Multiple solutions for the scalar curvature problem on the sphere

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ABSTRACT. We study the existence of solutions for the prescribed scalar curvature problem on S^n with $n \geq 3$. We prove that given an arbitrary $K_0 \in C^2(S^n), K_0 > 0$, any positive ϵ, any α in $(0,1)$, and any integer l we can find $\tilde{K} \in C^2(S^n)$ such that $\left\| K_0 - \tilde{K} \right\|_{C^{1,\alpha}} < \epsilon$ and the equation

$$(1) \qquad \begin{cases} \Delta u - \frac{n(n-2)}{2} u + \tilde{K} u^{\frac{n+2}{n-2}} = 0, & \text{on } S^n \\ u > 0, \end{cases}$$

has at least l solutions.

Key Words: Critical exponent, Scalar curvature equation, Minimax method.

AMS subject classification: 35J20, 35J60

1 Introduction

In this paper we prove multiplicity results for the following equation:

$$(P_{\tilde{K}}) \qquad \begin{cases} \Delta u - \frac{n(n-2)}{2} u + \tilde{K} u^{\frac{n+2}{n-2}} = 0 & \text{on } S^n, \\ u > 0, \end{cases}$$

where $\tilde{K} \in C^2(S^n), \tilde{K} > 0, n \geq 3$.

This is motivated by the *prescribed scalar curvature problem* in differential geometry. In fact let (M, g_0) be an n-dimensional Riemmanian manifold and let $\bar{K}_0$ be its scalar curvature. The problem is to find a metric g conformal to g_0 such that the corresponding scalar curvature is a given function $\bar{K} : M \longrightarrow \mathbb{R}$. Letting $g = u^{\frac{4}{n-2}} g_0$, one is led to solve the elliptic equation

$$(2) \qquad \begin{cases} -\frac{4(n-1)}{n-2} \Delta u + \bar{K}_0 u - \bar{K} u^{\frac{n+2}{n-2}} = 0 & \text{on } M, \\ u > 0. \end{cases}$$

If $M = S^n$, g_0 the standard metric and $\bar{K} = \tilde{K}$, the problem becomes just $(P_{\tilde{K}})$ up to an influent constant. For $n = 2$ the analogous problem involves the Gauss curvature of S^2 and it is known as the Nirenberg problem.

Problem $(P_{\tilde{K}})$ can be tackled using variational methods, but in general it presents several difficulties. For example we have a lack of compactness due to the presence of the critical Sobolev exponent. In

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addition, there are more intrinsic obstructions. For example if u is a solution of $(P_{\tilde{K}})$ we have the Kazdan-Warner identity:

$$\int_{S^n} \nabla X_i \cdot \nabla K u^{2^*} = 0 \quad i = 1, .., n+1,$$

where X_i are the restriction of the coordinates of $\mathbb{R}^{n+1}$ to S^n

Hence for $\tilde{K}$ strictly monotone in one direction we do not have any solution. On the other hand for $n \geq 3$ $(P_{\tilde{K}})$ had been widely studied see e.g [1,2,4,5,6,7,8,9,12,15,17,19,20,21,22,23,24,25], and for the Nirenberg problem see e.g [9,10,11,13,14,16,18]. Some existence results have been obtained under some assumptions involving the Laplacian at the critical point of $\tilde{K}$, see [5] for the case $n = 3$, see [21] for the case $n \geq 4$ and see [10] for the case $n = 2$. For example in [5] it is assumed that $\tilde{K}$ is a positive Morse function and $\Delta \tilde{K}(x) \neq 0$ if $\nabla \tilde{K}(x) = 0$, then if $m(x)$ denotes the Morse index of the critical point x, of $\tilde{K}$ then $(P_{\tilde{K}})$ has a solution provided that:

$$\sum_{\nabla \tilde{K}(x)=0, \Delta \tilde{K}(x)<0} (-1)^{m(x)} \neq -1.$$

In this paper we exhibit a large class of functions on S^n which can be realized as scalar curvatures of (S^n, g), where g is conformal to g_0. More precisely we prove the following result:

Theorem 1.1 *Let $K_0 \in C^2(S^n), K_0 > 0$. Then for all $\epsilon > 0$ small, for all $\alpha \in (0,1)$, for all $l \in \mathbb{N}$, there exists $\tilde{K} \in C^2(S^n)$, $||K_0 - \tilde{K}||_{C^{1,\alpha}} < \epsilon$ such that $(P_{\tilde{K}})$ has at least l solutions.*

Remark 1.1 *(a) Theorem 1.1 extends to the case of any integer l the result proved in [21] Corollary 0.16.*
(b) A C^2 density result is false in general. For example it follows from the compactness results in [9,21] (respectively in [9]) that for $n = 3, 4$ (respectively $n = 2$) that we have no solution for $\tilde{K}$, C^2 close to $K_0 = x_{n+1} + 2$.

The main future of the result here is that, even if a given function K_0 cannot be realized as the scalar curvature of a metric g conformal to g_0, nevertheless we can find a function $\tilde{K}$ arbitraly close to K_0 in $C^{1,\alpha}$ which is the scalar curvature of as many conformal metrics to g_0 as we want. To prove Theorem 1.1 we adopt an approach similar to the one used in [8]. First of all, by the regularity result in [3] and the strong maximum principle, smooth solutions of $(P_{\tilde{K}})$ can be found as critical points of the functional $J_{\tilde{K}} : M^+ \subseteq H^1(S^n) \longrightarrow \mathbb{R}$, see Section 2 (with K replaced by $\tilde{K}$). Since $J_{\tilde{K}}$ has in general no global extrema to do so we use a sup-min argument. Given any positive $K_0 \in C^2(S^n)$, we first reduce ourselves to the situation in [8] via a small perturbation of K_0. More precisely we exhibit a function K_1, close to K_0 in $C^{1,\alpha}$ as much as we want, which has two global maxima x_0, x_1. We can also construct K_1 in such a way that it has only one critical point (of saddle type) x_2 for which $K_1(x_2)$ is close to maximum of K_1, and moreover there holds $\Delta K_1(x_2) > 0$.

Then following the procedure in [8] we define the class of admissible paths $\Gamma_{K_1}^{x_0,x_1}$ (see (11) with K replaced by K_1) connecting bubbles highly concentrated near the points x_0, x_1 respectively, see the notation in Section 2.

Considering the coresponding sup-inf value $C_{\Gamma_{K_1}^{x_0,x_1}}$ (see (12)), we can get estimates on it (both from above and from below) in terms of the structure of K_1. Denoting by S the best Sobolev constant for the critical Sobolev embedding, see (7), one finds $S^{-2^*} K_1(x_2) < C_{\Gamma_{K_1}^{x_0,x_1}} < S^{-2^*} \max_{S^n} K_1$.

Letting the admissible paths evolve by the positive gradient flow of J_{K_1}, we either obtain a critical point u_1 such that $S^{-2^*} K_1(x_2) < J_{K_1}(u_1) \leq C_{\Gamma_{K_1}^{x_0,x_1}}$ or, if there is a non compact trajectory, we obtain bubbling at some points of S^n. By an accurate analysis of the flow (derived in [8]) if K_1 satisfies appropriate assumptions, one can prove that the bubbling occurs at a single point x for which $\nabla K_1(x) = 0$ and $\Delta K_1(x) \leq 0$, see Lemma 2.5. By the above estimates on $C_{\Gamma_{K_1}^{x_0,x_1}}$ and the structure of K_1 (see (A1)-(A3) in Section 3), we can rule out the alternative of non compactness, so we get existence of a critical point. Hence setting $\tilde{K} = K_1$ we have that Theorem 1.1 is proved for $l = 1$.

Now to prove Theorem 1.1 for the case $l = 2$ we perturbe K_1 near x_1 in such a way to obtain a new function K_2 with two global maxima x_3, x_4 and a critical point (of saddle type) x_5 with positive Laplacian near x_1. If the next perturbation is small enough in $C^{1,\alpha}$, then the estimates on $C_{\Gamma^{x_0,x_1}_{K_1}}$ will persist for $C_{\Gamma^{x_0,x_1}_{K_2}}$ see Lemma 3.4 and the final part of Section 3. Hence we obtain a critical point u_1 of J_{K_2} such that $S^{-2^*}K_1(x_2) < J_{K_2}(u_1) \le C_{\Gamma^{x_0,x_1}_{K_2}}$. On the other hand we can take advantage of the presence of the new critical points x_3, x_4 of K_2 definding similary as before a new sup-inf value $C_{\Gamma^{x_3,x_4}_{K_2}}$.

Now, choosing K_2 suitably, one can prove that $S^{-2^*}K_2(x_5) > C_{\Gamma^{x_0,x_1}_{K_2}}$. On the other hand repeating the same procedure as in the case $l = 1$ with x_3, x_4, x_5, we obtain a critical point u_2 of J_{K_2} such that $S^{-2^*}K_2(x_5) < J_{K_2}(u_1) \le C_{\Gamma^{x_3,x_4}_{K_2}}$. Hence from $S^{-2^*}K_2(x_5) > C_{\Gamma^{x_0,x_1}_{K_2}}$, we have that the two critical points u_1, u_2 we found are distinct. Hence by setting $\tilde{K} = K_2$ we have that the result is proved for the case $l = 2$.

To prove the case for l arbitrary we proceed by iteration repeating the construction used to bring from K_1 into K_2.

The paper is organized as follows. In Section 2 we collect some useful materials which includes the notation and some properties of the gradient flow. In Section 3 we prove Theorem 1.1 by dividing it into two subsections. Subsection 3.1 describes the general sup-inf scheme employed in [8] and adapted for our purposes, while Subsection 3.2 deals with the multiplicity result.

2 Notations and Preliminaries

Let $Lu = \Delta u - \frac{n(n-2)}{2}u$ be the conformal Laplacian on S^n and for any $u \in H^1(S^n)$, set $\|u\|^2_{-L} = \int_{S^n}(|\nabla u|^2 + \frac{n(n-2)}{2})dv$ (where dv is the volume element of S^n with respect to the standard metric) and let $\langle\cdot,\cdot\rangle_{-L}$ be the corresponding scalar product.

Let $M^+ = \{u \in H^1(S^n) : ||u|||_{-L} = 1, u \ge 0\}$, and given $K \in C^2(S^n)$, $K > 0$, we define $J_K : M^+ \longrightarrow \mathbb{R}$ as follows

$$J_K(u) = \int_{S^n} K u^{2^*} dv, \qquad u \in M^+, \tag{3}$$

where $2^* = \frac{2n}{n-2}$.

We set $K_{max} = \max_{x \in S^n} K(x)$, and given $a > 0, b > 0$ we set also

$$K^a_b = \{x \in S^n : a \le K(x) \le b\}. \tag{4}$$

In the following $K^{'}$ denotes the gradient of K, and ΔK the Laplacian of K.

It is well known that the functional J_K is of class C^1 and by the Lagrange multiplier rule any of its critical point can thus be rescaled so that the resulting function is a soluton of

$$\begin{cases} \Delta u - \frac{n(n-2)}{2}u + Ku^{\frac{n+2}{n-2}} = 0; \\ u \in H^1(S^n), u > 0. \end{cases} \tag{5}$$

For $a \in S^n, \lambda > 0$ let

$$U_{a,\lambda} = c_0 [\frac{\lambda}{\lambda^2 + (1-\lambda^2)cosd(a,x)^2}]^{\frac{n-2}{2}}. \tag{6}$$

Here c_0 is a constant, depending only on the dimension n, such that $\|U_{a,\lambda}\| = 1$ and $d(\cdot,\cdot)$ is the geodesic distance in S^n (with respect to the standard metric). The functions $U_{a,\lambda}$ are the only extremals of the Sobolev embedding:

$$||v||_{-L} - S||v||_{L^{2^*}} \geq 0, \qquad v \in H^1(S^n), \tag{7}$$

where S is the optimal constant.

For a positive integer p and for $\epsilon > 0$ we define $W(p,\epsilon)$ as follows:

$$\begin{aligned} W(p,\epsilon) = \Bigg\{ u \in M^+ : \exists \lambda_1, ..., \lambda_p > 0, a_1, ..., a_p, \text{with } \left\| u - c\sum_{i=1}^{i=p} K(a_i)^{\frac{2-n}{4}} U_{a_i,\lambda_i} \right\|_{-L} < \epsilon \ , \\ \text{where } c = (\sum_{i=1}^{i=p} K(a_i)^{\frac{2-n}{4}})^{-\frac{1}{2}}, \ \lambda_i > \frac{1}{\epsilon}, \ \epsilon_{i,j}^{\frac{2}{2-n}} = \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j d(a_i,a_j) > \frac{1}{\epsilon} \Bigg\}. \end{aligned} \tag{8}$$

For each $u \in W(P,\epsilon)$ we optimize the approximation of u by $\sum_{i=1}^{i=p} \alpha_i U_{a_i,\lambda_i}$ introducing the minimization problem

$$u \in W(p,\epsilon) \min_{a_i \in S^n, \alpha_i > 0 \lambda_i > 0} \left\| u - \sum_{i=1}^{i=p} \alpha_i U_{a_i,\lambda_i} \right\|_{-L}. \tag{M}$$

Lemma 2.1 *For any positive integer p there exists $\epsilon(p)$ such that for all $\epsilon \leq \epsilon(p)$, for all $u \in W(p,\epsilon)$ the minimization problem (M) has a unique solution up to pemutations.*

This minimization problem has already been considered in various papers see e.g [5].
We recall that u_m is a *Palais-Smale* sequence for J_K if $J_K(u_m)$ is bounded and $\nabla J_K(u_m)$ converges to 0. The behaviour of a general Palais-smale sequence is described in the next two lemmas, which are the counterparts of lemmas 1.2 and 1.3 in [8] .

Lemma 2.2 *Let u_m be a Palais-smale sequence for the functional J_K.Then either*

(a) u_m is precompact, or

(b) there exists a subsequence of u_m (that we still denote by u_m), a sequence ϵ_m tending to 0, a positive integer p such that u_m tends weakly to u_0 and $u_m - u_0 \in W(p,\epsilon_m)$.

Moreover let $(a_i)_m$ be the points associated to $u_m - u_0$ via the minimizatiom problem (M)(see above) and $a_i = \lim_m (a_i)_m$. Then one has

$$J_K(u_m) \longrightarrow J_K(u_0) + (\sum_{i=1}^{i=p} K(a_i)^{\frac{2-n}{2}})^{\frac{2}{2-n}} S^{-2^*}.$$

Lemma 2.3 *Let u_m be a Palais-smale sequence for the functional J_K and suppose we have that $\limsup_m J_K(u_m) > S^{-2^*} 2^{\frac{2}{2-n}} K_{max}$. Then either u_m is precompact, or the alternative (b) in the previous theorem holds with $p = 1$.*

The next result states that the Sobolev inequality improves when v is orthogonal to the affine subspace spanned by $U_{a,\lambda}$ and its partial derivatives with respect to its parameters at some point. More precisely we have the following.

Lemma 2.4 *Let $a_0 \in S^n$ and $\lambda_0 \in (0,\infty)$. Then there exists $c>0$ such that for all $v \in H^1(S^n)$ such that:*

$$< v, U_{a_0,\lambda_0} >_{-L} = < v, \frac{\partial U_{a,\lambda}}{\partial \lambda}|_{a_0,\lambda_0} >_{-L} = < v, \frac{\partial U_{a,\lambda}}{\partial a}|_{a_0,\lambda_0} >_{-L} = 0,$$

we have

$$\|v\|^2_{-L} - (2^*-1)S^{2^*}\int_{S^n} U_{a_0,\lambda_0} v^2 dv \geq c\,\|v\|^2_{-L}.$$

The functional here is the inverse of the Euler functional employed in [8] so we use a positive gradient flow instead of a negative one and therefore we have the analogue of Lemma 1.4 in [8].

Lemma 2.5 *Let $u_0 \in H^1(S^n)$ and let $u(t)$ be the solution of the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t}(t) = \nabla J_K(u(t)), \\ u(0) = u_0. \end{cases} \tag{9}$$

Suppose there exists an integer p such that for any $\epsilon > 0$ $u(t) \in W(p,\epsilon)$ for t sufficiently large, and assume that $\lim_{t\to+\infty} J_K(u(t)) > S^{-2^*} 2^{\frac{-2}{n-2}} K_{max}$.

Then $p = 1$ and if $a(t)$ denotes the point associated to $u(t)$ via (M), one has that $a(t)$ converges to some point $a \in S^n$ with $K^{'}(a) = 0$ and $\Delta K(a) \leq 0$.

Another useful proprety of the flow is given in the following result which is the counterpart of Lemma 1.5 in [8].

Lemma 2.6 *Let $u_0 \in H^1(S^n)$ and let $u(t)$ be the solution of the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t}(t) = \nabla J_K(u(t)), \\ u(0) = u_0, \end{cases} \tag{10}$$

then $\lim_{t\to+\infty} \|\nabla J_K(u(t))\|_{-L} = 0$.

3 Proof of Theorem 1.1

We begin this section by introducing the general supinf scheme which we are going to use in order to prove multiplicity results. We devote the first subsection to describe the existence argument, then in Subsection 3.2 we adapt the argument in order to obtain multiple solutions.

3.1 The supinf argument for existence of solutions

In this subsection we describe the general supinf argument used in order to prove existence results. It is convenient to recall the main steps since here some modifications will be necessary.

The main assumptions on the function K are the following.

$(A1)$ there exists $x_0 \neq x_1$ (strict) non degenerate global maxima for K (with $\Delta K(x_0) < 0, \Delta K(x_1) < 0$).

$(A2)$ there exists $\gamma : (-\infty,+\infty) \longrightarrow S^n$ such that $\lim_{s\to+\infty} \gamma(t) = x_1$, $\lim_{s\to+\infty} \gamma(t) = x_0$ and there exists $x_2 \in S^n$, $s_2 \in (0,1)$ such that $K^{'}(x_2) = 0$, $x_2 = \gamma(s_2)$, $K(x_2) = \min_{s\in[0,1]} K(\gamma(s))$ and x_2 is a nondegenerate critical point of K for which $\Delta K(x_2) > 0$.

$(A3)$ there exists $\eta > 0$ such that $K(x_2) > K_{max} - \eta$ and K has no critical points in $K^{K_{max}}_{K_{max}-\eta}$ except for x_0, x_1, x_2.

Now we introduce the set of admissible class to run the supinf scheme. In the notation below, the point a_s is the one given by the minimization problem (M). Given $x_0, x_1 \in S^n$ as in $(A1)$, we set

$$\Gamma_K^{x_0,x_1} = \{\gamma_s : (-\infty,+\infty) \longrightarrow M^+ \text{ such that } \forall\epsilon > 0\ \exists s_\epsilon \text{ such that } \gamma_s \in W(1,\epsilon) \text{ for } |s| > s_\epsilon \text{ and } a_s \to x_0 \text{ for } s \to -\infty\ (resp\ a_s \to x_1 \text{ for } s \to +\infty)\}; \tag{11}$$

$$C_{\Gamma_K^{x_0,x_1}} = \sup_{\gamma_s \in \Gamma_K^{x_0,x_1}} \inf_{s \in (-\infty,+\infty)} J_K(\gamma_s). \tag{12}$$

Now we have the following two results which can be proved as in Lemmas 2.3 and 2.4 in [8]. We recall the definition of the constant S in (7).

Lemma 3.1 *Assuming (A1) one has* $C_{\Gamma_K^{x_0,x_1}} < S^{-2^*} K_{max}$.

Lemma 3.2 *Assuming (A2) one has* $C_{\Gamma_K^{x_0,x_1}} > S^{-2^*} K(x_2)$.

The next result is based on Lemma 3.1 in [8], observing that to get the alternative in the statement of Lemma 3.3 it is sufficient to consider the presence of possible critical points of J_K in the set $\{S^{-2^*}K(x_2) < J_K \le C_{\Gamma_K^{x_0,x_1}}\}$.

Lemma 3.3 *Assume* $S^{-2^*}K(x_2) < C_{\Gamma_K^{x_0,x_1}} < S^{-2^*}K_{max}$. *Then either there exists* $\bar{u} \in H^1(S^n)$ *such that* $J_K(\phi(t,\bar{u})) \in (S^{-2^*}K(x_2), C_{\Gamma_K^{x_0,x_1}})$ *for all* t*, where* $\phi(\cdot,\bar{u})$ *is the solution of*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = \nabla J_K(u(t)), \\ u(0) = \bar{u}, \end{cases} \tag{13}$$

or there exists a critical point u *of* J_K *such that* $J_K(u) \in (S^{-2^*}K(x_2), C_{\Gamma_K^{x_0,x_1}}]$.

Next we show that the conclusion of *Lemma* 3.3 remains valid if, given K satisfying $(A1), (A2)$, we replace it by a (smooth) function $\tilde{K}$ which is close to K in the uniform topology. Fixing the two points x_0, x_1 corresponding to K, we define the class $\Gamma_{\tilde{K}}^{x_0,x_1}$ and the value $C_{\tilde{K}}^{x_0,x_1}$ as in (11) and (12) replacing K with $\tilde{K}$.

Lemma 3.4 *Let* $K \in C^2(S^n)$ *satisfy* $(A1)$ *and* $(A2)$*, and let* $\tilde{K} \in C^2(S^n)$ *be such that* $\left\|K - \tilde{K}\right\|_\infty < \delta$*. Then if* δ *is small enough then either there exists* $\bar{u} \in H^1(S^n)$ *such that* $J_K(\phi(t,\bar{u})) \in (S^{-2^*}\tilde{K}(x_2), C_{\Gamma_{\tilde{K}}^{x_0,x_1}})$ *for all* t *where* $\phi(\cdot,\bar{u})$ *is the solution of*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = \nabla J_{\tilde{K}}(u(t)), \\ u(0) = \bar{u}, \end{cases} \tag{14}$$

or there exists a critical point u *of* $J_{\tilde{K}}$ *such that* $J_{\tilde{K}}(u) \in (S^{-2^*}\tilde{K}(x_2), C_{\Gamma_{\tilde{K}}^{x_0,x_1}}]$.

PROOF. Since (A1) and (A2) holds for K we have by Lemma 3.1 and Lemma 3.2 $S^{-2^*}K(x_2) < C_{\Gamma_K^{x_0,x_1}} < S^{-2^*}K_{max}$.

On the other hand since $\left\|K - \tilde{K}\right\|_\infty < \delta$, and since the functionals $J_K, J_{\tilde{K}}$ are defined on M^+, by the Sobolev embedding we have

$$|J_K - J_{\tilde{K}}| \le \int_{S^n} |K - \tilde{K}| u^{2^*}\, dv \le \left\|K - \tilde{K}\right\|_\infty \|u\|_{L^{2^*}}^{2^*} < \delta S^{2^*}.$$

Hence for δ small enough we have $S^{-2^*}\tilde{K}(x_2) < C_{\Gamma^{x_0,x_1}_{\tilde{K}}} < S^{-2^*}\tilde{K}_{max}$, so Lemma 3.3 applied to $\tilde{K}$ concludes the proof. ■

The next lemma states that any flow for which the image under J_K remains in a (narrow) strip around $S^{-2^*}K_{max}$ stays compact. More precisely we have the following.

Lemma 3.5 *Let $K \in C^2(S^n), K > 0$, and assume K has no critical points with nonpositive laplacian in the interval K_a^b with $a > K_{max} - \delta$ and $b < K_{max}$. Then if δ is small enough the following property holds. Given $u \in H^1(S^n)$, if $\phi(\cdot, u)$ the solution of the Cauchy problem*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = \nabla J_K(u(t)); \\ u(0) = u, \end{cases} \tag{15}$$

satisfies $J_K(\phi(t,u)) \in (S^{-2^*}a, S^{-2^*}b)$ for all t, then $\phi(t,u)$ is relatively compact.

Proof. The proof is done by contradiction and we divide it into two steps.

Step1: We prove the following claim, namely if compactnes do not holds then $\forall \epsilon > 0$ $\exists t_\epsilon$ such that $\forall t \geq t_\epsilon$ *we have* $\phi(t,u) \in W(1,\epsilon)$.

For this itis sufficient to prove that $\forall\ t_m \to +\infty$ $\exists$ $\epsilon_m \to 0$ such that up to a subsequence $\phi(t_m,u) \in W(1,\epsilon_m)$.

Now let prove this. Since by assumption $J_K(\phi(t_m,u))$ is bounded, we have by Lemma 2.6 that $\phi(t_m,u)$ is a Palais-smale sequence . So by Lemma 2.2 we have that there exists a subsequence of $\phi(t_m,u)$ (that we still denote by $\phi(t_m,u)$), a sequence ϵ_m tending to 0, a positive integer p such that $\phi(t_m,u)$ tends weakly to u_0 and $\phi(t_m,u) - u_0 \in W(p,\epsilon_m)$.

Moreover let $(a_i)_m$ be the points associated to $\phi(t_m,u) - u_0$ via the minimizatiom problem (M) and $a_i = \lim_m (a_i)_m$. Then one has

$$J_K(\phi(t_m,u)) \longrightarrow J_K(u_0) + (\sum_{i=1}^{i=p} K(a_i)^{\frac{2-n}{2}})^{\frac{2}{2-n}} S^{-2^*}.$$

Now see (Lemma 2.2)from $J_K(\phi(t_m,u)) \longrightarrow J_K(u_0) + (\sum_{i=1}^{i=p} K(a_i)^{\frac{2-n}{2}})^{\frac{2}{2-n}} S^{-2^*}$,from the fact that $J_K(u_0) + (\sum_{i=1}^{i=p} K(a_i)^{\frac{2-n}{4}})^{\frac{2}{2-n}} S^{-2^*} \geq J_K(u_0) + K_{max}S^{2^*}$, and since $b < K_{max}$ then we have $u_0 = 0$.

On the other hand from the fact that $J_K(\phi(t_m,u)) \in (S^{-2^*}a, S^{-2^*}b)$, by passing to limit sup we have that $\limsup_{m\to+\infty} J_K(\phi(t_m,u)) \geq S^{-2^*}a > S^{-2^*}(K_{max} - \delta)$, which implies that for δ small enough we have $\limsup_{m\to+\infty} J_K(\phi(t_m,u)) > S^{-2^*}2^{\frac{2}{2-n}}K_{max}$, hence by Lemma 2.3 we have $p = 1$. So the claim is proved.

Step2: we prove the existence of a critical point with nonpositive laplacian of K in K_a^b , hence arriving to a contradiction to our hypothesis.

By assumption we have that $J_K(\phi(t,u))$ belongs to the interval $(S^{-2^*}a, S^{-2^*}b)$, hence we have that

$$\limsup_{t\to+\infty} J_K(\phi(t,u)) \geq S^{-2^*}a > S^{-2^*}(K_{max} - \delta),$$

which implies that for δ small enough we have $\limsup_{t\to+\infty} J_K(\phi(t,u)) > S^{-2^*}2^{\frac{2}{2-n}}K_{max}$. By Step 1 we are under the assumptions of Lemma 2.5 with $p = 1$, hence if $a(t)$ denotes the point associated to $\phi(t,u)$ via (M) we have that $a(t)$ converges to some point $a_1 \in S^n$ with $K^{'}(a_1) = 0$ and $\Delta K(a_1) \leq 0$. But we have $J_K(\phi(t,u)) \to K(a_1)S^{-2^*}$, hence $a_1 \in K_a^b$, so we are done. Hence the lemma is proved. ■

3.2 Proof of Theorem 1.1

The proof is divided into 3 cases.
Case $l = 1$: First we prove the following claim. Given $K_0 \in C^2(S^n), K_0 > 0, \alpha \in (0,1), \epsilon > 0$ there exists $K_1 \in C^2(S^n)$ satisfying (A1),(A2),(A3) such that $\|K_0 - K_1\|_{C^{1,\alpha}} < \epsilon$. The proof is done by explicit construction. Let $P_0 = (1, 0, ..., 0) \in \mathbb{R}^n$,$P_1 = (-1, 0, ..., 0) \in \mathbb{R}^n$, and let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined as

$$f(x) = -|x - P_0|^2 + 8 \qquad g(x) = -|x - P_1|^2 + 8.$$

Let also $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined as

$$h(x) = \max\{0, f(x), g(x)\}.$$

Let $\rho = \rho(|x|)$ be a nonnegative smooth radially decreasing cutoff function with $\int_{\mathbb{R}^n} \rho = 1$, and for $\sigma > 0$ let $\rho_\sigma(x) = \sigma^{-n}\rho(\frac{x}{\sigma})$. It is clear that for σ sufficiently small the function $h_\sigma = \rho_\sigma * h$ has only three critical points in $\{h_\sigma > 0\}$, two of which are strict maxima with negative Laplacian and one saddle point with positive Laplacian.
For simplicity we assume that the maximum of K_0 on S^n is attained at the north pole P_N. Then for $\varpi > 0$, if we set

$$\bar{K}_0 = K_0 + \frac{\varpi}{3} x_{n+1},$$

we have that that P_N is a non degenerate critical point for $\bar{K}_0$. Now let Π be the stereographic projection on S^n with respect to the south pole (P_S) and set

$$K_1 = \bar{K}_0 + \beta h_\sigma\left(\frac{\Pi(x) - \Pi(P)}{\beta^\omega}\right),$$

where $P \in S^n$ is close to P_N. Hence by the non degeneracy of P_N, we have by the implicit function theorem and the properties of h_σ that if β, ϖ are small enough and if the point P is chosen approprately then the function K_1 attains its maximum at exactly two points. Hence with elementary estimates one checks that K_1 verifies the properties (A1),(A2) and (A3) with

$$x_i = \Pi^{-1}(\Pi(P) + \beta^\omega P_i), \ i = 0, 1, \quad x_2 = P.$$

Moreover by straightforward computations we have that

$$\|K_1 - \bar{K}_0\|_{C^{1,\alpha}} \leq \beta^{1-(1+\alpha)\omega}.$$

Hence by taking β and ω small we have

$$\|K_1 - \bar{K}_0\|_{C^{1,\alpha}} \leq \frac{\epsilon}{2},$$

and hence

$$\|K_0 - K_1\|_{C^{1,\alpha}} < \epsilon,$$

if ϖ is chosen small enough.
We prove now Theorem 1.1 with $l = 1$ and $\tilde{K} = K_1$.
In fact since K_1 satisfies (A1), (A2) we have by Lemmas 3.1 and 3.2 that $S^{-2^*} K_1(x_2) < C_{\Gamma_{K_1}^{x_0,x_1}} < S^{-2^*}(K_1)_{max}$. Then by Lemma 3.3 we have eitherthere exist a critical point u_0 of J_{K_1} such that $J_{K_1}(u_0) \in (S^{-2^*}K_1(x_2), C_{\Gamma_{K_1}^{x_0,x_1}}]$, or that there exists $u \in H^1(S^n)$ such that $J_{K_1}(\phi(t,u)) \in (S^{-2^*}K_1(x_2), C_{\Gamma_{K_1}^{x_0,x_1}})$ for all t, where $\phi(\cdot, u)$ is the solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(t) = \nabla J_K(u(t)); \\ u(0) = u. \end{cases} \tag{16}$$

Assuming the second alternative holds, since K_1 verifies (A2) we have $S^{2^*}C_{\Gamma_{K_1}^{x_0,x_1}} < (K_1)_{max}$. On the other hand since K_1 satisfies (A3) we have $K_1(x_2) > (K_1)_{max} - \eta$ and K_1 has no critical point with nonpositive laplacian in $K_{K_1(x_2)}^{S^{2^*}C_{\Gamma_{K_1}^{x_0,x_1}}}$. Hence Lemma 3.5 implies that $\phi(t,u)$ is relatively compact, which gives rise to a critical point u of J_{K_1} such that $J_{K_1}(u) \in (S^{-2^*}K_1(x_2), C_{\Gamma_{K_1}^{x_0,x_1}}]$. So in any case there is a critical point of J_{K_1} in that strip, hence by setting $\tilde{K} = K_1$ the theorem is proved for $l = 1$.

Case $l = 2$:
As in the case $l = 1$ we have that there exists $K_1 \in C^2(S^n)$, satysfying (A1),(A2),(A3), and such that $\|K - K_1\|_{C^{1,\alpha}} < \frac{\epsilon}{2}$. Now, using the construction of the first case it is easy to prove that there exists $K_2 \in C^2(S^n)$ such that $\|K_1 - K_2\|_{C^{1,\alpha}} < \frac{\epsilon}{2}$, $K_2 = K_1$ on S^n except a neighborhood of x_1 and such that K_2 satisfies the following assumptions

$(A^{'}1)$ there exist $x_3 \neq x_4$ strict non degenerate global maxima of K_2 (with $\Delta K_2(x_3) < 0, \Delta K_2(x_4) < 0$)

$(A^{'}2)$ there exists $\tilde{\gamma} : (-\infty, +\infty) \longrightarrow S^n$ such that $\lim_{s\to+\infty} \tilde{\gamma}(t) = x_4$, $\lim_{s\to+\infty} \tilde{\gamma}(t) = x_3$ and there exist $x_5 \in S^n, s_5 \in (0,1)$ such that $K_2^{'}(x_5) = 0$, $x_5 = \tilde{\gamma}(s_5)$, $K(x_5) = \min_{s\in[0,1]} K_2(\tilde{\gamma}(s))$ and x_5 is a nondegenerate critical point of K_2 for which $\Delta K_2(x_5) > 0$.

$(A^{'}3)$ there exists $\tilde{\eta} > 0$ such that $K_2(x_5) > (K_2)_{max} - \tilde{\eta}$, $K_2(x_5) > (K_1)_{max}$ and K_2 has no critical points in $(K_2)_{(K_2)_{max}-\eta}^{(K_2)_{max}}$ except x_4, x_3, x_5.

Since K_2 is close to K_1 in $C^{1,\alpha}$, hence in C^0, we have by Lemma 3.4 that either there exists $u \in H^1(S^n)$ such that $J_K(\phi(t,u)) \in (S^{-2^*}K_2(x_2), C_{\Gamma_{K_2}^{x_0,x_1}})$ for all t where $\phi(\cdot,u)$ is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = \nabla J_{\tilde{K}}(u(t)), \\ u(0) = u, \end{cases} \tag{17}$$

or there exists a critical point u_0 of J_{K_2} such that $J_{K_2}(u_0) \in (S^{-2^*}K_2(x_2), C_{\Gamma_{K_2}^{x_0,x_1}}]$.
Suppose the first alternative holds. Then from the fact that $K_1 = K_2$ in a neighborhood of x_1 we deduce that K_2 has no critical points with nonpositive Laplacian in $(K_2)_{K_2(x_2)}^{S^{2^*}C_{\Gamma_{K_2}^{x_0,x_1}}}$. On the other hand since $K_1 = K_2$ in a neighborhood of x_1, and K_1 verifies (A2),(A3), we have that $K_2(x_2) = K_1(x_2) > (K_1)_{max} - \eta > (K_2)_{max} - \frac{\eta}{2}$ and $S^{2^*}C_{\Gamma_{K_2}^{x_0,x_1}} < (K_2)_{max}$ for ϵ small enough. Then Lemma 3.5 implies that the trajectory $\phi(\cdot,u)$ stays compact so gives a critical point u of J_{K_2} such that $J_{K_2}(u) \in (S^{-2^*}K_2(x_2), C_{\Gamma_{K_2}^{x_0,x_1}}]$. Hence in any case we have a critical point u_1 of J_{K_2} in this strip.
Now applying the same reasoning as in the case $l = 1$ to the path constructed using the points x_3, x_4, we get a critical point u_2 of J_{K_2} such that $J_{K_2}(u) \in (S^{-2^*}K_2(x_5), C_{\Gamma_{K_2}^{x_3,x_4}}]$. Since K_1 verifies (A1) we have $S^{-2^*}(K_1)_{max} > C_{\Gamma_{K_1}^{x_0,x_1}}$. On the other hand repeating the same argument as in the Lemma 3.4 we have for ϵ small enough

$$S^{-2^*}(K_1)_{max} > C_{\Gamma_{K_2}^{x_0,x_1}}.$$

We have also from K_2 verifying $(A^{'}3)$ that $K_2(x_5) > (K_1)_{max}$, hence we obtain $S^{-2^*}K_2(x_5) > C_{\Gamma_{K_2}^{x_0,x_1}}$. Since $J_{K_2}(u_1) \leq C_{\Gamma_{K_2}^{x_0,x_1}}$ and $J_{K_2}(u_2) > S^{-2^*}K_2(x_5)$ the two critical points u_1, u_2 we found are distinct. Hence by setting $\tilde{K} = K_2$ the theorem is proved for $l = 2$.

Case l arbitrary: We proceed by iteration repeating the procedure which transformed K_1 into K_2.

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