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Schrödinger operators on half-line with shrinking potentials at the origin

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Abstract. We discuss the general model of a Schrödinger quantum particle constrained on a straight half-line with given self-adjoint boundary condition at the origin and an interaction potential supported around the origin. We study the limit when the range of the potential scales to zero and its magnitude blows up. We show that in the limit the dynamics is generated by a self-adjoint negative Laplacian on the half-line, with a possible preservation or modification of the boundary condition at the origin, depending on the magnitude of the scaling and of the strength of the potential.

Keywords: point interaction, self-adjoint boundary conditions, singular scaling limits in Schrödinger operators, resolvent convergence, Konno–Kuroda resolvent formula

1. Introduction and main results

In this note we discuss similarities and differences between the well-studied phenomenon recalled here below for a non-relativistic quantum particle in $\mathbb{R}^d$, $d = 1, 2, 3$, subject to a peaked, short range potential, and the analogous model on a (straight) half-line.

If the particle is subject to a potential, centred around a point $x_0 \in \mathbb{R}^d$, of very short range and very strong magnitude, then an effective description for its dynamics is possible in terms of a so-called *zero-range interaction* (or *point interaction*), an interaction such that the particle is free as long as its wave-function is supported away from x_0, and that constrains the particle wave-function to fulfill a specific boundary condition at x_0. Only one simple piece of information concerning the potential, and not the whole amount of knowledge of it, enters the formula for the boundary condition (in $d = 3$, for instance, this is the s-wave scattering length of the potential), thus making the effective description much simpler. The small error made by replacing the true with the effective Hamiltonian is controlled by theorems that, generically, state that a sequence of Schrödinger operators with potentials centred around x_0 and scaling suitably so that in the limit their support shrinks to $\{x_0\}$ while their magnitude blows up, converges in an appropriate sense to a well-defined self-adjoint operator, the Hamiltonian of point interaction. Following an alternative approach, the Hamiltonian of point interaction can be independently conceived and constructed as *one* of the self-adjoint extensions of the free negative Laplacian restricted to regular functions on $\mathbb{R}^d$ vanishing in a neighbourhood of x_0 (thus encoding the idea that the particle is free when it is away from x_0). Thus, the above-mentioned limit theorems connect a mathematical construction with a limit of physical Hamiltonians: on the one hand they give a precise meaning to the idealisation of the zero-range interaction, and on the other hand they illustrate how *one* specific boundary condition (equivalently, *one* of the infinite possible self-adjoint extensions) is selected in the limit of shrinking potentials. At this level of generality we refer to [2] for a comprehensive overview of the point interaction theory; in this note we will discuss the case $d = 1$ in detail.

Our analysis will show that *a model of Schrödinger operators with shrinking potentials around the origin of the half-line admits a limit dynamics generated by a self-adjoint negative Laplacian on the half-line, with a possible preservation or modification of the boundary condition at the origin, depending on the magnitude of the scaling and of the strength of the potential.*

This conclusion can be easily extended to a model of Schrödinger operators on a metric graph with shrinking potentials around the graph's vertex. As we shall comment more in detail in the end of the Introduction, that this last fact has relevant consequences on the problem of determining how certain vertex conditions for a Schrödinger particle on a metric graph emerge from the Schrödinger dynamics on a tubular region around the graph in the limit where the transversal size of the "fat graph" goes to zero.

1.1. Set-up of the model

We consider a non-relativistic quantum particle on a half-line subject to an interaction potential V that is essentially supported around the origin and vanishes at infinity. The prototype we have in mind (although we will be more general in our final results) is an interaction in the form of a bump or a sequence of bumps in the vicinity of the origin, namely a (real-valued) function V supported, say, on $[0, 1]$ (V need *not* have a definite sign).

It is well known by a standard limit-point-limit-circle argument (see, e.g., [13], Theorems X.7 and X.11) that, for the formal Hamiltonian $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ to have an unambiguous self-adjoint realisation on $L^2(\mathbb{R}^+, \mathrm{d}x)$, a suitable boundary condition at the origin must be taken. In fact, as opposed to the case of $L^2(\mathbb{R})$, the symmetric Schrödinger operator

$$H_{(0)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \qquad \mathcal{D}(H_{(0)}) = C_0^\infty(0, +\infty) \tag{1.1}$$

(by $C_0^\infty(0, +\infty)$ we denote the C^∞-functions compactly supported away from $x = 0$) is *not* essentially self-adjoint on $L^2(\mathbb{R}^+, \mathrm{d}x)$ and admits an infinite family of self-adjoint extensions. Let us therefore make a short detour to discuss the set-up of the model (Theorem 1.1 below).

One way to determine each extension is the standard von Neumann extension theory. The equations for the deficiency subspaces of $H_{(0)}$ reduce to the classical ordinary differential equations $-f''+Vf = \pm\mathrm{i}f$, $f \in \mathcal{D}(H_{(0)}^*)$. From this it is straightforward to see that $H_{(0)}$ has deficiency indices $(1, 1)$ and hence a one-parameter family of extensions, each of which is a suitable restriction of $H_{(0)}^*$. As long as V is not explicit, though, it is practically cumbersome to come to an expression of the two deficiency spaces and of the unitary maps between them that, according to von Neumann's theory, identify each self-adjoint extension. Eventually the final conclusion that each extension is nothing but a restriction of $H_{(0)}^*$ to functions that satisfy a simple boundary condition at $x = 0$ would be obscured.

An equivalent way to give meaning to the formal Hamiltonian $-\frac{d^2}{dx^2} + V$ as a form sum in the sense of Kato's perturbation theory and of the KLMN Theorem (see, e.g., [14], Theorem 10.21). As the unperturbed operator we take any of the self-adjoint extensions of the symmetric operator on $L^2(\mathbb{R}^+, dx)$ defined by

$$\Delta_{(0)} f = f'', \quad f \in \mathcal{D}(\Delta_{(0)}) = C_0^\infty(0, +\infty). \tag{1.2}$$

It is known (see, e.g., [6], Section 6.2.2.1) that the collection of all self-adjoint extensions of $\Delta_{(0)}$ is the one-parameter family $\{\Delta_\nu | \nu \in (-\frac{\pi}{2}, \frac{\pi}{2}]\}$ of operators defined by

$$\Delta_\nu f = f'', \quad f \in \mathcal{D}(\Delta_\nu) = \left\{ f \in L^2(\mathbb{R}^+) \,\middle|\, \begin{array}{l} f, f' \in AC([0, +\infty)) \\ f'' \in L^2[0, +\infty) \\ f(0) \sin \nu = f'(0) \cos \nu \end{array} \right\}. \tag{1.3}$$

Thus, each self-adjoint extension is identified by a boundary condition at the origin of the form

$$f(0) \sin \nu = f'(0) \cos \nu, \quad \nu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{1.4}$$

The special cases $\nu = \frac{\pi}{2}$ and $\nu = 0$ correspond, respectively, to the extension with Dirichlet and with Neumann boundary conditions at $x = 0$: for them we use also the alternative notation Δ_D and Δ_N. We recall also (see, e.g., [6], Section 6.2.2.2) that the spectrum of each extension is

$$\sigma(-\Delta_\nu) = \sigma_{ac}(-\Delta_\nu) = [0, +\infty) \quad \text{if } \nu \in \left[0, \frac{\pi}{2}\right] \tag{1.5}$$

or

$$\begin{cases} \sigma(-\Delta_\nu) = \sigma_p(-\Delta_\nu) \cup \sigma_{ac}(-\Delta_\nu), \\ \sigma_p(-\Delta_\nu) = \{-(\tan \nu)^2\}, \\ \sigma_{ac}(-\Delta_\nu) = [0, +\infty) \end{cases} \quad \text{if } \nu \in \left(-\frac{\pi}{2}, 0\right). \tag{1.6}$$

In particular, each $-\Delta_\nu$ is bounded below.

To each self-adjoint free-particle Hamiltonian $-\Delta_\nu$ on $\mathbb{R}^+$ we add the potential V in the sense of a form perturbation. With the analysis discussed in Section 2 one proves the following.

Theorem 1.1. *Let $V \in L^p(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, real-valued, for some $p \in [1, 2]$. For each $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ let Δ_ν be the self-adjoint Laplacian on $L^2(\mathbb{R}^+)$ with boundary condition (1.4) at the origin. Then the operator $H^{(\nu)}$ acting as*

$$H^{(\nu)} = -\Delta_\nu + V \tag{1.7}$$

on the domain $\mathcal{D}(H^{(\nu)}) = \mathcal{D}(\Delta_\nu)$ is self-adjoint on $L^2(\mathbb{R}^+)$.

1.2. Scaling limits

Now that we have set up the model, let us go back to our original goal to discuss an interaction of strong magnitude and short range around the origin in the limit of zero range and infinite magnitude. We fix $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ and, for each $\varepsilon > 0$, we consider the self-adjoint Hamiltonian $H_\varepsilon^{(\nu)}$ on $L^2(\mathbb{R}^+)$ defined by

$$\begin{aligned} &H_\varepsilon^{(\nu)} = -\Delta_\nu + V_\varepsilon, \qquad \mathcal{D}\big(H_\varepsilon^{(\nu)}\big) = \mathcal{D}(-\Delta_\nu), \\ &V_\varepsilon(x) := \frac{\lambda(\varepsilon)}{\varepsilon^{1+\gamma}} V\left(\frac{x}{\varepsilon}\right), \end{aligned} \tag{1.8}$$

where V is real-valued, $V \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, and $\gamma \in \mathbb{R}$. The function $\varepsilon \mapsto \lambda(\varepsilon)$ is customarily introduced to "distort" the scaling: for convenience, and without loss of generality, we assume it to be continuous and with $\lambda(\varepsilon) \geqslant 0$, $\lambda(0) = \lambda(1) = 1$. It is clear by construction that as $\varepsilon \to 0$ the potential V_ε tends to spike up to a delta-like function in a region that shrinks more and more around $x = 0$.

For convenience we classify the degree of "squeezing" of V_ε as follows:

$$\begin{aligned} &\gamma < 0 \quad \textit{weak} \text{ scaling}, \\ &\gamma = 0 \quad \textit{canonical} \text{ scaling}, \\ &\gamma > 0 \quad \textit{strong} \text{ scaling}. \end{aligned} \tag{1.9}$$

Only in the canonical scaling is the L^1-norm of $x \mapsto \varepsilon^{-(1+\gamma)} V(x/\varepsilon)$ independent of ε; this norm, instead, vanishes in the weak scaling and blows up in the strong scaling as $\varepsilon \to 0$. By canonical we want also to emphasize these two features:

- when $\gamma = 0$, $V_\varepsilon(x)$ converges distributionally to $(\int V)\delta(x)$;
- the scaling for $\gamma = 0$ is precisely that scaling that produces in the limit $\varepsilon \to 0$ a point interaction on the straight line, namely an operator on $L^2(\mathbb{R})$ which is a self-adjoint extension of the free particle operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ defined on the C^∞-functions on $\mathbb{R}$ with compact support. (See Appendix for details.)

In analogy to the same question on $L^2(\mathbb{R}^d)$, $d = 1, 2, 3$, we want to discuss the limit $\varepsilon \to 0$ of the Hamiltonian $H_\varepsilon^{(\nu)}$ on $L^2(\mathbb{R}^+)$. As is customary in this context, we will do that *in the resolvent sense*, thus considering the bounded operators

$$\begin{aligned} &R_\varepsilon^{(\nu)}(k) := \big(-\Delta_\nu + V_\varepsilon - k^2\big)^{-1}, \\ &\quad k \in \mathbb{C}, k^2 \in \rho(-\Delta_\nu + V_\varepsilon), \mathfrak{Im}\, k > 0 \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &R^{(\nu)}(k) := \big(-\Delta_\nu - k^2\big)^{-1}, \\ &\quad k \in \mathbb{C}, k^2 \in \rho(-\Delta_\nu), \mathfrak{Im}\, k > 0. \end{aligned} \tag{1.11}$$

Note that, in view of (1.5) and (1.6), for each $k \in \mathbb{C}$ with $\mathfrak{Im}\, k > 0$, the condition $k^2 \in \rho(-\Delta_\nu)$ is equivalent to the condition $k \neq -\mathrm{i} \tan \nu$ when ν is negative.

As $\varepsilon \to 0$, $R_\varepsilon^{(\nu)}(k)$ turns out to have a limit that is a rank-one perturbation of $R^{(\nu)}(k)$. What we find in each of the three scaling regimes is stated by the following

Theorem 1.2. *For a fixed $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ and for each $\varepsilon > 0$, let $H_\varepsilon^{(\nu)} = -\Delta_\nu + V_\varepsilon$, $\mathcal{D}(H_\varepsilon^{(\nu)}) = \mathcal{D}(\Delta_\nu)$ be the self-adjoint Hamiltonian on $L^2(\mathbb{R}^+)$ defined in (1.8) with respect to a bounded, compactly supported, real-valued potential V and a scaling exponent $\gamma \in (-\infty, 1)$. In the case $\gamma \in (0,1)$ it is further assumed that $\int_{\mathbb{R}^+} V \neq 0$. Let $k \in \mathbb{C}$ with $\mathfrak{Im}\, k > 0$ and $k^2 \in \rho(-\Delta_\nu)$, and with the possible exception of the value $k = \mathrm{i}\beta_0$ in the case $\gamma = 0$, if there exists $\beta_0 > 0$ such that*

$$\beta_0 + \tan\nu + \int_{\mathbb{R}^+} V(x)\,\mathrm{d}x = 0. \tag{1.12}$$

Then, $k^2 \in \rho(-\Delta_\nu + V_\varepsilon)$ for $\varepsilon > 0$ sufficiently small, and

$$\big(-\Delta_\nu + V_\varepsilon - k^2\big)^{-1} \xrightarrow{\varepsilon\to 0} \big(-\Delta_\nu - k^2\big)^{-1} - \Theta_{\nu,V,k}(\gamma)\big|e^{\mathrm{i}kx}\big\rangle\big\langle\overline{e^{\mathrm{i}kx}}\big| \tag{1.13}$$

in the norm operator sense, where

$$\Theta_{\nu,V,k}(\gamma) = \begin{cases} 0 & \text{if } \gamma < 0 \text{ (weak)},\\ \frac{\int_{\mathbb{R}^+} V}{(\tan\nu - \mathrm{i}k)(\tan\nu - \mathrm{i}k + \int_{\mathbb{R}^+} V)} & \text{if } \gamma = 0 \text{ (canonical)},\\ (\tan\nu - \mathrm{i}k)^{-1} & \text{if } 0 < \gamma < 1 \text{ (strong)}. \end{cases} \tag{1.14}$$

We shall prove Theorem 1.2 in Section 5, after a preparatory discussion in Sections 3 and 4, by means of a convenient expression of the *difference* between the full resolvent $R_\varepsilon^{(\nu)}(k) = (-\Delta_\nu + V_\varepsilon - k^2)^{-1}$ and the free resolvent $R^{(\nu)}(k) = (-\Delta_\nu - k^2)^{-1}$, which allows for a control of the scaling with ε along the limit $\varepsilon \to 0$. This is quite a versatile scheme that was developed first by Albeverio, Gesztesy, and Høegh-Krohn [1] (see [2] and references therein for a general overview). In this note we adapt it to the case of a half-line, the three main technical differences being the interplay between the boundary condition at $x = 0$ (absent when the model is set up on a straight line) and the scaling with ε, the different expression of the integral kernel of the free resolvent $R^{(\nu)}(k)$ (that, unlike the case on the line, is *not* translation invariant), and the previously unexplored case of strong scaling, for which it is not enough to prove that certain compact operators appearing in the expression of $R_\varepsilon^{(\nu)}(k)$ converge as $\varepsilon \to 0$, as is the case for the model on a straight line, but instead a *quantitative* rate of convergence is needed.

1.3. Discussion of the limit: The problem of the limit dynamics

Upon a closer inspection, we observe that the "*pseudo-resolvent*"

$$\mathcal{R}_{\lim}^{(\nu)}(k) := \lim_{\varepsilon\to 0} R_\varepsilon^{(\nu)}(k) = \lim_{\varepsilon\to 0}\big(-\Delta_\nu + V_\varepsilon - k^2\big)^{-1} \tag{1.15}$$

obtained in Theorem 1.2 is in fact the resolvent, for all admissible k's, of another self-adjoint negative Laplacian on the half-line, with the noticeable feature of a possible *modification of the boundary condition at the origin* depending on the magnitude of the scaling and on the boundary condition before the limit.

Before formulating the complete result, let us anticipate first two conclusions that can be made by general arguments (see Section 6 for their proof). The first is a consequence of a well-known theorem of Kato:

Proposition 1.3. *The operator $\mathcal{R}^{(\nu)}_{\lim}(k) = \lim_{\varepsilon\to 0}(-\Delta_\nu + V_\varepsilon - k^2)^{-1}$ obtained under the conditions of Theorem 1.2 is, for all admissible k's, injective and therefore is the resolvent of a unique closed operator T_ν on $L^2(\mathbb{R}^+)$. Explicitly, the space $\mathrm{ran}(\mathcal{R}^{(\nu)}_{\lim}(k))$ is independent of the admissible k's, the domain $\mathcal{D}(T_\nu)$ is precisely this space, and $\forall f \in \mathcal{D}(T_\nu)$ one has $T_\nu f = k^2 f + \mathcal{R}^{(\nu)}_{\lim}(k)^{-1} f$.*

Remark 1.4. Since $\mathcal{R}^{(\nu)}_{\lim}(k) = (T_\nu - k^2)^{-1}$ and T_ν is *closed*, then necessarily the bounded operator-valued map $k^2 \mapsto \mathcal{R}^{(\nu)}_{\lim}(k)$ is *analytic* on the open subset of $\mathbb{C}$ consisting of the complex numbers k^2 given by the admissible k's (and more generally it is analytic on the resolvent set of T_ν). Note that, without the information that T_ν is closed, the sole norm-convergence $(-\Delta_\nu + V_\varepsilon - k^2)^{-1} \xrightarrow{\varepsilon\to 0} \mathcal{R}^{(\nu)}_{\lim}(k)$ and the fact that each resolvent $(-\Delta_\nu + V_\varepsilon - k^2)^{-1}$ is obviously analytic in k^2 would *not* be enough to conclude the analyticity of $k^2 \mapsto \mathcal{R}^{(\nu)}_{\lim}(k)$, analogously to the fact that a pointwise limit of complex-valued holomorphic functions need not be holomorphic.

In fact, such a closed operator T_ν is *self-adjoint*.

Proposition 1.5. *The operator $r^{(\nu)}(z) := \mathcal{R}^{(\nu)}_{\lim}(k) = \lim_{\varepsilon\to 0}(-\Delta_\nu + V_\varepsilon - k^2)^{-1}$, $z = k^2$, obtained under the conditions of Theorem 1.2 has the property $r^{(\nu)}(\overline{z}) = r^{(\nu)}(z)^*$ for all admissible k's. As a consequence, in the identity $\mathcal{R}^{(\nu)}_{\lim}(k) = (T_\nu - k^2)^{-1}$ established in Proposition 1.3 one has $T_\nu = T_\nu^*$.*

When we combine the result above with a closer inspection of Eqs (1.13)–(1.14), it is easy to come first of all to the following conclusion (see Section 6 for the proof):

Proposition 1.6. *The operator T_ν is one of the self-adjoint extension of the symmetric operator $-\Delta_{(0)}$ introduced in (1.2). It has to be therefore one of the negative self-adjoint Laplacians (1.3).*

This leads to the natural question on what is the self-adjointness boundary condition at the origin for T_ν and to the general result stated here.

Theorem 1.7. *Under the conditions of Theorem 1.2 the following convergences hold true in the norm-resolvent sense for all admissible k's:*

$$\begin{aligned}
&-\Delta_{\mathrm{D}} + V_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta_{\mathrm{D}}, \quad \gamma < 1,\\
&-\Delta_\nu + V_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta_\nu, \quad \gamma < 0, \nu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ or } \gamma = 0, \int_{\mathbb{R}^+} V = 0,\\
&-\Delta_\nu + V_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta_{\mathrm{D}}, \quad 0 < \gamma < 1, \nu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right], \int_{\mathbb{R}^+} V \neq 0,\\
&-\Delta_\nu + V_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta_{\widetilde{\nu}}, \quad \gamma = 0, \nu \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \int_{\mathbb{R}^+} V \neq 0,
\end{aligned} \tag{1.16}$$

where

$$\tan\widetilde{\nu} = \tan\nu + \int_{\mathbb{R}^+} V \,\mathrm{d}x. \tag{1.17}$$

Let us highlight a few comments about such findings.

Remark 1.8.

(i) Similarly to the analogous model on a straight line (see Theorems A.1 and A.2 in Appendix), suitable squeezings of the potential at the origin give rise to a well-defined limit dynamics (self-adjointly generated) that corresponds to a precise rank-one perturbation of the resolvent. The generator is again a negative self-adjoint Laplacian on the half-line.
(ii) The weak scaling is "too weak" to allow the squeezing of the potential to produce a different boundary condition at $x = 0$ in the limit $\varepsilon \to 0$ (the limit dynamics is that of a free Schrödinger particle with the same boundary condition).
(iii) Scaling canonically an interaction with zero strength does not affect any boundary condition ν in the limit.
(iv) Irrespective of whether the scaling is weak or strong (provided that $\gamma < 1$), the Dirichlet boundary condition is "too robust" to be modified by any squeezing potential.
(v) The strong scaling $\gamma \in (0, 1)$ always preserves or restores the Dirichlet boundary condition at the origin.
(vi) The canonical scaling applied to an initial boundary condition other than the Dirichlet one produces in general a *different* boundary condition, which is determined by the strength ($\int_{\mathbb{R}^+} V \,\mathrm{d}x$) of the potential.

We find particularly interesting what observed in the last point, which is in fact a mechanism for a modification of the boundary condition at the origin by means of a canonically scaling potential.

It is worth concluding our discussion on the quest for the limit dynamics by quoting previous results in the case of "*ultra-strong*" scaling $\gamma = 1$. In fact we did not cover this case here because our analysis of the convergence (see Section 5 below) is based on a factorisation of the difference

$$\left(-\Delta_\nu + V_\varepsilon - k^2\right)^{-1} - \left(-\Delta_\nu - k^2\right)^{-1}$$

into terms for each of which we know its limit as $\varepsilon \to 0$ and its rate of convergence, and only if $\gamma < 1$ is it possible to make a conclusion on the limit of the product of such terms based only on the information of the rate of convergence of each of them. If $\gamma = 1$, additional information is needed on the properties of each factor, and eventually on the potential V, in order to decide what limit the above difference has.

When such a strong scaling as $V_\varepsilon(x) = \varepsilon^{-2} V(x/\varepsilon)$ is applied, it is reasonable to expect that the singularity of the interaction as $\varepsilon \to 0$ is intense enough to create a self-adjoint boundary condition in the limit, at least under certain favourable conditions on the potential V. This is indeed what previous authors have found (Šeba [15], see also Golovaty and Hryniv [7,8]). The generic case is that this ultra-strong scaling in $-\Delta_{\mathrm{D}} + V_\varepsilon$ preserves Dirichlet boundary conditions and reproduces, in the norm-resolvent sense, the limit operator $-\Delta_{\mathrm{D}}$, while exceptionally, if the unscaled Hamiltonian $-\Delta_{\mathrm{D}} + V$ admits a zero-energy resonance, the limit $\varepsilon \to 0$ in $-\Delta_{\mathrm{D}} + V_\varepsilon$ may give rise to a different self-adjoint Laplacian $-\Delta_\nu$.

1.4. Application to the Schrödinger dynamics on graphs

At the end of this Introduction let us make a few comments on how our findings are related with the problem of the Schrödinger dynamics on a metric graph. This was in fact our original motivation and the reason why we extracted and studied the simplified model on the half-line.

In order to define a self-adjoint Laplacian on a metric graph one has to impose suitable boundary conditions at the vertices, precisely as one does at the origin of the half-line for the operators $-\Delta_\nu$ considered in (1.3) above. Which boundary condition is to be chosen for a realistic model of a quantum particle constrained on a metric graph is usually a matter of an ad hoc choice usually made to fit experimental data, [4]. For instance it is customary in Chemical Physics to choose vertex boundary conditions of Kirchhoff type, or sometimes also boundary conditions in which at each vertex the wave-function is continuous and the sum of the directional derivatives is proportional to the value that the function attains at the vertex itself.

These choices are seldom motivated theoretically and a major problem is to show how certain vertex conditions actually arise, given that a quantum graph is an idealisation of a physical system constrained (e.g., by confining forces) to a very small tubular neighbourhood of a graph-like structure. (For example the density of conducting electrons in graphene has the form of a "fat graph" with a hole at the vertices.)

For the three-dimensional, graph-shaped physical system, the corresponding Schrödinger operator is well-defined, be it of the form of the free negative Laplacian with boundary condition at the surface of the tubular graph, or the free negative Laplacian plus a strong confining potential that constrains the particle in a small region around the graph-like structure. It is therefore of interest to see if and in which sense a limit can be taken when the width of the "fat graph" tends to zero and, in the case that the emerging dynamics on the limit metric graph is generated by the negative Laplacian on the graph itself, what the origin is of the plurality of self-adjoint boundary conditions at the vertices.

It can be argued that this shrinking limit depends crucially on the geometry of the vertex region of the "fat graph" (and the possible occurrence of a zero-energy resonance for the free Laplacian on the "fat graph") – for a general discussion on how to address this limit problem "fat graph" → "thin graph" we refer to our recent note [5] and to the references therein. A convenient way to model the vertex effects is to introduce a fictitious potential supported in the vicinity of the junctions of the tubes of the "fat" graph, which scales to a delta-like profile as the "fat" graph's width squeezes to zero. If such an additional potential is only supported on the tubes close to the junction, but not inside the junction itself, this boils down to a model of a Schrödinger operator on the "thin" graph where a squeezing potential is added around the vertex: in this case the question is whether this squeezing limit selects a self-adjoint Laplacian on "thin" graph.

It is in this respect that our present analysis brings a new insight. Indeed, the discussion that we developed here for the model on a half-line can be easily exported to the case of a star graph and, with a further analysis, to the case of a graph with also internal edges. The whole family of self-adjoint Laplacians on a metric graph is well known, with explicit formulas for the resolvent and the vertex boundary conditions (see, e.g., [11,12], as well as the general discussion in Appendix K.4.2 of [2] and references therein). This allows for a direct generalisation of Theorems 1.2 and 1.7: their consequences, as discussed in Section 1.3 above, remain virtually the same.

2. Schrödinger operators on $L^2(\mathbb{R}^+)$ constructed perturbatively

In this section we prove Theorem 1.1 and hence the construction, for each self-adjoint free particle Hamiltonian $-\Delta_\nu$ on the positive half-line, of the self-adjoint Schrödinger operator $-\Delta_\nu + V$. We shall make use of the fact that, as a consequence of (1.3), the energy form associated with $-\Delta_\nu$ is the bounded below, closed, quadratic form

$$\begin{aligned} (-\Delta_\nu)[f,g] &= \begin{cases} \overline{f(0)}g(0)\tan\nu + \int_{\mathbb{R}^+} \overline{f'(x)}g'(x)\,\mathrm{d}x & \text{if } \nu \in (-\frac{\pi}{2},\frac{\pi}{2}), \\ \int_{\mathbb{R}^+} \overline{f'(x)}g'(x)\,\mathrm{d}x & \text{if } \nu = \frac{\pi}{2}, \end{cases} \\ \mathcal{D}[-\Delta_\nu] &= \begin{cases} H^1(\mathbb{R}^+) & \text{if } \nu \in (-\frac{\pi}{2},\frac{\pi}{2}), \\ H^1_0(\mathbb{R}^+) & \text{if } \nu = \frac{\pi}{2}. \end{cases} \end{aligned} \tag{2.1}$$

(Note that the boundary term is absent both in the Dirichlet ($\nu = \frac{\pi}{2}$) and in the Neumann ($\nu = 0$) case.) Moreover,

$$(-\Delta_\nu)[f,g] = \langle f, (-\Delta_\nu)g\rangle, \qquad \begin{array}{l} f \in \mathcal{D}[-\Delta_\nu], \\ g \in \mathcal{D}(-\Delta_\nu). \end{array} \tag{2.2}$$

In fact, by means of the KLMN Theorem we will show that Theorem 1.1 follows from the following two propositions.

Proposition 2.1. *Let $V \in L^p(\mathbb{R}^+)$, real-valued, for some $p \in [1,2]$. For each $\nu \in (-\frac{\pi}{2},\frac{\pi}{2}]$ let Δ_ν be the self-adjoint Laplacian on $L^2(\mathbb{R}^+)$ with boundary condition (1.4) at the origin and let $k \in \mathbb{C}$ be such that $\mathfrak{Im}\, k > 0$ and, in the case of negative ν, $k \neq -\mathrm{i}\tan\nu$. Then k^2 belongs to the resolvent set of $-\Delta_\nu$ and the operator*

$$|V|^{1/2}\big(-\Delta_\nu - k^2\big)^{-1}|V|^{1/2}$$

is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^+)$.

Proposition 2.2. *Let $V \in L^p(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, real-valued, for some $p \in [1,2]$. For each $\nu \in (-\frac{\pi}{2},\frac{\pi}{2}]$ let Δ_ν be the self-adjoint Laplacian on $L^2(\mathbb{R}^+)$ with boundary condition (1.4) at the origin. Then the multiplication operator V is infinitesimally form bounded with respect to $-\Delta_\nu$, i.e., for any $\alpha > 0$ there exists $\beta > 0$ such that*

$$\big|\langle f, Vf\rangle\big| \leqslant \alpha(-\Delta_\nu)[f,f] + \beta\|f\|_2^2 \quad \forall f \in \mathcal{D}[-\Delta_\nu]. \tag{2.3}$$

Let us first of all show how Theorem 1.1 follows Proposition 2.2 (which is in turn a consequence of Proposition 2.1).

Proof of Theorem 1.1. The boundedness of V guarantees that the symmetric quadratic form $\psi \mapsto \langle\psi, V\psi\rangle$ is well defined on $\mathcal{D}[-\Delta_\nu]$ and this, together with (2.3), gives precisely the assumption needed to apply the KLMN Theorem ([14], Theorem 10.21), which states that there exists a unique self-adjoint operator S with quadratic form

$$\begin{aligned} &\mathcal{D}[S] = \mathcal{D}[-\Delta_\nu], \\ &S[f,g] = (-\Delta_\nu)[f,g] + \langle f, Vg\rangle. \end{aligned} \tag{2.4}$$

Correspondingly, the operator domain and the action of S are given by

$$\begin{aligned} &\mathcal{D}(S) = \{ f \in \mathcal{D}[S] \mid \exists h_f \in L^2(\mathbb{R}^+) \text{ with } S[f,g] = \langle h_f, g\rangle \ \forall g \in \mathcal{D}[S] \}, \\ &Sf = h_f. \end{aligned}$$

By (2.4), this is the same as ($k_f \equiv h_f - Vf$)

$$\begin{aligned} &\mathcal{D}(S) = \{ f \in \mathcal{D}[-\Delta_\nu] \mid \exists k_f \in L^2(\mathbb{R}^+) \text{ with } (-\Delta_\nu)[f,g] = \langle k_f, g\rangle \ \forall g \in \mathcal{D}[-\Delta_\nu] \}, \\ &Sf = k_f + Vf. \end{aligned}$$

By (2.2), each element $f \in \mathcal{D}(S)$ must therefore satisfy the condition

$$\langle f, (-\Delta_\nu) g\rangle = \langle k_f, g\rangle \quad \forall g \in \mathcal{D}(-\Delta_\nu),$$

whence necessarily $f \in \mathcal{D}(-\Delta_\nu^*) = \mathcal{D}(-\Delta_\nu)$ (since $-\Delta_\nu$ is self-adjoint) and $k_f = -\Delta_\nu f$. The conclusion is that the operator S defined by

$$\begin{aligned} &\mathcal{D}(S) = \mathcal{D}(-\Delta_\nu), \\ &Sf = -\Delta_\nu f + Vf \end{aligned}$$

is self-adjoint. □

In the remaining part of this section we turn to the proofs of Propositions 2.1 and 2.2. An important tool will be the integral kernel of the resolvent of $-\Delta_\nu$ at the point k^2, namely the bounded operator

$$R^{(\nu)}(k) := (-\Delta_\nu - k^2)^{-1} \quad (k^2 \in \rho(-\Delta_\nu), \mathfrak{Im}\, k > 0). \tag{2.5}$$

Note that the assumption $k \in \mathbb{C}$, $\mathfrak{Im}\, k > 0$, and $k \neq -\mathrm{i} \tan \nu$ when ν is negative, made in Proposition 2.1, guarantees precisely that $k^2 \in \rho(-\Delta_\nu)$. The integral kernel associated with $R^{(\nu)}(k)$, i.e., the $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ measurable function $R^{(\nu)}(k)(x,y)$ such that

$$(R^{(\nu)}(k) f)(x) = \int_{\mathbb{R}^+} R^{(\nu)}(k)(x,y) f(y)\, \mathrm{d}y,$$

is given by

$$R^{(\nu)}(k)(x,y) = \frac{\mathrm{i}}{2k} \left[e^{\mathrm{i}k|x-y|} - \frac{\sin\nu + \mathrm{i}k\cos\nu}{\sin\nu - \mathrm{i}k\cos\nu} e^{\mathrm{i}k(x+y)} \right] \tag{2.6}$$

(see, e.g., [6], Section 6.2.2.2, or also [12], Section 4, with the notation $A = \sin\nu$, $B = -\cos\nu$). Note also here that for any $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ the quantity $\sin\nu - ik\cos\nu$ is invertible for all complex k's with $\mathfrak{Im}\, k > 0$ and $k \neq -\mathrm{i}\tan\nu$ when ν is negative.

Proof of Proposition 2.1. As already argued, $(-\Delta_\nu - k^2)$ is indeed invertible (with bounded inverse) and its inverse has integral kernel (2.6). Correspondingly, the operator

$$\mathcal{K}_k := |V|^{1/2}\big(-\Delta_\nu - k^2\big)^{-1}|V|^{1/2}$$

has integral kernel

$$\mathcal{K}_k(x,y) = \sqrt{\big|V(x)\big|}R^{(\nu)}(k)(x,y)\sqrt{\big|V(y)\big|}.$$

The Hilbert–Schmidt norm of $\mathcal{K}_k$ is given by the L^2-norm of its integral kernel:

$$\begin{aligned}
\|\mathcal{K}_k\|^2_{\mathrm{H.S.}} &= \big\|\mathcal{K}_k(\cdot,\cdot)\big\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^+)} \\
&= \iint_{\mathbb{R}^+\times\mathbb{R}^+}\big|V(x)\big|\frac{1}{4|k|^2}\left|e^{\mathrm{i}k|x-y|} - \frac{\sin\nu + ik\cos\nu}{\sin\nu - ik\cos\nu}e^{\mathrm{i}k(x+y)}\right|^2\big|V(y)\big|\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant \frac{1}{2|k|^2}\bigg(\iint_{\mathbb{R}^+\times\mathbb{R}^+}\big|V(x)\big|e^{-2\,\mathfrak{Im}\,k|x-y|}\big|V(y)\big|\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \left|\frac{\sin\nu + ik\cos\nu}{\sin\nu - ik\cos\nu}\right|^2\iint_{\mathbb{R}^+\times\mathbb{R}^+}\big|V(x)\big|e^{-2\,\mathfrak{Im}\,k(x+y)}\big|V(y)\big|\,\mathrm{d}x\,\mathrm{d}y\bigg).
\end{aligned} \tag{2.7}$$

The two integrals in the r.h.s. of (2.7) are estimated, respectively, as

$$\begin{aligned}
\iint_{\mathbb{R}^+\times\mathbb{R}^+}\big|V(x)\big|e^{-2(\mathfrak{Im}\,k)|x-y|}\big|V(y)\big|\,\mathrm{d}x\,\mathrm{d}y &\leqslant c_p\|V\|^2_p\big\|e^{-2\,\mathfrak{Im}\,k|\cdot|}\big\|_{(2-\frac{2}{p})^{-1}} \\
&= \frac{c_p\|V\|^2_p}{(\frac{p}{p-1}\,\mathfrak{Im}\,k)^{2-2/p}}
\end{aligned} \tag{2.8}$$

(Young's inequality for generic $p \in [1,2]$) and as

$$\begin{aligned}
\iint_{\mathbb{R}^+\times\mathbb{R}^+}\big|V(x)\big|e^{-2\,\mathfrak{Im}\,k(x+y)}\big|V(y)\big|\,\mathrm{d}x\,\mathrm{d}y &= \bigg(\int_{\mathbb{R}^+}\big|V(x)\big|e^{-2\,\mathfrak{Im}\,kx}\,\mathrm{d}x\bigg)^2 \\
&\leqslant \frac{\|V\|^2_p}{(\frac{p}{p-1}\,\mathfrak{Im}\,k)^{2-2/p}}
\end{aligned} \tag{2.9}$$

for any $p \in [1,\infty]$. Plugging (2.8) and (2.9) into (2.7) yields

$$\|\mathcal{K}_k\|_{\mathrm{H.S.}} \leqslant c_{pk}\|V\|_p, \qquad p \in [1,2], \tag{2.10}$$

with c_{pk} finite for any of the chosen p, k (and ν). □

Proof of Proposition 2.2. By dominated convergence, we deduce from (2.7) that

$$\big\| |V|^{1/2}\big(-\Delta_\nu - k^2\big)^{-1}|V|^{1/2}\big\|_{\mathrm{H.S.}} \xrightarrow[k^2\in\rho(-\Delta_\nu),\,\mathfrak{Im}\,k>0]{|k|\to\infty} 0. \tag{2.11}$$

In particular, along the complex positive imaginary axis (more precisely for $k = \mathrm{i}\beta$ as $\beta \to +\infty$, the finite exceptional value $\beta = -\tan\nu$, for $\nu < 0$, obviously does not affect the limit),

$$\big\| |V|^{1/2}\big(-\Delta_\nu + \beta^2\big)^{-1}|V|^{1/2}\big\|_{\mathrm{H.S.}} \xrightarrow{\beta\to+\infty} 0. \tag{2.12}$$

Each $-\Delta_\nu$ is bounded below and hence $-\Delta_\nu + \beta^2$ is eventually a positive operator as $\beta \to +\infty$. This allows us to re-write

$$\big\| |V|^{1/2}\big(-\Delta_\nu + \beta^2\big)^{-1}|V|^{1/2}\big\|_{\mathrm{H.S.}} = \big\| \big(-\Delta_\nu + \beta^2\big)^{-1/2}|V|\big(-\Delta_\nu + \beta^2\big)^{-1/2}\big\|_{\mathrm{H.S.}} \tag{2.13}$$

and hence to deduce from (2.12) that

$$\forall\varepsilon > 0, \exists\beta_\varepsilon > 0 \quad \text{such that} \quad \big\| \big(-\Delta_\nu + \beta_\varepsilon^2\big)^{-1/2}|V|\big(-\Delta_\nu + \beta_\varepsilon^2\big)^{-1/2}\big\| \leqslant \varepsilon. \tag{2.14}$$

(We used the operator norm, that is controlled by the (larger) Hilbert–Schmidt norm.) The last inequality implies

$$\big\langle g, \big(\big(-\Delta_\nu + \beta_\varepsilon^2\big)^{-1/2}|V|\big(-\Delta_\nu + \beta_\varepsilon^2\big)^{-1/2}\big)g\big\rangle \leqslant \varepsilon\langle g, g\rangle \quad \forall g \in L^2\big(\mathbb{R}^+\big). \tag{2.15}$$

We re-write (2.15) setting $f := (-\Delta_\nu + \beta_\varepsilon^2)^{-1/2} g$ and noting that $f \in \mathcal{D}[-\Delta_\nu]$ for every $g \in L^2(\mathbb{R}^+)$; moreover, since $|V|$ is bounded then any such f belongs to the form domain of the multiplication operator $|V|$. The result is

$$\begin{aligned}\big|\langle f, Vf\rangle\big| &\leqslant \varepsilon\big\langle\big(-\Delta_\nu + \beta_\varepsilon^2\big)^{1/2} f, \big(-\Delta_\nu + \beta_\varepsilon^2\big)^{1/2} f\big\rangle \\ &= \varepsilon\big(-\Delta_\nu + \beta_\varepsilon^2\big)[f, f] \\ &= \varepsilon(-\Delta_\nu)[f, f] + \varepsilon\beta_\varepsilon^2\langle f, f\rangle.\end{aligned} \tag{2.16}$$

This is precisely (2.3) with $\alpha \equiv \varepsilon$ and $\beta \equiv \varepsilon\beta_\varepsilon^2$. $\square$

3. Resolvent identities and scaling operators

In this section we discuss the two main technical tools for the proof of Theorem 1.2.

The first tool is a convenient resolvent identity that allows us to express the complete resolvent $R_\varepsilon^{(\nu)}(k) = (-\Delta_\nu + V_\varepsilon - k^2)^{-1}$ in terms of the free resolvent $R^{(\nu)}(k) = (-\Delta_\nu - k^2)^{-1}$, see (1.10) and (2.5) respectively. Since V_ε is infinitesimally form bounded with respect to $-\Delta_\nu$ (Proposition 2.2) and the operator $|V_\varepsilon|^{1/2}(-\Delta_\nu - k^2)^{-1}|V_\varepsilon|^{1/2}$ is compact (Proposition 2.1), then the so-called *Konno–Kuroda resolvent formula* is applicable ([10]; see also [2], Theorem B.1, and [16], Theorem II.34) and one has

$$R_\varepsilon^{(\nu)}(k) = R^{(\nu)}(k) - R^{(\nu)}(k)v_\varepsilon\big(1 + u_\varepsilon R^{(\nu)}(k)v_\varepsilon\big)^{-1}u_\varepsilon R^{(\nu)}(k),$$
$$k^2 \in \rho(-\Delta_\nu + V_\varepsilon) \cap \rho(-\Delta_\nu), \mathfrak{Im}\, k > 0, \tag{3.1}$$

where

$$v_\varepsilon(x) := \sqrt{\big|V_\varepsilon(x)\big|}, \qquad u_\varepsilon(x) := \sqrt{\big|V_\varepsilon(x)\big|}\,\mathrm{sgn}\big(V(x)\big). \tag{3.2}$$

In order to control the limit $\varepsilon \to 0$ in the resolvent identity (3.1) we shall make use of the second main tool, namely the following scaling operators defined for $\varepsilon > 0$:

$$U_\varepsilon : L^2\big(\mathbb{R}^+\big) \to L^2\big(\mathbb{R}^+\big), \qquad (U_\varepsilon f)(x) := \varepsilon^{-1/2} g(x/\varepsilon). \tag{3.3}$$

Each U_ε is a unitary map on $L^2(\mathbb{R}^+)$, whose adjoint acts as

$$\big(U_\varepsilon^* f\big)(x) = \varepsilon^{1/2} f(\varepsilon x). \tag{3.4}$$

The domain of the free Hamiltonian $-\Delta_{\nu'}$ on the half-line is *not* invariant under U_ε because U_ε changes the boundary condition (1.4) valid for functions $f \in \mathcal{D}(\Delta_{\nu'})$ into

$$(U_\varepsilon f)(0) \sin\nu' = (U_\varepsilon f)'(0)\varepsilon \cos\nu' \quad \big(f \in \mathcal{D}(\Delta_{\nu'})\big), \tag{3.5}$$

while of course the regularity of such functions remain the same. The boundary condition (3.5) is precisely the condition satisfied at $x = 0$ by functions in $\mathcal{D}(\Delta_\nu)$ with ν uniquely determined by $\tan\nu' = \varepsilon \tan\nu$. Therefore

$$\begin{aligned} &U_\varepsilon \mathcal{D}(\Delta_{\nu'}) \cong \mathcal{D}(\Delta_\nu), \\ &U_\varepsilon^* \mathcal{D}(\Delta_\nu) \cong \mathcal{D}(\Delta_{\nu'}) \end{aligned} \tag{3.6}$$

with

$$\tan\nu' = \varepsilon \tan\nu, \quad \nu, \nu' \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{3.7}$$

From (3.3) and (3.6) we deduce straightforwardly a number of relevant transformations under unitary scaling, tacitly assuming in the following that ν and ν' are always related by (3.7). We have

$$U_\varepsilon^*(-\Delta_\nu)U_\varepsilon = \varepsilon^{-2}(-\Delta_{\nu'}) \tag{3.8}$$

as an identity on $\mathcal{D}(-\Delta_{\nu'})$, whence also

$$U_\varepsilon^*\big(-\Delta_\nu - k^2\big)^{-1}U_\varepsilon = \varepsilon^2\big(-\Delta_{\nu'} - (\varepsilon k)^2\big)^{-1} \quad \big(k^2 \in \rho(-\Delta_\nu)\big) \tag{3.9}$$

as an identity on the whole $L^2(\mathbb{R}^+)$. Moreover, setting $v \equiv v_1$ and $u \equiv u_1$ in (3.2), i.e.,

$$v(x) := \sqrt{|V(x)|}, \qquad u(x) := \sqrt{|V(x)|}\,\mathrm{sgn}\big(V(x)\big) \tag{3.10}$$

(recall that we set for convenience $\lambda(1) = 1$), we also have

$$\begin{aligned} U_\varepsilon^* v_\varepsilon U_\varepsilon &= \sqrt{\lambda(\varepsilon)}\varepsilon^{-(1+\gamma)/2} v, \\ U_\varepsilon^* u_\varepsilon U_\varepsilon &= \sqrt{\lambda(\varepsilon)}\varepsilon^{-(1+\gamma)/2} u \end{aligned} \tag{3.11}$$

on the whole $L^2(\mathbb{R}^+)$.

By means of the scaling operators introduced above, we proceed to re-write the Konno–Kuroda resolvent identity (3.1) as follows: we plug $U_\varepsilon U_\varepsilon^* = \mathbb{1}$ into the second summand in the r.h.s. of (3.1) and then we exploit the scaling transformations (3.9) and (3.11). We thus obtain

$$\begin{aligned} R_\varepsilon^{(\nu)}(k) - R^{(\nu)}(k) &= -R^{(\nu)}(k) v_\varepsilon U_\varepsilon \big(1 + \lambda(\varepsilon)\varepsilon^{1-\gamma} u R^{(\nu')}(\varepsilon k) v\big)^{-1} U_\varepsilon^* u_\varepsilon R^{(\nu)}(k) \\ &= -A_\varepsilon(k)\big(\varepsilon^\gamma + B_\varepsilon(k)\big)^{-1} C_\varepsilon(k), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} A_\varepsilon(k) &:= \varepsilon^{\gamma/2} R^{(\nu)}(k) v_\varepsilon U_\varepsilon, \\ B_\varepsilon(k) &:= \lambda(\varepsilon)\varepsilon u R^{(\nu')}(\varepsilon k) v, \\ C_\varepsilon(k) &:= \varepsilon^{\gamma/2} U_\varepsilon^* u_\varepsilon R^{(\nu)}(k) \end{aligned} \tag{3.13}$$

and $k^2 \in \rho(-\Delta_\nu + V_\varepsilon) \cap \rho(-\Delta_\nu)$, $\mathfrak{Im}\, k > 0$. (The dependence on ν of the three operators defined in (3.13) is for convenience omitted in the symbols used to denote such operators.)

We shall study the limit $\varepsilon \to 0$ of the Hamiltonian $H_\varepsilon^{(\nu)} = -\Delta_\nu + V_\varepsilon$ in the resolvent sense using the expressions (3.12) and (3.13) for its resolvent.

4. Limit of the operators $A_\varepsilon(k)$, $B_\varepsilon(k)$, and $C_\varepsilon(k)$ as $\varepsilon \to 0$

This is another preparatory section for the proof of Theorem 1.2 and we discuss here some relevant properties of the operators $A_\varepsilon(k)$, $B_\varepsilon(k)$, and $C_\varepsilon(k)$ defined in (3.13) and of their limit as $\varepsilon \to 0$.

As a matter of fact, in order to prove the compactness of these operators and their convergence in ε, one has to require a stronger amount of decrease at infinity for the potential V, as compared with the general requirement $V \in L^p \cap L^\infty(\mathbb{R}^+)$, $p \in [0, 1]$, needed for the well-posedness of the model (Theorem 1.1). We shall content ourselves of making the assumptions of Propositions 4.1 and 4.2 below, having in mind an interaction potential V actually supported around the origin.

Let us start with $A_\varepsilon(k)$ and $C_\varepsilon(k)$ first.

Proposition 4.1. *Let the operators $A_\varepsilon(k)$ and $C_\varepsilon(k)$ be defined as in (3.13) with respect to a real-valued potential.*

(i) *If* $V \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, *then for each* $\varepsilon > 0$ *and each* $k \in \mathbb{C}$, $k^2 \in \rho(-\Delta_\nu + V_\varepsilon) \cap \rho(-\Delta_\nu)$, $\mathfrak{Im}\, k > 0$, *both* $A_\varepsilon(k)$ *and* $C_\varepsilon(k)$ *are Hilbert–Schmidt operators on* $L^2(\mathbb{R}^+)$ *with integral kernel, respectively,*

$$\begin{aligned} A_\varepsilon(k)(x,y) &= \sqrt{\lambda(\varepsilon)}\, R^{(\nu)}(k)(x,\varepsilon y) v(y), \\ C_\varepsilon(k)(x,y) &= \sqrt{\lambda(\varepsilon)}\, u(x) R^{(\nu)}(k)(\varepsilon x, y). \end{aligned} \tag{4.1}$$

(ii) *If* V *is bounded and with compact support, then there exist constants* $\omega_A = \omega_A(k,V)$ *and* $\omega_C = \omega_C(k,V)$ *such that*

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \big\| A_\varepsilon(k) - A(k) \big\|_{\mathrm{H.S.}} &= \omega_A, \\ \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \big\| C_\varepsilon(k) - A(k) \big\|_{\mathrm{H.S.}} &= \omega_C, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} A(k) &:= \left| \frac{e^{\mathrm{i}kx}}{\tan\nu - \mathrm{i}k} \right\rangle \langle v|, \\ C(k) &:= |u\rangle \left\langle \overline{\frac{e^{\mathrm{i}kx}}{\tan\nu - \mathrm{i}k}} \right|. \end{aligned} \tag{4.3}$$

In particular, $A_\varepsilon(k) \to A(k)$ *and* $C_\varepsilon(k) \to C(k)$ *as* $\varepsilon \to 0$ *in the Hilbert–Schmidt norm.*

Proof. The identities (4.1) follow immediately from

$$\begin{aligned} \big(A_\varepsilon(k) f\big)(x) &= \varepsilon^{\gamma/2} \int_{\mathbb{R}^+} \mathrm{d}y\, R^{(\nu)}(k)(x,y) \sqrt{\frac{\lambda(\varepsilon)}{\varepsilon^{1+\gamma}} \left| V\left(\frac{y}{\varepsilon}\right) \right|}\, \frac{1}{\sqrt{\varepsilon}} f\left(\frac{y}{\varepsilon}\right) \\ &= \int_{\mathbb{R}^+} \mathrm{d}y\, R^{(\nu)}(k)(x,\varepsilon y) \sqrt{\lambda(\varepsilon)}\, v(y) f(y), \qquad f \in L^2\big(\mathbb{R}^+\big), \end{aligned}$$

and from the analogous computation for $C_\varepsilon(k)$, where we used (3.2), (3.10), and (3.13). The fact that $A_\varepsilon(k)$ is a Hilbert–Schmidt operator follows from (4.1) and from the assumption $V \in L^1(\mathbb{R}^+)$, for

$$\iint_{\mathbb{R}^+\times\mathbb{R}^+} e^{-2(\mathfrak{Im}\, k)|x-\varepsilon y|} \big| V(y) \big|\, \mathrm{d}x\, \mathrm{d}y \leqslant \frac{c}{\varepsilon} \big\| e^{-2(\mathfrak{Im}\, k)|\cdot|} \big\|_1 \left\| V\left(\frac{\cdot}{\varepsilon}\right) \right\|_1 \leqslant \frac{c\|V\|_1}{2\,\mathfrak{Im}\, k}$$

(Young's inequality) and hence

$$\begin{aligned} \big\| A_\varepsilon(k) \big\|_{\mathrm{H.S.}}^2 &= \lambda(\varepsilon) \iint_{\mathbb{R}^+\times\mathbb{R}^+} \big| R^{(\nu)}(k)(x,\varepsilon y) \big|^2 \big| V(y) \big|\, \mathrm{d}x\, \mathrm{d}y \\ &\leqslant \frac{\lambda(\varepsilon)}{2|k|^2} \bigg(\iint_{\mathbb{R}^+\times\mathbb{R}^+} e^{-2(\mathfrak{Im}\, k)|x-\varepsilon y|} \big| V(y) \big|\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

$$
+ \left| \frac{\sin \nu + \mathrm{i}k \cos \nu}{\sin \nu - \mathrm{i}k \cos \nu} \right|^2 \iint_{\mathbb{R}^+ \times \mathbb{R}^+} e^{-2(\Im\mathrm{m}\, k)(x+\varepsilon y)} \left| V(y) \right| \mathrm{d}x \, \mathrm{d}y \Bigg)
$$

$$
\leqslant c(k,\nu)\lambda(\varepsilon) \|V\|_1
$$

for some constant $c(k,\nu)$, finite for any of the chosen k, ν. An analogous estimate shows that $C_\varepsilon(k)$ too is a Hilbert–Schmidt operator, and thus part (i) is proved. As for part (ii), first of all we observe from the definition (4.3) that $A(k) = C(k) = \mathbb{O}$ in case that Dirichlet boundary conditions $\nu = \frac{\pi}{2}$ are assumed in (3.13), whereas for all other boundary conditions $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $A(k)$ and $C(k)$ are rank-one operators on $L^2(\mathbb{R}^+)$. Using (2.6) we re-write

$$
R^{(\nu)}(k)(x,0) = R^{(\nu)}(k)(0,x) = (\tan \nu - \mathrm{i}k)^{-1} e^{\mathrm{i}kx}. \tag{4.4}
$$

Using (4.4) above, and again (4.1), we find

$$
\begin{aligned}
&\varepsilon^{-2} \left\| A_\varepsilon(k) - A(k) \right\|^2_{\mathrm{H.S.}} \\
&\quad = \varepsilon^{-2} \iint_{\mathbb{R}^+ \times \mathbb{R}^+} \left| \sqrt{\lambda(\varepsilon)} R^{(\nu)}(k)(x,\varepsilon y) - R^{(\nu)}(k)(x,0) \right|^2 \left| V(y) \right| \mathrm{d}x \, \mathrm{d}y \\
&\quad = \frac{\lambda(\varepsilon)}{2|k|^2} \iint_{\mathbb{R}^+ \times \mathbb{R}^+} \mathrm{d}x \, \mathrm{d}y \left| \frac{e^{\mathrm{i}k|x-\varepsilon y|} - e^{\mathrm{i}kx}}{\varepsilon} - d_{k,\nu} \frac{e^{\mathrm{i}k(x+\varepsilon y)} - e^{\mathrm{i}kx}}{\varepsilon} \right|^2 \left| V(y) \right| + O(\varepsilon)
\end{aligned}
$$

with $d_{k,\nu} := (\sin \nu + \mathrm{i}k \cos \nu)(\sin \nu - \mathrm{i}k \cos \nu)^{-1}$. One has

$$
\begin{aligned}
&\iint_{\mathbb{R}^+ \times \mathbb{R}^+} \mathrm{d}x \, \mathrm{d}y \left| \frac{e^{\mathrm{i}k(x+\varepsilon y)} - e^{\mathrm{i}kx}}{\varepsilon} \right|^2 \left| V(y) \right| \\
&\quad = \iint_{\mathbb{R}^+ \times \mathbb{R}^+} \mathrm{d}x \, \mathrm{d}y e^{-2(\Im\mathrm{m}\, k)x} \left| \frac{e^{\mathrm{i}k\varepsilon y} - 1}{\mathrm{i}k\varepsilon y} \right|^2 |k|^2 |y|^2 \left| V(y) \right| \\
&\quad \xrightarrow{\varepsilon \to 0} |k|^2 \int_{\mathbb{R}^+} \mathrm{d}x e^{-2(\Im\mathrm{m}\, k)x} \int_{\mathbb{R}^+} \mathrm{d}y |y|^2 \left| V(y) \right|,
\end{aligned}
$$

where the argument is the following: the y-integration ranges only on the (finite) support of V, on every finite disk of the complex plane the function $z \mapsto (e^z - 1)/z$ is bounded and converges to 1 as $z \to 0$, hence the limit follows by dominated convergence. Analogously,

$$
\begin{aligned}
&\iint_{\mathbb{R}^+ \times \mathbb{R}^+} \mathrm{d}x \, \mathrm{d}y \left| \frac{e^{\mathrm{i}k|x-\varepsilon y|} - e^{\mathrm{i}kx}}{\varepsilon} \right|^2 \left| V(y) \right| \\
&\quad = \iint_{\mathbb{R}^+ \times \mathbb{R}^+} \mathrm{d}x \, \mathrm{d}y e^{-2(\Im\mathrm{m}\, k)x} \left| \frac{e^{\mathrm{i}k(|x-\varepsilon y|-x)} - 1}{\mathrm{i}k(|x-\varepsilon y|-x)} \right|^2 |k|^2 \left| \frac{|x-\varepsilon y| - x}{\varepsilon} \right|^2 \left| V(y) \right| \\
&\quad \xrightarrow{\varepsilon \to 0} |k|^2 \int_{\mathbb{R}^+} \mathrm{d}x e^{-2(\Im\mathrm{m}\, k)x} \int_{\mathbb{R}^+} \mathrm{d}y |y|^2 \left| V(y) \right|.
\end{aligned}
$$

Combining the last two limits together one can easily argue that

$$\varepsilon^{-2}\big\|A_\varepsilon(k)-A(k)\big\|^2_{\mathrm{H.S.}}\xrightarrow{\varepsilon\to 0}\text{const.}$$

the constant depending only on k and V. The quantity $\varepsilon^{-2}\|C_\varepsilon(k)-C(k)\|^2_{\mathrm{H.S.}}$ can be treated in the same way, which leads to (4.2). □

We now turn to the analysis of the operators $B_\varepsilon(k)$, $\varepsilon>0$.

Proposition 4.2. *Let the operator $B_\varepsilon(k)$ be defined as in (3.13) with respect to a real-valued potential V.*

(i) *If $V\in L^p(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$, $p\in[1,2]$, then for each $\varepsilon>0$ and each $k\in\mathbb{C}$, $k^2\in\rho(-\Delta_\nu+V_\varepsilon)\cap\rho(-\Delta_\nu)$, $\mathfrak{Im}\,k>0$, the operator $B_\varepsilon(k)$ is Hilbert–Schmidt on $L^2(\mathbb{R}^+)$ with integral kernel*

$$B_\varepsilon(k)(x,y)=\lambda(\varepsilon)u(x)R^{(\nu)}(k)(\varepsilon x,\varepsilon y)v(y). \tag{4.5}$$

(ii) *If V is bounded an with compact support, then there exists a constant $\omega_B=\omega_B(k,V)$ such that*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big\|B_\varepsilon(k)-B(k)\big\|_{\mathrm{H.S.}}=\omega_B, \tag{4.6}$$

where

$$B(k):=(\tan\nu-\mathrm{i}k)^{-1}|u\rangle\langle v|. \tag{4.7}$$

Proof. From (2.6) and (3.7) one has

$$\begin{aligned}\varepsilon R^{(\nu')}(\varepsilon k)(x,y)&=\varepsilon\frac{\mathrm{i}}{2\varepsilon k}\left[e^{\mathrm{i}\varepsilon k|x-y|}-\frac{\sin\nu'+\mathrm{i}\varepsilon k\cos\nu'}{\sin\nu'-\mathrm{i}\varepsilon k\cos\nu'}e^{\mathrm{i}\varepsilon k(x+y)}\right]\\&=\frac{\mathrm{i}}{2k}\left[e^{\mathrm{i}k|\varepsilon x-\varepsilon y|}-\frac{\sin\nu+\mathrm{i}k\cos\nu}{\sin\nu-\mathrm{i}k\cos\nu}e^{\mathrm{i}k(\varepsilon x+\varepsilon y)}\right]\\&=R^{(\nu)}(k)(\varepsilon x,\varepsilon y),\end{aligned}$$

and hence

$$\begin{aligned}\big(B_\varepsilon(k)f\big)(x)&=\lambda(\varepsilon)\varepsilon u(x)\int_{\mathbb{R}^+}R^{(\nu')}(\varepsilon k)(x,y)v(y)f(y)\,\mathrm{d}y\\&=\lambda(\varepsilon)u(x)\int_{\mathbb{R}^+}R^{(\nu)}(k)(\varepsilon x,\varepsilon y)v(y)f(y)\,\mathrm{d}y,\qquad f\in L^2\big(\mathbb{R}^+\big),\end{aligned}$$

which proves (4.5). The proof that, for each $\varepsilon>0$, $B_\varepsilon(k)$ is Hilbert–Schmidt is precisely the same as the proof of Proposition 2.1, estimates (2.7)–(2.10). As for part (ii), by means of (4.5) and (4.7) we have

$$\begin{aligned}
&\varepsilon^{-2}\big\|B_\varepsilon(k)-B(k)\big\|^2_{\mathrm{H.S.}}\\
&\quad=\iint_{\mathbb{R}^+\times\mathbb{R}^+}\big|V(x)\big|\big|\lambda(\varepsilon)R^{(\nu)}(k)(\varepsilon x,\varepsilon y)-(\tan\nu-\mathrm{i}k)^{-1}\big|^2\big|V(y)\big|\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\frac{|\lambda(\varepsilon)|^2}{2|k|^2}\iint_{\mathbb{R}^+\times\mathbb{R}^+}\mathrm{d}x\,\mathrm{d}y\big|V(x)\big|\left|\frac{e^{\mathrm{i}k\varepsilon|x-y|}-1}{\varepsilon}-d_{k,\nu}\frac{e^{\mathrm{i}k\varepsilon(x+y)}-1}{\varepsilon}\right|^2\big|V(y)\big|+O(\varepsilon),
\end{aligned}$$

having set again $d_{k,\nu}:=(\sin\nu+\mathrm{i}k\cos\nu)(\sin\nu-\mathrm{i}k\cos\nu)^{-1}$. Since now V is assumed to be bounded and with compact support, the integration both on x and on y are limited to a finite interval around the origin, thus a dominated convergence argument yields

$$\begin{aligned}
&\iint_{\mathbb{R}^+\times\mathbb{R}^+}\mathrm{d}x\,\mathrm{d}y\big|V(x)\big|\left|\frac{e^{\mathrm{i}k\varepsilon\phi(x,y)}-1}{\varepsilon}\right|^2\big|V(y)\big|\\
&\quad=\iint_{\mathbb{R}^+\times\mathbb{R}^+}\mathrm{d}x\,\mathrm{d}y\big|V(x)\big|\left|\frac{e^{\mathrm{i}k\varepsilon\phi(x,y)}-1}{\mathrm{i}k\varepsilon\phi(x,y)}\right|^2|k|^2\big|\phi(x,y)\big|^2\big|V(y)\big|\\
&\quad\xrightarrow{\varepsilon\to 0}|k|^2\iint_{\mathbb{R}^+\times\mathbb{R}^+}\mathrm{d}x\,\mathrm{d}y\big|V(x)\big|\big|\phi(x,y)\big|^2\big|V(y)\big|
\end{aligned}$$

with $\phi(x,y)=|x-y|$ or $\phi(x,y)=x+y$, from which one can deduce that

$$\varepsilon^{-2}\big\|B_\varepsilon(k)-B(k)\big\|^2_{\mathrm{H.S.}}\xrightarrow{\varepsilon\to 0}\text{const.}$$

the constant depending only on k and V. □

Remark 4.3. Let us emphasize that the limit, as $\varepsilon\to 0$, of $A_\varepsilon(k)$, $B_\varepsilon(k)$, and $C_\varepsilon(k)$ always exists in the Hilbert–Schmidt norm, and it is in all three cases the zero operator if Dirichlet boundary conditions $\nu=\frac{\pi}{2}$ are assumed in (3.13), whereas for all other boundary conditions the limit is a rank-one operator on $L^2(\mathbb{R}^+)$.

Remark 4.4. In the literature of this field (see, e.g., [2] and references therein) it is customary to prove that operators of a form similar to our $A_\varepsilon(k)$, $B_\varepsilon(k)$, and $C_\varepsilon(k)$ have a limit as $\varepsilon\to 0$ in the Hilbert–Schmidt norm by showing (typically again by dominated convergence) that $A_\varepsilon(k)\to A(k)$ weakly in the operator sense and that $\|A_\varepsilon(k)\|_{\mathrm{H.S.}}\to\|A(k)\|_{\mathrm{H.S.}}$: by a general property of compact operators (see, e.g., [17], Theorem 2.21), this is a sufficient condition for the convergence $A_\varepsilon(k)\to A(k)$ to hold in the stronger sense of the Hilbert–Schmidt norm. In Propositions 4.1(ii) and 4.2(ii), instead, we computed the exact leading asymptotics as $\varepsilon\to 0$: this will be needed for the proof of Theorem 1.2 in the case of strong scaling.

5. Convergence results

We come now to the proof of Theorem 1.2. We shall control the limit $\varepsilon \to 0$ in the re-scaled Konno–Kuroda resolvent identity (3.12), that is,

$$R_\varepsilon^{(\nu)}(k) - R^{(\nu)}(k) = -A_\varepsilon(k)\big(\varepsilon^\gamma + B_\varepsilon(k)\big)^{-1} C_\varepsilon(k). \tag{5.1}$$

Here $A_\varepsilon(k)$, $B_\varepsilon(k)$, and $C_\varepsilon(k)$ are the Hilbert–Schmidt operators defined in (3.13).

First case: $\gamma = 0$ (canonical scaling). In this case (5.1) reads

$$R_\varepsilon^{(\nu)}(k) - R^{(\nu)}(k) = -A_\varepsilon(k)\big(\mathbb{1} + B_\varepsilon(k)\big)^{-1} C_\varepsilon(k). \tag{5.2}$$

The Hilbert–Schmidt-norm limit (4.6) proved in Proposition 4.2 implies

$$\mathbb{1} + B_\varepsilon(k) \xrightarrow{\varepsilon\to 0} \mathbb{1} + B(k) \tag{5.3}$$

in the operator norm sense, where $B(k)$ is the rank-one operator defined in (4.7). Whereas the invertibility of $\mathbb{1}+B_\varepsilon(k)$ is part of the proof of the Konno–Kuroda formula itself, the invertibility of $\mathbb{1}+B(k)$ follows by direct inspection. We have indeed:

Lemma 5.1. *Let $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ and a real-valued $V \in L^1(\mathbb{R}^+)$ be given. Correspondingly, let $B(k) = (\tan\nu - \mathrm{i}k)^{-1}|u\rangle\langle v|$ as in (4.7), with $k \in \mathbb{C}$, $k^2 \in \rho(-\Delta_\nu)$, $\mathfrak{Im}\, k > 0$, and with v and u defined in (3.10) with respect to V. Then, with the possible exception of the value $k = \mathrm{i}\beta_0$, if $\beta_0 > 0$ is a solution to*

$$\int_{\mathbb{R}^+} V(x)\,\mathrm{d}x = -(\tan\nu + \beta_0), \tag{5.4}$$

the operator $\mathbb{1} + B(k)$ is invertible (on its range).

Proof. The statement is trivial if $\nu = \frac{\pi}{2}$ (Dirichlet boundary condition in (1.4)), because in this case $B(k) = 0$, so let us proceed with $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2})$. If a non-zero $f \in L^2(\mathbb{R}^+)$ satisfies $B(k)f = -f$, then necessarily

$$f = -(\tan\nu - \mathrm{i}k)^{-1}\langle v, f\rangle u,$$

that is, f is not orthogonal to v and f is a multiple of u. Hence u itself must be an eigenfunction of $B(k)$ relative to the eigenvalue -1, which reads

$$u = -(\tan\nu - \mathrm{i}k)^{-1}\langle v, u\rangle u,$$

whence also

$$\int_{\mathbb{R}^+} V(x)\,\mathrm{d}x = -(\tan\nu - \mathrm{i}k). \tag{5.5}$$

If $\mathfrak{Re}\, k \neq 0$ then (5.5) is never satisfied, for the l.h.s. is real and the r.h.s. is not. If instead $k = \mathrm{i}\beta$ (the admissible β's are such that $\beta > 0$ and $\beta \neq -\tan\nu$ if $\nu < 0$), then (5.5) is only satisfied by that exceptional value of β determined by (5.4). Apart from such an exceptional value, $B(k)$ is therefore injective and thus invertible on its range. □

Thus, modulo the above-mentioned possible exceptional value, (5.3) implies

$$\big(\mathbb{1} + B_\varepsilon(k)\big)^{-1} \xrightarrow{\varepsilon\to 0} \big(\mathbb{1} + B(k)\big)^{-1}$$

in the operator norm sense and this, together with the Hilbert–Schmidt-norm limits (4.2) (Proposition 4.1), gives

$$\begin{aligned} -A_\varepsilon(k)\big(\mathbb{1} + B_\varepsilon(k)\big)^{-1} C_\varepsilon(k) &\xrightarrow{\varepsilon\to 0} -A(k)\big(\mathbb{1} + B(k)\big)^{-1} C(k) \\ &= -\left|\frac{e^{\mathrm{i}kx}}{\tan\nu - \mathrm{i}k}\right\rangle \langle v|\big(\mathbb{1} + B(k)\big)^{-1}|u\rangle \left\langle \overline{\frac{e^{\mathrm{i}kx}}{\tan\nu - \mathrm{i}k}}\right| \end{aligned} \tag{5.6}$$

in the operator norm sense, $A(k)$ and $C(k)$ being the rank-one operators (4.3). Since

$$\frac{\langle v|(\mathbb{1} + B(k))^{-1}|u\rangle}{(\tan\nu - \mathrm{i}k)^2} = \frac{\int_{\mathbb{R}^+} V}{(\tan\nu - \mathrm{i}k)(\tan\nu - \mathrm{i}k + \int_{\mathbb{R}^+} V)} = \Theta_{\nu,V,k}(\gamma)$$

(the second identity is definition (1.14)), then (5.6) reads

$$-A_\varepsilon(k)\big(\mathbb{1} + B_\varepsilon(k)\big)^{-1} C_\varepsilon(k) \xrightarrow{\varepsilon\to 0} -\Theta_{\nu,V,k}(\gamma)\big|e^{\mathrm{i}kx}\big\rangle\big\langle\overline{e^{\mathrm{i}kx}}\big|. \tag{5.7}$$

This, together with (5.2), gives precisely (1.13).

Second case: $\gamma < 0$ (weak scaling). In this case (5.1) reads

$$R_\varepsilon^{(\nu)}(k) - R^{(\nu)}(k) = -\varepsilon^{-\gamma} A_\varepsilon(k)\big(\mathbb{1} + \varepsilon^{-\gamma} B_\varepsilon(k)\big)^{-1} C_\varepsilon(k). \tag{5.8}$$

The Hilbert–Schmidt-norm limit (4.6) (Proposition 4.2) implies

$$\mathbb{1} + \varepsilon^{-\gamma} B_\varepsilon(k) \xrightarrow{\varepsilon\to 0} \mathbb{1},$$

whence also

$$\big(\mathbb{1} + \varepsilon^{-\gamma} B_\varepsilon(k)\big)^{-1} \xrightarrow{\varepsilon\to 0} \mathbb{1},$$

in the norm operator sense. This, together with the Hilbert–Schmidt-norm limits (4.2) (Proposition 4.1), and with $A(k)$ and $C(k)$ given by (4.3), yields

$$\begin{aligned} A_\varepsilon(k)\big(\mathbb{1} + \varepsilon^{-\gamma} B_\varepsilon(k)\big)^{-1} C_\varepsilon(k) &\xrightarrow{\varepsilon\to 0} A(k)C(k) \\ &= (\tan\nu - \mathrm{i}k)^{-2}\bigg(\int_{\mathbb{R}^+} V\bigg)\big|e^{\mathrm{i}kx}\big\rangle\big\langle\overline{e^{\mathrm{i}kx}}\big| \end{aligned}$$

in the operator norm sense. Plugging this into the r.h.s. of (5.8) yields

$$R_\varepsilon^{(\nu)}(k) - R^{(\nu)}(k) \xrightarrow{\varepsilon\to 0} 0$$

in the operator norm sense, that is, (1.13) with $\Theta_{\nu,V,k}(\gamma) = 0$.

Third case: $0 < \gamma < 1$ (strong scaling). In this case we re-write (5.1) as

$$\begin{aligned} R_\varepsilon^{(\nu)}(k) - R^{(\nu)}(k) = &-A(k)\big(\varepsilon^\gamma + B_\varepsilon(k)\big)^{-1}C(k) \\ &- A(k)\big(\varepsilon^\gamma + B_\varepsilon(k)\big)^{-1}\big(C_\varepsilon(k) - C(k)\big) \\ &- \big(A_\varepsilon(k) - A(k)\big)\big(\varepsilon^\gamma + B_\varepsilon(k)\big)^{-1}C(k) \\ &- (A_\varepsilon(k) - A(k)\big(\varepsilon^\gamma + B_\varepsilon(k)\big)^{-1}\big(C_\varepsilon(k) - C(k)\big). \end{aligned} \tag{5.9}$$

This allows to see that the first term in the r.h.s. of (5.9) is the leading one, all the others vanishing in operator norm as $\varepsilon \to 0$. To this aim, let us treat first the case in which the operators $A_\varepsilon(k)$, $B_\varepsilon(k)$, and $C_\varepsilon(k)$ are defined in (3.13) with respect to the resolvent of a Hamiltonian $H_\varepsilon^{(\nu)} = -\Delta_\nu + V_\varepsilon$ with boundary conditions at the origin other than the Dirichlet ones, namely $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and therefore their limit as $\varepsilon \to 0$ is a rank-one operator. We observe that

$$\varepsilon^\gamma + B_\varepsilon(k) \xrightarrow{\varepsilon\to 0} B(k) = (\tan\nu - \mathrm{i}k)^{-1}|u\rangle\langle v|$$

in the norm operator sense (owing to (4.6), Proposition 4.2), whence

$$\begin{aligned} &\big(\varepsilon^\gamma + B_\varepsilon(k)\big)u \xrightarrow{\varepsilon\to 0} (\tan\nu - \mathrm{i}k)^{-1}\bigg(\int_{\mathbb{R}^+} V\bigg)u, \\ &\big(\varepsilon^\gamma + B_\varepsilon(k)\big)v \xrightarrow{\varepsilon\to 0} (\tan\nu - \mathrm{i}k)^{-1}\bigg(\int_{\mathbb{R}^+} |V|\bigg)v, \\ &\big(\varepsilon^\gamma + B_\varepsilon(k)\big)v^\perp = \big(\varepsilon^\gamma + B_\varepsilon(k) - B(k)\big)v^\perp \xrightarrow{\varepsilon\to 0} 0 \end{aligned} \tag{5.10}$$

in the L^2-norm sense, if $v^\perp$ is any function orthogonal to v in $L^2(\mathbb{R}^+)$. (Note that $u \perp v$ only if $\int_{\mathbb{R}^+} V = 0$.) Therefore $(\varepsilon^\gamma + B_\varepsilon(k))^{-1}$ has a limit only on the subspace of $L^2(\mathbb{R}^+)$ corresponding to the linear span of u and v. A first consequence of (5.10) is

$$\langle v|\big(\varepsilon^\gamma + B_\varepsilon(k)\big)^{-1}|u\rangle \xrightarrow{\varepsilon\to 0} \tan\nu - \mathrm{i}k \tag{5.11}$$

provided that $\int_{\mathbb{R}^+} V \neq 0$. Thus,

$$\begin{aligned} &\big\|A(k)\big(\varepsilon^\gamma + B_\varepsilon(k)\big)^{-1}C(k) - (\tan\nu - \mathrm{i}k)^{-1}\big|e^{\mathrm{i}kx}\big\rangle\big\langle e^{\mathrm{i}kx}\big|\big\| \\ &\quad = \big|(\tan\nu - \mathrm{i}k)^{-2}\langle v|\big(\varepsilon^\gamma + B_\varepsilon(k)\big)^{-1}|u\rangle - (\tan\nu - \mathrm{i}k)^{-1}\big| \xrightarrow{\varepsilon\to 0} 0, \end{aligned}$$

and hence

$$A(k)\big(\varepsilon^{\gamma}+B_{\varepsilon}(k)\big)^{-1}C(k)\xrightarrow{\varepsilon\to 0}(\tan\nu-\mathrm{i}k)^{-1}\big|e^{\mathrm{i}kx}\big\rangle\big\langle e^{\mathrm{i}kx}\big| \tag{5.12}$$

in the norm operator sense, provided that $\int_{\mathbb{R}^+}V\neq 0$. As a second consequence of (5.10) and of the quantitative rate of convergence

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big\|B_{\varepsilon}(k)-B(k)\big\|_{\mathrm{H.S.}}=\omega_B$$

(see (4.6), Proposition 4.2), one also has

$$\varepsilon^{-\gamma}\big(\varepsilon^{\gamma}+B_{\varepsilon}(k)\big)v^{\perp}=\big(\mathbb{1}+\varepsilon^{-\gamma}\big(B_{\varepsilon}(k)-B(k)\big)\big)v^{\perp}\xrightarrow{\varepsilon\to 0}v^{\perp}$$

in the L^2-norm (here it is crucial that $\gamma<1$, for it guarantees that $\|\varepsilon^{-\gamma}(B_{\varepsilon}(k)-B(k))\|\leqslant\text{const.}\cdot\varepsilon^{1-\gamma}\to 0$ as $\varepsilon\to 0$). Thus,

$$\begin{aligned}&\big(\varepsilon^{\gamma}+B_{\varepsilon}(k)\big)^{-1}v\xrightarrow{\varepsilon\to 0}(\tan\nu-\mathrm{i}k)\bigg(\int_{\mathbb{R}^+}|V|\bigg)^{-1}v,\\&\varepsilon^{\gamma}\big(\varepsilon^{\gamma}+B_{\varepsilon}(k)\big)^{-1}v^{\perp}\xrightarrow{\varepsilon\to 0}v^{\perp}\end{aligned} \tag{5.13}$$

in the L^2-norm, and hence

$$\big\|\big(\varepsilon^{\gamma}+B_{\varepsilon}(k)\big)^{-1}\big\|\leqslant c\varepsilon^{-\gamma} \tag{5.14}$$

for some constant $c>0$. We also have

$$\big\|A_{\varepsilon}(k)-A(k)\big\|\leqslant c\varepsilon\quad\text{and}\quad\big\|C_{\varepsilon}(k)-C(k)\big\|\leqslant c\varepsilon \tag{5.15}$$

(owing to (4.2), Proposition 4.1), and by means of (5.14), (5.15) we see that the other terms in the r.h.s. of (5.9) vanish in norm as $\varepsilon\to 0$ (where we used again the crucial restriction $\gamma<1$). From this and from (5.12) above we conclude that the limit $\varepsilon\to 0$ in (5.9) is

$$R_{\varepsilon}^{(\nu)}(k)-R^{(\nu)}(k)\xrightarrow{\varepsilon\to 0}-(\tan\nu-\mathrm{i}k)^{-1}\big|e^{\mathrm{i}kx}\big\rangle\big\langle\overline{e^{\mathrm{i}kx}}\big|$$

in the operator norm sense, that is, (1.13) with $\Theta_{\nu,V,k}(\gamma)=\tan\nu-\mathrm{i}k$. The case in which a boundary condition $\nu=\frac{\pi}{2}$ is taken in the definition of (3.13) for $A_{\varepsilon}(k)$, $B_{\varepsilon}(k)$, and $C_{\varepsilon}(k)$, is treated through the same argument as above, with the further simplification that now $A(k)=B(k)=C(k)=\mathbb{O}$. There is no splitting in (5.9), and the bounds (5.14), (5.15) applied to $A_{\varepsilon}(k)(\varepsilon^{\gamma}+B_{\varepsilon}(k))^{-1}C_{\varepsilon}(k)$ give immediately the conclusion

$$R_{\varepsilon}^{(\nu)}(k)-R^{(\nu)}(k)\xrightarrow{\varepsilon\to 0}\mathbb{O}$$

in the operator norm sense, that is, (1.13) in the case $\nu=\frac{\pi}{2}$.

6. Analysis of the limit

We collect in this section the proofs of the propositions and of the theorem stated in Section 1.3 of the Introduction.

Proof of Proposition 1.3. Let us exclude the trivial cases when $\Theta_{\nu,V,k} = 0$ in (1.14). Had $\mathcal{R}^{(\nu)}_{\mathrm{lim}}(k) = (-\Delta_\nu - k^2)^{-1} + \Theta_{\nu,V,k}(\gamma)|e^{ikx}\rangle\langle\overline{e^{ikx}}|$ given by (1.13) a non-trivial kernel, say, $\mathcal{R}^{(\nu)}_{\mathrm{lim}}(k) f_0 = 0$ for some non-zero $f_0 \in L^2(\mathbb{R}^+)$, then f_0 could not be orthogonal to $\overline{e^{ikx}}$ and $\mathcal{R}^{(\nu)}(k) f_0 = 0$ would read

$$e^{ikx} = -\frac{1}{\Theta_{\nu,V,k}\langle\overline{e^{ikx}}, f_0\rangle}\big(-\Delta_\nu - k^2\big)^{-1}, \qquad f_0 \in \mathcal{D}(-\Delta_\nu),$$

which is contradicted by the fact that e^{ikx} does not satisfy (1.4) and hence does not belong to $\mathcal{D}(-\Delta_\nu)$. Thus, $\mathcal{R}^{(\nu)}(k)$ is invertible on its range. The rest of the conclusion then follows at once by a known theorem of Kato [9, Chapter VIII, Theorem 1.3]. $\square$

Proof of Proposition 1.5. The conclusion on T_ν is obvious once $r^{(\nu)}(\overline{z}) = r^{(\nu)}(z)^*$ is established. Theorem 1.2 states that $r^{(\nu)}_\varepsilon(z) \xrightarrow{\varepsilon\to 0} r^{(\nu)}(z)$ as a norm-limit of bounded and everywhere defined operators on $L^2(\mathbb{R}^+)$, where $z = k^2$, $r^{(\nu)}_\varepsilon(z) := (-\Delta_\nu + V_\varepsilon - k^2)^{-1}$, and $r^{(\nu)}(z) = \lim_{\varepsilon\to 0}(-\Delta_\nu + V_\varepsilon - k^2)^{-1}$. This also implies $r^{(\nu)}_\varepsilon(z)^* \xrightarrow{\varepsilon\to 0} r^{(\nu)}(z)^*$. Moreover, for $\varepsilon > 0$ one obviously has $r^{(\nu)}_\varepsilon(\overline{z}) = r^{(\nu)}_\varepsilon(z)^*$ because these are resolvents of self-adjoint operators, owing to Theorem 1.1. Therefore, $r^{(\nu)}_\varepsilon(z)^* = r^{(\nu)}_\varepsilon(\overline{z}) \xrightarrow{\varepsilon\to 0} r^{(\nu)}(\overline{z})$, whence necessarily $r^{(\nu)}(z)^* = r^{(\nu)}(\overline{z})$. Alternatively, we can check the latter identity by direct inspection on $r^{(\nu)}(z)$, using the explicit formulas (1.13)–(1.14). $\square$

Proof of Proposition 1.6. For any admissible k as specified in the assumptions of Theorem 1.2, it is clear that $e^{ikx} \in \ker(-\Delta^*_{(0)} - k^2)$, therefore from (1.13) one immediately deduces

$$\big(-\Delta^*_{(0)} - k^2\big)\big(T_\nu - k^2\big) = \mathbb{1}$$

as an identity on the whole $L^2[0,+\infty)$. Thus, for any $f \in \mathcal{D}(T_\nu)$, say, $f = (T_\nu - k^2)^{-1}u$ for some $u \in L^2[0,+\infty)$, one has $(-\Delta^*_{(0)} - k^2)(T_\nu - k^2)^{-1}u = u$, whence $f \in \mathcal{D}(-\Delta^*_{(0)})$ and $(-\Delta^*_{(0)} - k^2)f = (T_\nu - k^2)f$. This means that $T \subset -\Delta^*_{(0)}$, which implies in turn that $T_\nu = T_\nu^* \supset -\Delta^{**}_{(0)} = \overline{-\Delta_{(0)}}$. We conclude that T_ν must be a self-adjoint extension of $-\Delta_{(0)}$. $\square$

Proof of Theorem 1.7. In Eq. (1.16) the case of initial Dirichlet boundary condition with any scaling, the case $\gamma < 0$, and the case $\gamma = 0$ with $\int_{\mathbb{R}^+} V = 0$, all follow immediately from Eqs (1.13)–(1.14) in Theorem 1.2. One is thus left to prove the last two lines of (1.16). Let us first consider the case $\gamma \in (0,1)$, $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\int_{\mathbb{R}^+} V \neq 0$ ($\nu = \frac{\pi}{2}$ has been already discussed). In this case, for all admissible k's, namely $\forall k \in \widetilde{K}$ with

$$\widetilde{K} := \left\{ k \in \mathbb{C} \,\middle|\, \begin{array}{l} \mathfrak{Im}\, k > 0 \\ k \neq -\mathrm{i}\tan\nu \text{ if } \nu < 0 \end{array} \right\},$$

we find from formulas (1.13)–(1.14) for the limit resolvent $\mathcal{R}^{(\nu)}_{\mathrm{lim}}(k)$ and from the expression (2.6) for the integral kernel of $R^{(\nu)}(k) = (-\Delta_\nu - k^2)^{-1}$, the integral kernel

$$\begin{aligned}\mathcal{R}^{(\nu)}_{\mathrm{lim}}(k)(x,y) &= \frac{\mathrm{i}}{2k}\left(e^{\mathrm{i}k|x-y|} - \frac{\tan\nu + \mathrm{i}k}{\tan\nu - \mathrm{i}k}e^{\mathrm{i}k(x+y)}\right) - \frac{e^{\mathrm{i}k(x+y)}}{\tan\nu - \mathrm{i}k}\\ &= \frac{\mathrm{i}}{2k}\big(e^{\mathrm{i}k|x-y|} - e^{\mathrm{i}k(x+y)}\big),\end{aligned} \tag{6.1}$$

which is precisely the kernel of $(-\Delta_{\mathrm{D}} - k^2)^{-1}$. The last case to consider is $\gamma = 0$, $\nu \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\int_{\mathbb{R}^+} V \neq 0$, in which the admissible k's run in the set

$$K := \left\{ k \in \mathbb{C} \;\middle|\; \begin{array}{l} \mathfrak{Im}\,k > 0\\ k \neq -\mathrm{i}\tan\nu \text{ if } \nu < 0\\ k \neq \mathrm{i}\beta_0 \text{ if } \beta_0 > 0 \text{ is a solution to (1.12)}\end{array}\right\}.$$

Proposition (1.6) guarantees that $-\Delta_\nu + V_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta_{\widetilde{\nu}}$ in norm resolvent sense, and therefore

$$\mathcal{R}^{(\nu)}_{\mathrm{lim}}(k) = R^{(\widetilde{\nu})}(k) = \big(-\Delta_{\widetilde{\nu}} - k^2\big)^{-1} \quad \forall k \in K. \tag{6.2}$$

The goal now is to determine $\widetilde{\nu}$ that satisfies (6.2) for given ν. Both sides of (6.2) are analytic in k^2 (see Remark 1.4) so it is enough to exploit (6.2) with the (admissible) real k^2's, that is, for $k = \mathrm{i}\beta$, $\beta > 0$, with two possible exceptional values for β. In terms of the corresponding integral kernels (obtained again by (1.13)–(1.14) and by (2.6)), (6.2) then reads

$$\begin{aligned}&\frac{1}{2\beta}\left[e^{-\beta|x-y|} - \frac{\tan\nu - \beta}{\tan\nu + \beta}e^{-\beta(x+y)}\right] - \frac{\int_{\mathbb{R}^+} V}{(\tan\nu + \beta)(\tan\nu + \beta + \int_{\mathbb{R}^+} V)}e^{-\beta x}e^{-\beta x}\\ &\quad = \frac{1}{2\beta}\left[e^{-\beta|x-y|} - \frac{\tan\widetilde{\nu} - \beta}{\tan\widetilde{\nu} + \beta}e^{-\beta(x+y)}\right],\end{aligned}$$

for a.e. $x, y \in [0, +\infty)$, whence

$$-\frac{1}{2\beta}\frac{\tan\nu - \beta}{\tan\nu + \beta} - \frac{\int_{\mathbb{R}^+} V}{(\tan\nu + \beta)(\tan\nu + \beta + \int_{\mathbb{R}^+} V)} = -\frac{1}{2\beta}\frac{\tan\widetilde{\nu} - \beta}{\tan\widetilde{\nu} + \beta}. \tag{6.3}$$

It is easily checked that (6.3) is solved, *for all considered* β*'s*, by a unique $\widetilde{\nu} \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\tan\widetilde{\nu} = \tan\nu + \int_{\mathbb{R}^+} V$. This completes the last of the four cases of (1.16) and (1.17) as well. $\square$

Appendix. Point interaction on the straight line

We summarise in this appendix the main features of the point interaction model on a straight line which we have been referring to in this work.

Theorem A.1. *All self-adjoint extensions of the operator* $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ *on* $L^2(\mathbb{R})$ *with domain* $C_0^\infty(\mathbb{R}\setminus\{0\})$ *are given by the family* $\{-\Delta_\alpha \mid \alpha \in (-\infty, +\infty]\}$ *with*

$$-\Delta_\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}x^2},$$
$$\mathcal{D}(-\Delta_\alpha) = \{f \in W^{2,1}(\mathbb{R}) \cap H^2(\mathbb{R}\setminus\{0\}) \mid f'(0^+) - f'(0^-) = \alpha f(0)\}.$$

In the special case $\alpha = 0$, $-\Delta_0$ *is the negative Laplacian* $-\Delta$ *on* $L^2(\mathbb{R})$ *with domain of self-adjointness* $H^2(\mathbb{R})$. *The resolvent of* $-\Delta_\alpha$ *is given by*

$$(-\Delta_\alpha - k^2)^{-1} = (-\Delta - k^2)^{-1} - \frac{2\alpha k}{\mathrm{i}\alpha + 2k} |G_k\rangle\langle\overline{G_k}|$$

for $k^2 \in \rho(-\Delta_\alpha)$, $\mathfrak{Im}\, k > 0$, *where* $G_k(x) := \frac{\mathrm{i}}{2k} e^{\mathrm{i}k|x|}$, *i.e.,* $G_k(x-y)$ *is the integral kernel of* $(-\Delta - k^2)^{-1}$ *in* $L^2(\mathbb{R})$.

Theorem A.1 is part of an extensive, classical literature on the matter. We refer to Theorems I.3.1.1 and I.3.1.2 of [2] for the present formulation.

Theorem A.2. *Suppose* $V \in L^1(\mathbb{R})$ *is real-valued and* $\lambda : \mathbb{R} \to \mathbb{R}$ *is analytic around zero. For each* $\varepsilon > 0$ *consider the self-adjoint Schrödinger operator* $H_\varepsilon := -\Delta + \lambda(\varepsilon)\varepsilon^{-1} V(x/\varepsilon)$, *defined as operator form sum, and let* $k \in \mathbb{C}$, $\mathfrak{Im}\, k > 0$, $k^2 \in \rho(-\Delta_\alpha)$. *Then, for* $\varepsilon > 0$ *small enough, also* $k^2 \in \rho(H_\varepsilon)$ *and* H_ε *converges to* $-\Delta_\alpha$ *in norm-resolvent sense, i.e.,*

$$(H_\varepsilon - k^2)^{-1} \xrightarrow{\varepsilon \to 0} (-\Delta_\alpha - k^2)^{-1},$$

where $\alpha = \lambda(0)(\int_{\mathbb{R}^+} V)$.

Theorem A.2 was first proved in [3]. We refer to Theorem I.3.2.3 of [2] for the present formulation.
We remark that combining Theorems A.1 and A.2 one has the formula

$$(H_\varepsilon - k^2)^{-1} \xrightarrow{\varepsilon \to 0} (-\Delta - k^2)^{-1} - \frac{2\alpha k}{\mathrm{i}\alpha + 2k} |\overline{G_k}\rangle\langle G_k| = (-\Delta_\alpha - k^2)^{-1},$$

which is the analogue of our formula (1.13) in the canonical scaling.

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