

Adler-Gelfand-Dickey approach to classical \mathcal{W} -algebras within the theory of Poisson vertex algebras

Alberto De Sole¹, Victor G. Kac² and Daniele Valeri³

¹Dip.to Matematica, Università La Sapienza, P.le Aldo Moro 2, 00185 Rome, Italy, desole@mat.uniroma1.it, ² Department of Mathematics, MIT, 77 Mass. Avenue, Cambridge, MA 02139, USA, kac@math.mit.edu, and ³ SISSA, Via Bonomea 265, 34136 Trieste, Italy, dvaleri@sissa.it.

Correspondence to be sent to: desole@mat.uniroma1.it

We put the Adler-Gelfand-Dickey approach to classical \mathcal{W} -algebras in the framework of Poisson vertex algebras. We show how to recover the bi-Poisson structure of the KP hierarchy, together with its generalizations and reduction to the N -th KdV hierarchy, using the formal distribution calculus and the λ -bracket formalism. We apply the Lenard-Magri scheme to prove integrability of the corresponding hierarchies. We also give a simple proof of a theorem of Kupershmidt and Wilson in this framework. Based on this approach, we generalize all these results to the matrix case. In particular, we find (non-local) bi-Poisson structures of the matrix KP and the matrix N -th KdV hierarchies, and we prove integrability of the N -th matrix KdV hierarchy.

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0 Introduction

It is well known that classical \mathcal{W} -algebras play an important role in the theory of integrable bi-Hamiltonian equations. One of the most powerful approaches to classical \mathcal{W} -algebras is via the Drinfeld-Sokolov Hamiltonian reduction, [Drinfeld and Sokolov \[1985\]](#), associated to any simple Lie algebra and its principal nilpotent element. Later on this theory was extended to more general nilpotent elements, and it was put by [De Sole et al. \[2013-a\]](#) in the framework of Poisson vertex algebras (PVA).

However the most important classical \mathcal{W} -algebras, those associated to \mathfrak{sl}_N and its principal nilpotent element, have appeared prior to [Drinfeld and Sokolov \[1985\]](#) together with their deep connection to integrable systems. They were identified with the so-called second Poisson structure of the N -th (or generalized) KdV equations, the KdV equation corresponding to the case of $N = 2$. The first Poisson structure of the KdV equation was found by [Gardner \[1971\]](#) and [Zakharov and Faddeev \[1971\]](#). Later, it was shown by [Magri \[1978\]](#), that there exists not only a Poisson structure but a one-parameter family of compatible Poisson structures for the KdV equation. In this case we say that we have a bi-Poisson structure. The fact that the same equation could be written in two different Hamiltonian forms has become fundamental in proving integrability of such equations using a recurrence procedure, nowadays called the Lenard-Magri scheme of integrability.

The generalization to the N -th KdV hierarchy was suggested by [Gelfand and Dickey \[1976\]](#) who found a Poisson structure for these hierarchies, obtained as a Poisson algebra on a suitable space of scalar differential operators (N being the order of the operators). A candidate for a second Poisson structure (whence a bi-Poisson structure) was conjectured by [Adler \[1979\]](#) and proved by [Gelfand and Dickey \[1978\]](#). These Poisson structures are usually known as the *first and second Adler-Gelfand-Dickey (AGD) Poisson structures* and the second one is a special case of general classical \mathcal{W} -algebras.

Later on, all the N -th KdV hierarchies were “embedded” into one big hierarchy called the KP hierarchy, [Sato \[1981\]](#). A Poisson structure for this hierarchy (which was a slight modification of the AGD Poisson structure) was suggested by [Watanabe \[1983\]](#), and [Dickey \[1987\]](#) proved that there exists a second Poisson structure. In fact, it was shown by [Radul \[1987\]](#) that, for any $N \geq 1$, there is a bi-Poisson structure for the KP hierarchy which induces the bi-Poisson structure for the N -th KdV hierarchy. We refer to the book of [Dickey \[2003\]](#) for a detailed exposition of these topics and a large list of references.

The main goal of the present paper is to give a Poisson vertex algebra interpretation of the AGD approach to classical \mathcal{W} -algebras. This point of view greatly simplifies the theory and also allows to study its matrix generalization.

Let \mathcal{V} be a differential algebra with derivation ∂ . Following [Adler \[1979\]](#), given a pseudodifferential operator $L \in \mathcal{V}((\partial^{-1}))$, we define the corresponding *Adler map* $A^{(L)} : \mathcal{V}((\partial^{-1})) \rightarrow \mathcal{V}((\partial^{-1}))$ given by (2.1), and the associated skewadjoint matrix differential operator $H^{(L)}$ with coefficients in \mathcal{V} , given by (2.8). In [Theorem 2.7](#) we consider the algebra \mathcal{V}_N^∞ of differential polynomials on the differential variables $\{u_i\}_{i \geq -N}$ and the pseudodifferential operator $L = \partial^N + u_{-N}\partial^{N-1} + u_{-N+1}\partial^{N-2} + \dots$, and we prove that, in this case,

the corresponding 1-parameter family of matrices $H^{(L-c)}$ gives a bi-Poisson structure on \mathcal{V}_N^∞ . Theorem 2.10 is the analogous result for the algebra \mathcal{V}_N of differential polynomials in finitely many variables $\{u_i\}_{i=-N}^{-1}$ and $L = \partial^N + u_{-N}\partial^{N-1} + \cdots + u_{-1}$. Also in this case we get a bi-Poisson structure on \mathcal{V}_N . This is the classical \mathcal{W} -algebra associated to the Lie algebra \mathfrak{gl}_N and its principal nilpotent element. Furthermore, performing Dirac's reduction by the constraint $u_{-N} = 0$, we obtain bi-Poisson structures on $\mathcal{W}_N^\infty = \mathcal{V}_N^\infty / \langle u_{-N} \rangle$ and on $\mathcal{W}_N = \mathcal{V}_N / \langle u_{-N} \rangle$ (the latter is the classical \mathcal{W} -algebra for \mathfrak{sl}_N).

In Section 3 we associate to each of these bi-Poisson structures the corresponding integrable hierarchy of bi-Hamiltonian equations, using the Lenard-Magri scheme: on \mathcal{W}_N^∞ we obtain the KP hierarchy, while on \mathcal{W}_N we obtain the N -th KdV hierarchy.

Furthermore, in Section 2.7 we use the Adler map to define natural PVA homomorphisms $\mathcal{V}_{M+N}^\infty \rightarrow \mathcal{V}_M^\infty \otimes \mathcal{V}_N^\infty$ and $\mathcal{V}_{M+N} \rightarrow \mathcal{V}_M \otimes \mathcal{V}_N$, which are a generalization of the usual ‘‘Miura map’’, Miura [1968]. This allows us to give another proof of the Theorem of Kupershmidt and Wilson [1981].

A major advantage of our approach is that all the above constructions and proofs extend in a straightforward way to the case when the variables u_i are replaced by $m \times m$ matrices U_i . This is done in Section 4, where, as a result, we construct the matrix KP and N -th KdV bi-Hamiltonian equations. The main difference with the scalar case is that, in the matrix case, Dirac reduction (developed by De Sole et al. [2013-c]) by the constraint $U_{-N} = 0$ leads to a non-local bi-Poisson structure. These non-local Poisson structures have been discovered by Bilal [1995] and Olver and Sokolov [1998]. We then need to use the theory of non-local Poisson structures introduced by De Sole and Kac [2013-a], and the machinery of rational matrix pseudodifferential operators developed by Carpentier et al. [2013-b], to prove integrability of the matrix N -th KdV hierarchy.

1 Preliminaries

1.1 Some simple facts on formal distribution calculus

We briefly review here some basic facts on formal distribution calculus which will be used throughout the paper, cf. Kac [1996].

Given a vector space \mathcal{A} , an \mathcal{A} -valued formal distribution in z is a series of the form $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n$, where $a_n \in \mathcal{A}$. They form a vector space denoted by $\mathcal{A}[[z, z^{-1}]]$. An \mathcal{A} -valued formal distribution in two variables z and w is a series of the form $a(z, w) = \sum_{m, n \in \mathbb{Z}} a_{mn} z^m w^n$, where $a_{mn} \in \mathcal{A}$. They form a vector space denoted by $\mathcal{A}[[z, z^{-1}, w, w^{-1}]]$.

The δ -function is, by definition, the \mathbb{F} -valued formal distribution

$$\delta(z - w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n \in \mathbb{F}[[z, z^{-1}, w, w^{-1}]].$$

For every $a(z) \in \mathcal{A}[[z, z^{-1}]]$ we have

$$a(z)\delta(z - w) = a(w)\delta(z - w). \tag{1.1}$$

In particular, $\text{Res}_z a(z)\delta(z-w) = a(w)$, where Res_z denotes the coefficient of z^{-1} .

We denote by i_z the power series expansion for large $|z|$. For example,

$$i_z(z-w)^{-1} = \sum_{k \in \mathbb{Z}_+} z^{-k-1} w^k.$$

Using this notation, the δ -function can be rewritten as follows

$$\delta(z-w) = i_z(z-w)^{-1} - i_w(z-w)^{-1}. \quad (1.2)$$

For $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathcal{A}[[z, z^{-1}]]$, we denote $a(z)_+ = \sum_{n \in \mathbb{Z}_+} a_n z^n$ and $a(z)_- = \sum_{n < 0} a_n z^n$. It is easy to check that

$$\text{Res}_z a(z) i_z(z-w)^{-1} = a(w)_+. \quad (1.3)$$

1.2 The algebra of matrix pseudodifferential operators

By a *differential algebra* we mean a unital commutative associative algebra \mathcal{A} over a field \mathbb{F} of characteristic 0, with a derivation ∂ .

The algebra $\text{Mat}_{m \times m} \mathcal{A}$ of matrices with coefficients in \mathcal{A} is a unital associative algebra, with the obvious action of ∂ as a derivation. We consider the algebra $\text{Mat}_{m \times m} \mathcal{A}((\partial^{-1}))$ of $m \times m$ *matrix pseudodifferential operators* with coefficients in \mathcal{A} . Its product is determined by the following formula ($n \in \mathbb{Z}$, $A \in \text{Mat}_{m \times m} \mathcal{A}$):

$$\partial^n \circ A = \sum_{k \in \mathbb{Z}_+} \binom{n}{k} A^{(k)} \partial^{n-k}.$$

We say that a non-zero $A(\partial) = \sum_{n \leq N} A_n \partial^n \in \text{Mat}_{m \times m} \mathcal{A}((\partial^{-1}))$ has *order* $\text{ord}(A(\partial)) = N$ if $A_N \neq 0$. We denote by $\text{Mat}_{m \times m} \mathcal{A}((\partial^{-1}))_N$ the space of all matrix pseudodifferential operators of order less than or equal to N . We say that $A(\partial)$ is *monic* if $A_N = \mathbb{1}_m$.

The *residue* of $A(\partial) \in \text{Mat}_{m \times m} \mathcal{A}((\partial^{-1}))$ is, by definition,

$$\text{Res}_\partial A(\partial) = A_{-1} (= \text{coefficient of } \partial^{-1}). \quad (1.4)$$

The *adjoint* of $A(\partial) \in \text{Mat}_{m \times m} \mathcal{A}((\partial^{-1}))$ is, by definition,

$$A(\partial)^* = \sum_{n \leq N} (-\partial)^n \circ A_n^t,$$

where A_n^t denotes the transpose matrix of A_n .

We have the subalgebras $\text{Mat}_{m \times m} \mathcal{A}[\partial]$ of *matrix differential operators*, and $\text{Mat}_{m \times m} \mathcal{A}[[\partial^{-1}]]\partial^{-1}$ of *matrix integral operators*. Every matrix pseudodifferential operator $A(\partial)$ can be decomposed uniquely as $A(\partial) = A(\partial)_+ + A(\partial)_-$, where $A(\partial)_+ \in \text{Mat}_{m \times m} \mathcal{A}[\partial]$ and $A(\partial)_- \in \text{Mat}_{m \times m} \mathcal{A}[[\partial^{-1}]]\partial^{-1}$.

We let $\mathbb{F}[\partial, \partial^{-1}] \circ \text{Mat}_{m \times m} \mathcal{A} \subset \text{Mat}_{m \times m} \mathcal{A}((\partial^{-1}))$ be the space of all pseudodifferential operators of the form $\sum_{n \in \mathbb{Z}} \partial^n \circ A_n$, where all but finitely many elements $A_n \in \text{Mat}_{m \times m} \mathcal{A}$ are zero. We also let $\text{Mat}_{m \times m} \mathcal{A}[[\partial, \partial^{-1}]] \supset \text{Mat}_{m \times m} \mathcal{A}((\partial^{-1}))$ be the space of all formal series in ∂, ∂^{-1} with coefficients in $\text{Mat}_{m \times m} \mathcal{A}$ (it is not an algebra).

Recall the following simple facts about matrix pseudodifferential operators.

Proposition 1.1. Let $A(\partial)$ be a monic matrix pseudodifferential operator of order $N \geq 1$.

- (a) There exists exactly one monic matrix pseudodifferential operator of order one, which we denote $A^{\frac{1}{N}}(\partial)$, such that $(A^{\frac{1}{N}}(\partial))^N = A(\partial)$.
- (b) There exists exactly one monic matrix pseudodifferential operator of order $-N$, which we denote $A^{-1}(\partial)$, such that $A(\partial) \circ A^{-1}(\partial) = A^{-1}(\partial) \circ A(\partial) = \mathbb{1}_m$.

□

The *symbol* of a matrix pseudodifferential operator $A(\partial) = \sum_{n \leq N} A_n \partial^n \in \text{Mat}_{m \times m} \mathcal{A}((\partial^{-1}))$ is the Laurent series $A(z) = \sum_{n \leq N} A_n z^n \in \text{Mat}_{m \times m} \mathcal{A}((z^{-1}))$, where z is an indeterminate commuting with \mathcal{A} . This gives us a bijective map $\text{Mat}_{m \times m} \mathcal{A}((\partial^{-1})) \longrightarrow \text{Mat}_{m \times m} \mathcal{A}((z^{-1}))$ which is not an algebra homomorphism. For $A(\partial), B(\partial) \in \text{Mat}_{m \times m} \mathcal{A}((\partial^{-1}))$, we have

$$(A \circ B)(z) = A(z + \partial)B(z). \quad (1.5)$$

Here and further, for any $n \in \mathbb{Z}$, we expand $(z + \partial)^n$ in non-negative powers of ∂ (and we let the powers of ∂ act to the right, on the coefficients of $B(z)$), i.e. by $(z + \partial)^n$ we mean $i_z(z + \partial)^n$.

The following lemma will be used in Section 3.

Lemma 1.2. (a) If $a(z), b(z) \in \mathcal{A}((z^{-1}))$ are formal Laurent series with coefficients in an algebra \mathcal{A} , then

$$\text{Res}_z a(z)b(z-x) = \text{Res}_z a(z+x)b(z) \quad (\text{where we expand } (z \pm x)^n \text{ in non-negative powers of } x).$$

(b) If $A(\partial), B(\partial) \in \mathcal{A}((\partial^{-1}))$ are pseudodifferential operators over the differential algebra \mathcal{A} , then

$$\text{Res}_z A(z)B^*(-z + \lambda) = \text{Res}_z A(z + \lambda + \partial)B(z).$$

□

Proof. Part (a) follows from the formula of integration by parts, $\text{Res}_z A(z)\partial_z B(z) = -\text{Res}_z B(z)\partial_z A(z)$, and the Taylor expansion $a(z+x) = e^{x\partial_z} a(z)$. Part (b) is a special case of (a), when $x = \lambda + \partial$, acting on the coefficients of B . ■

1.3 Poisson vertex algebras

Definition 1.3. A λ -bracket on a differential algebra \mathcal{V} is an \mathbb{F} -linear map $\{\cdot, \cdot\}_\lambda : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}[\lambda]$ satisfying $(f, g, h \in \mathcal{V})$

(i) *sesquilinearity*: $\{\partial f, g\} = -\lambda\{f, g\}, \{f, \partial g\} = (\lambda + \partial)\{f, g\},$

(ii) *Leibniz rules*: $\{f, gh\} = \{f, g\}h + \{f, h\}g, \{fh, g\} = \{f, g\}h + \{h, g\}f.$

□

Here and further we use the following notation: if $\{f, g\} = \sum_{n \in \mathbb{Z}_+} \lambda^n c_n$, then $\{f, g\}_{\rightarrow h} = \sum_{n \in \mathbb{Z}_+} c_n (\lambda + \partial)^n h$.

Definition 1.4. A *Poisson vertex algebra* (PVA) is a differential algebra \mathcal{V} endowed with a λ -bracket $\{\cdot, \cdot\}_\lambda$ satisfying $(f, g, h \in \mathcal{V})$

(iii) *skew-symmetry*: $\{g, f\} = -\{f, g\}_{-\lambda - \partial},$

(iv) *Jacobi identity*: $\{f, \{g, h\}\} - \{g, \{f, h\}\} = \{\{f, g\}, h\}_{\lambda + \mu}.$

□

In the skew-symmetry we use the following notation: if $\{f, g\} = \sum_{n \in \mathbb{Z}_+} \lambda^n c_n$, then $\{f, g\}_{-\lambda - \partial} = \sum_{n \in \mathbb{Z}_+} (-\lambda - \partial)^n c_n$.

Definition 1.5. Let \mathcal{V} be a PVA. A *Virasoro element* $T \in \mathcal{V}$ with *central charge* $c \in \mathbb{F}$ is such that $\{T, T\} = (2\lambda + \partial)T + c\lambda^3$, a T -*eigenvector* $a \in \mathcal{V}$ of *conformal weight* $\Delta_a \in \mathbb{F}$ is such that $\{T, a\} = (\Delta_a \lambda + \partial)a + O(\lambda^2)$, and a *primary element* $a \in \mathcal{V}$ of conformal weight Δ_a is such that $\{T, a\} = (\Delta_a \lambda + \partial)a$. By definition, a PVA of *CFT* (conformal field theory) *type* is generated, as a differential algebra, by a Virasoro element and a finite number of primary elements. □

Let $\mathcal{I} \subset \mathcal{V}$ be a differential ideal. We say that \mathcal{I} is a *PVA ideal* if $\{\mathcal{I}, \mathcal{V}\} \subset \mathcal{I}[\lambda]$ (by skew-symmetry and the fact that \mathcal{I} is a differential ideal we also have $\{\mathcal{V}, \mathcal{I}\} \subset \mathcal{I}[\lambda]$). Given a PVA ideal \mathcal{I} , we can consider the induced Poisson vertex algebra structure over the differential algebra \mathcal{V}/\mathcal{I} , which is called the *quotient PVA*.

In this paper we consider PVA structures on the algebra $R_I = \mathbb{F}[u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$ of differential polynomials in the variables $\{u_i\}_{i \in I}$, where I is an index set (possibly infinite). The derivation ∂ is defined by $\partial(u_i^{(n)}) = u_i^{(n+1)}$, $i \in I, n \in \mathbb{Z}_+$. Note that on R_I we have the following commutation relations: $\left[\frac{\partial}{\partial u_i^{(n)}}, \partial \right] = \frac{\partial}{\partial u_i^{(n-1)}}$, where the RHS is considered to be zero if $n = 0$.

Theorem 1.6 ([Barakat et al., 2009, Theorem 1.15]). Let $\mathcal{V} = R_I$ and $H = (H_{ij}(\lambda))_{i, j \in I} \in \text{Mat}_{I \times I} \mathcal{V}[\lambda]$.

(a) There is a unique λ -bracket $\{\cdot, \cdot\}_H$ on \mathcal{V} , such that $\{u_i, u_j\}_H = H_{ji}(\lambda)$ for every $i, j \in I$, and it is given by the following Master Formula

$$\{f, g\}_H = \sum_{\substack{i, j \in I \\ m, n \in \mathbb{Z}_+}} \frac{\partial g}{\partial u_j^{(n)}} (\lambda + \partial)^n H_{ji}(\lambda + \partial) (-\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}}. \quad (1.6)$$

(b) The λ -bracket (1.6) on \mathcal{V} is skew-symmetric if and only if skew-symmetry holds on generators ($i, j \in I$):

$$\{u_i \lambda u_j\}_H = -\{u_j - \lambda - \partial u_i\}_H, \quad (1.7)$$

and this is equivalent to skewadjointness of H .

(c) Assuming that the skew-symmetry condition (1.7) holds, the λ -bracket (1.6) satisfies the Jacobi identity, thus making \mathcal{V} a PVA, provided that the Jacobi identity holds on any triple of generators ($i, j, k \in I$):

$$\{u_i \lambda \{u_{j\mu} u_k\}_H\}_H - \{u_{j\mu} \{u_i \lambda u_k\}_H\}_H = \{\{u_i \lambda u_j\}_H \lambda_{\lambda+\mu} u_k\}_H. \quad (1.8)$$

□

Remark 1.7. Theorem 1.6 holds in the more general situation when \mathcal{V} is any algebra of differential functions in the variables $\{u_i\}_{i \in I}$ (see Barakat et al. [2009] for a definition), but in the present paper we will only consider the case when $\mathcal{V} = R_I$. □

Definition 1.8. A *Poisson structure* on \mathcal{V} is a matrix differential operator $H(\partial) \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\partial]$ such that the corresponding λ -bracket $\{\cdot, \cdot\}_H$ defines a PVA structure on \mathcal{V} . □

Example 1.9. On $R_1 = \mathbb{F}[v, v', v'', \dots]$, we have the so-called *Gardner-Faddeev-Zakharov* (GFZ) PVA, given by

$$\{v \lambda v\} = \lambda, \quad (1.9)$$

and the corresponding Poisson structure is $H(\partial) = \partial$. In general, for $N \geq 1$, we consider the algebra of differential polynomials in N variables $R_N = \mathbb{F}[v_i^{(n)} \mid i \in \{1, \dots, N\}, n \in \mathbb{Z}_+]$ with the generalized GFZ λ -bracket

$$\{v_j \lambda v_i\} = s_{ij} \lambda, \quad i, j \in \{1, \dots, N\},$$

where $S = (s_{ij})_{i,j=1}^N$ is a symmetric matrix over \mathbb{F} . The corresponding Poisson structure is $H(\partial) = S\partial$. □

Example 1.10. On $R_1 = \mathbb{F}[u, u', u'', \dots]$ we have the *Virasoro-Magri* PVA, given by ($c \in \mathbb{F}$)

$$\{u \lambda u\} = (\partial + 2\lambda)u + c\lambda^3, \quad (1.10)$$

with Poisson structure $H(\partial) = u' + 2u\partial + c\partial^3$. □

1.4 Hamiltonian equations

The following proposition is immediate to check:

Proposition 1.11. Let \mathcal{V} be a PVA. Then we have a well defined Lie algebra bracket on the quotient space $\mathcal{V}/\partial\mathcal{V}$:

$$\{f, g\} = \int \{f\lambda g\}|_{\lambda=0}. \quad (1.11)$$

Here and further, $f : \mathcal{V} \rightarrow \mathcal{V}/\partial\mathcal{V}$ is the canonical quotient map. Moreover, we have a well defined Lie algebra action of $\mathcal{V}/\partial\mathcal{V}$ on \mathcal{V} by derivations of the commutative associative product on \mathcal{V} , commuting with ∂ , given by

$$\{f, g\} = \{f\lambda g\}|_{\lambda=0}.$$

□

Definition 1.12. Let \mathcal{V} be a PVA. The *Hamiltonian equation* with *Hamiltonian functional* $\int h \in \mathcal{V}/\partial\mathcal{V}$ is

$$\frac{du}{dt} = \{f, u\} \quad (1.12)$$

An *integral of motion* for the Hamiltonian equation (1.12) is an element $\int f \in \mathcal{V}/\partial\mathcal{V}$ such that $\{f, \int h\} = 0$ (or, equivalently, $\frac{d}{dt}\int f = 0$). Equation (1.12) is called *integrable* if there exists an infinite sequence $\int h_0 = \int h, \int h_1, \int h_2, \dots$, of linearly independent integrals of motion in involution: $\{\int h_m, \int h_n\} = 0$ for all $m, n \in \mathbb{Z}_+$.

The corresponding *integrable hierarchy of Hamiltonian equations* is

$$\frac{du}{dt_n} = \{\int h_n, u\}, \quad n \in \mathbb{Z}_+. \quad (1.13)$$

□

In the special case when $\mathcal{V} = R_\ell$ and $H \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\partial]$ is a Poisson structure, the Lie bracket (1.11) on $\mathcal{V}/\partial\mathcal{V}$ takes the usual form (see (1.6)):

$$\{\int f, \int g\}_H = \sum_{i, j \in I} \int \frac{\delta g}{\delta u_j} H_{ji}(\partial) \frac{\delta f}{\delta u_i},$$

where $\frac{\delta f}{\delta u_i}$ denotes the *variational derivative* of $f \in \mathcal{V}$ with respect to u_i ,

$$\frac{\delta f}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}, \quad (1.14)$$

(it is well defined on $\mathcal{V}/\partial\mathcal{V}$ since $\frac{\delta}{\delta u_i} \circ \partial = 0$), and the Hamiltonian equation associated to the Hamiltonian functional $\int h \in \mathcal{V}/\partial\mathcal{V}$ is, as usual,

$$\frac{du_i}{dt} = \sum_{j \in I} H_{ij}(\partial) \frac{\delta h}{\delta u_j}, \quad i \in I.$$

1.5 Bi-PVA and the Lenard-Magri scheme of integrability

Definition 1.13. Two PVA λ -brackets $\{\cdot \lambda \cdot\}_0$ and $\{\cdot \lambda \cdot\}_1$ on a differential algebra \mathcal{V} are *compatible* if any their linear combination (or, equivalently, their sum) is a PVA λ -bracket. We say in this case that \mathcal{V} is a *bi-PVA*. \square

For example, the GFZ λ -bracket (1.9) and the Virasoro-Magri λ -bracket (1.10) are compatible.

According to the *Lenard-Magri scheme of integrability*, Magri [1978], in order to obtain an integrable hierarchy of Hamiltonian equations, one needs to find a sequence $\{\int h_n\}_{n \in \mathbb{Z}_+} \subset \mathcal{V}/\partial\mathcal{V}$ spanning an infinite dimensional space, such that

$$\{\int h_n, u\}_1 = \{\int h_{n+1}, u\}_0 \quad \text{for } n \in \mathbb{Z}_+, u \in \mathcal{V}. \quad (1.15)$$

If this is the case, then $\{\int h_m, \int h_n\}_\delta = 0$, for all $m, n \in \mathbb{Z}_+$, $\delta = 0, 1$. Hence, we get the corresponding integrable hierarchy of Hamiltonian equations (1.13). Moreover, if $\{\int h_0, u\}_0 = 0$, and $\{\int g_n\}_{n \in \mathbb{Z}_+} \subset \mathcal{V}/\partial\mathcal{V}$ is another sequence satisfying the Lenard-Magri recursion $\{\int g_n, u\}_1 = \{\int g_{n+1}, u\}_0$, for all $n \in \mathbb{Z}_+$ and $u \in \mathcal{V}$, then the two sequences of integrals of motion are compatible: $\{\int h_m, \int g_n\}_\delta = 0$, for all $m, n \in \mathbb{Z}_+$, $\delta = 0, 1$, [Barakat et al., 2009, Sec.2.1].

1.6 Non-local Poisson vertex algebras

For a vector space V , we shall use the following notation:

$$V_{\lambda, \mu} := V[[\lambda^{-1}, \mu^{-1}, (\lambda + \mu)^{-1}][[\lambda, \mu],$$

namely, the quotient of the $\mathbb{F}[\lambda, \mu, \nu]$ -module $V[[\lambda^{-1}, \mu^{-1}, \nu^{-1}][[\lambda, \mu, \nu]$ by the submodule $(\nu - \lambda - \mu)V[[\lambda^{-1}, \mu^{-1}, \nu^{-1}][[\lambda, \mu, \nu]$. We have the natural embedding $\iota_{\mu, \lambda} : V_{\lambda, \mu} \hookrightarrow V((\lambda^{-1}))((\mu^{-1}))$ defined by expanding the negative powers of $\nu = \lambda + \mu$ by geometric series in the domain $|\mu| > |\lambda|$.

Let \mathcal{V} be a differential algebra. A *non-local λ -bracket* on \mathcal{V} is an \mathbb{F} -linear map $\{\cdot \lambda \cdot\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}((\lambda^{-1}))$ satisfying the sesquilinearity conditions and the Leibniz rules, as in Definition 1.3. It is called *skew-symmetric* if the skew-symmetry condition in Definition 1.4 holds as well. The term $\{g_{-\lambda - \partial} f\}$ in the RHS of the skewsymmetry condition should be interpreted as follows: we move $-\lambda - \partial$ to the left and we expand it in non-negative powers of ∂ , acting on the coefficients of the λ -bracket. Clearly, from skew-symmetry and the left Leibniz rule, we also have the right Leibniz rule, which should be interpreted in a similar way.

The non-local skew-symmetric λ -bracket $\{\cdot \lambda \cdot\}$ is called *admissible* if

$$\{f_\lambda \{g_\mu h\}\} \in \mathcal{V}_{\lambda, \mu} \quad \text{for all } f, g, h \in \mathcal{V}.$$

Here we are identifying the space $\mathcal{V}_{\lambda,\mu}$ with its image in $\mathcal{V}((\lambda^{-1}))((\mu^{-1}))$ via the embedding $\iota_{\mu,\lambda}$. Note that, from skew-symmetry, we also have that $\{g_\mu\{f_\lambda h\}\} \in \mathcal{V}_{\lambda,\mu}$ and $\{\{f_\lambda g\}_{\lambda+\mu} h\} \in \mathcal{V}_{\lambda,\mu}$. Therefore, the Jacobi identity can be understood as an equality in the space $\mathcal{V}_{\lambda,\mu}$.

Definition 1.14 (De Sole and Kac [2013-a]). A *non-local Poisson vertex algebra* (PVA) is a differential algebra \mathcal{V} endowed with an admissible non-local skew-symmetric λ -bracket $\{\cdot_\lambda \cdot\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}((\lambda^{-1}))$ satisfying the Jacobi identity. \square

1.7 Dirac reduction

Let \mathcal{V} be a (non-local) Poisson vertex algebra with λ -bracket $\{\cdot_\lambda \cdot\}$. Let $\theta_1, \dots, \theta_m$ be elements of \mathcal{V} , and let $\mathcal{I} = \langle \theta_1, \dots, \theta_m \rangle_{\mathcal{V}}$ be the differential ideal generated by them. Consider the matrix pseudodifferential operator $C(\partial) = (C_{\alpha\beta}(\partial))_{\alpha,\beta=1}^m \in \text{Mat}_{m \times m} \mathcal{V}((\partial^{-1}))$, whose symbol is

$$C_{\alpha\beta}(\lambda) = \{\theta_\beta \lambda \theta_\alpha\}. \quad (1.16)$$

By the skew-symmetry condition, the pseudodifferential operator $C(\partial)$ is skewadjoint. We shall assume that the matrix pseudodifferential operator $C(\partial)$ is invertible, and we denote its inverse by $C^{-1}(\partial) = ((C^{-1})_{\alpha\beta}(\partial))_{\alpha,\beta=1}^m$.

Definition 1.15. The *Dirac modification* of the PVA λ -bracket $\{\cdot_\lambda \cdot\}$, associated to the elements $\theta_1, \dots, \theta_m$, is the map $\{\cdot_\lambda \cdot\}^D : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}((\lambda^{-1}))$ given by ($a, b \in \mathcal{V}$):

$$\{a_\lambda b\}^D = \{a_\lambda b\} - \sum_{\alpha,\beta=1}^m \{\theta_\beta \lambda + \partial b\}_{\rightarrow} (C^{-1})_{\beta\alpha}(\lambda + \partial) \{a_\lambda \theta_\alpha\}. \quad (1.17)$$

\square

Theorem 1.16 ([De Sole et al., 2013-c, Theorem 2.2]). (a) The Dirac modification $\{\cdot_\lambda \cdot\}^D$ is a PVA λ -bracket on \mathcal{V} .

(b) All the elements θ_i , $i = 1, \dots, m$, are central with respect to the Dirac modified λ -bracket: $\{a_\lambda \theta_i\}^D = \{\theta_i \lambda a\}^D = 0$ for all $i = 1, \dots, m$ and $a \in \mathcal{V}$.

(c) The differential ideal $\mathcal{I} = \langle \theta_1, \dots, \theta_m \rangle_{\mathcal{V}} \subset \mathcal{V}$, generated by $\theta_1, \dots, \theta_m$, is such that $\{\mathcal{I}_\lambda \mathcal{V}\}^D, \{\mathcal{V}_\lambda \mathcal{I}\}^D \subset \mathcal{I}((\lambda^{-1}))$.

The quotient space \mathcal{V}/\mathcal{I} is a (non-local) PVA, with λ -bracket induced by $\{\cdot_\lambda \cdot\}^D$, which we call the *Dirac reduction* of \mathcal{V} by the constraints $\theta_1, \dots, \theta_m$. \square

Example 1.17. Let $N \geq 1$ and let us consider the generalized GFZ PVA R_N defined in Example 1.9 (associated to the symmetric matrix $S = (s_{ij})_{i,j=1}^N$). Let $\theta = v_1 + \dots + v_N \in R_N$. We have $\{\theta_\lambda \theta\} = \lambda s$, where $s = \sum_{h,k=1}^N s_{hk} \in \mathbb{F}$. Provided that $s \neq 0$, we can consider the Dirac reduction of R_N by θ , and we can identify it

with $R_{N-1} = \mathbb{F}[v_i^{(n)} \mid i \in \{1, \dots, N-1\}, n \in \mathbb{Z}_+]$. The corresponding Dirac modified λ -bracket (1.17) on R_{N-1} is given by

$$\{v_i \lambda v_j\}^D = \left(s_{ij} - \frac{s_i s_j}{s} \right) \lambda, \quad (1.18)$$

for all $i, j = 1, \dots, N-1$, where $s_i = \sum_{k=1}^N s_{ik} \in \mathbb{F}$. \square

Lemma 1.18. Let \mathcal{V}_1 and \mathcal{V}_2 be PVAs and let $\varphi : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ be a PVA homomorphism. Let $\theta_1, \dots, \theta_m \in \mathcal{V}_1$ be such that the matrix $C(\partial) \in \text{Mat}_{m \times m} \mathcal{V}_1((\partial^{-1}))$ defined in (1.16) is invertible. Then, φ induces a PVA homomorphism of the corresponding Dirac reduced PVAs

$$\varphi : \mathcal{V}_1 / \langle \theta_1, \dots, \theta_m \rangle_{\mathcal{V}_1} \rightarrow \mathcal{V}_2 / \langle \varphi(\theta_1), \dots, \varphi(\theta_m) \rangle_{\mathcal{V}_2}.$$

\square

Proof. Let $D_{\alpha\beta}(\lambda) = \{\varphi(\theta_\beta) \lambda \varphi(\theta_\alpha)\}_2 = \varphi(C_{\alpha\beta}(\lambda))$, $\alpha, \beta = 1, \dots, m$. Since φ is a differential algebra homomorphism, the matrix $D(\partial) \in \text{Mat}_{m \times m} \mathcal{V}_2((\partial^{-1}))$ is invertible, and $D^{-1}(\partial) = \varphi(C^{-1}(\partial))$. Hence, by (1.17) we have

$$\varphi(\{a_\lambda b\}_1^D) = \{\varphi(a) \lambda \varphi(b)\}_2^D,$$

for all $a, b \in \mathcal{V}_1$, as required. \blacksquare

In general, if we have two compatible PVA λ -brackets $\{\cdot \lambda \cdot\}_0$ and $\{\cdot \lambda \cdot\}_1$ on \mathcal{V} (recall Definition 1.13), and we take their Dirac reductions by a finite number of constraints $\theta_1, \dots, \theta_m$, we do NOT get compatible PVA λ -brackets on \mathcal{V}/\mathcal{I} , where $\mathcal{I} = \langle \theta_1, \dots, \theta_m \rangle_{\mathcal{V}}$. However, in the special case when the constraints $\theta_1, \dots, \theta_m$ are central with respect to the first λ -bracket $\{\cdot \lambda \cdot\}_0$ we have the following result.

Theorem 1.19 ([De Sole et al., 2013-c, Theorem 2.3]). Let \mathcal{V} be a differential algebra, endowed with two compatible PVA λ -brackets $\{\cdot \lambda \cdot\}_0, \{\cdot \lambda \cdot\}_1$. Let $\theta_1, \dots, \theta_m \in \mathcal{V}$ be central elements with respect to the first λ -bracket: $\{a \lambda \theta_i\}_0 = 0$ for all $i = 1, \dots, m, a \in \mathcal{V}$. Let $C(\partial) = (C_{\alpha,\beta}(\partial))_{\alpha,\beta=1}^m$ be the matrix pseudodifferential operator given by (1.16) for the second λ -bracket: $C_{\alpha,\beta}(\lambda) = \{\theta_\beta \lambda \theta_\alpha\}_1$. Suppose that the matrix $C(\partial)$ is invertible, and consider the Dirac modified PVA λ -bracket $\{\cdot \lambda \cdot\}_1^D$ given by (1.17). Then, $\{\cdot \lambda \cdot\}_0$ and $\{\cdot \lambda \cdot\}_1^D$ are compatible PVA λ -brackets on \mathcal{V} . Moreover, the differential algebra ideal $\mathcal{I} = \langle \theta_1, \dots, \theta_m \rangle_{\mathcal{V}}$ is a PVA ideal for both the λ -brackets $\{\cdot \lambda \cdot\}_0$ and $\{\cdot \lambda \cdot\}_1^D$, and we have the induced compatible PVA λ -brackets on \mathcal{V}/\mathcal{I} . \square

2 Adler type pseudodifferential operators, classical \mathcal{W} -algebras, and the Miura map

2.1 The Adler map for a scalar pseudodifferential operator

Let L be a scalar pseudodifferential operator of order $\text{ord}(L) = N \in \mathbb{Z}$, with coefficients in a differential algebra \mathcal{A} . The corresponding *Adler map* $A^{(L)} : \mathcal{A}((\partial^{-1})) \longrightarrow \mathcal{A}((\partial^{-1}))$ is given by (cf. Adler [1979])

$$A^{(L)}(F) = (LF)_+L - L(FL)_+ = L(FL)_- - (LF)_-L, \quad (2.1)$$

for any $F \in \mathcal{A}((\partial^{-1}))$. By the last expression in (2.1), we have that $A^{(L)}(F) \in \mathcal{A}((\partial^{-1}))_{N-1}$ for every $F \in \mathcal{A}((\partial^{-1}))$. Moreover, if $F \in \mathcal{A}((\partial^{-1}))_{-N-1}$, then $(FL)_+ = (LF)_+ = 0$, and therefore $A^{(L)}(F) = 0$. In conclusion, $A^{(L)}$ induces a map $A^{(L)} : \mathcal{A}((\partial^{-1}))/\mathcal{A}((\partial^{-1}))_{-N-1} \longrightarrow \mathcal{A}((\partial^{-1}))_{N-1}$. Note that $\mathcal{A}((\partial^{-1})) = \mathbb{F}[\partial, \partial^{-1}] \circ \mathcal{A} + \mathcal{A}((\partial^{-1}))_{-N-1}$, and $(\mathbb{F}[\partial, \partial^{-1}] \circ \mathcal{A}) \cap \mathcal{A}((\partial^{-1}))_{-N-1} = \partial^{-N-1}\mathbb{F}[\partial^{-1}] \circ \mathcal{A}$. Hence, we can canonically identify

$$\mathcal{A}((\partial^{-1}))/\mathcal{A}((\partial^{-1}))_{-N-1} \simeq (\mathbb{F}[\partial, \partial^{-1}] \circ \mathcal{A})/(\partial^{-N-1}\mathbb{F}[\partial^{-1}] \circ \mathcal{A}),$$

and we get the induced map

$$A^{(L)} : (\mathbb{F}[\partial, \partial^{-1}] \circ \mathcal{A})/(\partial^{-N-1}\mathbb{F}[\partial^{-1}] \circ \mathcal{A}) \longrightarrow \mathcal{A}((\partial^{-1}))_{N-1}. \quad (2.2)$$

Let $I = \{-N, -N+1, -N+2, \dots\} \subset \mathbb{Z}$. We have the identifications

$$(\mathbb{F}[\partial, \partial^{-1}] \circ \mathcal{A})/(\partial^{-N-1}\mathbb{F}[\partial^{-1}] \circ \mathcal{A}) \xrightarrow{\sim} \mathcal{A}^{\oplus I}, \quad \sum_{n=-N}^M \partial^n \circ F_n \mapsto (F_n)_{n \in I}, \quad (2.3)$$

and

$$\mathcal{A}((\partial^{-1}))_{N-1} \xrightarrow{\sim} \mathcal{A}^I, \quad \sum_{n=-N}^{\infty} P_n \partial^{-n-1} \mapsto (P_n)_{n \in I}. \quad (2.4)$$

Therefore the map $A^{(L)}$ in (2.2) induces a map

$$H^{(L)} : \mathcal{A}^{\oplus I} \longrightarrow \mathcal{A}^I. \quad (2.5)$$

This map is given by an $I \times I$ matrix differential operator $H^{(L)} = (H_{ij}^{(L)}(\partial))_{i,j \in I}$, which we compute explicitly in terms of the generating series of its entries:

$$H^{(L)}(\partial)(z, w) = \sum_{i,j \in I} H_{ij}^{(L)}(\partial) z^{-i-1} w^{-j-1}. \quad (2.6)$$

Lemma 2.1. We have

$$H^{(L)}(\partial)(z, w) = L(w)i_w(w - z - \partial)^{-1} \circ L(z) - L(z + \partial)i_w(w - z - \partial)^{-1} \circ L^*(-w + \partial), \quad (2.7)$$

where, as usual, we expand $L(z + \partial)$ and $L^*(-w + \partial)$ in non-negative powers of ∂ . \square

Proof. By definition, the matrix element $H_{ij}^{(L)}(\partial)$ is given by

$$H_{ij}^{(L)}(\partial)(f) = \text{Res}_\partial \left(A^{(L)} (\partial^j \circ f) \partial^i \right), \quad (2.8)$$

for all $f \in \mathcal{A}$ and $i, j \in I$. Note that, by the above observations on the Adler map $A^{(L)}$, the RHS of (2.8) is zero for i or j less than $-N$. Therefore, recalling the definition (1.2) of the δ -function,

$$\begin{aligned} H^{(L)}(\partial)(z, w)f &= \sum_{i, j \in \mathbb{Z}} \text{Res}_\partial \left(A^{(L)} (\partial^j \circ f) \partial^i \right) z^{-i-1} w^{-j-1} = \\ &= \text{Res}_\partial \left((L(\partial)\delta(w - \partial) \circ f)_+ L(\partial)\delta(z - \partial) - L(\partial)(\delta(w - \partial) \circ fL(\partial))_+ \delta(z - \partial) \right). \end{aligned} \quad (2.9)$$

By equation (1.1), we have

$$(L(\partial)\delta(w - \partial) \circ f)_+ = L(w)\delta(w - \partial)_+ \circ f = L(w)i_w(w - \partial)^{-1} \circ f. \quad (2.10)$$

Similarly,

$$(\delta(w - \partial) \circ fL(\partial))_+ = \delta(w - \partial)_+ \circ (L^*(-w + \partial)f) = i_w(w - \partial)^{-1} \circ (L^*(-w + \partial)f), \quad (2.11)$$

where $(L^*(-w + \partial)f) \in \mathcal{A}((w^{-1}))$ is obtained by applying the (non-negative) powers of ∂ to f (we put parentheses to denote this). Combining equations (2.9), (2.10) and (2.11), we get

$$\begin{aligned} H^{(L)}(\partial)(z, w)f &= \text{Res}_\partial \left(L(w)i_w(w - \partial)^{-1} \circ fL(\partial)\delta(z - \partial) \right. \\ &\quad \left. - L(\partial)i_w(w - \partial)^{-1} \circ (L^*(-w + \partial)f) \delta(z - \partial) \right). \end{aligned} \quad (2.12)$$

By equation (1.1), inside the residue we can replace ∂ , written on the right, by z (and therefore ∂ , written anywhere, by $z + \partial$, written in the same place). Hence, equation (2.12) gives (2.7). \blacksquare

In the last term of equation (2.7) $L^*(\partial)$ denotes the adjoint of the pseudodifferential operator $L(\partial)$, and $L^*(-w + \partial) \in \mathcal{A}[\partial]((w^{-1}))$ is obtained by replacing ∂ by $-w + \partial$ and expanding in non-negative powers of ∂ . This is not the same as the adjoint of the differential operator $L(-w + \partial)$. In fact, we have the identity

$$L(z + \partial)^* = L^*(-z + \partial). \quad (2.13)$$

Note that, while $H^{(L)}(\partial)(z, w)$ lies in $\mathcal{A}[\partial]((z^{-1}, w^{-1}))$, i.e. it has powers of z and w simultaneously bounded above (by construction), the two terms in the RHS of (2.7) do not: they lie in $\mathcal{A}[\partial]((z^{-1}))((w^{-1}))$ (and the powers of z are not bounded above).

2.2 Preliminary properties of the Adler map

Let \mathcal{V} be a differential algebra, let $L(\partial) \in \mathcal{V}((\partial^{-1}))$ be a pseudodifferential operator of order N , and let $H^{(L)}(\partial)(z, w)$ be as in (2.7).

Lemma 2.2. (a) $H^{(L)}(\partial)(z, w) = -H^{(L)}(\partial)^*(w, z)$.

(b) The following identity holds:

$$\begin{aligned}
& H^{(L)}(\lambda)(z_2, z_1)i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1}L(z_3) \\
& - H^{(L)}(\lambda)(z_3 + \mu + \partial, z_1)i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1}L^*(-z_2 + \mu) \\
& - L(z_1)i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1}H^{(L)}(\mu)(z_3, z_2) \\
& + L(z_3 + \lambda + \mu + \partial)i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1}\left(\left|_{x=\lambda+\mu+\partial} H^{(L)}(\mu)(z_1 - x, z_2)\right.\right) \\
& = H^{(L)}(\lambda + \mu + \partial)(z_3, z_1)i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1}L(z_2) \\
& - H^{(L)}(\lambda + \mu + \partial)(z_3, z_2 + \lambda + \partial)i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1}L^*(-z_1 + \lambda).
\end{aligned} \tag{2.14}$$

In the fourth term of the RHS of equation (2.14) we used the following notation: for a Laurent series $P(z) = \sum_{n=-\infty}^N c_n z^n \in \mathcal{A}((z^{-1}))$ and elements $a, b \in \mathcal{A}$, we let

$$a\left(\left|_{x=\nu+\partial} P(z+x)b\right.\right) = \sum_{n=-\infty}^N ai_z(z+\nu+\partial)^n(c_n b). \tag{2.15}$$

□

Proof. Taking the adjoint of (2.7) and using equation (2.13), we get

$$H^{(L)}(\partial)^*(z, w) = L(z)i_w(w - z + \partial)^{-1}L(w) - L(w + \partial)i_w(w - z + \partial)^{-1}L^*(-z + \partial). \tag{2.16}$$

Combining equations (2.7) and (2.16), and using equation (1.2), we get,

$$\begin{aligned}
& H^{(L)}(\partial)(z, w) + H^{(L)}(\partial)^*(w, z) \\
& = L(w)\delta(w - z - \partial)L(z) - L(z + \partial)\delta(w - z - \partial)L^*(-w + \partial).
\end{aligned} \tag{2.17}$$

By equation (1.1), the first term in the RHS of (2.17) is equal to $L(z + \partial)\delta(w - z - \partial)L(z)$. Moreover, if $L(z) = \sum_n a_n z^n$, then $L^*(-w + \partial) = \sum_n (w - \partial)^n a_n$. Therefore, by equation (1.1), $\delta(w - z - \partial)L^*(-w + \partial) = \delta(w - z - \partial)\sum_n z^n a_n = \delta(w - z - \partial)L(z)$. Hence, the second term in the RHS of (2.17) is equal to $L(z + \partial)\delta(w - z - \partial)L(z)$ as well, proving part (a).

Using (2.7), we can rewrite each of the six terms of equation (2.14). The first term is

$$\begin{aligned}
& H^{(L)}(\lambda)(z_2, z_1)i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1}L(z_3) \\
&= i_{z_1}(z_1 - z_2 - x)^{-1}i_{z_2}(z_2 - z_3 - y)^{-1}\left(L(z_1)\left(\left|_{x=\lambda+\partial}L(z_2)\right)\left(\left|_{y=\mu+\partial}L(z_3)\right)\right.\right. \\
&\quad \left.\left.-L(z_2+x)\left(\left|_{x=\lambda+\partial}L^*(-z_1+\lambda)\right)\left(\left|_{y=\mu+\partial}L(z_3)\right)\right)\right), \tag{2.18}
\end{aligned}$$

the second term is

$$\begin{aligned}
& H^{(L)}(\lambda)(z_3 + \mu + \partial, z_1)i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1}L^*(-z_2 + \mu) \\
&= i_{z_1}(z_1 - z_3 - x - y)^{-1}i_{z_2}(z_2 - z_3 - y)^{-1}\times \\
&\quad \times \left(L(z_1)\left(\left|_{x=\lambda+\partial}L(z_3+y)\right)\left(\left|_{y=\mu+\partial}L^*(-z_2+\mu)\right)\right.\right. \\
&\quad \left.\left.-L(z_3+x+y)\left(\left|_{x=\lambda+\partial}L^*(-z_1+\lambda)\right)\left(\left|_{y=\mu+\partial}L^*(-z_2+\mu)\right)\right)\right), \tag{2.19}
\end{aligned}$$

the third term is

$$\begin{aligned}
& L(z_1)i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1}H^{(L)}(\mu)(z_3, z_2) \\
&= i_{z_1}(z_1 - z_3 - x - y)^{-1}i_{z_2}(z_2 - z_3 - y)^{-1}\left(L(z_1)\left(\left|_{x=\lambda+\partial}L(z_2)\right)\left(\left|_{y=\mu+\partial}L(z_3)\right)\right.\right. \\
&\quad \left.\left.-L(z_1)\left(\left|_{x=\lambda+\partial}L(z_3+y)\right)\left(\left|_{y=\mu+\partial}L^*(-z_2+\mu)\right)\right)\right), \tag{2.20}
\end{aligned}$$

the fourth term is

$$\begin{aligned}
& L(z_3 + \lambda + \mu + \partial)i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1}\left(\left|_{x=\lambda+\mu+\partial}H^{(L)}(\mu)(z_1 - x, z_2)\right)\right) \\
&= i_{z_1}(z_1 - z_3 - x - y)^{-1}i_{z_2}(z_1 - z_2 - x)^{-1}L(z_3 + x + y) \\
&\quad \left(-\left(\left|_{x=\lambda+\partial}L(z_2)\right)\left(\left|_{y=\mu+\partial}i_{z_1}L(z_1 - x - y)\right)\right.\right. \\
&\quad \left.\left.+ \left(\left|_{x=\lambda+\partial}L^*(-z_1+\lambda)\right)\left(\left|_{y=\mu+\partial}L^*(-z_2+\mu)\right)\right)\right), \tag{2.21}
\end{aligned}$$

the fifth term is

$$\begin{aligned}
& H^{(L)}(\lambda + \mu + \partial)(z_3, z_1)i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1}L(z_2) \\
&= i_{z_1}(z_1 - z_3 - x - y)^{-1}i_{z_2}(z_1 - z_2 - x)^{-1}\left(L(z_1)\left(\left|_{x=\lambda+\partial}L(z_2)\right)\left(\left|_{y=\mu+\partial}L(z_3)\right)\right.\right. \\
&\quad \left.\left.-L(z_3+x+y)\left(\left|_{x=\lambda+\partial}L(z_2)\right)\left(\left|_{y=\mu+\partial}i_{z_1}L(z_1-x-y)\right)\right)\right), \tag{2.22}
\end{aligned}$$

and the last term is

$$\begin{aligned}
& H^{(L)}(\lambda + \mu + \partial)(z_3, z_2 + \lambda + \partial)i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1}L^*(-z_1 + \lambda) \\
&= i_{z_1}(z_1 - z_2 - x)^{-1}i_{z_2}(z_2 - z_3 - y)^{-1}\left(L(z_2+x)\left(\left|_{x=\lambda+\partial}L^*(-z_1+\lambda)\right)\right.\right. \\
&\quad \left.\left.\times \left(\left|_{y=\mu+\partial}L(z_3)\right)\right) - L(z_3+x+y)\left(\left|_{x=\lambda+\partial}L^*(-z_1+\lambda)\right)\left(\left|_{y=\mu+\partial}L^*(-z_2+\mu)\right)\right)\right). \tag{2.23}
\end{aligned}$$

Combining the second term in the RHS of (2.18) and the first term in the RHS of (2.23) we get 0, and combining the first term in the RHS of (2.19) and the second term in the RHS of (2.20) we also get 0. Next, combining

the first terms in the RHS of (2.18), (2.20) and (2.22), we get

$$\begin{aligned} & \left(i_{z_1}(z_1 - z_2 - x)^{-1} i_{z_2}(z_2 - z_3 - y)^{-1} - i_{z_1}(z_1 - z_3 - x - y)^{-1} i_{z_2}(z_2 - z_3 - y)^{-1} \right. \\ & \left. - i_{z_1}(z_1 - z_3 - x - y)^{-1} i_{z_1}(z_1 - z_2 - x)^{-1} \right) L(z_1) \left(\left|_{x=\lambda+\partial} L(z_2) \right) \left(\left|_{y=\mu+\partial} L(z_3) \right) \right), \end{aligned}$$

and this is zero by the obvious identity

$$a^{-1}b^{-1} - i_a(a+b)^{-1}b^{-1} - i_a(a+b)^{-1}a^{-1} = 0. \quad (2.24)$$

Next, combining the second terms in the RHS of (2.19), (2.21) and (2.23), we get

$$\begin{aligned} & \left(i_{z_1}(z_1 - z_3 - x - y)^{-1} i_{z_2}(z_2 - z_3 - y)^{-1} + i_{z_1}(z_1 - z_3 - x - y)^{-1} \times \right. \\ & \quad \times i_{z_2}(z_1 - z_2 - x)^{-1} - i_{z_1}(z_1 - z_2 - x)^{-1} i_{z_2}(z_2 - z_3 - y)^{-1} \left. \right) \times \\ & \quad \times L(z_3 + x + y) \left(\left|_{x=\lambda+\partial} L^*(-z_1 + \lambda) \right) \left(\left|_{y=\mu+\partial} L^*(-z_2 + \mu) \right) \right), \end{aligned}$$

which, by the identity $i_{z_2}(z_1 - z_2 - x)^{-1} = i_{z_1}(z_1 - z_2 - x)^{-1} - \delta(z_1 - z_2 - x)$ and equation (2.24), can be rewritten as follow:

$$\begin{aligned} & -i_{z_1}(z_1 - z_3 - x - y)^{-1} \delta(z_1 - z_2 - x) L(z_3 + x + y) \times \\ & \quad \times \left(\left|_{x=\lambda+\partial} L^*(-z_1 + \lambda) \right) \left(\left|_{y=\mu+\partial} L^*(-z_2 + \mu) \right) \right), \end{aligned} \quad (2.25)$$

Next, combining the first term in the RHS of (2.21) and the second term in the RHS of (2.22), we get the opposite of (2.25), proving the claim. \blacksquare

Lemma 2.3. If $L(\partial) \in \mathcal{V}[\partial]$, then $H^{(L)}(\partial)(z, w) \in \mathcal{V}[\partial][z, w]$. \square

Proof. By equation (2.7) $H^{(L)}(\partial)(z, w)$ has no negative powers of z , and by Lemma 2.2(a) it has no negative powers of w . \blacksquare

2.3 Adler type scalar pseudodifferential operators

Definition 2.4. Let \mathcal{V} be a differential algebra endowed with a λ -bracket $\{\cdot, \cdot\}_\lambda$. We call a pseudodifferential operator $L(\partial) \in \mathcal{V}((\partial^{-1}))$ of *Adler type* (for the λ -bracket $\{\cdot, \cdot\}_\lambda$) if the following identity holds in $\mathcal{V}[\lambda]((z^{-1}, w^{-1}))$:

$$\{L(z)_\lambda L(w)\} = H^{(L)}(\lambda)(w, z). \quad (2.26)$$

\square

Lemma 2.5. Let \mathcal{V} be a differential algebra, let $\{\cdot, \cdot\}_\lambda$ be a λ -bracket on \mathcal{V} , and let $L(\partial) \in \mathcal{V}((\partial^{-1}))$ be an Adler type pseudodifferential operator. Then:

(a) The following identity holds in $\mathcal{V}[\lambda]((z^{-1}, w^{-1}))$:

$$\{L(z)_\lambda L(w)\} = -\{L(w)_{-\lambda-\partial} L(z)\}. \quad (2.27)$$

(b) The following identity holds in $\mathcal{V}[\lambda, \mu]((z_1^{-1}, z_2^{-1}, z_3^{-1}))$:

$$\{L(z_1)_\lambda \{L(z_2)_\mu L(z_3)\}\} - \{L(z_2)_\mu \{L(z_1)_\lambda L(z_3)\}\} = \{\{L(z_1)_\lambda L(z_2)\}_{\lambda+\mu} L(z_3)\}. \quad (2.28)$$

□

Proof. In view of equation (2.26), part (a) follows from Lemma 2.2(a). Let us prove part (b). Using sesquilinearity, the left and right Leibniz rules and equation (2.7) we can rewrite each term of the equation (2.28) as follows. The first one is

$$\begin{aligned} & \{L(z_1)_\lambda \{L(z_2)_\mu L(z_3)\}\} \\ &= H^{(L)}(\lambda)(z_2, z_1) i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1} L(z_3) \end{aligned} \quad (2.29)$$

$$+ L(z_2) i_{z_2}(z_2 - z_3 - \lambda - \mu - \partial)^{-1} H^{(L)}(\lambda)(z_3, z_1) \quad (2.30)$$

$$- H^{(L)}(\lambda)(z_3 + \mu + \partial, z_1) i_{z_2}(z_2 - z_3 - \mu - \partial)^{-1} L^*(-z_2 + \mu) \quad (2.31)$$

$$- L(z_3 + \lambda + \mu + \partial) i_{z_2}(z_2 - z_3 - \lambda - \mu - \partial)^{-1} \left(\Big|_{x=\lambda+\mu+\partial} H^{(L)}(\lambda)(z_2 - x, z_1) \right); \quad (2.32)$$

the second term is

$$\begin{aligned} & \{L(z_2)_\mu \{L(z_1)_\lambda L(z_3)\}_H\}_H \\ &= H^{(L)}(\mu)(z_1, z_2) i_{z_1}(z_1 - z_3 - \lambda - \partial)^{-1} L(z_3) \end{aligned} \quad (2.33)$$

$$+ L(z_1) i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} H^{(L)}(\mu)(z_3, z_2) \quad (2.34)$$

$$- H^{(L)}(\mu)(z_3 + \lambda + \partial, z_2) i_{z_1}(z_1 - z_3 - \lambda - \partial)^{-1} L^*(-z_1 + \lambda) \quad (2.35)$$

$$- L(z_3 + \lambda + \mu + \partial) i_{z_1}(z_1 - z_3 - \lambda - \mu - \partial)^{-1} \left(\Big|_{x=\lambda+\mu+\partial} H^{(L)}(\mu)(z_1 - x, z_2) \right); \quad (2.36)$$

the third term is

$$\begin{aligned} & \{\{L(z_1)_\lambda L(z_2)\}_{H\lambda+\mu} L(z_3)\}_H \\ &= H^{(L)}(\lambda + \mu + \partial)(z_3, z_1) i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1} L(z_2) \end{aligned} \quad (2.37)$$

$$- H^{(L)}(\lambda + \mu + \partial)(z_3, z_2) i_{z_1}(z_2 - z_1 - \mu - \partial)^{-1} L(z_1) \quad (2.38)$$

$$- H^{(L)}(\lambda + \mu + \partial)(z_3, z_2 + \lambda + \partial) i_{z_1}(z_1 - z_2 - \lambda - \partial)^{-1} L^*(-z_1 + \lambda) \quad (2.39)$$

$$+ H^{(L)}(\lambda + \mu + \partial)(z_3, z_1 + \mu + \partial) i_{z_1}(z_2 - z_1 - \mu - \partial)^{-1} L^*(-z_2 + \mu). \quad (2.40)$$

By Lemma 2.2(b) we have the following identity

$$(2.29) + (2.31) - (2.34) - (2.36) = (2.37) + (2.39).$$

Hence, equation (2.28) is proved once we show that

$$(2.30) + (2.32) - (2.33) - (2.35) = (2.38) + (2.40). \quad (2.41)$$

Using equation (2.14) with λ and μ exchanged, and with z_1 and z_2 exchanged, we get that equation (2.41) is equivalent to the following equation

$$\begin{aligned} & H^{(L)}(\lambda + \mu + \partial)(z_3, z_2) \delta(z_2 - z_1 - \mu - \partial) L(z_1) \\ & - H^{(L)}(\lambda + \mu + \partial)(z_3, z_1 + \mu + \partial) \delta(z_2 - z_1 - \mu - \partial) L^*(-z_2 + \mu) = 0, \end{aligned}$$

which holds by the properties of the δ -function. ■

Remark 2.6. Let \mathcal{V} be a differential algebra, endowed with a λ -bracket $\{\cdot, \lambda \cdot\}$. Let $L(\partial) \in \mathcal{V}((\partial^{-1}))$ be a pseudodifferential operator, and let $\mathcal{U} \subset \mathcal{V}$ be the differential subalgebra generated by the coefficients of $L(\partial)$. Clearly, if $L(\partial)$ is of Adler type, then $\{\cdot, \lambda \cdot\}$ restricts to a λ -bracket of \mathcal{U} . Lemma 2.5, together with Theorem 1.6, is saying that \mathcal{U} , with this λ -bracket, is a PVA. □

2.4 The generic pseudodifferential operator of order N and the corresponding AGD bi-PVA \mathcal{V}_N^∞

Let N be a positive integer. Consider the algebra of differential polynomials $\mathcal{V}_N^\infty = \mathbb{F}[u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$, where, as before, $I = \{-N, -N+1, -N+2, \dots\}$. The *generic* pseudodifferential operator on \mathcal{V}_N^∞ is, by definition,

$$L(\partial) = \partial^N + u_{-N} \partial^{N-1} + u_{-N+1} \partial^{N-2} + \dots = \sum_{n \leq N} u_{-n-1} \partial^n \in \mathcal{V}_N^\infty((\partial^{-1})), \quad (2.42)$$

where $u_{-N-1} = 1$. For $c \in \mathbb{F}$, let $H^{(L-c)}(\partial)(z, w) \in \mathcal{V}_N^\infty[\partial]((z^{-1}, w^{-1}))$ be the corresponding series, as in equation (2.7).

Recall by Theorem 1.6(a) that a λ -bracket on \mathcal{V}_N^∞ is uniquely determined by assigning the λ -brackets on generators, $\{u_{i\lambda}u_j\}$, for all $i, j \in I$, or, equivalently, their generating series $\{L(z)_\lambda L(w)\}$. In particular, there exists a unique λ -bracket on \mathcal{V}_N^∞ such that $L(\partial) - c$ is of Adler type:

$$\{L(z)_\lambda L(w)\}_c = H^{(L-c)}(\lambda)(w, z). \quad (2.43)$$

It is clear from the definition (2.1) of the Adler map $A^{(L)}$ that $A^{(L-c)}$ is linear in c , and therefore

$$H^{(L-c)}(\partial) = H(\partial) - cK(\partial), \quad (2.44)$$

where $H = H^{(L)}$ is given by (2.6) and (2.7), and $K \in \text{Mat}_{I \times I} \mathcal{V}_N^\infty[\partial]$.

Theorem 2.7. The 1-parameter family of λ -brackets $\{\cdot_\lambda \cdot\}_c = \{\cdot_\lambda \cdot\}_H - c\{\cdot_\lambda \cdot\}_K$, $c \in \mathbb{F}$, defines a structure of bi-PVA on \mathcal{V}_N^∞ . \square

Proof. The statement is a special case of Remark 2.6. By Definition 1.8, we have to prove that the λ -bracket on \mathcal{V} defined by equation (2.43) defines a PVA structure on \mathcal{V} . By Theorem 1.6, this is the same as proving skew-symmetry (1.7) and Jacobi identity (1.8) on generators, which are the same as equations (2.27) and (2.28) respectively. \blacksquare

Definition 2.8. The AGD bi-PVA \mathcal{V}_N^∞ is given by the 1-parameter family of λ -brackets $\{\cdot_\lambda \cdot\}_c$, $c \in \mathbb{F}$. The operators K and H are usually called, respectively, the *first* and the *second Adler-Gelfand-Dickey Poisson structures*. \square

Using equation (2.7) we get the following explicit formulas for $\{L(z)_\lambda L(w)\}_c = \{L(z)_\lambda L(w)\}_H - c\{L(z)_\lambda L(w)\}_K$:

$$\begin{aligned} \{L(z)_\lambda L(w)\}_H &= L(z)i_z(z-w-\lambda-\partial)^{-1}L(w) \\ &\quad - L(w+\lambda+\partial)i_z(z-w-\lambda-\partial)^{-1}L^*(-z+\lambda) \end{aligned} \quad (2.45)$$

and

$$\begin{aligned} \{L(z)_\lambda L(w)\}_K &= i_z(z-w-\lambda)^{-1}(L(z) - L(w+\lambda)) \\ &\quad + i_z(z-w-\lambda-\partial)^{-1}(L(w) - L^*(-z+\lambda)). \end{aligned} \quad (2.46)$$

Expanding equations (2.45) and (2.46) in powers of z and w , we get the symbols of the Poisson structures H and K ($i, j \in I$):

$$\begin{aligned} H_{ji}(\lambda) &= \sum_{k, \alpha \in \mathbb{Z}_+} \binom{k}{\alpha} u_{i-k-1}(\lambda+\partial)^\alpha u_{j+k-\alpha} \\ &\quad - \sum_{k, \alpha, \beta \in \mathbb{Z}_+} (-1)^\alpha \binom{j}{\alpha} \binom{i-k-1}{\beta} u_{j+k-\alpha}(\lambda+\partial)^{\alpha+\beta} u_{i-\beta-k-1}, \\ K_{ji}(\lambda) &= \epsilon_{ij} \sum_{k \in \mathbb{Z}_+} \left(\binom{i}{k} (\lambda+\partial)^k - \binom{j}{k} (-\lambda)^k \right) u_{i+j-k}, \end{aligned} \quad (2.47)$$

where $\epsilon_{ij} = +1$ if $i, j \in \mathbb{Z}_+$, $\epsilon_{ij} = -1$ if $i, j < 0$, and $\epsilon_{ij} = 0$ otherwise. Note that the sums are all finite since $u_k = 0$ for $k < -N - 1$.

Remark 2.9. Let $R_+ = \bigoplus_{i \in \mathbb{Z}_+} \mathbb{F}[\partial]u_i \subset \mathcal{V}_N^\infty$ and $R_- = \left(\bigoplus_{i \in I_-} \mathbb{F}[\partial]u_i \right) \oplus \mathbb{F} \subset \mathcal{V}_N^\infty$. By the second equation in (2.47), R_+ and R_- are commuting Lie conformal subalgebras of \mathcal{V}_N^∞ for the λ -bracket $\{\cdot, \cdot\}_K: \{R_{\pm\lambda}R_{\pm}\}_K \subset R_{\pm}[\lambda]$, and $\{R_{\pm\lambda}R_{\mp}\}_K = 0$. In fact, the Lie conformal algebra R_+ is isomorphic to the general Lie conformal algebra \mathfrak{gc}_1 , via the map $u_i \mapsto J^i$ (see [Kac, 1996, Sec.2.10]). Recall from De Sole and Kac [2002] that we can identify $R_+ \simeq \mathfrak{gc}_1 \simeq \mathbb{F}[\partial, x]$ (the space of polynomials in two commuting variables ∂ and x) via the isomorphism $u_i^{(n)} \mapsto \partial^n x^i$. With this identification, the λ -bracket on \mathfrak{gc}_1 becomes:

$$\{A(\partial, x)_\lambda B(\partial, x)\}_K = A(-\lambda, x + \lambda + \partial)B(\lambda + \partial, x) - B(\lambda + \partial, x - \lambda)A(-\lambda, x).$$

In the same spirit, we can identify

$$R_- \simeq (\mathbb{F}[\partial, x^{-1}] \oplus \mathbb{F}) / \text{Span}_{\mathbb{F}}\{\partial^m x^{-N-1-n} - \delta_{m,0}\delta_{n,0}\}_{m,n \in \mathbb{Z}_+},$$

via the isomorphism $u_i^{(n)} \mapsto \partial^n x^i$, for all $i \in I_-$ and $n \in \mathbb{Z}_+$. Under this identification, the K - λ -bracket (2.47) on R_- becomes:

$$\{A(\partial, x)_\lambda B(\partial, x)\}_K = -i_x A(-\lambda, x + \lambda + \partial)B(\lambda + \partial, x) + i_x B(\lambda + \partial, x - \lambda)A(-\lambda, x).$$

In the same spirit, we can also rewrite the H - λ -bracket (2.47). We let $R = R_+ \oplus R_- = \left(\bigoplus_{i \in I} \mathbb{F}[\partial]u_i \right) \oplus \mathbb{F} \subset \mathcal{V}_N^\infty$. Note that $R \subset \mathcal{V}_N^\infty$ is not a Lie conformal subalgebra for the H - λ -bracket, since the expression of the λ -bracket of two generators is quadratic. On the other hand, we can represent homogeneous polynomials of degree 2 in the variables $u_i^{(n)}$ with polynomials in $\partial_1, x_1^{\pm 1}, \partial_2, x_2^{\pm 1}$, via the identification $\partial_1^m \partial_2^n x_1^i x_2^j \mapsto u_i^{(m)} u_j^{(n)}$. With this notation, the H - λ -bracket (2.47) becomes

$$\begin{aligned} \{A(\partial, x)_\lambda B(\partial, x)\}_K &= A(-\lambda, x_1)B(\lambda + \partial_1 + \partial_2, x_2) i_{x_1}(x_1 - x_2 - \partial_2 - \lambda)^{-1} \\ &\quad - i_{x_2} B(\lambda + \partial_1 + \partial_2, x_2 - \partial_1 - \lambda) i_{x_1} A(-\lambda, x_1 + \partial_1 + \lambda) i_{x_1}(x_1 - x_2 + \partial_1 + \lambda)^{-1}. \end{aligned}$$

□

2.5 The generic differential operator of order N and the corresponding AGD bi-PVA \mathcal{V}_N

Let $I_- = \{-N, \dots, -1\} \subset I$. Consider the subalgebra of differential polynomials $\mathcal{V}_N = \mathbb{F}[u_i^{(n)} \mid i \in I_-, n \in \mathbb{Z}_+]$ of \mathcal{V}_N^∞ , and the differential operator

$$L(\partial) = \partial^N + u_{-N} \partial^{N-1} + \dots + u_{-2} \partial + u_{-1} \in \mathcal{V}_N[\partial], \quad (2.48)$$

which we call *generic*. By Lemma 2.3, for all $c \in \mathbb{F}$ the corresponding operator $H^{(L-c)}(\partial)(z, w)$, given by equation (2.7), lies in $\mathcal{V}[\partial][z, w]$, i.e. $H_{ij}^{(L-c)}(\partial) = 0$ unless $i, j \in I_-$. Assign the (unique) λ -bracket on \mathcal{V}_N such that $L(\partial) - c$ is of Adler type:

$$\{L(z)_\lambda L(w)\}_c = H^{(L-c)}(\lambda)(w, z). \quad (2.49)$$

Theorem 2.10. The 1-parameter family of λ -brackets $\{\cdot_\lambda \cdot\}_c = \{\cdot_\lambda \cdot\}_H - c\{\cdot_\lambda \cdot\}_K$, $c \in \mathbb{F}$, defines a structure of bi-PVA on \mathcal{V}_N . \square

Proof. The same as the proof of Theorem 2.7. \blacksquare

Definition 2.11. The AGD bi-PVA \mathcal{V}_N is given by the 1-parameter family of λ -brackets $\{\cdot_\lambda \cdot\}_c$, $c \in \mathbb{F}$. \square

The PVA \mathcal{V}_N coincides with the classical \mathcal{W} -algebra associated to the Lie algebra \mathfrak{gl}_N and its principal nilpotent element, Drinfeld and Sokolov [1985]. The explicit formula for the λ -brackets of generators corresponding to H and K are the same as (2.45) and (2.46). The corresponding matrix elements $H_{ji}(\lambda)$ and $K_{ji}(\lambda)$ are the same as in (2.47), letting $u_i = 0$ for $i \geq 0$. Note that \mathcal{V}_N is a PVA subalgebra of \mathcal{V}_N^∞ for the Poisson structure K , but not for the Poisson structure H .

Example 2.12. For $N = 1$, we have $\mathcal{V}_1 = \mathbb{F}[u^{(n)} \mid n \in \mathbb{Z}_+]$, and $L(\partial) = \partial + u \in \mathcal{V}[\partial]$. In this case, $\{u_\lambda u\}_H = -\lambda$ and $K = 0$. \square

2.6 The PVA homomorphism φ_A

Proposition 2.13. Let \mathcal{V} be a PVA, with λ -bracket $\{\cdot_\lambda \cdot\}$. Let

$$A(\partial) = \partial^N + \sum_{i \in I} a_i \partial^{-i-1} \in \mathcal{V}((\partial^{-1})) \quad \left(\text{resp. } A(\partial) = \partial^N + \sum_{i \in I_-} a_i \partial^{-i-1} \in \mathcal{V}[\partial] \right)$$

be an Adler type pseudodifferential (resp. differential) operator with respect to the λ -bracket $\{\cdot_\lambda \cdot\}$. Then we have a PVA homomorphism

$$\begin{aligned} \varphi_A : \mathcal{V}_N^\infty = \mathbb{F}[u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+] &\rightarrow \mathcal{V} \\ \left(\text{resp. } \varphi_A : \mathcal{V}_N = \mathbb{F}[u_i^{(n)} \mid i \in I_-, n \in \mathbb{Z}_+] &\rightarrow \mathcal{V} \right), \end{aligned}$$

from the AGD PVA \mathcal{V}_N^∞ (resp. \mathcal{V}_N) for $c = 0$ to \mathcal{V} , given by $\varphi_A(u_i) = a_i$, for all i . \square

Proof. The map φ_A is defined, in terms of generating series, by $\varphi_A(L(z)) = A(z)$. Hence, we only need to check that $\varphi_A(\{L(z)_\lambda L(w)\}_H) = \{A(z)_\lambda A(w)\}$, or, equivalently, $\varphi_A(H^{(L)}(\lambda)(w, z)) = H^{(A)}(\lambda)(w, z)$. This is clear by the definitions of $H^{(L)}$, $H^{(A)}$ and φ_A . \blacksquare

2.7 Product of Adler type pseudodifferential operators and the generalized Miura map

Proposition 2.14. Let \mathcal{V} be a PVA and let $A(\partial), B(\partial) \in \mathcal{V}((\partial^{-1}))$ be Adler type pseudodifferential operators such that $\{A(z)_\lambda B(w)\} = 0$. Then $A(\partial) \circ B(\partial)$ is an Adler type pseudodifferential operator. \square

Proof. By (1.5) we need to show that

$$\{A(z + \partial)B(z)_\lambda A(w + \partial)B(w)\} = H^{(A \circ B)}(\lambda)(w, z).$$

By the sesquilinearity and Leibniz rules, and by the assumption that A and B are of Adler type, and that $\{A(z)_\lambda B(w)\} = 0$, we have

$$\begin{aligned} & \{A(z + \partial)B(z)_\lambda A(w + \partial)B(w)\} \\ &= \{A(z + x)_{\lambda+x} A(w + y)\} \left(\Big|_{x=\partial} B(z) \right) \left(\Big|_{y=\partial} B(w) \right) \\ &+ A(w + \lambda + \partial) \{B(z)_{\lambda+\partial} B(w)\} \rightarrow A^*(-z + \lambda) \\ &= (A(z + \partial)B(z)) i_z(z - w - \lambda - \partial)^{-1} A(w + \partial)B(w) \\ &- A(w + \lambda + \partial)B(w + \lambda + \partial) i_z(z - w - \lambda - \partial)^{-1} B^*(-z + \lambda + \partial) A^*(-z + \lambda), \end{aligned}$$

which is the same as $H^{(A \circ B)}(\lambda)(w, z)$. \blacksquare

Let M and N be positive integers. Consider the AGD PVAs (for $c = 0$) $\mathcal{V}_M^\infty = \mathbb{F}[a_i^{(m)} \mid i \geq -M, m \in \mathbb{Z}_+]$ (resp. $\mathcal{V}_M = \mathbb{F}[a_i^{(m)} \mid -M \leq i \leq -1, m \in \mathbb{Z}_+]$), $\mathcal{V}_N^\infty = \mathbb{F}[b_j^{(n)} \mid j \geq -N, n \in \mathbb{Z}_+]$ (resp. $\mathcal{V}_N = \mathbb{F}[b_j^{(n)} \mid -N \leq j \leq -1, n \in \mathbb{Z}_+]$), and $\mathcal{V}_{M+N}^\infty = \mathbb{F}[u_i^{(n)} \mid i \geq -M - N, n \in \mathbb{Z}_+]$ (resp. $\mathcal{V}_{M+N} = \mathbb{F}[u_i^{(n)} \mid -M - N \leq i \leq -1, n \in \mathbb{Z}_+]$), and the corresponding generic Adler type pseudodifferential (resp. differential) operators

$$\begin{aligned} A(\partial) &= \partial^M + \sum_{i=-M}^{\infty} a_i \partial^{-i-1} \quad \left(\text{resp. } A(\partial) = \partial^M + \sum_{i=-M}^{-1} a_i \partial^{-i-1} \right), \\ B(\partial) &= \partial^N + \sum_{j=-N}^{\infty} b_j \partial^{-j-1} \quad \left(\text{resp. } B(\partial) = \partial^N + \sum_{j=-N}^{-1} b_j \partial^{-j-1} \right), \\ L(\partial) &= \partial^{M+N} + \sum_{i=-M-N}^{\infty} u_i \partial^{-i-1} \quad \left(\text{resp. } L(\partial) = \partial^{M+N} + \sum_{i=-M-N}^{-1} u_i \partial^{-i-1} \right). \end{aligned}$$

Proposition 2.15. We have a PVA structure on the algebra of differential polynomials

$$\begin{aligned} & \mathcal{V}_M^\infty \otimes \mathcal{V}_N^\infty = \mathbb{F}[a_i^{(n)}, b_j^{(n)} \mid i \geq -M, j \geq -N, n \in \mathbb{Z}_+] \\ & \left(\text{resp. } \mathcal{V}_M \otimes \mathcal{V}_N = \mathbb{F}[a_i^{(n)}, b_j^{(n)} \mid -M \leq i \leq -1, -N \leq j \leq -1, n \in \mathbb{Z}_+] \right), \end{aligned}$$

given on generators by

$$\begin{aligned} \{A(z)_\lambda A(w)\} &= H^{(A)}(\lambda)(w, z), \quad \{B(z)_\lambda B(w)\} = H^{(B)}(\lambda)(w, z), \\ \{A(z)_\lambda B(w)\} &= \{B(z)_\lambda A(w)\} = 0, \end{aligned} \tag{2.50}$$

and a PVA homomorphism

$$\mu_{A,B} : \mathcal{V}_{M+N}^\infty \rightarrow \mathcal{V}_M^\infty \otimes \mathcal{V}_N^\infty \quad (\text{resp. } \mu_{A,B} : \mathcal{V}_{M+N} \rightarrow \mathcal{V}_M \otimes \mathcal{V}_N),$$

such that $\mu_{A,B}(u_i)$ is the coefficient of ∂^{-i-1} in $A(\partial) \circ B(\partial)$. In terms of generating series,

$$\mu_{A,B}(L(z)) = A(z + \partial)B(z).$$

□

Proof. It is immediate to check that formulas (2.50) define a structure of PVA on $\mathcal{V}_M^\infty \otimes \mathcal{V}_N^\infty$ (it is the tensor product of the PVAs \mathcal{V}_M^∞ and \mathcal{V}_N^∞). By construction, $A(\partial)$ and $B(\partial)$ are of Adler type in this PVA. Hence, by Proposition 2.14 $A(\partial) \circ B(\partial)$ is of Adler type as well. Therefore, by Proposition 2.13 we get the corresponding PVA homomorphism, which is exactly $\varphi_{A \circ B} = \mu_{A,B}$. ■

Definition 2.16. We call $\mu_{A,B}$ the *generalized Miura map* of type (M, N) . The same argument as in the proof of Proposition 2.15 can be applied to any number of factors, so we can talk about the *generalized Miura map* μ_{A_1, \dots, A_s} of type (N_1, \dots, N_s) . □

2.8 The classical \mathcal{W} -algebra \mathcal{W}_N

Lemma 2.17. In the AGD bi-PVA \mathcal{V}_N^∞ (resp. \mathcal{V}_N) we have:

- (a) $\{u_{-N\lambda}L(w)\}_H = L(w) - L(w + \lambda)$;
- (b) $\{L(z)_\lambda u_{-N}\}_H = L^*(-z + \lambda) - L(z)$;
- (c) $\{u_{-N\lambda}u_{-N}\}_H = -N\lambda$;
- (d) $\{u_{-N\lambda}L(w)\}_K = \{L(z)_\lambda u_{-N}\}_K = 0$.

□

Proof. By equation (2.45), we have

$$\begin{aligned} \{u_{-N\lambda}L(w)\}_H &= \text{Res}_z L(z) i_z (z - w - \lambda - \partial)^{-1} z^{-N} L(w) \\ &\quad - L(w + \lambda + \partial) \text{Res}_z i_z (z - w - \lambda - \partial)^{-1} L^*(-z + \lambda) z^{-N}. \end{aligned}$$

Note that $L(z)i_z(z-w-\lambda-\partial)^{-1}$ and $i_z(z-w-\lambda-\partial)^{-1}L^*(-z+\lambda)$ have order $N-1$ in z and their leading coefficient is $u_{-N-1} = 1$. Hence,

$$\operatorname{Res}_z L(z)i_z(z-w-\lambda-\partial)^{-1}z^{-N} = \operatorname{Res}_z i_z(z-w-\lambda-\partial)^{-1}L^*(-z+\lambda)z^{-N} = 1.$$

This proves part (a). Part (b) follows by the skew-symmetry (Lemma 2.5(a)). Part (c) follows from (a). Finally, for part (d), we have by (2.46)

$$\begin{aligned} \{u_{-N\lambda}L(w)\}_K &= \operatorname{Res}_z i_z(z-w-\lambda)^{-1}(L(z) - L(w+\lambda))z^{-N} \\ &+ \operatorname{Res}_z i_z(z-w-\lambda-\partial)^{-1}(L(w) - L^*(-z+\lambda))z^{-N} \\ &= \operatorname{Res}_z i_z(z-w-\lambda)^{-1}L(z)z^{-N} - \operatorname{Res}_z i_z(z-w-\lambda-\partial)^{-1}L^*(-z+\lambda)z^{-N}, \end{aligned}$$

which is zero by the same argument as before. \blacksquare

Since, by Lemma 2.17(c) and (d), $\{u_{-N\lambda}u_{-N}\}_H$ is not zero and u_{-N} is central with respect to the Poisson structure K , by Theorems 1.16 and 1.19 we can perform the Dirac reduction to get a bi-Poisson structure (H^D, K) on $\mathcal{V}_N^\infty/\langle u_{-N} \rangle \simeq \mathbb{F}[u_i^{(n)} \mid -N \neq i \in I, n \in \mathbb{Z}_+] =: \mathcal{W}_N^\infty$ (resp. $\mathcal{V}_N/\langle u_{-N} \rangle \simeq \mathbb{F}[u_i^{(n)} \mid -N \neq i \in I_-, n \in \mathbb{Z}_+] =: \mathcal{W}_N$).

Proposition 2.18. We have a local bi-Poisson structure (H^D, K) on \mathcal{W}_N^∞ (resp. \mathcal{W}_N), where H^D is defined, in terms of the generating series of the λ -brackets on generators, by

$$\begin{aligned} \{L(z)_\lambda L(w)\}_{H^D} &= L(z)i_z(z-w-\lambda-\partial)^{-1}L(w) \\ &- L(w+\lambda+\partial)i_z(z-w-\lambda-\partial)^{-1}L^*(-z+\lambda) \\ &- \frac{1}{N}(L(w+\lambda+\partial) - L(w))(\lambda+\partial)^{-1}(L^*(-z+\lambda) - L(z)), \end{aligned} \tag{2.51}$$

and K is given by (2.46). \square

Proof. Formula (2.51) follows from (1.17) using Lemma 2.17. Since $L^*(-z+\lambda) = (\big|_{x=\lambda+\partial} L(z-x))$, the term $(\lambda+\partial)^{-1}(L^*(-z+\lambda) - L(z))$ does not contain negative powers of $\lambda+\partial$, thus proving that the λ -bracket given in (2.51) is local. \blacksquare

Expanding the equation (2.51) in powers of z and w the matrix elements of H^D are (cf. equation (2.47)):

$$H_{ji}^D(\lambda) = H_{ji}(\lambda) - \frac{1}{N} \sum_{\alpha, \beta \geq 1} (-1)^\alpha \binom{j}{\alpha} \binom{i}{\beta} u_{j-\alpha}(\lambda+\partial)^{\alpha+\beta-1} u_{i-\beta}, \tag{2.52}$$

and $K_{ji}(\lambda)$ is the same as in equation (2.47). Letting $T = u_{-N+1}$, we have

$$\{T_\lambda T\}_{H^D} = H_{-N+1, -N+1}^D(\lambda) = (2\lambda+\partial)T + \frac{N^3 - N}{12}\lambda^3, \tag{2.53}$$

namely, T is a Virasoro element with central charge $\frac{N^3-N}{12}$ (cf. Definition 1.5). Furthermore, we have

$$\{T_\lambda u_j\}_{HD} = (\partial + (N + j + 1)\lambda)u_j + O(\lambda^2), \quad (2.54)$$

namely, u_j is a T -eigenvector of conformal weight $N + j + 1$, for every $j \neq -N$.

Definition 2.19. The *classical W-algebra* \mathcal{W}_N is given by the bi-Poisson structure (H^D, K) . \square

Remark 2.20. The classical W-algebras \mathcal{W}_N can be obtained by performing Drinfeld-Sokolov Hamiltonian reduction for the Lie algebra \mathfrak{sl}_N and its principal nilpotent element, [Drinfeld and Sokolov \[1985\]](#). \square

Example 2.21. For $N = 2$ we have $\mathcal{W}_2 = \mathbb{F}[u^{(n)} \mid n \in \mathbb{Z}_+]$, where $u = u_{-1}$, and the PVA structure $\{u_\lambda u\}_c = H^D(\lambda) - cK(\lambda)$, $c \in \mathbb{F}$, given by (2.51) and (2.46), becomes, in this case,

$$\{u_\lambda u\}_c = (2\lambda + \partial)u + \frac{1}{2}\lambda^3 - 2c\lambda. \quad (2.55)$$

This is the Virasoro-Magri PVA, which is the classical W-algebra for the Lie algebra \mathfrak{sl}_2 and its principal nilpotent element. \square

Example 2.22. For $N = 3$ we have $\mathcal{W}_3 = \mathbb{F}[u^{(n)}, v^{(n)} \mid n \in \mathbb{Z}_+]$, where $u = u_{-2}$ and $v = u_{-1}$, and the PVA structure is

$$\begin{aligned} \{u_\lambda u\}_c &= (2\lambda + \partial)u + 2\lambda^3, \\ \{u_\lambda v\}_c &= (3\lambda + \partial)v + u\lambda^2 + \lambda^4 - 3c\lambda \\ \{v_\lambda v\}_c &= (2\lambda + \partial) \left(\partial v - \frac{1}{2}\partial^2 u - \frac{1}{3}u^2 \right) - \frac{1}{6}(2\lambda + \partial)^3 u - \frac{2}{3}\lambda^5. \end{aligned} \quad (2.56)$$

This is known as the Zamolodchikov PVA, [Zamolodchikov \[1985\]](#), which is the classical W-algebra for \mathfrak{sl}_3 and its principal nilpotent element. Note that $T = u$ is a Virasoro element with central charge 2, and it is not hard to check that $w_3 = v - \frac{1}{2}\partial u$ is a primary element of conformal weight 3. In particular, \mathcal{W}_3 is a PVA of CFT type (cf. Definition 1.5). \square

For arbitrary $N \geq 1$, we let $T = u_{-N+1}$, and we have

$$\begin{aligned} \{T_\lambda T\}_{HD} &= (2\lambda + \partial)T + \frac{N^3 - N}{12}\lambda^3, \\ \{T_\lambda u_k\}_{HD} &= ((N + 1 + k)\lambda + \partial)u_k + O(\lambda^2), \quad k = -N + 2, \dots, -1. \end{aligned}$$

Hence, T is a Virasoro element, and u_k , $k = -N + 2, \dots, -1$, are T -eigenvectors. It was proved by [Balog et al. \[1990\]](#) and [Di Francesco et al. \[1991\]](#) that, in fact, \mathcal{W}_N is a PVA of CFT type.

2.9 The Kupershmidt-Wilson theorem and the Miura map

Consider the generalized GFZ PVA $R_N = \mathbb{F}[v_i^{(n)} \mid i = 1, \dots, N, n \in \mathbb{Z}_+]$ from Example 1.9, with λ -bracket $\{v_{i\lambda} v_j\} = -\delta_{ij}\lambda$ (we are taking $S = -\mathbb{1}_N$), and its Dirac reduction $R_N/\langle v_1 + \dots + v_N \rangle$ as in Example 1.17.

In this case, the Dirac reduced λ -bracket among generators (1.18) reads ($i, j = 1, \dots, N - 1$):

$$\{v_{i\lambda}v_j\}^D = \left(\frac{1}{N} - \delta_{ij}\right)\lambda. \quad (2.57)$$

Recall from Sections 2.5 and 2.8 the definitions of the AGD PVA \mathcal{V}_N , and of the classical \mathcal{W} -algebra \mathcal{W}_N , respectively. In this section we want to give another proof of the following theorem due to [Kupershmidt and Wilson \[1981\]](#), that we restate according to our formalism.

Theorem 2.23 (Kupershmidt-Wilson Theorem). (a) We have an injective PVA homomorphism $\mu : \mathcal{V}_N \hookrightarrow R_N$, given by

$$\mu(L(\partial)) = (\partial + v_N)(\partial + v_{N-1}) \cdots (\partial + v_1) \in R_N[\partial], \quad (2.58)$$

where $L(\partial) \in \mathcal{V}_N[\partial]$ is as in (2.48).

(b) The map μ in part (a) induces an injective PVA homomorphism

$$\mu : \mathcal{W}_N \hookrightarrow R_N / \langle v_1 + \cdots + v_N \rangle.$$

□

The map μ is called the Miura map ([Miura \[1968\]](#)). It allows us to express each differential variable u_i as a differential polynomial in R_N . The original proof was given by [Kupershmidt and Wilson \[1981\]](#). A shorter proof can be found in the work of [Dickey \[1982\]](#).

Proof. Recall from Example 2.12 that $\mathcal{V}_1 \simeq R_1 = \mathbb{F}[v^{(n)} \mid n \in \mathbb{Z}_+]$, with $\{v_\lambda v\} = -\lambda$. Hence, $R_N = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_1$ (N times). By Proposition 2.15 and Definition 2.16 we then have the corresponding generalized Miura map of type $(1, 1, \dots, 1)$, defined by (2.58). For the injectiveness of this map (and the induced map on the Dirac reductions) we refer to [Kupershmidt and Wilson \[1981\]](#). Part (b) follows immediately from Lemma 1.18. ■

Remark 2.24. The aim of the Kupershmidt-Wilson Theorem was to prove that the matrix differential operator $H^{(L)}(\partial)$, attached to a generic differential operator L , is a Poisson structure. Indeed, it was well known that the operator $-\mathbb{1}_N \partial$ is a Poisson structure (cf. Example 1.9) on R_N . The Kupershmidt-Wilson Theorem shows that $(\mathcal{V}_N, H^{(L)}) \subset (R_N, -\partial \mathbb{1}_N)$ is a PVA subalgebra. In particular, $H^{(L)}$ must be a Poisson structure. However, one cannot apply the same argument in the case of a generic pseudodifferential operator, since one does not have a factorization analogue to (2.58). □

Example 2.25. Recall from Example 2.21 that $\mathcal{W}_2 = \mathbb{F}[u^{(n)} \mid n \in \mathbb{Z}_+]$, with λ -bracket (2.55) (with $c = 0$). By Theorem 2.23(b) we have a PVA inclusion $\mathcal{W}_2 \hookrightarrow R_2 / \langle v_1 + v_2 \rangle_{R_2} \simeq \mathbb{F}[v^{(n)} \mid n \in \mathbb{Z}_+]$ (where $v = v_1$), with λ -bracket $\{v_\lambda v\} = -\frac{1}{2}\lambda$. The Miura map is given by $u = v' - v^2$, [Miura \[1968\]](#). □

3 Gelfand-Dickey integrable hierarchies

In this section we want to show how to apply the Lenard-Magri scheme of integrability (see Section 1.5) in order to obtain integrable hierarchies for the the bi-PVAs we constructed in Section 2.

3.1 Integrable hierarchies for the AGD bi-PVAs \mathcal{V}_N^∞ and \mathcal{V}_N

Recall from Section 2.4 the definition of the AGD bi-PVA $\mathcal{V}_N^\infty = \mathbb{F}[u_i^{(n)} \mid i \in I, n \in \mathbb{Z}_+]$ (as before, $I = \{-N, -N+1, -N+2, \dots\}$), associated to the generic pseudodifferential operator $L(\partial)$ as in (2.42), and recall from Section 2.5 the AGD bi-PVA $\mathcal{V}_N = \mathbb{F}[u_i^{(n)} \mid i \in I_-, n \in \mathbb{Z}_+]$ (where $I_- = \{-N, -N+1, \dots, -1\}$), associated to the generic differential operator $L(\partial)$ as in (2.48). Unless otherwise specified, throughout this section we let $\mathcal{V} = \mathcal{V}_N^\infty$ or \mathcal{V}_N , with its bi-Poisson structure (H, K) , and we let $L(\partial)$ as in (2.42) or (2.48). Let, for $k \geq 1$,

$$h_k = \frac{N}{k} \operatorname{Res}_z L^{\frac{k}{N}}(z) \in \mathcal{V}, \quad (3.1)$$

where $L^{\frac{1}{N}}(\partial) \in \mathcal{V}((\partial^{-1}))$ is uniquely defined by Proposition 1.1.

Theorem 3.1. We have an integrable hierarchy of bi-Hamiltonian equations in \mathcal{V} :

$$\frac{du}{dt_k} = \{ \int h_k, u \}_H = \{ \int h_{k+N}, u \}_K, \quad k \geq 1.$$

□

The remainder of this section will be the proof of Theorem 3.1.

Lemma 3.2. Let \mathcal{V} be an arbitrary differential algebra endowed with a λ -bracket $\{\cdot, \cdot\}_\lambda$. Let $L(\partial) \in \mathcal{V}((\partial^{-1}))$ be a monic pseudodifferential operator of order $N > 0$. Then, for all $k \geq 1$, the following identity holds in $\mathcal{V}((w^{-1}))$:

$$\operatorname{Res}_z \{ L^{\frac{k}{N}}(z)_\lambda L(w) \} \Big|_{\lambda=0} = \frac{k}{N} \operatorname{Res}_z \{ L(z+x)_x L(w) \} \Big|_{x=\partial} L^{\frac{k}{N}-1}(z). \quad (3.2)$$

□

Proof. Since, by (1.5), $L^{\frac{k}{N}}(z) = L^{\frac{1}{N}}(z+\partial)L^{\frac{1}{N}}(z+\partial)\dots L^{\frac{1}{N}}(z)$ (k times), we have, by sesquilinearity and the right Leibniz rule,

$$\{ L^{\frac{k}{N}}(z)_\lambda L(w) \} = \sum_{l=1}^k \{ L^{\frac{1}{N}}(z+x)_{\lambda+x+y} L(w) \} \Big|_{x=\partial} L^{\frac{k-l}{N}}(z) \Big|_{y=\partial} (L^*)^{\frac{l-1}{N}}(-z+\lambda). \quad (3.3)$$

Taking the residue of both sides of equation (3.3) and using Lemma 1.2(b), we get

$$\begin{aligned} & \operatorname{Res}_z \{ L^{\frac{k}{N}}(z)_\lambda L(w) \} \\ &= \operatorname{Res}_z \sum_{l=1}^k \{ L^{\frac{1}{N}}(z+\lambda+x+y)_{\lambda+x+y} L(w) \} \Big|_{x=\partial} L^{\frac{k-l}{N}}(z+\lambda+y) \Big|_{y=\partial} L^{\frac{l-1}{N}}(z), \end{aligned}$$

and setting $\lambda = 0$ we get

$$\operatorname{Res}_z \{L^{\frac{k}{N}}(z)_\lambda L(w)\} \Big|_{\lambda=0} = k \operatorname{Res}_z \{L^{\frac{1}{N}}(z+x)_x L(w)\} \left(\Big|_{x=\partial} L^{\frac{k-1}{N}}(z) \right). \quad (3.4)$$

On the other hand, letting $k = N$ in (3.3), we have

$$\{L(z)_\lambda L(w)\} = \sum_{l=1}^N \{L^{\frac{1}{N}}(z+x)_{\lambda+x+y} L(w)\} \left(\Big|_{x=\partial} L^{\frac{N-l}{N}}(z) \right) \left(\Big|_{y=\partial} (L^*)^{\frac{l-1}{N}}(-z+\lambda) \right). \quad (3.5)$$

If we replace, in equation (3.5), z by $z + \partial$ and λ by $\lambda + \partial$ acting on $L^{\frac{k}{N}-1}(z)$, we get

$$\begin{aligned} & \{L(z+x)_{\lambda+x} L(w)\} \left(\Big|_{x=\partial} L^{\frac{k}{N}-1}(z) \right) \\ &= \sum_{l=1}^N \{L^{\frac{1}{N}}(z+x)_{\lambda+x+y} L(w)\} \left(\Big|_{x=\partial} L^{\frac{k-l}{N}}(z) \right) \left(\Big|_{y=\partial} (L^*)^{\frac{l-1}{N}}(-z+\lambda) \right). \end{aligned} \quad (3.6)$$

Taking residues of both sides of equation (3.6) and using Lemma 1.2(b), we get

$$\begin{aligned} & \operatorname{Res}_z \{L(z+x)_{\lambda+x} L(w)\} \left(\Big|_{x=\partial} L^{\frac{k}{N}-1}(z) \right) \\ &= \operatorname{Res}_z \sum_{l=1}^N \{L^{\frac{1}{N}}(z+\lambda+x+y)_{\lambda+x+y} L(w)\} \left(\Big|_{x=\partial} L^{\frac{k-l}{N}}(z+\lambda+y) \right) \left(\Big|_{y=\partial} L^{\frac{l-1}{N}}(z) \right). \end{aligned} \quad (3.7)$$

In the second equality we used Lemma 1.2(b). Setting $\lambda = 0$ in both sides of equation (3.7), we get

$$\operatorname{Res}_z \{L(z+x)_x L(w)\} \left(\Big|_{x=\partial} L^{\frac{k}{N}-1}(z) \right) = N \operatorname{Res}_z \{L^{\frac{1}{N}}(z+x)_x L(w)\} \left(\Big|_{x=\partial} L^{\frac{k-1}{N}}(z) \right). \quad (3.8)$$

Equation (3.2) follows from equations (3.4) and (3.8). ■

Lemma 3.3. For every $i \in I$ (resp. I_-) and $k \geq 1$, we have in $\mathcal{V} = \mathcal{V}_N^\infty$ (resp. \mathcal{V}_N),

$$\frac{\delta h_k}{\delta u_i} = \operatorname{Res}_z (z + \partial)^{-i-1} L^{\frac{k}{N}-1}(z). \quad (3.9)$$

□

Proof. Let $\{\cdot_\lambda \cdot\}$ be any λ -bracket on \mathcal{V} , and let $A_{ij}(\partial) = \{u_j \lambda u_i\}$, $i, j \in I$ or I_- , be the associated matrix differential operator. Taking the coefficient of w^{-j-1} in both sides of equation (3.2) we have, by the definition (3.1) of h_k ,

$$\begin{aligned} & \{h_{k\lambda} u_j\} \Big|_{\lambda=0} = \operatorname{Res}_z \{L(z+x)_x u_j\} \left(\Big|_{x=\partial} L^{\frac{k}{N}-1}(z) \right) \\ &= \sum_i \operatorname{Res}_z \{u_i x u_j\} (z+x)^{-i-1} \left(\Big|_{x=\partial} L^{\frac{k}{N}-1}(z) \right) \\ &= \sum_i A_{ji}(\partial) \operatorname{Res}_z (z + \partial)^{-i-1} L^{\frac{k}{N}-1}(z). \end{aligned} \quad (3.10)$$

On the other hand, by the Master Formula (1.6) and the definition (1.14) of the variational derivative, we have

$$\{h_{k\lambda}u_j\}|_{\lambda=0} = \sum_i A_{ji}(\partial) \frac{\delta h_k}{\delta u_i}. \quad (3.11)$$

Equation (3.9) follows from (3.10) and (3.11), since the matrix differential operator $A(\partial)$ is arbitrary. \blacksquare

Lemma 3.4. The local functionals $\int h_k \in \mathcal{V}/\partial\mathcal{V}$, $k \in \mathbb{Z}_+ \setminus N\mathbb{Z}_+$, are all linearly independent. \square

Proof. By definition, \mathcal{V} is a polynomial algebra in the infinitely many variables $u_i^{(n)}$, for $i \in I$ or I_- and $n \in \mathbb{Z}_+$. Let $\mathcal{V} \rightarrow \mathbb{F}[u_{-N}]$, $f \mapsto \bar{f}$, be the evaluation homomorphism at $u_i^{(n)} = 0$, where $(i, n) \neq (-N, 0)$. We have $\overline{L(z)} = z^N + u_{-N}z^{N-1}$, and

$$\overline{L^{\frac{k}{N}-1}(z)} = \sum_{h \in \mathbb{Z}_+} \binom{\frac{k}{N}-1}{h} (u_{-N})^h z^{k-N-h}.$$

Hence, by equation (3.9), we get

$$\frac{\delta h_k}{\delta u_{-N}} = \binom{\frac{k}{N}-1}{k} (u_{-N})^k,$$

for all $k \geq 1$. The claim follows. \blacksquare

Remark 3.5. In fact, for $\mathcal{V} = \mathcal{V}_N^\infty$, it is not difficult to prove, using equation (3.9), that $\frac{\delta h_k}{\delta u_i} = 0$ for all $i \geq k - N + 1$, and $\frac{\delta h_k}{\delta u_{k-N}} = 1$. Therefore the elements $\frac{\delta h_k}{\delta u} \in \mathcal{V}_N^\infty \oplus I$, $k \geq 1$, and so the elements $\int h_k \in \mathcal{V}_N^\infty/\partial\mathcal{V}_N^\infty$, are all linearly independent. On the other hand, for $\mathcal{V} = \mathcal{V}_N$, $L(\partial)$ is a differential operator, and so $L^{\frac{k}{N}}(z)$ has non negative powers of z for all $k \in N\mathbb{Z}_+$. Hence, $\int h_k = 0$ for all $k \in N\mathbb{Z}_+$. \square

Lemma 3.6. For the AGD bi-PVA $\mathcal{V} = \mathcal{V}_N^\infty$ or \mathcal{V}_N with bi-Poisson structure (H, K) , we have

$$(a) \quad \{h_{k\lambda}L(w)\}_H|_{\lambda=0} = L^{\frac{k}{N}}(w + \partial)_+L(w) - L(w + \partial)L^{\frac{k}{N}}(w)_+;$$

$$(b) \quad \{h_{k\lambda}L(w)\}_K|_{\lambda=0} = L^{\frac{k}{N}-1}(w + \partial)_+L(w) - L(w + \partial)L^{\frac{k}{N}-1}(w)_+.$$

\square

Proof. By Lemma 3.2 and equation (2.45) we have

$$\begin{aligned} \{h_{k\lambda}L(w)\}_H|_{\lambda=0} &= \text{Res}_z L^{\frac{k}{N}}(z) i_z (z - w - \partial)^{-1} L(w) \\ &\quad - L(w + \partial) \text{Res}_z L^{\frac{k}{N}-1}(z) i_z (z - w - \partial)^{-1} L^*(-z). \end{aligned} \quad (3.12)$$

By equation (1.3), we have

$$\text{Res}_z L^{\frac{k}{N}}(z) i_z (z - w - \partial)^{-1} = L^{\frac{k}{N}}(w + \partial)_+, \quad (3.13)$$

while, by Lemma 1.2(b) and equation (1.3), we have

$$\begin{aligned} \operatorname{Res}_z L^{\frac{k}{N}-1}(z) i_z(z-w-\partial)^{-1} L^*(-z) &= \operatorname{Res}_z L^{\frac{k}{N}-1}(z+\partial) i_z(z-w)^{-1} L(z) \\ &= \operatorname{Res}_z L^{\frac{k}{N}}(z) i_z(z-w)^{-1} = L^{\frac{k}{N}}(w)_+. \end{aligned} \quad (3.14)$$

Combining equations (3.12), (3.13) and (3.14), we get part (a). Similarly, for part (b), we use Lemma 3.2 and equation (2.46) to get

$$\begin{aligned} \{h_{k\lambda} L(w)\}_K|_{\lambda=0} &= \operatorname{Res}_z i_z(z-w)^{-1} (L(z+\partial) - L(w+\partial)) L^{\frac{k}{N}-1}(z) \\ &+ \operatorname{Res}_z L^{\frac{k}{N}-1}(z) i_z(z-w-\partial)^{-1} (L(w) - L^*(-z)). \end{aligned} \quad (3.15)$$

By equation (1.3) and Lemma 1.2(b) we have

$$\begin{aligned} \operatorname{Res}_z i_z(z-w)^{-1} L(z+\partial) L^{\frac{k}{N}-1}(z) &= L^{\frac{k}{N}}(w)_+ \\ &= \operatorname{Res}_z L^{\frac{k}{N}-1}(z) i_z(z-w-\partial)^{-1} L^*(-z). \end{aligned} \quad (3.16)$$

Moreover, by equation (1.3) we also have

$$\operatorname{Res}_z i_z(z-w)^{-1} L(w+\partial) L^{\frac{k}{N}-1}(z) = L(w+\partial) L^{\frac{k}{N}-1}(w)_+, \quad (3.17)$$

and

$$\operatorname{Res}_z L^{\frac{k}{N}-1}(z) i_z(z-w-\partial)^{-1} L(w) = L^{\frac{k}{N}-1}(w+\partial)_+ L(w). \quad (3.18)$$

Combining equations (3.15), (3.16), (3.17), and (3.18), we get the claim. \blacksquare

Lemma 3.7. For every $k \geq 1$, we have the Lenard-Magri recursion

$$\{h_{k\lambda} u\}_H|_{\lambda=0} = \{h_{k+N\lambda} u\}_K|_{\lambda=0}, \quad \text{for all } u \in \mathcal{V}. \quad (3.19)$$

\square

Proof. By Lemma 3.6, the recursion (3.19) holds for $u = L(w)$, the generating series of the generators of \mathcal{V} . Hence, (3.19) holds for all $u \in \mathcal{V}$ by the Leibniz rule. \blacksquare

Lemma 3.8. For every $\varepsilon \in \{1, \dots, N\}$, we have

$$\{h_{\varepsilon\lambda} u\}_K|_{\lambda=0} = 0, \quad \text{for all } u \in \mathcal{V}. \quad (3.20)$$

\square

Proof. For $1 \leq \varepsilon < N$, we have $L^{\frac{\varepsilon}{N}-1}(w)_+ = 0$, and therefore equation (3.20) holds by Lemma Lemma 3.6(b). Moreover, $\{h_{N\lambda}L(w)\}_K|_{\lambda=0} = L(w) - L(w + \partial) \cdot 1 = 0$. ■

Proof of Theorem 3.1. According to the Lenard-Magri scheme of integrability (see Section 1.5), by Lemmas 3.7 and 3.8 we have that $\int h_k$, $k \geq 1$, are integrals of motion in involution: $\{\int h_m, \int h_n\}_{H,K} = 0$ for all $m, n \geq 1$. By Lemma 3.4 they span an infinite dimensional space, as required. ■

Remark 3.9. It follows from Lemma 3.6 and equation (1.5) that the Hamiltonian equation corresponding to the Hamiltonian functional $\int h_k$, $k \geq 1$, can be written as (in terms of generating series)

$$\frac{dL(w)}{dt_k} = [(L^{\frac{k}{N}})_+, L](w), \quad (3.21)$$

where on the RHS we have to take the symbol of the usual commutator of pseudodifferential operators. This equation is the symbol of the usual *Lax pair* representation of the AGD hierarchies of Hamiltonian equations. □

3.2 Integrable hierarchies for the \mathcal{W} -algebra \mathcal{W}_N^∞ and \mathcal{W}_N .

As in the previous section, let $\mathcal{V} = \mathcal{V}_N^\infty$ or \mathcal{V}_N . Let also (H, K) be the AGD bi-Poisson structure on \mathcal{V} , and let H^D be the Dirac modification of H by the constraint $\theta = u_{-N}$. The corresponding λ -bracket is given, in terms of generating series, by equation (2.51). Recall by Theorem 1.19 that (H^D, K) is also a bi-Poisson structure on \mathcal{V} .

Lemma 3.10. For any $k \geq 1$, we have, in \mathcal{V} ,

$$\{h_{k\lambda}L(w)\}_H|_{\lambda=0} = \{h_{k\lambda}L(w)\}_{H^D}|_{\lambda=0}.$$

□

Proof. By Lemma 3.2 and equation (2.51) we have

$$\begin{aligned} & \{h_{k\lambda}L(w)\}_{H^D}|_{\lambda=0} - \{h_{k\lambda}L(w)\}_H|_{\lambda=0} \\ &= -\frac{1}{N} (L(w + \partial) - L(w)) \partial^{-1} \text{Res}_z (L^*(-z) - L(z + \partial)) L^{\frac{k}{N}-1}(z). \end{aligned}$$

This is zero since, by Lemma 1.2(b), we have

$$\text{Res}_z L^*(-z) L^{\frac{k}{N}-1}(z) = \text{Res}_z L^{\frac{k}{N}}(z) = \text{Res}_z L(z + \partial) L^{\frac{k}{N}-1}(z).$$

■

Recall from Section 2.8 the definition of the classical \mathcal{W} -algebras $\mathcal{W}_N^\infty = \mathbb{F}[u_i^{(n)} \mid i \in I \setminus \{-N\}, n \in \mathbb{Z}_+] = \mathcal{V}_N^\infty / \langle u_{-N} \rangle$ and $\mathcal{W}_N = \mathbb{F}[u_i^{(n)} \mid i \in I_- \setminus \{-N\}, n \in \mathbb{Z}_+] = \mathcal{V}_N / \langle u_{-N} \rangle$, obtained from the AGD bi-PVAs \mathcal{V}_N^∞ and \mathcal{V}_N respectively, via Dirac reduction. We shall denote $\mathcal{W} = \mathcal{W}_N^\infty$ or \mathcal{W}_N , with its bi-Poisson structure (H^D, K) .

With an abuse of notation, we denote $h_k \in \mathcal{W}$, for $k \geq 1$, the image of (3.1) in the quotient space $\mathcal{W} = \mathcal{V} / \langle u_{-N} \rangle$. By Lemmas 3.7, 3.8, and 3.10, we have the Lenard-Magri recursions ($u \in \mathcal{W}$):

$$\begin{aligned} \{h_k \lambda u\}_K \Big|_{\lambda=0} &= 0 \quad \text{for all } k = 1, \dots, N, \\ \{h_k \lambda u\}_{H^D} \Big|_{\lambda=0} &= \{h_{k+N} \lambda u\}_K \Big|_{\lambda=0} \quad \text{for all } k \geq 1. \end{aligned}$$

Furthermore, with the same argument as in the proof of Lemma 3.4, we get

$$\begin{aligned} \frac{\overline{\delta h_k}}{\delta u_{-N+1}} &= \left(\frac{\frac{k}{N} - 1}{\frac{k-1}{2}} \right) (u_{-N+1})^{\frac{k-1}{2}} \quad \text{if } k \text{ is odd, and } 0 \text{ otherwise,} \\ \frac{\overline{\delta h_k}}{\delta u_{-N+2}} &= \left(\frac{\frac{k}{N} - 1}{\frac{k}{2} - 1} \right) (u_{-N+1})^{\frac{k}{2}-1} \quad \text{if } k \text{ is even, and } 0 \text{ otherwise,} \end{aligned}$$

where this time $f \mapsto \bar{f}$ denotes the evaluation map $\mathcal{W} \rightarrow \mathbb{F}[u_{-N+1}]$ at $u_i^{(n)} = 0$ for $(i, n) \neq (-N+1, 0)$. It follows, in particular, that the local functionals $\int h_k \in \mathcal{W} / \partial \mathcal{W}$, for $k \in \mathbb{Z}_+ \setminus N\mathbb{Z}_+$ are linearly independent. In conclusion, according to the Lenard-Magri scheme of integrability, we get the following

Theorem 3.11. We have an integrable hierarchy of Hamiltonian equations in \mathcal{W} :

$$\frac{du}{dt_k} = \{ \int h_k, u \}_{H^D} = \{ \int h_{k+N}, u \}_K, \quad k \geq 1.$$

□

Example 3.12 (\mathcal{W}_1^∞ : the KP hierarchy). On $\mathcal{W}_1^\infty = \mathbb{F}[u_i^{(n)} \mid i, n \in \mathbb{Z}_+]$, we have $L(\partial) = \partial + \sum_{i \in \mathbb{Z}_+} u_i \partial^{-i-1}$. It is not difficult to compute the first few integrals of motion $\int h_k$, $k \geq 1$, directly from the definition (3.1):

$$\int h_1 = \int u_0, \quad \int h_2 = \int u_1, \quad \int h_3 = \int u_2 + u_0^2, \quad \int h_4 = \int u_3 + 3u_0 u_1, \dots$$

To find the corresponding bi-Hamiltonian equations, we use Lemma 3.6. We have $L(w)_+ = w$, $L^2(w)_+ = w^2 + 2u_0$, $L^3(w)_+ = w^3 + 3u_0 w + 3(u_1 + u_0')$. Hence,

$$\begin{aligned} \frac{dL(w)}{dt_1} &= \partial L(w), \quad \frac{dL(w)}{dt_2} = \partial^2 L(w) + 2w \partial L(w) + 2(L(w) - L(w + \partial))u_0, \\ \frac{dL(w)}{dt_3} &= \partial^3 L(w) + 3w \partial^2 L(w) + 3w^2 \partial L(w) + 3u_0 \partial L(w), \\ &\quad + 3(L(w) - L(w + \partial))((w + \partial)u_0 + u_1) \dots \end{aligned} \tag{3.22}$$

Remark 3.13. Consider the first two equations in the second system of the hierarchy (3.22), and the first equation in the third system of (3.22). After eliminating the variables u_1 and u_2 and relabeling $t_1 = y$, $t_2 = t$

and $u = 2u_0$, we get

$$3u_{yy} = (4u_t - u''' - 6uu')', \quad (3.23)$$

which is known as the Kadomtsev-Petviashvili (KP) equation. \square

\square

Remark 3.14. In fact, we have infinitely many bi-Poisson structures for the KP equation, corresponding to the biPVA's $\mathcal{W}_N^\infty = \mathbb{F}[u_i^{(n)} \mid i \geq -N + 1, n \in \mathbb{Z}_+]$, for $N \geq 1$, Radul [1987]. An explicit differential algebra isomorphism $\varphi_N : \mathcal{W}_1^\infty \rightarrow \mathcal{W}_N^\infty$ is defined by the equation

$$\varphi_N(L(z)) = L_N^{\frac{1}{N}}(z),$$

where $L(\partial)$ is as in Example 3.12, and $L_N(\partial) = \partial^N + \sum_{i \geq -N+1} u_i \partial^{-i-1}$. This is not a PVA isomorphism (since $L_N^{\frac{1}{N}}(\partial)$ is not of Adler type). On the other hand, one can check that the integrable hierarchy in \mathcal{W}_N^∞ , given by Theorem 3.11, is the same for every choice of the positive integer N . Namely, we have, for every $N \geq 1$,

$$\varphi_N \left(\frac{dL^N(w)}{dt_k} \right) = \frac{dL_N(w)}{dt_{N,k}} = [(L_N^{\frac{k}{N}})_+, L_N](w).$$

\square

Example 3.15 (\mathcal{W}_2 : the KdV hierarchy). Recall from Example 2.21 that $\mathcal{W}_2 = \mathbb{F}[u^{(n)} \mid n \in \mathbb{Z}_+]$, with the bi-PVA structure as in (2.55). The first few fractional powers of $L(\partial) = \partial^2 + u \in \mathcal{W}_2[\partial]$ are

$$\begin{aligned} L^{\frac{1}{2}}(\partial) &= \partial + \frac{1}{2}u\partial^{-1} - \frac{1}{4}u'\partial^{-2} + \frac{1}{8}(u'' - u^2)\partial^{-3} - \frac{1}{16}(u''' - 6uu')\partial^{-4} + \dots, \\ L^{\frac{3}{2}}(\partial) &= \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u' + \frac{1}{8}(3u^2 - u'')\partial^{-1} + \dots, \end{aligned}$$

from which we get $L^{\frac{1}{2}}(w)_+ = w$, $L^{\frac{3}{2}}(w)_+ = w^3 + \frac{3}{4}(2w + \partial)u$, and $\int h_1 = \int u$, $\int h_3 = \int \frac{1}{4}u^2$. By Lemma 3.6 we get the corresponding Hamiltonian equations (1.13): $\frac{du}{dt_1} = u'$ and the Korteweg-de Vries equation

$$\frac{du}{dt_3} = \frac{1}{4}(u''' + 6uu').$$

\square

Example 3.16 (\mathcal{W}_3 : the Boussinesq hierarchy). Recall from Example 2.22 that $\mathcal{W}_3 = \mathbb{F}[u^{(n)}, v^{(n)} \mid n \in \mathbb{Z}_+]$, with the bi-PVA structure as in (2.56). The first few fractional powers of $L(\partial) = \partial^3 + u\partial + v \in \mathcal{W}_3[\partial]$ are

$$\begin{aligned} L^{\frac{1}{3}}(\partial) &= \partial + \frac{1}{3}u\partial^{-1} - \frac{1}{3}(u' - v)\partial^{-2} + \frac{1}{9}(2u'' - 3v' - u^2)\partial^{-3} + \dots, \\ L^{\frac{2}{3}}(\partial) &= \partial^2 + \frac{2}{3}u\partial + \frac{1}{3}(2v - u')\partial^{-1} + \dots \end{aligned}$$

Hence, $L^{\frac{1}{3}}(w)_+ = w$, $L^{\frac{2}{3}}(w)_+ = w^2 + \frac{2}{3}u$, and $\int h_1 = \int u$, $\int h_2 = \int v$. The corresponding Hamiltonian equations are $\frac{du}{dt_1} = u'$, $\frac{dv}{dt_1} = v'$, and

$$\frac{du}{dt_2} = -u'' + 2v', \quad \frac{dv}{dt_2} = v'' - \frac{2}{3}u''' - \frac{2}{3}uu'.$$

After eliminating v from this system, we get the Boussinesq equation:

$$u_{tt} = -\frac{1}{3}(u^{(4)} - 4(uu')').$$

□

4 Generalization to the matrix case

4.1 Adler type matrix pseudodifferential operators

Let \mathcal{V} be a differential algebra, and let $\mathcal{M} = \text{Mat}_{m \times m} \mathcal{V}$. Let $L = (L_{ab}(\partial))_{a,b=1}^m \in \mathcal{M}((\partial^{-1}))$ be a matrix pseudodifferential operator of order $\text{ord}(L) = N \in \mathbb{Z}$. As in the scalar case, we can define the corresponding Adler map $A^{(L)} : \mathcal{M}((\partial^{-1})) \rightarrow \mathcal{M}((\partial^{-1}))$ given by (2.1), and with identifications analogous to (2.3) and (2.4), we get the corresponding map $H^{(L)} : \mathcal{M}^{\oplus I} \rightarrow \mathcal{M}^I$ (cf. (2.5)), where, as before, $I = \{-N, -N+1, \dots\}$. This map is represented by a tensor $H^{(L)} = (H_{ij;abcd}^{(L)}(\partial))_{i,j \in I; a,b,c,d \in \{1, \dots, m\}}$, where $H_{ij;abcd}^{(L)}(\partial) \in \mathcal{V}[\partial]$. As in Lemma 2.1, we can write an explicit formula for $H^{(L)}$, in terms of the generating series $H_{abcd}^{(L)}(\partial)(z, w) = \sum_{i,j \in I} H_{ij;abcd}^{(L)}(\partial) z^{-i-1} w^{-j-1}$. We have (cf. equation (2.7)):

$$\begin{aligned} H_{abcd}^{(L)}(\partial)(z, w) &= L_{ad}(w) i_w (w - z - \partial)^{-1} \circ L_{cb}(z) \\ &\quad - L_{ad}(z + \partial) i_w (w - z - \partial)^{-1} \circ L_{cb}^*(-w + \partial). \end{aligned} \tag{4.1}$$

Let \mathcal{V} be a differential algebra endowed with a λ -bracket $\{\cdot \lambda \cdot\}$. As in the scalar case, we say that a matrix pseudodifferential operator $L(\partial) \in \mathcal{M}((\partial^{-1}))$ is of *Adler type* (for the λ -bracket $\{\cdot \lambda \cdot\}$) if the following identity holds in $\mathcal{V}[\lambda]((z^{-1}, w^{-1}))$:

$$\{L_{ab}(z) \lambda L_{cd}(w)\} = H_{cdab}^{(L)}(\lambda)(w, z), \tag{4.2}$$

for all $a, b, c, d = 1, \dots, m$. The analogue of Lemma 2.2 still holds in the matrix case. As a consequence, we get (cf. Lemma 2.5):

Lemma 4.1. Let \mathcal{V} be a differential algebra, let $\{\cdot \lambda \cdot\}$ be a λ -bracket on \mathcal{V} , and let $L(\partial) \in \mathcal{M}((\partial^{-1}))$ be an Adler type matrix pseudodifferential operator. Then:

(a) The following identity holds in $\mathcal{V}[\lambda]((z^{-1}, w^{-1}))$:

$$\{L_{ab}(z) \lambda L_{cd}(w)\} = -\{L_{cd}(w)_{-\lambda-\partial} L_{ab}(z)\}, \tag{4.3}$$

for all $a, b, c, d = 1, \dots, m$.

(b) The following identity holds in $\mathcal{V}[\lambda, \mu]((z_1^{-1}, z_2^{-1}, z_3^{-1}))$:

$$\begin{aligned} & \{L_{ab}(z_1)_\lambda \{L_{cd}(z_2)_\mu L_{ef}(z_3)\}\} - \{L_{cd}(z_2)_\mu \{L_{ab}(z_1)_\lambda L_{ef}(z_3)\}\} \\ &= \{\{L_{ab}(z_1)_\lambda L_{cd}(z_2)\}_{\lambda+\mu} L_{ef}(z_3)\}. \end{aligned} \quad (4.4)$$

for all $a, b, c, d, e, f = 1, \dots, m$.

□

4.2 The generic matrix pseudodifferential operator of order N and the corresponding AGD bi-PVA

For $N, m \geq 1$, let $\mathcal{V}_{N,m}^\infty = \mathbb{F}[u_{i,ab}^{(n)} \mid i \in I, a, b = 1, \dots, m, n \in \mathbb{Z}_+]$, and $\mathcal{M}_{N,m}^\infty = \text{Mat}_{m \times m} \mathcal{V}_{N,m}^\infty$, and let $\mathcal{V}_{N,m} = \mathbb{F}[u_{i,ab}^{(n)} \mid i \in I_-, a, b = 1, \dots, m, n \in \mathbb{Z}_+]$, and $\mathcal{M}_{N,m} = \text{Mat}_{m \times m} \mathcal{V}_{N,m}$. The *generic* matrix pseudodifferential operator on $\mathcal{V}_{N,m}^\infty$ (resp. $\mathcal{V}_{N,m}$) is

$$\begin{aligned} L(\partial) &= \partial^N \mathbb{1}_m + U_{-N} \partial^{N-1} + U_{-N+1} \partial^{-N-2} + \dots \in \mathcal{M}_{N,m}^\infty((\partial^{-1})), \\ &\left(\text{resp. } L(\partial) = \partial^N \mathbb{1}_m + U_{-N} \partial^{N-1} + \dots + U_{-1} \in \mathcal{M}_{N,m}[\partial] \right). \end{aligned} \quad (4.5)$$

where $U_i = (u_{i,ab})_{a,b=1}^m \in \mathcal{M}_{N,m}^\infty$ for all $i \in I$ (resp. $U_i \in \mathcal{M}_{N,m}$ for all $i \in I_-$). There is a unique λ -bracket $\{\cdot, \cdot\}_c$ on $\mathcal{V}_{N,m}^\infty$ (resp. $\mathcal{V}_{N,m}$) such that $L(\partial) - c\mathbb{1}_m$ is of Adler type:

$$\{L_{ab}(z)_\lambda L_{cd}(w)\}_c = H_{cdab}^{(L-c\mathbb{1}_m)}(\lambda)(w, z), \quad (4.6)$$

for all $a, b, c, d = 1, \dots, m$. Letting, as in Section 2.4, $H^{(L-c\mathbb{1}_m)}(\partial) = H(\partial) - cK(\partial)$, we have, as a consequence of Lemma 4.1, two compatible Poisson structures K and H on $\mathcal{V}_{N,m}^\infty$ (resp. $\mathcal{V}_{N,m}$), which we call, respectively, the *first* and the *second matrix AGD Poisson structures* (cf. Definition 2.8). Using equation (4.1) we get the following explicit formulas for the λ -brackets associated to H and K ($a, b, c, d = 1, \dots, m$):

$$\begin{aligned} \{L_{ab}(z)_\lambda L_{cd}(w)\}_H &= L_{cb}(z) i_z (z - w - \lambda - \partial)^{-1} L_{ad}(w) \\ &\quad - L_{cb}(w + \lambda + \partial) i_z (z - w - \lambda - \partial)^{-1} L_{ad}^*(-z + \lambda) \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} \{L_{ab}(z)_\lambda L_{cd}(w)\}_K &= \delta_{ad} i_z (z - w - \lambda)^{-1} (L_{cb}(z) - L_{cb}(w + \lambda)) \\ &\quad + \delta_{cb} i_z (z - w - \lambda - \partial)^{-1} (L_{ad}(w) - L_{ad}^*(\lambda - z)). \end{aligned} \quad (4.8)$$

Expanding equations (2.45) and (2.46) in powers of z and w , we get ($i, j \in I$):

$$\begin{aligned} H_{ji;cdab}(\lambda) &= \sum_{k, \alpha \in \mathbb{Z}_+} \binom{k}{\alpha} u_{i-k-1,cb}(\lambda + \partial)^\alpha u_{j+k-\alpha,ad} \\ &- \sum_{k, \alpha, \beta \in \mathbb{Z}_+} (-1)^\alpha \binom{j}{\alpha} \binom{i-k-1}{\beta} u_{j+k-\alpha,cb}(\lambda + \partial)^{\alpha+\beta} u_{i-\beta-k-1,ad}, \\ K_{ji;cdab}(\lambda) &= \epsilon_{ij} \sum_{k \in \mathbb{Z}_+} \left(\binom{i}{k} \delta_{cb}(\lambda + \partial)^k u_{i+j-k,ad} - \binom{j}{k} \delta_{ad}(-\lambda)^k u_{i+j-k,cb} \right), \end{aligned} \quad (4.9)$$

where, as in (2.47), $\epsilon_{ij} = +1$ if $i, j \in \mathbb{Z}_+$, $\epsilon_{ij} = -1$ if $i, j < 0$, and $\epsilon_{ij} = 0$ otherwise.

Remark 4.2. As in Remark 2.9, the $\mathbb{F}[\partial]$ -submodule $R_+ \subset \mathcal{V}_{N,m}^\infty$ generated by $u_{i,ab}$, $i \in I$, $a, b \in \{1, \dots, m\}$ is closed with respect to the K - λ -bracket, and it is a Lie conformal algebra isomorphic to \mathfrak{gc}_N (see Kac [1996]). \square

Example 4.3. For $N = 1$, we have $\mathcal{V}_{1,m} = \mathbb{F}[u_{ab}^{(n)} \mid a, b = 1, \dots, m, n \in \mathbb{Z}_+]$ and $L(\partial) = \partial \mathbb{1}_m + U \in \mathcal{M}_{1,m}[\partial]$.

In this case we have

$$\{u_{ab\lambda}u_{cd}\}_H = \delta_{bc}u_{ad} - \delta_{da}u_{cb} - \delta_{ad}\delta_{cb}\lambda,$$

for any $a, b, c, d = 1, \dots, m$ and $K = 0$. This is the affine PVA $S(\mathbb{F}[\partial]\mathfrak{gl}_m)$ associated to the Lie algebra \mathfrak{gl}_m and its trace form. \square

4.3 The classical matrix \mathcal{W} -algebras $\mathcal{W}_{N,m}$

Lemma 4.4. In the AGD bi-PVA $\mathcal{V}_{N,m}^\infty$ (resp. $\mathcal{V}_{N,m}$) we have, for all $a, b, c, d = 1, \dots, m$:

- (a) $\{u_{-N,ab\lambda}L_{cd}(w)\}_H = \delta_{cb}L_{ad}(w) - \delta_{ad}L_{cb}(w + \lambda)$;
- (b) $\{L_{ab}(z)\lambda u_{-N,cd}\}_H = \delta_{cb}L_{ad}^*(-z + \lambda) - \delta_{ad}L_{cb}(z)$;
- (c) $\{u_{-N,ab\lambda}u_{-N,cd}\}_H = \delta_{cb}u_{-N,ad} - \delta_{ad}u_{-N,cb} - \delta_{ad}\delta_{cb}N\lambda$;
- (d) $\{u_{-N,ab\lambda}L_{cd}(w)\}_K = \{L_{ab}(z)\lambda u_{-N,cd}\}_K = 0$.

\square

Proof. Same as the proof of Lemma 2.17. \blacksquare

Let $C_{abcd}(\lambda) = \{u_{-N,cd\lambda}u_{-N,ab}\}_H$, $a, b, c, d \in \{1, \dots, m\}$. By Lemma 4.4(c) and Proposition 1.1(b), the corresponding matrix differential operator

$$C(\partial) = (C_{abcd}(\partial))_{abcd=1}^m \in \text{Mat}_{m^2 \times m^2} \mathcal{V}_{N,m}, \quad (4.10)$$

is invertible. Furthermore, by 4.4(d), the elements $u_{-N,ab}$ are central with respect to the Poisson structure K , for all $a, b = 1, \dots, m$. Therefore, by Theorems 1.16 and 1.19 we can perform the Dirac reduction to get a bi-Poisson

structure (H^D, K) on

$$\begin{aligned} \overline{\mathcal{V}}_{N,m}^\infty / \langle u_{-N,ab} \rangle_{a,b=1}^m &\simeq \mathbb{F} \left[u_{i,ab}^{(n)} \mid \substack{-N \neq i \in I, n \in \mathbb{Z}_+ \\ a, b \in \{1, \dots, m\}} \right] =: \mathcal{W}_{N,m}^\infty \\ \left(\text{resp. } \mathcal{V}_{N,m} / \langle u_{-N,ab} \rangle_{a,b=1}^m &\simeq \mathbb{F} \left[u_{i,ab}^{(n)} \mid \substack{-N \neq i \in I_-, n \in \mathbb{Z}_+ \\ a, b \in \{1, \dots, m\}} \right] =: \mathcal{W}_{N,m} \right). \end{aligned}$$

We can compute the non-local λ -brackets corresponding to H^D using Lemma 4.4. In terms of the generating series, we have $(a, b, c, d = 1, \dots, m)$

$$\begin{aligned} \{L_{ab}(z)_\lambda L_{cd}(w)\}_{H^D} &= L_{cb}(z) i_z (z - w - \lambda - \partial)^{-1} L_{ad}(w) \\ &\quad - L_{cb}(w + \lambda + \partial) i_z (z - w - \lambda - \partial)^{-1} L_{ad}^*(-z + \lambda) \\ &\quad - \frac{1}{N} L_{cb}(w + \lambda + \partial) (\lambda + \partial)^{-1} L_{ad}^*(-z + \lambda) - \frac{1}{N} L_{ad}(w) (\lambda + \partial)^{-1} L_{cb}(z) \\ &\quad + \frac{1}{N} \sum_{k=1}^m \delta_{ad} L_{ck}(w + \lambda + \partial) (\lambda + \partial)^{-1} L_{kb}(z) \\ &\quad + \frac{1}{N} \sum_{k=1}^m \delta_{cb} L_{kd}(w) (\lambda + \partial)^{-1} L_{ak}^*(-z + \lambda). \end{aligned} \tag{4.11}$$

It is not hard to show that $T = \text{Tr Res}_z L(z) z^{-N+1}$ is a Virasoro element with central charge $\frac{m(N^3-N)}{12}$ (cf. Definition 1.5). Moreover, we have, for all $a, b = 1, \dots, m$

$$\{T \lambda u_{j,ab}(w)\}_{H^D} = (\partial + (N + j + 1)\lambda) u_{j,ab} + O(\lambda^2),$$

i.e. $u_{j,ab}$ is a T -eigenvector of conformal weight $N + j + 1$. It was proved by Bilal [1995] that $\mathcal{W}_{N,m}$ has a differential basis given by T and primary elements, i.e. it is a non-local PVA of CFT type.

Example 4.5. For $N = 2$ we have $\mathcal{W}_{2,m} = \mathbb{F}[u_{ab}^{(n)} \mid a, b \in \{1, \dots, m\}, n \in \mathbb{Z}_+]$. The formula of the λ -brackets (4.11) and (4.8) is $(a, b, c, d = 1, \dots, m)$:

$$\begin{aligned} \{u_{ab} \lambda u_{cd}\}_{H^D} &= \delta_{ad} \delta_{cb} \frac{\lambda^3}{2} + \delta_{cb} \left(\lambda + \frac{\partial}{2} \right) u_{ad} + \delta_{ad} \left(\lambda + \frac{\partial}{2} \right) u_{cb} - \frac{1}{2} u_{ad} (\lambda + \partial)^{-1} u_{cb} \\ &\quad - \frac{1}{2} u_{cb} (\lambda + \partial)^{-1} u_{ad} + \frac{1}{2} \sum_{k=1}^m \left(\delta_{ad} u_{ck} (\lambda + \partial)^{-1} u_{kb} + \delta_{cb} u_{kd} (\lambda + \partial)^{-1} u_{ak} \right), \\ \{u_{ab} \lambda u_{cd}\}_K &= 2 \delta_{ad} \delta_{bc} \lambda. \end{aligned} \tag{4.12}$$

This is the same bi-Poisson structure considered by Olver and Sokolov [1998] when studying the non-commutative KdV equation. For $m = 2$, let $T = u_{11} + u_{22}$, $v = u_{11} - u_{22}$, $v_+ = u_{12}$ and $v_- = u_{21}$. Then (cf.

Bilal [1995])

$$\begin{aligned}
\{T_\lambda T\}_c &= (2\lambda + \partial)T + \lambda^3 - 4c\lambda, \quad \{T_\lambda v\}_c = (2\lambda + \partial)v, \quad \{T_\lambda v_\pm\}_c = (2\lambda + \partial)v_\pm, \\
\{v_\lambda v\}_c &= 2v_+(\lambda + \partial)^{-1}v_- + 2v_-(\lambda + \partial)^{-1}v_+ + (2\lambda + \partial)T + \lambda^3 - 4c\lambda, \\
\{v_\lambda v_\pm\}_c &= -v(\lambda + \partial)^{-1}v_\pm, \quad \{v_\pm \lambda v_\pm\}_c = -v_\pm(\lambda + \partial)^{-1}v_\pm, \\
\{v_+ \lambda v_-\}_c &= \frac{1}{2}v(\lambda + \partial)^{-1}v + v_-(\lambda + \partial)^{-1}v_+ + \frac{1}{2}(2\lambda + \partial)T + \frac{1}{2}\lambda^3 - 2c\lambda.
\end{aligned}$$

Note that T is a Virasoro element with central charge 1 and v, v_\pm are primary elements of conformal weight 2. Hence $\mathcal{W}_{2,2}$ is a non-local PVA of CFT type. \square

Remark 4.6. The following matrix analogue of the Kupershmidt-Wilson Theorem 2.23 holds. Let $R_{N,m}$ be the tensor product of N -copies of the affine Poisson vertex algebra for \mathfrak{gl}_m : $R_{N,m} = \mathbb{F}[v_{i,ab}^{(n)} \mid i \in \{1, \dots, N\}, a, b \in \{1, \dots, m\}, n \in \mathbb{Z}_+]$, with the λ -bracket $(i, j \in \{1, \dots, N\}, a, b, c, d \in \{1, \dots, m\})$

$$\{v_{i,ab} v_{j,cd}\} = \delta_{ij}(\delta_{bc}v_{ad} - \delta_{ad}v_{cb} + \delta_{ad}\delta_{cb}\lambda).$$

We have a PVA homomorphism $\mu: \mathcal{V}_{N,m} \rightarrow R_{N,m}$, given by

$$\mu(L(\partial)) = (\partial + V_N)(\partial + V_{N-1}) \cdots (\partial + V_1) \in \text{Mat}_{m \times m} R_{N,m}[\partial], \quad (4.13)$$

where $L(\partial) \in \mathcal{M}_{N,m}[\partial]$ is as in (4.5), and $V_i = (v_{i,ab})_{a,b=1}^m$. Moreover, we can consider the Dirac reduction of $R_{N,m}$ by the constraints $\theta_{ab} = \sum_{k=1}^N v_{k,ab}$, for $a, b \in \{1, \dots, m\}$. Then the map (4.13) induces a PVA homomorphism on the Dirac reductions, called the *matrix Miura map*:

$$\mu: \mathcal{W}_{N,m} \rightarrow R_{N,m}/\langle \theta_{ab} \mid a, b = 1, \dots, m \rangle.$$

\square

4.4 Gelfand-Dickey Integrable hierarchies in the matrix case

Throughout this section we let $\mathcal{V} = \mathcal{V}_{N,m}^\infty$ or $\mathcal{V}_{N,m}$, endowed with its AGD bi-Poisson structure (H, K) , and we let $\mathcal{M} = \text{Mat}_{m \times m} \mathcal{V}$. Let $L(\partial)$ as in (4.5). In analogy with Theorem 3.1, we shall prove that the sequence of local functionals $\{\int h_k\}_{k \geq 1}$, where

$$h_k = \frac{N}{k} \text{Tr Res}_z L^{\frac{k}{N}}(z) \in \mathcal{V},$$

satisfies the Lenard-Magri recursive condition (1.15), and it spans an infinite dimensional subspace of $\mathcal{V}/\partial\mathcal{V}$. Hence, we get the corresponding integrable hierarchy of bi-Hamiltonian equations

$$\frac{du_{i,ab}}{dt_k} = \{\int h_k, u_{i,ab}\}_H = \{\int h_{k+N}, u_{i,ab}\}_K.$$

This is a consequence of the following results, which are proved in the same way as in Section 3.1, for the scalar case.

Lemma 4.7. Let \mathcal{V} be a differential algebra endowed with a λ -bracket $\{\cdot, \lambda \cdot\}$, and let $\mathcal{M} = \text{Mat}_{m \times m} \mathcal{V}$. Let $L(\partial) \in \mathcal{M}((\partial^{-1}))$ be a monic matrix pseudodifferential operator of order $N > 0$. Then, for all $k \geq 1$ and $a, b \in \{1, \dots, m\}$, the following identity holds in $\mathcal{V}((w^{-1}))$:

$$\begin{aligned} & \text{Res}_z \{ \text{Tr } L^{\frac{k}{N}}(z)_{\lambda} L_{ab}(w) \} \Big|_{\lambda=0} \\ &= \frac{k}{N} \sum_{c,d=1}^m \text{Res}_z \{ L_{cd}(z+x)_x L_{ab}(w) \} \Big|_{x=\partial} (L^{\frac{k}{N}-1})_{dc}(z). \end{aligned} \quad (4.14)$$

□

Lemma 4.8. For every $i \in I$ (resp. I_-), $a, b = 1, \dots, m$, and $k \geq 1$, we have, in $\mathcal{V} = \mathcal{V}_{N,m}^{\infty}$ (resp. $\mathcal{V}_{N,m}$),

$$\frac{\delta h_k}{\delta u_{i,ab}} = \text{Res}_z (z + \partial)^{-i-1} (L^{\frac{k}{N}-1})_{ba}(z). \quad (4.15)$$

□

Lemma 4.9. The local functionals $\int h_k \in \mathcal{V}/\partial\mathcal{V}$, $k \in \mathbb{Z}_+ \setminus N\mathbb{Z}_+$, are all linearly independent. □

Lemma 4.10. For the AGD bi-PVA $\mathcal{V} = \mathcal{V}_{N,m}^{\infty}$ or $\mathcal{V}_{N,m}$, with bi-Poisson structure (H, K) , we have ($a, b = 1, \dots, m$, $k \geq 1$)

$$(a) \quad \{h_{k\lambda} L_{ab}(w)\}_H \Big|_{\lambda=0} = \sum_{c=1}^m \left((L^{\frac{k}{N}})_{ac}(w+\partial)_+ L_{cb}(w) - L_{ac}(w+\partial) (L^{\frac{k}{N}})_{cb}(w)_+ \right);$$

$$(b) \quad \{h_{k\lambda} L_{ab}(w)\}_K \Big|_{\lambda=0} = \sum_{c=1}^m \left((L^{\frac{k}{N}-1})_{ac}(w+\partial)_+ L_{cb}(w) - L_{ac}(w+\partial) (L^{\frac{k}{N}-1})_{cb}(w)_+ \right).$$

□

Lemma 4.11. For every $k \geq 1$, we have the Lenard-Magri recursion

$$\{h_{k\lambda} u\}_H \Big|_{\lambda=0} = \{h_{k+N\lambda} u\}_K \Big|_{\lambda=0}, \quad \text{for all } u \in \mathcal{V}. \quad (4.16)$$

□

Lemma 4.12. For every $\varepsilon \in \{1, \dots, N\}$, we have

$$\{h_{\varepsilon\lambda} u\}_K \Big|_{\lambda=0} = 0, \quad \text{for all } u \in \mathcal{V}. \quad (4.17)$$

□

Remark 4.13. As already noted in Remark 3.9 in the scalar case, it follows from Lemma 4.10 and (1.5) that the Hamiltonian equation corresponding to the Hamiltonian functional $\int h_k$, $k \geq 1$, can be written as

$$\frac{dL(w)}{dt_k} = [(L^{\frac{k}{N}})_+, L](w), \quad (4.18)$$

in the space $\mathcal{M}((w^{-1}))$. □

4.5 Integrable hierarchies for the reduction to the case $U_{-N} = 0$

Let, as before, $\mathcal{V} = \mathcal{V}_{N,m}$ and let \mathcal{K} be the (differential) field of fractions of \mathcal{V} . We shall identify the space $\text{Mat}_{m \times m} \mathcal{V}$ of $m \times m$ matrices with coefficients in \mathcal{V} with the space \mathcal{V}^{m^2} in the obvious way. Let, as in (4.5),

$$L(\partial) = \partial^N \mathbb{1}_m + U_{-N} \partial^{N-1} + \dots + U_{-1} \in \text{Mat}_{m \times m} \mathcal{V}[\partial], \quad (4.19)$$

where $U_i = (u_{i,ab})_{a,b=1}^m$ and the $u_{i,ab}$'s are the generators of the algebra of differential polynomials \mathcal{V} .

Recall from Section 4.2 that we have a bi-Poisson structure $H, K \in \text{Mat}_{Nm^2 \times Nm^2} \mathcal{V}[\partial]$ on \mathcal{V} , given by (4.9) with indices running in I_- and $u_{i,ab} = 0$ for $i \geq 0$. Consider the Dirac modification H^D of H by the constraints $u_{-N,ab}$, $a, b = 1, \dots, m$, given by (4.11). Note that H^D is a non-local Poisson structure, hence the analogue of the Lenard-Magri recursion condition (4.16) has to be expressed in terms of association relations:

$$\frac{\delta h_k}{\delta u} \xleftrightarrow{H^D} P_k \xleftrightarrow{K} \frac{\delta h_{k+N}}{\delta u}, \quad (4.20)$$

for some $P_k \in \mathcal{V}^{Nm^2}$. Recall from De Sole and Kac [2013-a] that, for $X \in \text{Mat}_{Nm^2 \times Nm^2} \mathcal{V}((\partial^{-1}))$ and $F, P \in \mathcal{V}^{Nm^2}$, the association relation $F \xleftrightarrow{X} P$ means that X admits a fractional decomposition of the form $X = AB^{-1}$, where A and B are matrix differential operators with coefficients in \mathcal{V} and B is nondegenerate, and there exists $F_1 \in \mathcal{K}^{Nm^2}$ such that $F = BF_1$ and $P = AF_1$. In this section we shall prove the following (cf. [De Sole et al., 2013-c, Ansatz 7.3]):

Theorem 4.14. The Lenard-Magri recursive conditions (4.20) hold. As a consequence, we get the induced integrable hierarchy of bi-Hamiltonian equations on $\mathcal{W}_{N,m}$: $\frac{du_{i,ab}}{dt_k} = (P_k)_{i,ab}$, $i \in \{-N+1, \dots, -1\}$, $a, b \in \{1, \dots, m\}$, $k \geq 1$. □

In order to prove Theorem 4.14 we apply the theory of singular degree of rational matrix pseudodifferential operators, developed in Carpentier et al. [2013-b]. Recall that we can write (see De Sole et al. [2013-c])

$$H^D = H + BC^{-1}B^*, \quad (4.21)$$

where $B \in \text{Mat}_{Nm^2 \times m^2} \mathcal{V}[\partial]$ has entries $B_{i,ab;cd}(\partial) = \text{Res}_w \{u_{-N,cd\partial} L_{ab}(w)\}_H w^i$, and $C \in \text{Mat}_{m^2 \times m^2} \mathcal{V}[\partial]$ has entries $C_{ab;cd}(\partial) = \{u_{-N,cd\partial} u_{-N,ab}\}_H$. Let $B(\lambda; \partial)_{ab,cd} = \sum_{i=-N}^{-1} B_{i,ab;cd}(\partial) \lambda^{-i-1}$. By Lemma 4.4(a) and (c),

we get explicit formulas for the matrices $B(\lambda; \partial)$ and $C(\partial)$, considered as differential operators on \mathcal{V}^{m^2} :

$$B(\lambda; \partial)F = [F^t, L](\lambda), \quad C(\partial)F = [F^t, U_{-N}] - N\partial F^t, \quad (4.22)$$

where $[F^t, L](\lambda)$ is the symbol of the differential operator $[L(\partial), X] = F^t L(\partial) - L(\partial) \circ F^t$.

The proof of Theorem 4.14 will be based on the following 7 Lemmas.

Lemma 4.15. Let $N > 1$. The only solutions $X \in \text{Mat}_{m \times m} \tilde{\mathcal{K}}$, where $\tilde{\mathcal{K}}$ is any differential field extension of \mathcal{K} , of the system of differential equations $[L(\partial), X] = 0$ are constant scalar multiples of the identity matrix. \square

Proof. Equating to zero the coefficients of ∂^{N-1} and ∂^{N-2} of $[L(\partial), X]$, we get the following system of differential equations:

$$\begin{aligned} X' &= \frac{1}{N}[X, U_{-N}], \\ \frac{N(N-1)}{2}X'' + (N-1)U_{-N}X' + [U_{-N+1}, X] &= 0. \end{aligned} \quad (4.23)$$

Using the first equation, the second equation of (4.23) becomes

$$\left[X, \frac{N-1}{2}U'_{-N} + \frac{N-1}{2N}U_{-N}^2 - U_{-N+1} \right] = 0. \quad (4.24)$$

In some differential field extension $\tilde{\mathcal{K}}$ of \mathcal{K} , there exists a non-degenerate matrix $E \in \text{Mat}_{m \times m} \tilde{\mathcal{K}}$ such that

$$E' = \frac{1}{N}EU_{-N}. \quad (4.25)$$

(The rows of E form a basis of the space of solutions of the linear system $Y' = \frac{1}{N}YU_{-N}$, for $Y \in \tilde{\mathcal{K}}^m$.) It is immediate to see that, in this case,

$$(E^{-1})' = -\frac{1}{N}U_{-N}E^{-1}. \quad (4.26)$$

Then, all solutions $X \in \text{Mat}_{m \times m} \tilde{\mathcal{K}}$ of the first equation in (4.23) are of the form

$$X = E^{-1}CE, \quad (4.27)$$

where C is an arbitrary matrix with constant entries: $\partial C = 0$. (It is immediate to check, using (4.25) and (4.26), that all the matrices X as in (4.27) solve the first equation in (4.23). On the other hand, the space of solutions of the first equation in (4.23) is a vector space over the field of constants of dimension less than or equal to m^2 , De Sole and Kac [2013-b].) Let then $V_{-N} = EU_{-N}E^{-1}$, and $V_{-N+1} = EU_{-N+1}E^{-1}$. We immediately have by (4.25) and (4.26) that $(V_{-N})' = EV'_{-N}E^{-1}$. Hence, after conjugating by E , equation (4.24) becomes

$$\left[C, \frac{N-1}{2}V'_{-N} + \frac{N-1}{2N}V_{-N}^2 - V_{-N+1} \right] = 0. \quad (4.28)$$

Let $\tilde{\mathcal{C}} \subset \tilde{\mathcal{K}}$ be the subfield of constants in $\tilde{\mathcal{K}}$. Consider the map $\tilde{\mathcal{C}}[V_{-N,ab}, V'_{-N,ab}, V_{-N+1,ab} \mid a, b \in \{1, \dots, m\}] \rightarrow \tilde{\mathcal{K}}$ given by conjugation by E . It is obviously injective. Hence, we can view (4.28) as a system of equations in the polynomial algebra $\tilde{\mathcal{C}}[V_{-N,ab}, V'_{-N,ab}, V_{-N+1,ab} \mid a, b \in \{1, \dots, m\}]$. The coefficient of $V_{-N+1,ab}$ in the LHS of (4.28) is $[E_{ab}, C]$. Therefore, equation (4.28) implies that $C = c\mathbb{1}$, for some $c \in \tilde{\mathcal{C}}$. The claim follows. ■

Lemma 4.16. The kernel of the operator $B(\lambda; \partial)$ on $\tilde{\mathcal{K}}^{m^2}$ consists of constant scalar multiples of the identity matrix, and it is contained in the kernel of $C(\partial)$. □

Proof. The first claim is obvious by the definition (4.29) of $B(\lambda; \partial)$ and Lemma 4.15. For the second claim we just need to observe that $C(\partial)$ is the coefficient of λ^{N-1} in $B(\lambda; \partial)$. ■

Lemma 4.17. Let $Q(\partial) : \mathcal{V}^{m^2} \rightarrow \mathcal{V}^{m^2}$ be the following differential operator

$$Q(\partial)F = F - F_{mm}\mathbb{1} + \partial F_{mm}E_{mm}. \quad (4.29)$$

Then $B(\lambda; \partial)$ and $C(\partial)$ are both divisible on the right by $Q(\partial)$, i.e. $B(\lambda; \partial) = B_1(\lambda; \partial) \circ Q(\partial)$, $C(\partial) = C_1(\partial) \circ Q(\partial)$. Moreover, the kernels of $B_1(\lambda; \partial)$ and $C_1(\partial)$ have zero intersection in $\tilde{\mathcal{K}}^{m^2}$ for any differential field extension $\tilde{\mathcal{K}}$ of \mathcal{K} . □

Proof. The operator $Q(\partial)$ is a non-degenerate matrix differential operator of degree 1, and its kernel is $\mathbb{F}\mathbb{1}$. The claim follows by [Carpentier et al., 2013-a, Thm.4.4]. ■

Lemma 4.18. We have $C_1^*(\partial)\mathbb{1} \neq 0$, where $C_1(\partial) : \mathcal{V}^{m^2} \rightarrow \mathcal{V}^{m^2}$ is as in Lemma 4.17, and $C_1^*(\partial)$ is its formal adjoint. □

Proof. It is straightforward to check that $C(\partial) = C_1(\partial) \circ Q(\partial)$, where $C_1(\partial)$ is the following matrix differential operator

$$C_1(\partial)F = [F^t, U_{-N}] - N\partial F^t - F_{mm}(N\mathbb{1} + [E_{mm}, U_{-N}]) + NF'_{mm}E_{mm}.$$

Its formal adjoint is

$$C_1^*(\partial)F = [U_{-N}, F^t] + N\partial F^t - \text{Tr}(F^t(N\mathbb{1} + [E_{mm}, U_{-N}]))E_{mm} - NF'_{mm}E_{mm}.$$

Hence, $C_1^*(\partial)\mathbb{1} = -NmE_{mm} \neq 0$. ■

Lemma 4.19. The kernels of the operators $B(\lambda, \partial)$ and $C_1^*(\partial)$ have zero intersection in $\tilde{\mathcal{K}}^{m^2}$, for any differential field extension $\tilde{\mathcal{K}}$ of \mathcal{K} . □

Proof. It follows from Lemmas 4.16 and 4.18. ■

Lemma 4.20. $H^D = H + B_1C_1^{-1}B^*$ is a minimal rational expression for H^D . □

Proof. It follows from [Carpentier et al., 2013-b, Cor.4.11]. Indeed, the space \mathcal{E} defined in [Carpentier et al., 2013-b, Eq.(4.30)] is, in this case, $\mathcal{E} = \ker(B_1) \cap \ker(C_1) \subset \tilde{\mathcal{K}}^{m^2}$, which is zero by Lemma 4.17, and the space \mathcal{E}^* defined in [Carpentier et al., 2013-b, Eq.(4.31)] is, in this case, $\mathcal{E}^* = \ker(B) \cap \ker(C_1^*) \subset \tilde{\mathcal{K}}^{m^2}$, which is zero by Lemma 4.19. ■

Lemma 4.21. $B^*(\partial) \frac{\delta h_k}{\delta u} = 0$ for every $k \geq 1$. □

Proof. The symbols of the matrix elements of B^* are $(i \in \{-N, \dots, -1\}, a, b, c, d \in \{1, \dots, m\})$

$$B_{ab;i,cd}^*(\lambda) = \text{Res}_z (\delta_{ad} L_{cb}^*(-z + \lambda) - \delta_{cb} L_{ad}(z)) w^i.$$

Using Lemma 4.8 we have

$$\begin{aligned} (B^*(\partial) \frac{\delta h_k}{\delta u_i})_{ab} &= \sum_{c=1}^m \text{Res}_z \text{Res}_w L_{cb}^*(-z + \partial) \delta(z - w - \partial) (L^{\frac{k}{N}-1})_{ac}(w) \\ &\quad - \sum_{d=1}^m \text{Res}_z \text{Res}_w L_{ad}(z) \delta(z - w - \partial) (L^{\frac{k}{N}-1})_{db}(w). \end{aligned}$$

Using equation (1.1) and Lemma 1.2(b) we can rewrite

$$\begin{aligned} &\sum_{c=1}^m \text{Res}_z \text{Res}_w L_{cb}^*(-z + \partial) \delta(z - w - \partial) (L^{\frac{k}{N}-1})_{ac}(w) \\ &= \sum_{c=1}^m \text{Res}_w L_{cb}^*(-w) (L^{\frac{k}{N}-1})_{ac}(w) = \text{Res}_w (L^{\frac{k}{N}})_{ab}(w) \end{aligned}$$

and

$$\begin{aligned} &\sum_{d=1}^m \text{Res}_z \text{Res}_w L_{ad}(z) \delta(z - w - \partial) (L^{\frac{k}{N}-1})_{db}(w) \\ &= \sum_{d=1}^m \text{Res}_w L_{ad}(w + \partial) (L^{\frac{k}{N}-1})_{db}(w) = \text{Res}_w (L^{\frac{k}{N}})_{ab}(w). \end{aligned}$$

This shows that $(B^*(\partial) \frac{\delta h_k}{\delta u_i})_{ab} = 0$ for all $a, b = 1, \dots, m$ and $k \geq 1$. Note that, in fact, this Lemma follows from a general result on Dirac reduction, see [De Sole et al., 2013-c, Lem.5.2(b)]. ■

Proof of Theorem 4.14. Condition $\frac{\delta h_k}{\delta u} \xrightarrow{K} P_k$ holds by assumption. By Lemma 4.20 and [Carpentier et al., 2013-b, Theorem 4.12], condition $\frac{\delta h_k}{\delta u} \xrightarrow{H^D} P_k$ is equivalent to $B^*(\partial) \frac{\delta h_k}{\delta u} = 0$, which holds by Lemma 4.21. As in the scalar case, it is not difficult to show that the elements $\frac{\delta h_k}{\delta u}$, $k \in \mathbb{Z}_+$, are linearly independent. The claim follows. ■

Example 4.22 ($\mathcal{W}_{2,m}$: the Korteweg-de Vries hierarchy). Recall from Example 4.5 that $\mathcal{W}_{2,m} = \mathbb{F}[u_{ab}^{(n)} \mid a, b \in \{1, \dots, m\}, n \in \mathbb{Z}_+]$, with the bi-PVA structure as in (4.12). The first few fractional powers of $L(\partial) = \partial^2 + U$,

where $U = (u_{ab})_{a,b=1}^m$, are

$$\begin{aligned} L^{\frac{1}{2}}(\partial) &= \partial + \frac{1}{2}U\partial^{-1} - \frac{1}{4}U'\partial^{-2} + \frac{1}{8}(U'' - U^2)\partial^{-3} + \dots, \\ L^{\frac{3}{2}}(\partial) &= \partial^3 + \frac{3}{2}U\partial + \frac{3}{4}U' + \frac{1}{8}(3U^2 - U'')\partial^{-1} + \dots, \end{aligned}$$

from which we get $L^{\frac{1}{2}}(w)_+ = w$, $L^{\frac{3}{2}}(w)_+ = w^3 + \frac{3}{4}(2w + \partial)U$, and $\int h_1 = \int \text{Tr } U$, $\int h_3 = \int \frac{1}{4} \text{Tr } U^2$. The corresponding Hamiltonian equations (1.13) are $\frac{dU}{dt_1} = U'$ and the *matrix Korteweg-de Vries equation*

$$\frac{dU}{dt_3} = \frac{1}{4}(U''' + 3UU' + 3U'U). \quad (4.30)$$

Thus we proved, in particular, that this equation is integrable, as was conjectured by [Olver and Sokolov \[1998\]](#). For $m = 2$ (see [Example 4.5](#) for notation), the Hamiltonian functionals, expressed in terms of the matrix elements of U , become $\int h_0 = \int T$ and $\int h_1 = \frac{1}{2} \int (T^2 + 2v_+v_- + v^2)$, and the matrix equation (4.30) becomes the system

$$\begin{cases} \frac{dT}{dt_3} = \frac{1}{4}(T''' + 3(T^2 + v_+v_- + v^2)'), \\ \frac{dv}{dt_3} = \frac{1}{4}(v''' + 6(vT)'), \\ \frac{dv_{\pm}}{dt_3} = \frac{1}{4}(v'_{\pm} + 6(v_{\pm}T)'). \end{cases}$$

□

Remark 4.23. The results of [Carpentier et al. \[2013-b\]](#) only apply to the case of matrices of finite size. Hence, the proof of [Theorem 4.14](#) does not work for the bi-Poisson structure (H^D, K) on $\mathcal{W}_{N,m}^{\infty}$. However, we expect that the claim of [Theorem 4.14](#) holds also in this case. □

Example 4.24 ($\mathcal{W}_{1,m}^{\infty}$: the matrix KP hierarchy). On $\mathcal{W}_{1,m}^{\infty} = \mathbb{F}[u_{i,ab}^{(n)} \mid i, n \in \mathbb{Z}_+, a, b \in \{1, \dots, m\}]$, we have $L(\partial) = \partial + \sum_{i \in \mathbb{Z}_+} U_i \partial^{-i-1}$, where $U_i = (u_{i,ab})_{a,b=1}^m$. The first few integrals of motion $\int h_k$, $k \geq 1$, are

$$\begin{aligned} \int h_1 &= \int \text{Tr } U_0 = \int \sum_{a=1}^m u_{aa;0}, & \int h_2 &= \int \text{Tr } U_1 = \int \sum_{a=1}^m u_{aa;1}, \\ \int h_3 &= \int \text{Tr}(U_2 + U_0^2) = \int \sum_{a=1}^m u_{aa;2} + \sum_{a,b=1}^m u_{ab;0}u_{ba;0}, \dots \end{aligned}$$

To find the corresponding bi-Hamiltonian equations, we use [Lemma 4.10](#). We have $L(w)_+ = w$, $L^2(w)_+ = w^2 + 2U_0$ and $L^3(w)_+ = w^3 + 3U_0w + 3(U_1 + U_0')$. Hence,

$$\begin{aligned} \frac{dL(w)}{dt_1} &= \partial L(w), & \frac{dL(w)}{dt_2} &= \partial^2 L(w) + 2w\partial L(w) + 2U_0L(w) - 2L(w + \partial)U_0, \\ \frac{dL(w)}{dt_3} &= \partial^3 L(w) + 3w\partial^2 L(w) + 3w^2\partial L(w) + 3U_0\partial L(w) \\ &+ 3w(U_0L(w) - L(w + \partial)U_0) + 3((U_1 + U_0')L(w) - L(w + \partial)(U_1 + U_0')). \end{aligned} \quad (4.31)$$

As done in Remark 3.13, we can eliminate the variable U_2 from the first two equations in the second system of the hierarchy (4.31), and the first equation in the third system of (4.31). After relabeling $t_1 = y$, $t_2 = t$, $U = U_0$ and $W = 2U_1 + U'_0$, we get the system

$$\begin{cases} W' = U_y, \\ 3W_y = 4U_t - U''' - 6(U^2)' + 6[U, W], \end{cases} \quad (4.32)$$

which we call the *matrix Kadomtsev-Petviashvili equation* (when $m = 1$ it reduces to the usual Kadomtsev-Petviashvili equation (3.23)). According to Remark 4.23, we expect that equation (4.32) belongs to an infinite hierarchy of integrable bi-Hamiltonian equations. \square

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