GLOBALLY STABLE QUASISTATIC EVOLUTION FOR A COUPLED ELASTOPLASTIC–DAMAGE MODEL

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Abstract. We show the existence of globally stable quasistatic evolutions for a material model with elastoplasticity and incomplete damage, in small strain assumptions. The main feature of our model is that the scalar internal variable which describes the damage affects both the elastic tensor and the plastic yield surface. It is also possible to require that the history of plastic strain up to the current state influences the future evolution of damage.

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Contents

1. Introduction

In this paper we study the problem of quasistatic evolution for a material model with elastoplasticity and incomplete damage, in small strain assumptions. The damage is described by a scalar internal variable, which affects both the elastic tensor and the plastic yield surface.

Models for elastic materials where the bulk energy depends on a scalar damage variable were considered for instance in [17, 25, 24, 4, 32, 15, 31, 20, 21] (without plasticity). In contrast, in the elastoplastic setting of e.g. [7, 8, 9, 10] the plastic dissipation is function of a scalar internal variable, but the elastic tensor is constant.

The present formulation accounts for both these dependences and takes inspiration from [2], where a variational model for complete damage was proposed and certain closed-form solutions were given in dimension one. (See also [1] for a numerical analysis in dimension two.) In particular, the material exhibits a softening behavior: as damage increases, the stiffness decreases and the plastic yield surface shrinks.

In order to prove the existence of globally stable quasistatic evolutions, we follow the so-called energetic approach (cf. e.g. [23] and references therein), as common in the study of plasticity, damage, as well as fracture (see for instance [6, 7], [4, 32, 31], and [12, 13, 11], respectively).

We now briefly describe the problem, formulated in a reference configuration $\Omega \subset \mathbb{R}^n$. The evolution is driven by a time-dependent loading limited for simplicity to a hard device,

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that is by a (sufficiently smooth) displacement $w(t)$ acting on the whole boundary $\partial\Omega$ of the domain during a time interval $[0,T]$. As in other models with small strain elastoplasticity, the linearized strain Eu, defined as the symmetric part of the spatial gradient of the displacement u, is decomposed as the sum $Eu = e + p$, where e and p are the elastic and plastic strains. We assume $p \in \mathbb{M}^{n\times n}_D$, the space of trace free $n \times n$ symmetric matrices, as usual for materials which are insensitive to pressure. The damage variable α takes values in $[0,1]$ (as a function from $[0,T]\times\Omega$); here $\alpha = 1$ stands for no damage and $\alpha = 0$ for maximal damage. We require explicitly an irreversibility condition, namely α is nonincreasing in time.

In the framework of the energetic approach to quasistatic evolutions, we introduce a total energy and a plastic dissipation.

The total energy $\mathcal{E}$ is made up of three terms: a continuous dissipation functional D and a regularizing gradient term, both depending only on α, and the stored elastic energy, quadratic form $\mathcal{Q} = \frac{1}{2}\langle \mathbb{C}(\alpha)e, e\rangle_{L^2}$ of the elastic strain with $e \mapsto \mathbb{C}(\alpha)e : e$ nondecreasing in α for every e and equicoercive (indeed only incomplete damage is considered here). For a technical reason (see Lemma 2.3), it is not sufficient to take a regularizing term of the type $\|\nabla\alpha\|^2_{L^2}$ as done in [2] (where the setting is one dimensional). Thus, we choose $\|\nabla\alpha\|^\gamma_{L^\gamma}$ with $\gamma > n$, a regularization present also in [24] and more recently in [21], for example. Hence,

$$\mathcal{E}(\alpha, e) := \mathcal{Q}(\alpha, e) + D(\alpha) + \|\nabla\alpha\|^\gamma_{L^\gamma}\,.$$

In the abstract scheme of [23] the irreversibility is forced by taking a dissipative term $\mathcal{R}$ positively one-homogeneous and unbounded in the damage variable. We remark that in the present formulation the role of $\mathcal{R}$ is played by D and by the irreversibility condition; in our model we allow also for nonhomogeneous damage dissipations.

In order to define the plastic dissipation, we consider a family $(K(\alpha))_{\alpha\in\mathbb{R}}$ of closed convex subsets of $\mathbb{M}^{n\times n}_D$ such that

$$B_r(0) \subset K(0) \subset K(\alpha_1) \subset K(\alpha_2) \subset K(1) \subset B_R(0)\,,$$

for every $\alpha_1 \le \alpha_2$, with $0 < r < R$, and $\alpha \mapsto \sup_{\sigma\in K(\alpha)} \sigma\colon \xi$ is continuous for every $\xi \in \mathbb{M}^{n\times n}_D$. These sets are called the constraint sets because we will see in Corollary 5.3 that, during the evolution, $\sigma_D(t,x) \in K(\alpha(t,x))$ for every $t \in [0,T]$ and a.e. $x \in \Omega$, σ_D being the deviatoric part of the elastic stress $\sigma := \mathbb{C}(\alpha)e$. Under suitable regularity assumption, the rate of the plastic strain belongs to the normal cone to $K(\alpha(t,x))$ at $\sigma_D(t,x)$ (see Proposition 5.6): this justifies the name of plastic yield surfaces for the boundaries of the constraint sets.

The properties of $\alpha \mapsto K(\alpha)$ imply that the function $H\colon \mathbb{R}\times\mathbb{M}^{n\times n}_D \to \mathbb{R}^+\cup\{0\}$ defined by

$$H(\alpha,\xi) := \sup_{\sigma\in K(\alpha)} \sigma\colon \xi\,,$$

namely such that $\xi \mapsto H(\alpha,\xi)$ is the support function of $K(\alpha)$, is convex and positively one-homogeneous in ξ, and continuous. In some cases, we require also that $\xi \mapsto H(\alpha_2,\xi) - H(\alpha_1,\xi)$ is convex for every $\alpha_1 \le \alpha_2$. As a particular case, we can choose a multiplicative definition for $K(\alpha)$ (see Remark 2.1).

In the energetic formulation of elastoplastic problems it is natural to assume that p belongs to $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, the space of $\mathbb{M}^{n\times n}_D$-valued Borel measures on $\overline{\Omega}$, since the plastic dissipation is one-homogeneous in p. Then, in accordance to the theory of convex functions of measures developed in [18], we define the plastic potential as

$$\mathcal{H}(\alpha,p) := \int_{\overline{\Omega}} H\Big(\alpha(x), \frac{\mathrm{d}p}{\mathrm{d}\mu}(x)\Big)\,\mathrm{d}\mu(x)\,,$$

where $\mu \in M_b(\overline{\Omega})^+$ is any measure such that $p \ll \mu$. In particular it is convex, positively one-homogeneous, and weakly* lower semicontinuous with respect to $p \in M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$.

Given two evolutions of damage and plastic strain $t \mapsto \alpha(t)$ and $t \mapsto p(t)$, the plastic dissipation is then defined as

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;s,t) := \sup_{\mathcal{P}} \mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;s,t), \quad \mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha,p;s,t) := \sum_{j=1}^N \mathcal{H}(\alpha(t_j), p(t_j) - p(t_{j-1}))\,,$$

where $\mathcal{P} = \{t_j\}_{j=0}^N$ is a partition of $[s,t]$, and the supremum is taken over the partitions. Notice that for each subinterval $[t_{i-1}, t_i]$ one considers $\alpha(t_i)$ as a "weight" for this sort of variation. When p is sufficiently smooth in time and $t \mapsto \alpha(x,t)$ is nonincreasing, we can say (see Remark 5.2) that

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; s, t) = \int_s^t \mathcal{H}(\alpha(\tau), \dot{p}(\tau))\,\mathrm{d}\tau\,. \tag{1.1}$$

In Theorem 4.3 we prove an existence result for a quasistatic evolution $t \mapsto (\alpha(t), u(t), e(t), p(t))$ that satisfies the following conditions for a given parameter $\lambda \in [0,1]$:

(qs0) *irreversibility* : for every $x \in \Omega$

$$t \in [0,T] \mapsto \alpha(t,x) \quad \text{is nonincreasing}\,;$$

(qs1) *global stability*: the function $t \mapsto p(t)$ from $[0,T]$ into $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ has bounded variation, $(u(t), e(t), p(t)) \in A(w(t))$ for every $t \in [0,T]$, and

$$\mathcal{E}(\alpha(t), e(t)) + \lambda\mathcal{V}_{\mathcal{H}}(\alpha(t), p; 0, t) \leq \mathcal{E}(\beta, e) + \lambda\mathcal{V}_{\mathcal{H}}(\beta, p; 0, t) + \mathcal{H}(\beta, q - p(t))$$

for every $\beta \leq \alpha(t)$ and $(u, e, q) \in A(w(t))$, where

$$A(w(t)) := \{(u,e,p) | Eu = e + p \text{ in } \Omega, p = (w(t) - u) \odot \nu\, \mathcal{H}^{n-1} \text{ on } \partial\Omega\}\,;$$

(qs2) *energy balance*: for every $t \in [0,T]$

$$\mathcal{E}(\alpha(t), e(t)) + \lambda\mathcal{V}_{\mathcal{H}}(\alpha(t), p; 0, t) + (1-\lambda)\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) = \mathcal{E}(\alpha(0), e(0)) + \int_0^t \langle \sigma(s), E\dot{w}(s)\rangle\,\mathrm{d}s\,,$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$.

Choosing the parameter λ in this formulation, one can account for different interactions between plasticity and damage. Indeed, if $\lambda = 0$ this evolution is the counterpart of the one considered in [6] in the case of perfect plasticity. When $\lambda = 1$ we recover instead the analogous of the evolution proposed in [1] and [2].

We point out that in the latter case the damage process from an instant t is affected by the history of the plastic strain up to t. Let us examine the condition (qs1): due to the term $\lambda\mathcal{V}_{\mathcal{H}}$, at a given instant t it is easier to damage a part of the material more affected by plastic strain's changes until t. Indeed, if the plastic strain is prescribed in a time interval $[0,t]$, in order to minimize $\beta \mapsto \mathcal{V}_{\mathcal{H}}(\beta, p; 0, t)$ it is convenient that β is smaller where p has had more changes up to t (see also Remark 5.2). Tuning between zero and one the parameter λ one can account for different effects of the plasticity on the damage process.

The proof of Theorem 4.3 is based on time discretization and on approximation by means of solutions to incremental minimum problems, following a method common in the study of quasistatic evolutions. As a technical note, we remark just that the monotonicity in time of α and the softening property of $\mathcal{H}$ allow us to prove that $\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha, p; 0, t)$ is indeed nondecreasing with respect to refinements of the partition $\mathcal{P}$ of $[0,t]$ (Lemma A.1). This is crucial in order to pass to the limit in the energy balance, see (4.18), and to recover (1.1) when p is more regular in time.

The structure of the paper is the following: in Section 2 we set the notation, describe the assumptions of the model, and introduce the energy and the dissipation terms with their main properties; Section 3 includes the results needed in order to solve the incremental problems and to assure convergence of the stability properties in the continuous time limit; Section 4 is devoted to prove the existence result; in Section 5 we show qualitative properties of the evolution. Finally, in the Appendix we analyse the particular variation used to define the plastic dissipation and show a property of increasing functions with values in L^p spaces, employed to prove more regularity for the evolution.

2. Preliminaries

Mathematical preliminaries. The Lebesgue measure on $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. The space of bounded X-valued Radon measures on B is denoted by $M_b(B;X)$, for a locally compact subset B of $\mathbb{R}^n$ and a finite dimensional Hilbert space X. The indication of the space X is omitted when $X=\mathbb{R}$. The space $M_b(B;X)$ is endowed with the norm $\|\mu\|_1 := |\mu|(B)$, where $|\mu| \in M_b(B)$ is the variation of the measure μ, and it is identified with the dual of $C_0(B;X)$, the space of continuous functions $\varphi\colon B \to X$ such that $\{|\varphi| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$, by the Riesz Representation Theorem (see, e.g., [27, Theorem 6.19]). The weak* topology of $M_b(B;X)$ is defined by the duality.

The space $L^1(B;X)$ of X-valued $\mathcal{L}^n$-integrable functions is regarded as a subspace of $M_b(B;X)$, with the induced norm. The L^p norm, $1 \leq p \leq \infty$ is denoted by $\|\cdot\|_p$, while the brackets $\langle\cdot,\cdot\rangle$ denote the duality product between conjugate L^p spaces.

The space of *symmetric* $n\times n$ *matrices* is denoted by $\mathbb{M}^{n\times n}_{sym}$; it is endowed with the Euclidean scalar product $\xi\colon\eta := \sum_{ij}\xi_{ij}\eta_{ij}$ and with the corresponding Euclidean norm $|\xi| := (\xi\colon\xi)^{1/2}$. The symbol for the space of trace free matrices in $\mathbb{M}^{n\times n}_{sym}$ is $\mathbb{M}^{n\times n}_{D}$. For every $\xi \in \mathbb{M}^{n\times n}_{sym}$ the orthogonal projection of ξ on $\mathbb{R}I$ is $\frac{1}{n}\mathrm{tr}\,(\xi)I$. Therefore the orthogonal projection on $\mathbb{M}^{n\times n}_{D}$, called the *deviator* of ξ, is

$$\xi_D := \xi - \frac{1}{n}(\mathrm{tr}\,\xi)I\,.$$

The *symmetrized tensor product* $a\odot b$ of two vectors a, $b \in \mathbb{R}^n$ is the symmetric matrix with entries $(a_ib_j + a_jb_i)/2$. If X_1, X_2 are Banach spaces, $Lin(X_1;X_2)$ is the space of linear operators from X_1 into X_2, endowed with the usual operator norm.

For every $u \in L^1(U;\mathbb{R}^n)$, with U open in $\mathbb{R}^n$, let Eu be the $\mathbb{M}^{n\times n}_{sym}$-valued distribution on U whose components are defined by $E_{ij}u = \frac{1}{2}(D_ju_i + D_iu_j)$. The space $BD(U)$ of functions with *bounded deformation* is the space of all $u \in L^1(U;\mathbb{R}^n)$ such that $Eu \in M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space with the norm $\|u\|_1 + \|Eu\|_1$. It is possible to prove that $BD(U)$ is the dual of a normed space (see [30] and [22]), and this defines the weak* topology of $BD(U)$. A sequence u_k converges to u weakly* in $BD(U)$ if and only if $u_k \to u$ strongly in $L^1(U;\mathbb{R}^n)$ and $Eu_k \rightharpoonup Eu$ weakly* in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. If U is a bounded open set with Lipschitz boundary, for every function $u \in BD(U)$ the trace of u on ∂U belongs to $L^1(\partial U;\mathbb{R}^n)$. It will always be denoted by the same symbol u. If u_k, $u \in BD(U)$, $u_k \to u$ strongly in $L^1(U;\mathbb{R}^n)$, and $\|Eu_k\|_1 \to \|Eu\|_1$, then $u_k \to u$ strongly in $L^1(\partial U;\mathbb{R}^n)$ (see [29, Chapter II, Theorem 3.1]). Moreover (see [29, Proposition 2.4 and Remark 2.5]), there exists a constant $C>0$, depending on U, such that

$$\|u\|_{1,U} \leq C\,\|u\|_{1,\partial U} + C\,\|Eu\|_{1,U}\,, \tag{2.1}$$

$\|\cdot\|_{p,B}$ being the L^p norm of a function with domain a Borel set B. For the general properties of $BD(U)$ we refer to [29].

The reference configuration. Throughout the paper the *reference configuration* Ω is a *bounded connected open set* in $\mathbb{R}^n$, $n \geq 2$, with *Lipschitz boundary.* On $\partial\Omega$ we shall prescribe only Dirichlet boundary conditions, to simplify the presentation. This will be done by assigning a function $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, whose trace on $\partial\Omega$ (again denoted by w) is the prescribed boundary value. This choice is motivated by the fact that we do not want to impose "discontinuous" boundary data, so that, if the displacement develops sharp discontinuities, this is a result of energy minimization.

In our problem $u \in BD(\Omega;\mathbb{R}^n)$ represents the *displacement* of an elasto-plastic body and Eu is the corresponding linearized *strain.* We now introduce the coupled elastoplastic damage model. As for modeling plasticity, we follow [6] and use the corrisponding notations.

The elastic and plastic strains. Given a displacement $u \in BD(\Omega;\mathbb{R}^n)$ and a boundary datum $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, the *elastic* and *plastic strains* $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $p \in M_b(\overline{\Omega};\mathbb{M}^{n\times n}_{D})$

satisfy the equations (weak kinematic compatibility conditions)

$$Eu = e + p \quad \text{in } \Omega\,, \tag{2.2a}$$

$$p = (w - u) \odot \nu\, \mathcal{H}^{n-1} \quad \text{on } \partial\Omega\,. \tag{2.2b}$$

Given $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, the *set of admissible displacements and strains* for the boundary datum w on $\partial\Omega$ is defined, with the same meaning and notation of [6], as

$$A(w) := \{(u,e,p) \in BD(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D) |\, (2.2)\,\text{hold}\,\}\,.$$

We shall also use *the space of admissible plastic strains*

$$\Pi(\Omega) := \{p \in M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D) |\, \exists\, (u,w,e) \in BD(\Omega;\mathbb{R}^n) \times H^1(\mathbb{R}^n;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \\ \text{s.t.}\, (u,e,p) \in A(w)\}\,.$$

The damage variable and the associated dissipation. In addition to u, e, and p, we consider an internal variable $\alpha\colon \Omega \to \mathbb{R}$, which represents the damage state of the body. Actually this variable will take values in $[0,1]$ during the evolution. At a given point $x \in \Omega$, as $\alpha(x)$ decreases from 1 to 0, the material point x passes from a sound state to a fully damaged one.

For technical reasons we will introduce in the total energy a regularizing term $\|\nabla\alpha\|_\gamma^\gamma$, with $\gamma > n$, on the damage variable, cf. (2.7). In particular, whenever the enegy is finite the damage variable will be in $W^{1,\gamma}(\Omega) \subset C(\overline{\Omega})$. (Recall that this embedding is compact.) Therefore, in the following we define the other energy terms for $\alpha \in C(\overline{\Omega})$. Notice that the quantities depending on the damage variable are defined also for negative α, in order to consider variations with respect to α in the proof of Proposition 5.4.

We shall denote the *admissible damage states from a given* α_0 by

$$\mathcal{D}(\alpha_0) := \{\alpha \in C(\overline{\Omega}) |\, \alpha \le \alpha_0 \text{ in } \overline{\Omega}\}\,.$$

We assume that the damage process is irreversible, i.e., if α_0 is the current damage state, then the future damage states are in $\mathcal{D}(\alpha_0)$. Let us remark that

$$\mathcal{D}(\alpha_2) \subset \mathcal{D}(\alpha_1) \text{ for every } \alpha_2 \in \mathcal{D}(\alpha_1)\,. \tag{2.3}$$

In the total energy we consider a term which accounts for the energy dissipated by the body during the damage process. The *dissipation term* is defined for every $\alpha \in C(\Omega)$ by

$$D(\alpha) := \int_\Omega d(\alpha(x))\,\mathrm{d}x, \text{ with } d \in C(\mathbb{R};\mathbb{R}^+ \cup \{0\}),\ d(x) > d(0) \text{ for } x < 0\,, \tag{2.4}$$

We do not require that d is nonincreasing or positively one-homogeneous, because it is not needed to prove our result. However, such assumptions would be natural, since D represents a dissipation.

The stored elastic energy. For every $(\alpha, e) \in C(\overline{\Omega}) \times L^2(\Omega,\mathbb{M}^{n\times n}_{sym})$, the *stored elastic energy* is given by

$$\mathcal{Q}(\alpha,e) := \frac{1}{2}\int_\Omega \mathbb{C}(\alpha(x))e(x) : e(x)dx = \frac{1}{2}\langle \mathbb{C}(\alpha)e, e\rangle_{L^2(\Omega;\mathbb{M}^{n\times n}_{sym})}\,.$$

We assume the following properties for the dependence of $\mathbb{C}$ on the damage variable:

$$\mathbb{C}\colon \mathbb{R} \to Lin(\mathbb{M}^{n\times n}_{sym};\mathbb{M}^{n\times n}_{sym}) \text{ is Lipschitz and } \mathbb{C}(\mathbb{R}^-) = \{\mathbb{C}(0)\}\,, \tag{2.5a}$$

$$\alpha \mapsto \mathbb{C}(\alpha)\xi : \xi \quad \text{is nondecreasing for every } \xi \in \mathbb{M}^{n\times n}_{sym}\,, \tag{2.5b}$$

$$\mathbb{C}(\alpha)\xi := \mathbb{C}_D(\alpha)\xi_D + \kappa(\alpha)(\operatorname{tr}\xi)I \text{ with } \mathbb{C}_D \in L^\infty(\mathbb{R};Sym(\mathbb{M}^{n\times n}_D;\mathbb{M}^{n\times n}_D)),\ \kappa \in L^\infty(\mathbb{R}), \tag{2.5c}$$

$$\gamma_1|\xi|^2 \le \mathbb{C}(\alpha)\xi : \xi \le \gamma_2|\xi|^2 \quad \text{for every } \alpha \in \mathbb{R}, \xi \in \mathbb{M}^{n\times n}_{sym}\,, \tag{2.5d}$$

where γ_1, γ_2 are positive constants independent of α and $Sym(\mathbb{M}^{n\times n}_D;\mathbb{M}^{n\times n}_D)$ is the set of symmetric endomorphisms on $\mathbb{M}^{n\times n}_D$. In particular, this implies

$$|\mathbb{C}(\alpha)\xi| \le 2\gamma_2|\xi|\,. \tag{2.6}$$

Assumption (2.5b) is reasonable since in applications the stiffness decreases as the material passes from the sound to the fully damaged state.

It is well known that for a given $\alpha \in C(\overline{\Omega})$ the function $e \mapsto \mathcal{Q}(\alpha, e)$ is weakly lower semicontinuous on $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$.

The total energy. We are now in a position to define the *total energy* of the body corresponding to a damage state α and an elastic strain e.

We set

$$\mathcal{E}(\alpha, e) := \mathcal{Q}(\alpha, e) + D(\alpha) + \|\nabla\alpha\|_\gamma^\gamma\,. \tag{2.7}$$

It is easy to see that $\mathcal{E}$ is lower semicontinuous with respect to the uniform convergence of α_k and the weak*-$L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ convergence of e_k, i.e.,

$$\mathcal{E}(\alpha, e) \leq \liminf_{k\to\infty} \mathcal{E}(\alpha_k, e_k) \tag{2.8}$$

for every sequences α_k and e_k converging to α uniformly in $\overline{\Omega}$ and to e weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, respectively.

The constraint sets and their support functions. Let $(K(\alpha))_{\alpha\in\mathbb{R}}$ be a family of subsets of $\mathbb{M}^{n\times n}_D$ such that:

$$K(\alpha) \text{ is closed and convex, for every } \alpha \in \mathbb{R}\,, \tag{2.9a}$$

$$U \subset \mathbb{M}^{n\times n}_D \text{ open } \Longrightarrow \{\alpha \in \mathbb{R} \,|\, K(\alpha) \cap U \neq \emptyset\} \text{ and } \{\alpha \in \mathbb{R} \,|\, K(\alpha) \subset U\} \text{ open} \tag{2.9b}$$

$$B_r(0) \subset K(0) \subset K(\alpha_1) \subset K(\alpha_2) \subset K(1) \subset B_R(0), \text{ for every } \alpha_1 \leq \alpha_2\,, \tag{2.9c}$$

with $0 < r < R$. In particular we have that $K(\alpha) = K(0)$ for $\alpha \leq 0$ and $K(\alpha) = K(1)$ for $\alpha \geq 1$, When (2.9b) holds we say that the multifunction $\alpha \mapsto K(\alpha)$ is continuous.

The sets above are called the constraint sets because we will see (see Corollary 5.3) that, during the evolution, $\sigma_D(t, x) \in K(\alpha(t, x))$ for every $t \in [0, T]$ and a.e. $x \in \Omega$, $\sigma := \mathbb{C}(\alpha)e$ being the *elastic stress*.

Let us consider the function $H\colon \mathbb{R} \times \mathbb{M}^{n\times n}_D \to \mathbb{R}^+ \cup \{0\}$ defined by

$$H(\alpha, \xi) := \sup_{\sigma\in K(\alpha)} \sigma : \xi \;\text{ for every } \alpha \in \mathbb{R}\,,$$

namely $\xi \mapsto H(\alpha, \xi)$ is the support function of $K(\alpha)$. Arguing as in [28, Proposition 2.4], we can show that (2.9b) implies that

$$\alpha \mapsto H(\alpha, \xi) \text{ is continuous for every } \xi \in \mathbb{M}^{n\times n}_D\,. \tag{2.10}$$

Then we get, from (2.9), that the four conditions below are simultaneously satisfied:

$$H \text{ is continuous}\,, \tag{2.11a}$$

$$\alpha \mapsto H(\alpha, \xi) \text{ is nondecreasing for every } \xi \in \mathbb{M}^{n\times n}_D\,, \tag{2.11b}$$

$$\xi \mapsto H(\alpha, \xi) \text{ is convex and positively one-homogeneous for every } \alpha \in \mathbb{R}\,, \tag{2.11c}$$

$$r|\xi| \leq H(\alpha, \xi) \leq R|\xi| \text{ for every } \alpha \in \mathbb{R} \text{ and every } \xi \in \mathbb{M}^{n\times n}_D\,. \tag{2.11d}$$

Indeed, by [19, Theorem 5], we have that (2.9a) and (2.9c) are equivalent to (2.11b), (2.11c), and (2.11d). Since the functions $\xi \mapsto H(\alpha, \xi)$ are convex with respect to ξ for every α and locally equi-bounded with respect to α by (2.11d), condition (2.10) is equivalent to (2.11a).

In some of the results we will make the additional assumption that

$$\xi \mapsto H(\alpha_2, \xi) - H(\alpha_1, \xi) \text{ is convex, for every } \alpha_1 \leq \alpha_2\,. \tag{2.12}$$

Remark 2.1. Let us consider a multiplicative setting for the constraint sets, i.e., let us define

$$K(\alpha) := V(\alpha)K(1)\,,$$

where $B_r(0) \subset K(1) \subset B_R(0)$, $K(1)$ is closed and convex, and

$$V\colon \mathbb{R} \to [m, M] \;\text{ is Lipschitz, nondecreasing, and constant in } (-\infty, 0] \text{ and } [1, +\infty)$$

with r, R, m, M positive constants.

Then (2.9) and (2.12) holds.

The plastic potential. Basing on the theory of convex functions of measures developed in [18], we define the *plastic potential* $\mathcal{H}\colon C(\overline{\Omega})\times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)\to\mathbb{R}$ by

$$\mathcal{H}(\alpha,p):=\int_{\overline{\Omega}} H\Big(\alpha(x),\frac{\mathrm{d}p}{\mathrm{d}\mu}(x)\Big)\,\mathrm{d}\mu(x)\,, \tag{2.13}$$

where $\mu\in M_b(\overline{\Omega})^+$ is any measure such that $p\ll\mu$ and $\frac{\mathrm{d}p}{\mathrm{d}\mu}$ denotes the Radon-Nikodym derivative of p with respect to μ; note that the homogeneity of H with respect to ξ implies that the integral does not depend on μ. When $\dot p$ is the rate of plastic strain and α the internal variable, $\mathcal{H}(\alpha,\dot p)$ represents the rate of plastic dissipation. Moreover, applying [3, Proposition 2.37], we get that

$$p\mapsto\mathcal{H}(\alpha,p)\text{ is convex and positively one-homogeneous for every }\alpha\in C(\overline{\Omega})\,,$$

which implies

$$\mathcal{H}(\alpha,p_1+p_2)\le\mathcal{H}(\alpha,p_1)+\mathcal{H}(\alpha,p_2) \tag{2.14}$$

for every $\alpha\in C(\overline{\Omega})$ and $p_1,p_2\in M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$.

Remark 2.2. Since

$$\Big|\frac{\mathrm{d}p}{\mathrm{d}|p|}(x)\Big|=1\quad\text{for }|p|\text{-a.e. }x\in\overline{\Omega}\,, \tag{2.15}$$

from (2.11d) it follows immediately that for every $\alpha\in C(\overline{\Omega})$

$$r\|p\|_1\le\mathcal{H}(\alpha,p)\le R\|p\|_1\,. \tag{2.16}$$

Moreover, by continuity of H there exists a modulus of continuity ω, namely an increasing function defined on $\mathbb{R}^+\cup\{0\}$ which vanishes at 0, such that

$$|H(\alpha_1(x),\xi)-H(\alpha_2(x),\xi)|\le\omega(|\alpha_1(x)-\alpha_2(x)|)\,, \tag{2.17}$$

for every α_1, $\alpha_2\in C(\overline{\Omega})$, $x\in\Omega$, and $\xi\in\mathbb{M}^{n\times n}_D$ with $|\xi|=1$. Then, from (2.15) we obtain

$$|\mathcal{H}(\alpha_2,p)-\mathcal{H}(\alpha_1,p)|\le\omega(\|\alpha_1-\alpha_2\|_\infty)\|p\|_1 \tag{2.18}$$

for every α_1, $\alpha_2\in C(\overline{\Omega})$.

Lemma 2.3. *Let α_k and p_k be sequences in $C(\overline{\Omega})$ and $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ such that $\alpha_k\to\alpha$ uniformly and $p_k\rightharpoonup p$ weakly* in $M_b(\Omega;\mathbb{M}^{n\times n}_D)$. Then*

$$\mathcal{H}(\alpha,p)\le\liminf_{k\to\infty}\mathcal{H}(\alpha_k,p_k)\,. \tag{2.19}$$

Proof. From (2.18) we obtain

$$\mathcal{H}(\alpha_k,p_k)\ge\mathcal{H}(\alpha,p_k)-\omega(\|\alpha_k-\alpha\|_\infty)\|p_k\|_1\,.$$

The lower semicontinuity result follows now from the weak* convergence of p_k and Reshetnyak's Lower Semicontinuity Theorem (see [26, Theorem 2]). □

Stress-strain duality. Let

$$\Sigma(\Omega):=\{\sigma\in L^2(\Omega;M^{n\times n}_{sym})\,|\,\mathrm{div}\,\sigma\in L^n(\Omega;\mathbb{R}^n),\ \sigma_D\in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)\}\,.$$

If $\sigma\in\Sigma(\Omega)$, then $\sigma\in L^r(\Omega;\mathbb{M}^{n\times n}_{sym})$ for every $1\le r<\infty$ by [16, Proposition 6.1]. For $\sigma\in\Sigma(\Omega)$ and $p\in\Pi(\Omega)$ such that $(u,e,p)\in A(w)$, we define, as in [16, Section 6], the distribution $[\sigma_D:p]$ on $\mathbb{R}^n$ by

$$\langle[\sigma_D:p],\varphi\rangle:=-\int_\Omega\varphi\sigma\cdot(e-Ew)\,\mathrm{d}x-\int_\Omega\sigma\cdot[(u-w)\odot\nabla\varphi]\,\mathrm{d}x-\int_\Omega\varphi(\mathrm{div}\,\sigma)\cdot(u-w)\,\mathrm{d}x\,, \tag{2.20}$$

for every $\varphi\in C^\infty_c(\mathbb{R}^n)$. Notice that this is indeed a well defined distribution since $u\in L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, being in $BD(\Omega;\mathbb{R}^n)$. Moreover, in [16, Theorem 6.2 and Remark 6.3] it is proved that $[\sigma_D:p]$ is a bounded Radon measure such that

$$\|[\sigma_D:p]\|_1\le\|\sigma_D\|_\infty\|p\|_1\quad\text{in }\mathbb{R}^n\,.$$

In particular, we can consider its restriction, as a measure, to $\overline{\Omega}$, that we denote in the same way. We also define

$$\langle\sigma_D\,|\,p\rangle:=[\sigma_D:p](\overline{\Omega})\,.$$

By (2.20), for $\varphi \in C_c^\infty(\mathbb{R}^n)$, $\varphi \equiv 1$ on $\overline{\Omega}$, we obtain the integration by parts formula

$$\langle \sigma_D \,|\, p \rangle = -\langle \sigma, e - Ew \rangle - \langle \operatorname{div} \sigma, u - w \rangle \,. \tag{2.21}$$

For a given $\alpha \in C(\overline{\Omega})$ let

$$\mathcal{K}_\alpha(\Omega) := \{\sigma \in L^2(\Omega; M^{n\times n}_{sym}) \,|\, \operatorname{div} \sigma \in L^n(\Omega; \mathbb{R}^n),\ \sigma_D(x) \in K(\alpha(x)) \text{ for a.e. } x \in \Omega\} \,.$$

Since the multifunction $\alpha \in [0,1] \mapsto K(\alpha)$ is continuous, that is (2.9b) holds, [16, Proposition 3.9] (which holds also if div σ is not identically 0) implies that for every $\sigma \in \mathcal{K}_\alpha(\Omega)$:

$$H\left(\alpha, \frac{\mathrm{d}p}{\mathrm{d}|p|}\right)|p| \geq [\sigma_D : p] \quad \text{as measures on } \overline{\Omega}, \tag{2.22}$$

and, arguing as in [28, Theorem 3.6 and Corollary 3.8], we deduce that for every $p \in \Pi(\Omega)$:

$$\mathcal{H}(\alpha, p) = \sup_{\sigma \in \mathcal{K}_\alpha(\Omega)} \langle \sigma_D \,|\, p \rangle \,. \tag{2.23}$$

The plastic dissipation. We introduce now a term which represents the plastic dissipation in a given time interval.

A function $p\colon [0,T] \to M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ will be regarded as a function defined on the time interval $[0,T]$ with values in the dual of the separable Banach space $C(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$, $\overline{\Omega} \subset \mathbb{R}^n$ being compact. For every $s, t \in [0,T]$ with $s \leq t$ the total variation of p on $[s,t]$ is defined by

$$\mathcal{V}(p; s, t) = \sup\left\{ \sum_{j=1}^N \|p(t_j) - p(t_{j-1})\|_1 \,\Big|\, s = t_0 < t_1 < \dots < t_N = t,\ N \in \mathbb{N} \right\}.$$

Let $\alpha\colon [0,T] \to C(\overline{\Omega})$. For every partition $\mathcal{P}$ of $[s,t]$, namely $\mathcal{P} := \{t_i\}_{0\leq i\leq N}$ with $s = t_0 < t_1 < \dots < t_N = t$, we define

$$\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha, p; s, t) := \sum_{i=1}^N \mathcal{H}(\alpha(t_i), p(t_i) - p(t_{i-1})) \,.$$

The *$\mathcal{H}$-variation of p with respect to α* on $[s,t]$, which will play the role of the plastic dissipation in the time interval $[s,t]$, is denoted by $\mathcal{V}_{\mathcal{H}}(\alpha, p; s, t)$ and is defined through

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(\alpha, p; s, t) &:= \sup\left\{ \sum_{j=1}^N \mathcal{H}(\alpha(t_j), p(t_j) - p(t_{j-1})) \,\Big|\, s = t_0 < t_1 < \dots < t_N = t,\ N \in \mathbb{N} \right\} \\ &= \sup\left\{ \mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha, p; s, t) \,|\, \mathcal{P} \text{ partition of } [s,t] \right\}. \end{aligned} \tag{2.24}$$

Lemma A.1 in the Appendix states some properties of $\mathcal{V}_{\mathcal{H}}$ when the functions $t \mapsto \alpha(t,x)$ are nonincreasing for every $x \in \Omega$.

Remark 2.4. When α is constant in time, we fit in the notion of $\mathcal{H}$-variation of p on $[s,t]$ introduced in [6, Appendix]. There the $\mathcal{G}$-variation of p on $[s,t]$ is defined, when p takes values in $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$, as

$$\begin{aligned} \mathcal{V}_{\mathcal{G}}(p; s, t) &:= \sup\left\{ \sum_{i=1}^N \mathcal{G}(p(t_i) - p(t_{i-1})) \Big|\, s = t_0 < t_1 < \dots < t_N = t,\ N \in \mathbb{N} \right\} \\ &= \sup\{ \mathcal{V}^{\mathcal{P}}_{\mathcal{G}}(p; s, t) \,|\, \mathcal{P} \text{ partition of } [s,t] \}, \end{aligned}$$

where $\mathcal{G}(p) = \sup_{\varphi \in \mathcal{K}} \langle p, \varphi \rangle$ is the support function of a bounded closed convex set $\mathcal{K} \subset C(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ and $\mathcal{V}^{\mathcal{P}}_{\mathcal{G}}(p; s, t)$ is defined similarly to $\mathcal{V}^{\mathcal{P}}_{\mathcal{H}}(\alpha, p; s, t)$. By (2.23), it suffices to take $\mathcal{K} = \mathcal{K}_\alpha(\Omega)$ in order to obtain $\mathcal{G} = \mathcal{H}(\alpha, \cdot)$ and

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; s, t) = \mathcal{V}_{\mathcal{G}}(p; s, t) \,.$$

The prescribed boundary displacement. In this paper the external loading will consist only in Dirichlet boundary conditions, for the sake of simplicity. However, similar results to those showed here hold also in the presence of external forces, under suitable regularity assumptions on $\partial\Omega$ and uniform safe load conditions, like the ones in [9, Section 2].

We assume that the prescribed boundary displacement w depends on time and satisfies the regularity assumption

$$w \in AC([0,T]; H^1(\mathbb{R}^n;\mathbb{R}^n))\,, \tag{2.25}$$

so that the time derivative $t \mapsto \dot w(t)$ belongs to $L^1(0,T;H^1(\mathbb{R}^n;\mathbb{R}^n))$ and its strain $t\mapsto E\dot w(t)$ belongs to $L^1(0,T;L^2(\mathbb{R}^n;\mathbb{M}^{n\times n}_{sym}))$. For the main properties of absolutely continuous functions with values in reflexive Banach spaces we refer to [5, Appendix].

3. The minimum problem

In this section we study the minimum problem employed in the incremental formulation of the quasistatic evolution corresponding to a given parameter $\lambda \in [0,1]$. Therefore we deal with a problem of the type

$$\min_{(\alpha,(u,e,p))\in\mathcal{D}(\overline\alpha)\times A(w)} \{\mathcal{E}_\lambda(\alpha,e;\overline q,t) + \mathcal{H}(\alpha,p-\overline p)\}\,, \tag{3.1}$$

where

$$\mathcal{E}_\lambda(\alpha,e;\overline q,t) := \mathcal{E}(\alpha,e) + \lambda\mathcal{V}_{\mathcal{H}}(\alpha,\overline q;0,t)\,. \tag{3.2}$$

The data are the current values $\overline\alpha \in W^{1,\gamma}(\Omega)$ and $\overline p \in \Pi(\Omega)$ of the damage variable and the plastic strain, and the updated value $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$ of the boundary displacement; if $\lambda > 0$ we consider as an additional datum a function $\overline q\colon [0,t] \to M_b(\overline\Omega;\mathbb{M}^{n\times n}_D)$ with bounded variation, which represents the evolution of the plastic strain up to the current time t. Solving this problem, we get the updated values α, u, e, and p of damage, displacement, elastic and plastic strain.

First we show the existence and the main properties of the solutions to (3.1). The second part of the section is devoted to prove a stability property with respect to variations of the data. Throughout this section, we suppose that (2.4), (2.5), (2.9), and (2.25) hold when $\lambda = 0$; when $\lambda > 0$ we will assume also (2.12).

Let us prove the existence of a solution to (3.1).

Theorem 3.1 (Existence of solutions to the incremental problem)**.** *Let $w \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\overline\alpha \in W^{1,\gamma}(\Omega)$, $\overline p \in \Pi(\Omega)$, and $\overline q\colon [0,t] \to M_b(\overline\Omega;\mathbb{M}^{n\times n}_D)$ with bounded variation. Then* (3.1) *has a solution. Moreover, if $\overline\alpha \in W^{1,\gamma}(\Omega;[0,1])$, then for every $(\alpha,(u,e,p))$ solution of* (3.1) *we have that $\alpha \in W^{1,\gamma}(\Omega;[0,1])$.*

Proof. Let $(\alpha_k,(u_k,e_k,p_k)) \in \mathcal{D}(\overline\alpha)\times A(w)$ be a minimizing sequence for problem (3.1).

By (2.5d) and (2.16) the sequences α_k, e_k, and p_k are bounded in $W^{1,\gamma}(\Omega)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $M_b(\overline\Omega;\mathbb{M}^{n\times n}_D)$, respectively. Since $Eu_k = e_k + p_k$ in Ω, it follows that Eu_k is bounded in $M_b(\Omega;\mathbb{M}^{n\times n}_{sym})$. Since $(w-u_k)\odot\nu\,\mathcal{H}^{n-1} = p_k$ is bounded in $M_b(\partial\Omega;\mathbb{M}^{n\times n}_D)$, the traces of u_k are bounded in $L^1(\partial\Omega;\mathbb{R}^n)$. Therefore u_k is bounded in $BD(\Omega;\mathbb{R}^n)$ by (2.1).

Up to extracting a subsequence, we may assume that $u_k \rightharpoonup u$ weakly* in $BD(\Omega;\mathbb{R}^n)$, $e_k \rightharpoonup e$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $p_k \rightharpoonup p$ weakly* in $M_b(\overline\Omega;\mathbb{M}^{n\times n}_D)$. By [6, Lemma 2.1], we have $(u,e,p) \in A(w)$.

The existence of solutions to (3.1) now follows from the lower semicontinuity of $\mathcal{E}$ (see (2.8)) and $\mathcal{H}$ (see (2.19)). Notice that if $\alpha \neq \alpha^+ := \alpha \vee 0$ then

$$\mathcal{E}_\lambda(\alpha^+,e;\overline q,t) = \mathcal{E}(\alpha^+,e) + \lambda\mathcal{V}_{\mathcal{H}}(\alpha^+,\overline q;0,t) < \mathcal{E}_\lambda(\alpha,e;\overline q,t) = \mathcal{E}(\alpha,e) + \lambda\mathcal{V}_{\mathcal{H}}(\alpha,\overline q;0,t)\,,$$

and this is enough to conclude that α takes values in $[0,1]$ if $\overline\alpha \in W^{1,\gamma}(\Omega;[0,1])$. □

The following remark is not only useful in Lemma 3.3 below, but also in the proof of the stability for the approximate solutions in Theorem 4.3, when $\lambda = 0$.

Lemma 3.2. *If* $(\alpha,(u,e,p))$ *solves* (3.1) *then*

$$\mathcal{E}_\lambda(\alpha,e;\overline{q},t)\le\mathcal{E}_\lambda(\tilde\alpha,\tilde e;\overline{q},t)+\mathcal{H}(\tilde\alpha,\tilde p-p)\,, \tag{3.3}$$

for every $(\tilde\alpha,(\tilde u,\tilde e,\tilde p))\in\mathcal{D}(\alpha)\times A(w)$.

Proof. Let $(\tilde\alpha,(\tilde u,\tilde e,\tilde p))\in\mathcal{D}(\alpha)\times A(w)$. Then, by (2.3), this quadruple belongs to $\mathcal{D}(\overline{\alpha})\times A(w)$ too. From our hypotesis, $\mathcal{E}_\lambda(\alpha,e;\overline{q},t)\le\mathcal{E}_\lambda(\tilde\alpha,\tilde e;\overline{q},t)+\mathcal{H}(\tilde\alpha,\tilde p-\overline{p})-\mathcal{H}(\alpha,p-\overline{p})$, and by (2.14) and (2.11b), $\mathcal{H}(\tilde\alpha,\tilde p-\overline{p})\le\mathcal{H}(\tilde\alpha,\tilde p-p)+\mathcal{H}(\tilde\alpha,p-\overline{p})\le\mathcal{H}(\tilde\alpha,\tilde p-p)+\mathcal{H}(\alpha,p-\overline{p})$. Thus we conclude. ☐

We now derive some differential conditions for a triple (u,e,p) such that $(\alpha,(u,e,p))$ solves (3.1), from a characterization of the solutions to (3.3).

Lemma 3.3. *Let* $w\in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\alpha,\overline{\alpha}\in W^{1,\gamma}(\Omega)$, $(u,e,p)\in A(w)$, $\overline{p}\in\Pi(\Omega)$, *and* $\overline{q}\colon[0,t]\to M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ *with bounded variation. Then the following conditions are equivalent:*

(a) $(\alpha,(u,e,p))$ *satisfies* (3.3)*;*

(b) $-\mathcal{H}(\alpha,q)\le\langle\mathbb{C}(\alpha)e,\eta\rangle\le\mathcal{H}(\alpha,-q)$ *for every* $(v,\eta,q)\in A(0)$*;*

(c) $\mathbb{C}(\alpha)e\in\mathcal{K}_\alpha(\Omega)$, $\mathrm{div}(\mathbb{C}(\alpha)e)=0$ *in* Ω.

Proof. Let us assume (a), and fix $(v,\eta,q)\in A(0)$. Since for every $\varepsilon\in\mathbb{R}$

$$(\alpha,(u+\varepsilon v,e+\varepsilon\eta,p+\varepsilon q))\in\mathcal{D}(\alpha)\times A(w)\,,$$

we have

$$\mathcal{Q}(\alpha,e+\varepsilon\eta)+\mathcal{H}(\alpha,\varepsilon q)\ge\mathcal{Q}(\alpha,e)\ \text{ for every }\varepsilon\in\mathbb{R}\,.$$

The positive homogeneity of $\mathcal{H}$ implies

$$\mathcal{Q}(\alpha,e\pm\varepsilon\eta)+\varepsilon\mathcal{H}(\alpha,\pm q)\ge\mathcal{Q}(\alpha,e)\ \text{ for every }\varepsilon>0\,.$$

Dividing by ε and passing to the limit as $\varepsilon\to0$, we recover (b). By convexity, (a) and (b) are equivalent.

Assuming (b), in order to get (c) we can argue as in the first part of [6, Proposition 3.5], using the integration by parts formula (2.21). By (2.23) and (2.9b), arguing as in [6, Theorem 3.6] and [28, Theorem 3.10], we can deduce that (c) implies (b). ☐

The following lemma shows, for pairs $(\alpha,(u,e,p))$ that satisfy (3.3), the Hölder dependence of u and e on α, p, and w.

Lemma 3.4. *For* $i=1,2$ *let* $w_i\in H^1(\mathbb{R}^n,\mathbb{R}^n)$. *Suppose that* $(\alpha_i,(u_i,e_i,p_i))$ *satisfies* (3.3) *with boundary datum* $w=w_i$, *and let*

$$\omega_{12}:=\|\alpha_2-\alpha_1\|_\infty+\|p_2-p_1\|_1^{1/2}+\|Ew_2-Ew_1\|_2\,.$$

Then

$$\|e_2-e_1\|_2\le C\,\omega_{12}\,, \tag{3.4}$$

where C *is a positive constant depending on* $\|e_1\|_2$, R, γ_1, γ_2, *and* Ω.

Proof. We modify the proof of [6, Theorem 3.8], considering that here $\mathbb{C}$ depends on α. Let

$$\begin{aligned}v&:=(u_2-w_2)-(u_1-w_1),\\ \eta&:=(e_2-Ew_2)-(e_1-Ew_1),\\ q&:=p_2-p_1\,.\end{aligned}$$

Since $(v,\eta,q)\in A(0)$, by Lemma 3.3 it follows that

$$\begin{aligned}-\mathcal{H}(\alpha_1,p_2-p_1)\le&\langle\mathbb{C}(\alpha_1)e_1,\eta\rangle,\\ &\langle\mathbb{C}(\alpha_2)e_2,\eta\rangle\le\mathcal{H}(\alpha_2,p_1-p_2)\,.\end{aligned}$$

Adding term by term and using (2.11d), we obtain

$$\langle\mathbb{C}(\alpha_2)(e_2-e_1),\eta\rangle)\le\langle[\mathbb{C}(\alpha_1)-\mathbb{C}(\alpha_2)]e_1,\eta\rangle+2R\|p_2-p_1\|_1\,.$$

Remark that above we have put an extra term $-\langle \mathbb{C}(\alpha_2)e_1, \eta\rangle$ on both sides. From the definition of η,

$$\begin{aligned}\langle \mathbb{C}(\alpha_2)(e_2-e_1), e_2-e_1)\rangle \leq& \langle \mathbb{C}(\alpha_2)(e_2-e_1), Ew_2-Ew_1\rangle + \langle [\mathbb{C}(\alpha_1)-\mathbb{C}(\alpha_2)]e_1, e_2-e_1\rangle \\ &+ \langle [\mathbb{C}(\alpha_1)-\mathbb{C}(\alpha_2)]e_1, Ew_1-Ew_2\rangle + 2R\|p_2-p_1\|_1\,.\end{aligned}$$

By (2.5), this implies

$$\begin{aligned}2\gamma_1\|e_2-e_1\|_2^2 \leq& \, 2\gamma_2\|e_2-e_1\|_2\|Ew_2-Ew_1\|_2 \\ &+ \|e_1\|_2\|\alpha_2-\alpha_1\|_\infty(\|e_2-e_1\|_2+\|Ew_2-Ew_1\|_2) + 2R\|p_2-p_1\|_1\,,\end{aligned}$$

which yields (3.4) by the Cauchy inequality. □

Remark 3.5. We can also deduce the continuous dependence on α, p, and w of u, expressed (with the same notation as above) by

$$\|Eu_2-Eu_1\|_1 \leq C\,(\omega_{12}+\|p_2-p_1\|_1)\,,$$
$$\|u_2-u_1\|_1 \leq C\,(\omega_{12}+\|p_2-p_1\|_1+\|w_2-w_1\|_2)\,,$$

arguing as in the final part of [6, Theorem 3.8].

We now show some stability results for the minimum problems of the type (3.1) with respect to the weak convergence of the data. To ease the reading we first consider, in Theorem 3.6, the case $\lambda=0$, and then we study, in Lemma 3.7, the additional term present when $\lambda>0$. The result for the case $\lambda>0$ (Theorem 3.8) follows from this lemma, arguing as in Theorem 3.6.

Theorem 3.6 (Stability, case $\lambda=0$)**.** *Let $w_k\in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\alpha_k\in W^{1,\gamma}(\Omega)$, and $(u_k,e_k,p_k)\in A(w_k)$ for every k. Assume that $\alpha_k \rightharpoonup \alpha_\infty$ weakly in $W^{1,\gamma}(\Omega)$, $u_k\rightharpoonup u_\infty$ weakly* in $BD(\Omega;\mathbb{R}^n)$, $e_k\rightharpoonup e_\infty$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p_k\rightharpoonup p_\infty$ weakly* in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, $w_k\rightharpoonup w_\infty$ weakly in $H^1(\mathbb{R}^n;\mathbb{R}^n)$. Then $(u_\infty,e_\infty,p_\infty)\in A(w_\infty)$. If, in addition,*

$$\mathcal{E}(\alpha_k,e_k)\leq \mathcal{E}(\tilde\alpha_k,\tilde e_k)+\mathcal{H}(\tilde\alpha_k,\tilde p_k-p_k) \tag{3.6}$$

for every k and every $(\tilde\alpha_k,(\tilde u_k,\tilde e_k,\tilde p_k))\in\mathcal{D}(\alpha_k)\times A(w_k)$, then

$$\mathcal{E}(\alpha_\infty,e_\infty)\leq \mathcal{E}(\alpha,e)+\mathcal{H}(\alpha,p-p_\infty) \tag{3.7}$$

for every $(\alpha,(u,e,p))\in\mathcal{D}(\alpha_\infty)\times A(w_\infty)$.

Proof. The fact that $(u_\infty,e_\infty,p_\infty)\in A(w_\infty)$ follows by [6, Lemma 2.1].

We fix $\alpha\in\mathcal{D}(\alpha_\infty)$ and $(u,e,p)\in A(w_\infty)$, and test (3.6) by

$$\begin{aligned}\tilde\alpha_k &:= \alpha\wedge\alpha_k\,,\\ \tilde u_k &:= u-u_\infty+u_k\,,\\ \tilde e_k &:= e-e_\infty+e_k\,,\\ \tilde p_k &:= p-p_\infty+p_k\,.\end{aligned}$$

Then $\tilde\alpha_k\rightharpoonup\alpha$ and $\alpha\vee\alpha_k\rightharpoonup\alpha_\infty$ weakly in $W^{1,\gamma}(\Omega)$, $\tilde u_k\rightharpoonup u$ weakly* in $BD(\Omega;\mathbb{R}^n)$, $\tilde e_k\rightharpoonup e$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $\tilde p_k\rightharpoonup p$ weakly* in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$.

Since for every $\alpha\in W^{1,\gamma}(\Omega)$ and every $e_1,e_2\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ we have

$$\mathcal{Q}(\alpha,e_1)-\mathcal{Q}(\alpha,e_2)=\frac{1}{2}\langle\mathbb{C}(\alpha)(e_1+e_2),e_1-e_2\rangle \tag{3.8}$$

and for every $\alpha,\beta\in W^{1,\gamma}(\Omega)$

$$\|\nabla(\alpha\vee\beta)\|_\gamma^\gamma+\|\nabla(\alpha\wedge\beta)\|_\gamma^\gamma=\|\nabla\alpha\|_\gamma^\gamma+\|\nabla\beta\|_\gamma^\gamma\,,$$

(3.6) can be rewritten, adding to both sides the term $-\mathcal{Q}(\tilde\alpha_k,e_k)$, as

$$\begin{aligned}\gamma_k :=&\,\mathcal{Q}(\alpha_k,e_k)-\mathcal{Q}(\tilde\alpha_k,e_k)+D(\alpha_k)+\|\nabla(\alpha\vee\alpha_k)\|_\gamma^\gamma-\|\nabla\alpha\|_\gamma^\gamma\\ \leq&\,\frac{1}{2}\langle\mathbb{C}(\tilde\alpha_k)(e-e_\infty+2e_k),e-e_\infty\rangle+D(\tilde\alpha_k)+\mathcal{H}(\tilde\alpha_k,p-p_\infty)=:\eta_k\,.\end{aligned}$$

From (2.5a), for every $\alpha_1,\alpha_2\in C(\Omega)$ and $e\in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$

$$|\mathcal{Q}(\alpha_1,e)-\mathcal{Q}(\alpha_2,e)|\leq \mathrm{Lip}(\mathbb{C})\|\alpha_1-\alpha_2\|_\infty\|e\|_2^2\,.$$

Therefore,

$$\liminf_{k\to\infty} \mathcal{Q}(\alpha_k, e_k) - \mathcal{Q}(\tilde{\alpha}_k, e_k) = \liminf_{k\to\infty} \mathcal{Q}(\alpha_\infty, e_k) - \mathcal{Q}(\alpha, e_k)$$
$$= \liminf_{k\to\infty} \frac{1}{2}\langle[\mathbb{C}(\alpha_\infty) - \mathbb{C}(\alpha)]e_k, e_k\rangle\,.$$

Since $\alpha \in \mathcal{D}(\alpha_\infty)$, by (2.5b) we have that $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \mapsto [\mathbb{C}(\alpha_\infty) - \mathbb{C}(\alpha)]e : e$ is a positive semidefinite quadratic form. Hence, by lower semicontinuity,

$$\liminf_{k\rightharpoonup\infty} \gamma_k \geq \mathcal{Q}(\alpha_\infty, e_\infty) - \mathcal{Q}(\alpha, e_\infty) + D(\alpha_\infty) + \|\nabla(\alpha_\infty)\|_\gamma^\gamma - \|\nabla\alpha\|_\gamma^\gamma\,.$$

On the other hand,

$$\lim_{k\rightharpoonup\infty} \eta_k = \frac{1}{2}\langle\mathbb{C}(\alpha)(e + e_\infty), e - e_\infty\rangle + D(\alpha) + \mathcal{H}(\alpha, p - p_\infty)$$
$$= \mathcal{Q}(\alpha, e) - \mathcal{Q}(\alpha, e_\infty) + D(\alpha) + \mathcal{H}(\alpha, p - p_\infty)\,.$$

This concludes the proof. □

From now on we treat the case $\lambda > 0$.

Lemma 3.7. *In addition to* (2.4), (2.5), (2.9), *and* (2.25), *let us assume also* (2.12). *Let* β_k *and* $\tilde{\beta}_k$ *be two sequences in* $C(\overline{\Omega})$ *such that* $\beta_k \to \beta_\infty$ *and* $\tilde{\beta}_k \to \beta$ *uniformly in* $\overline{\Omega}$, *and* $\tilde{\beta}_k \in \mathcal{D}(\beta_k)$ *for every* k. *Moreover let* q_k, q *be functions from* $[0,t]$ *into* $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ *such that* $q_k(s) \rightharpoonup q(s)$ *weakly* in* $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ *for every* $s \in [0,t]$. *Then*

$$\mathcal{V}_\mathcal{H}(\beta_\infty, q; 0, t) - \mathcal{V}_\mathcal{H}(\beta, q; 0, t) \leq \liminf_{k\to\infty} [\mathcal{V}_\mathcal{H}(\beta_k, q_k; 0, t) - \mathcal{V}_\mathcal{H}(\tilde{\beta}_k, q_k; 0, t)]\,. \tag{3.9}$$

Proof. Let us consider the functionals $\widetilde{\mathcal{H}}_k$ and $\widetilde{\mathcal{H}}$ from $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ into $\mathbb{R}^+ \cup \{0\}$ defined, for every $p \in M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$, by

$$\widetilde{\mathcal{H}}(p) := \mathcal{H}(\beta_\infty, p) - \mathcal{H}(\beta, p)\,,$$
$$\widetilde{\mathcal{H}}_k(p) := \mathcal{H}(\beta_k, p) - \mathcal{H}(\tilde{\beta}_k, p)\,.$$

By (2.12), $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}}_k$ are convex, positively one-homogeneous (and consequently subadditive), and weakly* lower semicontinuous, thanks to Reshetnyak's Lower Semicontinuity Theorem. We now show that

$$\mathcal{V}_\mathcal{H}(\beta_\infty, q; 0, t) - \mathcal{V}_\mathcal{H}(\beta, q; 0, t) = \mathcal{V}_{\widetilde{\mathcal{H}}}(q; 0, t)\,, \tag{3.10}$$
$$\mathcal{V}_\mathcal{H}(\beta_k, q; 0, t) - \mathcal{V}_\mathcal{H}(\tilde{\beta}_k, q; 0, t) = \mathcal{V}_{\widetilde{\mathcal{H}}_k}(q; 0, t)\,, \tag{3.11}$$

for every k. Indeed, let us fix $\varepsilon > 0$ and let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be three partitions of $[0,t]$ such that

$$\mathcal{V}^{\mathcal{P}_1}_\mathcal{H}(\beta_\infty, q; 0, t) > \mathcal{V}_\mathcal{H}(\beta_\infty, q; 0, t) - \varepsilon\,,$$
$$\mathcal{V}^{\mathcal{P}_2}_\mathcal{H}(\beta, q; 0, t) > \mathcal{V}_\mathcal{H}(\beta, q; 0, t) - \frac{\varepsilon}{2}\,,$$
$$\mathcal{V}^{\mathcal{P}_3}_{\widetilde{\mathcal{H}}}(q; 0, t) > \mathcal{V}_{\widetilde{\mathcal{H}}}(q; 0, t) - \frac{\varepsilon}{2}\,.$$

It follows that

$$\mathcal{V}_{\widetilde{\mathcal{H}}}(q; 0, t) \geq \mathcal{V}^{\mathcal{P}_1}_{\widetilde{\mathcal{H}}}(q; 0, t) = \mathcal{V}^{\mathcal{P}_1}_\mathcal{H}(\beta_\infty, q; 0, t) - \mathcal{V}^{\mathcal{P}_1}_\mathcal{H}(\beta, q; 0, t) > \mathcal{V}_\mathcal{H}(\beta_\infty, q; 0, t) - \varepsilon - \mathcal{V}_\mathcal{H}(\beta, q; 0, t)\,.$$

On the other hand, we get

$$\mathcal{V}_\mathcal{H}(\beta_\infty, q; 0, t) - \mathcal{V}_\mathcal{H}(\beta, q; 0, t) > \mathcal{V}^{\mathcal{P}_2\cup\mathcal{P}_3}_\mathcal{H}(\beta_\infty, q; 0, t) - \mathcal{V}^{\mathcal{P}_2}_\mathcal{H}(\beta, q; 0, t) - \frac{\varepsilon}{2}$$
$$\geq \mathcal{V}^{\mathcal{P}_2\cup\mathcal{P}_3}_\mathcal{H}(\beta_\infty, q; 0, t) - \mathcal{V}^{\mathcal{P}_2\cup\mathcal{P}_3}_\mathcal{H}(\beta, q; 0, t) - \frac{\varepsilon}{2}$$
$$= \mathcal{V}^{\mathcal{P}_2\cup\mathcal{P}_3}_{\widetilde{\mathcal{H}}}(q; 0, t) - \frac{\varepsilon}{2} > \mathcal{V}_{\widetilde{\mathcal{H}}}(q; 0, t) - \varepsilon\,,$$

where the second inequality follows from Lemma A.1(1) and the last one comes from the subadditivity of $\widetilde{\mathcal{H}}$. This concludes the verification of (3.10). The proof of (3.11) is analogous.

Arguing as in Lemma 2.3 we have that

$$\widetilde{\mathcal{H}}(p) \le \liminf_{k\to\infty} \widetilde{\mathcal{H}}_k(p_k)\,,$$

for every $p_k \rightharpoonup p$ weakly* in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$. Hence

$$\mathcal{V}_{\widetilde{\mathcal{H}}}(q;0,t) \le \liminf_{k\to\infty} \mathcal{V}_{\widetilde{\mathcal{H}}_k}(q_k;0,t)\,,$$

and we conclude by (3.10) and (3.11). □

Theorem 3.8 (Stability, case $\lambda > 0$)**.** *Besides* (2.4)*,* (2.5)*,* (2.9)*, and* (2.25)*, assume also* (2.12)*. Let $w_k \in H^1(\mathbb{R}^n;\mathbb{R}^n)$, $\alpha_k \in W^{1,\gamma}(\Omega)$, $(u_k,e_k,p_k) \in A(w_k)$, and q_k be functions from $[0,t]$ into $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ of bounded variation, for every k. Suppose that $\alpha_k \rightharpoonup \alpha_\infty$ weakly in $W^{1,\gamma}(\Omega)$, $u_k \rightharpoonup u_\infty$ weakly* in $BD(\Omega;\mathbb{R}^n)$, $e_k \rightharpoonup e_\infty$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $p_k \rightharpoonup p_\infty$ weakly* in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, $w_k \rightharpoonup w_\infty$ weakly in $H^1(\mathbb{R}^n;\mathbb{R}^n)$, and $q_k(s) \rightharpoonup q(s)$ weakly* in $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ for every $s\in[0,t]$. Then $(u_\infty,e_\infty,p_\infty)\in A(w_\infty)$. If, in addition,*

$$\mathcal{E}(\alpha_k,e_k) + \lambda\mathcal{V}_{\mathcal{H}}(\alpha_k,q_k;0,t) \le \mathcal{E}(\tilde\alpha_k,\tilde e_k) + \lambda\mathcal{V}_{\mathcal{H}}(\tilde\alpha_k,q_k;0,t) + \mathcal{H}(\tilde\alpha_k,\tilde p_k - p_k)$$

for every k and every $(\tilde\alpha_k,(\tilde u_k,\tilde e_k,\tilde p_k)) \in \mathcal{D}(\alpha_k)\times A(w_k)$, then

$$\mathcal{E}(\alpha_\infty,e_\infty) + \lambda\mathcal{V}_{\mathcal{H}}(\alpha_\infty,q;0,t) \le \mathcal{E}(\alpha,e) + \lambda\mathcal{V}_{\mathcal{H}}(\alpha,q;0,t) + \mathcal{H}(\alpha,p-p_\infty)\,,$$

for every $(\alpha,(u,e,p)) \in \mathcal{D}(\alpha_\infty)\times A(w_\infty)$.

Proof. We can argue as in the proof of Theorem 3.6, choosing the same test functions, and adding to γ_k the term $\lambda(\mathcal{V}_{\mathcal{H}}(\alpha_k,q_k;0,t) - \mathcal{V}_{\mathcal{H}}(\tilde\alpha_k,q_k;0,t))$. The sequence of these terms is lower semicontinuous by Lemma 3.7 and this is enough to conclude. □

4. Quasistatic evolution

Fixed $\lambda\in[0,1]$, we now consider the problem of existence of globally minimizing quasistatic evolutions, where the time-dependent data are (only) Dirichlet boundary conditions $w\in AC([0,T];H^1(\mathbb{R}^n;\mathbb{R}^n))$. The functions α, u, e, p will be then functions from $[0,T]$ into $W^{1,\gamma}(\Omega;[0,1])$, $BD(\Omega;\mathbb{R}^n)$, $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$, respectively.

If $\lambda=0$ we get the counterpart of the model considered in [6] in the case of perfect plasticity. When $\lambda=1$ we recover the analogous of the evolution studied in [1] and [2].

The existence of quasistatic evolutions is shown in Theorem 4.3, the main result of the paper.

Definition 4.1. Let $\lambda\in[0,1]$. A *quasistatic evolution (corresponding to λ)* is a function $t\mapsto(\alpha(t),u(t),e(t),p(t))$ from $[0,T]$ into $W^{1,\gamma}(\Omega;[0,1])\times BD(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ that satisfies the following conditions:

(qs0) *irreversibility* : for every $x\in\Omega$

$$t\in[0,T]\mapsto \alpha(t,x) \quad \text{is nonincreasing;} \tag{4.1}$$

(qs1) *global stability*: the function $t\mapsto p(t)$ from $[0,T]$ into $M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ has bounded variation, $(u(t),e(t),p(t))\in A(w(t))$ for every $t\in[0,T]$, and

$$\mathcal{E}(\alpha(t),e(t)) + \lambda\mathcal{V}_{\mathcal{H}}(\alpha(t),p;0,t) \le \mathcal{E}(\beta,e) + \lambda\mathcal{V}_{\mathcal{H}}(\beta,p;0,t) + \mathcal{H}(\beta,q-p(t)) \tag{4.2}$$

for every $(\beta,(u,e,q))\in\mathcal{D}(\alpha(t))\times A(w(t))$;

(qs2) *energy balance*: for every $t\in[0,T]$

$$\begin{aligned}\mathcal{E}(\alpha(t),e(t)) + \lambda\mathcal{V}_{\mathcal{H}}(\alpha(t),p;0,t) + (1-\lambda)\mathcal{V}_{\mathcal{H}}(\alpha,p;0,t)&\\ = \mathcal{E}(\alpha(0),e(0)) + \int_0^t \langle\sigma(s),E\dot w(s)\rangle\,\mathrm{d}s\,,&\end{aligned} \tag{4.3}$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$.

Notice that, according to (2.24), in the conditions above:

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) = \sup\Big\{\sum_{i=1}^{N} \mathcal{H}(\alpha(t_i), p(t_i) - p(t_{i-1})) : 0 = t_0 < t_1 < \cdots < t_N = t,\, N \in \mathbb{N}\Big\}, \text{ while}$$

$$\mathcal{V}_{\mathcal{H}}(\alpha(t), p; 0, t) = \sup\Big\{\sum_{i=1}^{N} \mathcal{H}(\alpha(t), p(t_i) - p(t_{i-1})) : 0 = t_0 < t_1 < \cdots < t_N = t,\, N \in \mathbb{N}\Big\}.$$

Remark 4.2. The integral in (4.3) is well defined.

Indeed, from (4.2) it follows that $t \mapsto (\alpha(t), u(t), e(t), p(t))$ is a solution to the minimum problem

$$\min_{(\beta,(u,e,q))\in\mathcal{D}(\alpha(t))\times A(w(t))} \{\mathcal{E}_\lambda(\beta, e; p, t) + \mathcal{H}(\beta, q - p(t))\},$$

for every $t \in [0, T]$, where $\mathcal{E}_\lambda$ is defined in (3.2). In view of Lemma 3.4, choosing $e_2 = e(t)$ for every $t \in [0, T]$ and $e_1 = e(0)$, we can observe that

$$\sup_{t\in[0,T]} \|e(t)\|_2 \leq C, \tag{4.4}$$

where C is independent of time.

Let us now verify the measurability of $t \mapsto e(t)$. This follows from Lemma 3.4 if we show that $t \mapsto \alpha(t)$ is continuous for a.e. t with respect to the uniform convergence in $\overline{\Omega}$, since $t \mapsto p(t)$ is strongly continuous into $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ for a.e. t, having bounded variation. Now, by the irreversibility condition and the fact that for every $t \in [0, T]$ the function $\alpha(t)$ takes values in $[0, 1]$ we have, using Lemma A.2, that there exists a countable set $E \subset [0, T]$ such that α is continuous in every $t \in [0, T] \setminus E$ with respect to the L^p norm, with $1 \leq p < \infty$. In other words, for every $t \in [0, T] \setminus E$

$$\alpha(s) \to \alpha(t) \text{ in } L^p(\Omega) \text{ as } s \to t\,.$$

From the stability condition, choosing $\beta \equiv 0$ and $(u, e, q) = (u(t), e(t), p(t))$, and using (4.4), it follows that

$$\sup_{t\in[0,T]} \|\nabla\alpha(t)\|_\gamma^\gamma < C$$

with C independent of $t \in [0, T]$. Then, by the Urysohn Property, α is continuous in every $t \in [0, T] \setminus E$ with respect to the weak convergence in $W^{1,\gamma}$, i.e., for every $t \in [0, T] \setminus E$

$$\alpha(s) \rightharpoonup \alpha(t) \text{ weakly in } W^{1,\gamma}(\Omega) \text{ as } s \to t\,.$$

The above convergence is uniform in $\overline{\Omega}$ by the compact embedding.

Then e and σ belong to $L^\infty(0, T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$. Finally, by (2.25), it follows that $\dot{w} \in L^1(0, T; H^1(\mathbb{R}^n; \mathbb{R}^n))$, and we conclude.

Theorem 4.3 (Existence of quasistatic evolutions)**.** *Let* $\lambda \in [0, 1]$ *and assume* (2.4), (2.5), (2.9)*, and* (2.25)*. If* $\lambda > 0$ *assume also* (2.12)*. Let* $(\alpha_0, (u_0, e_0, p_0)) \in W^{1,\gamma}(\Omega; [0, 1]) \times A(w(0))$ *satisfy the stability condition*

$$\mathcal{E}(\alpha_0, e_0) \leq \mathcal{E}(\beta, e) + \mathcal{H}(\beta, p - p_0) \tag{4.5}$$

for every $(\beta, (u, e, p)) \in \mathcal{D}(\alpha_0) \times A(w(0))$*. Then there exists a quasistatic evolution* $t \mapsto (\alpha(t), u(t), e(t), p(t))$ *corresponding to* λ *such that* $\alpha(0) = \alpha_0$*,* $u(0) = u_0$*,* $e(0) = e_0$*,* $p(0) = p_0$*.*

Proof. The proof is based on discrete time approximation and is subdivided into several steps.

Approximate solutions. Let us fix a sequence of subdivisions $(t^i_k)_{0\leq i\leq k}$ of the interval $[0, T]$, with

$$0 = t^0_k < t^1_k < \cdots < t^{k-1}_k < t^k_k = T\,,$$
$$\lim_{k\to\infty} \max_{1\leq i\leq k} (t^i_k - t^{i-1}_k) = 0\,.$$

For every k we define the approximate solutions α_k, u_k, e_k, and p_k by induction as follows. We set $(\alpha_k^0, (u_k^0, e_k^0, p_k^0)) := (\alpha_0, (u_0, e_0, p_0)) \in W^{1,\gamma}(\Omega;[0,1]) \times A(w(0))$ and for $i = 1, \dots, k$ we define $(\alpha_k^i, (u_k^i, e_k^i, p_k^i))$ as a solution to the incremental problem

$$\min_{(\beta,(u,e,q))\in\mathcal{D}(\alpha_k^{i-1})\times A(w_k^i)} \{\mathcal{E}_\lambda(\beta, e; p_k, t_k^{i-1}) + \mathcal{H}(\beta, q - p_k^{i-1})\}\,, \tag{4.6}$$

where $w_k^i := w(t_k^i)$ and, according to (3.2) and using Lemma A.1(2),

$$\mathcal{E}_\lambda(\beta, e; p_k, t_k^{i-1}) = \mathcal{E}(\beta, e) + \lambda \sum_{j=1}^{i-1} \mathcal{H}(\beta, p_k^j - p_k^{j-1})\,,$$

with $p_k(t) := p_k^h$, h being the largest integer such that $t_k^h \le t$. The existence of a solution to this problem and the fact that $\alpha_k^i \in W^{1,\gamma}(\Omega;[0,1])$ for every $k \in \mathbb{N}$ and $0 \le i \le k$ follow from Theorem 3.1.

For every $t \in [0,T]$ we define the piecewise constant interpolations

$$\alpha_k(t) := \alpha_k^i,\;\; u_k(t) := u_k^i,\;\; e_k(t) := e_k^i,\;\; \sigma_k(t) := \mathbb{C}(\alpha_k^i)e_k^i,\;\; w_k(t) := w_k^i\,, \tag{4.7}$$

where i is the largest integer such that $t_k^i \le t$. By definition $t \mapsto \alpha_k(t)$ is nonincreasing, $\alpha_k(t) \in W^{1,\gamma}(\Omega;[0,1])$ and $(u_k(t), e_k(t), p_k(t)) \in A(w_k(t))$ for every $t \in [0,T]$, and by (4.6) we have that

$$\begin{aligned} \mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^i) + (1-\lambda)\mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) &= \mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^{i-1}) + \mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) \\ &\le \mathcal{E}_\lambda(\beta, e; p_k, t_k^{i-1}) + \mathcal{H}(\beta, q - p_k^{i-1}) \end{aligned} \tag{4.8}$$

for every k, $1 \le i \le k$, and $(\beta, (u,e,q)) \in \mathcal{D}(\alpha_k^i) \times A(w_k^i)$. Since

$$\begin{aligned} \mathcal{H}(\beta, q - p_k^{i-1}) &\le \mathcal{H}(\beta, p_k^i - p_k^{i-1}) + \mathcal{H}(\beta, q - p_k^i) \\ &\le \lambda\mathcal{H}(\beta, p_k^i - p_k^{i-1}) + (1-\lambda)\mathcal{H}(\alpha_k^i, p_k^i - p_k^{i-1}) + \mathcal{H}(\beta, q - p_k^i)\,, \end{aligned}$$

from (4.8) we get the discrete formulation of global stability

$$\mathcal{E}_\lambda(\alpha_k(t), e_k(t); p_k, t) \le \mathcal{E}_\lambda(\beta, e; p_k, t) + \mathcal{H}(\beta, q - p_k(t)) \tag{4.9}$$

for every k, $t \in [0,T]$, and $(\beta, (u,e,q)) \in \mathcal{D}(\alpha_k(t)) \times A(w_k(t))$. Notice that if $\lambda = 0$ the equation (4.9) follows directly from Lemma 3.2.

The discrete energy inequality. We now derive an energy estimate for the solutions of the incremental problems. Let us fix $i \in \{1, \dots, k\}$ and for a given integer h with $1 \le h \le i$ let $u := u_k^{h-1} - w_k^{h-1} + w_k^h$ and $e := e_k^{h-1} - Ew_k^{h-1} + Ew_k^h$.

Since $(\alpha_k^{h-1}, (u, e, p_k^{h-1})) \in \mathcal{D}(\alpha_k^{h-1}) \times A(w_k^h)$, by the minimality condition (4.6) we obtain

$$\begin{aligned} &\mathcal{E}_\lambda(\alpha_k^h, e_k^h; p_k, t_k^{h-1}) + \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) \le \mathcal{E}_\lambda(\alpha_k^{h-1}, e_k^{h-1}; p_k, t_k^{h-1}) \\ &\quad + \mathcal{Q}(\alpha_k^{h-1}, Ew_k^h - Ew_k^{h-1}) + \langle \mathbb{C}(\alpha_k^{h-1})e_k^{h-1}, Ew_k^h - Ew_k^{h-1}\rangle\,, \end{aligned} \tag{4.10}$$

where we have used the identity

$$\mathcal{Q}(\alpha, e_1 + e_2) = \mathcal{Q}(\alpha, e_1) + \mathcal{Q}(\alpha, e_2) + \langle \mathbb{C}(\alpha)e_1, e_2\rangle$$

for every $\alpha \in W^{1,\gamma}(\Omega)$ and e_1, $e_2 \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. From the absolute continuity of w with respect to t we obtain

$$w_k^h - w_k^{h-1} = \int_{t_k^{h-1}}^{t_k^h} \dot w(t)\,\mathrm{d}t\,,$$

where we use a Bochner integral of a function with values in $H^1(\mathbb{R}^n;\mathbb{R}^n)$. This implies that

$$Ew_k^h - Ew_k^{h-1} = \int_{t_k^{h-1}}^{t_k^h} E\dot w(t)\,\mathrm{d}t\,, \tag{4.11}$$

where the integral is again in the sense of Bochner and the target space is $L^2(\mathbb{R}^n;\mathbb{M}^{n\times n}_{sym})$. By (2.5d) and (4.11) we get

$$\mathcal{Q}(\alpha_k^{h-1}, Ew_k^h - Ew_k^{h-1}) \le \gamma_2\Big(\int_{t_k^{h-1}}^{t_k^h} \|E\dot w(t)\|_2\,\mathrm{d}t\Big)^2\,.$$

From (4.10) and (4.11) it follows that

$$\begin{aligned}\mathcal{E}_\lambda(\alpha_k^h, e_k^h; p_k, t_k^h) + (1-\lambda)\mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) &\le \mathcal{E}_\lambda(\alpha_k^{h-1}, e_k^{h-1}; p_k, t_k^{h-1}) \\ &+ \int_{t_k^{h-1}}^{t_k^h} \langle \mathbb{C}(\alpha_k^{h-1}) e_k^{h-1}, E\dot{w}(t)\rangle \,\mathrm{d}t + \omega_k \int_{t_k^{h-1}}^{t_k^h} \|E\dot{w}(t)\|_2 \,\mathrm{d}t\,,\end{aligned} \tag{4.12}$$

where

$$\omega_k := \gamma_2 \max_{1\le h\le k} \int_{t_k^{h-1}}^{t_k^h} \|E\dot{w}(t)\|_2 \,\mathrm{d}t \to 0$$

by the absolute continuity of the integral. Iterating now inequality (4.12) for $1 \le h \le i$, we have

$$\mathcal{E}_\lambda(\alpha_k^i, e_k^i; p_k, t_k^i) + (1-\lambda)\sum_{h=1}^{i} \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) \le \mathcal{E}(\alpha_0, e_0) + \int_0^{t_k^i} \langle \sigma_k(s), E\dot{w}(s)\rangle \,\mathrm{d}s + \delta_k\,, \tag{4.13}$$

with $\delta_k := \omega_k \int_0^T \|E\dot{w}(t)\|_2 \,\mathrm{d}t \to 0$.

A priori estimates. Using the hypoteses (2.5d) and (2.11d) in the left-hand side, as well as (2.6) and the fact that the function $t \mapsto \|E\dot{w}(t)\|_2$ is integrable on $[0,T]$ in the right-hand side, we find

$$\begin{aligned}&\gamma_1 \|e_k(t)\|_2^2 + D(\alpha_k(t)) + \|\nabla\alpha_k(t)\|_\gamma^\gamma + r(1-\lambda)\sum_{h=1}^{i} \|p_k^h - p_k^{h-1}\|_1 \\ &\le \mathcal{E}(\alpha_0, e_0) + 2\gamma_2 \sup_{t\in[0,T]} \|e_k(t)\|_2 \int_0^T \|E\dot{w}(s)\|_2 \,\mathrm{d}s + \delta_k\end{aligned}$$

for every k and $t \in [t_k^1, T]$, where i is the largest integer such that $t_k^i \le t$.

Thus, by the Cauchy inequality,

$$\sup_{t\in[0,T]} \|e_k(t)\|_2 \le C\,. \tag{4.14}$$

Henceforth, C denotes a suitable constant depending only on γ_1, γ_2, r, and on the functions α_0, e_0, and w. We immediately deduce that

$$\sup_{t\in[0,T]} \|\nabla\alpha_k(t)\|_\gamma^\gamma \le C\,, \tag{4.15}$$

and, from the fact that $t \mapsto p_k(t)$ is constant on the intervals $[t_k^{h-1}, t_k^h[$, that

$$\mathcal{V}(p_k; 0, T) = \sum_{i=1}^{k} \|p_k^i - p_k^{i-1}\|_1 \le C\,. \tag{4.16}$$

Passage to the limit. Since the functions α_k are nonincreasing in time and take values in $[0,1]$, by virtue of (4.15) we can apply the generalized version of the classical Helly Theorem given in [14, Helly Theorem] to conclude that there exist a subsequence, still denoted α_k, and a function $\alpha\colon [0,T] \to W^{1,\gamma}(\Omega; [0,1])$ nonincreasing in time such that $\alpha_k(t) \to \alpha(t)$ strongly in $L^1(\Omega)$ for every $t \in [0,T]$. By (4.15) and the Urysohn Property we have weak convergence in $W^{1,\gamma}(\Omega)$ and thus uniform convergence in $\overline{\Omega}$. In particular (qs0) holds.

In the same way, using now (4.16) and [6, Lemma 7.2], we can assume that there exists $p\colon [0,T] \to M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$ with bounded variation on $[0,T]$ such that $p_k(t) \rightharpoonup p(t)$ weakly* in $M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$, for every $t \in [0,T]$.

Following the same argument used in the proof of Theorem 3.1, by (4.14) and (4.16) we can deduce that

$$\sup_{t\in[0,T]} \|u_k(t)\|_{BD(\Omega;\mathbb{R}^n)} \le C\,. \tag{4.17}$$

Let us fix $t \in [0,T]$. From (4.14) and (4.17) it follows that there exist an increasing sequence k_j (possibly depending on t) and two functions $u(t) \in BD(\Omega;\mathbb{R}^n)$ and $e(t) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ such that $u_{k_j}(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega;\mathbb{R}^n)$ and $e_{k_j}(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. By

(4.9) we can apply Theorem 3.8 (or Theorem 3.6 if $\lambda = 0$) and find that $(\alpha(t), (u(t), e(t), p(t)))$ is a solution to the minimum problem

$$\min_{(\beta,(u,e,q))\in\mathcal{D}(\alpha(t))\times A(w(t))} \{\mathcal{E}_\lambda(\beta, e; p, t) + \mathcal{H}(\beta, q - p(t))\} .$$

Since by convexity the functional $e \mapsto \mathcal{E}(\alpha(t), e)$ has a unique minimizer, then the convergence result holds for the whole sequence, namely $u_k(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega;\mathbb{R}^n)$ and $e_k(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. Then (qs1) holds.

To prove that $t \mapsto (\alpha(t), u(t), e(t), p(t))$ is a quasistatic evolution it remains to show the energy balance (qs2).

Energy balance. We consider now the asymptotics of the discrete energy inequality (4.13). Later we will show that also the equality holds in the limit.

Since p_k is piecewise constant and continuous from the right, α_k is nonincreasing, and (2.11b) holds, by Lemma A.1(2) we have

$$\mathcal{V}_\mathcal{H}(\alpha_k, p_k; 0, t) = \sum_{h=1}^{i} \mathcal{H}(\alpha_k^h, p_k^h - p_k^{h-1}) , \tag{4.18}$$

where i is the largest integer such that $t_k^i \le t$. From the lower semicontinuity of $\mathcal{H}$ (Lemma 2.3) and the definition of plastic dissipation (2.24) it follows that

$$\mathcal{V}_\mathcal{H}(\alpha, p; 0, t) \le \liminf_{k\to\infty} \mathcal{V}_\mathcal{H}(\alpha_k, p_k; 0, t), \text{ and } \mathcal{V}_\mathcal{H}(\alpha(t), p; 0, t) \le \liminf_{k\to\infty} \mathcal{V}_\mathcal{H}(\alpha_k(t), p_k; 0, t) . \tag{4.19}$$

Notice that in the first inequality α and α_k are functions from $[0, T]$ into $W^{1,\gamma}(\Omega)$, while in the second one we consider the pointwise value of α and α_k at a given time t.

Moreover, since $\alpha_k(t) \to \alpha(t)$ uniformly in $\overline{\Omega}$ and $e_k(t) \rightharpoonup e(t)$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ for every $t \in [0, T]$, by (2.5), (2.25), (4.14), and the Dominated Convergence Theorem we get

$$\int_0^t \langle\sigma(s), E\dot{w}(s)\rangle\, \mathrm{d}s = \lim_{k\to\infty} \int_0^{t_k^i} \langle\sigma_k(s), E\dot{w}(s)\rangle\, \mathrm{d}s , \tag{4.20}$$

where $\sigma(s) := \mathbb{C}(\alpha(s))e(s)$ for every $s \in [0, T]$.

Collecting (4.18)–(4.20), from (4.13) and the lower semicontinuity of the remaining terms the inequality

$$\mathcal{E}_\lambda(\alpha(t), e(t); p, t) + (1-\lambda)\mathcal{V}_\mathcal{H}(\alpha, p; 0, t) \le \mathcal{E}(\alpha(0), e(0)) + \int_0^t \langle\sigma(s), E\dot{w}(s)\rangle\, \mathrm{d}s$$

follows, for every $t \in [0, T]$.

Conversely, let us fix $t \in [0, T]$ and let $(s_k^i)_{0\le i\le k}$ be a sequence of subdivisions of the interval $[0, t]$ satisfying

$$0 = s_k^0 < s_k^1 < \cdots < s_k^{k-1} < s_k^k = t ,$$
$$\lim_{k\to\infty} \max_{1\le i\le k} (s_k^i - s_k^{i-1}) = 0 .$$

For every $i = 1, \ldots, k$ let $u := u(s_k^i) - w(s_k^i) + w(s_k^{i-1})$ and $e := e(s_k^i) - Ew(s_k^i) + Ew(s_k^{i-1})$. Since $(\alpha(s_k^i), (u, e, p(s_k^i))) \in \mathcal{D}(\alpha(s_k^{i-1})) \times A(w(s_k^{i-1}))$, by the global stability (4.2) we have

$$\begin{aligned}
\mathcal{E}_\lambda(\alpha(s_k^{i-1}), e(s_k^{i-1}); p, s_k^{i-1}) &\le \mathcal{E}_\lambda(\alpha(s_k^i), e(s_k^i); p, s_k^{i-1}) + \mathcal{Q}(\alpha(s_k^i), Ew(s_k^{i-1}) - Ew(s_k^i)) \\
&\quad - \langle\sigma(s_k^i), Ew(s_k^i) - Ew(s_k^{i-1})\rangle + \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1})) \\
&= \mathcal{E}_\lambda(\alpha(s_k^i), e(s_k^i); p, s_k^i) + \mathcal{Q}(\alpha(s_k^i), Ew(s_k^{i-1}) - Ew(s_k^i)) \\
&\quad - \langle\sigma(s_k^i), Ew(s_k^i) - Ew(s_k^{i-1})\rangle + (1-\lambda)\mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1}))
\end{aligned}$$

Now, following the same argument used in (4.12), we find that there exists a sequence $\omega_k \to 0^+$ such that

$$\mathcal{E}_\lambda(\alpha(s_k^{i-1}),\, e(s_k^{i-1}); p, s_k^{i-1}) \;\le\; \mathcal{E}_\lambda(\alpha(s_k^i), e(s_k^i); p, s_k^i) + (1-\lambda)\mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1}))$$
$$- \int_{s_k^{h-1}}^{s_k^h} \langle \sigma(s_k^i), E\dot w(t)\rangle dt + \omega_k \int_{s_k^{h-1}}^{s_k^h} \|E\dot w(t)\|_2 \,\mathrm{d}t\,.$$

On $[0,t]$ we define the piecewise constant function $\overline{\sigma}_k(s) := \sigma(s_k^i)$, where i is the smallest index such that $s \le s_k^i$.

Since $\sum_i \mathcal{H}(\alpha(s_k^i), p(s_k^i) - p(s_k^{i-1})) \le \mathcal{V}_\mathcal{H}(\alpha, p; 0, t)$, iterating the last inequality for $1 \le i \le k$ we obtain

$$\mathcal{E}(\alpha(0), e(0)) \le \mathcal{E}_\lambda(\alpha(t), e(t); p, t) + (1-\lambda)\mathcal{V}_\mathcal{H}(\alpha, p; 0, t) - \int_0^t \langle \overline{\sigma}_k(s), E\dot w(s)\rangle ds + \delta_k\,,$$

where $\delta_k := \omega_k \int_0^T \|E\dot w(s)\|_2 \,\mathrm{d}s$. As in (4.20) we get

$$\int_0^t \langle \sigma(s), E\dot w(s)\rangle \,\mathrm{d}s = \lim_{k\to\infty} \int_0^{t_k^i} \langle \overline{\sigma}_k(s), E\dot w(s)\rangle \,\mathrm{d}s\,,$$

which concludes the proof. □

5. Qualitative properties of quasistatic evolutions

In this section we show some qualitative properties of quasistatic evolutions, whose existence is proved in Theorem 4.3.

First, in Proposition 5.1, we deduce that $t \mapsto u(t)$, $t \mapsto e(t)$, and $t \mapsto p(t)$ are continuous, with respect to the norms of their spaces, at the continuity points for $t \mapsto \alpha(t)$ with respect to the uniform convergence in $\overline{\Omega}$. Then the time discontinuities of the quasistatic evolution are at most countable, by Remark 4.2. This regularity in time of α also permits to say that $\mathcal{H}(\alpha(\bar t), \dot p(\bar t))$ represents the rate of plastic dissipation at $\bar t$, and then to understand the physical meaning of the term in λ in (qs1) (cf. Remark 5.2).

In Corollary 5.3 we derive from (qs1) Euler conditions with respect to the variation of u, e, and p. In the last part of the section we assume suitable regularity properties on $\mathbb{C}$, D and $\mathcal{H}$, and absolute continuity of the evolutions. In Proposition 5.4 is shown an Euler condition for α and the differential counterpart of the energy balance (qs2): together with the irreversibility, these can be seen as Kuhn-Tucker conditions for the damage variable α (in damage models is usual to deal with such conditions, see e.g. [25]); moreover, it is recovered in (5.11) the Hill's maximum plastic work principle.

Finally we show that, if p is regular enough, we can recover the flow rule present in the classical formulation of evolutions for elastoplastic models (see e.g. [6, Section 1] and [16, Definition 3.1]).

Throughout this section, we suppose that (2.4), (2.5), (2.9), and (2.25) hold when $\lambda = 0$; when $\lambda > 0$ we will assume also (2.12).

Except for countable many instants, every quasistatic evolution is continuous in time, as shown in the following result.

Proposition 5.1. *Every quasistatic evolution* $t \mapsto (\alpha(t), u(t), e(t), p(t))$ *is strongly continuous from* $[0,T]$ *into* $C(\overline{\Omega}; [0,1]) \times BD(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times M_b(\overline{\Omega};\mathbb{M}^{n\times n}_D)$ *except for a countable subset of* $[0,T]$*, which is the set of discontinuity points of* α *with respect to the uniform convergence in* $\overline{\Omega}$.

Proof. From the energy balance condition (qs2), written for a time interval $[s,t]$, we deduce

$$Q(\alpha(t), e(t)) - Q(\alpha(s), e(s)) + \mathcal{H}(\alpha(t), p(t) - p(s))$$
$$\le \int_s^t \langle \sigma(\tau), E\dot w(\tau)\rangle \,\mathrm{d}\tau + D(\alpha(s)) - D(\alpha(t)) + \|\nabla(\alpha(s))\|_\gamma^\gamma - \|\nabla(\alpha(t))\|_\gamma^\gamma\,,$$

using also (1) of Lemma A.1 both for $(1-\lambda)\mathcal{V}_\mathcal{H}(\alpha, p; s, t)$ and for $\lambda\mathcal{V}_\mathcal{H}(\alpha(t), p; s, t)$.

Notice now that

$$D(\alpha(s)) - D(\alpha(t)) + \|\nabla(\alpha(s))\|_\gamma^\gamma - \|\nabla(\alpha(t))\|_\gamma^\gamma \leq 0\,.$$

Indeed, if the term above were strictly positive, from (2.11b) and (2.5b) we would have

$$\mathcal{E}(\alpha(t), e(s)) + \lambda \mathcal{V}_{\mathcal{H}}(\alpha(t), p; 0, t) < \mathcal{E}(\alpha(s), e(s)) + \lambda \mathcal{V}_{\mathcal{H}}(\alpha(s), p; 0, t)\,,$$

which contradicts (qs1) since $(\alpha(t), (u(s), e(s), p(s))) \in \mathcal{D}(\alpha(s)) \times A(w(s))$.

Then

$$Q(\alpha(t), e(t)) - Q(\alpha(s), e(s)) + \mathcal{H}(\alpha(t), p(t) - p(s)) \leq \int_s^t \langle \sigma(\tau), E\dot{w}(\tau)\rangle\,\mathrm{d}\tau \tag{5.1}$$

Now, by Lemma 3.3 it follows that

$$-\langle \sigma(s), e(t) - e(s) - (Ew(t) - Ew(s))\rangle \leq \mathcal{H}(\alpha(s), p(t) - p(s))\,, \tag{5.2}$$

because $(u(t)-u(s)-(w(t)-w(s)), e(t)-e(s)-(Ew(t)-Ew(s)), p(t)-p(s)) \in A(0)$. Summing (5.1) and (5.2) we get

$$\begin{aligned} Q(\alpha(s), e(t) - e(s)) \leq \frac{1}{2}\langle \big[\mathbb{C}(\alpha(s)) - \mathbb{C}(\alpha(t))\big] e(t), e(t)\rangle - \langle \sigma(s), Ew(t) - Ew(s)\rangle \\ + \int_s^t \langle \sigma(\tau), E\dot{w}(\tau)\rangle\,\mathrm{d}\tau + \mathcal{H}(\alpha(s), p(t) - p(s)) - \mathcal{H}(\alpha(t), p(t) - p(s)) \end{aligned}$$

which implies

$$\|e(t) - e(s)\|_2^2 \leq C\Big(\|\alpha(t) - \alpha(s)\|_\infty + \omega(\|\alpha(t) - \alpha(s)\|_\infty) + \|Ew(t) - Ew(s)\|_2\Big)\,, \tag{5.3}$$

where ω was introduced in (2.17) and C depends on $\mathrm{Lip}(\mathbb{C})$, γ_1, γ_2, and $\sup_t \|e(t)\|_2$ (recall that, from (qs2), the variation of p is bounded by such a C).

By (5.1), (2.16), and (5.3), we obtain

$$\|p(t) - p(s)\|_2^2 \leq \widetilde{C}\Big(\|\alpha(t) - \alpha(s)\|_\infty + \omega(\|\alpha(t) - \alpha(s)\|_\infty) + \|Ew(t) - Ew(s)\|_2\Big)\,,$$

$\widetilde{C}$ depending on C, r, and $\sup_t \|Ew(t)\|_2$. An analogous estimate holds for u, arguing as in [6, Theorem 3.8]. Then we conclude by Remark 4.2, where it is stated that the discontinuity points of $t \mapsto \alpha(t)$ with respect to the uniform convergence in $\overline{\Omega}$ are countable many. □

In order to establish the differential formulation of the energy balance the following Remark turns to be useful.

Remark 5.2. If in addition $p \in AC([0,T]; M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D))$ then

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) = \int_0^t \mathcal{H}(\alpha(s), \dot{p}(s))\,\mathrm{d}s \tag{5.4}$$

for every $t \in [0,T]$.

Indeed, since $\alpha\colon [0,T] \to \mathbb{C}(\overline{\Omega}; [0,1])$ has at most countable many discontinuity points, it can be approximated by functions $\alpha_k\colon [0,T] \to \mathbb{C}(\overline{\Omega}; [0,1])$ piecewise constant and nonincreasing in time, such that $\alpha_k(t) \to \alpha(t)$ uniformly and increasingly in $\overline{\Omega}$ for $\mathcal{L}^1$-a.e. $t \in [0,T]$. Therefore we apply Lemma A.1(4) and conclude (5.4).

In the light of (5.4), we point out that the term in λ in (qs1) makes it easier to damage, at a given instant t, a part of the material more affected by plastic evolution until t: indeed, if $p \in AC([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_D))$ and $\alpha \in C(\overline{\Omega}; [0,1])$, we get that

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) = \int_\Omega \int_0^t H(\alpha(x), \dot{p}(s,x))\,\mathrm{d}s\,\mathrm{d}x\,.$$

To fix the ideas, let us consider the simplest case of a multiplicative setting (see Remark 2.1) where $K(1) = B(1)$, the unit ball of $\mathbb{M}^{n\times n}_D$. Here the above formula reads as

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) = \int_\Omega V(\alpha(x))\Big(\int_0^t |\dot{p}(s,x)|\,\mathrm{d}s\Big)\,\mathrm{d}x\,.$$

By the monotonicity property of V, in order to minimize $\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t)$ it is convenient to take α smaller when the variation of plastic strain $\int_0^t |\dot{p}(s, \cdot)|\,\mathrm{d}s$ is greater.

The stability condition (qs1) and Lemma 3.3 imply the following result, which states Euler conditions with respect to variations of u, e, and p; in particular, (5.5a) gives a constraint for the elastic stress.

Corollary 5.3. *Let $t \in [0, T] \mapsto (\alpha(t), u(t), e(t), p(t))$ be a quasistatic evolution corresponding to $\lambda \in [0, 1]$. Then we have that for every $t \in [0, T]$:*

$$\sigma(t) \in \mathcal{K}_{\alpha(t)}(\Omega)\,, \tag{5.5a}$$

$$\operatorname{div}(\sigma(t)) = 0 \text{ in } \Omega\,. \tag{5.5b}$$

Let us now assume the multiplicative setting of Remark 2.1, C^1 regularity for $\mathbb{C}$, D, V, and absolute continuity for the quasistatic evolution. Then we can obtain a differential condition also for the damage variable α and a differential formulation of the energy balance.

Proposition 5.4. *Besides the assumptions* (2.4)*,* (2.5)*, and* (2.9)*, let us assume that*

$$d \in C^1(\mathbb{R})\,, \tag{5.6}$$

$$\mathbb{C} \in C^1(\mathbb{R}; Lin(\mathbb{M}^{n\times n}_{sym}; \mathbb{M}^{n\times n}_{sym}))\,, \tag{5.7}$$

$$K(\alpha) = V(\alpha)K(1), \text{ with } K(1) \text{ closed and convex, } B_r(0) \subset K(1) \subset B_R(0),\ V \in C^1(\mathbb{R})\,. \tag{5.8}$$

Let $t \in [0, T] \mapsto (\alpha(t), u(t), e(t), p(t))$ be a quasistatic evolution corresponding to $\lambda \in [0, 1]$ absolutely continuous into $W^{1,\gamma}(\Omega; [0, 1]) \times BD(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times M_b(\overline{\Omega}; \mathbb{M}^{n\times n}_D)$. Then both D and $\beta \mapsto \mathcal{V}_{\mathcal{H}}(\beta, p; 0, t)$ belong to $C^1(C(\overline{\Omega}))$ and the following hold:

$$\begin{aligned} &\frac{1}{2}\langle \mathbb{C}'(\alpha(t))\beta e(t), e(t)\rangle + \langle D'(\alpha(t)), \beta\rangle_{C(\overline{\Omega})} + \gamma \int_\Omega |\nabla\alpha(t)|^{\gamma-2}\nabla\alpha(t)\cdot\nabla\beta\,\mathrm{d}x \\ &\qquad + \lambda\langle \frac{\partial}{\partial\alpha}\mathcal{V}_{\mathcal{H}}(\alpha(t), p; 0, t), \beta\rangle_{C(\overline{\Omega})} \geq 0 \end{aligned} \tag{5.9}$$

for every $t \in [0, T]$ and $\beta \in W^{1,\gamma}(\Omega)$, $\beta \leq 0$ in Ω,

$$\begin{aligned} &\frac{1}{2}\langle \mathbb{C}'(\alpha(t))\dot{\alpha}(t) e(t), e(t)\rangle + \langle D'(\alpha(t)), \dot{\alpha}(t)\rangle_{C(\overline{\Omega})} + \gamma \int_\Omega |\nabla\alpha(t)|^{\gamma-2}\nabla\alpha(t)\cdot\nabla\dot{\alpha}(t)\,\mathrm{d}x \\ &\qquad + \lambda\langle \frac{\partial}{\partial\alpha}\mathcal{V}_{\mathcal{H}}(\alpha(t), p; 0, t), \dot{\alpha}(t)\rangle_{C(\overline{\Omega})} = 0\,, \end{aligned} \tag{5.10}$$

and

$$\mathcal{H}(\alpha(t), \dot{p}(t)) = \langle (\sigma(t))_D \,|\, \dot{p}(t)\rangle\,, \tag{5.11}$$

for a.e. $t \in (0, T)$, with $\sigma(t) := \mathbb{C}(\alpha(t))e(t)$.

Proof. The fact that $\beta \mapsto \mathcal{V}_{\mathcal{H}}(\beta, p; 0, t) \in C^1(C(\overline{\Omega}))$ follows from (5.8) and Dominated Convergence Theorem. Let $t \in [0, T]$ and $\beta \in W^{1,\gamma}(\Omega)$, with $\beta \leq 0$ in Ω. For every $h > 0$, using $(\alpha(t) + h\beta, (u(t), e(t), p(t)))$ as a test pair in (qs1), we get

$$\frac{\mathcal{E}_\lambda(\alpha(t) + h\beta, e(t); p, t) - \mathcal{E}_\lambda(\alpha(t), e(t); p, t)}{h} \geq 0\,. \tag{5.12}$$

Now (5.6) and (5.7) imply that the left hand side of (5.9) is the limit as $h \to 0$ of the one of (5.12), and this immediately gives (5.9).

By [6, Lemma 5.5] we have that for a.e. $t \in (0, T)$

$$(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in A(\dot{w}(t))\,. \tag{5.13}$$

Thus, by (5.5b), (5.13), and the integration by parts formula (2.21) we get

$$\langle (\sigma(t))_D \,|\, \dot{p}(t)\rangle = \langle \sigma(t), E\dot{w}(t) - \dot{e}(t)\rangle \tag{5.14}$$

and by (qs2), recalling (5.4), it follows that for a.e. $t \in (0, T)$

$$\begin{aligned} &\langle \sigma(t), \dot{e}(t)\rangle + \mathcal{H}(\alpha(t), \dot{p}(t)) + \frac{1}{2}\langle \mathbb{C}'(\alpha(t))\dot{\alpha}(t)e(t), e(t)\rangle + \langle D'(\alpha(t)), \dot{\alpha}(t)\rangle \\ &\quad + \gamma\int_\Omega |\nabla\alpha(t)|^{\gamma-2}\nabla\alpha(t)\cdot\nabla\dot{\alpha}(t)\,\mathrm{d}x + \lambda\langle \frac{\partial}{\partial\alpha}\mathcal{V}_{\mathcal{H}}(\alpha(t), p; 0, t), \dot{\alpha}(t)\rangle = \langle \sigma(t), E\dot{w}(t)\rangle \end{aligned} \tag{5.15}$$

for a.e. $t \in (0,T)$. From (5.14) and (5.15) we obtain that

$$\begin{aligned}&\mathcal{H}(\alpha(t),\dot p(t)) - \langle(\sigma(t))_D\,|\,\dot p(t)\rangle + \frac{1}{2}\langle\mathbb{C}'(\alpha(t))\dot\alpha(t)e(t),e(t)\rangle + \langle D'(\alpha(t)),\dot\alpha(t)\rangle\\&+\gamma\int_\Omega|\nabla\alpha(t)|^{\gamma-2}\nabla\alpha(t)\cdot\nabla\dot\alpha(t)\,\mathrm{d}x + \lambda\langle\frac{\partial}{\partial\alpha}\mathcal{V}_\mathcal{H}(\alpha(t),p;0,t),\dot\alpha(t)\rangle = 0\end{aligned}\tag{5.16}$$

for a.e. $t \in (0,T)$. By (5.5a) and (2.22),

$$\mathcal{H}(\alpha(t),\dot p(t)) - \langle(\sigma(t))_D\,|\,\dot p(t)\rangle \geq 0\,,\tag{5.17}$$

while (5.9) gives

$$\begin{aligned}&\frac{1}{2}\langle\mathbb{C}'(\alpha(t))\dot\alpha(t)e(t),e(t)\rangle + \langle D'(\alpha(t)),\dot\alpha(t)\rangle\\&+\gamma\int_\Omega|\nabla\alpha(t)|^{\gamma-2}\nabla\alpha(t)\cdot\nabla\dot\alpha(t)\,\mathrm{d}x + \lambda\langle\frac{\partial}{\partial\alpha}\mathcal{V}_\mathcal{H}(\alpha(t),p;0,t),\dot\alpha(t)\rangle \geq 0\end{aligned}\tag{5.18}$$

for a.e. $t \in (0,T)$. Therefore we conclude by (5.16), (5.17) and (5.18). □

We can now use the maximal dissipation property (5.11) (also called Hill's maximum plastic work principle) to show the validity of the elastoplastic flow rule $\mathcal{L}^n$–a.e. on the support $\{|\dot p(t)|>0\}$ of the measure $\dot p(t)$. The following Remark is useful to prove Proposition 5.6.

Remark 5.5. From (5.5a), (2.22), and (5.11) we deduce that for a.e. $t \in (0,T)$

$$H\Big(\alpha(t),\frac{\mathrm{d}\dot p(t)}{\mathrm{d}|\dot p(t)|}\Big)|\dot p(t)| = [\sigma_D(t):\dot p(t)] \quad \text{as measures on } \overline{\Omega},\tag{5.19}$$

where the measure denoted by square brackets is defined in (2.20).

Proposition 5.6 (Flow rule). *In the hypoteses of Proposition 5.4, for a.e. $t \in (0,T)$*

$$\frac{\mathrm{d}\dot p(t)}{\mathrm{d}|\dot p(t)|}(x) \in N_{K(\alpha(t,x))}(\sigma_D(t,x)) \quad \textit{for } \mathcal{L}^n\textit{–a.e. } x \in \{|\dot p(t)|>0\}\,,\tag{5.20}$$

where $\sigma_D(t,x)$ denotes the value of $\sigma_D(t)$ at the point x and $N_{K(\alpha(t,x))}(\sigma_D(t,x))$ is the normal cone to the closed convex set $K(\alpha(t,x))$ at $\sigma_D(t,x)$. In particular, if $\dot p(t) \in L^1(\Omega)$ for a.e. $t \in (0,T)$, we have that

$$\dot p(t,x) \in N_{K(\alpha(t,x))}(\sigma_D(t,x)) \quad \textit{for } \mathcal{L}^n\textit{–a.e. } x\,.\tag{5.21}$$

Proof. It is enough to argue as in the proof of [16, Theorem 3.13]. □

A. Auxiliary results

In this Appendix we analyse the particular variation used to define the plastic dissipation and show a property of increasing functions with values in L^p spaces.

A.1. **A "weighted" variation.** Let X be a Banach space, F a set, and $\mathcal{H}\colon F\times X\to\mathbb{R}^+\cup\{0\}$. Given $\alpha\colon[0,T]\to F$, $p\colon[0,T]\to X$, $a,b\in[0,T]$ with $a<b$, and $\mathcal{P}:=\{t_i\}_{0\leq i\leq N}$ with $a=t_0<t_1<\dots<t_N=b$, we define

$$\mathcal{V}^\mathcal{P}_\mathcal{H}(\alpha,p;a,b) := \sum_{i=1}^N \mathcal{H}(\alpha(t_i),p(t_i)-p(t_{i-1}))\,.$$

and the *$\mathcal{H}$-variation of p with respect to α on* $[a,b]$ as

$$\begin{aligned}\mathcal{V}_\mathcal{H}(\alpha,p;a,b) :=& \sup\Big\{\sum_{i=1}^N\mathcal{H}(\alpha(t_i),p(t_i)-p(t_{i-1})) : a=t_0<t_1<\dots<t_N=b,\ N\in\mathbb{N}\Big\}\\=&\sup\Big\{\mathcal{V}^\mathcal{P}_\mathcal{H}(\alpha,p;a,b)\,|\,\mathcal{P}\text{ partition of }[a,b]\Big\}\,.\end{aligned}\tag{A.1}$$

Let us assume that

$$\mathcal{H}(\alpha(t_2), f) \leq \mathcal{H}(\alpha(t_1), f), \text{ for every } \quad 0 \leq t_1 \leq t_2 \leq T,\ f \in X\,, \tag{A.2a}$$

$$\mathcal{H}(\beta, 0) = 0, \text{ for every } \quad \beta \in F\,, \tag{A.2b}$$

$$\mathcal{H}(\beta, f_1 + f_2) \leq \mathcal{H}(\beta, f_1) + \mathcal{H}(\beta, f_2), \quad \text{for every } \beta \in F,\ f_1, f_2 \in X\,. \tag{A.2c}$$

Lemma A.1. *With the notations and assumptions above, it follows that:*

(1) *If $\mathcal{P}_1, \mathcal{P}_2$ are partitions of $[a,b]$, with $\mathcal{P}_1 \subset \mathcal{P}_2$, then*

$$\mathcal{V}_{\mathcal{H}}^{\mathcal{P}_1}(\alpha, p; a, b) \leq \mathcal{V}_{\mathcal{H}}^{\mathcal{P}_2}(\alpha, p; a, b)\,.$$

(2) *For every $p\colon [a,b] \to X$ piecewise constant and continuous from the right, with discontinuities at the points $s_1, \dots, s_N$ with $a < s_1 < s_2 < \dots < s_N \leq b$,*

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; a, b) = \sum_{i=1}^{N} \mathcal{H}(\alpha(s_i), p(s_i) - p(s_{i-1}))\,,$$

where $s_0 := a$.

(3) *For every $a \leq t_1 < t_2 < t_3 \leq b$,*

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; t_1, t_3) = \mathcal{V}_{\mathcal{H}}(\alpha, p; t_1, t_2) + \mathcal{V}_{\mathcal{H}}(\alpha, p; t_2, t_3)\,.$$

(4) *Assume in addition that F is a measurable topological space, X is the dual of a separable Banach space Y, $p \in AC([a,b];X)$, $\alpha\colon [a,b] \to F$ is continuous for a.e. $t \in [a,b]$, and*

$$\mathcal{H}(\beta, tf) = t\mathcal{H}(\beta, f) \text{ for every } \beta \in F,\ f \in X, \text{ and } t > 0\,, \tag{A.3a}$$

$$f \mapsto \mathcal{H}(\beta, f) \text{ is weakly}^* \text{ lower semicontinuous in } X \text{ for every } \beta \in F\,, \tag{A.3b}$$

$$\mathcal{H}(\beta_k, f) \to \mathcal{H}(\beta, f) \quad \text{for every } \beta_k \to \beta \text{ in } F \text{ and } f \in X\,. \tag{A.3c}$$

Then $t \mapsto \mathcal{H}(\alpha(t), \dot{p}(t))$ is measurable and

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; a, b) = \int_a^b \mathcal{H}(\alpha(t), \dot{p}(t))\, \mathrm{d}t\,. \tag{A.4}$$

Proof. (1) It is enough to see that, for every $a \leq t_1 \leq t_2 \leq t_3 \leq b$,

$$\mathcal{H}(\alpha(t_3), p(t_3) - p(t_1)) \leq \mathcal{H}(\alpha(t_3), p(t_3) - p(t_2)) + \mathcal{H}(\alpha(t_2), p(t_2) - p(t_1))\,.$$

This is true because, by (A.2c), $\mathcal{H}(\alpha(t_3), p(t_3) - p(t_1)) \leq \mathcal{H}(\alpha(t_3), p(t_3) - p(t_2)) + \mathcal{H}(\alpha(t_3), p(t_2) - p(t_1))$; apply then (A.2a) to the second term in the right-hand side.

(2) Observe firstly that given a partition $\mathcal{P} := \{t_i\}_{0 \leq i \leq N}$ of $[a,b]$ it is possible to choose a set of indices $1 \leq i_1 < i_2 < \dots < i_k \leq N$ such that

$$\mathcal{V}_{\mathcal{H}}^{\mathcal{P}}(\alpha, p; a, b) \leq \sum_{j=1}^{k} \mathcal{H}(\alpha(s_{i_j}), p(s_{i_j}) - p(s_{i_j - 1}))\,. \tag{A.5}$$

In fact, if $s_i \leq t_j < t_{j+1} < s_{i+1}$, then

$$\mathcal{H}(\alpha(t_{j+1}), p(t_{j+1}) - p(t_j)) = \mathcal{H}(\alpha(t_{j+1}), p(s_i) - p(s_i)) = 0\,,$$

while if $s_i \leq t_j < s_{i+1} < \dots < s_{i+l} \leq t_{j+1} < s_{i+l+1}$ it follows that

$$\mathcal{H}(\alpha(t_{j+1}), p(t_{j+1}) - p(t_j)) = \mathcal{H}(\alpha(t_{j+1}), p(s_{i+l}) - p(s_i)) \leq \mathcal{H}(\alpha(s_{i+l}), p(s_{i+l}) - p(s_i))\,,$$

by (A.2b) and (A.2a). From (1) and (A.5), for every $\mathcal{P}$ partition of $[a,b]$ the inequalities

$$\mathcal{V}_{\mathcal{H}}^{\mathcal{P}}(\alpha, p; a, b) \leq \sum_{i=1}^{N} \mathcal{H}(\alpha(s_i), p(s_i) - p(s_{i-1})) \leq \mathcal{V}_{\mathcal{H}}(\alpha, p; a, b)$$

holds. The conclusion follows by taking the supremum over the partitions of $[a,b]$.

(3) It is always true that $\mathcal{V}_{\mathcal{H}}(\alpha, p; t_1, t_3) \geq \mathcal{V}_{\mathcal{H}}(\alpha, p; t_1, t_2) + \mathcal{V}_{\mathcal{H}}(\alpha, p; t_2, t_3)$ because for every partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[t_1, t_2]$ and $[t_2, t_3]$, $\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2$ is a partition of $[t_1, t_3]$. On the other hand, for every $\mathcal{P}$ partition of $[t_1, t_3]$, $\tilde{\mathcal{P}} := \mathcal{P} \cup \{t_2\}$ is the union of two partitions of $[t_1, t_2]$ and $[t_2, t_3]$ respectively; since, by (1),

$$\mathcal{V}_{\mathcal{H}}^{\mathcal{P}}(\alpha, p; a, b) \leq \mathcal{V}_{\mathcal{H}}^{\tilde{\mathcal{P}}}(\alpha, p; a, b)\,,$$

the latter inequality holds.

(4) From (A.2c), (A.3a), and (A.3b), we have that for every $\beta \in F$ the function $f \mapsto \mathcal{H}(\beta, f)$ is weakly* lower semicontinuous, convex and positively one-homogeneous. Then, by [19, Theorem 5], for every $\beta \in F$ there exists a bounded closed convex set $\mathcal{K}_\beta \subset Y$ such that

$$\mathcal{H}(\beta, f) = \sup_{y \in \mathcal{K}_\beta} \langle y, f \rangle\,,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and Y. Being Y separable, we get $\mathcal{H}(\beta, f) = \sup_{y \in \mathcal{K}_\beta^0} \langle y, f \rangle$, where $\mathcal{K}_\beta^0$ is a countable dense subset of $\mathcal{K}_\beta$.

Since $p \in AC([a, b]; X)$, the weak*-limit

$$\dot{p}(t) := w^*\text{-}\lim_{s \to t} \frac{p(s) - p(t)}{s - t}$$

exists for a.e. $t \in [a, b]$, and then the function $t \mapsto \langle y, \dot{p}(t) \rangle$ is measurable for every $y \in Y$. Therefore $t \mapsto \mathcal{H}(\beta, \dot{p}(t))$ is measurable for every $\beta \in F$. Moreover, from [6, Theorem 7.1],

$$\mathcal{V}_{\mathcal{H}}(\beta, p; t_1, t_2) = \int_{t_1}^{t_2} \mathcal{H}(\beta, \dot{p}(t))\, \mathrm{d}t\,, \tag{A.6}$$

for every $a \leq t_1 < t_2 \leq b$ and every $\beta \in F$.

Let us fix $\varepsilon > 0$. There exist points $t_0, \dots, t_N$, with $a = t_0 < t_1 < t_2 < \dots < t_N \leq b$, such that

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) - \varepsilon \leq \sum_{i=1}^{N} \mathcal{H}(\alpha(t_i), p(t_i) - p(t_{i-1}))\,. \tag{A.7}$$

For every $k \in \mathbb{N}$ we consider the set $\big(a + i\frac{b-a}{k}\big)_{i=0}^{k} \cup (t_j)_{j=1}^{N} =: s_0^k < s_1^k < \dots < s_{M(k)}^K$, with $s_0^k = a$, and we define α_k as

$$\alpha_k(t) := \alpha(s_{j+1}) \quad \text{when } t \in (s_j, s_{j+1}]$$

and $\alpha_k(a) = \alpha(a)$. In other words α_k is the left-continuous piecewise constant interpolation of α with nodes $(s_j)_j$. By construction

$$\alpha_k(t_j) = \alpha(t_j) \quad \text{for every } j \in \{1, \dots, N\} \tag{A.8}$$

and by (A.2a) and (A.3c) we get that for every $f \in X$

$$\mathcal{H}(\alpha_k(s), f) \leq \mathcal{H}(\alpha_{k+1}(s), f) \leq \mathcal{H}(\alpha(s), f) \tag{A.9}$$

for every $s \in [a, b]$, and

$$\mathcal{H}(\alpha_k(s), f) \to \mathcal{H}(\alpha(s), f) \tag{A.10}$$

for every s continuity point of α.

Since the functions α_k are piecewise constant, from the point (3) and (A.6) we have that

$$\begin{aligned} \mathcal{V}_{\mathcal{H}}(\alpha_k, p; a, b) &= \sum_{j=1}^{M(k)} \mathcal{V}_{\mathcal{H}}(\alpha_k, p; s_{j-1}^k, s_j^k) = \sum_{j=1}^{M(k)} \mathcal{V}_{\mathcal{H}}(\alpha_k(s_j^k), p; s_{j-1}^k, s_j^k) \\ &= \sum_{j=1}^{M(k)} \int_{s_{j-1}}^{s_j} \mathcal{H}(\alpha_k(s_j^k), \dot{p}(t))\, \mathrm{d}t = \int_a^b \mathcal{H}(\alpha_k(t), \dot{p}(t))\, \mathrm{d}t\,. \end{aligned} \tag{A.11}$$

Moreover the fact that α is continuous for a.e. $t \in [a, b]$ and (A.10) imply that

$$t \mapsto \mathcal{H}(\alpha(t), \dot{p}(t)) \text{ is measurable}\,,$$

as well as

$$\int_a^b \mathcal{H}(\alpha(t),\dot p(t))\,\mathrm{d}t = \lim_{k\to\infty}\int_a^b \mathcal{H}(\alpha_k(t),\dot p(t))\,\mathrm{d}t\,, \tag{A.12}$$

using the Monotone Convergence Theorem.

By (A.7), (A.8), and (A.9) we obtain

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;a,b)-\varepsilon \le \sum_{i=1}^N \mathcal{H}(\alpha_k(t_i),p(t_i)-p(t_{i-1})) \le \mathcal{V}_{\mathcal{H}}(\alpha_k,p;a,b) \le \mathcal{V}_{\mathcal{H}}(\alpha,p;a,b)\,,$$

and using (A.11) and (A.12) we can pass to the limit as $k\to\infty$ and get

$$\mathcal{V}_{\mathcal{H}}(\alpha,p;a,b)-\varepsilon \le \int_a^b \mathcal{H}(\alpha(t),\dot p(t))\,\mathrm{d}t \le \mathcal{V}_{\mathcal{H}}(\alpha,p;a,b)\,.$$

We therefore conclude since ε is arbitrary.

□

A.2. **A remark about increasing functions from time into L^p spaces.**

Lemma A.2. *Let (X,μ) a measure space with $\mu(X)<\infty$, and $\alpha\colon[0,T]\to L^\infty(X,\mu)$ such that $\|\alpha(t)\|_\infty\le M$ for every $t\in[0,T]$ and*

$$\alpha(t_2)\le\alpha(t_1)\ \mu\text{-a.e. in } X \text{ for every } t_1\le t_2\,. \tag{A.13}$$

Then there exists a countable set $E\subset[0,T]$ such that for every $1\le p<\infty$ the function α is continuous in every $t\in[0,T]\setminus E$ with respect to the $L^p(X,\mu)$ norm.

Proof. For every $s\in(0,T]$ and $t\in[0,T)$ we define

$$\alpha^-(s):=\inf_{n\in\mathbb{N}}\alpha(t_n^-),\quad \alpha^+(t):=\sup_{n\in\mathbb{N}}\alpha(t_n^+)\,,$$

where $t_n^-<s$ and $t<t_n^+$ are sequences in $[0,T]$ convergent to s and t, and

$$\alpha^-(0):=\alpha(0),\quad \alpha^+(T):=\alpha(T)\,.$$

By (A.13) these definitions are well posed. Indeed, let for instance $t<s_n^+$ be a sequence that converges to t, and $\tilde\alpha(t^+):=\sup_{n\in\mathbb{N}}\alpha(s_n^+)$. For every $m\in\mathbb{N}$, there exists n_m such that $t<s_n^+\le t_m^+$ for every $n\ge n_m$: therefore $\tilde\alpha(t^+)\ge\alpha(s_n^+)\ge\alpha(t_m^+)$ for every m, and $\tilde\alpha(t^+)\ge\alpha(t^+)$, taking the supremum over m. The opposite inequality follows by interchanging the two sequences. Moreover for every $t\in[0,T]$

$$\alpha(t_n^+)\to\alpha^+(t),\quad \alpha(t_n^-)\to\alpha^-(t)\quad \text{strongly in } L^p(X,\mu)\,, \tag{A.14}$$

by Monotone Convergence Theorem and (A.13) again, and

$$\alpha^-(t)\ge\alpha(t)\ge\alpha^+(t)\,,$$

for every $t\in[0,T]$. Let us consider now the function

$$g(t):=\int_X\big(\alpha^-(t)-\alpha^+(t)\big)\,\mathrm{d}\mu\,.$$

It takes values in $\mathbb{R}^+\cup\{0\}$ and for every $t_1<\dots<t_k\in E:=\{t\in[0,T]|\,g(t)>0\}$ we get, using in particular (A.13), that

$$\sum_{i=1}^k g(t_i)\le\int_X\big(\alpha^-(t_1)-\alpha^+(t_k)\big)\,\mathrm{d}\mu\le 2M\mu(X)\,.$$

By a standard argument, we deduce that E is a countable set. By definition of E, $\alpha^+(t)=\alpha^-(t)=\alpha(t)$ μ-a.e. for every $t\in[0,T]\setminus E$ and we conclude by (A.14). □

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