|})|\lambda^{-1} \leq |y||\phi(y)| \leq C\langle y \rangle^{-1}$. Thus, their parts on

$$\{(x,y)\colon ||x|-|y|| < 1\} \cup \{(x,y)\colon ||x|^2-|y|^2| < 1\} \cup \{(x,y)\colon |x|+|y| < 1\} \quad (4.8)$$

are admissible kernels and we again ignore this part in the following consideration.

Lemma 4.4. *Modulo an admissible kernel we have*

$$Z_1(x,y) \equiv \sum_{\pm} \frac{i}{\pi} \frac{|x||y|\phi(x)\phi(y)}{(|x| \pm |y|)^2}$$

and Z_1 is an L^p-bounded kernel for $1 < p < \infty$. If $\phi \in \mathcal{E}$ satisfies $\langle V, x^\alpha \phi \rangle = 0$ for $|\alpha| \leq 1$, $Z_1(x,y)$ is admissible.

Proof. By virtue of (4.6), we have

$$\begin{aligned} Z_1(x,y) = &-\frac{i}{\pi} \int \left(\sum_{\pm} \int_0^\infty e^{i\lambda(|x|\pm|y|)} r_1(\lambda,x,z) r_1(\pm\lambda,w,y) \lambda F(\lambda) d\lambda \right) \\ &\times \frac{(|x-z|-|x|)(|w-y|-|y|)(V\phi)(z)(V\phi)(w)}{16\pi^2|x-z||w-y|} dwdz \\ = &-\frac{i}{\pi} \int W_1(x,z,w,y) \frac{(|x-z|-|x|)(|w-y|-|y|)(V\phi)(z)(V\phi)(w)}{16\pi^2|x-z||w-y|} dwdz, \end{aligned}$$

where the definition of W_1 should be obvious. We apply integration by parts twice to $W_1(x,z,w,y)$ and obtain

$$W_1(x,z,w,y) = \sum_{\pm} \frac{-1}{(|x| \pm |y|)^2} + Y_1(x,z,w,y), \quad (4.9)$$

$$Y_1 = \sum_{\pm} \frac{-1}{(|x| \pm |y|)^2} \int_0^\infty e^{i\lambda(|x|\pm|y|)} (r_1(\lambda,x,z) r_1(\pm\lambda,w,y)\lambda F(\lambda))'' d\lambda.$$

By using $\int (V\phi)(x)dx = 0$ and (1.5), we have

$$\int \frac{(|x-z|-|x|)(V\phi)(z)}{4\pi|x-z|} dz = |x|\phi(x) \quad (4.10)$$

and the like for the integral involving $(V\phi)(w)$. Thus, the contribution to $Z_1(x,y)$ of the boundary term in (4.9) is given by

$$\frac{i}{\pi} \sum_{\pm} \frac{|x||y|\phi(x)\phi(y)}{(|x| \pm |y|)^2}. \quad (4.11)$$

By virtue of Lemma 3.1 both + and − terms of (4.11) are L^p bounded kernels for all $1 < p < \infty$ and they are admissible if $\int_{\mathbb{R}^3} x_j (V\phi)(x)dx = 0$ for $j = 1, 2, 3$. Further integration by parts and (4.7) imply

$$Y_1(x, z, w, y) \leq_{|\cdot|} C \sum_{\pm} \frac{(1 + |z| + |w|)^3}{(|x| \pm |y|)^3}.$$

Thus, the contribution of Y_1 to $Z_1(x, y)$ produces the kernel which is bounded in modulus by

$$C \sum_{\pm} \int \frac{|z||w|(1 + |z| + |w|)^3 |(V\phi)(z)(V\phi)(w)|}{|x - z||w - y|(|x| \pm |y|)^3} dwdz$$
$$\leq \sum_{\pm} \frac{C}{\langle x \rangle \langle y \rangle (|x| \pm |y|)^3}$$

and, hence, is admissible. This proves the lemma. □

Lemma 4.5. *Modulo an admissible kernel*

$$Z_2(x, y) \equiv \frac{2i}{\pi} \frac{|x|\phi(x)|y|\phi(y)}{|x|^2 \pm |y|^2} \tag{4.12}$$

and Z_2 is bounded for all $1 < p < \infty$. If $\phi \in \mathcal{E}$ satisfies $\int_{\mathbb{R}^3} x_j (V\phi)(x)dx = 0$ for $j = 1, 2, 3$, in addition, then $Z_2(x, y)$ is admissible.

Proof. The proof goes in parallel with that of Lemma 4.4. Using (4.6), we write $Z_2(x, y)$ in the form

$$\frac{1}{\pi} \int \left(\sum_{\pm} \int_0^\infty e^{i\lambda(|x| \pm |y|)} r_1(\pm\lambda, w, y) F(\lambda) d\lambda \right) \frac{(|w - y| - |y|)\phi(x)(V\phi)(w)}{4\pi|w - y|} dw.$$

Write $W_2 = W_2(x, w, y)$ for the function inside the parenthesis. By integration by parts, we have

$$W_2 = \sum_{\pm} \frac{i}{|x| \pm |y|} + \sum_{\pm} \frac{i}{|x| \pm |y|} \int_0^\infty e^{i\lambda(|x| \pm |y|)} (r_1(\pm\lambda, w, y)F(\lambda))' d\lambda.$$

The contribution to $Z_2(x, y)$ of the boundary term for W_2 is given by

$$\sum_{\pm} \frac{i}{\pi} \int \frac{(|w - y| - |y|)\phi(x)(V\phi)(w)}{(|x| \pm |y|) \cdot 4\pi|w - y|} dw = \sum_{\pm} \frac{i}{\pi} \frac{\phi(x)|y|\phi(y)}{|x| \pm |y|},$$

which is equal to (4.12). Further integration by parts shows the integral term for W_2 is given by

$$Y_2(x,w,y) = \sum_{\mp i} \frac{\mp(|w-y|-|y|)}{2(|x|\pm|y|)^2} + O\left(\frac{C(1+|w|)^3}{(|x|\pm|y|)^3}\right).$$

The second term on the right produces in $Z_2(x,y)$ the admissible kernel bounded outside (4.8) by

$$C\frac{|\phi(x)|\langle y\rangle^{-1}}{\langle |x|-|y|\rangle^3}$$

and, the boundary term in Y_2 does in $Z_2(x,y)$ the kernel

$$\sum_{\pm i} \frac{\mp 1}{2(|x|\pm|y|)^2}\int_{\mathbb{R}^3} \frac{\phi(x)(|w-y|-|y|)^2(V\phi)(w)}{4\pi^2|w-y|}dw \leq_{|\cdot|} \frac{C|\phi(x)||x||y|}{\langle y\rangle(|x|^2-|y|^2)^2},$$

which is again admissible outside (4.8). If ϕ satisfies $\langle V, x_j\phi\rangle = 0$ for $j = 1,2,3$, then $|\phi(x)| \leq C\langle x\rangle^{-3}$ and (4.12) also becomes admissible. This proves the lemma. □

Lemma 4.6. *Modulo an admissible kernel*

$$Z_3(x,y) \equiv \frac{-2i}{\pi}\frac{|x|\phi(x)|y|\phi(y)}{|x|^2-|y|^2} \tag{4.13}$$

and Z_3 is bounded for all $1<p<\infty$. If $\phi \in \mathcal{E}$ satisfies $\int_{\mathbb{R}^3} x_j(V\phi)(x)dx = 0$ for $j = 1,2,3$ in addition, then $Z_2(x,y)$ is an admissible kernel.

Proof. The proof goes in parallel with that of Lemma 4.5. By using (4.6) and (4.10) once more we write $Z_3(x,y)$ in the form

$$Z_3(x,y) = \frac{1}{\pi}\int_{\mathbb{R}^3} W_3(x,z,y)\frac{(|x-z|-|x|)(V\phi)(z)\phi(y)}{4\pi|x-z|}dz. \tag{4.14}$$

$$W_3(x,z,y) = \sum_{\pm} \pm \int_0^\infty e^{i\lambda(|x|\pm|y|)} r_1(\lambda,x,z)F(\lambda)d\lambda. \tag{4.15}$$

Application of integration by parts shows that

$$W_3(x,z,y) = \sum_{\pm} \frac{\pm i}{|x|\pm|y|} + Y_3(x,z,y), \tag{4.16}$$

$$Y_3(x,z,y) = \sum_{\pm} \frac{\pm i}{|x|\pm|y|}\int_0^\infty e^{i\lambda(|x|\pm|y|)}(r_1(\lambda,x,z)F(\lambda))'d\lambda. \tag{4.17}$$

The contribution to $Z_3(x,y)$ of the boundary term of W_3 in (4.16) is given by virtue of (4.10) by

$$\sum_{\pm}\frac{\pm i}{\pi}\int_{\mathbb{R}^3}\frac{(|x-z|-|x|)(V\phi)(z)\phi(y)}{(|x|\pm|y|)4\pi|x-z|}dz=\sum_{\pm}\frac{\pm i}{\pi}\frac{|x|\phi(x)\phi(y)}{|x|\pm|y|} \tag{4.18}$$

Further integration by parts implies

$$Y_3(x,z,y)=\sum_{\pm}\frac{\mp i(|x-z|-|x|)}{2(|x|\pm|y|)^2}+O\left(\frac{(1+|z|)^3}{(|x|\pm|y|)^3}\right) \tag{4.19}$$

and substituting Y_3 for W_3 in (4.14) produces an admissible kernel. Indeed, from the boundary term on the right of (4.19) we have

$$\sum\frac{\mp i}{2(|x|\pm|y|)^2}\int_{\mathbb{R}^3}\frac{(|x-z|-|x|)^2(V\phi)(z)\phi(y)}{4\pi|x-z|}dz\leq_{|\cdot|}\frac{C|x||y||\phi(y)|}{(|x|^2-|y|^2)^2\langle x\rangle}$$

which is admissible and, from the second term the kernel bounded by

$$C\frac{|\phi(y)|\langle x\rangle^{-1}}{\langle|x|-|y|\rangle^3},$$

which is also admissible. If $\phi\in\mathcal{E}$ satisfies $\int_{\mathbb{R}^3}x_j(V\phi)(x)dx=0$ for $j=1,2,3$ in addition, it is obvious that (4.13) becomes an admissible kernel. This concludes the proof of the lemma. □

For proving the following Lemma 4.8, we use the next lemma.

Lemma 4.7. (1) *Suppose that* $K(x,y)$ *satisfies*

$$K(x,y)\leq_{|\cdot|}C\int_{-1}^{1}\langle x\rangle^{-2}\langle y\rangle^{-1}\langle|x|-\theta|y|\rangle^{-\delta}d\theta,\quad x,y\in\mathbb{R}^3, \tag{4.20}$$

for some constants $\delta>1$ *and* $C>0$. *Then,* K *is* L^p*-bounded kernel for* $1\leq p<3$.

(2) *If* K *satisfies* (4.20) *with* $\langle x\rangle^{-2}\langle y\rangle^{-\kappa}$, $\kappa>2$ *in place of* $\langle x\rangle^{-2}\langle y\rangle^{-1}$, *then* K *is admissible.*

Proof. (1) It suffices show that K is bounded in L^1 and L^p for $2<p<3$ by virtue of the interpolation theorem. By using polar coordinates, we estimate

$$\int_{\mathbb{R}^3}|K(x,y)|dx\leq C\int_0^1\left(\int_0^\infty\frac{dr}{\langle r-\theta|y|\rangle^\delta}\right)\langle y\rangle^{-1}d\theta\leq C\langle y\rangle^{-1} \tag{4.21}$$

and K is bounded in L^1. We next let $2 < p < 3$. Minkowski's inequality implies

$$\left(\int_{\mathbb{R}^3} |K(x,y)|^p dx\right)^{1/p} \leq C\langle y\rangle^{-1} \int_0^1 \left(\int_0^\infty \frac{r^2\langle r\rangle^{-2p}}{\langle r-\theta|y|\rangle^{p\delta}} dr\right)^{1/p} d\theta. \tag{4.22}$$

We denote the integrand $(\cdots)^{1/p}$ with respect to θ by $G(\theta|y|)$. It is obvious that $G(\theta|y|) \leq C$ if $\theta|y| \leq 2$, hence (4.22) $\leq C$ for $|y| \leq 2$. Let $\theta|y| \geq 2$ and $|y| \geq 2$. We split as $(0,\infty) = I_1 \cup I_2$,

$$I_1 = \{r > 0 \colon \theta|y|/2 < r < 3\theta|y|/2\} \text{ and } I_2 = \{r > 0 \colon |r - \theta|y|| \geq \theta|y|/2\}$$

and estimate $r^2\langle r\rangle^{-2p} \leq C\langle\theta|y|\rangle^{2-2p}$ on I_1 and $\langle r-\theta|y|\rangle^{-p\delta} \leq C\langle\theta|y|\rangle^{-p\delta}$ on I_2. Then,

$$G(\theta|y|) \leq \left\{\left(\int_{I_1} + \int_{I_2}\right) \frac{r^2\langle r\rangle^{-2p}}{\langle r-\theta|y|\rangle^{p\delta}} dr\right\}^{1/p} \leq C(\langle\theta|y|\rangle^{\frac{2}{p}-2} + \langle\theta|y|\rangle^{-\delta}).$$

Since $\delta > 1$ and $2 < p < 3$, we obtain for $|y| \geq 2$ that

$$\int_0^1 \chi(\theta|y| \geq 2) G(\theta|y|) d\theta \leq C\left(\int_{2/|y|}^1 (\langle\theta|y|\rangle^{\frac{2}{p}-2} + \langle\theta|y|\rangle^{-\delta}) d\theta\right) \leq \frac{C}{\langle y\rangle}.$$

Thus, for $2 < p < 3$, we have by Minkowski's and Hölder inequalities that

$$\|Ku\|_p \leq \int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} |K(x,y)|^p dx\right)^{1/p} |u(y)| dy \leq C\|\langle y\rangle^{-2}\|_{p'}\|u\|_p \leq C\|u\|_p,$$

where the dual exponent $p' = p(p-1)^{-1} > 3/2$.
(2) We have $\langle x\rangle^{-2}\langle |x| - \theta|y|\rangle^{-\delta} \leq C\langle\theta|y|\rangle^{-\min(2,\delta)}$ and

$$\sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} |K(x,y)| dy \leq C \int_{\mathbb{R}^3} \langle y\rangle^{-\kappa-1} dy < \infty.$$

This with (4.21) implies K is admissible. □

Lemma 4.8. (1) *Modulo an L^p-bounded kernel for $1 \leq p < 3$,*

$$Z_4(x,y) \equiv -\frac{i}{\pi}\phi(x)\phi(y)|y| \cdot \int_{-1}^1 \frac{\chi(||x| + \theta|y|| > 1)}{|x| + \theta|y|} d\theta. \tag{4.23}$$

(2) *Z_4 is bounded in L^p for $3/2 < p < 3$.*

(3) *If* $\langle V, x^\alpha\phi\rangle = 0$ *for* $|\alpha| \le 1$, *then* (4.23) *holds modulo the kernel which is* L^p*-bounded for all* $1 \le p \le \infty$.

Proof. We have

$$\begin{aligned} Z_4(x,y) &= \frac{i}{\pi}\phi(x)\phi(y)\int_0^\infty e^{i\lambda|x|}(e^{i\lambda|y|}-e^{-i\lambda|y|})\frac{F(\lambda)}{\lambda}d\lambda \\ &= -\frac{1}{\pi}\phi(x)\phi(y)|y|\int_{-1}^1\left(\int_0^\infty e^{i\lambda(|x|+\theta|y|)}F(\lambda)d\lambda\right)d\theta. \qquad (4.24)\end{aligned}$$

By inserting $1 = \chi(||x|+\theta|y|| \le 1) + \chi(||x|+\theta|y|| > 1)$ in front of $d\theta$ of (4.24), we split $Z_4(x,y)$ into two parts:

$$Z_4(x,y) = Z_4^{\le}(x,y) + Z_4^{>}(x,y).$$

We clearly have

$$Z_4^{\le}(x,y) \le_{|\cdot|} \frac{2}{\pi}\int_0^1 |\phi(x)\phi(y)||y|\chi(||x|-\theta|y|| \le 1)d\theta$$

and Lemma 4.7 implies $Z_4^{\le}$ is bounded in L^p for $1 \le p < 3$ in general and, is admissible if ϕ satisfies $|\phi(x)| \le C\langle x\rangle^{-3}$. We next consider $Z_4^{>}(x,y)$. By integration by parts, we have that

$$\int_0^\infty e^{i\lambda(|x|+\theta|y|)}F(\lambda)d\lambda = \frac{i}{|x|+\theta|y|} + i\int_0^\infty \frac{e^{i\lambda(|x|+\theta|y|)}F'(\lambda)}{|x|+\theta|y|}d\lambda \qquad (4.25)$$

and, as $F' \in C_0^\infty((0,\infty))$, the integral term is bounded for any $N = 1, 2, \ldots$ by $C\langle |x|+\theta|y|\rangle^{-N}$ when $||x|+\theta|y|| \ge 1$. It follows that its contribution of to $Z_4^{>}(x,y)$ produces the kernel bounded in modulus by

$$C\int_{-1}^1 \frac{|\phi(x)\phi(y)||y|}{\langle |x|+\theta|y|\rangle^N}d\theta$$

and Lemma 4.7 implies it enjoys the same property of $Z_4^{\le}$ stated above. Since the contribution of the boundary term in (4.25) to $Z_4^{>}(x,y)$ is given by (4.23), statements (1) and (3) of the lemma are proved.
(2) Since $Z_4(x,y) \le_{|\cdot|} C|y|\phi(x)\phi(y)$, it is obvious that $\chi(|y|<2)Z_4(x,y)$ produces bounded operator in L^p for $3/2 < p < 3$ and we may ignore the part $\{y \in \mathbb{R}^3 \colon |y| < 2\}$ of $Z_4(x,y)$. We estimate

$$\begin{aligned} &\int_{-1}^1 \frac{\chi(||x|+\theta|y||>1)}{|x|+\theta|y|}d\theta = \frac{1}{|y|}\int_{|x|-|y|}^{|x|+|y|}\frac{\chi(|\theta|>1)}{\theta}d\theta \\ &\quad \le \frac{1}{|y|}\int_1^{|x|+|y|}\frac{d\theta}{\theta} = \frac{\log(|x|+|y|)}{|y|} \le \frac{\log(1+|x|)+\log(1+|y|)}{|y|}. \qquad (4.26)\end{aligned}$$

Then we have $Z_4(x,y) \le_{|\cdot|} C(\log\langle x\rangle + \log\langle y\rangle)\phi(x)\phi(y)$ and $\log\langle x\rangle\phi(x) \in L^p(\mathbb{R}^p)$ for any $3/2 < p$. Statement (2) follows. □

Completion of the proof of Lemma 4.3. Combining the results of previous four lemmas and observing that the boundary terms in (4.12) and (4.13) cancel each other, we see that $Z(x,y)$ is modulo the kernel which is bounded in L^p for $1 \le p < 3$ equals to the sum $K_0(x,y) = K_{01}(x,y) + K_{02}(x,y)$ where

$$K_{01}(x,y) = \sum_{\pm} \frac{i}{\pi} \frac{\chi(||x|-|y|| \ge 1)|x||y|\phi(x)\phi(y)}{(|x| \pm |y|)^2} \tag{4.27}$$

and

$$K_{02}(x,y) = -\frac{i}{\pi}\phi(x)\phi(y)|y| \cdot \int_{-1}^{1} \frac{\chi(||x|+\theta|y|| > 1)}{|x|+\theta|y|} d\theta. \tag{4.28}$$

Since Lemmas 4.4 and 4.5 already prove that K_0 is bounded in L^p for $3/2 < p < 3$, it suffices to show by virtue of interpolation ([2]) that K_0 is also bounded in L^1. We write $|x| = |x| \pm |y| \mp |y|$ in (4.27). Then,

$$K_{01}(x,y) = \sum_{\pm} \frac{i}{\pi} \frac{\chi(||x|-|y|| \ge 1)|y|\phi(x)\phi(y)}{|x| \pm |y|} \tag{4.29}$$

$$- \frac{i}{\pi} \frac{4\chi(||x|-|y|| \ge 1)|x||y|^3\phi(x)\phi(y)}{(|x|+|y|)^2(|x|-|y|)^2}. \tag{4.30}$$

We denote (4.30) by $K_a(x,y)$. It is then evident that

$$K_a(x,y) \le_{|\cdot|} C \frac{\chi(||x|-|y|| \ge 1)|\phi(x)||y|^2\phi(y)}{(|x|-|y|)^2}$$

and it produces a bounded operator in L^1. Thus, for concluding the proof of Lemma 4.3, it suffices show that

$$K_e(x,y) = \frac{i}{\pi}\phi(x)\phi(y)|y| \left(\sum_{\pm} \frac{\chi(||x|-|y|| \ge 1)}{|x| \pm |y|} - \int_{-1}^{1} \frac{\chi(||x|+\theta|y|| > 1)}{|x|+\theta|y|} d\theta \right)$$

is an L^1-bounded integral kernel. We first remark that

$$\phi(x)\phi(y)|y| \left(\frac{\chi(||x|-|y|| \ge 1)}{|x| \pm |y|} - \frac{1}{\langle |x| \pm |y| \rangle} \right)$$
$$= \frac{\phi(x)\phi(y)|y|\chi(||x|-|y|| \ge 1)}{(|x| \pm |y|)\langle |x| \pm |y| \rangle \{(|x| \pm |y|) + \langle |x| \pm |y| \rangle\}} - \frac{\chi(||x| \pm |y|| \le 1)}{\langle |x| \pm |y| \rangle}$$

is an L^1 bounded integral kernel. We can likewise show that

$$\int_{-1}^{1} \phi(x)\phi(y)|y| \left(\frac{\chi(||x|+\theta|y|| > 1)}{|x|+\theta|y|} - \frac{1}{\langle |x|+\theta|y| \rangle} \right) d\theta$$

is an L^1-bounded kernel. Finally, observing that

$$\frac{1}{\langle a\rangle} - \frac{1}{\langle b\rangle} = \frac{\langle b\rangle - \langle a\rangle}{\langle a\rangle\langle b\rangle} = \frac{b^2 - a^2}{\langle a\rangle\langle b\rangle(\langle a\rangle + \langle b\rangle)} \leq_{|\cdot|} \frac{|b-a|}{\langle a\rangle\langle b\rangle},$$

we estimate

$$\frac{i}{\pi}\phi(x)\phi(y)|y|\left(\sum_{\pm}\frac{1}{\langle |x| \pm |y|\rangle} - \int_{-1}^{1}\frac{d\theta}{\langle |x| + \theta|y|\rangle}\right) \tag{4.31}$$

$$= \sum_{\pm}\frac{i}{\pi}\phi(x)\phi(y)|y|\int_0^1\left(\frac{1}{\langle |x| \pm |y|\rangle} - \frac{1}{\langle |x| \pm \theta|y|\rangle}\right)d\theta \tag{4.32}$$

$$\leq_{|\cdot|} \sum_{\pm}\frac{i}{\pi}\phi(x)\phi(y)|y|\int_0^1\frac{(1-\theta)|y|}{\langle |x| \pm |y|\rangle\langle |x| \pm \theta|y|\rangle}d\theta. \tag{4.33}$$

Then, we have

$$\int_{\mathbb{R}^3}\frac{\phi(x)\phi(y)|y|^2}{\langle |x| \pm |y|\rangle\langle |x| \pm \theta|y|\rangle}dx \leq C\int_0^\infty\frac{dr}{\langle r \pm |y|\rangle\langle r \pm \theta|y|\rangle}$$
$$\leq \frac{C}{2}\int_{-\infty}^{\infty}\left(\frac{1}{\langle r \pm |y|\rangle^2} + \frac{1}{\langle r \pm \theta|y|\rangle^2}\right)dr \leq \pi C.$$

This shows that (4.31) is an L^1-bounded kernel and we conclude the proof of the lemma. □

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