

# FRACTURE MODELS FOR ELASTO-PLASTIC MATERIALS AS LIMITS OF GRADIENT DAMAGE MODELS COUPLED WITH PLASTICITY: THE ANTIPLANE CASE

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**ABSTRACT.** We study the asymptotic behavior of a variational model for damaged elasto-plastic materials in the case of antiplane shear. The energy functionals we consider depend on a small parameter  $\varepsilon$ , which forces damage concentration on regions of codimension one. We determine the  $\Gamma$ -limit as  $\varepsilon$  tends to zero and show that it contains an energy term involving the crack opening.

**Keywords:** Damage problems, gradient damage models, elasto-plasticity, cohesive fracture,  $\Gamma$ -convergence

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## 1. INTRODUCTION

Alessi, Marigo, and Vidoli have recently proposed a gradient damage model coupled with plasticity to describe the evolution of an elasto-plastic material that undergoes a damage process (see [2, 3]).

In the simplest situation of antiplane shear for an isotropic and homogeneous material, the model can be described as follows. The reference configuration is a bounded open set  $\Omega \subset \mathbb{R}^n$  ( $n = 2$  in the case of three-dimensional bodies) and the displacement is described by a scalar function  $u: \Omega \rightarrow \mathbb{R}$ . As usual in the theory of small strain elasto-plasticity, the displacement gradient is decomposed as

$$\nabla u = e + p,$$

where  $e$  and  $p$  are vector functions, representing the elastic and the plastic part of the strain, respectively. The damage variable is a scalar function  $\alpha: \Omega \rightarrow [\alpha_{min}, 1]$ , with 1 corresponding to the sound material, and  $\alpha_{min} \in (0, 1)$  representing the maximum possible damage. We assume that the stress depends only on the elastic part of the strain, through the formula

$$\sigma = \alpha e.$$

Since the elastic energy is given by

$$\mathcal{Q}(e, \alpha) := \frac{1}{2} \int_{\Omega} \sigma \cdot e \, dx = \frac{1}{2} \int_{\Omega} \alpha |e|^2 \, dx,$$

for a prescribed elastic strain the stored elastic energy decreases when the damage variable  $\alpha$  decreases (indicating a more damaged material).

The stress constraint is given by

$$|\sigma| \leq \kappa(\alpha),$$

where  $\kappa: [0, 1] \rightarrow \mathbb{R}$  is a continuous nondecreasing function with  $0 \leq \kappa(0) \leq \kappa(1) < +\infty$ , and  $\kappa(\beta) > 0$  for  $\beta > 0$ . It follows that the plastic potential, which is related to the energy dissipated by the plastic strain, is given by

$$\mathcal{H}(p, \alpha) := \int_{\Omega} \kappa(\alpha) |p| \, dx.$$

The energy dissipated by the damage process is

$$\mathcal{W}(\alpha) := b \int_{\Omega} W(\alpha) \, dx + \ell \int_{\Omega} |\nabla \alpha|^2 \, dx$$

where the function  $W: [0, 1] \rightarrow \mathbb{R}$  is continuous and decreasing,  $W(1) = 0$ , and  $b, \ell > 0$ . The gradient term has a regularizing effect and prevents sharp transitions of the damage.

The quasistatic evolution model introduced in [2, 3] and developed in [11, 12] is based on the minimization of the total energy

$$\mathcal{E}(e, p, \alpha) := \mathcal{Q}(e, \alpha) + \mathcal{H}(p, \alpha) + \mathcal{W}(\alpha),$$

under the constraint  $\nabla u = e + p$ , with  $u$  satisfying prescribed boundary conditions.

The aim of this paper is to study the behavior of the functional  $\mathcal{E}(e, p, \alpha)$  when the minimum problem forces the damage to be concentrated on sets of codimension one. To obtain this behavior, we assume that the three constants  $\alpha_{min}$ ,  $b$ , and  $\ell$  depend on a small parameter  $\varepsilon > 0$  in the following way:

$$\alpha_{min} = \delta_{\varepsilon}, \quad b = \frac{1}{\varepsilon}, \quad \ell = \varepsilon,$$

with  $\delta_{\varepsilon}/\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . With this choice the total energy becomes

$$(1.1) \quad \mathcal{E}_{\varepsilon}(e, p, \alpha) := \mathcal{Q}(e, \alpha) + \mathcal{H}(p, \alpha) + \mathcal{W}_{\varepsilon}(\alpha),$$

where

$$(1.2) \quad \mathcal{W}_{\varepsilon}(\alpha) := \int_{\Omega} \frac{W(\alpha)}{\varepsilon} \, dx + \varepsilon \int_{\Omega} |\nabla \alpha|^2 \, dx.$$

Some comments are now in order. Let  $(u_{\varepsilon}, e_{\varepsilon}, p_{\varepsilon}, \alpha_{\varepsilon})$  be a minimizer of  $\mathcal{E}_{\varepsilon}$  with prescribed boundary conditions. The term  $1/\varepsilon$  in the integral of  $W(\alpha_{\varepsilon})$  implies that  $\alpha_{\varepsilon} \rightarrow 1$  a.e. in  $\Omega$  as  $\varepsilon \rightarrow 0$ . On the other hand, to make  $\mathcal{Q}(e_{\varepsilon}, \alpha_{\varepsilon})$  small it might be convenient to force  $\alpha_{\varepsilon}$  to be close to 0 around some lower dimensional set  $S \subset \Omega$ . The interaction between the two terms of  $\mathcal{W}_{\varepsilon}$  implies that  $\mathcal{W}_{\varepsilon}(\alpha_{\varepsilon})$  approximates a suitable multiple of the  $(n-1)$ -dimensional measure of  $S$  (see [25] and [6]).

As for the dependence of  $\alpha_{min}$  on  $\varepsilon$ , the hypothesis  $\delta_{\varepsilon}/\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$  is crucial. Indeed, if  $\delta_{\varepsilon}$  is replaced by  $\varepsilon$ , the limit problem is different (see [14], [23], and [20]) and will not be considered in this paper.

Since  $\mathcal{H}$  has linear growth, it is useful to extend  $\mathcal{H}$  to the space of bounded measures  $\mathcal{M}_b(\Omega; \mathbb{R}^n)$ , which has better compactness properties, by setting

$$\mathcal{H}(p, \alpha) := \int_{\Omega} \kappa(\tilde{\alpha}) d|p|,$$

where  $\tilde{\alpha}$  denotes the quasicontinuous representative of  $\alpha \in H^1(\Omega)$  (see [18, Section 4.8]) and  $|p|$  is the total variation of the vector measure  $p$ . This leads us to consider the displacement  $u$  in the space  $BV(\Omega)$  of functions of bounded variation in  $\Omega$ .

To describe the asymptotic behavior of  $\mathcal{E}_{\varepsilon}$  as  $\varepsilon \rightarrow 0$ , it is convenient to define the functionals  $\mathcal{F}_{\varepsilon}: BV(\Omega) \times H^1(\Omega) \rightarrow [0, +\infty]$ , depending only on the displacement  $u$  and on the damage variable  $\alpha$ , by

$$(1.3) \quad \mathcal{F}_{\varepsilon}(u, \alpha) := \min_{e, p} \{ \mathcal{E}_{\varepsilon}(e, p, \alpha) : e \in L^2(\Omega; \mathbb{R}^n), p \in \mathcal{M}_b(\Omega; \mathbb{R}^n), Du = e + p \}$$

if  $\delta_\varepsilon \leq \alpha \leq 1$   $\mathcal{L}^n$ -a.e. in  $\Omega$ , and  $\mathcal{F}_\varepsilon(u, \alpha) = +\infty$  otherwise. The functional  $\mathcal{F}_\varepsilon$  represents the energy of the optimal additive decomposition of the displacement gradient. Note (see Proposition 2.1) that the minimum in (1.3) is achieved at a unique pair  $(e, p)$ , and that  $\mathcal{F}_\varepsilon$  can be written explicitly in an integral form as

$$(1.4) \quad \mathcal{F}_\varepsilon(u, \alpha) = \int_{\Omega} f_\varepsilon(\alpha, |\nabla u|) dx + \int_{\Omega} \kappa(\tilde{\alpha}) d|D^s u| + \mathcal{W}_\varepsilon(\alpha),$$

where  $\nabla u$  is the density of the absolutely continuous part of the measure  $Du$  with respect to the Lebesgue measure and  $D^s u$  is the singular part of  $Du$ . In order to define the integrand  $f_\varepsilon$  which appears in (1.4), we first introduce the function  $f$  defined for every  $\beta \in (0, 1]$  and  $t \geq 0$  by

$$(1.5) \quad f(\beta, t) := \min_{0 \leq s \leq t} \left[ \frac{1}{2} \beta s^2 + \kappa(\beta)(t - s) \right] = \begin{cases} \frac{1}{2} \beta t^2 & \text{if } t \leq \frac{\kappa(\beta)}{\beta}, \\ \kappa(\beta)t - \frac{\kappa(\beta)^2}{2\beta} & \text{if } t \geq \frac{\kappa(\beta)}{\beta}, \end{cases}$$

and then we set  $f_\varepsilon(\beta, t) := f(\beta, t)$  if  $\delta_\varepsilon \leq \beta \leq 1$  and  $f_\varepsilon(\beta, t) := +\infty$  otherwise.

The limit functional  $\mathcal{F}$  is defined in the space  $GBV(\Omega)$  (see [4]) of generalized functions of bounded variation by

$$(1.6) \quad \mathcal{F}(u) := \int_{\Omega} f(1, |\nabla u|) dx + \kappa(1)|D^c u|(\Omega) + \int_{J_u} \Psi(|[u]|) d\mathcal{H}^{n-1},$$

where  $J_u$  is the jump set of  $u$ ,  $[u]$  is the difference between the traces of  $u$  on both sides of  $J_u$ , and  $|D^c u|$  is the Cantor part of  $|Du|$  (see also the Appendix), while for every  $t \geq 0$

$$(1.7) \quad \Psi(t) := \min \left\{ \min_{0 \leq \beta \leq 1} [\kappa(\beta)t + \gamma_W(\beta)], \gamma_W(0) \right\}, \quad \text{with}$$

$$\gamma_W(\beta) := 4 \int_{\beta}^1 \sqrt{W(s)} ds, \quad \beta \in [0, 1].$$

The asymptotic behavior of the functionals  $\mathcal{F}_\varepsilon$  is obtained by studying their  $\Gamma$ -limit in the space  $L^1(\Omega) \times L^1(\Omega)$  as  $\varepsilon \rightarrow 0$ . For the definition and properties of  $\Gamma$ -convergence we refer to [13, 9]. The choice of the topology is suggested by the compactness properties of sequences  $(u_\varepsilon, \alpha_\varepsilon)$  with equibounded energies  $\mathcal{F}_\varepsilon(u_\varepsilon, \alpha_\varepsilon)$  (see Theorem 5.2). Therefore the functionals  $\mathcal{F}_\varepsilon$  defined in (1.3) are extended to  $L^1(\Omega) \times L^1(\Omega)$  by setting  $\mathcal{F}_\varepsilon(u, \alpha) := +\infty$  if  $u \notin BV(\Omega)$  or  $\alpha \notin H^1(\Omega)$ .

To describe the  $\Gamma$ -limit we introduce the functional  $\mathcal{F}_0: L^1(\Omega) \times L^1(\Omega) \rightarrow [0, +\infty]$  defined by

$$\mathcal{F}_0(u, \alpha) = \begin{cases} \mathcal{F}(u) & \text{if } u \in GBV(\Omega) \text{ and } \alpha = 1 \text{ } \mathcal{L}^n\text{-a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

The following theorem is the main result of this paper.

**Theorem 1.1.** *The functionals  $\mathcal{F}_\varepsilon$   $\Gamma$ -converge to  $\mathcal{F}_0$  in  $L^1(\Omega) \times L^1(\Omega)$  as  $\varepsilon \rightarrow 0$ .*

To study the minimum problems with Dirichlet boundary conditions, we assume in addition that  $\Omega$  has a Lipschitz boundary and we fix a relatively open subset  $\partial_D \Omega$  of  $\partial \Omega$ , where we prescribe the displacement.

We would like to analyze the asymptotic behaviour of the solutions of the minimum problems

$$(1.8) \quad \min \{ \mathcal{F}_\varepsilon(u, \alpha) : u \in BV(\Omega), \alpha \in H^1(\Omega), u = w, \alpha = 1 \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \partial_D \Omega \},$$

where  $w \in L^\infty(\partial_D \Omega)$ . Unfortunately these problems, in general, have no solutions. As for many other variational problems with linear growth in  $Du$ , the difficulty is given by the attainment of the boundary condition  $u = w$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial_D \Omega$ .

However, for every  $\eta > 0$ , it is always possible to consider an  $\eta$ -minimizer of (1.8), defined as a pair  $(u_\varepsilon, \alpha_\varepsilon) \in BV(\Omega) \times H^1(\Omega)$ , with  $u_\varepsilon = w$  and  $\alpha_\varepsilon = 1$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial_D \Omega$ , such that

$$\mathcal{F}_\varepsilon(u_\varepsilon, \alpha_\varepsilon) < \mathcal{I}_\varepsilon + \eta,$$

where  $\mathcal{I}_\varepsilon$  is the infimum in (1.8).

Since the functional  $\mathcal{F}_\varepsilon(\cdot, \alpha)$  decreases by truncation, for every  $w \in L^\infty(\partial_D \Omega)$  and for every  $\eta > 0$  the minimum problem (1.8) always has an  $\eta$ -minimizer  $(u_\varepsilon, \alpha_\varepsilon)$  satisfying

$$(1.9) \quad \|u_\varepsilon\|_{L^\infty(\Omega)} \leq \|w\|_{L^\infty(\partial_D \Omega)}.$$

In Section 5, we obtain the following result.

**Theorem 1.2.** *Let  $w \in L^\infty(\partial_D \Omega)$  and let  $\eta_\varepsilon \searrow 0$ . For every  $\varepsilon > 0$ , let  $(u_\varepsilon, \alpha_\varepsilon) \in BV(\Omega) \times H^1(\Omega)$  be a  $\eta_\varepsilon$ -minimizer of problem (1.8) satisfying (1.9). Then  $\alpha_\varepsilon \rightarrow 1$  in  $L^1(\Omega)$  and a subsequence of  $u_\varepsilon$  converges in  $L^1(\Omega)$  to a minimizer  $u \in BV(\Omega)$  of the problem*

$$(1.10) \quad \min \left\{ \mathcal{F}(u) + \int_{\partial_D \Omega} \Psi(|u - w|) d\mathcal{H}^{n-1} : u \in BV(\Omega) \right\}.$$

Note that in the limit problem the boundary condition  $u = w$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial_D \Omega$  is relaxed. Indeed, it is replaced by the term  $\int_{\partial_D \Omega} \Psi(|u - w|) d\mathcal{H}^{n-1}$ , which penalizes the non attainment of the prescribed boundary value. This is a typical feature of functionals with linear growth in the gradient.

The reason why we have studied the  $\Gamma$ -limit of  $\mathcal{F}_\varepsilon$ , rather than the  $\Gamma$ -limit of the functionals  $\mathcal{E}_\varepsilon$  introduced in (1.1), is that sequences  $(u_\varepsilon, \alpha_\varepsilon)$  with  $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, \alpha_\varepsilon) < +\infty$  are relatively compact, while this is not true for sequences  $(e_\varepsilon, p_\varepsilon, \alpha_\varepsilon)$  with  $\sup_\varepsilon \mathcal{E}_\varepsilon(e_\varepsilon, p_\varepsilon, \alpha_\varepsilon) < +\infty$ , since there is no a priori bound on  $e_\varepsilon$ .

A similar problem for elastic materials with damage and without plastic slips can be solved using the approximation results for the Mumford-Shah functional obtained by Ambrosio and Tortorelli in [5] and [6]. For the asymptotic analysis of other damage problems without plasticity see also [19], [23], [20], [10]. For the applications to the numerical approximation of fracture problems see [8] and the references therein.

The results of [5] and [6] correspond formally to our problem with  $\kappa(\beta) = +\infty$  for every  $0 \leq \beta \leq 1$ . Indeed, in this case  $f(1, t) = \frac{1}{2}t^2$ ,  $\Psi(t) = \gamma_W(0)$  for  $t > 0$ , and  $\Psi(0) = 0$ . Therefore  $|D^c u| = 0$  whenever the  $\Gamma$ -limit is finite and the term depending on  $J_u$  reduces to  $\gamma_W(0)\mathcal{H}^{n-1}(J_u)$ , which corresponds to Griffith's model in fracture mechanics for a brittle material with toughness  $\gamma_W(0)$ .

In the special case  $\kappa(\beta) = \kappa(1) < +\infty$  for every  $0 \leq \beta \leq 1$ , the function  $\Psi$  reduces to

$$\Psi(t) = \min\{\kappa(1)t, \gamma_W(0)\},$$

which describes Dugdale's cohesive model in fracture mechanics. A different approximation of this model has been obtained in [10].

In the general case  $J_u$  represents a crack (in the reference configuration) and the term

$$(1.11) \quad \int_{J_u} \Psi(|[u]|) d\mathcal{H}^{n-1}$$

in (1.6), depending on the crack opening, can be seen as the energy dissipated in the process of crack growth. The set  $\{x \in \partial_D \Omega : u(x) \neq w(x)\}$  can be interpreted as a crack on the Dirichlet part of the boundary of  $\Omega$  and the integral on  $\partial_D \Omega$  in (1.10) is the corresponding dissipated energy.

It turns out that  $\Psi$  satisfies the following properties (see Fig. 1):

- $\Psi$  is concave, nondecreasing, and  $\Psi(t) > 0$  for  $t > 0$ ;
- $\Psi(1) \min\{t, 1\} \leq \Psi(t) \leq \min\{\kappa(1)t, \gamma_W(0)\}$ ;
- $\Psi'(0) = \kappa(1)$ ;

$$\cdot \Psi(t) = \gamma_W(0) \text{ if } \kappa(0)t \geq \gamma_W(0).$$

Since the force between the crack lips is given by the derivative of  $\Psi$ , the above properties show that this force is always present when the crack opening is small and vanishes when the crack opening is large enough, provided  $\kappa(0) > 0$ . We refer to Section 2 for a detailed description of the behavior of  $\Psi$  when  $\kappa(0) = 0$ .

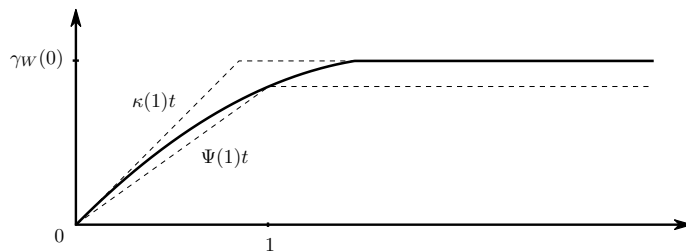


Figure 1. Graph of the crack energy density  $\Psi(t)$ .

For every  $u \in GBV(\Omega)$ , the functional  $\mathcal{F}$  can be written as

$$\mathcal{F}(u) = \min_{e,p} \left\{ \frac{1}{2} \int_{\Omega} |e|^2 dx + \kappa(1) |p|(\Omega \setminus J_u) + \int_{J_u} \Psi(|[u]|) d\mathcal{H}^{n-1} \right\}$$

where the minimum is taken among all  $e \in L^2(\Omega; \mathbb{R}^n)$ ,  $p \in \mathcal{M}_b(\Omega; \mathbb{R}^n)$  such that  $\nabla u \mathcal{L}^n + D^c u = e + p$  as measures on  $\Omega \setminus J_u$  (see Proposition 2.2). Moreover, this minimum is attained at a unique pair. In conclusion the functional  $\mathcal{F}$  can be interpreted as the total energy of an elasto-plastic material with a cohesive fracture.

The paper is organized as follows. In Section 2 we fix the notation, we list some useful properties of the function  $f$ , and we describe in detail the density  $\Psi$  of the crack energy of the limit problem. Section 3 is devoted to the proof of the main theorem in the one-dimensional case. The general case is studied in Section 4, where the  $\Gamma$ -liminf inequality is proved by a slicing argument, whereas the  $\Gamma$ -limsup inequality is obtained by using an integral representation result. In Section 5 we establish the convergence of minimizers of some model problems. Finally, in the Appendix, we make precise the notion of Cantor part  $D^c u$  for a function  $u \in GBV(\Omega)$ .

## 2. NOTATION AND PRELIMINARIES

Throughout the paper  $\Omega$  is a bounded open set of  $\mathbb{R}^n$  and  $\mathcal{A}(\Omega)$  denotes the class of all open subsets contained in  $\Omega$ . The notation  $A \Subset B$  means that  $A$  is relatively compact in  $B$ .

Since in Theorems 1.1, 1.2 it is enough to prove the result along every sequence  $\varepsilon_k \rightarrow 0$ , we fix once and for all a sequence  $\varepsilon_k \rightarrow 0$  and we use the shorthand notation  $\delta_k := \delta_{\varepsilon_k}$ ,  $f_k := f_{\varepsilon_k}$ ,  $\mathcal{F}_k := \mathcal{F}_{\varepsilon_k}$ ,  $\mathcal{W}_k := \mathcal{W}_{\varepsilon_k}$ , and  $\mathcal{E}_k := \mathcal{E}_{\varepsilon_k}$ .

**Functional setting.** The space of bounded  $\mathbb{R}^n$ -valued Radon measures on  $\Omega$  is denoted by  $\mathcal{M}_b(\Omega; \mathbb{R}^n)$ . It can be regarded as the dual of the Banach space  $\mathcal{C}_0(\Omega; \mathbb{R}^n)$  of continuous functions on  $\bar{\Omega}$  vanishing on  $\partial\Omega$ . The notion of weak\* convergence on  $\mathcal{M}_b(\Omega; \mathbb{R}^n)$  always refers to this duality.

For the definition and the main properties of the space of functions of bounded variation  $BV(\Omega)$  see, e.g., [4, 18]. We recall that for every function  $u \in BV(\Omega)$ , the Lebesgue decomposition of the bounded measure  $Du \in \mathcal{M}_b(\Omega; \mathbb{R}^n)$  is given by

$$Du = D^a u + D^s u = \nabla u \mathcal{L}^n + D^s u$$

where  $\nabla u$  is the approximate gradient of  $u$ ,  $\mathcal{L}^n$  is the Lebesgue measure on  $\mathbb{R}^n$ , and the measure  $D^s u$  is singular with respect to the Lebesgue measure. It can be further decomposed as

$$D^s u = D^c u + D^j u = D^c u + [u] \nu_u \mathcal{H}^{n-1} \llcorner J_u$$

where  $\mathcal{H}^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure,  $J_u$  is the jump set of  $u$ ,  $\nu_u$  is the normal to the  $\mathcal{H}^{n-1}$ -rectifiable set  $J_u$ ,  $[u] = u^+ - u^-$  is the jump of  $u$ , and  $D^c u$  is the Cantor part of  $Du$ , which is a singular measure with respect to the Lebesgue measure and vanishes on all Borel sets  $B \subset \mathbb{R}^n$  with  $\mathcal{H}^{n-1}(B) < +\infty$ . The precise representative  $\tilde{u}(x)$  of  $u$  is defined for  $\mathcal{H}^{n-1}$ -a.e.  $x$  in  $\Omega \setminus J_u$  as the limit of the averages of  $u$  on the balls  $B_\rho(x)$  as  $\rho \rightarrow 0^+$ .

A function  $u$  is in  $GBV(\Omega)$  if the truncated functions  $u_\lambda := \min\{\max\{-\lambda, u\}, \lambda\}$  belong to  $BV_{loc}(\Omega)$  for every  $\lambda > 0$ . The fine properties of generalized functions of bounded variation are listed in [4, Theorem 4.34]. In particular the approximate gradient is well-defined  $\mathcal{L}^n$ -a.e. on  $\Omega$ . Under suitable assumptions (see the Appendix), we can define a unique  $\mathbb{R}^n$ -valued Borel measure  $D^c u$  on  $\Omega \setminus J_u$  such that

$$(D^c u)(B) = (D^c u^\lambda)(B)$$

for every  $\lambda > 0$  and for every Borel set  $B \subset \{|\tilde{u}| < \lambda\} \setminus J_u$ .

**Capacity and quasicontinuous representatives.** For the notion of capacity we refer, e.g., to [18, 22, 24, 26]. Here we just recall some useful definitions and properties.

A property is said to hold Cap-quasi everywhere (abbreviated as Cap-q.e.) if it holds except for a subset of capacity zero.

A function  $\beta : \Omega \rightarrow \mathbb{R}$  is Cap-quasicontinuous if for every  $\varepsilon > 0$  there exists a set  $E_\varepsilon$  with  $\text{Cap}(E_\varepsilon) < \varepsilon$  such that  $\beta|_{\Omega \setminus E_\varepsilon}$  is continuous. For every function  $\alpha \in H^1(\Omega)$  there exists a Cap-quasicontinuous representative  $\tilde{\alpha}$ , i.e., a Cap-quasicontinuous function  $\tilde{\alpha}$  such that  $\tilde{\alpha} = \alpha$   $\mathcal{L}^n$ -a.e. in  $\Omega$ . The Cap-quasicontinuous representative is essentially unique, that is, if  $\beta$  is another Cap-quasicontinuous representative of  $\alpha$ , then  $\beta = \tilde{\alpha}$  Cap-q.e. in  $\Omega$ . Moreover it can be proved that (see [18, Theorem 4.8.1])

$$(2.1) \quad \lim_{\rho \rightarrow 0^+} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |\alpha(y) - \tilde{\alpha}(x)| dy = 0 \quad \text{for Cap-a.e. } x \in \Omega.$$

We recall that if  $E \subset \mathbb{R}^n$  is such that  $\text{Cap}(E) = 0$ , then its  $s$ -dimensional Hausdorff measure  $\mathcal{H}^s(E)$  vanishes for every  $s > n-2$ . As a consequence, the Cap-quasicontinuous representative of a function  $\alpha \in H^1(\Omega)$  is well defined  $\mathcal{H}^{n-1}$ -a.e. in  $\Omega$ . Therefore for every  $\alpha \in H^1(\Omega)$  the integral

$$\int_{\Omega} \kappa(\tilde{\alpha}) d\mu$$

makes sense for every measure  $\mu \in \mathcal{M}_b(\Omega)$  which vanishes on  $\mathcal{H}^{n-1}$ -negligible sets.

**The energy of the optimal decomposition.** We now provide an explicit expression for the minimum value in (1.3).

**Proposition 2.1.** *Let  $\mathcal{F}_k$  be the functional defined in (1.3). Then for every  $u \in BV(\Omega)$  and for every  $\alpha \in H^1(\Omega)$ , with  $\delta_k \leq \alpha \leq 1$ , there exists a unique pair  $(e, p)$  with  $e \in L^2(\Omega; \mathbb{R}^n)$  and  $p \in \mathcal{M}_b(\Omega; \mathbb{R}^n)$  such that  $Du = e + p$  and*

$$\mathcal{F}_k(u, \alpha) = \mathcal{E}_k(e, p, \alpha).$$

Moreover

$$\mathcal{F}_k(u, \alpha) = \int_{\Omega} f(\alpha, |\nabla u|) dx + \int_{\Omega} \kappa(\tilde{\alpha}) d|D^s u| + \mathcal{W}_k(\alpha),$$

where  $f$  is the function defined in (1.5).

*Proof.* The proof of the existence of a minimizing pair  $(e, p)$  is straightforward, and the uniqueness follows from the strict convexity of the  $L^2$  norm.

Let us prove the integral formula for  $\mathcal{F}_k$ . The inequality

$$\mathcal{F}_k(u, \alpha) \geq \int_{\Omega} f(\alpha, |\nabla u|) dx + \int_{\Omega} \kappa(\tilde{\alpha}) d|D^s u| + \mathcal{W}_k(\alpha)$$

is trivial. To prove the opposite inequality, we fix  $u \in BV(\Omega)$ ,  $\alpha \in H^1(\Omega)$  with  $\delta_k \leq \alpha \leq 1$ , and we define

$$e(x) := \begin{cases} \nabla u(x) & \text{if } |\nabla u(x)| \leq \frac{\kappa(\alpha(x))}{\alpha(x)}, \\ \frac{\kappa(\alpha(x))}{\alpha(x)} \frac{\nabla u(x)}{|\nabla u(x)|} & \text{if } |\nabla u(x)| \geq \frac{\kappa(\alpha(x))}{\alpha(x)}, \end{cases}$$

so that  $e \in L^2(\Omega; \mathbb{R}^n)$  and

$$\frac{1}{2} \alpha(x) |e(x)|^2 + \kappa(\alpha(x)) |\nabla u(x) - e(x)| = f(\alpha(x), |\nabla u(x)|)$$

for a.e.  $x \in \Omega$ . Let  $p := Du - e \in \mathcal{M}_b(\Omega; \mathbb{R}^n)$ , whose Lebesgue decomposition is

$$p = (\nabla u - e) \mathcal{L}^n + D^s u.$$

We have

$$\begin{aligned} \mathcal{F}_k(u, \alpha) &\leq \frac{1}{2} \int_{\Omega} \alpha |e|^2 dx + \int_{\Omega} \kappa(\tilde{\alpha}) d|p| + \mathcal{W}_k(\alpha) \\ &= \frac{1}{2} \int_{\Omega} \alpha |e|^2 + \kappa(\alpha) |\nabla u - e| dx + \int_{\Omega} \kappa(\tilde{\alpha}) d|D^s u| + \mathcal{W}_k(\alpha) \\ &= \int_{\Omega} f(\alpha, |\nabla u|) dx + \int_{\Omega} \kappa(\tilde{\alpha}) d|D^s u| + \mathcal{W}_k(\alpha). \end{aligned}$$

This concludes the proof.  $\square$

The same argument can be used to prove the following characterization of the functional  $\mathcal{F}$ .

**Proposition 2.2.** *Let  $\mathcal{F}$  be the functional defined in (1.6). Then for every  $u \in GBV(\Omega)$  with  $\mathcal{F}(u) < +\infty$  we have*

$$(2.2) \quad \mathcal{F}(u) = \min_{e, p} \left\{ \frac{1}{2} \int_{\Omega} |e|^2 dx + \kappa(1) |p|(\Omega \setminus J_u) + \int_{J_u} \Psi(|[u]|) d\mathcal{H}^{n-1} \right\},$$

where the minimum in (2.2) is taken among all  $e \in L^2(\Omega; \mathbb{R}^n)$ ,  $p \in \mathcal{M}_b(\Omega; \mathbb{R}^n)$  such that  $\nabla u \mathcal{L}^n + D^c u = e + p$  as measures on  $\Omega \setminus J_u$ . Moreover, the minimum is attained at a unique pair  $(e, p)$ .

We conclude with some remarks on the function  $f$  used in Proposition 2.1. From the very definition of  $f$  (see (1.5)) it follows that  $f(\beta, t)$  is increasing with respect to  $\beta$  and convex with respect to  $t$ . Moreover, from the explicit formula it is immediate to deduce that there exists a constant  $C > 0$  such that

$$(2.3) \quad \frac{1}{C}(t-1) \leq f(1, t) \leq Ct$$

for all  $t \geq 0$ . Finally, we notice that

$$(2.4) \quad f(\beta, \lambda t) \leq \lambda^2 f(\beta, t)$$

for every  $\lambda \geq 1$ ,  $\beta \in (0, 1]$ , and  $t \geq 0$ .

**Semicontinuity of the functionals  $\mathcal{F}_k$ .** In the next result, for every  $k$  we discuss the semicontinuity properties of the functional  $\mathcal{F}_k$  introduced in (1.3).

**Proposition 2.3.** *Let  $u_j, u \in BV(\Omega)$  and  $\alpha_j, \alpha \in H^1(\Omega)$ ,  $\delta_k \leq \alpha_j \leq 1$  be such that*

$$\begin{aligned} u_j &\rightarrow u && \text{strongly in } L^1(\Omega), \\ \alpha_j &\rightharpoonup \alpha && \text{weakly in } H^1(\Omega), \end{aligned}$$

as  $j \rightarrow +\infty$ . Then

$$(2.5) \quad \mathcal{F}_k(u, \alpha) \leq \liminf_{j \rightarrow +\infty} \mathcal{F}_k(u_j, \alpha_j).$$

*Proof.* In a first instance, let us prove the theorem in the case  $\|u_j\|_{L^\infty(\Omega)} \leq M$ . Moreover, let us assume that  $\kappa$  is a Lipschitz function.

We may assume that the liminf in (2.5) is finite and, up to extracting a subsequence, that  $\mathcal{F}_k(u_j, \alpha_j)$  is equibounded with respect to  $j$ . Let us fix  $e_j \in L^2(\Omega; \mathbb{R}^n)$  and  $p_j \in \mathcal{M}_b(\Omega; \mathbb{R}^n)$  such that  $Du_j = e_j + p_j$  and  $\mathcal{F}_k(u_j, \alpha_j) = \mathcal{E}_k(e_j, p_j, \alpha_j)$  (see Proposition 2.1). Since  $e_j$  is bounded in  $L^2(\Omega; \mathbb{R}^n)$  and  $p_j$  is bounded in  $\mathcal{M}_b(\Omega; \mathbb{R}^n)$ , we have that

$$\begin{aligned} e_j &\rightharpoonup e && \text{weakly in } L^2(\Omega; \mathbb{R}^n) \\ p_j &\overset{*}{\rightharpoonup} p && \text{weakly* in } \mathcal{M}_b(\Omega; \mathbb{R}^n), \end{aligned}$$

up to a subsequence. This implies that  $Du = e + p$ . It is not restrictive to assume that  $\alpha_j \rightarrow \alpha$   $\mathcal{L}^n$ -a.e. in  $\Omega$ . Then, since the sequence  $\alpha_j$  is uniformly bounded in  $L^\infty(\Omega)$ , we have that

$$\sqrt{\alpha_j} e_j \rightharpoonup \sqrt{\alpha} e \quad \text{weakly in } L^2(\Omega; \mathbb{R}^n)$$

as  $j \rightarrow +\infty$ . This implies

$$\int_{\Omega} \alpha |e|^2 dx \leq \liminf_{j \rightarrow +\infty} \int_{\Omega} \alpha_j |e_j|^2 dx.$$

Thus, to conclude the proof of (2.5), it suffices to show that

$$\int_{\Omega} \kappa(\tilde{\alpha}) d|p| \leq \liminf_{j \rightarrow +\infty} \int_{\Omega} \kappa(\tilde{\alpha}_j) d|p_j|,$$

since the other terms of the functional can be treated in a simple way. In order to prove this inequality, we just need to show that

$$(2.6) \quad \kappa(\tilde{\alpha}_j) p_j \overset{*}{\rightharpoonup} \kappa(\alpha) p \quad \text{weakly* in } \mathcal{M}_b(\Omega; \mathbb{R}^n).$$

Let us start by noticing that  $\kappa(\alpha_j) u_j, \kappa(\alpha) u \in BV(\Omega)$  and

$$\begin{aligned} D(\kappa(\alpha_j) u_j) &= \nabla(\kappa(\alpha_j)) u_j + \kappa(\tilde{\alpha}_j) Du_j \\ D(\kappa(\alpha) u) &= \nabla(\kappa(\alpha)) u + \kappa(\tilde{\alpha}) Du. \end{aligned}$$

Indeed, since  $u_j$  is bounded in  $L^\infty$  and  $\kappa$  is a Lipschitz function, the formulas above are true if  $\alpha_j$  and  $\alpha$  are  $\mathcal{C}^1$  functions. Then they can be extended to the case  $\alpha_j, \alpha \in H^1(\Omega)$  by an approximation argument, based on the fact that strong convergence in  $H^1(\Omega)$  implies Cap-q.e. pointwise convergence (for a subsequence) of the quasicontinuous representatives, which implies  $Du$ -a.e. convergence.

The measures  $\kappa(\tilde{\alpha}_j) Du_j$  are uniformly bounded in  $\mathcal{M}_b(\Omega; \mathbb{R}^n)$  and  $\nabla(\kappa(\alpha_j)) u_j \rightharpoonup \nabla(\kappa(\alpha)) u$ , weakly in  $L^2(\Omega; \mathbb{R}^n)$ . Hence the measures  $D(\kappa(\alpha_j) u_j)$  are uniformly bounded in  $\mathcal{M}_b(\Omega; \mathbb{R}^n)$ . Since  $\kappa(\alpha_j) u_j \rightarrow \kappa(\alpha) u$  in  $L^1(\Omega)$ , we have that  $D(\kappa(\alpha_j) u_j) \overset{*}{\rightharpoonup} D(\kappa(\alpha) u)$  weakly\* in  $\mathcal{M}_b(\Omega; \mathbb{R}^n)$ , and therefore, by difference,  $\kappa(\tilde{\alpha}_j) Du_j \overset{*}{\rightharpoonup} \kappa(\tilde{\alpha}) Du$  weakly\* in  $\mathcal{M}_b(\Omega; \mathbb{R}^n)$ . We conclude that (2.6) holds, taking into account that  $\kappa(\alpha_j) e_j \rightharpoonup \kappa(\alpha) e$  weakly in  $L^2(\Omega; \mathbb{R}^n)$ .



To remove the assumption that  $\kappa$  is Lipschitz, we approximate  $\kappa$  from below with Lipschitz functions  $\kappa_h \nearrow \kappa$ . By applying the previous step, we deduce that

$$\int_{\Omega} \kappa_h(\tilde{\alpha}) \, d|p| \leq \liminf_{j \rightarrow +\infty} \int_{\Omega} \kappa_h(\tilde{\alpha}_j) \, d|p_j| \leq \liminf_{j \rightarrow +\infty} \int_{\Omega} \kappa(\tilde{\alpha}_j) \, d|p_j|$$

and then we pass to the limit in  $h$ . This concludes the proof of (2.5) when  $u_j$  is bounded in  $L^\infty(\Omega)$ .

The extension to the unbounded case is obtained by a truncation argument.  $\square$

**The density of the crack energy.** In this subsection we study the main qualitative properties of the function  $\Psi$  defined in (1.7). It is convenient to introduce the function  $\Phi: [0, +\infty) \rightarrow \mathbb{R}$  defined by

$$(2.7) \quad \Phi(t) := \min_{0 \leq \beta \leq 1} [\kappa(\beta)t + \gamma_W(\beta)],$$

so that

$$(2.8) \quad \Psi(t) = \min\{\Phi(t), \gamma_W(0)\}.$$

Since  $\kappa(\beta) > 0$  for  $\beta > 0$  and  $\gamma_W(0) > 0$ , we have  $\Phi(t) \geq \Psi(t) > 0$  for every  $t > 0$ . Since  $\Phi$  and  $\Psi$  are obtained as minimum of nondecreasing affine functions, they are concave and nondecreasing. Therefore the inequality  $\Psi(t) > 0$  implies that

$$(2.9) \quad \Psi(1) \min\{t, 1\} \leq \Psi(t).$$

For every  $\beta \in [0, 1]$  we have  $\kappa(0)t \leq \kappa(\beta)t + \gamma_W(\beta)$ , hence  $\kappa(0)t \leq \Phi(t)$ . Moreover, the equality  $\gamma_W(1) = 0$  implies that  $\Phi(t) \leq \kappa(1)t$ . Therefore we have

$$(2.10) \quad \kappa(0)t \leq \Phi(t) \leq \kappa(1)t \quad \text{for every } t \geq 0,$$

which gives

$$(2.11) \quad \min\{\kappa(0)t, \gamma_W(0)\} \leq \Psi(t) \leq \min\{\kappa(1)t, \gamma_W(0)\} \quad \text{for every } t \geq 0.$$

In particular, if  $\kappa(0) > 0$ , then

$$(2.12) \quad \Psi(t) = \gamma_W(0) \quad \text{for } t \geq \frac{\gamma_W(0)}{\kappa(0)}.$$

When  $\kappa(0) = 0$ , we always have

$$\Phi(t) \leq \kappa(0)t + \gamma_W(0) = \gamma_W(0),$$

so that, in this case,

$$(2.13) \quad \Psi(t) = \Phi(t) \quad \text{for every } t \geq 0.$$

In the following proposition we show that, in any case, the function  $\Psi$  approaches the value  $\gamma_W(0)$  at infinity.

**Proposition 2.4.** *We have that*

$$(2.14) \quad \lim_{t \rightarrow +\infty} \Psi(t) = \gamma_W(0).$$

*Proof.* Since  $\Psi$  is nondecreasing, it suffices to prove the proposition when  $\Psi(t) < \gamma_W(0)$  for every  $t \geq 0$ . In this case

$$\Psi(t) = \Phi(t) = \kappa(\beta_t)t + \gamma_W(\beta_t)$$

for some  $\beta_t \in (0, 1]$ . Let us prove that  $\beta_t \rightarrow 0$  as  $t \rightarrow +\infty$ . If  $\limsup_t \beta_t =: \ell > 0$ , then there would exist a sequence  $t_j \rightarrow +\infty$  such that  $\beta_{t_j} \geq \ell/2$ , in turn implying that

$$\kappa(\ell/2)t_j \leq \kappa(\beta_{t_j})t_j + \gamma_W(\beta_{t_j}) = \Psi(t_j) < \gamma_W(0).$$

This would lead to a contradiction as  $j \rightarrow +\infty$ , and therefore  $\beta_t \rightarrow 0$  as  $t \rightarrow +\infty$ . Since

$$\kappa(\beta_t)t \leq \kappa(\beta_t)t + \gamma_W(\beta_t) \leq \sup_{s \geq 0} \Psi(s) \leq \lim_{s \rightarrow +\infty} \Psi(s),$$

by letting  $t \rightarrow +\infty$  we obtain  $\gamma_W(0) \leq \lim_{s \rightarrow +\infty} \Psi(s)$ .  $\square$

In general, when  $\kappa(0) = 0$ , it may happen that  $\Psi(t) < \gamma_W(0)$  for every  $t \geq 0$ , as the following example shows.

*Example 2.5.* Let us consider the functions

$$\kappa(\beta) := \beta^2 \quad \text{and} \quad W(\beta) := \frac{(1-\beta)^2}{4}.$$

In this way  $\gamma_W(\beta) = (1-\beta)^2$ . Then it is immediate to see that

$$\Psi(t) = \min_{0 \leq \beta \leq 1} [(1+t)\beta^2 - 2\beta + 1] = \frac{t}{1+t} < 1 = \gamma_W(0).$$

Nevertheless, if  $\kappa(\beta)$  tends to zero slowly enough as  $\beta \rightarrow 0$ , we still have  $\Psi(t) = \gamma_W(0)$  for some  $t > 0$ , as shown in the following proposition.

**Proposition 2.6.** *Assume that*

$$(2.15) \quad \liminf_{\beta \rightarrow 0^+} \frac{\kappa(\beta)}{\beta} > 0.$$

*Then there exists  $t_0$  such that  $\Psi(t) = \gamma_W(0)$  for  $t \geq t_0$ .*

*Proof.* Suppose, by contradiction, that for every  $j \in \mathbb{N}$  there exists  $\beta_j \in (0, 1]$  such that

$$(2.16) \quad \kappa(\beta_j)j + \gamma_W(\beta_j) < \gamma_W(0).$$

Arguing as in the proof of Proposition 2.4, we get that  $\beta_j \rightarrow 0$  as  $j \rightarrow +\infty$ .

From (2.16) it follows that

$$j \leq \frac{\gamma_W(0) - \gamma_W(\beta_j)}{\kappa(\beta_j)},$$

which implies, by (2.15), that

$$\begin{aligned} +\infty &= \limsup_{j \rightarrow +\infty} \frac{\gamma_W(0) - \gamma_W(\beta_j)}{\kappa(\beta_j)} \leq \limsup_{\beta \rightarrow 0^+} \frac{\gamma_W(0) - \gamma_W(\beta)}{\beta} \frac{\beta}{\kappa(\beta)} \\ &= \gamma'_W(0) \limsup_{\beta \rightarrow 0^+} \frac{\beta}{\kappa(\beta)} = 4\sqrt{W(0)} \limsup_{\beta \rightarrow 0^+} \frac{\beta}{\kappa(\beta)} < +\infty. \end{aligned}$$

This contradiction concludes the proof of the proposition.  $\square$

We now investigate the regularity properties of  $\Phi$  and  $\Psi$ . Since these functions are concave they admit left and right derivatives at every point. The following proposition provides the connection between  $\kappa$  and the derivatives of  $\Phi$ . For every function  $g(t)$ , the left and right derivatives are denoted by  $g'_-(t)$  and  $g'_+(t)$ , respectively.

**Proposition 2.7.** *Let  $t \in [0, +\infty)$  and let  $\beta_t^{\min}, \beta_t^{\max} \in [0, 1]$  be the smallest and the greatest solution of the minimum problem (2.7) which defines  $\Phi(t)$ . Then*

$$(2.17) \quad \Phi'_+(t) = \kappa(\beta_t^{\min}) \quad \text{for } t \geq 0 \quad \text{and} \quad \Phi'_-(t) = \kappa(\beta_t^{\max}) \quad \text{for } t > 0.$$

*If  $\beta_t^{\min} = \beta_t^{\max}$ , then  $\Phi$  is differentiable at  $t$  and  $\Phi'(t) = \kappa(\beta_t^{\min})$ . If  $\beta_t^{\min} < \beta_t^{\max}$ , then  $\Phi$  is not differentiable at  $t$ .*

*Proof.* Let us fix  $t > 0$  and let  $\beta_t$  be such that  $\Phi(t) = \kappa(\beta_t)t + \gamma_W(\beta_t)$ . First of all we prove that

$$(2.18) \quad \Phi'_+(t) \leq \kappa(\beta_t) \leq \Phi'_-(t).$$

By the definition of  $\Phi$ , for every  $s \geq 0$  we have  $\Phi(s) \leq \kappa(\beta_t)s + \gamma_W(\beta_t)$ . By the choice of  $\beta_t$ , this implies  $\Phi(s) \leq \kappa(\beta_t)(s-t) + \Phi(t)$ , which leads immediately to (2.18).

To prove the first equality in (2.17), let us now fix  $t \geq 0$ . Since  $\Phi$  is concave, there exists a decreasing sequence  $t_j \rightarrow t$  such that  $\Phi$  is differentiable at every  $t_j$ . Let  $\beta_j \in$

$[0, 1]$  be such that  $\Phi(t_j) = \kappa(\beta_j)t_j + \gamma_W(\beta_j)$ . A subsequence of  $\beta_j$  converges to some  $\beta^*$ . Passing to the limit in the previous equality, by the continuity of  $\kappa$ ,  $\Phi$ , and  $\gamma_W$  we get  $\Phi(t) = \kappa(\beta^*)t + \gamma_W(\beta^*)$ , which implies  $\beta_t^{\min} \leq \beta^*$ , and hence  $\kappa(\beta_t^{\min}) \leq \kappa(\beta^*)$ . As  $\Phi'_-(t_j) = \Phi'_+(t_j)$ , by (2.18) we have that  $\Phi'(t_j) = \kappa(\beta_j) \rightarrow \kappa(\beta^*)$ . Using the monotonicity of the difference quotients of  $\Phi$ , it is easy to prove that  $\Phi'(t_j) \rightarrow \Phi'_+(t)$  as  $j \rightarrow +\infty$ . This implies that  $\Phi'_+(t) = \kappa(\beta^*)$ . Therefore, the inequality  $\kappa(\beta_t^{\min}) \leq \kappa(\beta^*)$  together with (2.18) gives  $\Phi'_+(t) = \kappa(\beta_t^{\min})$ , which concludes the proof of the first part of (2.17). The proof of the second part is analogous.

The statement about the differentiability of  $\Phi$  is an obvious consequence of (2.17). As for the last statement, if  $\beta_t^{\min} < \beta_t^{\max}$  we have  $\kappa(\beta_t^{\min})t + \gamma_W(\beta_t^{\min}) = \kappa(\beta_t^{\max})t + \gamma_W(\beta_t^{\max})$ . Since  $\gamma_W$  is injective, we have also  $\gamma_W(\beta_t^{\min}) \neq \gamma_W(\beta_t^{\max})$ , which excludes the case  $t = 0$  and implies  $\kappa(\beta_t^{\min}) \neq \kappa(\beta_t^{\max})$ . Then (2.17) gives  $\Phi'_+(t) < \Phi'_-(t)$ , hence  $\Phi$  is not differentiable at  $t$ .  $\square$

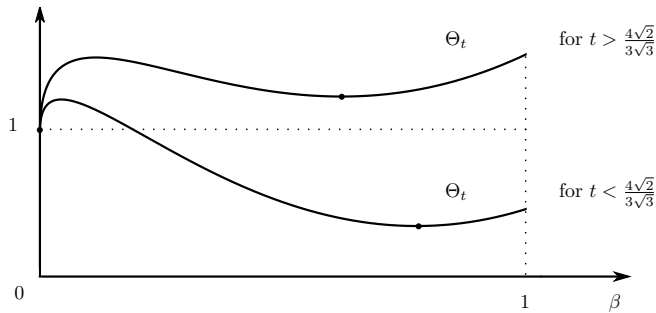
*Remark 2.8.* If  $\kappa(0) > 0$ , by (2.12) there exists  $t_0 > 0$  such that  $\Psi(t) = \Phi(t) < \gamma_W(0)$  for  $0 \leq t < t_0$  and  $\Phi(t) = \gamma_W(0)$  for  $t \geq t_0$ . It is clear that  $\Psi'_-(t_0) = \Phi'_-(t_0)$  and  $\Psi'_+(t_0) = 0$ . By Proposition 2.7 we have that  $\Phi'_-(t_0) = \kappa(\beta_{t_0}^{\max}) > 0$ , so that  $\Psi'_-(t_0) > \Psi'_+(t_0)$ .

We also provide an example in which  $\kappa(0) = 0$  and  $\Psi$ , equal to  $\Phi$  by (2.13), is not everywhere differentiable.

*Example 2.9.* Let us define

$$\kappa(\beta) := \sqrt{\beta} \quad \text{and} \quad W(\beta) := \frac{(1 - \beta)^2}{4},$$

so that  $\gamma_W(\beta) = (1 - \beta)^2$ . For  $0 < t < \frac{8}{3\sqrt{3}}$ , the function  $\Theta_t(\beta) := \kappa(\beta)t + \gamma_W(\beta)$  has exactly two local minimum points in  $[0, 1]$ : the first one is 0, whereas the second one is a point  $\alpha_t \in (0, 1)$ . For  $0 < t < \frac{4\sqrt{2}}{3\sqrt{3}}$ , the global minimum of  $\Theta_t$  is attained only at  $\alpha_t$ ; for  $t > \frac{4\sqrt{2}}{3\sqrt{3}}$ , the global minimum of  $\Theta_t$  is attained only at 0 (see Figure 2). For  $t_0 = \frac{4\sqrt{2}}{3\sqrt{3}}$ , there are two different global minimum points: 0 and  $\alpha_{t_0}$ . By the last statement of Proposition 2.7,  $\Phi$  is not differentiable at  $t_0$ . The previous analysis shows that  $\Phi(t) = \Theta_t(\alpha_t) < \Theta_t(0) = \gamma_W(0)$  for  $t < t_0$ , while  $\Phi(t) = \Theta_t(0) = \gamma_W(0)$  for  $t \geq t_0$ . So, in this example, the function  $\Phi$  is not differentiable at the first point where it attains the constant value  $\gamma_W(0)$ .



**Figure 2.** Graph of  $\Theta_t(\beta)$  for different values of  $t$ .

*Remark 2.10.* If for every  $t \geq 0$  the minimum problem (2.7) in the definition of  $\Phi$  has a unique solution, then  $\Phi$  is differentiable everywhere. Since it is concave, we conclude that it is of class  $\mathcal{C}^1([0, +\infty))$ . The uniqueness of the solution of (2.7) is always satisfied if  $\kappa$  is convex. Indeed  $\gamma_W$  is strictly convex, because its derivative  $-4\sqrt{W}$  is increasing. If  $\kappa$  is convex,  $\kappa(0) = 0$  and  $\kappa'_+(0) > 0$ , then  $\Psi = \Phi$  by (2.13),  $\Psi$  is differentiable by the previous analysis, and by Proposition 2.6 there exists  $t_0 > 0$  such that  $\Psi(t) < \gamma_W(0)$  for  $t < t_0$  and

$\Psi(t) = \gamma_W(0)$  for  $t \geq t_0$ . Note that in this case  $\Psi$  is differentiable at the first point in which it attains the constant value  $\gamma_W(0)$ .

### 3. PROOF OF THE MAIN RESULT IN THE ONE-DIMENSIONAL CASE

In this section we prove Theorem 1.1 when  $n = 1$ . We recall that in dimension one all Sobolev functions have a continuous representative. Without specifying it further again, we will always identify a function  $\alpha \in H^1(\Omega)$  with its continuous representative.

We start with the proof of the  $\Gamma$ -liminf inequality. Let us fix a sequence  $(u_k, \alpha_k)$  in  $L^1(\Omega) \times L^1(\Omega)$  and  $u \in L^1(\Omega)$  such that

$$(u_k, \alpha_k) \rightarrow (u, 1) \quad \text{in} \quad L^1(\Omega) \times L^1(\Omega).$$

We want to prove that

$$(3.1) \quad \mathcal{F}_0(u, 1) \leq \liminf_{k \rightarrow +\infty} \mathcal{F}_k(u_k, \alpha_k).$$

It is not restrictive to assume that the liminf in (3.1) is finite, hence that  $u_k \in BV(\Omega)$ ,  $\alpha_k \in H^1(\Omega)$ , and  $\delta_k \leq \alpha_k \leq 1$ .

To obtain an estimate from below of the liminf, we will carry out a careful analysis of the regions on which the damage is concentrating as  $\varepsilon_k \rightarrow 0$ . To do this, we will study the  $\Gamma$ -convergence of the sequence of functions  $\alpha_k$  defined on the space  $\Omega$  endowed with the topology induced by  $\mathbb{R}$ . This notion will be denoted by  $\Gamma(\mathbb{R})$ -convergence.

It is enough to prove (3.1) when  $\Omega$  is an interval, since the liminf is superadditive. Let  $e_k \in L^2(\Omega)$  and  $p_k \in \mathcal{M}_b(\Omega)$  be two sequences such that

$$Du_k = e_k + p_k \quad \text{in} \quad \Omega.$$

We will prove that  $u \in BV(\Omega)$  and

$$(3.2) \quad \mathcal{F}(u) \leq \liminf_{k \rightarrow +\infty} \mathcal{E}_k(e_k, p_k, \alpha_k).$$

We may assume that the liminf in (3.2) is finite and, up to extracting a subsequence, that it is actually a limit, so that

$$(3.3) \quad \mathcal{E}_k(e_k, p_k, \alpha_k) \leq c \quad \text{for every } k,$$

for some  $c \in \mathbb{R}$ . We now extract a subsequence of  $\alpha_k$ , not relabeled, which  $\Gamma(\mathbb{R})$ -converges to some function  $\alpha : \Omega \rightarrow [0, 1]$ .

*Remark 3.1.* For every  $\lambda \in [0, 1)$  the set  $\{\alpha \leq \lambda\}$  is finite. Indeed, let  $E = \{x_1, \dots, x_r\}$  be any finite subset of  $\{\alpha \leq \lambda\}$  and let  $\sigma > 0$  be such that the intervals  $[x_i - \sigma, x_i + \sigma]$ ,  $i = 1, \dots, r$ , are pairwise disjoint and contained in  $\Omega$ . Since  $\alpha_k$   $\Gamma(\mathbb{R})$ -converges to  $\alpha$ , for every  $i$  there exists a recovery sequence  $x_k^i \in (x_i - \sigma/2, x_i + \sigma/2)$  converging to  $x_i$  and such that  $\alpha_k(x_k^i) \rightarrow \alpha(x_i)$  as  $k \rightarrow +\infty$ . Moreover, since  $\alpha_k(x) \rightarrow 1$  for a.e.  $x \in \Omega$ , it is possible to find  $x_i - \sigma < y_1^i < x_k^i < y_2^i < x_i + \sigma$  such that  $\alpha_k(y_1^i) \rightarrow 1$ ,  $\alpha_k(y_2^i) \rightarrow 1$ . Using Young's inequality, from (3.3) we deduce that

$$\begin{aligned} c &\geq \sum_{i=1}^r \int_{x_i - \sigma}^{x_i + \sigma} \left[ \frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\alpha_k'|^2 \right] dx \\ &\geq 2 \sum_{i=1}^r \left[ \int_{y_1^i}^{x_k^i} \sqrt{W(\alpha_k)} |\alpha_k'| dx + \int_{x_k^i}^{y_2^i} \sqrt{W(\alpha_k)} |\alpha_k'| dx \right] \\ &= 2 \sum_{i=1}^r \left[ \int_{\alpha_k(x_k^i)}^{\alpha_k(y_1^i)} \sqrt{W(s)} ds + \int_{\alpha_k(x_k^i)}^{\alpha_k(y_2^i)} \sqrt{W(s)} ds \right] \end{aligned}$$

and letting  $k \rightarrow +\infty$

$$c \geq 4 \sum_{i=1}^r \int_{\alpha(x_i)}^1 \sqrt{W(s)} \, ds = \sum_{i=1}^r \gamma_W(\alpha(x_i)) \geq \mathcal{H}^0(E) \gamma_W(\lambda),$$

since  $\mathcal{H}^0$  is the counting measure. It follows that

$$(3.4) \quad \mathcal{H}^0(E) \leq \frac{c}{\gamma_W(\lambda)}.$$

Since  $E$  was an arbitrary finite subset of  $\{\alpha \leq \lambda\}$  and the estimate (3.4) does not depend on  $E$ , we conclude that  $\{\alpha \leq \lambda\}$  is finite.

*Remark 3.2.* Let  $\lambda \in [0, 1)$  and let  $K$  be a compact set such that  $K \subset \{\alpha > \lambda\}$ . Since  $\alpha$  is lower semicontinuous, we have that  $\lambda < \min_K \alpha \leq \liminf_k \min_K \alpha_k$ , where the last inequality follows from the lower semicontinuity with respect to the  $\Gamma(\mathbb{R})$ -convergence of the minimum on compact sets. It follows that  $\alpha_k > \lambda$  on  $K$  for  $k$  large enough.

**Lemma 3.3.** *The function  $u$  belongs to  $BV(\Omega)$  and there exist a subsequence of  $(e_k, p_k)$  (not relabeled), a function  $e \in L^2(\Omega)$ , and a measure  $p \in \mathcal{M}_b(\Omega)$  such that*

$$(3.5) \quad Du = e + p \quad \text{in } \Omega,$$

$$(3.6) \quad e_k \rightharpoonup e \quad \text{weakly in } L^2(A),$$

$$(3.7) \quad p_k \xrightarrow{*} p \quad \text{weakly* in } \mathcal{M}_b(A)$$

for every open set  $A \Subset \{\alpha > 0\}$ .

*Proof.* Since  $\{\alpha \leq \frac{1}{2}\}$  is finite by Remark 3.1, we can find  $\lambda \in (0, \frac{1}{2})$  such that

$$(3.8) \quad \lambda < \min\{\alpha(x) : \alpha(x) \leq \frac{1}{2}, \alpha(x) > 0\}.$$

From (3.8) it follows that  $\{\alpha \leq \lambda\} = \{\alpha = 0\}$ . Let us consider a sequence of open sets  $A_j$  such that

$$(3.9) \quad A_j \Subset A_{j+1}, \quad \bigcup_{j=1}^{+\infty} A_j = \{\alpha > \lambda\}.$$

Fix  $j \geq 1$ . By Remark 3.2,  $\alpha_k > \lambda$  on  $A_j$  for  $k$  large enough. From (3.3) we deduce

$$c \geq \int_{A_j} \alpha_k |e_k|^2 \, dx \geq \lambda \int_{A_j} |e_k|^2 \, dx$$

and hence  $\|e_k\|_{L^2(A_j)}^2 \leq c/\lambda$ . Therefore there exists a subsequence, which we do not relabel, such that

$$e_k \rightharpoonup e^j \quad \text{weakly in } L^2(A_j),$$

and by a diagonal argument it is possible to extract a subsequence, not depending on  $j$ , such that

$$e_k \rightharpoonup e^j \quad \text{weakly in } L^2(A_j) \quad \text{for every } j \geq 1.$$

By the lower semicontinuity of the norm, we have  $\|e^j\|_{L^2(A_j)}^2 \leq c/\lambda$ . Therefore there exists a function  $e \in L^2(\Omega)$  such that  $e = e^j$  on  $A_j$ , for every  $j$ . It follows that

$$(3.10) \quad e_k \rightharpoonup e \quad \text{weakly in } L^2(A_j).$$

On the other hand, since  $\kappa$  is nondecreasing and by (3.3),

$$c \geq \int_{A_j} \kappa(\alpha_k) \, d|p_k| \geq \kappa(\lambda) |p_k|(A_j),$$

from which it follows that  $p_k$  is bounded in  $\mathcal{M}_b(A_j)$ . Thus there exists a subsequence (which we do not relabel) and a measure  $p^j \in \mathcal{M}_b(A_j)$  such that

$$p_k \xrightarrow{*} p^j \quad \text{weakly* in } \mathcal{M}_b(A_j).$$

By a diagonal argument, there exists a subsequence, not depending on  $j$ , such that

$$p_k \xrightarrow{*} p^j \quad \text{weakly* in } \mathcal{M}_b(A_j) \quad \text{for every } j \geq 1.$$

By the lower semicontinuity of the total variation, it follows that  $|p^j|(A_j) \leq c/\kappa(\lambda)$ , and hence there exists a measure  $p \in \mathcal{M}_b(\{\alpha > 0\})$  such that  $p \llcorner A_j = p^j$ , for every  $j$ . This yields

$$(3.11) \quad p_k \xrightarrow{*} p \quad \text{weakly* in } \mathcal{M}_b(A_j).$$

From (3.10) and (3.11), it follows that  $u \in BV(A_j)$ ,  $Du = e + p$  in  $A_j$ , and  $Du_k \xrightarrow{*} Du$  in  $\mathcal{M}_b(A_j)$ , for every  $j \geq 1$ . Since

$$\|e\|_{L^2(A_j)}^2 \leq \frac{c}{\lambda} \quad \text{and} \quad |p|(A_j) \leq \frac{c}{\kappa(\lambda)} \quad \text{for every } j \geq 1,$$

we deduce that  $u \in BV(\{\alpha > 0\})$ , with  $Du = e + p$  in the open set  $\{\alpha > 0\}$ . Since the set  $\{\alpha = 0\}$  is finite and the right and left limits  $u^+$  and  $u^-$  are well defined and finite on each point of  $\{\alpha = 0\}$ , we conclude that  $u \in BV(\Omega)$ . The measure  $p \in \mathcal{M}_b(\Omega)$  extended to  $\Omega$  by

$$p := p \llcorner \{\alpha > 0\} + (u^+ - u^-)\mathcal{H}^0 \llcorner \{\alpha = 0\}$$

satisfies (3.7) and (3.5).  $\square$

We are now able to prove (3.2).

**Proposition 3.4.** *Let  $e \in L^2(\Omega)$  and  $p \in \mathcal{M}_b(\Omega)$  be given by Lemma 3.3, in such a way that (3.5), (3.6), and (3.7) hold. Then*

$$(3.12) \quad \frac{1}{2} \int_{\Omega} |e|^2 dx \leq \liminf_{k \rightarrow +\infty} \frac{1}{2} \int_{\Omega} \alpha_k |e_k|^2 dx,$$

$$(3.13) \quad \kappa(1)|p|(\Omega \setminus J_u) + \sum_{x \in J_u} \Psi(|u|(x)) \leq \liminf_{k \rightarrow +\infty} [\mathcal{H}(p_k, \alpha_k) + \mathcal{W}_k(\alpha_k)].$$

Moreover, (3.2) holds.

*Proof.* Let us fix  $\eta \in (0, 1]$ . By Remark 3.1, the set  $\{\alpha \leq 1 - \eta\}$  is finite, hence we can write  $\{\alpha \leq 1 - \eta\} = \{x_1, \dots, x_r\}$  with  $x_1 < \dots < x_r$ . Moreover, let  $\partial\Omega = \{x_0, x_{r+1}\}$ . Finally, let  $\sigma_0 > 0$  be such that the intervals  $[x_i - \sigma_0, x_i + \sigma_0]$ ,  $i = 0, \dots, r + 1$ , are pairwise disjoint. For  $\sigma \in (0, \sigma_0)$ , let

$$A_\sigma := \Omega \setminus \left( \bigcup_{i=0}^{r+1} [x_i - \sigma, x_i + \sigma] \right).$$

Since  $A_\sigma \Subset \{\alpha > 1 - \eta\}$ , we have  $\alpha_k > 1 - \eta$  for  $k$  large enough, by Remark 3.2. Moreover (3.6) and (3.7) hold with  $A = A_\sigma$ . By the lower semicontinuity of the norm in  $L^2(A_\sigma)$  and in  $\mathcal{M}_b(A_\sigma)$ , it follows that

$$(3.14) \quad \frac{1 - \eta}{2} \int_{A_\sigma} |e|^2 dx \leq \liminf_{k \rightarrow +\infty} \frac{1 - \eta}{2} \int_{A_\sigma} |e_k|^2 dx \leq \liminf_{k \rightarrow +\infty} \frac{1}{2} \int_{\Omega} \alpha_k |e_k|^2 dx,$$

$$(3.15) \quad \kappa(1 - \eta)|p|(A_\sigma) \leq \liminf_{k \rightarrow +\infty} \kappa(1 - \eta)|p_k|(A_\sigma) \leq \liminf_{k \rightarrow +\infty} \int_{A_\sigma} \kappa(\alpha_k) d|p_k|.$$

Let  $i = 1, \dots, r$ . Arguing as in Remark 3.1, we can find a sequence  $x_k^i \rightarrow x_i$ , with  $\alpha_k(x_k^i) \rightarrow \alpha(x_i)$ , and  $x_i - \sigma < y_1^i < x_k^i < y_2^i < x_i + \sigma$  such that  $\alpha_k(y_1^i) \rightarrow 1$ ,  $\alpha_k(y_2^i) \rightarrow 1$ , yielding

$$(3.16) \quad \begin{aligned} & \liminf_{k \rightarrow +\infty} \int_{x_i - \sigma}^{x_i + \sigma} \left[ \frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\alpha_k'|^2 \right] dx \\ & \geq \liminf_{k \rightarrow +\infty} 2 \left[ \int_{y_1^i}^{x_k^i} \sqrt{W(\alpha_k)} |\alpha_k'| dx + \int_{x_k^i}^{y_2^i} \sqrt{W(\alpha_k)} |\alpha_k'| dx \right] \geq \gamma_W(\alpha(x_i)). \end{aligned}$$

If  $\alpha(x_i) = 0$ , then we estimate from below simply by

$$(3.17) \quad \liminf_{k \rightarrow +\infty} \int_{x_i - \sigma}^{x_i + \sigma} \kappa(\alpha_k) d|p_k| \geq 0.$$

If, instead,  $\alpha(x_i) > 0$ , we can fix  $\omega > 0$  such that  $\alpha(x_i) - \omega > 0$ . Since  $\alpha_k$   $\Gamma(\mathbb{R})$ -converges to  $\alpha$ , then

$$\alpha(x_i) = \sup_{\rho > 0} \liminf_{k \rightarrow +\infty} \inf_{|x - x_i| < \rho} \alpha_k(x_i),$$

and therefore there exists  $\rho_i > 0$  such that

$$\alpha(x_i) - \omega < \liminf_{k \rightarrow +\infty} \inf_{|x - x_i| < \rho_i} \alpha_k(x_i),$$

from which it follows that for  $k$  large enough

$$\alpha(x_i) - \omega < \inf_{|x - x_i| < \rho_i} \alpha_k(x_i).$$

Hence, if  $\sigma_0 = \sigma_0(\omega) > 0$  is small enough, by (3.7) we obtain

$$(3.18) \quad \begin{aligned} & \kappa(\alpha(x_i) - \omega) |p(\{x_i\})| \leq \kappa(\alpha(x_i) - \omega) |p((x_i - \sigma, x_i + \sigma))| \\ & \leq \liminf_{k \rightarrow +\infty} \kappa(\alpha(x_i) - \omega) |p_k((x_i - \sigma, x_i + \sigma))| \leq \liminf_{k \rightarrow +\infty} \int_{[x_i - \sigma, x_i + \sigma]} \kappa(\alpha_k) d|p_k|, \end{aligned}$$

for every  $\sigma \in (0, \sigma_0)$ . Summing (3.15)–(3.18), by the superadditivity of the liminf we deduce that

$$(3.19) \quad \begin{aligned} & \kappa(1 - \eta) |p(A_\sigma)| + \sum_{x \in \{0 < \alpha \leq 1 - \eta\}} \kappa(\alpha(x) - \omega) |p(\{x\})| + \sum_{x \in \{\alpha \leq 1 - \eta\}} \gamma_W(\alpha(x)) \\ & \leq \liminf_{k \rightarrow +\infty} [\mathcal{H}(p_k, \alpha_k) + \mathcal{W}_k(\alpha_k)]. \end{aligned}$$

Letting  $\sigma \rightarrow 0^+$ ,  $\omega \rightarrow 0^+$ , and then  $\eta \rightarrow 0^+$  in (3.14) and (3.19), we obtain (3.12) and

$$(3.20) \quad \begin{aligned} & \kappa(1) |p(\{\alpha = 1\})| + \sum_{x \in \{0 < \alpha < 1\}} \kappa(\alpha(x)) |p(\{x\})| + \sum_{x \in \{\alpha < 1\}} \gamma_W(\alpha(x)) \\ & \leq \liminf_{k \rightarrow +\infty} [\mathcal{H}(p_k, \alpha_k) + \mathcal{W}_k(\alpha_k)]. \end{aligned}$$

By (3.5) and by the general properties of the Cantor part of  $Du$ , we have  $p(B \setminus J_u) = 0$  for every countable set  $B$ . Using the definition of  $\Psi$  (see (1.7)) and the inequality  $\Psi(z) \leq$

$\kappa(1)|z|$ , we get

$$\begin{aligned}
& \kappa(1)|p|(\Omega \setminus J_u) + \sum_{x \in J_u} \Psi(|[u](x)|) \\
&= \kappa(1)|p|(\{\alpha = 1\}) + \sum_{x \in \{\alpha < 1\}} \Psi(|p(\{x\})|) - \kappa(1)|p|(J_u \cap \{\alpha = 1\}) + \sum_{x \in J_u \cap \{\alpha = 1\}} \Psi(|[u](x)|) \\
&\leq \kappa(1)|p|(\{\alpha = 1\}) + \sum_{x \in \{\alpha < 1\}} \Psi(|p(\{x\})|) \\
&\leq \kappa(1)|p|(\{\alpha = 1\}) + \sum_{x \in \{0 < \alpha < 1\}} \kappa(\alpha(x))|p(\{x\})| + \sum_{x \in \{\alpha < 1\}} \gamma_W(\alpha(x)),
\end{aligned}$$

which, together with (3.20), gives (3.13).

By (2.2) we have

$$\mathcal{F}(u) \leq \frac{1}{2} \int_{\Omega} |e|^2 dx + \kappa(1)|p|(\Omega \setminus J_u) + \sum_{x \in J_u} \Psi(|[u](x)|),$$

so that (3.12) and (3.13) yield (3.2). This concludes the proof.  $\square$

*Remark 3.5.* With respect to (3.13), inequality (3.20) proved in Proposition 3.4 gives a more precise estimate from below, which takes into account the asymptotic values of the damage variable on sets of Lebesgue measure zero. Unfortunately, it is not clear how to extend this result to dimension  $n > 1$ .

Inequality (3.1) now simply follows from (3.2) by choosing  $e_k \in L^2(\Omega)$  and  $p_k \in \mathcal{M}_b(\Omega)$  such that  $Du_k = e_k + p_k$  in  $\Omega$  and

$$\mathcal{F}_k(u_k, \alpha_k) = \mathcal{E}_k(e_k, p_k, \alpha_k).$$

We now prove the  $\Gamma$ -limsup inequality. We start with the following preliminary result concerning the domain of the limit functional in the one-dimensional setting.

**Proposition 3.6.** *Let  $\Omega \subset \mathbb{R}$  be a bounded open set. Let  $u \in GBV(\Omega) \cap L^1(\Omega)$  be such that  $\mathcal{F}(u) < +\infty$ . Then  $u \in BV(\Omega)$ .*

*Proof.* For every open set  $A \subset \Omega$  we define

$$\mathcal{G}(u; A) := \int_A |u'| dx + |D^c u|(A) + \sum_{x \in (J_u \setminus J_u^1) \cap A} |[u](x)| + \mathcal{H}^0(J_u^1 \cap A),$$

where  $J_u^1 := \{x \in J_u : |[u](x)| \geq 1\}$ . By (2.3) and (2.9), there exists a constant  $c > 0$  such that  $\mathcal{G}(u; \Omega) \leq c(\mathcal{F}(u) + 1) < +\infty$ .

*Step 1:* Let us assume that  $\Omega$  is a bounded interval and that all the jumps of  $u$  are smaller than 1, i.e.,  $J_u^1 = \emptyset$ . Then for every  $\lambda > 0$ , the truncated functions  $u^\lambda$  belong to  $BV(\Omega)$  and  $|Du^\lambda|(\Omega) \leq \mathcal{G}(u; \Omega)$ , which implies that  $u \in BV(\Omega)$  and  $|Du|(\Omega) \leq \mathcal{G}(u; \Omega)$ .

*Step 2:* Let us assume that  $\Omega$  is a bounded interval. Since  $\mathcal{G}(u; \Omega) < +\infty$ , the set  $J_u^1$  is finite. Therefore  $\Omega \setminus J_u^1$  is the union of a finite number of open intervals  $\Omega_i$ . By *Step 1*, we have  $u \in BV(\Omega_i)$  with  $|Du|(\Omega_i) \leq \mathcal{G}(u; \Omega_i)$ , because all jump points of  $u$  in  $\Omega_i$  are smaller than 1. We conclude that  $u \in BV(\Omega)$  and  $|Du|(\Omega) \leq \mathcal{G}(u; \Omega) + \sum_{x \in J_u^1} |[u](x)|$ .

*Step 3:* Let us assume that  $\Omega$  is a bounded open set in  $\mathbb{R}$ . Then  $\Omega$  is the union of a family of pairwise disjoint open intervals  $\Omega_i$ . Since  $\mathcal{G}(u; \Omega) < +\infty$ , the set  $J_u^1$  is finite, hence there exists a finite set of indices  $I$  such that  $J_u^1 \subset \bigcup_{i \in I} \Omega_i$ . Arguing as in *Step 1* for  $i \notin I$  and as in *Step 2* for  $i \in I$ , we get that  $u \in BV(\Omega_i)$  for every  $i$  and

$$|Du|(\Omega) \leq \sum_{i \notin I} \mathcal{G}(u; \Omega_i) + \sum_{i \in I} \mathcal{G}(u; \Omega_i) + \sum_{x \in J_u^1} |[u](x)| = \mathcal{G}(u; \Omega) + \sum_{x \in J_u^1} |[u](x)| < +\infty,$$

hence  $u \in BV(\Omega)$ .  $\square$



We now construct a recovery sequence. More precisely, we prove the following result.

**Proposition 3.7.** *For every  $u \in BV(\Omega)$  with  $\mathcal{F}(u) < +\infty$ , there exist  $u_k \in BV(\Omega)$ ,  $e_k \in L^2(\Omega)$ ,  $p_k \in \mathcal{M}_b(\Omega)$ , and  $\alpha_k \in H^1(\Omega)$  such that*

$$(3.21) \quad \begin{aligned} (u_k, \alpha_k) &\rightarrow (u, 1) \quad \text{in } L^1(\Omega) \times L^1(\Omega), \\ Du_k &= e_k + p_k \quad \text{in } \Omega, \\ \limsup_{k \rightarrow +\infty} \mathcal{E}_k(e_k, p_k, \alpha_k) &\leq \mathcal{F}(u). \end{aligned}$$

*Proof.* Let us fix  $u \in BV(\Omega)$  with  $\mathcal{F}(u) < +\infty$ . By Proposition 2.2 there exist  $e \in L^2(\Omega)$  and  $p \in \mathcal{M}_b(\Omega)$  such that  $Du = e + p$  in  $\Omega \setminus J_u$  and

$$(3.22) \quad \mathcal{F}(u) = \frac{1}{2} \int_{\Omega} |e|^2 dx + \kappa(1)|p|(\Omega \setminus J_u) + \sum_{x \in J_u} \Psi(|[u](x)|).$$

For every  $\lambda > 0$ , the set

$$J_u^\lambda := \{x \in J_u : |[u](x)| \geq \lambda\},$$

is finite. Let  $\eta > 0$  and let us choose  $\lambda > 0$  such that

$$(3.23) \quad \kappa(1)|p|(\Omega \setminus J_u^\lambda) \leq \kappa(1)|p|(\Omega \setminus J_u) + \eta.$$

For simplicity, let us assume for the moment that  $J_u^\lambda = \{x_0\}$ . From the definition of  $\Psi$  in (1.7), we have that there exists a value  $\alpha_0 \in [0, 1]$  such that

$$\Psi(|[u](x_0)|) = \begin{cases} \kappa(\alpha_0)|[u](x_0)| + \gamma_W(\alpha_0) & \text{if } \alpha_0 > 0, \\ \gamma_W(0) & \text{if } \alpha_0 = 0. \end{cases}$$

If  $\alpha_0 = 1$ , then we have trivially

$$\limsup_{k \rightarrow +\infty} \mathcal{E}_k(e, p, 1) \leq \mathcal{F}(u),$$

since  $\Psi(|[u](x_0)|) = \kappa(1)|[u](x_0)|$ .

Let us discuss the case  $\alpha_0 < 1$ . We define now a suitable infinitesimal sequence  $\tau_k$ , as in the proof of [23, Theorem 3.3]. Let  $h_1(\tau) := W(1 - \tau)$ ,  $h_2(\tau) := (\int_{\alpha_0}^{1-\tau} W(s)^{-\frac{1}{2}} ds)^{-1}$ . The function  $(h_1 h_2)^{\frac{1}{2}}$  is strictly increasing and infinitesimal in 0, and  $h_1/h_2$  is infinitesimal in 0. Indeed, since  $W$  is decreasing,

$$\frac{h_1(\tau)}{h_2(\tau)} = W(1 - \tau) \int_{\alpha_0}^{1-\tau} W(s)^{-\frac{1}{2}} ds \leq (1 - \tau)W(1 - \tau)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } \tau \rightarrow 0.$$

Let  $\tau_k$  be such that  $(h_1(\tau_k)h_2(\tau_k))^{\frac{1}{2}} = \varepsilon_k$ . In this way

$$(3.24) \quad \frac{W(1 - \tau_k)}{\varepsilon_k} = \frac{h_1(\tau_k)}{\varepsilon_k} \rightarrow 0 \quad \text{and} \quad \zeta_k := \varepsilon_k \int_{\alpha_0 + \delta_k}^{1 - \tau_k} W(s)^{-\frac{1}{2}} ds = \frac{\varepsilon_k}{h_2(\tau_k)} + o(\varepsilon_k) \rightarrow 0$$

as  $k \rightarrow +\infty$ .

Let us consider the solution  $\psi_k$  of the differential equation

$$(3.25) \quad \begin{cases} \psi_k' = \frac{1}{\varepsilon_k} \sqrt{W(\psi_k)}, \\ \psi_k(0) = \alpha_0 + \delta_k. \end{cases}$$

The solution of (3.25) is given by the inverse of the function

$$z \in [\alpha_0 + \delta_k, 1 - \tau_k] \mapsto \varepsilon_k \int_{\alpha_0 + \delta_k}^z W(s)^{-\frac{1}{2}} ds \in [0, \zeta_k].$$

Moreover, let  $\sigma_k$  be an infinitesimal sequence such that

$$(3.26) \quad \frac{\sigma_k}{\varepsilon_k} \rightarrow 0 \quad \text{and} \quad \frac{\delta_k}{\sigma_k} \rightarrow 0.$$

Let  $A_k := [x_0 - \sigma_k, x_0 + \sigma_k]$  and  $B_k := [x_0 - \sigma_k - \zeta_k, x_0 - \sigma_k] \cup [x_0 + \sigma_k, x_0 + \sigma_k + \zeta_k]$ . It is not restrictive to assume that  $\partial A_k \cap J_u = \emptyset$  for every  $k$ , so that the precise values  $\tilde{u}(x_0 - \sigma_k)$  and  $\tilde{u}(x_0 + \sigma_k)$  are well defined. Let  $u_k \in BV(\Omega)$  be the affine interpolation between  $\tilde{u}(x_0 - \sigma_k)$  and  $\tilde{u}(x_0 + \sigma_k)$  on  $A_k$ , while  $u_k := u$  out of  $A_k$ . Finally, let  $\alpha_k \in H^1(\Omega)$  be defined as

$$\alpha_k(x) := \begin{cases} 1 - \tau_k & \text{if } x \in \Omega \setminus (A_k \cup B_k), \\ \alpha_0 + \delta_k & \text{if } x \in A_k, \\ \psi_k(|x - x_0| - \sigma_k) & \text{if } x \in B_k. \end{cases}$$

Let us notice that  $\delta_k \leq \alpha_k \leq 1$ .

Let us discuss the case  $\alpha_0 > 0$  first. In this case, let  $e_k \in L^2(\Omega)$ ,  $p_k \in \mathcal{M}_b(\Omega)$  be defined by

$$e_k := \begin{cases} e & \text{in } \Omega \setminus A_k, \\ 0 & \text{in } A_k, \end{cases} \quad p_k := \begin{cases} p & \text{in } \Omega \setminus A_k, \\ u'_k \mathcal{L}^1 & \text{in } A_k. \end{cases}$$

Let us estimate  $\mathcal{F}_k(u_k, \alpha_k)$ :

$$(3.27) \quad \int_{\Omega} \alpha_k |e_k|^2 dx = \int_{\Omega \setminus A_k} \alpha_k |e|^2 dx \leq \int_{\Omega \setminus A_k} |e|^2 dx.$$

$$(3.28) \quad \int_{\Omega \setminus A_k} \kappa(\alpha_k) d|p_k| \leq \kappa(1) |p|(\Omega \setminus A_k).$$

Since  $u_k$  is linear in  $A_k$ , we have

$$(3.29) \quad \begin{aligned} \int_{A_k} \kappa(\alpha_k) d|p_k| &= \kappa(\alpha_0 + \delta_k) \int_{A_k} |u'_k(x)| dx \\ &= \kappa(\alpha_0 + \delta_k) \int_{A_k} \left| \frac{\tilde{u}(x_0 + \sigma_k) - \tilde{u}(x_0 - \sigma_k)}{2\sigma_k} \right| dx \\ &= \kappa(\alpha_0 + \delta_k) |\tilde{u}(x_0 + \sigma_k) - \tilde{u}(x_0 - \sigma_k)|. \end{aligned}$$

Moreover

$$(3.30) \quad \int_{\Omega \setminus (A_k \cup B_k)} \frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\alpha'_k|^2 dx = \frac{W(1 - \tau_k)}{\varepsilon_k} \mathcal{L}^1(\Omega),$$

is infinitesimal as  $k \rightarrow +\infty$  by (3.24), and

$$(3.31) \quad \int_{A_k} \frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\alpha'_k|^2 dx = \int_{A_k} \frac{W(\alpha_0 + \delta_k)}{\varepsilon_k} dx = W(\alpha_0 + \delta_k) \frac{2\sigma_k}{\varepsilon_k}$$

goes to 0 as  $k \rightarrow +\infty$  by (3.26).

Finally, from the definition of  $\alpha_k$  in  $B_k$  it turns out that the equality in Young's inequality holds, and hence

$$(3.32) \quad \int_{B_k} \frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\alpha'_k|^2 dx = 2 \int_{B_k} \sqrt{W(\alpha_k)} |\alpha'_k| dx = 4 \int_{\alpha_0 + \delta_k}^{1 - \tau_k} \sqrt{W(s)} ds.$$

By (3.26), summing (3.27)–(3.32), and passing to the limsup we obtain

$$\begin{aligned} \limsup_{k \rightarrow +\infty} \mathcal{E}_k(e_k, p_k, \alpha_k) &\leq \frac{1}{2} \int_{\Omega} |e|^2 dx + \kappa(1)|p|(\Omega \setminus \{x_0\}) + \kappa(\alpha_0)|[u](x_0)| + \gamma_W(\alpha_0) \\ &= \frac{1}{2} \int_{\Omega} |e|^2 dx + \kappa(1)|p|(\Omega \setminus \{x_0\}) + \Psi(|[u](x_0)|). \end{aligned}$$

Let us discuss the case  $\alpha_0 = 0$ . Let us define this time

$$e_k(x) := \begin{cases} e(x) & \text{if } x \in \Omega \setminus A_k, \\ u'_k(x) & \text{if } x \in A_k, \end{cases} \quad p_k := \begin{cases} p & \text{in } \Omega \setminus A_k, \\ 0 & \text{in } A_k. \end{cases}$$

The term

$$(3.33) \quad \int_{\Omega \setminus A_k} \alpha_k |e_k|^2 dx \leq \int_{\Omega \setminus A_k} |e|^2 dx$$

can be treated as in (3.27). Moreover

$$(3.34) \quad \begin{aligned} \int_{A_k} \alpha_k |e_k|^2 dx &= \int_{A_k} \alpha_k |u'_k|^2 dx \leq \int_{A_k} \delta_k \left| \frac{\tilde{u}(x_0 + \sigma_k) - \tilde{u}(x_0 - \sigma_k)}{2\sigma_k} \right|^2 dx \\ &\leq \frac{\delta_k}{2\sigma_k} |\tilde{u}(x_0 + \sigma_k) - \tilde{u}(x_0 - \sigma_k)|^2. \end{aligned}$$

By (3.26), by summing (3.28), (3.30)–(3.34), and passing to the limsup, we obtain

$$\begin{aligned} \limsup_{k \rightarrow +\infty} \mathcal{E}_k(e_k, p_k, \alpha_k) &\leq \frac{1}{2} \int_{\Omega} |e|^2 dx + \kappa(1)|p|(\Omega \setminus \{x_0\}) + \gamma_W(0) \\ &= \frac{1}{2} \int_{\Omega} |e|^2 dx + \kappa(1)|p|(\Omega \setminus \{x_0\}) + \Psi(|[u](x_0)|). \end{aligned}$$

Arguing in this way for all the elements of  $J_u^\lambda$ , by the choice of  $\lambda$  made in (3.23), and by (3.22) we get

$$\begin{aligned} \limsup_{k \rightarrow +\infty} \mathcal{E}_k(e_k, p_k, \alpha_k) &\leq \frac{1}{2} \int_{\Omega} |e|^2 dx + \kappa(1)|p|(\Omega \setminus J_u^\lambda) + \sum_{x \in J_u^\lambda} \Psi(|[u](x)|) \\ &\leq \mathcal{F}(u) + \eta, \end{aligned}$$

which yields (3.21) by letting  $\eta \rightarrow 0$ .  $\square$

#### 4. PROOF OF THE MAIN RESULT IN THE GENERAL CASE

To study the  $n$ -dimensional case, we shall use the localized version of the functionals introduced in (1.2) and (1.4): they are defined for every open set  $A \subset \Omega$ , for every  $u \in BV(A)$ , and for every  $\alpha \in H^1(A)$  by

$$\begin{aligned} \mathcal{W}_k(\alpha; A) &:= \int_A \left[ \frac{W(\alpha)}{\varepsilon_k} + \varepsilon_k |\nabla \alpha|^2 \right] dx, \\ \mathcal{F}_k(u, \alpha; A) &:= \int_A f_k(\alpha, |\nabla u|) dx + \int_A \kappa(\tilde{\alpha}) d|D^s u| + \mathcal{W}_k(\alpha; A), \end{aligned}$$

and extended to  $+\infty$  otherwise in  $L^1(\Omega)$  and  $L^1(\Omega) \times L^1(\Omega)$  respectively. For the localized version of the  $\Gamma$ -limits, we adopt the notation

$$\mathcal{F}'(\cdot, \cdot; A) := \Gamma\text{-}\liminf_{k \rightarrow +\infty} \mathcal{F}_k(\cdot, \cdot; A) \quad \text{and} \quad \mathcal{F}''(\cdot, \cdot; A) := \Gamma\text{-}\limsup_{k \rightarrow +\infty} \mathcal{F}_k(\cdot, \cdot; A).$$

We omit the indication of the set when  $A = \Omega$ .

We shall use a slicing argument to reduce the proof of the  $\Gamma$ -liminf inequality to the one-dimensional case. For every  $\xi \in \mathbb{S}^{n-1}$  (playing the role of the slicing direction) and for every set  $B \subset \mathbb{R}^n$ , we define

$$\Pi^\xi := \{z \in \mathbb{R}^n : z \cdot \xi = 0\} \quad \text{and} \quad B_y^\xi := \{t \in \mathbb{R} : y + t\xi \in B\},$$

for every  $y \in \Pi^\xi$ . If  $w: \Omega \rightarrow \mathbb{R}$  is a scalar function and  $v: \Omega \rightarrow \mathbb{R}^n$  is a vector function, we define their slices  $w_y^\xi: \Omega_y^\xi \rightarrow \mathbb{R}$  and  $\hat{v}_y^\xi: \Omega_y^\xi \rightarrow \mathbb{R}$  by

$$w_y^\xi(t) := w(y + t\xi) \quad \text{and} \quad \hat{v}_y^\xi := (v \cdot \xi)_y^\xi,$$

respectively. We recall that a function  $u \in L^1(\Omega)$  belongs to  $BV(\Omega)$  if and only if, for every direction  $\xi \in \mathbb{S}^{n-1}$ , we have

$$(4.1) \quad u_y^\xi \in BV(\Omega_y^\xi) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi \quad \text{and} \quad \int_{\Omega_y^\xi} |Du_y^\xi|(\Omega_y^\xi) d\mathcal{H}^{n-1}(y) < +\infty.$$

For all details about this characterization of  $BV$  functions, we refer to [4, Section 3.11].

**Proposition 4.1.** *For every  $u \in L^1(\Omega)$  we have  $\mathcal{F}_0(u, 1) \leq \mathcal{F}'(u, 1)$ .*

*Proof.* Let us first prove the proposition under the additional assumption that  $\|u\|_{L^\infty(\Omega)} \leq \lambda$  for some constant  $\lambda > 0$ . Let us consider a sequence  $(u_k, \alpha_k) \in L^1(\Omega) \times L^1(\Omega)$  and  $u \in L^1(\Omega)$  such that  $(u_k, \alpha_k) \rightarrow (u, 1)$  in  $L^1(\Omega) \times L^1(\Omega)$  and

$$(4.2) \quad \mathcal{F}'(u, 1) = \liminf_{k \rightarrow +\infty} \mathcal{F}_k(u_k, \alpha_k).$$

We can always assume that the liminf in (4.2) is a limit and that  $\mathcal{F}_k(u_k, \alpha_k)$  is bounded, and hence  $u_k \in BV(\Omega)$ ,  $\alpha_k \in H^1(\Omega)$ , and  $\delta_k \leq \alpha_k \leq 1$ . Let  $e_k \in L^2(\Omega; \mathbb{R}^n)$  and  $p_k \in \mathcal{M}_b(\Omega; \mathbb{R}^n)$  be such that  $Du_k = e_k + p_k$  and

$$(4.3) \quad \mathcal{E}_k(e_k, p_k, \alpha_k) = \mathcal{F}_k(u_k, \alpha_k) \leq c.$$

Let us fix  $\xi \in \mathbb{S}^{n-1}$ . Then there exists a subsequence (not relabeled), possibly depending on  $\xi$ , such that

$$((u_k)_y^\xi, (\alpha_k)_y^\xi) \rightarrow (u_y^\xi, 1) \quad \text{in } L^1(\Omega_y^\xi) \times L^1(\Omega_y^\xi) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y \in \Pi^\xi.$$

Since  $u_k \in BV(\Omega)$ , we know that  $(u_k)_y^\xi \in BV(\Omega_y^\xi)$  for  $\mathcal{H}^{n-1}$ -a.e.  $y \in \Pi^\xi$  and that the measures  $Du_k \cdot \xi$  and  $|Du_k \cdot \xi|$  are decomposed as

$$\begin{aligned} Du_k \cdot \xi(B) &= \int_{\Pi^\xi} (D(u_k)_y^\xi)(B_y^\xi) d\mathcal{H}^{n-1}(y), \\ |Du_k \cdot \xi|(B) &= \int_{\Pi^\xi} |D(u_k)_y^\xi|(B_y^\xi) d\mathcal{H}^{n-1}(y), \end{aligned}$$

for every Borel set  $B \subset \Omega$ . Since  $Du_k = e_k + p_k$ , it is immediate to deduce that

$$\begin{aligned} p_k \cdot \xi(B) &= \int_{\Pi^\xi} (\hat{p}_k)_y^\xi(B_y^\xi) d\mathcal{H}^{n-1}(y), \\ |p_k \cdot \xi|(B) &= \int_{\Pi^\xi} |(\hat{p}_k)_y^\xi|(B_y^\xi) d\mathcal{H}^{n-1}(y), \end{aligned}$$

where the measures  $(\hat{p}_k)_y^\xi \in \mathcal{M}_b(\Omega_y^\xi)$  are defined by  $(\hat{p}_k)_y^\xi := D(u_k)_y^\xi - (\hat{e}_k)_y^\xi$ .

To apply the results of the one dimensional case, we first have to check that  $(\tilde{\alpha}_k)_y^\xi$  coincides with the continuous representative of  $(\alpha_k)_y^\xi \in H^1(\Omega_y^\xi)$  for  $\mathcal{H}^{n-1}$ -a.e.  $y \in \Pi^\xi$ . Indeed,  $\tilde{\alpha}_k$  is the precise representative of  $\alpha_k$ , in the sense of (2.1) and this implies, by [4, Theorem 3.108], that for  $\mathcal{H}^{n-1}$ -a.e.  $y \in \Pi^\xi$  the function  $(\tilde{\alpha}_k)_y^\xi$  is a good representative of  $(\alpha_k)_y^\xi$ , meaning

that its pointwise total variation coincides with the total variation of  $(\alpha_k)_y^\xi$ . This implies that  $(\tilde{\alpha}_k)_y^\xi$  must be the continuous representative of  $(\alpha_k)_y^\xi$ .

From the Fubini Theorem it follows that

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \alpha_k |e_k|^2 dx &\geq \frac{1}{2} \int_{\Omega} \alpha_k |e_k \cdot \xi|^2 dx = \frac{1}{2} \int_{\Pi^\xi \Omega_y^\xi} (\alpha_k)_y^\xi |(\hat{e}_k)_y^\xi|^2 dt d\mathcal{H}^{n-1}(y), \\ &\int_{\Omega} \kappa(\tilde{\alpha}_k) d|p_k| + \int_{\Omega} \left[ \frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\nabla \alpha_k|^2 \right] dx \\ &\geq \int_{\Omega} \kappa(\tilde{\alpha}_k) d|p_k \cdot \xi| + \int_{\Omega} \left[ \frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\nabla \alpha_k \cdot \xi|^2 \right] dx \\ &= \int_{\Pi^\xi} \left\{ \int_{\Omega_y^\xi} \kappa((\tilde{\alpha}_k)_y^\xi) d|(\hat{p}_k)_y^\xi| + \int_{\Omega_y^\xi} \left[ \frac{W((\alpha_k)_y^\xi)}{\varepsilon_k} + \varepsilon_k |((\alpha_k)_y^\xi)'|^2 \right] dt \right\} d\mathcal{H}^{n-1}(y). \end{aligned}$$

Summing the previous inequalities and using (4.3) we obtain that

$$(4.4) \quad \int_{\Pi^\xi} \mathcal{E}_k^{\xi,y}((\hat{e}_k)_y^\xi, (\hat{p}_k)_y^\xi, (\alpha_k)_y^\xi) d\mathcal{H}^{n-1}(y) \leq \mathcal{F}_k(u_k, \alpha_k) \leq c,$$

where  $\mathcal{E}_k^{\xi,y}$  is defined by

$$\mathcal{E}_k^{\xi,y}(e, p, \alpha) := \frac{1}{2} \int_{\Omega_y^\xi} \alpha |e|^2 dt + \int_{\Omega_y^\xi} \kappa(\alpha) d|p| + \int_{\Omega_y^\xi} \left[ \frac{W(\alpha)}{\varepsilon_k} + \varepsilon_k |\alpha'|^2 \right] dt$$

for every  $e \in L^2(\Omega_y^\xi)$ ,  $p \in \mathcal{M}_b(\Omega_y^\xi)$ , and  $\alpha \in H^1(\Omega_y^\xi)$  with  $\delta_k \leq \alpha \leq 1$ . By the Fatou Lemma we have that

$$(4.5) \quad \liminf_{k \rightarrow +\infty} \mathcal{E}_k^{\xi,y}((\hat{e}_k)_y^\xi, (\hat{p}_k)_y^\xi, (\alpha_k)_y^\xi) < +\infty$$

for  $\mathcal{H}^{n-1}$ -a.e.  $y \in \Pi^\xi$ ,

Let us fix  $y \in \Pi^\xi$  such that (4.5) holds. Up to a subsequence, possibly depending on  $y$ , we can suppose that the liminf in (4.5) is actually a limit. By Lemma 3.3 and Proposition 3.4, we have that  $u_y^\xi \in BV(\Omega_y^\xi)$  and there exist  $e_{\xi,y} \in L^2(\Omega_y^\xi)$ ,  $p_{\xi,y} \in \mathcal{M}_b(\Omega_y^\xi)$  such that  $Du_y^\xi = e_{\xi,y} + p_{\xi,y}$  and

$$(4.6) \quad \frac{1}{2} \int_{\Omega_y^\xi} |e_{\xi,y}(t)|^2 dt \leq \liminf_{k \rightarrow +\infty} \frac{1}{2} \int_{\Omega_y^\xi} (\alpha_k)_y^\xi(t) |(\hat{e}_k)_y^\xi(t)|^2 dt,$$

$$(4.7) \quad \begin{aligned} &\kappa(1) |p_{\xi,y}|(\Omega_y^\xi \setminus J_{u_y^\xi}) + \sum_{t \in J_{u_y^\xi}} \Psi(|u_y^\xi(t)|) \\ &\leq \liminf_{k \rightarrow +\infty} \left\{ \int_{\Omega_y^\xi} \kappa((\alpha_k)_y^\xi) d|(\hat{p}_k)_y^\xi| + \int_{\Omega_y^\xi} \left[ \frac{W((\alpha_k)_y^\xi)}{\varepsilon_k} + \varepsilon_k |((\alpha_k)_y^\xi)'|^2 \right] dt \right\}. \end{aligned}$$

We now prove that  $u \in BV(\Omega)$  by showing that (4.1) holds. From the additive decomposition of  $Du_y^\xi$  we get

$$(4.8) \quad \begin{aligned} |Du_y^\xi|(\Omega_y^\xi) &\leq \int_{\Omega_y^\xi} |e_{\xi,y}| dt + |p_{\xi,y}|(\Omega_y^\xi) \\ &\leq \frac{1}{2} \mathcal{L}^1(\Omega_y^\xi) + \frac{1}{2} \int_{\Omega_y^\xi} |e_{\xi,y}|^2 dt + |p_{\xi,y}|(\Omega_y^\xi \setminus J_{u_y^\xi}) + |p_{\xi,y}|(J_{u_y^\xi}) \end{aligned}$$

Let us estimate the last term in the sum. By (2.9), using the a priori bound  $\|u\|_{L^\infty(\Omega)} \leq \lambda$ , we obtain

$$(4.9) \quad \begin{aligned} |p_{\xi,y}|(J_{u_y^\xi}) &= \sum_{t \in \{|[u_y^\xi]| < 1\}} |[u_y^\xi](t)| + \sum_{t \in \{|[u_y^\xi]| \geq 1\}} |[u_y^\xi](t)| \\ &\leq \sum_{t \in \{|[u_y^\xi]| < 1\}} \frac{1}{\Psi(1)} \Psi(|[u_y^\xi](t)|) + \sum_{t \in \{|[u_y^\xi]| \geq 1\}} \frac{2\lambda}{\Psi(1)} \Psi(|[u_y^\xi](t)|) \leq c \sum_{t \in J_{u_y^\xi}} \Psi(|[u_y^\xi](t)|). \end{aligned}$$

By (4.6)–(4.9), by Fatou Lemma, and by (4.4) it follows that

$$\begin{aligned} \int_{\Pi^\xi} |Du_y^\xi|(\Omega_y^\xi) d\mathcal{H}^{n-1}(y) &\leq C \left[ 1 + \int_{\Pi^\xi} \liminf_{k \rightarrow +\infty} \mathcal{E}_k^{\xi,y}((\hat{e}_k)_y^\xi, (\hat{p}_k)_y^\xi, (\alpha_k)_y^\xi) d\mathcal{H}^{n-1}(y) \right] \\ &\leq C \left[ 1 + \liminf_{k \rightarrow +\infty} \int_{\Pi^\xi} \mathcal{E}_k^{\xi,y}((\hat{e}_k)_y^\xi, (\hat{p}_k)_y^\xi, (\alpha_k)_y^\xi) d\mathcal{H}^{n-1}(y) \right] \\ &\leq C \left[ 1 + \liminf_{k \rightarrow +\infty} \mathcal{F}_k(u_k, \alpha_k) \right] < +\infty. \end{aligned}$$

This proves that  $u \in BV(\Omega)$ .

We can now go back to the proof of the estimate from below  $\mathcal{F}(u) \leq \mathcal{F}'(u, 1)$ . Summing (4.6) and (4.7), we obtain that

$$\begin{aligned} \int_{\Omega_y^\xi} f(1, |(u_y^\xi)'|) dt + \kappa(1) |D^c u_y^\xi|(\Omega_y^\xi) + \sum_{t \in J_{u_y^\xi}} \Psi(|[u_y^\xi](t)|) \\ \leq \liminf_{k \rightarrow +\infty} \mathcal{E}_k^{\xi,y}((\hat{e}_k)_y^\xi, (\hat{p}_k)_y^\xi, (\alpha_k)_y^\xi). \end{aligned}$$

Integrating the inequality above with respect to  $y \in \Pi^\xi$  and using the Fatou Lemma, from (4.4) and (4.2) we obtain

$$(4.10) \quad \int_{\Omega} f(1, |\nabla u \cdot \xi|) dt + \kappa(1) |D^c u \cdot \xi|(\Omega) + \int_{J_u} \Psi(|[u]|) |\nu_u \cdot \xi| d\mathcal{H}^{n-1} \leq \mathcal{F}'(u, 1).$$

To get rid of  $\xi$ , we use a localization argument. Let  $(\xi_i)_i$  be a dense sequence in  $\mathbb{S}^{n-1}$  and let

$$\mu := \mathcal{L}^n + |D^c u| + \mathcal{H}^{n-1} \llcorner J_u.$$

Let  $\Sigma$  be a Borel set containing  $J_u$  such that  $\mathcal{L}^n(\Sigma) = 0$  and  $|D^s u|(\Omega \setminus \Sigma) = 0$ . For every  $\xi$ , we define the function

$$\varphi_\xi := f(1, |\nabla u \cdot \xi|) 1_{\Omega \setminus \Sigma} + \kappa(1) |\beta_u \cdot \xi| 1_{\Sigma \setminus J_u} + \Psi(|[u]|) |\nu_u \cdot \xi| 1_{J_u},$$

where  $\beta_u = \frac{dD^c u}{d|D^c u|}$  is the density of the measure  $D^c u$  with respect to its total variation. It is immediate to obtain estimate (4.10) on every open set contained in  $\Omega$ . This implies that

$$\int_{A_i} \varphi_{\xi_i} d\mu \leq \mathcal{F}'(u, 1; A_i)$$

for every  $i$  and for every open set  $A_i \subset \Omega$ . Since  $\mathcal{F}'(u, 1; \cdot)$  is superadditive, we obtain

$$\sum_i \int_{A_i} \varphi_{\xi_i} d\mu \leq \sum_i \mathcal{F}'(u, 1; A_i) \leq \mathcal{F}'(u, 1)$$

for every sequence  $A_i$  of pairwise disjoint open sets contained in  $\Omega$ . By [9, Lemma 15.2], the supremum of the left hand side is given by

$$\int_{\Omega} \sup_i \varphi_{\xi_i} d\mu.$$

Since

$$\sup_i f(1, |\nabla u \cdot \xi_i|) = f(1, |\nabla u|), \quad \sup_i |\beta_u \cdot \xi_i| = 1, \quad \sup_i |\nu_u \cdot \xi_i| = 1,$$

this concludes the proof in the case  $\|u\|_{L^\infty(\Omega)} \leq \lambda$ .

The general case is treated with a truncation argument. Let  $\lambda > 0$  be any positive constant. Let us consider the functions

$$u_{k,\lambda} := \min\{\max\{-\lambda, u_k\}, \lambda\} \quad \text{and} \quad u_\lambda := \min\{\max\{-\lambda, u\}, \lambda\}.$$

Notice that  $u_{k,\lambda} \rightarrow u_\lambda$  in  $L^1(\Omega)$ ,  $\alpha_k \rightarrow 1$  in  $L^1(\Omega)$ , and

$$\mathcal{F}_k(u_{k,\lambda}, \alpha_k) \leq \mathcal{F}_k(u_k, \alpha_k) \leq c,$$

since the functionals are decreasing by truncation. From the bounded case, it follows that  $u_\lambda \in BV(\Omega)$  and

$$\mathcal{F}(u_\lambda) \leq \liminf_{k \rightarrow +\infty} \mathcal{F}_k(u_{k,\lambda}, \alpha_k) \leq \liminf_{k \rightarrow +\infty} \mathcal{F}_k(u_k, \alpha_k) = \mathcal{F}'(u, 1).$$

By letting  $\lambda \rightarrow +\infty$  we conclude that  $u \in GBV(\Omega)$  and  $\mathcal{F}(u) \leq \mathcal{F}'(u, 1)$ .  $\square$

To prove the  $\Gamma$ -limesup inequality, we shall apply an integral representation result to the limit functional. In order to do this, we use the notion of  $\bar{\Gamma}$ -convergence, for which we refer to [13]. Given a metric space  $X$  and a sequence of functionals  $F_k: X \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$  increasing on  $\mathcal{A}(\Omega)$ , we recall that the sequence  $F_k$   $\bar{\Gamma}$ -converges to a functional  $F: X \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$  if  $F$  coincides with the inner regular envelope of both functionals  $F'$  and  $F''$ . The  $\bar{\Gamma}$ -limit of a sequence of increasing functionals is increasing, inner regular and lower semicontinuous. Moreover, if the functionals  $F_k$  are superadditive, then also  $F$  is superadditive (see [13, Proposition 16.12]).

We start with a rough estimate of the  $\Gamma$ -limesup.

**Proposition 4.2.** *There exists a constant  $C > 0$  such that for all  $u \in BV(\Omega)$  and for every open set  $A$  we have*

$$\mathcal{F}''(u, 1; A) \leq C|Du|(A).$$

*Proof.* Let us choose  $u_k = u$  and  $\alpha_k = 1$  for every  $k$ . In this way

$$(4.11) \quad \begin{aligned} \mathcal{F}''(u, 1; A) &\leq \limsup_{k \rightarrow +\infty} \mathcal{F}_k(u, 1; A) \\ &\leq \int_A f(1, |\nabla u|) dx + \kappa(1)|D^s u|(A) \leq C|Du|(A), \end{aligned}$$

where we used (2.3) in the last inequality.  $\square$

We now use the De Giorgi slicing and averaging argument to prove the weak subadditivity of the  $\Gamma$ -limesup.

**Proposition 4.3.** *Let  $u \in L^1(\Omega)$ , let  $A', A, B$  be open subset of  $\Omega$  with  $A' \Subset A$ . Then*

$$\mathcal{F}''(u, 1; A' \cup B) \leq \mathcal{F}''(u, 1; A) + \mathcal{F}''(u, 1; B).$$

*Proof.* Let  $(u_k^A, \alpha_k^A), (u_k^B, \alpha_k^B) \in L^1(\Omega) \times L^1(\Omega)$  be such that

$$(u_k^A, \alpha_k^A), (u_k^B, \alpha_k^B) \rightarrow (u, 1) \quad \text{in } L^1(\Omega) \times L^1(\Omega)$$

and

$$\limsup_{k \rightarrow +\infty} \mathcal{F}_k(u_k^A, \alpha_k^A; A) = \mathcal{F}''(u, 1; A), \quad \limsup_{k \rightarrow +\infty} \mathcal{F}_k(u_k^B, \alpha_k^B; B) = \mathcal{F}''(u, 1; B).$$

We can assume that both  $\mathcal{F}''(u, 1; A)$  and  $\mathcal{F}''(u, 1; B)$  are finite, otherwise the statement is trivial. In particular  $u_k^A \in BV(A), u_k^B \in BV(B)$ ,  $\alpha_k^A \in H^1(A), \alpha_k^B \in H^1(B)$ , and  $\delta_k \leq \alpha_k^A, \alpha_k^B \leq 1$ . Let  $d := \text{dist}(A', \partial A) > 0$  and let  $h \in \mathbb{N}$ . Let  $A_0 := A'$  and  $A_{h+1} := A$ . We consider a chain of open sets  $A_1, \dots, A_h$  such that  $A_i \Subset A_{i+1}$  and  $\text{dist}(A_i, \partial A_{i+1}) \leq d/(h+1)$

for every  $0 \leq i \leq h-1$ . Let  $\varphi_i \in C_c^1(\Omega)$  be a cut-off function between  $A_i$  and  $A_{i+1}$ , i.e.,  $0 \leq \varphi_i \leq 1$ ,  $\text{supp}(\varphi_i) \subset A_{i+1}$ , and  $\varphi_i = 1$  in a neighborhood of  $\overline{A_i}$ . We assume in addition that  $\|\nabla \varphi_i\|_{L^\infty(\Omega)} \leq 2(h+1)/d$ . We set

$$u_k^i := \varphi_i u_k^A + (1 - \varphi_i) u_k^B \in BV(A' \cup B),$$

and we define the functions  $\alpha_k^i \in H^1(A' \cup B)$  as in [10, Lemma 6.2]:

$$\alpha_k^i := \begin{cases} \varphi_{i-1} \alpha_k^A + (1 - \varphi_{i-1})(\alpha_k^A \wedge \alpha_k^B) & \text{in } A_i, \\ \alpha_k^A \wedge \alpha_k^B & \text{in } A_{i+1} \setminus A_i, \\ \varphi_{i+1}(\alpha_k^A \wedge \alpha_k^B) + (1 - \varphi_{i+1})\alpha_k^B & \text{in } \Omega \setminus A_{i+1}, \end{cases}$$

where  $a \wedge b = \min\{a, b\}$ . Let us notice that  $\delta_k \leq \alpha_k^i \leq 1$ . Let  $1 \leq i \leq h-1$ . We estimate  $\mathcal{F}_k$  on  $A' \cup B$  in the following way

$$\begin{aligned} \mathcal{F}_k(u_k^i, \alpha_k^i; A' \cup B) &\leq \mathcal{F}_k(u_k^i, \alpha_k^i; (A' \cup B) \cap A_{i-1}) + \mathcal{F}_k(u_k^i, \alpha_k^i; B \setminus A_{i+2}) \\ &\quad + \mathcal{F}_k(u_k^i, \alpha_k^i; B \cap (A_{i+2} \setminus A_{i-1})) \\ (4.12) \quad &\leq \mathcal{F}_k(u_k^A, \alpha_k^A; A_{i-1}) + \mathcal{F}_k(u_k^B, \alpha_k^B; B \setminus A_{i+2}) \\ &\quad + \mathcal{F}_k(u_k^i, \alpha_k^i; B \cap (A_{i+2} \setminus A_{i-1})). \end{aligned}$$

We need only to bound the last term:

$$\mathcal{F}_k(u_k^i, \alpha_k^i; B \cap (A_{i+2} \setminus A_{i-1})) \leq \mathcal{F}_k(u_k^i, \alpha_k^i; S_{i+1}) + \mathcal{F}_k(u_k^i, \alpha_k^i; S_i) + \mathcal{F}_k(u_k^i, \alpha_k^i; S_{i-1}),$$

where  $S_i = B \cap (A_{i+1} \setminus A_i)$  for  $0 \leq i \leq h-1$ . Since  $\alpha_k^i \geq \alpha_k^A \wedge \alpha_k^B$ , we have

$$\int_{S_{i+1}} \frac{W(\alpha_k^i)}{\varepsilon_k} dx \leq \int_{S_{i+1}} \frac{W(\alpha_k^A \wedge \alpha_k^B)}{\varepsilon_k} dx \leq \mathcal{F}_k(u_k^A, \alpha_k^A; S_{i+1}) + \mathcal{F}_k(u_k^B, \alpha_k^B; S_{i+1}).$$

Moreover

$$\begin{aligned} \int_{S_{i+1}} |\nabla \alpha_k^i|^2 dx &= \int_{S_{i+1}} |\nabla \varphi_{i+1}((\alpha_k^A \wedge \alpha_k^B) - \alpha_k^B) + \varphi_{i+1} \nabla(\alpha_k^A \wedge \alpha_k^B) + (1 - \varphi_{i+1}) \nabla \alpha_k^B|^2 dx \\ &\leq \int_{S_{i+1}} 2\|\nabla \varphi_{i+1}\|_{L^\infty(\Omega)}^2 |(\alpha_k^A \wedge \alpha_k^B) - \alpha_k^B|^2 + 2|\nabla(\alpha_k^A \wedge \alpha_k^B)|^2 + 2|\nabla \alpha_k^B|^2 dx \\ &\leq \frac{c(h+1)^2}{d^2} \int_{S_{i+1}} |\alpha_k^A - \alpha_k^B|^2 dx + c \int_{S_{i+1}} |\nabla \alpha_k^A|^2 dx + c \int_{S_{i+1}} |\nabla \alpha_k^B|^2 dx \end{aligned}$$

and hence, using the fact that  $\alpha_k^i \leq \alpha_k^B$  (and  $\tilde{\alpha}_k^i \leq \tilde{\alpha}_k^B$ ) in  $S_{i+1}$  and the monotonicity of  $\kappa$  and of  $f$  with respect to the first variable, we get

$$\begin{aligned} \mathcal{F}_k(u_k^i, \alpha_k^i; S_{i+1}) &= \int_{S_{i+1}} f(\alpha_k^i, |\nabla u_k^i|) dx + \int_{S_{i+1}} \kappa(\tilde{\alpha}_k^i) d|D^s u_k^i| + \mathcal{W}(\alpha_k^i; S_{i+1}) \\ (4.13) \quad &\leq \int_{S_{i+1}} f(\alpha_k^B, |\nabla u_k^B|) dx + \int_{S_{i+1}} \kappa(\tilde{\alpha}_k^B) d|D^s u_k^B| + \mathcal{W}(\alpha_k^i; S_{i+1}) \\ &\leq c[\mathcal{F}_k(u_k^A, \alpha_k^A; S_{i+1}) + \mathcal{F}_k(u_k^B, \alpha_k^B; S_{i+1})] \\ &\quad + \frac{c(h+1)^2}{d^2} \varepsilon_k \int_{S_{i+1}} |\alpha_k^A - \alpha_k^B|^2 dx. \end{aligned}$$



In the same way, we estimate

$$(4.14) \quad \begin{aligned} \mathcal{F}_k(u_k^i, \alpha_k^i; S_{i-1}) &\leq c[\mathcal{F}_k(u_k^A, \alpha_k^A; S_{i-1}) + \mathcal{F}_k(u_k^B, \alpha_k^B; S_{i-1})] \\ &\quad + \frac{c(h+1)^2}{d^2} \varepsilon_k \int_{S_{i-1}} |\alpha_k^A - \alpha_k^B|^2 dx. \end{aligned}$$

It remains to bound  $\mathcal{F}_k(u_k^i, \alpha_k^i; S_i)$ . This time we use the fact that in  $S_i$  we have

$$Du_k^i = \nabla \varphi_i(u_k^A - u_k^B) + \varphi_i Du_k^A + (1 - \varphi_i) Du_k^B,$$

from which it follows that

$$\begin{aligned} \nabla u_k^i &= \nabla \varphi_i(u_k^A - u_k^B) + \varphi_i \nabla u_k^A + (1 - \varphi_i) \nabla u_k^B, \\ D^s u_k^i &= \varphi_i D^s u_k^A + (1 - \varphi_i) D^s u_k^B. \end{aligned}$$

Using the convexity of  $f$  with respect to the second variable and (2.4), this implies that

$$(4.15) \quad \begin{aligned} \mathcal{F}_k(u_k^i, \alpha_k^i; S_i) &= \int_{S_i} f(\alpha_k^i, |\nabla u_k^i|) dx + \int_{S_i} \kappa(\tilde{\alpha}_k^i) d|D^s u_k^i| + \mathcal{W}(\alpha_k^i; S_i) \\ &\leq \int_{S_i} 2f(\alpha_k^A \wedge \alpha_k^B, |\nabla \varphi_i(u_k^A - u_k^B)|) dx + \int_{S_i} 2f(\alpha_k^A, |\nabla u_k^A|) dx \\ &\quad + \int_{S_i} 2f(\alpha_k^B, |\nabla u_k^B|) dx + \int_{S_i} \kappa(\tilde{\alpha}_k^A) d|D^s u_k^A| + \int_{S_i} \kappa(\tilde{\alpha}_k^B) d|D^s u_k^B| \\ &\quad + \mathcal{W}(\alpha_k^A; S_i) + \mathcal{W}(\alpha_k^B; S_i) \\ &\leq c[F_k(u_k^A, \alpha_k^A; S_i) + F_k(u_k^B, \alpha_k^B; S_i)] + \frac{c(h+1)}{d} \int_{S_i} |u_k^A - u_k^B| dx, \end{aligned}$$

where we used (2.3). Summing (4.12)–(4.15), we obtain

$$\begin{aligned} \mathcal{F}_k(u_k^i, \alpha_k^i, A' \cup B) &\leq \mathcal{F}_k(u_k^A, \alpha_k^A; A) + \mathcal{F}_k(u_k^B, \alpha_k^B; B) \\ &\quad + c[\mathcal{F}_k(u_k^A, \alpha_k^A; B \cap (A_{i+2} \setminus A_{i-1})) + \mathcal{F}_k(u_k^B, \alpha_k^B; B \cap (A_{i+2} \setminus A_{i-1}))] \\ &\quad + \frac{c(h+1)^2}{d^2} \varepsilon_k \int_{B \cap (A_{i+2} \setminus A_{i-1})} |\alpha_k^A - \alpha_k^B|^2 dx + \frac{c(h+1)}{d} \int_{B \cap (A_{i+2} \setminus A_{i-1})} |u_k^A - u_k^B| dx. \end{aligned}$$

Now, summing on  $i$  between 1 and  $h-1$  and taking the average, we obtain that for every  $k$  there exists an index  $i_k$  such that

$$\begin{aligned} \mathcal{F}_k(u_k^{i_k}, \alpha_k^{i_k}, A' \cup B) &\leq \mathcal{F}_k(u_k^A, \alpha_k^A; A) + \mathcal{F}_k(u_k^B, \alpha_k^B; B) \\ &\quad + \frac{c}{h-1} [\mathcal{F}_k(u_k^A, \alpha_k^A; B \cap (A \setminus A')) + \mathcal{F}_k(u_k^B, \alpha_k^B; B \cap (A \setminus A'))] \\ &\quad + \frac{c(h+1)^2}{d^2(h-1)} \varepsilon_k \int_{B \cap (A \setminus A')} |\alpha_k^A - \alpha_k^B|^2 dx + \frac{c(h+1)}{d(h-1)} \int_{B \cap (A \setminus A')} |u_k^A - u_k^B| dx. \end{aligned}$$

We conclude by letting  $k \rightarrow +\infty$  and then  $h \rightarrow +\infty$ .  $\square$

**Proposition 4.4.** *Let  $\mathcal{F}_{k_j}$  be a subsequence of  $\mathcal{F}_k$   $\bar{\Gamma}$ -converging to some functional  $\hat{\mathcal{F}} : L^1(\Omega) \times L^1(\Omega) \rightarrow [0, +\infty]$ . Then for every  $u \in BV(\Omega)$  the set function  $\hat{\mathcal{F}}(u, 1; \cdot)$  is the restriction to open sets of a Radon measure on  $\Omega$ . Moreover,  $\hat{\mathcal{F}}$  is local, i.e., for every open set  $A \subset \Omega$  we have  $\hat{\mathcal{F}}(u, 1; A) = \hat{\mathcal{F}}(v, 1; A)$  if  $u = v$  a.e. in  $A$ .*

*Proof.* We have already observed that  $\hat{\mathcal{F}}(u, 1; \cdot)$  is increasing, inner regular and superadditive. Subadditivity follows from Proposition 4.3, taking inner regularity into account. We can now apply an extension theorem (see [17] and [13, Theorem 14.23]) to construct a Borel measure which coincides with  $\hat{\mathcal{F}}(u, 1; \cdot)$  on all open sets. This measure is bounded thanks to Proposition 4.2. The locality of  $\hat{\mathcal{F}}$  is trivial.  $\square$

**Proposition 4.5.** *For every  $u \in L^1(\Omega)$  we have  $\mathcal{F}''(u, 1) \leq \mathcal{F}_0(u, 1)$ .*

*Proof.* Let us fix a subsequence of  $\mathcal{F}_k$ , which we do not relabel,  $\bar{\Gamma}$ -converging to some functional  $\hat{\mathcal{F}} : L^1(\Omega) \times L^1(\Omega) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty]$ . By Proposition 4.4, for every  $u \in BV(\Omega)$  the set function  $\hat{\mathcal{F}}(u, 1; \cdot)$  is the restriction to open sets of a Radon measure on  $\Omega$ . We notice that  $\hat{\mathcal{F}}$  coincides with the  $\Gamma$ -limit of the sequence  $\mathcal{F}_k$  on the space  $BV(\Omega)$ , by [13, Proposition 18.6].

First of all, let us prove that for every  $u \in BV(\Omega)$  we have  $\hat{\mathcal{F}}(u, 1) \leq \mathcal{F}(u)$ . Let us define the functional  $\mathcal{G} : BV(\Omega) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty)$  by

$$\mathcal{G}(u; A) := \hat{\mathcal{F}}(u, 1; A),$$

The functional  $\mathcal{G}$  satisfies the following properties for every  $u \in BV(\Omega)$  and for every  $A \in \mathcal{A}(\Omega)$ :

1.  $\mathcal{G}(\cdot; A)$  is  $L^1$ -lower semicontinuous on  $BV(\Omega)$ ;
2.  $\mathcal{G}$  is local;
3.  $\mathcal{G}(u; A) = \hat{\mathcal{F}}(u, 1; A) \leq C|Du|(A)$ ;
4.  $\mathcal{G}(u; \cdot)$  is the restriction to open sets of a Radon measure;
5.  $\mathcal{G}(u(\cdot - z) + b; z + A) = \mathcal{G}(u; A)$  for all  $b \in \mathbb{R}$  and  $z \in \mathbb{R}^n$  such that  $z + A \subset \Omega$ .

We now want to apply the integral representation result in [7], which requires also an estimate from below. To this aim, for every  $\lambda > 0$  we consider the functional

$$\mathcal{G}_\lambda(u; A) := \mathcal{G}(u; A) + \lambda|Du|(A).$$

By [7, Theorem 3.12 and Remark 3.8], there exist three Borel functions  $g_\lambda : \mathbb{R}^n \rightarrow [0, +\infty)$ ,  $h_\lambda : \mathbb{R}^n \rightarrow [0, +\infty)$ , and  $\psi_\lambda : \mathbb{R} \times \mathbb{S}^{n-1} \rightarrow [0, +\infty)$  such that

$$\mathcal{G}_\lambda(u; A) = \int_A g_\lambda(\nabla u) \, dx + \int_A h_\lambda\left(\frac{D^c u}{|D^c u|}\right) \, d|D^c u| + \int_{J_u \cap A} \psi_\lambda([u], \nu_u) \, d\mathcal{H}^{n-1}$$

for every  $u \in BV(\Omega)$  and for every  $A \in \mathcal{A}(\Omega)$ .

By (4.11), we have that

$$\mathcal{G}_\lambda(u; A) \leq \int_A (f(1, |\nabla u|) + \lambda|\nabla u|) \, dx + (\kappa(1) + \lambda)|D^c u|(A) + \int_{A \cap J_u} (\kappa(1) + \lambda)|[u]| \, d\mathcal{H}^{n-1},$$

from which it follows in particular that

$$(4.16) \quad g_\lambda(\xi) \leq f(1, |\xi|) + \lambda|\xi|, \quad h_\lambda(\xi) \leq \kappa(1)|\xi| + \lambda|\xi|,$$

for every  $\xi \in \mathbb{R}^n$ .

As for the surface term, for every  $a \in \mathbb{R}$  and for every  $\nu \in \mathbb{S}^{n-1}$  the value of  $\psi_\lambda(a, \nu)$  can be characterized by means of minimum problems related to the pure jump functions  $u_a^\nu$  defined by

$$u_a^\nu(y) := \begin{cases} a & \text{if } y \cdot \nu > 0, \\ 0 & \text{if } y \cdot \nu < 0. \end{cases}$$

More precisely, let  $Q_\rho^\nu$  be a cube of side  $\rho$  centered at the origin and with a face orthogonal to  $\nu$ . By [7, Theorem 3.12] we have that

$$\begin{aligned} \psi_\lambda(a, \nu) &= \lim_{\rho \rightarrow 0^+} \frac{\inf\{\mathcal{G}_\lambda(v; Q_\rho^\nu) : v \in BV(Q_\rho^\nu), v = u_a^\nu \text{ on } \partial Q_\rho^\nu\}}{\rho^{n-1}} \\ &\leq \limsup_{\rho \rightarrow 0^+} \frac{\mathcal{G}_\lambda(u_a^\nu; Q_\rho^\nu)}{\rho^{n-1}} \leq \limsup_{\rho \rightarrow 0^+} \frac{\mathcal{G}(u_a^\nu; Q_\rho^\nu)}{\rho^{n-1}} + \lambda|a|. \end{aligned}$$

We claim that

$$(4.17) \quad \limsup_{\rho \rightarrow 0^+} \frac{\mathcal{G}(u_a^\nu; Q_\rho^\nu)}{\rho^{n-1}} \leq \Psi(|a|).$$

This will conclude the proof of the  $\Gamma$ -limsup inequality when  $u \in BV(\Omega)$ . Indeed, combining (4.16) and (4.17), we obtain that

$$\mathcal{G}(u; \Omega) \leq \mathcal{G}_\lambda(u; \Omega) \leq \mathcal{F}(u) + \lambda|Du|(\Omega)$$

and the result follows by letting  $\lambda \rightarrow 0^+$ .

To prove (4.17), we construct a suitable approximating sequence. Without loss of generality, let us assume that  $\nu = e_n$ , so that  $Q_\rho^\nu$  is the cube  $Q_\rho$  of side  $\rho$  centered at the origin with faces orthogonal to the axes. The corresponding function  $u_a^\nu$  will be denoted simply by  $u_a$ . Let  $\tau_k$ ,  $\zeta_k$ ,  $\sigma_k$ , and  $\psi_k$  be as in the construction in the one-dimensional case, i.e., as in (3.24)–(3.26). Let

$$\begin{aligned} A_k &:= \{x_n = 0\} \times (-\sigma_k, \sigma_k), \\ B_k &:= \{x_n = 0\} \times ((-\sigma_k - \zeta_k, -\sigma_k) \cup (\sigma_k, \sigma_k + \zeta_k)). \end{aligned}$$

We define  $u_k$  as  $u_a$  outside  $A_k$ , and by linking linearly the values 0 and  $a$  inside  $A_k$ . Let  $\alpha_0 \in [0, 1]$  be such that

$$\Psi(|a|) = \begin{cases} \kappa(\alpha_0)|a| + \gamma_W(\alpha_0) & \text{if } \alpha_0 > 0, \\ \gamma_W(0) & \text{if } \alpha_0 = 0. \end{cases}$$

If  $\alpha_0 = 1$ , we simply put  $\alpha_k = 1$ . Otherwise, let

$$\alpha_k(x', x_n) := \begin{cases} 1 - \tau_k & \text{if } |x_n| \geq \zeta_k + \sigma_k, \\ \psi_k(|x_n| - \sigma_k) & \text{if } t \in (-\sigma_k - \zeta_k, -\sigma_k) \cup (\sigma_k, \sigma_k + \zeta_k), \\ \alpha_0 + \delta_k & \text{if } |x_n| \leq \sigma_k, \end{cases}$$

where  $x' = (x_1, \dots, x_{n-1})$ .

In the case  $0 < \alpha_0 < 1$ , we define  $e_k = 0$  and

$$p_k := \begin{cases} 0 & \text{in } Q_\rho \setminus A_k, \\ \nabla u_k \mathcal{L}^n & \text{in } Q_\rho \cap A_k. \end{cases}$$

Let us estimate all the terms in  $\mathcal{F}_k(u_k, \alpha_k; Q_\rho)$ :

$$\begin{aligned} \int_{Q_\rho \cap A_k} \kappa(\alpha_k) d|p_k| &= \kappa(\alpha_0 + \delta_k) \int_{Q_\rho \cap A_k} |\nabla u_k| dx = \kappa(\alpha_0 + \delta_k) |a| \rho^{n-1}, \\ \int_{Q_\rho \setminus (A_k \cup B_k)} \left[ \frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\nabla \alpha_k|^2 \right] dx &\leq \frac{W(1 - \tau_k)}{\varepsilon_k} \rho^n, \\ \int_{Q_\rho \cap A_k} \left[ \frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\nabla \alpha_k|^2 \right] dx &= \int_{Q_\rho \cap A_k} \frac{W(\alpha_0 + \delta_k)}{\varepsilon_k} dx = \frac{2\sigma_k}{\varepsilon_k} W(\alpha_0 + \delta_k) \rho^{n-1}, \\ \int_{Q_\rho \cap B_k} \left[ \frac{W(\alpha_k)}{\varepsilon_k} + \varepsilon_k |\nabla \alpha_k|^2 \right] dx &= \int_{Q'_\rho} \int_0^{\zeta_k} 4\sqrt{W(\psi_k(t))} |\psi'_k(t)| dt dx' = 4\rho^{n-1} \int_{\alpha_0 + \delta_k}^{1 - \tau_k} \sqrt{W(s)} ds, \end{aligned}$$

where  $Q'_\rho$  is the corresponding cube in  $\mathbb{R}^{n-1}$ . Summing all the above inequalities and letting  $k \rightarrow +\infty$ , we obtain

$$\mathcal{G}(u_a; Q_\rho) = \hat{\mathcal{F}}(u_a, 1; Q_\rho) \leq \limsup_{k \rightarrow +\infty} \mathcal{F}_k(u_k, \alpha_k; Q_\rho) \leq (\kappa(\alpha_0) |a| + \gamma_W(\alpha_0)) \rho^{n-1} = \Psi(|a|) \rho^{n-1},$$

from which (4.17) follows.

In the case  $\alpha_0 = 0$ , let  $p_k = 0$  and

$$e_k(x) := \begin{cases} 0 & \text{if } x \in Q_\rho \setminus A_k, \\ \nabla u_k(x) & \text{if } x \in Q_\rho \cap A_k, \end{cases}$$

With this choice,

$$\int_{Q_\rho \cap A_k} \alpha_k |e_k|^2 dx \leq \int_{Q_\rho \cap A_k} \delta_k |\nabla u_k|^2 dx = \frac{\delta_k}{2\sigma_k} |a|^2 \rho^{n-1}$$

and therefore

$$\mathcal{G}(u_a; Q_\rho) = \hat{\mathcal{F}}(u_a, 1; Q_\rho) \leq \limsup_{k \rightarrow +\infty} \mathcal{F}_k(u_k, 1; Q_\rho) \leq \gamma_W(0) \rho^{n-1} = \Psi(|a|) \rho^{n-1},$$

from which (4.17) follows.

We have proved that  $\mathcal{F}''(u, 1) = \hat{\mathcal{F}}(u, 1) \leq \mathcal{F}(u)$  for all  $u \in BV(\Omega)$ . Assume now that  $u \in GBV(\Omega)$ . For every  $\lambda$  we consider the truncated functions  $u_\lambda := \min\{\max\{-\lambda, u\}, \lambda\} \in BV_{loc}(\Omega)$ . We want to prove that

$$(4.18) \quad \mathcal{F}''(u_\lambda, 1) \leq \mathcal{F}(u_\lambda).$$

It is not restrictive to assume that  $\mathcal{F}(u_\lambda) < +\infty$ . From (2.3) and (2.9), we obtain

$$(4.19) \quad \int_{\Omega} |\nabla u_\lambda| dx + |D^c u_\lambda|(\Omega) + \int_{J_{u_\lambda} \setminus J_{u_\lambda}^1} |[u_\lambda]| d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(J_{u_\lambda}^1) < +\infty,$$

where  $J_{u_\lambda}^1 := \{|[u_\lambda]| \geq 1\}$ . Since  $\|u_\lambda\|_{L^\infty(\Omega)} \leq \lambda$ , we have

$$|Du_\lambda|(J_{u_\lambda}) \leq \int_{J_{u_\lambda} \setminus J_{u_\lambda}^1} |[u_\lambda]| d\mathcal{H}^{n-1} + 2\lambda \mathcal{H}^{n-1}(J_{u_\lambda}^1),$$

so that (4.19) implies  $|Du_\lambda|(\Omega) < +\infty$ . Therefore  $u_\lambda \in BV(\Omega)$  and (4.18) follows from the previous step of the proof. Letting  $\lambda \rightarrow +\infty$ , we obtain  $\mathcal{F}''(u, 1) \leq \mathcal{F}(u)$ , thanks to the lower semicontinuity of  $\mathcal{F}''(\cdot, 1)$ .  $\square$

## 5. CONVERGENCE OF MINIMIZERS

In this section we study the convergence of  $\eta_\varepsilon$ -minimizers of problem (1.8) with Dirichlet boundary conditions. To this aim, for every  $w \in L^\infty(\partial_D\Omega)$  we introduce the functionals  $\mathcal{F}_k^w, \mathcal{F}_0^w$  defined on the space  $L^1(\Omega) \times L^1(\Omega)$  by

$$(5.1) \quad \mathcal{F}_k^w(u, \alpha) = \begin{cases} \mathcal{F}_k(u, \alpha) & \text{if } u = w \text{ and } \alpha = 1 \text{ } \mathcal{H}^{n-1}\text{-a.e. on } \partial_D\Omega, \\ +\infty & \text{otherwise,} \end{cases}$$

$$(5.2) \quad \mathcal{F}_0^w(u, \alpha) = \begin{cases} \mathcal{F}(u) + \int_{\partial_D\Omega} \Psi(|u - w|) d\mathcal{H}^{n-1} & \text{if } u \in GBV(\Omega) \text{ and} \\ & \alpha = 1 \text{ } \mathcal{L}^n\text{-a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

We begin by proving the following result.

**Theorem 5.1.** *Let  $w \in L^\infty(\partial_D\Omega)$ . Then the functionals  $\mathcal{F}_k^w$   $\Gamma$ -converge to  $\mathcal{F}_0^w$ , as  $k \rightarrow +\infty$  in  $L^1(\Omega) \times L^1(\Omega)$ .*

*Proof.* Let us prove the  $\Gamma$ -liminf inequality. Given a sequence  $(u_k, \alpha_k)$  converging to  $(u, 1)$  in  $L^1(\Omega) \times L^1(\Omega)$ , we want to show that

$$(5.3) \quad \mathcal{F}_0^w(u, 1) \leq \liminf_{k \rightarrow +\infty} \mathcal{F}_k^w(u_k, \alpha_k),$$

where  $\mathcal{F}_0^w$  is defined by (5.2). By Gagliardo's Theorem (see [21, Theorem 2.16]), there exists a function  $v \in W^{1,1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$  whose trace on  $\partial_D\Omega$  coincides with  $w$ . We can assume that the liminf is finite and it is actually a limit, hence  $u_k \in BV(\Omega)$  with  $u_k = w$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial_D\Omega$ , and  $\alpha_k \in H^1(\Omega)$  with  $\delta_k \leq \alpha_k \leq 1$  and  $\alpha_k = 1$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial_D\Omega$ . Since  $\partial_D\Omega$  is relatively open in  $\partial\Omega$ , there exists a bounded open set  $U \subset \mathbb{R}^n$  such that  $\partial_D\Omega = U \cap \partial\Omega$ . Let  $\tilde{\Omega} := \Omega \cup U$ . We can extend the functions  $u_k$  and  $\alpha_k$  to  $\tilde{\Omega}$  by putting  $u_k := v$  and  $\alpha_k := 1$  in  $U \setminus \Omega$ , respectively. Moreover, we extend  $u$  to  $\tilde{\Omega}$  by defining  $u := v$  in  $U \setminus \Omega$ . Since  $(u_k, \alpha_k) \rightarrow (u, 1)$  in  $L^1(\tilde{\Omega}) \times L^1(\tilde{\Omega})$  and the functionals  $\mathcal{F}_k(\cdot, \cdot; \tilde{\Omega})$   $\Gamma$ -converge to  $\mathcal{F}_0(\cdot, \cdot; \tilde{\Omega})$  by Theorem 1.1 (applied to  $\tilde{\Omega}$ ), we have that  $u \in GBV(\tilde{\Omega})$  and

$$\mathcal{F}(u; \tilde{\Omega}) \leq \liminf_{k \rightarrow +\infty} \mathcal{F}_k(u_k, \alpha_k; \tilde{\Omega}).$$

On the other hand

$$\begin{aligned} \mathcal{F}(u; \tilde{\Omega}) &= \mathcal{F}(u; \Omega) + \int_{\partial_D\Omega} \Psi(|u - w|) d\mathcal{H}^{n-1} + \int_{U \setminus \Omega} f(1, |\nabla v|) dx, \\ \mathcal{F}_k(u_k, \alpha_k; \tilde{\Omega}) &= \mathcal{F}_k(u_k, \alpha_k; \Omega) + \int_{U \setminus \Omega} f(1, |\nabla v|) dx, \end{aligned}$$

and therefore

$$\mathcal{F}(u; \Omega) + \int_{\partial_D\Omega} \Psi(|u - w|) d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow +\infty} \mathcal{F}_k(u_k, \alpha_k; \Omega).$$

This concludes the proof of (5.3).

To prove the  $\Gamma$ -limsup inequality, it is enough to consider the case  $u \in BV(\Omega)$ . Indeed, if  $u \in GBV(\Omega)$ , we can argue by approximation as in the proof of Proposition 4.5. We have to construct a sequence  $(u_k, \alpha_k)$  converging to  $(u, 1)$  in  $L^1(\Omega) \times L^1(\Omega)$  and satisfying the boundary conditions  $u_k = w$ ,  $\alpha_k = 1$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial_D\Omega$ . We extend the function  $w$  to the whole boundary  $\partial\Omega$  by putting  $w$  equal to the trace of  $u$  on  $\partial\Omega \setminus \partial_D\Omega$ . By [21, Theorem 2.16], there exists a function  $v \in W^{1,1}(\mathbb{R}^n)$  whose trace on  $\partial\Omega$  is  $w$ . By [16, Proposition 1.2], for every  $\eta > 0$  it is possible to find a  $\mathcal{C}^\infty$  function  $r_\eta : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $r_\eta(\tilde{\Omega}) \subset \Omega$ ,

$r_\eta - Id$  has compact support, and  $r_\eta - Id \rightarrow 0$  in  $C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$  as  $\eta \rightarrow 0$ , where  $Id$  is the identity map. Let us fix  $\eta > 0$  and let us consider the function  $u_\eta$  defined by

$$u_\eta(x) := \begin{cases} u(x) & \text{if } x \in \Omega_\eta := r_\eta(\Omega), \\ v(x) & \text{if } x \in \Omega \setminus \Omega_\eta. \end{cases}$$

Let us fix  $\hat{\Omega}$  such that  $\Omega_\eta \Subset \hat{\Omega} \Subset \Omega$ . By Proposition 4.5, there exists a recovery sequence  $(\hat{u}_k, \hat{\alpha}_k) \rightarrow (u_\eta, 1)$  in  $L^1(\hat{\Omega}) \times L^1(\hat{\Omega})$  such that

$$\mathcal{F}(u_\eta; \hat{\Omega}) = \limsup_{k \rightarrow +\infty} \mathcal{F}_k(\hat{u}_k, \hat{\alpha}_k; \hat{\Omega}).$$

We now modify the sequence  $(\hat{u}_k, \hat{\alpha}_k)$  using the De Giorgi slicing and averaging argument in such a way that the boundary conditions are satisfied. Let  $d := \text{dist}(\Omega_\eta, \partial\hat{\Omega})$ . As in the proof of Proposition 4.3, we consider a finite chain of open sets  $\Omega_\eta = A_0 \Subset A_1 \Subset \dots \Subset A_h \Subset A_{h+1} = \hat{\Omega}$  such that  $\text{dist}(A_i, \partial A_{i+1}) \leq d/(h+1)$ . Then we consider  $\varphi_i \in C_c^1(\mathbb{R}^n)$  such that  $0 \leq \varphi_i \leq 1$ ,  $\text{supp}(\varphi_i) \subset A_{i+1}$ ,  $\varphi_i = 1$  on an open neighborhood of  $\bar{A}_i$  and  $\|\nabla \varphi_i\|_{L^\infty(\Omega)} \leq 2(h+1)/d$  and we define

$$u_k^i := \varphi_i \hat{u}_k + (1 - \varphi_i)v, \quad \alpha_k^i := \varphi_{i+1} \hat{\alpha}_k + (1 - \varphi_{i+1}).$$

We have that  $u_k^i = w$  and  $\alpha_k^i = 1$   $\mathcal{H}^{n-1}$ -a.e. on  $\partial_D \Omega$ . With computations similar to those made in the proof of Proposition 4.3, it is possible to deduce the following estimate

$$\begin{aligned} \mathcal{F}_k(u_k^i, \alpha_k^i; \Omega) &\leq \mathcal{F}_k(\hat{u}_k, \hat{\alpha}_k; \hat{\Omega}) + \mathcal{F}_k(v, 1; \Omega \setminus \Omega_\eta) \\ &\quad + c[\mathcal{F}_k(\hat{u}_k, \hat{\alpha}_k; A_{i+2} \setminus A_i) + \mathcal{F}_k(v, 1; A_{i+2} \setminus A_i)] \\ &\quad + \frac{c(h+1)}{d} \int_{A_{i+2} \setminus A_i} |\hat{u}_k - v| dx + \frac{c(h+1)^2}{d^2} \varepsilon_k \int_{A_{i+2} \setminus A_i} |\hat{\alpha}_k - 1|^2 dx \end{aligned}$$

for every  $i \in \{0, \dots, h-1\}$ , and therefore, by taking averages, there exists  $i_k \in \{0, \dots, h-1\}$  such that

$$\begin{aligned} \mathcal{F}_k(u_k^{i_k}, \alpha_k^{i_k}; \Omega) &\leq \mathcal{F}_k(\hat{u}_k, \hat{\alpha}_k; \hat{\Omega}) + \mathcal{F}_k(v, 1; \Omega \setminus \Omega_\eta) \\ &\quad + \frac{c}{h} [\mathcal{F}_k(\hat{u}_k, \hat{\alpha}_k; \hat{\Omega} \setminus \Omega_\eta) + \mathcal{F}_k(v, 1; \hat{\Omega} \setminus \Omega_\eta)] \\ &\quad + \frac{c(h+1)}{dh} \int_{\hat{\Omega} \setminus \Omega_\eta} |\hat{u}_k - v| dx + \frac{c(h+1)^2}{d^2 h} \varepsilon_k \int_{\hat{\Omega} \setminus \Omega_\eta} |\hat{\alpha}_k - 1|^2 dx. \end{aligned}$$

Letting  $k \rightarrow +\infty$  and then  $h \rightarrow +\infty$ , we obtain

$$\limsup_{k \rightarrow +\infty} \mathcal{F}_k(u_k^{i_k}, \alpha_k^{i_k}; \Omega) \leq \mathcal{F}(u_\eta; \hat{\Omega}) + \mathcal{F}(v; \Omega \setminus \Omega_\eta).$$

By the arbitrariness of  $\hat{\Omega}$ , we have

$$(\Gamma\text{-}\limsup_{k \rightarrow +\infty} \mathcal{F}_k^w)(u_\eta, 1) \leq \limsup_{k \rightarrow +\infty} \mathcal{F}_k(u_k^{i_k}, \alpha_k^{i_k}; \Omega) \leq \mathcal{F}(u_\eta; \bar{\Omega}_\eta) + \mathcal{F}(v; \Omega \setminus \Omega_\eta) = \mathcal{F}(u_\eta; \Omega).$$

By the lower semicontinuity of the  $\Gamma$ -limsup, to conclude the proof it is enough to show that

$$(5.4) \quad \mathcal{F}(u_\eta; \Omega) \rightarrow \mathcal{F}(u; \Omega) + \int_{\partial_D \Omega} \Psi(|u - w|) d\mathcal{H}^{n-1} \quad \text{as } \eta \rightarrow 0.$$

We observe that

$$\mathcal{F}(u_\eta; \Omega) = \mathcal{F}(u; \Omega_\eta) + \int_{\partial \Omega_\eta} \Psi(|u_{\Omega_\eta} - v|) d\mathcal{H}^{n-1} + \mathcal{F}(v; \Omega \setminus \Omega_\eta),$$

where  $u_{\Omega_\eta}$  is the trace on  $\partial\Omega_\eta$  of  $u|_{\Omega_\eta}$ . Since  $\mathcal{F}(v; \Omega \setminus \Omega_\eta) \rightarrow 0$  and  $\mathcal{F}(u; \Omega_\eta) \rightarrow \mathcal{F}(u; \Omega)$ , to prove (5.4) we only need to show that

$$(5.5) \quad \int_{\partial\Omega_\eta} \Psi(|u_{\Omega_\eta} - v|) d\mathcal{H}^{n-1} \rightarrow \int_{\partial\Omega} \Psi(|u - w|) d\mathcal{H}^{n-1} = \int_{\partial_D\Omega} \Psi(|u - w|) d\mathcal{H}^{n-1}.$$

By making the change of variables  $z = r_\eta(x)$ , we obtain

$$(5.6) \quad \int_{r_\eta(\partial\Omega)} \Psi(|u_{\Omega_\eta}(z) - v(z)|) d\mathcal{H}^{n-1}(z) = \int_{\partial\Omega} \Psi(|(u_\eta^*)_\Omega(x) - v_\eta^*(x)|)(1 + \omega_\eta(x)) d\mathcal{H}^{n-1}(x)$$

where  $u_\eta^* := u \circ r_\eta$  and  $v_\eta^* := v \circ r_\eta$ . The term  $(1 + \omega_\eta(x))$  is due to the Generalized Area Formula (see [4, Theorem 2.91]) and  $\omega_\eta \rightarrow 0$  uniformly since  $r_\eta$  is converging to the identity map in  $\mathcal{C}^\infty$ . Since  $v \in W^{1,1}(\mathbb{R}^n)$ , it is easy to see that  $v_\eta^* \rightarrow v$  in  $L^1(\partial\Omega)$ . To prove the same result for  $u_\eta^*$  we start by computing its total variation. If  $u$  is  $\mathcal{C}^1$ , we have

$$(5.7) \quad \begin{aligned} |Du_\eta^*(\Omega)| &= \int_{\Omega} |\nabla u_\eta^*(x)| dx = \int_{\Omega} |\nabla u(r_\eta(x)) \nabla r_\eta(x)| dx \\ &= \int_{r_\eta(\Omega)} |\nabla u(z) \nabla r_\eta(r_\eta^{-1}(z))| \frac{1}{|\det(\nabla r_\eta(r_\eta^{-1}(z)))|} dz \\ &\leq (1 + \omega'_\eta) \int_{\Omega_\eta} |\nabla u| dz \leq (1 + \omega'_\eta) |Du|(\Omega_\eta), \end{aligned}$$

with  $\omega'_\eta \rightarrow 0$ . By approximation we obtain that (5.7) holds for an arbitrary  $u \in BV(\Omega)$ . Formula (5.7) in particular implies that

$$\limsup_{\eta \rightarrow 0} |Du_\eta^*(\Omega)| \leq |Du|(\Omega).$$

From the convergence  $u_\eta^* \rightarrow u$  in  $L^1(\Omega)$ , we conclude that  $|Du_\eta^*(\Omega)| \rightarrow |Du|(\Omega)$ . Since the trace is continuous with respect to this kind of convergence, we deduce that  $(u_\eta^*)_\Omega \rightarrow u_\Omega$  in  $L^1(\partial\Omega)$ . Therefore we can pass to the limit in (5.6) and eventually obtain (5.5). This concludes the proof.  $\square$

Another ingredient in the proof of the convergence of  $\eta_\varepsilon$ -minimizers with Dirichlet boundary conditions is the following compactness result.

**Theorem 5.2.** *Let  $M, c > 0$  and let  $(u_k, \alpha_k) \in BV(\Omega) \times H^1(\Omega)$  be such that  $\|u_k\|_{L^\infty(\Omega)} \leq M$  and*

$$\mathcal{F}_k(u_k, \alpha_k) \leq c.$$

*Then  $\alpha_k \rightarrow 1$  in  $L^1(\Omega)$  and there exists a subsequence of  $u_k$  and a function  $u \in BV(\Omega)$  such that  $u_k \rightarrow u$  in  $L^1(\Omega)$ .*

*Proof.* Let us start with the proof of the theorem in the case  $n = 1$ . As in the proof of Lemma 3.3, we extract a subsequence from  $\alpha_k$  such that  $\alpha_k$   $\Gamma(\mathbb{R})$ -converges to some function  $\alpha$ , and we consider the set  $\{\alpha = 0\}$ , which is finite by Remark 3.1. Let  $A_j$ ,  $j \geq 1$ , be open sets as in (3.9). By repeating the proof of Lemma 3.3, we obtain that the sequence  $u_k$  is bounded in  $BV(A_j)$ , uniformly with respect to  $k$  and  $j$ . Therefore, by a diagonal argument, it is possible to extract a subsequence from  $u_k$  converging to some  $u \in L^1(\Omega)$  strongly in  $L^1(\Omega)$ . Moreover  $u \in BV(\Omega)$ .

To prove the theorem in the case  $n > 1$ , we make use of [1, Theorem 6.6] to reduce the problem to the one dimensional case. In order to apply that result, we consider the family  $\mathcal{U} = (u_k)$ , which is by hypotheses equibounded in  $L^\infty(\Omega)$ . To prove that  $\mathcal{U}$  is relatively compact in  $L^1(\Omega)$ , it suffices to prove that there exist  $n$  linearly independent vectors  $\xi$  satisfying the following property: for every  $\eta > 0$ , there exists an equibounded subset  $\mathcal{U}_\eta$  of

$L^\infty(\Omega)$  lying in a  $\eta$ -neighborhood of  $\mathcal{U}$  with respect to the  $L^1(\Omega)$  topology, and such that  $(\mathcal{U}_\eta)^\xi := \{w_y^\xi : w \in \mathcal{U}_\eta\}$  is relatively compact in  $L^1(\Omega_y^\xi)$  for  $\mathcal{H}^{n-1}$ -a.e.  $y \in \Pi^\xi$ . To prove this, we fix  $\xi \in \mathbb{R}^n$  and we consider the set

$$A_k := \{y \in \Pi^\xi : \mathcal{F}_k^{\xi,y}((u_k)_y^\xi, (\alpha_k)_y^\xi) \leq L\},$$

where  $\mathcal{F}_k^{\xi,y} : BV(\Omega_y^\xi) \times H^1(\Omega_y^\xi) \rightarrow [0, +\infty]$  is the one-dimensional functional defined by

$$\mathcal{F}_k^{\xi,y}(u, \alpha) := \int_{\Omega_y^\xi} f_k(\alpha, |u'|) dt + \kappa(1) |D^s u|(\Omega_y^\xi) + \int_{\Omega_y^\xi} \left[ \frac{W(\alpha)}{\varepsilon_k} + \varepsilon_k |\alpha'|^2 \right] dt,$$

and  $L$  is a suitable constant that we will choose later. By the Chebyshev Inequality, we have

$$L\mathcal{H}^{n-1}(\Omega^\xi \setminus A_k) \leq \int_{\Omega^\xi \setminus A_k} \mathcal{F}_k^{\xi,y}((u_k)_y^\xi, (\alpha_k)_y^\xi) d\mathcal{H}^{n-1}(y) \leq \mathcal{F}_k(u_k, \alpha_k) \leq c,$$

where  $\Omega^\xi$  is the projection of  $\Omega$  on  $\Pi^\xi$ . Let us define the function  $w_k$  in such a way that

$$(w_k)_y^\xi := \begin{cases} (u_k)_y^\xi & \text{if } y \in A_k \\ 0 & \text{otherwise.} \end{cases}$$

Letting  $\mathcal{U}_\eta := (w_k)$ , we have that  $\mathcal{U}_\eta$  lies in a  $\eta$ -neighborhood of  $\mathcal{U}$  for a suitable choice of  $L$ , since

$$\|w_k - u_k\|_{L^1(\Omega)} = \int_{\Omega^\xi \setminus A_k} \int_{\Omega_y^\xi} |(u_k)_y^\xi| dt d\mathcal{H}^{n-1}(y) \leq \frac{c}{L} \text{diam}(\Omega) M \leq \eta,$$

if  $L \geq \eta^{-1} c \text{diam}(\Omega) M$ . Moreover  $(\mathcal{U}_\eta)^\xi$  is relatively compact in  $L^1(\Omega_y^\xi)$  by the previous step. This proves that  $\mathcal{U}$  is relatively compact and therefore there exists a subsequence of  $u_k$  converging to some  $u \in L^1(\Omega)$ . Following the proof of Proposition 4.1, we deduce that  $u \in BV(\Omega)$ .  $\square$

*Proof of Theorem 1.2.* The result is an immediate consequence of Theorem 5.1, Theorem 5.2, and of the general theory developed in [13, Corollary 7.20].  $\square$

We conclude this section with an application in which the limit problem is actually defined on the space  $GBV(\Omega)$  and not just on  $BV(\Omega)$ . We omit the proof, since it follows the arguments in [14] with obvious modifications.

**Theorem 5.3.** *Let  $q > 1$  and let  $g \in L^q(\Omega)$ . For every  $k$ , let  $(u_k, \alpha_k) \in BV(\Omega) \times H^1(\Omega)$  be a minimizer of the problem*

$$\min \left\{ \mathcal{F}_k(u, \alpha) + \int_{\Omega} |u - g|^q dx : u \in BV(\Omega), \alpha \in H^1(\Omega), \delta_k \leq \alpha \leq 1 \right\}.$$

*Then  $\alpha_k \rightarrow 1$  in  $L^1(\Omega)$  and a subsequence of  $u_k$  converges in  $L^q(\Omega)$  to a minimizer  $u \in GBV(\Omega)$  of the problem*

$$\min \left\{ \mathcal{F}(u) + \int_{\Omega} |u - g|^q dx : u \in GBV(\Omega) \right\}.$$



## 6. APPENDIX

The following proposition shows that we can define the Cantor part of the gradient of a  $GBV$  function under suitable assumptions.

**Proposition 6.1.** *Let  $u \in GBV(\Omega)$  be such that*

$$(6.1) \quad \sup_{\lambda > 0} \left[ \int_{\Omega} |\nabla u_{\lambda}| \, dx + |D^c u_{\lambda}|(\Omega) + \int_{J_{u_{\lambda}}} \min\{|[u_{\lambda}]|, 1\} \, d\mathcal{H}^{n-1} \right] < +\infty,$$

where  $u_{\lambda} := \min\{\max\{-\lambda, u\}, \lambda\}$ . Then there exists a unique  $\mathbb{R}^n$ -valued Borel measure on  $\Omega \setminus J_u$ , denoted by  $D^c u$ , such that

$$(6.2) \quad (D^c u)(B) = (D^c u_{\lambda})(B)$$

for every  $\lambda > 0$  and for every Borel set  $B \subset \{|\tilde{u}| < \lambda\} \setminus J_u$ . Moreover, we have

$$(6.3) \quad |D^c u|(B) = \sup_{\lambda > 0} |D^c u_{\lambda}|(B \cap \{|\tilde{u}| < \lambda\})$$

for every Borel set  $B \subset \Omega \setminus J_u$ .

*Proof.* For every Borel set  $B \subset \Omega$  we define

$$\mu(B) := \sup_{\lambda > 0} \left[ \int_B |\nabla u_{\lambda}| \, dx + |D^c u_{\lambda}|(B) + \int_{J_{u_{\lambda}} \cap B} \min\{|[u_{\lambda}]|, 1\} \, d\mathcal{H}^{n-1} \right]$$

By (6.1), the set function  $\mu$  is a bounded Borel measure on  $\Omega$ .

We observe that for every  $\lambda > 0$  we have  $\tilde{u}_{\lambda} = \tilde{u}$   $\mathcal{H}^{n-1}$ -a.e. on  $\{|\tilde{u}| < \lambda\} \setminus J_u$ . Applying [15, Theorem A1], for every  $0 < \lambda < \lambda^*$ , we have  $(D^c u_{\lambda})(B) = (D^c u_{\lambda^*})(B)$  for every Borel set  $B \subset \{|\tilde{u}| < \lambda\} \setminus J_u$ . For these sets we can define  $(D^c u)(B) := (D^c u_{\lambda})(B)$  and the definition does not depend on  $\lambda$ . Since  $\mu$  is a bounded measure, we have  $\mu(\{\lambda \leq |\tilde{u}| < +\infty\} \setminus J_u) \rightarrow 0$  as  $\lambda \rightarrow +\infty$ . By the Cauchy criterion, this implies that the limit

$$(6.4) \quad \lim_{\lambda \rightarrow +\infty} (D^c u_{\lambda})(B \cap \{|\tilde{u}| < \lambda\})$$

exists and is finite for every Borel set  $B \subset \Omega \setminus J_u$ . We now define  $(D^c u)(B)$  by (6.4). Using again the upper bound  $|D^c u_{\lambda}|(B) \leq \mu(B)$ , we can prove that  $D^c u$  is a bounded  $\mathbb{R}^n$ -valued Borel measure on  $\Omega \setminus J_u$ . Equality (6.3) follows easily from (6.4), while the uniqueness is a consequence of the fact that  $|\tilde{u}|$  is finite  $\mathcal{H}^{n-1}$ -a.e. on  $\Omega \setminus J_u$ . This last property can be obtained arguing as in the proof of [4, Theorem 4.40], replacing  $S_u^*$  by  $\{|[u]| \geq 1\}$ .  $\square$

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