HILBERT SCHEMES OF POINTS OF $\mathcal{O}_{\mathbb{P}^1}(-n)$ AS QUIVER VARIETIES

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Abstract. Relying on a representation of framed torsion-free sheaves on Hirzebruch surfaces in terms of monads, we construct ADHM data for the Hilbert scheme of points of the total space of the line bundle $\mathcal{O}(-n)$ on $\mathbb{P}^1$. This ADHM description is then used to realize these Hilbert schemes as quiver varieties.

1. Introduction

Let X be a smooth quasi-projective irreducible surface over $\mathbb{C}$. The Hilbert scheme of points $\operatorname{Hilb}^c(X)$, which parameterizes 0-dimensional subschemes of X of length c, is quasi-projective [9] and smooth of dimension $2c$ [5]. Hilbert schemes of points of surfaces has been extensively studied from many perspectives over the past two decades (see e.g. [18, 14, 19]). There are nevertheless few cases where an explicit description has been worked out: significant examples are the spaces $\operatorname{Hilb}^c(\mathbb{C}^2)$, which can be described by means of linear data, the so-called ADHM (Atiyah-Drinfel'd-Hitchin-Manin) data [18]. Also the Hilbert schemes of points of multi-blowups of $\mathbb{C}^2$ admit an ADHM description, as provided by the work of A.A. Henni [10] specialized to the rank one case.

The ADHM description of $\operatorname{Hilb}^c(\mathbb{C}^2)$ is virtually equivalent to the realization of these spaces as quiver varieties, in the sense defined by Hiraku Nakajima in the groundbreaking papers [16, 17]. Many of the essential features of the varieties $\operatorname{Hilb}^c(\mathbb{C}^2)$ — including their symplectic structure — are captured and reinterpreted in the more general framework of the theory of quiver varieties, as illustrated in [7].

In the first part of this paper we construct ADHM data for the Hilbert schemes of points of the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$. To achieve this aim, the space $\operatorname{Hilb}^c(\operatorname{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n)))$ is identified with the moduli space $\mathcal{M}^n(1,0,c)$ of framed sheaves on the Hirzebruch surface Σ_n that have rank 1, vanishing first Chern class, and second Chern class $c_2 = c$ (the framing is a fixed isomorphism with the trivial rank 1 bundle on a divisor linearly equivalent to the section of $\Sigma_n \to \mathbb{P}^1$ of positive self-intersection). By exploiting the description of $\mathcal{M}^n(1,0,c)$ in terms of

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monads given in [1], we prove (Theorem 3.1) that the moduli space $\mathcal{M}^n(1,0,c)$ is isomorphic to the quotient $P^n(c)/\operatorname{GL}(c,\mathbb{C})\times\operatorname{GL}(c,\mathbb{C})$, where $P^n(c)$ is a quasi-affine variety contained in the linear space $\operatorname{End}(\mathbb{C}^c)^{\oplus n+2}\oplus\operatorname{Hom}(\mathbb{C}^c,\mathbb{C})$. This result relies on the fact that the partial quotient $P^n(c)/\operatorname{GL}(c,\mathbb{C})$ can be assembled by glueing $c+1$ open sets, each one isomorphic to the space of ADHM data for $\operatorname{Hilb}^c(\mathbb{C}^2)$ (Propositions 3.2 and 3.6).

In the second part of this paper we show (Theorem 4.1) that these Hilbert schemes are connected components of quiver varieties, namely, they are naturally embedded as connected components into varieties of representations of a quiver naturally associated with the ADHM data describing the Hilbert schemes, for a suitable choice of the stability parameter. Our result includes the particular case of the Hilbert scheme of points of the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$, which is isomorphic, as a complex variety, to the ALE space A_1. Kuznetsov has provided, from a different point of view, a description of the Hilbert schemes of the ALE spaces A_n as quiver varieties [12]. We check indeed (Corollary 4.6) that for $n=2$ our representation coincides with that of Kuznetsov.

Finally, Appendix A is devoted to proving the rather technical Proposition 3.2.

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2. BACKGROUND MATERIAL

The construction of the ADHM data is based on the description of the moduli spaces of framed sheaves on Σ_n in terms of monads worked out in [1]. We briefly review the basic ingredients of that construction. Let Σ_n be the n-th Hirzebruch surface, i.e., the projective closure of the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$; we shall assume $n>0$. We denote by F the class in $\operatorname{Pic}(\Sigma_n)$ of the fibre of the natural ruling $\Sigma_n\longrightarrow\mathbb{P}^1$, and by H and E the classes of the sections squaring to n and $-n$, respectively. We shall denote $\mathcal{O}_{\Sigma_n}(p,q)=\mathcal{O}_{\Sigma_n}(pH+qF)$. We fix a curve $\ell_\infty\simeq\mathbb{P}^1$ in Σ_n belonging to the class H and call it the "line at infinity". A framed sheaf on Σ_n is a pair $(\mathcal{E},\theta)$, where $\mathcal{E}$ is a torsion-free sheaf trivial along ℓ_∞, and $\theta\colon\mathcal{E}|_{\ell_\infty}\xrightarrow{\sim}\mathcal{O}_{\ell_\infty}^{\oplus r}$ is an isomorphism, r being the rank of $\mathcal{E}$. A morphism between framed sheaves $(\mathcal{E},\theta)$, $(\mathcal{E}',\theta')$ is by definition a morphism $\Lambda\colon\mathcal{E}\longrightarrow\mathcal{E}'$ such that $\theta'\circ\Lambda|_{\ell_\infty}=\theta$. The moduli space parameterizing isomorphism classes of framed sheaves $(\mathcal{E},\theta)$ on Σ_n with $\operatorname{ch}(\mathcal{E})=(r,aE,-c-\frac{1}{2}na^2)$ will be denoted by $\mathcal{M}^n(r,a,c)$. We normalize the framed sheaves so that $0\leq a\leq r-1$.

As proved in [1], a framed sheaf $(\mathcal{E},\theta)$ on Σ_n, having invariants (r,a,c), is isomorphic to the cohomology of a monad

$$M(\alpha,\beta):\qquad 0\longrightarrow\mathcal{U}_{\vec{k}}\xrightarrow{\alpha}\mathcal{V}_{\vec{k}}\xrightarrow{\beta}\mathcal{W}_{\vec{k}}\longrightarrow 0\ , \tag{2.1}$$

where $\vec{k}$ denotes the quadruple (n,r,a,c), and we have set

$$\mathcal{U}_{\vec{k}}:=\mathcal{O}_{\Sigma_n}(0,-1)^{\oplus k_1},\quad \mathcal{V}_{\vec{k}}:=\mathcal{O}_{\Sigma_n}(1,-1)^{\oplus k_2}\oplus\mathcal{O}_{\Sigma_n}^{\oplus k_4},\quad \mathcal{W}_{\vec{k}}:=\mathcal{O}_{\Sigma_n}(1,0)^{\oplus k_3}\ ,$$

with

$$k_1 = c + \frac{1}{2}na(a-1), \quad k_2 = k_1 + na, \quad k_3 = k_1 + (n-1)a, \quad k_4 = k_1 + r - a\,.$$

The set $L_{\vec{k}}$ of pairs in $\mathrm{Hom}(\mathcal{U}_{\vec{k}}, \mathcal{V}_{\vec{k}}) \oplus \mathrm{Hom}(\mathcal{V}_{\vec{k}}, \mathcal{W}_{\vec{k}})$ fitting into the complex (2.1) is a smooth algebraic variety. One can introduce a principal $\mathrm{GL}(r,\mathbb{C})$-bundle $P_{\vec{k}}$ over $L_{\vec{k}}$, whose fibre at a point (α, β) is identified with the space of framings for the cohomology of (2.1). The algebraic group $G_{\vec{k}} = \mathrm{Aut}(\mathcal{U}_{\vec{k}}) \times \mathrm{Aut}(\mathcal{V}_{\vec{k}}) \times \mathrm{Aut}(\mathcal{W}_{\vec{k}})$ of automorphisms of the monads of the form (2.1) acts freely on $P_{\vec{k}}$, and the moduli space $\mathcal{M}^n(r,a,c)$ can be described as the quotient $P_{\vec{k}}/G_{\vec{k}}$ [1, Theorem 3.4]. This space is nonempty if and only if $c + \frac{1}{2}na(a-1) \geq 0$, and when nonempty, it is a smooth algebraic variety of dimension $2rc + (r-1)na^2$.

When $r = 1$ we can assume $a = 0$, so that the double dual $\mathcal{E}^{**}$ of $\mathcal{E}$ is isomorphic to the structure sheaf $\mathcal{O}_{\Sigma_n}$. As a consequence, since $\mathcal{E}$ is trivial on ℓ_∞, the mapping carrying $\mathcal{E}$ to the schematic support of $\mathcal{O}_{\Sigma_n}/\mathcal{E}$ yields an isomorphism

$$\mathcal{M}^n(1,0,c) \simeq \mathrm{Hilb}^c(\Sigma_n \setminus \ell_\infty) = \mathrm{Hilb}^c(\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-n)))\,.$$

In the following, we shall denote the moduli space $\mathcal{M}^n(1,0,c)$ simply by $\mathcal{M}^n(c)$.

We also fix some notation about quiver representations (see [7] for details). A quiver $\mathcal{Q}$ is a finite oriented graph, given by a set of vertices I and a set of arrows E. The path algebra $\mathbb{C}\mathcal{Q}$ is the $\mathbb{C}$-algebra with basis the paths in $\mathcal{Q}$ and with product given by composition of paths whenever possible, zero otherwise. A (complex) representation of a quiver $\mathcal{Q}$ is a pair (V, X), where $V = \bigoplus_{i\in I} V_i$ is an I-graded complex vector space and $X = (X_a)_{a\in E}$ is a collection of linear maps such that $X_a \in \mathrm{Hom}_{\mathbb{C}}(V_i, V_j)$ whenever the arrow a starts at the vertex i and terminates at the vertex j. We say that a representation (V, X) is supported by V, and denote by $\mathrm{Rep}(\mathcal{Q}, V)$ the space of representations of $\mathcal{Q}$ supported by V. Morphisms and direct sum of representations are defined in an obvious way; it can be shown that the abelian category of complex representations of Q is equivalent to the category of left $\mathbb{C}\mathcal{Q}$-modules. In particular, a subrepresentation of a given representation (V, X) is a pair (S, Y), where S is an I-graded subspace of V which is preserved by the linear maps X, and Y is the restriction of X to S.

We consider only finite-dimensional representations. If $\dim_{\mathbb{C}} V_i = v_i$, a representation (V, X) of $\mathcal{Q}$ is said to be $\mathbf{v}$-dimensional, where $\mathbf{v} = (v_i)_{i\in I} \in \mathbb{N}^I$. With an abuse of notation, after fixing a $\mathbf{v}$-dimensional vector space V, we write $\mathrm{Rep}(\mathcal{Q}, \mathbf{v})$ instead of $\mathrm{Rep}(\mathcal{Q}, V)$.

More generally one can define the representations of a quotient algebra $B = \mathbb{C}\mathcal{Q}/J$, for some ideal J of the path algebra $\mathbb{C}\mathcal{Q}$. We denote by $\mathrm{Rep}(B, \mathbf{v})$ the space of representations of B supported by a given $\mathbf{v}$-dimensional vector space V. There is a natural action of $\prod_i \mathrm{GL}(v_i)$ on $\mathrm{Rep}(B, \mathbf{v})$ given by change of basis. One would like to consider the space of isomorphism classes of $\mathbf{v}$-dimensional representations of B, but unfortunately this space is in most cases "badly behaved". To overcome this drawback, following A. King's approach [11], one introduces a notion of (semi)stability depending on the choice of a parameter ϑ, considers the subset $\mathrm{Rep}^{ss}_\vartheta(B, \mathbf{v})$ of $\mathrm{Rep}(B, \mathbf{v})$ consisting of semistable representations, and finally takes the corresponding GIT quotient $\mathrm{Rep}^{ss}_\vartheta(B, \mathbf{v})/\!/_\vartheta \prod_i \mathrm{GL}(v_i)$.

3. ADHM DATA

In this section we construct ADHM data for the Hilbert scheme of points of the total spaces of the line bundles $\mathcal{O}_{\mathbb{P}^1}(-n)$. To achieve that, first we show that the Hilbert schemes can be covered by open subsets, each of which is isomorphic to the Hilbert scheme of $\mathbb{C}^2$, and therefore admits Nakajima's ADHM description; then we prove that these "local data" can be glued together, and provide ADHM data for the Hilbert schemes of $\mathcal{O}_{\mathbb{P}^1}(-n)$.

We denote by $P^n(c)$ the subset of the vector space $\operatorname{End}(\mathbb{C}^c)^{\oplus n+2} \oplus \operatorname{Hom}(\mathbb{C}^c, \mathbb{C})$ whose points $(A_1, A_2; C_1, \ldots, C_n; e)$ satisfy the following conditions:

(P1)

$$\begin{cases} A_1C_1A_2 = A_2C_1A_1 & & \text{when } n = 1 \\ \\ A_1C_q = A_2C_{q+1} & \multirow{2}{*}{for $\quad q = 1, \ldots, n-1$} & \multirow{2}{*}{when $n > 1$;} \\ C_qA_1 = C_{q+1}A_2 & & \end{cases}$$

(P2) $A_1 + \lambda A_2$ is a *regular pencil* of matrices; equivalently, there exists $[\nu_1, \nu_2] \in \mathbb{P}^1$ such that $\det(\nu_1 A_1 + \nu_2 A_2) \neq 0$;

(P3) for all values of the parameters $([\lambda_1, \lambda_2], (\mu_1, \mu_2)) \in \mathbb{P}^1 \times \mathbb{C}^2$ such that

$$\lambda_1^n \mu_1 + \lambda_2^n \mu_2 = 0$$

there is no nonzero vector $v \in \mathbb{C}^c$ such that

$$\begin{cases} C_1A_2v = -\mu_1 v \\ C_nA_1v = (-1)^n \mu_2 v \\ v \in \ker e \end{cases} \qquad \text{and} \qquad (\lambda_2 A_1 + \lambda_1 A_2)\, v = 0\,.$$

The action of group $\mathrm{GL}(c, \mathbb{C}) \times \mathrm{GL}(c, \mathbb{C})$ on $P^n(c)$ is given by

$$(A_i,\, C_j, e) \mapsto (\phi_2 A_i \phi_1^{-1},\, \phi_1 C_j \phi_2^{-1},\, e\phi_1^{-1}) \tag{3.1}$$

for $i = 1, 2$, $j = 1, \ldots, n$, $(\phi_1, \phi_2) \in \mathrm{GL}(c, \mathbb{C}) \times \mathrm{GL}(c, \mathbb{C})$.

Theorem 3.1. *$P^n(c)$ is a principal $\mathrm{GL}(c, \mathbb{C}) \times \mathrm{GL}(c, \mathbb{C})$-bundle over $\mathcal{M}^n(c)$.*

The remainder of this Section is devoted to proving Theorem 3.1. At first, we provide an ADHM description for each open set of an open cover of $\mathcal{M}^n(c)$. If we fix $c + 1$ distinct fibres $F_0, \ldots, F_c \in F$, for any $[(\mathcal{E}, \theta)] \in \mathcal{M}^n(c)$ there exists at least one $m \in \{0, \ldots, c\}$ such that $\mathcal{E}|_{F_m} \simeq \mathcal{O}_{F_m}$. We choose the fibres F_m as the closed subvarieties cut in

$$\Sigma_n = \left\{([y_1, y_2], [x_1, x_2, x_3]) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid x_1 y_1^n = x_2 y_2^n\right\} \tag{3.2}$$

by the equations

$$F_m = \{[y_1, y_2] = [c_m, s_m]\}, \qquad m = 0, \ldots, c\,,$$

where

$$c_m = \cos\left(\pi \frac{m}{c+1}\right), \qquad s_m = \sin\left(\pi \frac{m}{c+1}\right)\,. \tag{3.3}$$

We get an open cover $\{\mathcal{M}^n(c)_m\}_{m=0,\dots,c}$ for $\mathcal{M}^n(c)$ by letting

$$\mathcal{M}^n(c)_m := \left\{[(\mathcal{E},\theta)] \in \mathcal{M}^n(c) \;\middle|\; \mathcal{E}_{|F_m} \simeq \mathcal{O}_{F_m}\right\}.$$

Each of these spaces is isomorphic to the Hilbert scheme of points of $\mathbb{C}^2$, so that it admits Nakajima's ADHM description [18] in terms of two $c \times c$ matrices b_1, b_2 and a row c-vector e, satisfying the conditions

(T1) $[b_1, b_2] = 0$;

(T2) for all $(z, w) \in \mathbb{C}^2$ there is no nonzero vector $v \in \mathbb{C}^c$ such that

$$\begin{cases} b_1 = zv \\ b_2 = wv \\ v \in \ker e\,. \end{cases}$$

The space of triples (b_1, b_2, e) satisfying the previous two conditions will be denoted by $\mathcal{T}(c)$. Elements ϕ of the group $\mathrm{GL}(c,\mathbb{C})$ act on $\mathcal{T}(c)$ according to the rule

$$(b_1, b_2, e) \mapsto (\phi\, b_1\, \phi^{-1}, \phi\, b_2\, \phi^{-1}, e\,\phi^{-1}).$$

The ADHM data for the open set $\mathcal{M}^n(c)_m$ will be denoted by (b_{1m}, b_{2m}, e_m); the next Proposition gives the transition functions on the intersections.

Proposition 3.2. *The intersection $\mathcal{M}^n(c)_{ml} = \mathcal{M}^n(c)_m \cap \mathcal{M}^n(c)_l$ is characterized by the condition $\det(c_{m-l}\mathbf{1}_c - s_{m-l}b_{1m}) \neq 0$, where c_m and s_m are the numbers defined in eq.* (3.3). *On any of these intersections, the ADHM data are related by the equations*

$$\begin{cases} b_{1l} &= (c_{m-l}\mathbf{1}_c - s_{m-l}b_{1m})^{-1}\,(s_{m-l}\mathbf{1}_c + c_{m-l}b_{1m}) \\ b_{2l} &= (c_{m-l}\mathbf{1}_c - s_{m-l}b_{1m})^{n}\, b_{2m} \\ e_l &= e_m\,. \end{cases}$$

Proof. The proof of this result is given in the Appendix A. □

We introduce the matrices

$$\begin{aligned} A_{1m} &= c_m A_1 - s_m A_2\,, \qquad A_{2m} = s_m A_1 + c_m A_2\,, \\ E_m &= \left[\sum_{q=1}^{n} \binom{n-1}{q-1} c_m^{n-q} s_m^{q-1} C_q\right] A_{2m}\,, \end{aligned} \tag{3.4}$$

for $m = 0, \dots, c$. Since the polynomial $\det(\nu_1 A_1 + \nu_2 A_2)$ has at most c distinct roots in $\mathbb{P}^1$, the $\mathrm{GL}(c,\mathbb{C}) \times \mathrm{GL}(c,\mathbb{C})$-invariant open subsets

$$P^n(c)_m = \{(A_1, A_2; C_1, \dots, C_n; e) \in P^n(c)\,|\det A_{2m} \neq 0\}\,, \qquad m = 0, \dots, c\,,$$

cover $P^n(c)$. If we also define the matrices $B_m = A_{2m}^{-1} A_{1m}$, the linear data $(B_m, E_m, e; A_{2m})$ provide local affine coordinates for $P^n(c)$.

Proposition 3.3. *The morphism*

$$\begin{array}{rccc} \zeta_m\colon & P^n(c)_m & \longrightarrow & \left[\mathrm{End}(\mathbb{C}^c)^{\oplus 2} \oplus \mathrm{Hom}(\mathbb{C}^c,\mathbb{C})\right] \times \mathrm{GL}(c,\mathbb{C}) \\ & (A_1, A_2; C_1, \dots, C_n; e) & \longmapsto & (B_m, E_m, e; A_{2m}) \end{array}$$

is an isomorphism onto $\mathcal{T}(c) \times \mathrm{GL}(c,\mathbb{C})$. *The induced* $\mathrm{GL}(c,\mathbb{C}) \times \mathrm{GL}(c,\mathbb{C})$*-action is given by*

$$(B_m, E_m, e; A_{2m}) \mapsto (\phi_1 B_m \phi_1^{-1},\, \phi_1 E_m \phi_1^{-1},\, e\phi_1^{-1};\, \phi_2 A_{2m} \phi_1^{-1})\,.$$

We divide the proof of Proposition 3.3 into a few steps. First we define the matrices $\sigma^h_m = (\sigma^h_{m;pq})_{0\le p,q\le h}$ for all $h \ge 0$ and $m \in \mathbb{Z}$ by means of the equations

$$(s_m\mu_1 + c_m\mu_2)^p (c_m\mu_1 - s_m\mu_2)^{h-p} = \sum_{q=0}^{h} \sigma^h_{m;pq} \mu_2^q \mu_1^{h-q} \tag{3.5}$$

for any $(\mu_1,\mu_2) \in \mathbb{C}^2$ and $p = 0,\dots,h$. Notice that $\sigma^h_m \sigma^h_l = \sigma^h_{m+l}$ and $\sigma^h_0 = \mathbf{1}_{h+1}$. In particular, σ^h_m is invertible for all $h \ge 0$ and $m \in \mathbb{Z}$.

Lemma 3.4. *Assume* $n > 1$. *If the matrices* $A_1, A_2 \in \mathrm{End}(\mathbb{C}^c)$ *satisfy condition* (P2)*, the system* $A_1 C_q = A_2 C_{q+1}$, $q = 1,\dots,n-1$, *with* $C_q \in \mathrm{End}(\mathbb{C}^c)$*, has maximal rank, namely,* $(n-1)c^2$. *In particular, if* $\det A_{2m} \neq 0$*, the general solution is*

$$\begin{pmatrix} C_1 \\ \vdots \\ \vdots \\ C_n \end{pmatrix} = (\sigma^{n-1}_m \otimes \mathbf{1}_c) \begin{pmatrix} \mathbf{1}_c \\ B_m \\ \vdots \\ B_m^{n-1} \end{pmatrix} D_m\,, \tag{3.6}$$

where we have chosen as free parameter the matrix

$$D_m = \sum_{q=1}^{n} \binom{n-1}{q-1} c_m^{n-q} s_m^{q-1} C_q\,.$$

Proof. A proof can be found in [13]. □

Since $E_m = D_m A_{2m}$, the morphism ζ_m is injective.

Next we prove that $\mathrm{Im}\,\zeta_m \subseteq \mathcal{T}(c) \times \mathrm{GL}(c,\mathbb{C})$. This follows from the next Lemma.

Lemma 3.5. *1. For all* $(B_m, E_m, e; A_{2m}) \in \mathrm{Im}\,\zeta_m$*, one has* $[B_m, E_m] = 0$.

2. Let $(A_1, A_2; C_1,\dots,C_n; e) \in \mathrm{End}(\mathbb{C}^c)^{\oplus(n+2)} \oplus \mathrm{Hom}(\mathbb{C}^c,\mathbb{C})$ *be an* $(n+3)$*-tuple such that condition* (P1) *is satisfied and* $\det A_{2m} \neq 0$. *Then*

- *if* $[\lambda_1,\lambda_2] = [c_m, s_m]$*, condition* (P3) *is trivially satisfied;*
- *if* $[\lambda_1,\lambda_2] \neq [c_m, s_m]$*, condition* (P3) *holds if and only if condition* (T2) *holds for the triple* (B_m, E_m, e).

Proof. A proof is given in [13]. □

Finally, we prove that $\mathcal{T}(c) \times \mathrm{GL}(c,\mathbb{C}) \subseteq \mathrm{Im}\,\zeta_m$. Let $(b_1, b_2, e; A) \in \mathcal{T}(c) \times \mathrm{GL}(c,\mathbb{C})$; if

$$A_1 = A(c_m b_1 + s_m \mathbf{1}_c)\,, \quad A_2 = A(-s_m b_1 + c_m \mathbf{1}_c)\,,$$

$$\begin{pmatrix} C_1 \\ \vdots \\ \vdots \\ C_n \end{pmatrix} = (\sigma^{n-1}_m \otimes \mathbf{1}_c) \begin{pmatrix} \mathbf{1}_c \\ b_1 \\ \vdots \\ b_1^{n-1} \end{pmatrix} b_2 A^{-1}\,, \tag{3.7}$$

then $(A_1, A_2; C_1, \ldots, C_n; e) \in P^n(c)_m$ and $\zeta_m(A_1, A_2; C_1, \ldots, C_n; e) = (b_1, b_2, e; A)$. It is easy to verify by substitution that condition (P1) holds. Notice now that by substituting (3.7) into eq. (3.4) one gets

$$A_{1m} = Ab_1, \qquad A_{2m} = A, \qquad E_m = b_2\,.$$

This shows that A_{2m} is invertible, and in particular, condition (P2) holds. Since $B_m = b_1$, by Lemma 3.5 condition (P3) holds as a consequence of (T2). This concludes the proof of Proposition 3.3.

We now compute the transition functions on the intersections $P^n(c)_{ml} = P^n(c)_m \cap P^n(c)_l$, for $m, l = 0, \ldots, c$. First observe that

$$\zeta_m\left(P^n(c)_{ml}\right) = \mathcal{T}(c)_{m,l} \times \mathrm{GL}(c, \mathbb{C})$$

as a consequence of the identity

$$A_{2l} = \begin{pmatrix} s_l \mathbf{1}_c & c_l \mathbf{1}_c \end{pmatrix} \begin{pmatrix} c_m \mathbf{1}_c & s_m \mathbf{1}_c \\ -s_m \mathbf{1}_c & c_m \mathbf{1}_c \end{pmatrix} \begin{pmatrix} A_{1m} \\ A_{2m} \end{pmatrix} = A_{2m}(c_{m-l}\mathbf{1}_c - s_{m-l}B_m) \tag{3.8}$$

(the notation $\mathcal{T}(c)_{m,l}$ in introduced in eq. (A.3) in Appendix A).

Proposition 3.6. *One has the commutative triangle*

$$\begin{array}{ccc} & P^n(c)_{ml} & \\ {}^{\zeta_{m,l}}\swarrow & & \searrow^{\zeta_{l,m}} \\ \mathcal{T}(c)_{m,l} \times \mathrm{GL}(c, \mathbb{C}) & \xrightarrow{\ \omega_{lm}\ } & \mathcal{T}(c)_{l,m} \times \mathrm{GL}(c, \mathbb{C})\,, \end{array}$$

where $\zeta_{m,l}$ and $\zeta_{l,m}$ are the restrictions of ζ_m and ζ_l, respectively, and

$$\omega_{lm}(B_m, E_m, e; A_{2m}) = \left(\tilde{\varphi}_{lm}(B_m, E_m, e), A_{2m}(c_{m-l}\mathbf{1}_c - s_{m-l}B_m)\right)\,,$$

the functions $\tilde{\varphi}_{lm}$ being defined analogously to the transition functions in Proposition 3.2. The transition functions ω_{lm} are $\mathrm{GL}(c, \mathbb{C}) \times \mathrm{GL}(c, \mathbb{C})$-equivariant.

Proof. We want to express $(B_l, E_l, e; A_{2l})$ in terms of $(B_m, E_m, e; A_{2m})$. We already have eq. (3.8); analogously, one can prove $A_{1l} = A_{2m}(s_{m-l}\mathbf{1}_c + c_{m-l}B_m)$. It follows that $B_l = (c_{m-l}\mathbf{1}_c - s_{m-l}B_m)^{-1}(s_{m-l}\mathbf{1}_c + c_{m-l}B_m)$. As for E_l, one has

$$E_l = \left[\sum_{p=1}^{n} \sigma^{n-1}_{-l;0,p-1} C_p\right] A_{2l} = \left[\sum_{p=0}^{n-1} \sigma^{n-1}_{m-l;0p} B_m^p\right] E_m A_{2m}^{-1} A_{2l} = (c_{l-m}\mathbf{1}_c - s_{l-m}B_m)^n E_m\,,$$

where we have used eq. (3.6), the relation $\sigma^{n-1}_{m-l} = \sigma^{n-1}_{-l}\sigma^{n-1}_m$ and Lemma 3.5.

The equivariance of ω_{lm} is straightforward, and this completes the proof. □

From Proposition 3.3 we have

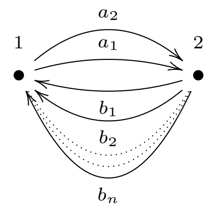
$$P^n(c)_m / \mathrm{GL}(c, \mathbb{C}) \times \mathrm{GL}(c, \mathbb{C}) \simeq \mathcal{T}(c) / \mathrm{GL}(c, \mathbb{C}) \simeq \mathcal{M}^n(c)_m\,;$$

moreover, there is an equivariant isomorphism $P^n(c)_m \simeq \mathcal{T}(c) \times \mathrm{GL}(c, \mathbb{C})$. As $\mathcal{T}(c)$ is a principal $\mathrm{GL}(c, \mathbb{C})$-bundle over $\mathcal{T}(c)/\mathrm{GL}(c, \mathbb{C})$, the space $P^n(c)_m$ turns out to be a principal $\mathrm{GL}(c, \mathbb{C}) \times \mathrm{GL}(c, \mathbb{C})$-bundle over $\mathcal{M}^n(c)_m$. Propositions 3.2 and 3.6 now imply that $P^n(c)$ is a principal $\mathrm{GL}(c, \mathbb{C}) \times \mathrm{GL}(c, \mathbb{C})$-bundle, and this completes the proof of Theorem 3.1.

4. Hilbert schemes as quiver varieties

In this last section we shall prove that Hilbert schemes of points of the total space of $\mathcal{O}_{\mathbb{P}^1}(-n)$ are isomorphic to suitable moduli spaces of quiver representations.

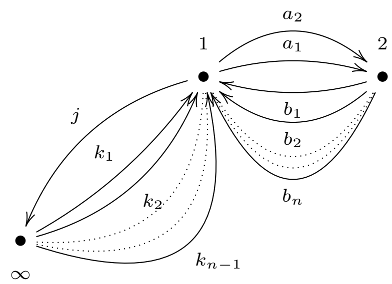
For $n \geq 1$, let $\mathcal{Q}_n$ be the quiver



.

We add to $\mathcal{Q}_n$ a "framing vertex," in a sense we shall explain later:

(4.1)



thus obtaining a new quiver $\mathcal{Q}_n^{\mathrm{fr}}$. Let I_n be the ideal of the path algebra $\mathbb{C}\mathcal{Q}_n^{\mathrm{fr}}$ generated by the relations

$$
\begin{cases} a_1 b_1 a_2 = a_2 b_1 a_1 & & \text{when } n = 1 \\ \\ a_1 b_q = a_2 b_{q+1} & \multirow{2}{*}{for $\quad q = 1, \dots, n-1$} & \multirow{2}{*}{when $n \geq 2$} \\ b_q a_1 + k_q j = b_{q+1} a_2 & & \end{cases} \tag{4.2}
$$

and let $B_n^{\mathrm{fr}} = \mathbb{C}\mathcal{Q}_n^{\mathrm{fr}}/I_n$. For every $\mathbf{v} = (v_1, v_2, v_\infty) \in \mathbb{N}^3$, a $\mathbf{v}$-dimensional representation of B_n^{fr} is given by the choice of three $\mathbb{C}$-vector spaces, V_1, V_2 and V_∞ with $\dim V_i = v_i$, together with an element $(A_1, A_2; C_1, \dots, C_n; e; f_1, \dots, f_{n-1})$ of

$$
\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)^{\oplus 2} \oplus \mathrm{Hom}_{\mathbb{C}}(V_2, V_1)^{\oplus n} \oplus \mathrm{Hom}_{\mathbb{C}}(V_1, V_\infty) \oplus \mathrm{Hom}_{\mathbb{C}}(V_\infty, V_1)^{\oplus n-1}
$$

compatible with the relations in (4.2). We will refer to the totality of the equations induced by (4.2) at the representation level as "condition (Q1)". The vertex ∞ is interpreted as a framing vertex because we regard $\mathrm{Rep}\left(B_n^{\mathrm{fr}}, \mathbf{v}\right)$ as a $\mathrm{GL}(v_1, \mathbb{C}) \times \mathrm{GL}(v_2, \mathbb{C})$-variety, avoiding the change of basis action of $\mathrm{GL}(v_\infty, \mathbb{C})$.

Fix $\vartheta \in \mathbb{Z}^2$. According to the general theory, a $\mathbf{v}$-dimensional representation is said to be *ϑ-semistable* if, for any subrepresentation $S = (S_1, S_2)$, one has:

$$
\text{if } S_1 \subseteq \ker e, \text{ then } \vartheta \cdot (\dim S_1, \dim S_2) \leq 0\,; \tag{4.3}
$$

$$
\text{if } S_1 \supseteq \mathrm{Im}\, f_i \text{ for } i = 1, \dots, n-1, \text{ then } \vartheta \cdot (\dim S_1, \dim S_2) \leq \vartheta \cdot (v_1, v_2)\,. \tag{4.4}
$$

A ϑ-semistable representation is *ϑ-stable* if strict inequality holds in (4.3), or in (4.4)), whenever $S \neq 0$, or, respectively, $S \neq (V_1, V_2)$. We denote by $/\!/_\vartheta$ the GIT quotient associated with the parameter ϑ.

We shall prove the following result:

Theorem 4.1. *For every $n, c \geq 1$, the variety $\operatorname{Hilb}^c(\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-n))$ is isomorphic to a connected component of the quotient*

$$\operatorname{Rep}^{ss}_{\vartheta_c}\left(B_n^{\mathrm{fr}}, \mathbf{v}_c\right) /\!/_{\vartheta_c} \operatorname{GL}(c,\mathbb{C}) \times \operatorname{GL}(c,\mathbb{C}),$$

where $\mathbf{v}_c = (c, c, 1)$ and $\vartheta_c = (2c, -2c+1)$.

More precisely, the connected component is given by the equations $f_1 = \cdots = f_{n-1} = 0$.

As it is clear from (4.2), the case $n = 1$ is special. For simplicity's sake, we shall give the proof of Theorem 4.1 assuming $n > 1$, but the result holds true for $\operatorname{Hilb}^c(\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-1))$ as well [13, Chapter 4].

From now on we fix $V_1 = V_2 = \mathbb{C}^c$. For brevity we denote by $\mathcal{R}_n(c)$ the space $\operatorname{Rep}^{ss}_{\vartheta_c}\left(B_n^{\mathrm{fr}}, \mathbf{v}_c\right)$. The following Lemma is a direct consequence of the semistability conditions (4.3) and (4.4).

Lemma 4.2. *An element $(A_1, A_2; C_1, \ldots, C_n; e; f_1, \ldots, f_{n-1}) \in \mathcal{R}_n(c)$ is ϑ_c-semistable if and only if*

- (Q2) *for all subrepresentations $S = (S_1, S_2)$ such that $S_1 \supseteq \operatorname{Im} f_i$, for $i = 1, \ldots, n-1$, one has $\dim S_1 \leq \dim S_2$;*
- (Q3) *for all subrepresentations $S = (S_1, S_2)$ such that $S_1 \subseteq \ker e$, one has $\dim S_1 \leq \dim S_2$, and, if $\dim S_1 = \dim S_2$, then $S = 0$.*

Furthermore, ϑ_c-semistability and ϑ_c-stability are equivalent.

Corollary 4.3. *If $(A_1, A_2; C_1, \ldots, C_n; e; f_1, \ldots, f_{n-1}) \in \mathcal{R}_n(c)$, the map e is not zero.*

Proof. If e were the zero map, the subrepresentation $S = (\mathbb{C}^c, \mathbb{C}^c)$ would violate condition (Q3). □

We observe that the action of $\operatorname{GL}(c,\mathbb{C}) \times \operatorname{GL}(c,\mathbb{C})$ on $\mathcal{R}_n(c)$ is compatible with the action of the same group on the space of ADHM data $P^n(c)$ we have defined in eq. (3.1). Thus, to prove Theorem 4.1 we can work directly on $P^n(c)$ and $\mathcal{R}_n(c)$ without taking into consideration such actions.

We denote by $Z_n(c)$ the closed subvariety of $\mathcal{R}_n(c)$ cut by the equations $f_1 = \cdots = f_{n-1} = 0$. We start by proving that $P^n(c) = Z_n(c)$. First, observe that in $Z_n(c)$ condition (P1) expresses exactly the constraints in (4.2), and (Q2) amounts to say that $\dim S_1 \leq \dim S_2$ *for all subrepresentations* $S = (S_1, S_2)$. We begin by showing that $P^n(c) \subseteq Z_n(c)$, i.e., that any element of $P^n(c)$ satisfies conditions (Q2) and (Q3).

Lemma 4.4. *The matrices A_1, A_2 satisfy condition* (P2) *if and only if they satisfy the requirement*

(Q2*) *for any subspace $S_1 \subseteq \mathbb{C}^c$, $\dim(A_1(S_1) + A_2(S_1)) \geq \dim S_1$.*

Proof. Suppose that condition (P2) is satisfied by A_1, A_2. Let S_1 be any subspace, and let $\{v_1, \ldots, v_k\}$ be a basis for it. Then, for suitable $[\nu_1, \nu_2] \in \mathbb{P}^1$, $\{(\nu_1 A_1 + \nu_2 A_2)v_j\}_{j=1}^k$ is a set of linearly indipendent vectors in $A_1(S_1) + A_2(S_1)$. So (Q2*) is also satisfied.

Now, for the converse, suppose that condition (P2) is not satisfied, so that the pencil $A_1 + \lambda A_2$ is singular. Let we consider a polynomial solution of minimal degree ε for that pencil,[2]

$$v(\lambda) = v_0 - \lambda v_1 + \lambda^2 v_2 + \cdots + (-1)^\varepsilon \lambda^\varepsilon v_\varepsilon, \qquad \text{with} \quad v_\varepsilon \neq 0\,. \tag{4.5}$$

Introduce the subspace $S_1 := \langle v_0, \ldots, v_\varepsilon \rangle$. The vectors $v_0, \ldots, v_\varepsilon$ are linearly indipendent (see [6, XII, Proof of Theorem 4]), so $\dim S_1 = \varepsilon + 1$. Now,

$$A_1(S_1) + A_2(S_1) = \langle A_1 v_0, \ldots, A_1 v_\varepsilon, A_2 v_0, \ldots, A_2 v_\varepsilon \rangle\,. \tag{4.6}$$

By substituting (4.5) in the equation $(A_1 + \lambda A_2)v(\lambda) = 0$ and by equating to zero the coefficients of the powers of λ, we get the $\varepsilon + 2$ relations

$$A_1 v_0 = 0\,, \quad A_2 v_0 - A_1 v_1 = 0\,, \;\ldots\;, \quad A_2 v_{\varepsilon-1} - A_1 v_\varepsilon = 0\,, \quad A_2 v_\varepsilon = 0\,. \tag{4.7}$$

Hence the maximum number of linearly indipendent vectors in (4.6) is

$$2\varepsilon + 2 - (\varepsilon + 2) = \varepsilon < \varepsilon + 1\,.$$

□

Since condition (Q2*) is clearly stronger than (Q2), Lemma 4.4 entails that condition (Q2) holds in $P^n(c)$. Let $S = (S_1, S_2)$ be a subrepresentation which makes condition (Q3) false. In particular, $\dim S_1 = \dim S_2 > 0$, and $S_1 \subseteq \ker e$. By exploiting [13, Remark 2.3.5], which works for the restrictions $A_1|_{S_1}$, $A_2|_{S_1}$, $C_1|_{S_2}$, $\ldots$, $C_n|_{S_2}$ as well, we can produce a vector $0 \neq v \in S_1$ and parameters $\lambda_1, \lambda_2, \mu_1, \mu_2$ that fail to satisfy condition (P3). As a consequence, also condition (Q3) holds in $P^n(c)$.

As for the opposite inclusion, $Z_n(c) \subseteq P^n(c)$, we have to show that any element of $Z_n(c)$ satisfies conditions (P2) and (P3).

Lemma 4.5. *If* $(A_1, A_2; C_1, \ldots, C_n; e; f_1, \ldots, f_{n-1}) \in \mathcal{R}_n(c)$, $\ker A_1 \cap \ker A_2 = \{0\}$.

Proof. Suppose that there exists a nonzero vector $v \in \mathbb{C}^c$ such that $A_i(v) = 0$ for $i = 1, 2$. If $v \in \ker e$, the subrepresentation $(\langle v \rangle, \{0\})$ violates condition (Q2); if $v \notin \ker e$, one has $\operatorname{Im} f_q = \langle f_q e(v) \rangle$, for $q = 1, \ldots, n-1$, but $f_q e(v) = (f_q e + C_q A_1 - C_{q+1} A_2)(v) = 0$, so that the subrepresentation $(\langle v \rangle, \{0\})$ violates (Q3). □

Suppose that the pencil $A_1 + \lambda A_2$ is singular. Let $v(\lambda)$ be a polynomial solution of minimal degree for the pencil defined in eq.(4.5). Lemma 4.5 implies that $\varepsilon \geq 1$. Set

$$\begin{aligned} V_0 &:= \langle v_0, \ldots, v_\varepsilon \rangle\,, \\ V_1 &:= A_1(V_0) + A_2(V_0)\,, \\ V_2 &:= \sum_{q=1}^{n} C_q(V_1)\,. \end{aligned}$$

We know that $\dim V_0 = \varepsilon + 1$ and $\dim V_1 = \varepsilon$ [6, XII, Proof of Theorem 4]. It is not difficult to show that $V_2 = 0$. The basic idea is that, if $V_2 \neq 0$, by using repeatedly condition (P1) and the relations in (4.7), one can produce a polynomial solution of degree smaller than ε (for more details, see [13, Proof of Theorem 4.2.1]). Thus, the subrepresentation $S = (V_0, V_1)$ fails to satisfy (Q2). As a consequence, condition (P2) holds in $Z_n(c)$.

[2]By *polynomial solution* we mean a solution $v(\lambda)$ of the equation $(A_1 + \lambda A_2)v(\lambda) = 0$ which is polynomial in λ. Such a solution always exists (see [6, p.29]).

Finally, let $v \in \mathbb{C}^c$ be a vector violating condition (P3). Set

$$S_1 := \langle v \rangle \,, \quad S_2 := \langle A_1 v, A_2 v \rangle \,.$$

In particular, $v \neq 0$, so $\dim S_1 = 1$. The conditions $\lambda_2 A_1 v + \lambda_1 A_2 v = 0$ and (P2) together imply that $\dim S_2 = 1$. We claim that

$$S_3 := \sum_{q=1}^{n} C_q(S_2) \subseteq \langle v \rangle \,.$$

Suppose that $S_2 = \langle A_1 v \rangle$. Then $A_2 v = \lambda A_1 v$ for some $\lambda \in \mathbb{C}$. This implies that

$$S_3 = \langle C_1 A_1 v, \dots, C_n A_1 v \rangle \,.$$

Now, by hypothesis $C_n A_1 v \in \langle v \rangle$; we get

$$C_q A_1 v = C_{q+1} A_2 v = \lambda C_{q+1} A_1 v \qquad \text{for } q = 1, \dots, n-1 \,,$$

so that by induction one gets $C_q A_1 v = \lambda^{n-q} C_n A_1 v \subseteq \langle v \rangle$, for $q = 1, \dots, n-1$. The case $S_2 = \langle A_2 v \rangle$ is completely analogous. Thus, the claim is proved, and because $S_1 \subseteq \ker e$ by hypothesis, (S_1, S_2) is a subrepresentation violating condition (Q3). So $P^n(c) = Z_n(c)$. Notice that condition (P3) holds on the whole of $\mathcal{R}_n(c)$.

To conclude the proof of Theorem 4.1 we have just to show that $Z_n(c)$ is a connected component of $\mathcal{R}_n(c)$. This goal is reached by proving that $Z_n(c)$ is closed and open at the same time as a subset of $\mathcal{R}_n(c)$, and that it is connected. This last statement follows easily from the fact that $Z_n(c) = P^n(c)$ and from the connectedness of $\mathrm{Hilb}^c(\mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^1}(-n))$ [5, Prop. 2.3].

We claim that the closed subvariety $Z_n(c)$ coincides with the open subset of $\mathcal{R}_n(c)$ where condition (P2) is satisfied, that is, we assume condition (P2) and prove that $f_1 = \cdots = f_{n-1} = 0$.

Given an element $(A_1, A_2; C_1, \dots, C_n; e; f_1, \dots, f_{n-1}) \in \mathcal{R}_n(c)$, we introduce the matrices A_{1m}, A_{2m} and E_m, as in eq. (3.4); by virtue of condition (P2), we can choose m such that $\det A_{2m} \neq 0$. After introducing the matrix $B_m = A_{2m}^{-1} A_{1m}$, we define

$$u_m := \sum_{q=1}^{n-1} \binom{n-2}{q-1} s_m^{n-1-q} c_m^{q-1} f_q \,,$$

and we set

$$b_1 = {}^t B_m \,, \qquad b_2 = {}^t E_m \,, \qquad i = {}^t e \,, \qquad j = {}^t u_m \,.$$

The data b_1, b_2, i, j satisfy:

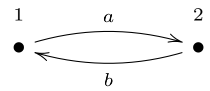
(i) $[b_1, b_2] + ij = 0$;
(ii) there exists no proper subspace $S \subsetneq \mathbb{C}^c$ such that $b_\alpha(S) \subseteq S$ $(\alpha = 1, 2)$ and $\mathrm{Im}\, i \subseteq S$.

Indeed, relation (i) follows by direct computation, suitably manipulating condition (Q1) and the expressions for the C_q's given in eq. (3.6). As for condition (ii), it suffices to observe that the second statement of Lemma 3.5, which works here as well, is equivalent to the maximality of the rank of

$$\begin{pmatrix} -(b_2 - w\mathbf{1}_c) & (b_1 - z\mathbf{1}_c) & i \end{pmatrix} \,,$$

so that we can apply [18, Lemma 2.7 (2)]. By [18, Proposition 2.8 (1)], one has $u_m = 0$, which implies that B_m and E_m commute; by using this fact in combination with condition (Q1), one shows that $f_q e = 0$, for $q = 1, \ldots, n-1$. Corollary 4.3 allows to conclude that $f_1 = \cdots = f_{n-1} = 0$, and this completes the proof of Theorem 4.1.

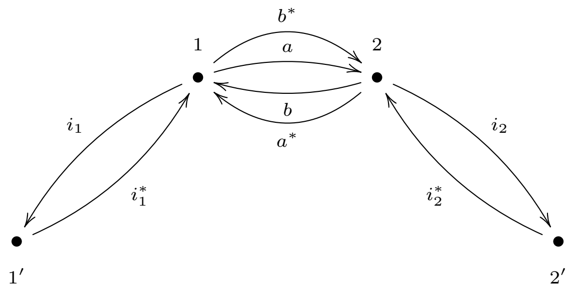
We can recover a result due to Kuznetsov [12] as an easy consequence of Theorem 4.1. We recall that, according to Nakajima [16], any quiver $\mathcal{Q}$, having vertex set I, is associated with a quiver variety $\mathcal{M}_{\lambda,\vartheta}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$, where $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$, $\lambda \in \mathbb{C}^I$ and $\vartheta \in \mathbb{Z}^I$ (see [7] for details). Let us consider the quiver $\mathcal{K}$:



Corollary 4.6. *For every $c \geq 1$, the variety $\mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-2))$ is isomorphic to the Nakajima quiver variety $\mathcal{M}_{0,\vartheta_c}(\mathcal{K}, \mathbf{v}_c, \mathbf{w})$, where $\vartheta_c = (2c, -2c+1)$, $\mathbf{v}_c = (c, c)$ and $\mathbf{w} = (1, 0)$.*

Proof. The construction of the Nakajima variety $\mathcal{M}_{\lambda,\vartheta}(\mathcal{K}, \mathbf{v}, \mathbf{w})$ relies on the definition of a new quiver $\mathcal{K}'$:

(4.8)



The choice $\mathbf{w} = (1, 0)$ implies that the linear morphisms associated with i_2 and i_2^* are zero morphisms, and this enables one to construct $\mathcal{M}_{\lambda,\vartheta}(\mathcal{K}, \mathbf{v}, \mathbf{w})$ using $\mathcal{Q}_2^{\mathrm{fr}}$ (see (4.1)). By a general result proved by Crawley-Boevey [4], the variety $\mathcal{M}_{0,\vartheta_c}(\mathcal{K}, \mathbf{v}_c, \mathbf{w})$ is connected (see also [7, Thm. 5.2.2] for some comments). The thesis then follows from Theorem 4.1. □

Remark 4.7. For any quiver $\mathcal{Q}$, the corresponding Nakajima variety $\mathcal{M}_{\lambda,\vartheta}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ carries a symplectic structure. This is essentially due to the fact that $\mathcal{M}_{\lambda,\vartheta}(\mathcal{Q}, \mathbf{v}, \mathbf{w})$ is concocted from a new quiver $\mathcal{Q}'$, associated with $\mathcal{Q}$, with the property of being a "double", that is, for any of its arrows $i \xrightarrow{a} j$ there is an opposite arrow $j \xrightarrow{a^*} i$ (cf. diagram (4.8)). It follows that every $\mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-2))$ is a symplectic variety (consistently with [18, Thm. 1.17]). On the contrary, no quiver $\mathcal{Q}_n^{\mathrm{fr}}$ is a double for $n \neq 2$. A result of Bottacin [2] implies, however, that $\mathrm{Hilb}^c(\mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-n))$ carries, for all $n \geq 1$, a Poisson structure whose rank is generically maximal.

APPENDIX A. PROOF OF PROPOSITION 3.2

We observe that $\mathrm{GL}(c, \mathbb{C})$ can be embedded as a closed subgroup of $G_{\vec{k}}$ by

$$\iota\colon \phi \mapsto \left({}^t\phi^{-1},\, \mathrm{diag}({}^t\phi^{-1},\, {}^t\phi^{-1},\, 1),\, {}^t\phi^{-1}\right) . \tag{A.1}$$

Let $\{P_{\vec{k},m}\}$ be the open cover of $P_{\vec{k}}$ given by the inverse image of the open cover of $\mathcal{M}^n(c)$. We want as a first step to prove the following claim: *for each* $m = 0, \ldots, c$ *there is a* $\mathrm{GL}(c, \mathbb{C})$-*equivariant closed immersion* $j_m \colon \mathcal{T}(c) \hookrightarrow P_{\vec{k},m}$ *which induces an isomorphism*

$$\eta_m \colon \mathcal{T}(c)/\,\mathrm{GL}(c,\mathbb{C}) \longrightarrow P_{\vec{k},m}/G_{\vec{k}} \simeq \mathcal{M}^n(c)_m\,. \tag{A.2}$$

Let us prove this claim. We have $r = 1$ so that $a = 0$. We begin by constructing the immersion j_m for any fixed $m \in \{0, \ldots, c\}$. To this aim, after fixing homogeneous coordinates $[y_1, y_2]$ for $\mathbb{P}^1$ (cfr. eq. (3.2)), we introduce additional c pairs of coordinates

$$[y_{1m}, y_{2m}] = [c_m y_1 + s_m y_2, -s_m y_1 + c_m y_2] \qquad m = 0, \ldots, c\,,$$

where c_m and s_m are the real numbers defined in eq. (3.3). We call s_E and s_∞, respectively, the (unique up to homotheties) global section of $\mathcal{O}_{\Sigma_n}(E)$, and the global section of $\mathcal{O}_{\Sigma_n}(1,0)$ that vanishes on ℓ_∞. We put $V_{\vec{k}} := H^0(\mathcal{V}_{\vec{k}}|_{\ell_\infty})$. These notations enable us to define the morphism

$$\begin{aligned}\tilde{\jmath}_m \colon \ \mathrm{End}(\mathbb{C}^c)^{\oplus 2} \oplus \mathrm{Hom}(\mathbb{C}^c, \mathbb{C}) &\longrightarrow \mathrm{Hom}(\mathcal{U}_{\vec{k}}, \mathcal{V}_{\vec{k}}) \oplus \mathrm{Hom}(\mathcal{V}_{\vec{k}}, \mathcal{W}_{\vec{k}}) \oplus \mathrm{Hom}(\mathbb{C}^r, V_{\vec{k}}) \\ (b_1, b_2, e) &\longmapsto (\alpha, \beta, \xi)\,,\end{aligned}$$

where

$$\alpha = \begin{pmatrix} \mathbf{1}_c(y_{2m}^n s_E) + {}^t b_2 s_\infty \\ \mathbf{1}_c y_{1m} + {}^t b_1 y_{2m} \\ 0 \end{pmatrix}\,,$$

$$\beta = \begin{pmatrix} \mathbf{1}_c y_{1m} + {}^t b_1 y_{2m}, & -\left(\mathbf{1}_c(y_{2m}^n s_E) + {}^t b_2 s_\infty\right), & {}^t e\, s_\infty \end{pmatrix}\,,$$

$$\xi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\,,$$

and j_m is the restriction of $\tilde{\jmath}_m$ to $\mathcal{T}(c)$.

Lemma A.1. *The morphism* j_m *is a* $\mathrm{GL}(c,\mathbb{C})$-*equivariant closed immersion of* $\mathcal{T}(c)$ *into* $P_{\vec{k},m}$.

Proof. Since it is clear that $\tilde{\jmath}_m$ is a closed immersion, it is enough to prove that

$$\mathrm{Im}\,\tilde{\jmath}_m \cap P_{\vec{k},m} = \mathrm{Im}\,j_m\,.$$

Let $(\alpha, \beta, \xi) = \tilde{\jmath}_m(b_1, b_2, e)$ be a point in the intersection $\mathrm{Im}\,\tilde{\jmath}_m \cap P_{\vec{k},m}$; the equation $\beta \circ \alpha = 0$ implies that the triple (b_1, b_2, e) satisfies condition (T1), while the fact that $\beta \otimes k(x)$ has maximal rank for all $x \in \Sigma_n$ entails condition (T2). It follows that

$$\mathrm{Im}\,\tilde{\jmath}_m \cap P_{\vec{k},m} \subseteq \mathrm{Im}\,j_m\,.$$

To get the opposite inclusion, note that for all $(\alpha, \beta, \xi) \in \mathrm{Im}\,\tilde{\jmath}_m$:

- (i) the morphism $\alpha \otimes k(x)$ fails to have maximal rank at most at a finite number of points $x \in \Sigma_n$; hence, α is injective;
- (ii) the morphisms $\alpha \otimes k(x)$ and $\beta \otimes k(x)$ have maximal rank for all points $x \in \ell_\infty \cup F_m$;
- (iii) the natural morphism $\Phi \colon H^0((\mathrm{coker}\,\alpha)_{|\ell_\infty}(-1)) \to H^0(\mathcal{W}_{\vec{k}}|_{\ell_\infty}(-1))$ is invertible;
- (iv) $\beta_1|_{F_m} = \mathbf{1}_c$, where $\beta_1 \colon \mathcal{O}_{\Sigma_n}(1,-1)^{\oplus k_2} \longrightarrow \mathcal{O}_{\Sigma_n}(1,0)^{\oplus k_3}$ is the first component of β;
- (v) the morphism ξ has maximal rank.

If $(\alpha,\beta,\xi)\in\operatorname{Im} j_m$, condition (T2) implies that $\beta\otimes k(x)$ has maximal rank for all $x\in\Sigma_n\backslash(\ell_\infty\cup F_m)$; by (ii) this is enough to ensure that β is surjective. Condition (T1) implies that $\beta\circ\alpha=0$, so that we can define $\mathcal{E}=\ker\beta/\operatorname{Im}\alpha$. By (i) $\mathcal{E}$ is torsion free, by (ii) and (iii) it is trivial at infinity, and by (iv) $\mathcal{E}|_{F_m}$ is trivial as well. The $\mathrm{GL}(c,\mathbb{C})$-equivariance of j_m is readily checked. □

Weare now in a position to prove that j_m induces an isomorphism between the quotients of $\mathcal{T}(c)$ and $P_{\vec{k},m}$ under the actions of $\mathrm{GL}(c,\mathbb{C})$ and $\mathrm{GL}(c,\mathbb{C})\times\mathrm{GL}(c,\mathbb{C})$, respectively. Indeed, it suffices to show that for any $G_{\vec{k}}$-orbit $O_{G_{\vec{k}}}$ in $P_{\vec{k},m}$, the intersection $O_{G_{\vec{k}}}\cap\operatorname{Im} j_m$ is not empty and its stabilizer in $G_{\vec{k}}$ coincides with $\operatorname{Im}\iota$ (the latter morphism was defined in eq. (A.1)). A detailed proof of this fact is provided in [13].

As a next step in the proof, we introduce the open subsets

$$\mathcal{T}(c)_{m,l}=j_m^{-1}\left(\operatorname{Im} j_m\cap P_{\vec{k},l}\right)\qquad\text{for}\quad m,l=0,\dots,c\,. \tag{A.3}$$

It is not difficult to see that

$$\mathcal{T}(c)_{m,l}=\{(b_1,b_2,e)\in\mathcal{T}(c)\,|\det\left(c_{m-l}\mathbf{1}_c-s_{m-l}b_1\right)\neq 0\}\ .$$

The second claim we need to prove is the following: *the map*

$$\begin{array}{cccc}\tilde{\varphi}_{lm}\colon & \mathcal{T}(c)_{m,l} & \longrightarrow & \mathcal{T}(c)_{l,m}\\ & \begin{pmatrix}b_1\\ b_2\\ e\end{pmatrix} & \longmapsto & \begin{pmatrix}\left(c_{m-l}\mathbf{1}_c-s_{m-l}b_1\right)^{-1}\left(s_{m-l}\mathbf{1}_c+c_{m-l}b_1\right)\\ \left(c_{m-l}\mathbf{1}_c-s_{m-l}b_1\right)^n b_2\\ e\end{pmatrix}\end{array} \tag{A.4}$$

is $\mathrm{GL}(c,\mathbb{C})$*-equivariant, and induces an isomorphism*

$$\varphi_{lm}\colon\mathcal{T}(c)_{m,l}/\,\mathrm{GL}(c,\mathbb{C})\longrightarrow\mathcal{T}(c)_{l,m}/\,\mathrm{GL}(c,\mathbb{C})\,,$$

such that $\eta_{m,l}=\eta_{l,m}\circ\varphi_{lm}$*, where* $\eta_{m,l}$ *is the restriction of* η_m *to* $\mathcal{T}(c)_{m,l}/\,\mathrm{GL}(c,\mathbb{C})$ *(see eq.* (A.2)*).*

To conclude our reasoning we need one more Lemma.

Lemma A.2. *For any* $l,m=0,\dots,c$ *and for any point* $\vec{b}_m=(b_{1m},b_{2m},e_m)\in\mathcal{T}(c)_{m,l}$*, there exists a unique element* $\psi_l(\vec{b}_m)=(\phi,\psi,\chi)\in G_{\vec{k}}$ *such that* $\chi=\mathbf{1}_c$*, and the point* $(\alpha',\beta',\xi')=\psi_l(\vec{b}_m)\cdot j_m(\vec{b}_m)$ *lies in the image of* j_l*. If we set*

$$(b_{1l},b_{2l},e_l)=j_l^{-1}(\alpha',\beta',\xi')\,,$$

we have

$$\begin{cases}b_{1l}=\left(c_{m-l}\mathbf{1}_c-s_{m-l}b_{1m}\right)^{-1}\left(s_{m-l}\mathbf{1}_c+c_{m-l}b_{1m}\right)\\ b_{2l}=\left(c_{m-l}\mathbf{1}_c-s_{m-l}b_{1m}\right)^n b_{2m}\\ \ e_l=e_m\,.\end{cases} \tag{A.5}$$

Proof. If we set $(\alpha,\beta,\xi)=j_m(\vec{b}_m)$, by expressing the coordinates $[y_{1m},y_{2m}]$ as functions of $[y_{1l},y_{2l}]$ we get

$$\alpha=\begin{pmatrix}\sum_{q=0}^n(\sigma_q\mathbf{1}_c)(y_{2l}^q y_{1l}^{n-q}s_E)+{}^tb_{2m}s_\infty\\ d_{1m}y_{1m}+d_{2m}y_{2m}\\ 0\end{pmatrix}\,,$$

$$\beta=\left(d_{1m}y_{1m}+d_{2m}y_{2m},\quad-\sum_{q=0}^n(\sigma_q\mathbf{1}_c)(y_{2l}^q y_{1l}^{n-q}s_E)-{}^tb_{2m}s_\infty,\quad {}^te_m s_\infty\right)\,,$$

where

$$d_{1m} = c_{m-l}\mathbf{1}_c - s_{m-l}{}^t b_{1m} \qquad\qquad d_{2m} = s_{m-l}\mathbf{1} + c_{m-l}{}^t b_{1m}$$

and we have put $\sigma_q = \sigma^n_{l-m;nq}$ for $q = 0, \dots, n$ (see eq. (3.5)). The explicit form of $\psi_l(\vec{b}_m)$ is obtained by imposing the equality

$$(\phi, \psi, \mathbf{1}_c) \cdot (\alpha, \beta, \xi) = j_l(b_{1l}, b_{2l}, e_l) \tag{A.6}$$

for some $(b_{1l}, b_{2l}, e_l) \in \mathcal{T}(c)_l$. One gets

$$\phi = d_{1m}^{-(n-1)}$$

$$\psi = \begin{pmatrix} d_{1m} & \psi_{12,1} & 0 \\ 0 & d_{1m}^{-n} & 0 \\ 0 & 0 & \mathbf{1}_r \end{pmatrix},$$

$$\text{where} \qquad \psi_{12,1} = -\sum_{q=0}^{n-1}\sum_{p=0}^{q} \sigma_{q-p}\left(-d_{2m}d_{1m}^{-1}\right)^p y_{1l}^q y_{2l}^{n-1-q}.$$

Eq. (A.5) follows from eq. (A.6). □

Since j_m and j_l are injective, the map $\vec{b}_m \longmapsto \psi_l(\vec{b}_m) \cdot \vec{b}_m$ induces the morphism $\tilde{\varphi}_{lm}$ in eq. (A.4). This completes the proof of Proposition 3.2.

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