

Wild Quiver Gauge Theories

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Abstract

We study $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theories coupled to non-Lagrangian superconformal field theories induced by compactifying the six dimensional A_1 (2,0) theory on Riemann surfaces with irregular punctures. These are naturally associated to Hitchin systems with wild ramification whose spectral curves provide the relevant Seiberg-Witten geometries. We propose that the prepotential of these gauge theories on the Ω -background can be obtained from the corresponding irregular conformal blocks on the Riemann surfaces via a generalization of the coherent state construction to the case of higher order singularities.

1 Introduction

In the last two years an intriguing relation between non-rational conformal field theories in two dimensions and $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions [1] has been under scrutiny by a variously composed community of physicists and mathematicians. This correspondence can be obtained by considering the $A_{N-1} (2, 0)$ theory on $\mathbb{R}^4 \times \mathcal{C}$ – where \mathcal{C} is a punctured Riemann surface – and a related Hitchin integrable system [2, 3].

In the context of this correspondence, strongly coupled Argyres-Douglas points [4, 5, 6] and their generalizations correspond to specific singular geometries of \mathcal{C} as already realized in [7]. A complementary picture was developed in [8] in terms of quiver representations related to particular triangulations of the curve.

Most of the analysis of the correspondence put forward so far had to do with the gauge theory in a weakly coupled regime where a Lagrangian frame can be found and used to extract relevant physical quantities, including the instanton partition function [9], to compare with the CFT prediction in the limit of degenerate complex structure of the corresponding Riemann surface.

In this paper we will use the CFT approach in order to access a class of $SU(2)$ gauge theories coupled to superconformal field theories (SCFTs) without Lagrangian descriptions, obtained from the compactification of the $A_1 (2, 0)$ theory on a Riemann surface with irregular singularities. The Nekrasov partition function can be defined [9] as the small radius limit of a twisted index of a five dimensional theory on the circle. In order to compute it in the gauge theory, one should have a definition of the proper Hilbert space and Hamiltonian or a Lagrangian description of the theory. For strongly coupled matter sectors, as for the SCFTs under consideration here, this direct approach looks hard. Instead, we implement an extension of the Seiberg-Witten geometric construction to compute the solution. This extension is provided by the AGT correspondence. We conjecture that the relevant full prepotential in the Ω -background can be obtained by constructing the generalized coherent states in the Verma module of the Virasoro algebra corresponding to the SCFT sectors.

In section 2, we construct $SU(2)$ quiver gauge theory coupled to SCFTs. The basic building block, called D_n theory [8], is a deformation of the D -type superconformal fixed point in the classification of [10]. This theory has global $SU(2)$ flavor symmetry and, by gauging it, we get the $SU(2)$ gauge theory coupled to the D_n SCFT. The Seiberg-Witten curve of the D_n theory can be described as a double cover of a sphere with an irregular and a regular puncture, which extends Gaiotto's construction in [7] for the A_1 case. To obtain the generic $SU(2)$ quiver coupled to D_n SCFTs one further building block is needed, namely the T_2 theory of [7] which consists of four free hypermultiplets and is associated to the sphere with three regular

punctures. We call the resulting theories $SU(2)$ wild quiver gauge theories because of the link with Hitchin systems with wild ramification that will be explained below.

In section 3, we relate these gauge theories with the Hitchin integrable system. The D_n theory corresponds to the Hitchin system with wild ramification: we describe how the most general irregular singularity encodes the relevant deformation parameters from the corresponding superconformal fixed point and provide a precise prescription defining the moduli of the theory, i.e. the vevs of relevant operators.

In section 4, we consider the CFT approach to the wild quiver gauge theory. We first find that the CFT counterpart of the D_n theory is a generalization of the coherent state [11] in the Verma module. These states correspond to the operators creating irregular singularities of the stress-energy tensor on the Riemann surface. Therefore, we propose that the corresponding irregular conformal blocks describe the partition functions of the wild quiver gauge theories. We provide evidence of this proposal by reproducing the correct prepotential of $SU(2)$ gauge theories coupled to one or two D_n SCFTs. By studying the insertions of degenerate fields on the irregular conformal blocks we then obtain the quantization of the $SL(2, \mathbb{C})$ Hitchin system with wild ramification.

We conclude with several open questions in section 5. In appendix A, we show the calculation of the integral of the Seiberg-Witten differential which is used in the comparison with the CFT approach.

2 Gauge theories with strongly coupled sectors

In [8] (see also [12]), a class of $\mathcal{N} = 2$ gauge theories associated with a Riemann surface with higher order singularities was discussed. Such a class of theories is specified by BPS quiver diagrams related with triangulations of the Riemann surface. Among them, we focus on the so-called D_n theory which is associated with a sphere with a regular puncture of degree 2 and an irregular puncture of degree $n + 2$.

In section 2.1 we will see that this theory is obtained as a deformation of the maximally superconformal fixed point of $\mathcal{N} = 2$, $SU(n - 1)$ gauge theory with two fundamental hypermultiplets [13, 10]. Since D_n theories have an $SU(2)$ flavor symmetry corresponding to the regular puncture, we can gauge it to obtain $SU(2)$ gauge theories coupled to them. In section 2.2, we will see that for one $SU(2)$ gauge group, at most two D_n theories can be coupled and the corresponding Riemann surface is a sphere with at most two irregular punctures. We will call this as $\hat{A}_{m,n}$ theory following [8]. Finally, we will discuss more generic situations, namely $SU(2)$ wild quiver gauge theory associated with a Riemann surface with various regular and

irregular punctures in section 2.3. The analysis in this section is purely field-theoretical. The geometric viewpoint from string and M-theory will be analyzed in the next section.

2.1 D_n theories

Let us first see that the Seiberg-Witten curve of the D_n theory is realized as a double cover of a sphere with an irregular puncture of degree $n + 2$ and a regular one of degree 2. Let us start with the Seiberg-Witten curve of $SU(n - 1)$ gauge theory with two fundamental hypermultiplets which has $U(2)$ flavor symmetry [14, 15, 16, 17]

$$y^2 = (\hat{x}^{n-1} + \hat{u}_2 \hat{x}^{n-3} + \dots + \hat{u}_{n-1})^2 - \Lambda^{2n-4} \prod_{i=1,2} (\hat{x} + m_i), \quad (2.1)$$

where m_i are the mass parameters of the fundamentals and \hat{u}_i are the Coulomb moduli. We follow the procedure used in [18] to obtain the maximally conformal point. Let us define $u_1 = -\frac{n-1}{2}(m_1 + m_2)$ and $C_2 = -\frac{1}{4}(m_1 - m_2)^2$ which are related with the mass parameters associated with the $U(1)$ and the $SU(2)$ flavor symmetries respectively. We first shift $\hat{x} = x + \frac{u_1}{n-1}$ to obtain

$$y^2 = (x^{n-1} + u_1 x^{n-2} + u_2 x^{n-3} + \dots + u_{n-1})^2 - \Lambda^{2n-4} (x^2 + C_2), \quad (2.2)$$

where u_i with $i = 2, \dots, n - 1$ are defined to include the shifts due to the change of the coordinate x . Then, this curve can be written as

$$y^2 = (x^{n-1} + \dots + \tilde{u}_{n-2} x + u_{n-1})(x^{n-1} + \dots + (\tilde{u}_{n-2} - 2\Lambda^{n-2})x + u_{n-1}) - \Lambda^{2n-4} C_2, \quad (2.3)$$

where $\tilde{u}_{n-2} = u_{n-2} + \Lambda^{n-2}$. We can easily see that when the moduli u_i , \tilde{u}_{n-2} and C_2 are small compared with Λ the curve around $x = 0$ degenerates to

$$y^2 \sim -2\Lambda^{n-2} x^n, \quad (2.4)$$

which indicates the maximally conformal point. In this limit, the Seiberg-Witten differential can be written as

$$\lambda = 2\hat{x} \frac{P^2}{M} d\left(\frac{y}{P}\right) \sim \frac{1}{\Lambda^{n-2}} \frac{y dx}{x}, \quad (2.5)$$

where $P = \hat{x}^{n-1} + \hat{u}_2 \hat{x}^{n-3} + \dots + \hat{u}_{n-1}$ and $M = P^2 - y^2$.

To see a small deformation from this point, we define $\tilde{y}^2 = \frac{y^2}{\Lambda^{2n-4} x^2}$. The Seiberg-Witten curve of the deformed theory is

$$\tilde{y}^2 = x^{n-2} + c_1 x^{n-3} + \dots + c_{n-2} + \frac{c_{n-1}}{x} + \frac{C_2}{x^2}, \quad (2.6)$$

with Seiberg-Witten differential $\lambda = \tilde{y}dx$. The parameters c_i descend from the moduli parameters u_i of the original curve. Note that we have rescaled x and u_i to get rid of the dynamical scale. It is obvious that the quadratic differential $\lambda^2 = \tilde{y}^2(dx)^2$ has a pole of degree 2 at $x = 0$ and a pole of degree $n + 2$ at $x = \infty$. Thus, we can see that similarly to Gaiotto's construction, the Seiberg-Witten curve (2.6) is a double cover of the sphere with one regular puncture and one degree $n + 2$ puncture, which is the curve of the D_n theory. The topology of the curve is as follows: if we define $\tilde{y}^2 = x^{-2}P_n(z)$, then the branch points are at the roots of P_n for even n and at the roots of P_n and ∞ for odd n . Therefore, the genus of the Seiberg-Witten curve is $\frac{n}{2} - 1$ for n even and $\frac{n-1}{2}$ for n odd.

The scaling dimensions of the deformation parameters can be easily determined by demanding $\Delta(\lambda) = 1$. It follows that

$$\Delta(C_2) = 2, \quad \Delta(c_i) = \frac{2i}{n}, \quad i = 1, \dots, n-1 \quad (2.7)$$

Let us consider the meaning of these dimensions. In general, from a non-trivial superconformal fixed point on the Coulomb branch in an $\mathcal{N} = 2$ gauge theory, one can consider a deformation by adding to the Lagrangian $\delta\mathcal{L} = \int d^2\theta_1 d^2\theta_2 mV$, where $\theta_{1,2}$ are the superspace variables of $\mathcal{N} = 2$ supersymmetry and V is the $\mathcal{N} = 2$ vector superfield whose lowest component is the chiral primary field v . Since we consider a non-trivial CFT, v is a relevant operator if $1 < \Delta(v) \leq 2$, and equivalently m is a relevant parameter if $0 \leq \Delta(m) < 1$. Then, returning to the D_n theory, one can find a remarkable relation between the dimensions: $\Delta(c_i) + \Delta(c_{n-i}) = 2$ and when n is even $\Delta(c_{n/2}) = 1$. This implies that c_i for $i = [\frac{n}{2}] + 1, \dots, n-1$, where $\Delta(c_i) > 1$, are the vevs of the relevant operators while for $i = 1, \dots, [\frac{n}{2}]$, where $\Delta(c_i) \leq 1$, the coefficients c_i are their corresponding couplings and a dimension 1 parameter when n is even*. The number of relevant operators is therefore $[\frac{n-1}{2}]$. In order to make the difference explicit, we rename the vevs as v_a where $a = 1, \dots, [\frac{n-1}{2}]$ in the following.

The parameter C_2 , whose dimension is 2, comes from the Casimir of the mass parameter associated with the $SU(2)$ flavor symmetry. Since the superconformal fixed point keeps this $SU(2)$ flavor symmetry, C_2 is still the Casimir mass parameter. Indeed, for a mass parameter associated with a non-Abelian flavor symmetry, the scaling dimension does not acquire an anomaly [13]. Note that when $n = 3$ the original flavor symmetry is enhanced to $SO(4) \sim SU(2) \times SU(2)$. However, the fixed point preserves only one of the $SU(2)$'s, because of $u_1 \sim \Lambda$ at that point. (Note that we have defined $\tilde{u}_1 = u_1 + \Lambda$ for $n = 3$.)

Note also that these fixed points are in the same universality class of the ones which can be obtained from $\mathcal{N} = 2$, $SO(2n)$ pure Yang-Mills theory as maximally conformal points called as

*[s] is the integer part of s .

MD_n in [10]. In fact, the dimensions of the parameters (2.7) are the same as those in the table 3 in [10]. The name D_n seems to be more suitable from this viewpoint. However, from the $SO(2n)$ gauge theory viewpoint, it is not trivial to see how the $SU(2)$ flavor symmetry arises, though this was trivially seen in the analysis in this subsection.

For future reference, let us consider the curve (2.6) in the $w = \frac{1}{x}$ coordinate:

$$y^2 = \frac{1}{w^{n+2}} + \frac{c_1}{w^{n+1}} + \dots + \frac{C_{[\frac{n}{2}]}}{w^{[\frac{n+3}{2}]+1}} + \frac{v_{[\frac{n-1}{2}]}}{w^{[\frac{n+3}{2}]}} + \dots + \frac{v_1}{w^3} + \frac{C_2}{w^2}, \quad (2.8)$$

with the Seiberg-Witten differential $\lambda = ydw$. We call this as D_n curve. Note that we have assumed from the beginning that $n \geq 3$. However, even for $n = 1, 2$ the curve exists, although this does not describe a nontrivial theory. We will see these explicitly below.

The flavor central charges of these theories have been computed in [19, 18] as

$$k = \frac{4(n-1)}{n}. \quad (2.9)$$

We will use this later.

2.2 $\hat{A}_{m,n}$ theories

Now, we consider $\hat{A}_{m,n}$ theories. As already stated in [8], this is an $SU(2)$ gauge theory coupled to two SCFTs D_m and D_n . Let us first consider the small m and n cases.

$\hat{A}_{1,1}$ theory

The simplest one is $\hat{A}_{1,1}$ which is $SU(2)$ pure super Yang-Mills theory. The curve is $x^2 = \phi_2$ with [2]

$$\phi_2 = \frac{\Lambda^2}{z^3} + \frac{u}{z^2} + \frac{\Lambda}{z}, \quad (2.10)$$

where u is the Coulomb moduli parameter. The Seiberg-Witten differential is denoted as $\lambda = xdz$. The quadratic differential $\phi_2(dz)^2 = \lambda^2$ has poles of degree 3 at $z = 0$ and ∞ . The Seiberg-Witten curve is a double cover of the sphere with two irregular punctures of degree 3.

We can obtain the D_1 curve by taking the weak coupling limit $\Lambda \rightarrow 0$ of the above. By redefining $z = \Lambda^2 w$, we obtain $\tilde{x}^2 = \tilde{\phi}_2 = \Lambda^4 \phi_2 \rightarrow \frac{1}{w^3} + \frac{u}{w^2}$, where the differential is once again $\lambda = \tilde{x}dw$. This is the D_1 curve (2.8). Conversely, we can obtain the $SU(2)$ pure Yang-Mills theory by gauging the diagonal $SU(2)$ flavor symmetries of two D_1 theories. However, these theories are empty and do not contribute to the beta function of the coupled $SU(2)$ gauge theory.

$\hat{A}_{1,2}$ theory

This is $SU(2)$ gauge theory with one fundamental hypermultiplet. The curve is given by [2]

$$\phi_2 = \frac{\Lambda^2}{z^4} + \frac{m\Lambda}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z}. \quad (2.11)$$

The quadratic differential has poles of degree 4 at $z = 0$ and of degree 3 at $z = \infty$. Thus, this theory is associated with the sphere with two punctures of degree 4 and 3.

Similarly to the previous case, we can obtain the D_2 curve from the above one by redefining $z = \Lambda w$ and taking $\Lambda \rightarrow 0$. In other words, by looking at the region near $z = 0$ we get $\tilde{\phi}_2 = \Lambda^2 \phi_2 \rightarrow \frac{1}{w^4} + \frac{m}{w^3} + \frac{u}{w^2}$. Indeed the D_2 theory is that of four free half-hypermultiplets and contributes to the one-loop beta function coefficient by -1 once the $SU(2)$ symmetry is gauged.

$\hat{A}_{1,3}$ theory

This is the first example which includes a nontrivial SCFT. By generalizing the curve (2.11), we obtain the Seiberg-Witten curve $x^2 = \phi_2$ where

$$\phi_2 = \frac{\Lambda^2}{z^5} + \frac{\Lambda^{\frac{4}{3}}c_1}{z^4} + \frac{\Lambda^{\frac{2}{3}}v_1}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z}. \quad (2.12)$$

This theory is the $SU(2)$ gauge theory coupled to the D_3 theory. Indeed, by redefining $z = \Lambda^{2/3}w$ and taking the weak coupling limit $\Lambda \rightarrow 0$, we obtain

$$\tilde{\phi}_2 = \Lambda^{4/3}\phi_2 \rightarrow \frac{1}{w^5} + \frac{c_1}{w^4} + \frac{v_1}{w^3} + \frac{u}{w^2}, \quad (2.13)$$

which is the D_3 curve (2.8). As can be expected, the Coulomb moduli parameter u corresponds to the mass parameter C_2 associated with the $SU(2)$ flavor symmetry after decoupling.

The Seiberg-Witten curve of the $\hat{A}_{1,3}$ theory (2.12) is a double cover of the sphere with two punctures of degree 5 and 3. The branch points are at $z = 0, \infty$ and at the four roots of the polynomial $P_4(z)$, where we defined $\phi_2 = z^{-5}P_4(z)$. Therefore the genus of the Seiberg-Witten curve is two. This matches with the fact that this theory can be seen as a deformation of the superconformal point of a parent $SU(2) \times SU(2)$ quiver gauge theory with one bifundamental hypermultiplet, whose Seiberg-Witten curve is indeed genus 2.

One can also see that the derivatives of the Seiberg-Witten differential λ with respect to u and $\Lambda^{\frac{2}{3}}v_1$ give a basis of holomorphic differentials on the curve. Note that the derivative with respect to c_1 does not give an independent holomorphic differential since c_1 corresponds to a mass parameter not associated to the moduli of the original $SU(2) \times SU(2)$ theory.

Since the D_3 sector contributes to the one-loop beta function coefficient of the $SU(2)$ gauge theory by $-\frac{k}{2} = -\frac{4}{3}$, where k is given by (2.9), this coupled theory has $b_0 = \frac{8}{3}$. This fractional number reflects the fractional power of Λ in (2.12).

$\hat{A}_{m,n}$ theory

It is straightforward to generalize the above construction to the $\hat{A}_{1,n}$, that is to the $SU(2)$ gauge theory coupled to the D_n SCFT. The curve is

$$\phi_2 = \frac{\Lambda^2}{z^{n+2}} + \frac{\Lambda^{\frac{2n-2}{n}} c_1}{z^{n+1}} + \frac{\Lambda^{\frac{2n-4}{n}} c_2}{z^n} + \dots + \frac{\Lambda^{\frac{2}{n}} v_2}{z^4} + \frac{\Lambda^{\frac{2}{n}} v_1}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z} \quad (2.14)$$

which is associated to a sphere with punctures of degree $n+2$ and 3. Indeed, by taking the weak coupling limit with $z = \Lambda^{\frac{2}{n}} w$, we reproduce the D_n curve (2.8). Let us check the genus of this curve. We define $\phi_2 = z^{-(n+2)} P_{n+1}(z)$ where P_{n+1} is a polynomial of degree $n+1$. When n is even, the branch points of the curve are at the roots of P_{n+1} and at $z = \infty$, thus the genus is $\frac{n}{2}$. When n is odd, the branch points are at the roots and $z = 0, \infty$, leading to genus $\frac{n+1}{2}$. This is greater than the genus of the D_n curve by one, as can be expected.

Again, the one-loop beta function coefficient of this $SU(2)$ gauge theory is

$$b_0 = \frac{2n+2}{n}. \quad (2.15)$$

We can expect that this theory can be obtained as a fixed point of $SU(n-1) \times SU(2)$ quiver gauge theory with a bifundamental field in $(\mathbf{n}-1, \mathbf{2})$ representation.

Now, we consider the most general case $\hat{A}_{m,n}$ which is the $SU(2)$ gauging of the D_n and D_m SCFTs. The corresponding curve is

$$\begin{aligned} \phi_2 = & \frac{\Lambda^2}{z^{n+2}} + \frac{\Lambda^{\frac{2n-2}{n}} c_1}{z^{n+1}} + \frac{\Lambda^{\frac{2n-4}{n}} c_2}{z^n} + \dots + \frac{\Lambda^{\frac{2}{n}} v_1}{z^3} \\ & + \frac{u}{z^2} + \frac{\Lambda^{\frac{2}{m}} \tilde{v}_1}{z} + \dots + \Lambda^{\frac{2m-2}{m}} \tilde{c}_1 z^{m-3} + \Lambda^2 z^{m-2}, \end{aligned} \quad (2.16)$$

where c_i and v_a (\tilde{c}_j and \tilde{v}_b) are the relevant parameters and the vevs of the relevant operators of the D_n (D_m) theory respectively. The one-loop beta function coefficient is $b_0 = 2(\frac{1}{n} + \frac{1}{m})$.

At this stage, let us count the number of moduli of this theory. Obviously there is a single Coulomb modulus while the other parameters are associated to the vevs of the relevant operators v_a . As found in the previous subsection, the number of them for the D_n theory is $[\frac{n-1}{2}]$. Therefore, the total number of the moduli is

$$(\# \text{ of moduli}) = 1 + \left[\frac{m-1}{2} \right] + \left[\frac{n-1}{2} \right]. \quad (2.17)$$

When $n=2$ or $m=2$, the coupled sector is just a fundamental hypermultiplet. Hence the $\hat{A}_{n,2}$ theory is an $SU(2)$ gauge theory coupled to the D_n SCFT and one fundamental hypermultiplet. The parameter \tilde{c}_1 in this case is nothing but the mass parameter of the hypermultiplet.

Finally, when $n = m = 2$, this reduces to the $SU(2)$ gauge theory with two fundamental hypermultiplets whose Seiberg-Witten curve is described by [2]

$$\phi_2 = \frac{\Lambda^2}{z^4} + \frac{m\Lambda}{z^3} + \frac{u}{z^2} + \frac{\tilde{m}\Lambda}{z} + \Lambda^2. \quad (2.18)$$

Of course, the number of moduli (2.17) is one for the $m = n = 2$ case.

In summary, we constructed the Seiberg-Witten curve of the $SU(2)$ gauge theory coupled to two SCFTs by using as building blocks the D_n theories, the theory being associated to the sphere with two irregular punctures. It is also possible to consider gauge theories associated to generic Riemann surfaces with many irregular and regular punctures. This turns out to be an $SU(2)$ wild quiver gauge theory. We will see this below.

2.3 Generalization to wild quivers

In this subsection we generalize the above analysis to Riemann surface with ℓ regular punctures and k irregular punctures with degree $n_\alpha + 2$ ($\alpha = 1, \dots, k$) which we denote as $\mathcal{C}_{g,\ell,\{n_\alpha\}}$. By definition, $n_\alpha \geq 1$.

The basic building blocks are $\mathcal{C}_{0,3}$ and $\mathcal{C}_{0,1,\{n\}}$ where $\mathcal{C}_{0,3} = \mathcal{C}_{0,3,\{\emptyset\}}$ and \emptyset denotes the absence of irregular punctures. The theory corresponding to $\mathcal{C}_{0,3}$ is that of four free hypermultiplets which was called T_2 theory in [7]. On the other hand, as analyzed in the previous section, the latter is the new ingredient inducing the D_n theory. We can construct a large class of $SU(2)$ wild quiver gauge theories with (bi and tri)fundamental hypermultiplets and coupled to SCFTs, by gauging $SU(2)$ flavor symmetries of the T_2 and D_n theories. This gauging process corresponds to connect regular punctures of $\mathcal{C}_{0,3}$'s and $\mathcal{C}_{0,1,\{n\}}$'s by thin tubes. In this way one can get any Riemann surface $\mathcal{C}_{g,\ell,\{n_\alpha\}}$.

Let us consider these quiver gauge theories more explicitly starting with the $g = 0$ case. According to our general construction, the (effective) beta function coefficients could be vanishing or positive. Actually, connecting two $\mathcal{C}_{0,3}$'s leads to an $SU(2)$ gauge group whose one-loop beta function coefficient is zero. Connecting $\mathcal{C}_{0,3}$ and $\mathcal{C}_{0,1,\{n\}}$ gives rise to an asymptotically free gauge group. Therefore, the number of the asymptotically free $SU(2)$ gauge groups is k , the number of the irregular singularities[†]. Then, there are $\ell + k - 3$ $SU(2)$ gauge groups which have vanishing beta function coefficients. The (bi and tri)fundamental hypermultiplets couple with these gauge groups preserving a total flavor symmetry $SU(2)^\ell$.

Correspondingly, $\ell + k - 3$ complex structure moduli of the Riemann surface $\mathcal{C}_{0,\ell,\{n_\alpha\}}$ are identified with the gauge coupling constants of the gauge groups with vanishing beta function

[†]This counting changes in the $\hat{A}_{m,n}$ case which is obtained by connecting two $\mathcal{C}_{0,1,\{m\}}$ and $\mathcal{C}_{0,1,\{n\}}$.

coefficients $q_i = e^{2\pi i \tau_i}$. The Seiberg-Witten curve is the double cover of this sphere: $x^2 = \phi_2(z)$. The quadratic differential locally has the following structures: at the regular punctures $z = z_f$ ($f = 1, \dots, \ell$),

$$\phi_2 \sim \frac{m_f^2}{(z - z_f)^2} + \dots, \quad (2.19)$$

The residues of λ at $z = z_f$ are the mass parameters associated with the $SU(2)^\ell$ flavor symmetry. Near the irregular punctures, $z = z_\alpha$ ($\alpha = 1, \dots, k$)

$$\phi_2 \sim \frac{\Lambda_\alpha^2}{(z - z_\alpha)^{n_\alpha+2}} + \frac{\Lambda_\alpha^{\frac{2n_\alpha-2}{n_\alpha}} c_1^\alpha}{(z - z_\alpha)^{n_\alpha+1}} + \frac{\Lambda_\alpha^{\frac{2n_\alpha-4}{n_\alpha}} c_2^\alpha}{(z - z_\alpha)^{n_\alpha}} + \dots + \frac{\Lambda_\alpha^{\frac{2}{n_\alpha}} v_1^\alpha}{(z - z_\alpha)^3} + \frac{u_\alpha}{(z - z_\alpha)^2} + \dots \quad (2.20)$$

where u_α and Λ_α are, respectively, the Coulomb moduli and the dynamical scale of the gauge group which couples to the D_{n_α} theory, and c_i^α ($i = 1, \dots, [\frac{n_\alpha}{2}]$) and v_a^α ($a = 1, \dots, [\frac{n_\alpha-1}{2}]$) are the parameters labeling the deformations from the fixed point. The other Coulomb moduli parameters are encoded in the less singular terms in ϕ_2 .

The case with $\ell = 0$ and $k = 2$ is exceptional in the sense that the above counting of the number of the gauge groups is invalid. This corresponds to the $\hat{A}_{m,n}$ theory analyzed in the previous section. The case with $\ell = 2$ and $k = 1$ was analyzed in [8] where it was called \hat{D}_n theory. These are the only two cases having one asymptotically free $SU(2)$ gauge group.

For $g > 0$, the construction is similar to the above one. The number of asymptotically free gauge groups is still equal to k , and the number of gauge groups with vanishing beta function coefficient is $3g - 3 + \ell + k$, which agrees with the number of complex structure moduli of the Riemann surface $\mathcal{C}_{g,\ell,\{n_\alpha\}}$. There are in total $3g - 3 + \ell + 2k$ Coulomb moduli of the $SU(2)$ gauge groups. Let us now include in the counting the vevs of the relevant operators v_a^α . Therefore, the total number of moduli is

$$(\# \text{ of the moduli}) = 3g - 3 + \ell + 2k + \sum_{\alpha=1}^k \left[\frac{n_\alpha - 1}{2} \right]. \quad (2.21)$$

The Seiberg-Witten curve is given by a double cover of $\mathcal{C}_{g,\ell,\{n_\alpha\}}$ and its local behavior at the singularities is that of (2.19) or (2.20).

3 Geometric interpretation

So far, we considered four-dimensional $SU(2)$ wild quiver gauge theories from a purely field theoretical point of view. Linear and elliptic quivers are induced as world-volume theories of an appropriate intersecting D4-NS5 brane system. Its M-theory lift leads to two M5-branes

wrapping the corresponding Riemann surface which is the base of the Seiberg-Witten double cover [20, 7]. As found in [7], even if type IIA brane configuration does not exist, a large class of superconformal quiver gauge theories can be obtained by wrapping M5-branes on $\mathcal{C}_{g,\ell,\{\emptyset\}}$, that is the one with only regular singularities. More precisely, the theory is superconformal only at the origin of the moduli space and with vanishing masses. The analysis in the previous section suggests that the $SU(2)$ wild quiver gauge theory with D_n sectors can also be obtained from two M5-branes compactified on $\mathcal{C}_{g,\ell,\{n_\alpha\}}$.

The low energy dynamics of two M5-branes, after decoupling the center of mass mode, is governed by the $\mathcal{N} = (2, 0)$ A_1 theory in six dimensions. Thus, the gauge theory constructed in the previous section is given by compactifying the A_1 $(2, 0)$ theory on $\mathbb{R}^{1,3} \times \mathcal{C}_{g,\ell,\{n_\alpha\}}$.

In this section, we develop the geometrical interpretation of $SU(2)$ wild quiver gauge theories. In order to do that, it is crucial to find the related integrable system. As discussed in [21] and more recently in [2, 22], for a large class of $\mathcal{N} = 2$ superconformal quiver gauge theories obtained from the $(2, 0)$ theory on $\mathbb{R}^{1,3} \times \mathcal{C}_{g,\ell,\{\emptyset\}}$, the Seiberg-Witten fibration was identified with the Hitchin integrable system (or Hitchin fibration) associated to $\mathcal{C}_{g,\ell,\{\emptyset\}}$ [23, 24, 25, 26, 27]. The singularity in the $SU(2)$ superconformal case is the mildest one: we allow at most double poles of the quadratic differential ϕ_2 . In terms of the Hitchin moduli space, this corresponds to tame ramifications where the gauge and Higgs fields have simple poles. However, we can allow a higher order singularity of the Higgs field, which is called wild ramification [28]. Therefore, it is natural to expect that our theory associated with $\mathcal{C}_{g,\ell,\{n_\alpha\}}$ where we allow higher order singularities is related to the Hitchin moduli space with wild ramifications. The case with $g = 0$, $\ell = 0$ and $n_\alpha \leq 2$ has been already discussed in [29].

Let us explain why such a connection with the Hitchin moduli space appears, starting from the $(2, 0)$ theory in six dimensions. As argued above, the $(2, 0)$ theory compactified on a Riemann surface induces a four-dimensional $\mathcal{N} = 2$ gauge theory. Furthermore, let us consider its compactification on $\mathbb{R}^{1,1} \times S^1 \times S^1$. By compactifying on S^1 , we get a three-dimensional gauge theory whose Coulomb moduli space \mathcal{M} is an hyper-Kähler manifold. This, in a particular complex structure, is the Seiberg-Witten fibration of the four-dimensional theory [30]. By further compactifying on S^1 , we are led to a two-dimensional $\mathcal{N} = (4, 4)$ sigma model with target space \mathcal{M} . Let us go back to the $(2, 0)$ theory and reverse the order of the compactifications, namely we first compactify the $(2, 0)$ theory on $S^1 \times S^1$, which leads to $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions. Then, by compactifying on the Riemann surface with a suitable twist, we get a sigma model whose target space is the Hitchin moduli space \mathcal{M}_H [31, 32, 33, 34, 35]. Therefore, comparing the two perspectives of the compactification of the $(2, 0)$ theory suggests a relation between the low energy physics of four-dimensional $\mathcal{N} = 2$

gauge theory and the Hitchin moduli space. Note that a similar argument for the case of $\mathbb{R}^{1,2} \times S^1$ also leads to the same conclusion [36, 37, 2].

We will see below that wild quiver gauge theories are indeed related to Hitchin systems with wild ramifications, and that this gives a further geometric understanding of the gauge theory. We first give a review of the Hitchin system with wild ramifications in section 3.1. Then, we discuss the correspondence with the D_n theories in section 3.2 and finally, in section 3.3, we describe the wild quiver gauge theory in terms of the Hitchin systems with wild ramifications.

3.1 Hitchin system with wild ramifications

In this subsection, we review the Hitchin system with wild ramifications. While the gauge theory considered above corresponds to the Hitchin moduli space with the gauge group $SU(2)$, here we discuss the case of a generic gauge group.

First of all, we consider the case without ramification. Let G be a Lie group whose algebra is denoted by \mathfrak{g} . Let E be a G -bundle on a Riemann surface $\mathcal{C}_g \equiv \mathcal{C}_{g,0,\{\emptyset\}}$ with a connection A and ϕ be a one-form valued in $\text{ad}(E)$. The space parametrized by (A, ϕ) has an hyper-Kähler structure, with three complex structures conventionally written as I, J and K satisfying $IJ = K$. In particular, in the complex structure I , $A_{\bar{z}}$ and ϕ_z are holomorphic, $A_{\bar{z}}$ and ϕ_z being the $(0,1)$ and $(1,0)$ components of A and ϕ respectively. In the complex structure J instead, $A_z + i\phi_z$ and $A_{\bar{z}} + i\phi_{\bar{z}}$ are holomorphic. In other words, the $G_{\mathbb{C}}$ valued connection $\mathcal{A} = A + i\phi$ is holomorphic. This implies that J does not depend on the complex structure of \mathcal{C}_g .[‡] Correspondingly, there are three symplectic structures ω_I, ω_J and ω_K . Let us define $\Omega_I = \omega_J + i\omega_K$, and Ω_J and Ω_K as its cyclic permutations. In this notation, Ω_I and Ω_J can be written as

$$\Omega_I \sim \int_{\mathcal{C}_g} d^2z \text{Tr} \delta\phi_z \wedge \delta A_{\bar{z}}, \quad \Omega_J \sim \int_{\mathcal{C}_g} d^2z \text{Tr} \delta\mathcal{A} \wedge \delta\mathcal{A}, \quad (3.1)$$

where δ denotes the exterior derivative on the space of (A, ϕ) . These are holomorphic $(2,0)$ forms in the complex structures I and J respectively.

The Hitchin equations are

$$\begin{aligned} F - \phi \wedge \phi &= 0, \\ D\phi &= D \star \phi = 0 \end{aligned} \quad (3.2)$$

where F is the curvature of the connection and \star is the Hodge star. The Hitchin moduli space $\mathcal{M}_{\text{H}}^{\text{reg}}$ is the set of regular solutions to (3.2) divided by G gauge transformations.

[‡]We are following here the notation in [33].

Let us consider $\mathcal{M}_{\mathbb{H}}^{reg}$ in the complex structure I . The Hitchin equations in the second line of (3.2) are equivalent to $D_{\bar{z}}\phi_z = 0$ and its complex conjugate. This is holomorphic in I since this only depends on $A_{\bar{z}}$ and ϕ_z . (This equation is equivalent to the vanishing of the moment map associated with Ω_I .) On the other hand, the first equation is a ‘‘real’’ equation (which is equivalent to the vanishing of the real moment map with ω_I). It turns out to be convenient to treat these equations separately. Let us define $\varphi = \phi_z dz$ and call it as the Higgs field. The equation $D_{\bar{z}}\phi_z = 0$ means that φ is a holomorphic section of $K_{\mathcal{C}_g} \otimes \text{ad}(E)$ where $K_{\mathcal{C}_g}$ is the canonical line bundle on \mathcal{C}_g . Thus, the solutions to the holomorphic equation are described by a pair (E, φ) where E is a holomorphic G -bundle determined by $A_{\bar{z}}$.

Then, we impose the real equation $F - \phi \wedge \phi = 0$. It can be shown that imposing this equation and quotienting by G is equivalent to quotienting the pair (E, φ) by complexified $G_{\mathbb{C}}$ gauge transformations, modulo stability. Summarizing, the Hitchin moduli space $\mathcal{M}_{\mathbb{H}}$, in the complex structure I , is the pair (E, φ) divided by $G_{\mathbb{C}}$.

A similar analysis can be applied to the system in the complex structure J . The result is that $\mathcal{M}_{\mathbb{H}}^{reg}$ is the space of $G_{\mathbb{C}}$ flat connections \mathcal{A} divided by $G_{\mathbb{C}}$ gauge transformations, again modulo stability. Indeed, the vanishing of the moment map associated with Ω_J is equivalent to the flatness condition. The complex dimension of the moduli space is $\dim_{\mathbb{C}} \mathcal{M}_{\mathbb{H}}^{reg} = 2(g-1) \dim(G)$. Since \mathcal{A} is a flat connection, the moduli space can be specified by the monodromies around the independent A and B cycles of \mathcal{C}_g , A_{α} and B_{α} ($\alpha = 1, \dots, g$). These are $G_{\mathbb{C}}$ valued and satisfy the condition:

$$1 = A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1}. \quad (3.3)$$

By dividing the $G_{\mathbb{C}}$ gauge transformations, one can obtain the dimension above.

Let us now go back to the complex structure I and describe the so-called Hitchin fibration as a completely integrable system [23]. For simplicity and for the purpose of this paper, we choose $G = SU(2)$. Then, we consider the space of gauge invariant polynomials of φ . In the $SU(2)$ case, this is generated by $\text{Tr} \varphi^2$, a holomorphic quadratic differential parametrizing $\mathcal{Q} = H^0(\mathcal{C}_g, K_{\mathcal{C}}^2)$. The Hitchin fibration is specified by a map $\mathcal{M}_{\mathbb{H}}^{reg} \rightarrow \mathcal{Q}$. The complex dimension of the base space \mathcal{Q} is simply given by $3(g-1)$ which is one half of the complex dimension of $\mathcal{M}_{\mathbb{H}}^{reg}$.

The commuting Hamiltonians H_p ($p = 1, \dots, 3(g-1)$) can be constructed from $\text{Tr} \varphi^2$ as

$$H_p = \int_{\mathcal{C}_g} \alpha_p \wedge \text{Tr} \varphi^2, \quad (3.4)$$

where α_p are basis of Beltrami differentials. We can easily show that these Hamiltonians commute with each other with respect to the holomorphic symplectic form Ω_I (3.1), because

(3.4) depends only on ϕ_z . The spectral curve of the integrable system is defined by

$$x^2 = \text{Tr } \varphi^2, \quad (3.5)$$

where we omitted $(dz)^2$ and considered $\text{Tr } \varphi^2$ as the coefficient of the quadratic differential. This is identified with the Seiberg-Witten curve of the corresponding $SU(2)$ quiver gauge theory. In particular, $\text{Tr } \varphi^2$ is identified with ϕ_2 in the Seiberg-Witten curve in section 2.

Wild ramification

Let us consider now the singular solutions. We focus on the case with one singularity of degree m at $z = 0$ where z is a local coordinate on the Riemann surface. The generalization to more singularities is straightforward, as we will see in subsequent subsections.

Let \mathbb{T} be the maximal torus of G . Let also \mathfrak{t} and $\mathfrak{t}_{\mathbb{C}}$ be the Lie algebras of \mathbb{T} and its complexification respectively. We define the parameters $t_i \in \mathfrak{t}_{\mathbb{C}}$ and $\alpha \in \mathfrak{t}$. The singular solution which we focus on here is

$$\begin{aligned} A &= \alpha d\theta + \dots, \\ \phi &= dz \left(\frac{t_m}{z^m} + \dots + \frac{t_1}{z} \right) + \dots + c.c., \end{aligned} \quad (3.6)$$

where the ellipsis denotes the regular part. The moduli space \mathcal{M}_{H} is given by a space parametrized by (A, ϕ) divided by G gauge transformations preserving the singular structure (3.6). As in the case of the regular solutions, we consider the moduli space \mathcal{M}_{H} in the complex structures I and J .

In the complex structure J , the moduli space is a space of a $G_{\mathbb{C}}$ valued flat connection \mathcal{A} with a singularity at $z = 0$. \mathcal{A} can be transformed to the local form $\mathcal{A} = dz \left(\frac{2t_m}{z^m} + \dots + \frac{2t_1}{z} - i \frac{\alpha - i2\text{Im}t_1}{z} \right)$. The moduli space is again parametrized by the monodromies. However, compared to the regular case, the inclusion of the singularity induces additional monodromy factors to (3.3) which are basically written in terms of the Stokes matrices. By counting the dimension of them, one obtains [35]

$$\dim_{\mathbb{C}} \mathcal{M}_{\text{H}} = 2(g - 1) \dim(G) + m(\dim(G) - r). \quad (3.7)$$

where r is the rank of G . Note that the contribution of the singularity corresponds to the second term in (3.7).

In the complex structure I , we give a local trivialization of E which reduces $\bar{D} = d\bar{z}(\partial_{\bar{z}} + A_{\bar{z}})$ to the $\bar{\partial}$ operator. The Higgs field φ is given by the holomorphic part of (3.6)

$$\varphi = dz \left(\frac{t_m}{z^m} + \dots + \frac{t_1}{z} \right) + \dots \quad (3.8)$$

Therefore, the moduli space is described by the pair (E, φ) where E is a holomorphic G -bundle. This moduli space in this complex structure I varies holomorphically with the parameters t_1, \dots, t_m .

Now, let us consider the Hitchin fibration $\mathcal{M}_H \rightarrow \mathcal{Q}$ where \mathcal{Q} is the space of a quadratic differential, focusing again to the $SU(2)$ case. The quadratic differential is locally

$$\mathrm{Tr} \varphi^2 = \frac{\mathrm{Tr} t_m^2}{z^{2m}} + \frac{2 \mathrm{Tr} t_m t_{m-1}}{z^{2m-1}} + \dots + \frac{2 \mathrm{Tr} t_m t_1 + \dots}{z^{m+1}} + \dots \quad (3.9)$$

Note that the terms less singular than $1/z^{m+1}$ depend on the regular terms of the Higgs field. This will be very important to make a connection to the gauge theory. The base space \mathcal{Q} is of complex dimension $3g - 3 + m$. Indeed, this can be seen as follows: the parameters in \mathcal{Q} are the ones needed to specify the last dots in (3.9), since the more singular terms are fixed by t_i 's. This is one half of the complex dimension of \mathcal{M}_H counted above.

3.2 D_n theory and Hitchin system

We are ready to describe the singularity structure of the Higgs field which corresponds to the D_n theory. Here we focus on the Hitchin fibration in the complex structure I .

Before going into the D_n theory, let us briefly recall the case for $\mathcal{N} = 2$ superconformal $SU(2)$ gauge theories associated to the Riemann surface $\mathcal{C}_{g,\ell}$. In this theory the relation with the Hitchin system is as follows: the Coulomb moduli space, parametrized by $u_i = \langle \mathrm{tr} \phi_i^2 \rangle$, is identified with the base space \mathcal{Q} of the Hitchin fibration, as in [21]. Note that mass parameters do not correspond to the variables parametrizing \mathcal{Q} . Indeed, the quadratic differential has at most double poles which correspond to regular singularities of degree 1 of the Higgs field. Thus, the parameter t_1 of the Higgs field is related to the mass parameter which is the residue of the Seiberg-Witten differential. As argued above, the Hitchin fibration depends holomorphically on this parameter. Thus, the Coulomb moduli and the mass parameters are on different footings.

We will see below that these relations are slightly modified in the D_n theory case. First of all, let us consider $n = 2m$. Our claim is that the Hitchin moduli space associated with the D_{2m} theory is the one in which the Higgs field has a singularity of order $m + 1$ at $z = 0$

$$\varphi \sim dz \left(\frac{t_{m+1}}{z^{m+1}} + \dots + \frac{t_1}{z} + \dots \right), \quad (3.10)$$

and of order 1 at $z = z_\infty$

$$\varphi \sim dz \left(\frac{\tilde{t}_1}{z - z_\infty} + \dots \right). \quad (3.11)$$

More explicitly, by comparing with (2.8), we identify the parameters as

$$t_{m+1} = \sigma_3, \quad t_m = \frac{c_1}{2}\sigma_3, \quad t_{m-1} = \frac{1}{2} \left(c_2 - \frac{c_1^2}{4} \right) \sigma_3, \quad \dots \quad (3.12)$$

where $\sigma_3 = \text{diag}(1, -1)/\sqrt{2}$.

Let us first see the dimension of the moduli space \mathcal{M}_H and interpret it from the gauge theory point of view. The complex dimension of \mathcal{M}_H with the required singularity structure is $\dim_{\mathbb{C}} \mathcal{M}_H = -6 + 2(m+1) + 2 = 2(m-1)$. Correspondingly, the complex dimension of \mathcal{Q} is $m-1$. With $m=1$, this has to correspond to the D_2 theory. Indeed, the D_2 theory is simply that of the four free half-hypermultiplets and does not have any modulus in agreement with $\dim_{\mathbb{C}} \mathcal{Q} = 0$. What are the $m-1$ moduli in the D_{2m} theory for $m > 1$? As discussed at the end of section 2.1, this theory has $m-1$ parameters v_a which are the vevs of relevant operators. Therefore, we identify them with the moduli parametrizing the base space \mathcal{Q} of the Hitchin fibration.

Indeed, this can be made more concrete by comparing the spectral curve and the Seiberg-Witten curve. Let us write down the local behavior of the spectral curve at $z=0$

$$\text{Tr} \varphi^2 \sim \frac{\text{Tr} t_{m+1}^2}{z^{2m+2}} + \dots + \frac{2 \text{Tr} t_{m+1} t_1 + \dots}{z^{m+2}} + \dots \quad (3.13)$$

As already noted in (3.9), the terms of order higher than $1/z^{m+1}$ depend only on the parameters t_i which specify the singularity of the Higgs field. The regular part of the Higgs field will enter the equation from the order $1/z^{m+1}$ on. In the Seiberg-Witten curve of the D_n theory, the vevs of the relevant operators v_a will also enter from the order $1/z^{m+1}$ as in (2.8). Therefore, the geometric meaning of the parameters of the D_{2m} theory now becomes clear: the relevant parameters c_i correspond to the parameters t_i of the Higgs field specifying the singular part. The Hitchin fibration varies holomorphically with them. On the other hand, the vevs of the relevant operators v_a correspond to the moduli of the base space of the Hitchin fibration. So, we see that the parameters are on different footings, as in the case with regular singularities.

Let us now discuss the case $n = 2m - 1$ ($m \geq 1$). At first sight, we encounter a contradiction because the most singular term of the spectral curve is always of even degree. Therefore, it is impossible to describe this case from solutions of section 3.1. Notice however that when we wrote down (3.6) we restricted to $t_i \in \mathfrak{t}_{\mathbb{C}}$. Thus, we relax this condition and allow t_m to be a nilpotent element of $G_{\mathbb{C}}$, which, in the $SL(2, \mathbb{C})$ case, corresponds to $\sigma_{\pm} = \sigma_1 \pm i\sigma_2$.

As discussed in [35], after a gauge transformation, one can recover an analogous local behavior to the previous case but on the double cover of the z -plane, namely

$$A = 0, \quad \phi = dt \left(\frac{s_m}{t^{2m}} + \frac{s_{m-1}}{t^{2(m-1)}} + \dots + \frac{s_1}{t^2} \right) + \dots + c.c. \quad (3.14)$$

where $s_i \in \mathfrak{t}_{\mathbb{C}}$ and $t^2 = z$. The dimension of the $SL(2, \mathbb{C})$ Hitchin moduli space is $\dim_{\mathbb{C}} \mathcal{M}_{\text{H}} = 6(g-1) + 2(m+1)$ and, in the complex structure I , the Hitchin fibration varies holomorphically with respect to the parameters s_i . By going back to the z -coordinate, we get

$$A = 0, \quad \phi = dz \left(\frac{s_m}{z^{m+1/2}} + \frac{s_{m-1}}{z^{m-1/2}} + \dots + \frac{s_1}{z^{3/2}} \right) + \dots + c.c., \quad (3.15)$$

where the fractional power indicates the presence of a cut in the z -plane.

Now, our claim is that the D_{2m-1} theory can be described by the Hitchin fibration with a singularity as above at $z = 0$ and a regular singularity as in (3.11) at $z = z_{\infty}$. The complex dimension of the Hitchin moduli space in this case is $-6 + 2(m+1) + 2 = 2(m-1)$. Indeed, the spectral curve near to $z = 0$ is

$$\text{Tr } \varphi^2 \sim \frac{\text{Tr } v_m^2}{z^{2m+1}} + \frac{2 \text{Tr } v_m v_{m-1}}{z^{2m}} + \dots + \frac{2 \text{Tr } v_m v_1 + \dots}{z^{m+2}} + \dots \quad (3.16)$$

As in the previous case, the terms less singular than $1/z^{m+2}$ include the regular terms, and the dimension of the space \mathcal{Q} of these quadratic differentials is $m-1$. So, the parameters s_i are related with the relevant deformation parameters c_i and the moduli of \mathcal{Q} corresponds to the vevs of the $m-1$ relevant operators v_a . Therefore, the geometric meaning of the gauge theory parameters is the same as in the $n = 2m$ case.

3.3 Wild quiver gauge theories and Hitchin systems

In this subsection, we shortly consider the Hitchin moduli space corresponding to $SU(2)$ wild quiver gauge theories in section 2.3. Associated with $\mathcal{C}_{g,\ell,\{n_{\alpha}\}}$ where $\alpha = 1, \dots, k$, we constructed $SU(2)$ quiver gauge theory with k strongly coupled sectors $D_{n_{\alpha}}$. Here n_{α} is the degree minus 2 of the singularity of the quadratic differential ϕ_2 (2.20).

The corresponding Hitchin moduli space is formulated on a genus g Riemann surface with k irregular and ℓ regular singularities. The singularity structure of the Higgs field is specified by (3.10) when $n_{\alpha} = 2m$ and (3.15) when $n_{\alpha} = 2m-1$, and by (3.11) for regular singularities. Let us check the dimension of the Hitchin moduli space. As in the previous subsection, the singular behavior of the Higgs field, corresponding to an irregular singularity of the D_n theory, contributes to the complex dimension by $2\lceil \frac{n+3}{2} \rceil$. Moreover, each regular singularity contributes by 2 to the dimension of the Hitchin moduli space. Therefore, the complex dimension of the Hitchin moduli space is $\dim_{\mathbb{C}} \mathcal{M}_{\text{H}} = 6(g-1) + 2\ell + 2 \sum_{\alpha=1}^k \lceil \frac{n_{\alpha}+3}{2} \rceil$ and correspondingly,

$$\dim_{\mathbb{C}} \mathcal{Q} = 3g - 3 + \ell + \sum_{\alpha=1}^k \left\lceil \frac{n_{\alpha} + 3}{2} \right\rceil. \quad (3.17)$$

This agrees with the counting of the moduli on the gauge theory side (2.21).

4 Irregular conformal blocks

In [1], it was proposed that the instanton partition function of a weakly coupled $\mathcal{N} = 2$, $SU(2)$ gauge theory associated with a particular marking of the Riemann surface $\mathcal{C}_{g,\ell}$ can be obtained from the Virasoro conformal block on $\mathcal{C}_{g,\ell}$. In this section we claim that the partition function of the wild quiver gauge theory can be obtained from irregular conformal blocks on $\mathcal{C}_{g,\ell,\{n_\alpha\}}$. To make this statement more precise, we have to specify what corresponds to the basic building block $\mathcal{C}_{0,1,\{n\}}$. It is already known that for $n = 1, 2$ cases, this can be described by the coherent state in the Verma module [11][§]. Here, we need to find the generalization of this state, which we will refer to as $|G_n\rangle$ corresponding to $\mathcal{C}_{0,1,\{n\}}$.

First of all, let us review the properties of the state for $n = 1, 2$. For $n = 1$ the state is specified by the coherent condition:

$$L_1 |G_1\rangle = \hat{\Lambda}^2 |G_1\rangle, \quad L_n |G_1\rangle = 0, \quad (\text{for } n > 1) \quad (4.1)$$

in the Verma module of conformal weight Δ . Equivalently, this state can be written as [45]

$$|G_1\rangle = \sum_{k=0}^{\infty} \hat{\Lambda}^{2k} Q_{\Delta}^{-1}(1^k; Y) L_{-Y} |\Delta\rangle, \quad (4.2)$$

where $|\Delta\rangle$ is the primary state of weight Δ and Q_{Δ}^{-1} is the inverse of the Shapovalov matrix: $Q_{\Delta}(W; Y) = \langle \Delta | L_W L_{-Y} | \Delta \rangle$ and we use the notation $L_{-Y} = (L_{-1})^{m_1} (L_{-2})^{m_2} \dots$ for $Y = \{Y_1, Y_2, \dots\} = [1^{m_1} 2^{m_2} \dots]$. Indeed, one can show that $\langle \Delta | L_Y | G_1 \rangle = \hat{\Lambda}^{2k} \delta_{Y, 1^k}$, and this implies (4.1).

The inclusion of the mass parameter, that is the state corresponding to the D_2 theory, is specified by

$$L_1 |G_2\rangle = \hat{m} \hat{\Lambda} |G_2\rangle, \quad L_2 |G_2\rangle = \hat{\Lambda}^2 |G_2\rangle, \quad L_n |G_2\rangle = 0, \quad (\text{for } n > 2). \quad (4.3)$$

This state can be written as

$$|G_2\rangle = \sum_{k=0}^{\infty} \sum_{p=0}^k \hat{m}^{k-2p} \hat{\Lambda}^k Q_{\Delta}^{-1}(2^p 1^{k-2p}; Y) L_{-Y} |\Delta\rangle, \quad (4.4)$$

where the sum over p is taken such that $k - 2p$ is not negative. As above, one can check that $\langle \Delta | L_Y | G_2 \rangle = \hat{m}^{k-2p} \hat{\Lambda}^k \delta_{Y, 2^p 1^{k-2p}}$, which leads to (4.3).

By using these states, one can write the Nekrasov partition function of $SU(2)$ gauge theories with $N_f = 0, 1, 2$, which are the $\hat{A}_{m,n}$ theories with $m, n = 1, 2$. For the $\hat{A}_{1,1}$ theory, it was

[§]Analogous coherent states were discussed in [38, 39, 40, 41, 42, 43, 44] for various different conformal algebras. These states are also called Whittaker vectors in mathematics.

checked in [11] that the partition function is just the norm of the simplest state:

$$Z_{Nek}^{\hat{A}_{1,1}} = \langle G_1 | G_1 \rangle. \quad (4.5)$$

Moreover, for the $\hat{A}_{1,2}$ and $\hat{A}_{2,2}$ theories we have $Z_{Nek}^{\hat{A}_{1,2}} = \langle G_1 | G_2 \rangle$ and $Z_{Nek}^{\hat{A}_{2,2}} = \langle G_2 | G_2 \rangle$. The identification of the parameters is as follows. First of all, since the parameters in the conformal block are dimensionless, we have to supply a scale which we denote by \hbar . The Nekrasov deformation parameters are then identified as

$$\epsilon_1 = b\hbar, \quad \epsilon_2 = -\hbar/b. \quad (4.6)$$

In other words, $\hbar^2 = -\epsilon_1\epsilon_2$. The mass parameters and the vev of the scalar multiplet are identified as $m = \hbar\hat{m}$, $ia = \hbar\alpha$ and $\Lambda = \hbar\hat{\Lambda}$, where α is the internal momentum and the conformal dimension is $\Delta = \frac{Q^2}{4} - \alpha^2$. Note that we identified the scalar multiplet vev up to an i factor for later convenience. The relation (4.5) was proved in [46] by using recursion relations [47, 48, 49, 50].

In [1], it was found that the Seiberg-Witten curve can be obtained from the classical limit $\epsilon_{1,2} \rightarrow 0$ of the vev of the energy-momentum tensor in the conformal block. This works also in the above examples [11]:

$$-\epsilon_1\epsilon_2 \frac{\langle G_1 | T(z) | G_1 \rangle}{\langle G_1 | G_1 \rangle} \rightarrow \frac{\Lambda^2}{z^3} + \frac{U}{z^2} + \frac{\Lambda^2}{z} \equiv \phi_2^{\text{CFT}}, \quad (4.7)$$

where we have taken the gauge theory limit $\epsilon_{1,2} \rightarrow 0$. U is denoted by

$$U = \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} (-\epsilon_1\epsilon_2) \frac{\langle G_1 | L_0 | G_1 \rangle}{\langle G_1 | G_1 \rangle} = a^2 - \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \frac{\epsilon_1\epsilon_2}{4} \frac{\partial \ln \langle G_1 | G_1 \rangle}{\partial \ln \Lambda} = a^2 + \frac{1}{4} \frac{\partial F}{\partial \ln \Lambda}, \quad (4.8)$$

where we have defined $\langle G_1 | G_1 \rangle = e^{-\frac{F}{\epsilon_1\epsilon_2} + \dots}$. This U therefore coincides with the Coulomb moduli u in the Seiberg-Witten curve by using the Matone relation [51]. Thus, (4.7) agrees with (2.10). Similarly, it is easy to see that

$$\begin{aligned} -\epsilon_1\epsilon_2 \frac{\langle G_1 | T(z) | G_2 \rangle}{\langle G_1 | G_2 \rangle} &\rightarrow \frac{\Lambda^2}{z^4} + \frac{m\Lambda}{z^3} + \frac{U}{z^2} + \frac{\Lambda^2}{z}, \\ -\epsilon_1\epsilon_2 \frac{\langle G_2 | T(z) | G_2 \rangle}{\langle G_2 | G_2 \rangle} &\rightarrow \frac{\Lambda^2}{z^4} + \frac{m\Lambda}{z^3} + \frac{U}{z^2} + \frac{\tilde{m}\Lambda}{z} + \Lambda^2, \end{aligned} \quad (4.9)$$

which are (2.11) and (2.18).

4.1 Generalization of the coherent states

The form of the Seiberg-Witten curve and the above observations suggest that the partition functions of the $\hat{A}_{m,n}$ theory and of more general wild quiver gauge theories can be obtained by

generalizing the above coherent state. However, the naive generalization like $L_k |G_n\rangle \sim |G_n\rangle$ for $k \leq k_0$, and $L_k |G_n\rangle = 0$ for $k > k_0$ is inconsistent with the Virasoro algebra. Thus we can no longer use the coherent condition to describe these theories. It turns out to be easier to approach the problem using the explicit expression of the state like (4.2) and (4.4).

As a generalization of the states $|G_1\rangle$ and $|G_2\rangle$, we introduce

$$|G_n\rangle = \sum_{k=0}^{\infty} \sum_{\ell_p} \hat{\Lambda}^{2k/n} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \hat{c}_i^{\ell_{n-i}} \prod_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} \hat{v}_a^{\ell_a} Q_{\Delta}^{-1}(n^{\ell_n}(n-1)^{\ell_{n-1}} \dots 2^{\ell_2} 1^{\ell_1}; Y) L_{-Y} |\Delta\rangle, \quad (4.10)$$

with $\ell_1 = k - \sum_{m=2}^n m \ell_m$, which is associated with the D_n theory. The parameters $\hat{\Lambda}$, \hat{c}_i and \hat{v}_a are identified with the dynamical scale of the theory and the parameters of the D_n theory as $\hbar \hat{\Lambda} = \Lambda$, $\hbar^{2i/n} \hat{c}_i = c_i$ and $\hbar^{2(n-a)/n} \hat{v}_a = v_a$.

Let us derive the conditions satisfied by this state. It is easy to see that

$$\langle \Delta | L_Y | G_n \rangle = \begin{cases} \hat{\Lambda}^{2k/n} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \hat{c}_i^{\ell_{n-i}} \prod_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} \hat{v}_a^{\ell_a}, & \text{for } Y = n^{\ell_n}(n-1)^{\ell_{n-1}} \dots 2^{\ell_2} 1^{\ell_1} \\ 0, & \text{otherwise} \end{cases} \quad (4.11)$$

which implies that

$$L_1 |G_n\rangle = \hat{\Lambda}^{\frac{2}{n}} \hat{v}_1 |G_3\rangle, \quad L_k |G_n\rangle = 0 \quad \text{for } k > n. \quad (4.12)$$

To compute the action of L_n , we first observe that $L_Y L_n = L_{Y'} + \dots$, where the dots denote terms involving L_k with $k > n$ and $Y' = n^{\ell_n+1}(n-1)^{\ell_{n-1}} \dots 2^{\ell_2} 1^{\ell_1}$. Therefore, we obtain $\langle \Delta | L_Y L_n | G_n \rangle = \langle \Delta | L_{Y'} | G_n \rangle = \hat{\Lambda}^2 \langle \Delta | L_Y | G_n \rangle$ which implies

$$L_n |G_n\rangle = \hat{\Lambda}^2 |G_n\rangle. \quad (4.13)$$

We note that the state (4.10) is not an eigenfunction of L_k with $1 < k < n$. For example, for $k = n-1$ the argument goes as follows. Let us observe that $L_Y L_{n-1} = L_{Y''} + (2-n)\ell_1 L_{Y'''} + \dots$, where again the dots denote terms involving L_k with $k > n$. Also, $Y'' = n^{\ell_n}(n-1)^{\ell_{n-1}+1} \dots 2^{\ell_2} 1^{\ell_1}$ and $Y''' = n^{\ell_n+1}(n-1)^{\ell_{n-1}} \dots 2^{\ell_2} 1^{\ell_1-1}$. Therefore, we obtain

$$\begin{aligned} \langle \Delta | L_Y L_{n-1} | G_n \rangle &= \hat{\Lambda}^{2(n-1)/n} \hat{c}_1 \langle \Delta | L_Y | G_n \rangle + (2-n)\ell_1 \hat{\Lambda}^{2(k+n-1)/n} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \hat{c}_i^{\ell_{n-i}} \prod_{a=2}^{\lfloor \frac{n-1}{2} \rfloor} \hat{v}_a^{\ell_a} \hat{v}_1^{\ell_1-1} \\ &= \hat{\Lambda}^{2(n-1)/n} \left(\hat{c}_1 + (2-n) \frac{\partial}{\partial \hat{v}_1} \right) \langle \Delta | L_Y | G_n \rangle, \end{aligned} \quad (4.14)$$

which implies that

$$L_{n-1} |G_n\rangle = \hat{\Lambda}^{2(n-1)/n} \left(\hat{c}_1 + (2-n) \frac{\partial}{\partial \hat{v}_1} \right) |G_n\rangle. \quad (4.15)$$

The action of the other L_k 's can be calculated in a similar way, for example

$$L_{n-2} |G_n\rangle = \hat{\Lambda}^{2(n-2)/n} \left(\hat{c}_2 + (3-n)\hat{c}_1 \frac{\partial}{\partial \hat{v}_1} + \frac{(2-n)(3-n)}{2} \frac{\partial^2}{\partial \hat{v}_1^2} + (4-n) \frac{\partial}{\partial \hat{v}_2} \right) |G_n\rangle, \quad (4.16)$$

and so on. A generic feature is that the action of L_{n-k} starts with a linear term in the corresponding parameter and the remaining terms, although involved, can be written as linear differential operators in the parameters.

4.2 Irregular conformal block on $\mathcal{C}_{g,\ell,\{n_\alpha\}}$

Now we are ready to consider the conformal block on $\mathcal{C}_{g,\ell,\{n_\alpha\}}$ (with a particular marking), by using the sewing procedure. Without irregular punctures, the only building block corresponding to $\mathcal{C}_{0,3}$ is the chiral three-point function. Connecting two $\mathcal{C}_{0,3}$'s through a tube generates the four-point conformal block

$$\sum_{Y,W} \langle \Delta_1 | V_{\alpha_2} L_{-Y} | \Delta \rangle Q_{\Delta}^{-1}(Y; W) \langle \Delta | L_W V_{\alpha_3} | \Delta_4 \rangle. \quad (4.17)$$

By repeatedly applying this procedure we can in principle construct the conformal block on any $\mathcal{C}_{g,\ell,\{\emptyset\}}$. This was proposed to be equivalent to the Nekrasov partition function of a weakly coupled $\mathcal{N} = 2$ gauge theory with vanishing beta function, associated with the same (marking of) $\mathcal{C}_{g,\ell,\{\emptyset\}}$ [1]. (See [52, 53] for higher genus case).

The inclusion of the second building block $\mathcal{C}_{0,1,\{n\}}$ is easily understood as follows: connecting two regular punctures of $\mathcal{C}_{0,3}$ and $\mathcal{C}_{0,1,\{n\}}$ can be denoted by the three-point function

$$\langle \Delta_1 | V_{\alpha_2} | G_n \rangle, \quad (4.18)$$

where $|G_n\rangle$ is the state constructed in the previous subsection. In the gauge theory, this denotes an $SU(2)$ gauge theory coupled with two fundamental flavors, corresponding to two regular punctures, and one strongly coupled sector D_n (denoted as \hat{D}_n theory in [8]). Note that in the case with $n = 1, 2$, this corresponds to $SU(2)$ gauge theory with two and three flavors respectively and was already analyzed in [11]. By further connecting this with other building blocks as in (4.17) we obtain the generic conformal block on $\mathcal{C}_{g,\ell,\{n_\alpha\}}$. This describes the wild quiver gauge theory constructed in section 2.3.

One exception of this construction is the case $\mathcal{C}_{0,0,\{m,n\}}$, namely a sphere with two irregular punctures. This is simply the scalar product of the states $|G_m\rangle$ and $|G_n\rangle$, and corresponds to the $\hat{A}_{m,n}$ gauge theory. Namely, we conjecture that the partition function of the $\hat{A}_{m,n}$ theory

is the scalar product of the generalized states:

$$Z_{Nek}^{\hat{A}_{m,n}} = \langle G_m | G_n \rangle, \quad (4.19)$$

in the appropriate identification of the parameters. A weaker statement, which we will check in the present paper, is that the prepotential of the gauge theory can be obtained from $F = -\lim_{\epsilon_1, \epsilon_2 \rightarrow 0} (\epsilon_1 \epsilon_2) \log \langle G_m | G_n \rangle$.

For a check of this relation we can see that the insertion of the energy-momentum tensor can be written as

$$\begin{aligned} -\epsilon_1 \epsilon_2 \frac{\langle G_m | T(z) | G_n \rangle}{\langle G_m | G_n \rangle} \rightarrow \phi_2^{\text{CFT}} &= \frac{\Lambda^2}{z^{n+2}} + \frac{\Lambda^{\frac{2n-2}{d_2}} (c_1 + s_1)}{z^{n+1}} + \frac{\Lambda^{\frac{2n-4}{d_2}} (c_2 + s_2)}{z^{d_2}} + \dots + \frac{\Lambda^{\frac{2}{n}} v_1}{z^3} \\ &+ \frac{U}{z^2} + \frac{\Lambda^{\frac{2}{m}} \tilde{v}_1}{z} + \dots + \Lambda^{\frac{2m-2}{m}} (\tilde{c}_1 + \tilde{s}_1) z^{m-3} + \Lambda^2 z^{m-2} \end{aligned} \quad (4.20)$$

where c_i , v_a and \tilde{c}_i , \tilde{v}_a are the parameters in the $|G_m\rangle$ and $|G_n\rangle$ states and we supplied the dimension to the parameters. s_i and \tilde{s}_i can be written in terms of the derivatives of F with respect to v_a and \tilde{v}_a , which comes from the derivative terms in the definition of the state (4.15) and (4.16). We defined the coefficient of the double pole as

$$U = a^2 + \frac{1}{b_0} \frac{\partial F}{\partial \ln \Lambda}. \quad (4.21)$$

where b_0 is the one-loop beta function coefficient of the $\hat{A}_{m,n}$ theory. This shows that ϕ_2^{CFT} has a similar structure as that of ϕ_2 . Let us below consider a few example more explicitly and check the agreement with the gauge theory.

Irregular conformal block for the $\hat{A}_{1,3}$ theory

The state $|G_3\rangle$ is given by

$$|G_3\rangle = \sum_{k=0}^{\infty} \sum_{p,q} \hat{\Lambda}^{2k/3} \hat{c}_1^q \hat{v}_1^{k-3p-2q} Q_{\Delta}^{-1}(3^p 2^q 1^{k-3p-2q}; Y) L_{-Y} |\Delta\rangle, \quad (4.22)$$

which satisfies $\langle \Delta | L_Y | G_3 \rangle = \hat{\Lambda}^{2k/3} \hat{c}_1^q \hat{v}_1^{k-3p-2q}$ for $Y = 3^p 2^q 1^{k-3p-2q}$. From the general argument above, this state is also specified by

$$L_1 |G_3\rangle = \hat{\Lambda}^{\frac{2}{3}} \hat{v}_1 |G_3\rangle, \quad L_2 |G_3\rangle = \hat{\Lambda}^{\frac{4}{3}} \left(\hat{c}_1 - \frac{\partial}{\partial \hat{v}_1} \right) |G_3\rangle, \quad L_3 |G_3\rangle = \hat{\Lambda}^2 |G_3\rangle, \quad (4.23)$$

and $L_k |G_3\rangle = 0$ for $k > 3$. We will consider the scalar product $\langle G_1 | G_3 \rangle$.

The insertion of the energy-momentum tensor can be written as

$$-\epsilon_1 \epsilon_2 \frac{\langle G_1 | T(z) | G_3 \rangle}{\langle G_1 | G_3 \rangle} \rightarrow \phi_2^{\text{CFT}} = \frac{\Lambda^2}{z^5} + \frac{\Lambda^{4/3} (c_1 + s)}{z^4} + \frac{\Lambda^{2/3} v_1}{z^3} + \frac{U}{z^2} + \frac{\Lambda^2}{z}, \quad (4.24)$$

where we supplied the dimensions to the parameters and introduced

$$s = -\frac{\partial F}{\partial v_1}. \quad (4.25)$$

Note that s comes from the derivative term in the L_2 action on $|G_3\rangle$ (4.23).

Let us then compute the scalar product explicitly. By expanding in Λ , this is written as

$$\langle G_1|G_3\rangle = \sum_{k=0} \Lambda^{8k/3} \mathcal{B}_k, \quad (4.26)$$

where $\Lambda = \hbar \hat{\Lambda}$, $\mathcal{B}_0 = 1$ and the lowest orders are

$$\mathcal{B}_1 = \frac{v_1}{2(\epsilon_1 \epsilon_2)^2 \Delta}, \quad \mathcal{B}_2 = \frac{(8\Delta + c)v_1^2 - 12\Delta c_1}{4(\epsilon_1 \epsilon_2)^4 \Delta (2\Delta(8\Delta - 5) + (1 + 2\Delta)c)}, \quad \dots \quad (4.27)$$

Then, it is easy to get

$$F = \frac{v_1}{2a^2} \Lambda^{8/3} + \frac{5v_1^2 - 12c_1 a^2}{64a^6} \Lambda^{16/3} + \frac{9v_1^3 - 28c_1 v_1 a^2 + 32a^4}{192a^{10}} \Lambda^8 + \dots \quad (4.28)$$

One may think that this might be equivalent to the prepotential of the $\hat{A}_{1,3}$ theory. However, there is a subtlety associated to the presence of s in (4.24). As we will see in detail in a moment, this provides a redefinition of the c_1 modulus of the CFT curve which gives back the u modulus of the Seiberg-Witten curve. We will postpone the discussion on the agreement with the prepotential to the next subsection.

At this stage, we can however check at least the equivalence between ϕ_2 and ϕ_2^{CFT} . In order to get the same expression, we have to identify the parameter c_1 in the gauge theory with $c_1 + s$ in (4.24). A subtlety here is that s is an infinite series in Λ . However, we can do the identification order by order in the Λ expansion. Then U , which depends on c_1 , should be considered under this identification

$$\begin{aligned} U &= a^2 + \frac{3}{8} \frac{\partial F}{\partial \ln \Lambda} \Big|_{c_1 \rightarrow c_1 - s} \\ &= a^2 + \frac{v_1}{2a^2} \Lambda^{8/3} + \frac{5v_1^2 - 12c_1 a^2}{32a^6} \Lambda^{16/3} + \frac{9v_1^3 - 28c_1 v_1 a^2 + 20a^4}{64a^{10}} \Lambda^8 + \dots \end{aligned} \quad (4.29)$$

We can see that this agrees with the u modulus (A.7) calculated in the Appendix A from the Seiberg-Witten curve.

Irregular conformal block for the $\hat{A}_{1,4}$ theory

Let us next consider the state $|G_4\rangle$ which is characterized by

$$\begin{aligned} L_1 |G_4\rangle &= \hat{\Lambda}^{\frac{1}{2}} \hat{v}_1 |G_4\rangle, \quad L_2 |G_4\rangle = \hat{\Lambda} \left(\hat{c}_2 - \hat{c}_1 \frac{\partial}{\partial \hat{v}_1} + \frac{\partial^2}{\partial \hat{v}_1^2} \right) |G_4\rangle, \\ L_3 |G_4\rangle &= \hat{\Lambda}^{\frac{3}{2}} \left(\hat{c}_1 - 2 \frac{\partial}{\partial \hat{v}_1} \right) |G_4\rangle, \quad L_4 |G_4\rangle = \hat{\Lambda}^2 |G_4\rangle, \quad L_k |G_4\rangle = 0, \quad (k > 4) \end{aligned} \quad (4.30)$$

We consider the scalar product $\langle G_1|G_4\rangle$. The insertion of the energy-momentum tensor and the limit $\epsilon_{1,2} \rightarrow 0$ lead to

$$\phi_2^{\text{CFT}} = \frac{\Lambda^2}{z^6} + \frac{\Lambda^{3/2}(c_1 + s_1)}{z^5} + \frac{\Lambda(c_2 + s_2)}{z^4} + \frac{\Lambda^{1/2}v_1}{z^3} + \frac{U}{z^2} + \frac{\Lambda^2}{z}, \quad (4.31)$$

where

$$s_1 = -2\frac{\partial F}{\partial v_1}, \quad s_2 = -c_1\frac{\partial F}{\partial v_1} - \left(\frac{\partial F}{\partial v_1}\right)^2. \quad (4.32)$$

These are obtained from the derivative terms in (4.30). As explained for the previous case, in order to compare with the Seiberg-Witten curve we have to identify the gauge theory parameters c_i ($i = 1, 2$) with $c_i + s_i$ appearing in (4.31). The function F can be calculated to be

$$F = \frac{v_1}{2a^2}\Lambda^{5/2} + \frac{5v_1^2 - 12c_2a^2}{64a^6}\Lambda^5 + \frac{9v_1^3 - 28c_2v_1a^2 + 32c_1a^4}{192a^{10}}\Lambda^{15/2} + \dots \quad (4.33)$$

As in the previous case, the value of U after the shifting $c_i \rightarrow c_i - s_i$ ($i = 1, 2$) can be checked to agree with the u modulus (A.8) calculated from the gauge theory.

Irregular conformal block for the $\hat{A}_{2,3}$ theory

As a last example, we consider $\langle G_2|G_3\rangle$ corresponding to the $\hat{A}_{2,3}$ theory. As before, the energy-momentum tensor insertion in the $\epsilon_{1,2} \rightarrow 0$ limit lead to

$$\phi_2^{\text{CFT}} = \frac{\Lambda^2}{z^5} + \frac{\Lambda^{4/3}(c_1 + s)}{z^4} + \frac{\Lambda^{2/3}v_1}{z^3} + \frac{U}{z^2} + \frac{\Lambda m}{z} + \Lambda^2, \quad (4.34)$$

where s is expressed by (4.25), and F can be calculated as

$$F = \frac{mv_1}{2a^2}\Lambda^{5/3} + \frac{5m^2v_1^2 - 12(m^2c_1 + v_1^2)a^2 + 16c_1a^4}{64a^6}\Lambda^{10/3} + \frac{m(9m^2v_1^3 - 28(m^2c_1v_1 + v_1^3)a^2 + (80c_1v_1 + 32m^2)a^4 - 64a^6)}{192a^{10}}\Lambda^5 + \dots \quad (4.35)$$

Again by taking the existence of s in ϕ_2^{CFT} into account, we find that U agrees with the gauge theory result (A.8).

4.3 Insertion of degenerate field

In this subsection, we consider the insertion of a degenerate field in the conformal blocks. We concentrate on the degenerate field $\Phi_{1,2}$, which is the operator with Liouville momentum $-\frac{1}{2b}$ (and thus the dimension $\Delta_{1,2} = -\frac{1}{2} - \frac{3}{4b^2}$) and define the conformal block with the additional

degenerate field as $\Psi(z)$, e.g., in the case of the scalar product of the states that we considered in the previous subsection we define

$$\Psi(z) = \langle G_m | \Phi_{1,2}(z) | G_n \rangle. \quad (4.36)$$

We will obtain below the second order differential equation satisfied by (4.36) which follows from the null field condition $(b^{-2}L_{-2} + (L_{-1})^2)\Phi_{1,2}(z) = 0$ [54]. The equations for the case with $m = n = 1$ and with $m, n \leq 2$ were calculated in [55] and [56, 57] respectively. We will also consider the monodromies of Ψ along some non-contractible cycles of the Riemann surface. As in [58], this leads, in the $\epsilon_{1,2} \rightarrow 0$ limit, to the special geometry relation identified with the Seiberg-Witten one. By using this idea, we will calculate the prepotential of the gauge theory from the CFT analysis performed in the previous subsection.

In [58], it was claimed that the insertion of the degenerate field corresponds to the Nekrasov partition function in presence of a surface operator. This was checked and analyzed in [59, 60, 56, 57, 61]. It is natural to think that Ψ here describes the surface operator insertion in the wild quiver gauge theory.

First of all, we note that in the limit $\epsilon_{1,2} \rightarrow 0$, the semiclassical expansion of (4.36) dictates the dependence on z to start from the subleading order in \hbar as

$$\Psi = \exp\left(-\frac{1}{\epsilon_1\epsilon_2}(F + \frac{\hbar}{b}\mathcal{W}(z) + \mathcal{O}(\hbar^2))\right), \quad (4.37)$$

where the first term is the leading term in the scalar product that we computed in the previous subsection.

Let us then derive the differential equation for $\Psi(z)$. While this can be obtained in any irregular conformal block which includes several generalized states, we focus here on the case of (4.36). Let $\Delta = \Delta(\alpha - b/4)$ and $\Delta' = \Delta(\alpha + b/4)$ be the conformal dimensions of the level zero parts of $|G_m\rangle$ and $|G_n\rangle$, in accordance with the fusion rule. Then, what we need to calculate is $\langle G_m | L_{-2}\Phi_{1,2}(z) | G_n \rangle$. In order to do that, we consider the insertion of the energy momentum tensor:

$$\begin{aligned} & \langle G_m | T(w)\Phi_{1,2}(z) | G_n \rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{w^{n+2}} \langle G_m | [L_n, \Phi_{1,2}(z)] | G_n \rangle + \frac{1}{w^2} \langle G_m | \Phi_{1,2}(z)L_0 | G_n \rangle - \frac{1}{\epsilon_1\epsilon_2} \hat{\phi}_2^{\text{CFT}} \langle G_m | \Phi_{1,2}(z) | G_n \rangle \\ &= \left[\frac{z}{w(w-z)} \frac{\partial}{\partial z} + \frac{\Delta_{1,2}}{(w-z)^2} - \frac{1}{\epsilon_1\epsilon_2} \hat{\phi}_2^{\text{CFT}} \right. \\ & \quad \left. + \frac{2}{b_0 w^2} \left(-\frac{z}{m} \frac{\partial}{\partial z} - \frac{\Delta_{1,2}}{m} + \frac{1}{2} \frac{\partial}{\partial \ln \Lambda} + \frac{m\Delta + n\Delta'}{mn} \right) \right] \Psi(z), \end{aligned} \quad (4.38)$$

where we used $[L_n, \Phi_{1,2}(z)] = (z^{n+1}\partial_z + (n+1)z^n\Delta_{1,2})\Phi_{1,2}$ and

$$\frac{\partial}{\partial \ln \Lambda} \Psi(z) = \frac{2}{m} \langle G_m | [L_0, \Phi_{1,2}(z)] | G_n \rangle + b_0 \langle G_m | \Phi_{1,2}(z) L_0 | G_n \rangle - 2 \left(\frac{\Delta'}{m} + \frac{\Delta}{n} \right) \Psi(z),$$

in order to rewrite $\langle G_m | \Phi_{1,2}(z) L_0 | G_n \rangle$ in the second line. Moreover, we defined $\hat{\phi}_2^{\text{CFT}}$ as

$$\hat{\phi}_2^{\text{CFT}} = \phi_2^{\text{CFT}} - \frac{U}{z^2}. \quad (4.39)$$

By reading off the coefficients of $(w-z)^0$ in (4.38), we get $\langle G_m | L_{-2} \Phi_{1,2}(z) | G_n \rangle = \hat{L} \Psi(z)$ where

$$\hat{L} = -\frac{1}{\epsilon_1 \epsilon_2} \hat{\phi}_2^{\text{CFT}} - \frac{1}{z^2} \left(1 + \frac{2}{b_0 m} \right) z \frac{\partial}{\partial z} + \frac{2}{b_0 z^2} \left(-\frac{\Delta_{1,2}}{m} + \frac{1}{2} \frac{\partial}{\partial \ln \Lambda} + \frac{m\Delta + n\Delta'}{mn} \right). \quad (4.40)$$

Therefore, the differential equation is

$$\left(b^2 \frac{\partial^2}{\partial z^2} + \hat{L} \right) \Psi(z) = 0. \quad (4.41)$$

As discussed in [62, 56, 63, 64] (see also [65, 66]), this equation, in the $\epsilon_2 \rightarrow 0$ limit, is related to the quantization of the corresponding integrable system, namely the Hitchin system with wild ramification. The quantization of the related Gaudin model with irregular singularity was discussed e.g. in [67, 68, 69]. It would be interesting to study this direction further.

Here, we are interested in the limit $\epsilon_{1,2} \rightarrow 0$. It follows from the expansion $-\epsilon_1 \epsilon_2 \Delta = a^2 + \dots$ and a similar one for Δ' , that

$$\lim_{\epsilon_{1,2} \rightarrow 0} (-\epsilon_1 \epsilon_2) \hat{L} \Psi = \phi_2^{\text{CFT}} \Psi. \quad (4.42)$$

Therefore, we finally obtain in the scaling limit

$$\left((b\hbar)^2 \frac{\partial^2}{\partial z^2} + \phi_2^{\text{CFT}} \right) \Psi = 0. \quad (4.43)$$

By formally solving this, we get

$$\mathcal{W}(z) = \pm i \int^z \sqrt{\phi_2^{\text{CFT}}} dz', \quad (4.44)$$

where $\mathcal{W}(z)$ was defined (4.37). The \pm sign reflects the two-fold degeneracy of the solution to the quadratic differential equation. In what follows, we consider the case with the plus sign.

As found in [58], the monodromies of the conformal block with a degenerate field insertion along the A - and B -cycles correspond to Wilson and t' Hooft loop operators on the surface operator in the gauge theory. In [58], these monodromies have been calculated in the conformal field theory:

$$\Psi(a, z + A) = \exp\left(\frac{2\pi a}{\hbar b}\right) \Psi(a, z), \quad \Psi(a, z + B) = \Psi\left(a + \frac{i\hbar}{2b}, z\right), \quad (4.45)$$

where $\Psi(z + A(\text{or } B))$ denotes the monodromy along the $A(\text{or } B)$ -cycle. Since Ψ is expanded in \hbar as (4.37), after the semi-classical expansion we obtain

$$\oint_A \sqrt{\phi_2^{\text{CFT}}} dz = 2\pi ia, \quad \oint_B \sqrt{\phi_2^{\text{CFT}}} dz = \frac{i}{2} \frac{\partial F}{\partial a}. \quad (4.46)$$

Note that we have already checked that U in the integrand can be identified with the Coulomb modulus u computed from the A -cycle integral in the $\hat{A}_{1,3}$, $\hat{A}_{2,3}$ and $\hat{A}_{1,4}$ theories. We expect that this result is generic for all conformal blocks involving irregular states $|G_n\rangle$ and the corresponding wild quiver gauge theories. Since the integrand is the Seiberg-Witten differential, the result of the B -cycle integral is the same too.

However, from the conformal field theory side, we do not need to calculate the B -cycle integral, since it can be obtained directly from the derivative of F . The final caution is the shift in the c_i parameters found in the previous subsection. Indeed, only after taking into account this shift, the B-integral matches the computation from F , namely $\frac{\partial F}{\partial a}|_{c_i \rightarrow c_i - s_i}$ agrees with the B -cycle integral of the Seiberg-Witten differential. By using the definition (A.2), the prepotential \mathcal{F} is obtained as the primitive function in a of $\frac{\partial F}{\partial a}|_{c_i \rightarrow c_i - s_i}$.

E.g., from the scalar product $\langle G_1|G_3\rangle$, we get

$$\mathcal{F}_{\hat{A}_{1,3}} = \frac{8a^2}{3} \ln \Lambda + \frac{v_1}{2a^2} \Lambda^{8/3} + \frac{5v_1^2 - 12c_1 a^2}{64a^6} \Lambda^{16/3} + \frac{9v_1^3 - 28c_1 v_1 a^2 + 20a^4}{192a^{10}} \Lambda^8 + \dots \quad (4.47)$$

while from $\langle G_1|G_4\rangle$ and $\langle G_2|G_3\rangle$, we get

$$\mathcal{F}_{\hat{A}_{1,4}} = \frac{5a^2}{2} \ln \Lambda + \frac{v_1}{2a^2} \Lambda^{5/2} + \frac{5v_1^2 - 12c_2 a^2}{64a^6} \Lambda^5 + \frac{9v_1^3 - 28c_2 v_1 a^2 + 20c_1 a^4}{192a^{10}} \Lambda^{15/2} + \dots, \quad (4.48)$$

and

$$\begin{aligned} \mathcal{F}_{\hat{A}_{2,3}} = & \frac{5a^2}{3} \ln \Lambda + \frac{mv_1}{2a^2} \Lambda^{5/3} + \frac{5m^2 v_1^2 - 12(v_1^2 + m^2 c_1) a^2 + 16c_1 a^4}{64a^6} \Lambda^{10/3} \\ & + \frac{m(9m^2 v_1^3 - 28(v_1^3 + m^2 c_1 v_1) a^2 + 20(m^2 + 4c_1 v_1) a^4 - 52a^6)}{192a^{10}} \Lambda^5 + \dots \end{aligned} \quad (4.49)$$

respectively.

Note that the monodromies found in [58] are valid for the regular conformal blocks corresponding to the $\mathcal{N} = 2$ superconformal gauge theories. The monodromies of the irregular conformal block have not been calculated yet. However, in some cases we can verify this: for the $\hat{A}_{1,3}$ theory which is obtained from the $SU(2) \times SU(2)$ superconformal theory, the corresponding conformal block might be also obtained from the five-point regular conformal block. The limit which one takes to get the $\hat{A}_{1,3}$ theory does not affect the monodromies and therefore (4.46) is correct in this case.

5 Conclusions and discussions

In this paper we proposed a quantitative approach to calculate the full prepotential in the Ω -background of $SU(2)$ wild quiver gauge theories coupled to nontrivial SCFTs via the AGT correspondence.

It would be interesting to generalize the construction of wild quivers to the higher rank case. Indeed, when we consider the $A_{N-1}(2,0)$ theory on a Riemann surface, various types of singularities, labeled by Young diagrams, can be allowed [7, 70, 71, 72, 73, 74, 75]. These corresponds to $\mathcal{N} = 2$ quiver gauge theories with vanishing beta function coefficients. More in general, it is possible to consider irregular singularities also in the higher rank case. These correspond to asymptotically free gauge theories, as exemplified in [2, 29] and, in the A_1 case, reduce to the D_1 and the D_2 type singularities studied in our paper. Therefore, our results suggest to investigate more general singularities and the corresponding irregular conformal blocks which should give a generalization of the one found in [38] for the $SU(3)/\mathcal{W}_3$ case.

We observe that it would be useful to gain insight in the CFT on a more direct and geometrical construction of the coherent state and its generalizations that we discussed in section 4. In the specific case of the Liouville theory, the operator creating an irregular puncture is naturally induced by the boundary condition at the insertion point resulting by the solution of the classical Liouville field generating higher order singularities in the classical stress-energy tensor. The very definition of the operator is anyway independent on the specific CFT at hand and having a geometric counterpart of the state building recipe (4.10) would be interesting. Also the role of these states in the matrix model approach to AGT correspondence [76, 77, 78] should be clarified.

Our construction of the irregular conformal blocks is shown to be strictly related to Hitchin systems with wild ramification and provides a scheme to quantize them which should be further developed. This should be obtained by analyzing the irregular conformal block in the $\epsilon_2 \rightarrow 0$ limit [79, 80].

Last but not least our results pave the way towards a topological string interpretation of the strongly couples systems which would be very interesting to analyze. A useful tool in this context would be the study of the generalized holomorphic anomaly equation, as done for example in [81].

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Appendix

A Computation of $u(a)$

In this appendix we calculate the A -cycle integral of the Seiberg-Witten differential. As seen in section 4.3, it is enough to compute it in order to check the correspondence with the conformal block. While the way which will be explained here can be applied to the generic $\hat{A}_{m,n}$ case, we mainly consider the $\hat{A}_{1,3}$ theory for illustration. We also give the relevant results for the $\hat{A}_{1,4}$ and $\hat{A}_{2,3}$ theories.

Let us analyze the curve of the $\hat{A}_{1,3}$ theory which is $x^2 = \phi_2$ where

$$\phi_2 = \frac{\Lambda^2}{z^5} + \frac{\Lambda^{\frac{4}{3}}c_1}{z^4} + \frac{\Lambda^{\frac{2}{3}}v_1}{z^3} + \frac{u}{z^2} + \frac{\Lambda^2}{z} = \frac{\Lambda^2}{z^5}P_4(z). \quad (\text{A.1})$$

The corresponding Seiberg-Witten differential is $\lambda = xdz$. We want to calculate the A -cycle integral when the dynamical scale Λ is very small. This corresponds to the classical limit. In order to do that, we have to specify the A -cycle of the curve. As seen in section 2.2, the branch points are at the roots of $P_4(z)$ and $z = 0, \infty$. Among the four roots of P_4 , one of them, say a_1 , scales as Λ^{-2} and the others as $\Lambda^{2/3}$. Therefore, in the classical limit $\Lambda \rightarrow 0$, the root a_1 collapses to infinity, and the others collapse to $z = 0$. Thus, it is natural to take the A -cycle as the contour around the cut between a_1 and infinity. Then it is possible to deform the contour to the one around $z = 0$ with radius $r \simeq \mathcal{O}(\Lambda^0)$. Let this contour be C .

Now we consider the Seiberg-Witten relation

$$2\pi ia = \oint_A \lambda, \quad 2\pi ia_D = \oint_B \lambda, \quad (\text{A.2})$$

where the prepotential is given by

$$a_D = \frac{1}{4\pi} \frac{\partial \mathcal{F}}{\partial a}. \quad (\text{A.3})$$

It follows from the observation above that the A -cycle integral of the differential can be expanded as

$$2\pi ia = \oint_A \lambda = \oint_C \frac{\sqrt{u}}{z} \left(1 + \frac{X}{2u} - \frac{X^2}{8u^2} + \dots \right), \quad (\text{A.4})$$

where

$$X = \frac{\Lambda^2}{z^3} + \frac{\Lambda^{\frac{4}{3}}c_1}{z^2} + \frac{\Lambda^{\frac{2}{3}}v_1}{z} + \Lambda^2 z \quad (\text{A.5})$$

Note that this expansion is valid for our choice of the contour C . Since the integrand has a pole only at $z = 0$, what one has to do is to find out the coefficient in z^{-1} in each order in the expansion in Λ . This gives the result:

$$a = \sqrt{u} \left(1 - \frac{v_1}{4u^2} \Lambda^{8/3} + \frac{12c_1 u - 15v_1^2}{64u^4} \Lambda^{16/3} - \frac{40u^2 - 140c_1 v_1 u + 105v_1^3}{256u^6} \Lambda^8 + \dots \right). \quad (\text{A.6})$$

By inverting this equation, we obtain

$$u = a^2 + \frac{v_1}{2a^2} \Lambda^{8/3} + \frac{5v_1^2 - 12c_1 a^2}{32a^6} \Lambda^{16/3} + \frac{9v_1^3 - 28c_1 v_1 a^2 + 20a^4}{64a^{10}} \Lambda^8 + \dots \quad (\text{A.7})$$

This agrees with U calculated from $\langle G_1|G_3 \rangle$ in section 4.2.

In the same way, we can calculate the u 's of the $\hat{A}_{1,4}$ and $\hat{A}_{2,3}$ theories. The results are

$$\begin{aligned} u_{\hat{A}_{1,4}} &= a^2 + \frac{v_1}{2a^2} \Lambda^{5/2} + \frac{5v_1^2 - 12a^2 c_2}{32a^6} \Lambda^5 + \frac{9v_1^3 - 28c_2 v_1 a^2 + 20c_1 a^4}{64a^{10}} \Lambda^{15/2} + \dots, \\ u_{\hat{A}_{2,3}} &= a^2 + \frac{mv_1}{2a^2} \Lambda^{5/3} + \frac{5m^2 v_1^2 - 12(v_1^2 + m^2 c_1) a^2 + 16c_1 a^4}{32a^6} \Lambda^{10/3} \\ &\quad + \frac{m(9m^2 v_1^3 - 28(v_1^3 + m^2 c_1 v_1) a^2 + 20(m^2 + 4c_1 v_1) a^4 - 48a^6)}{64a^{10}} \Lambda^5 + \dots \end{aligned} \quad (\text{A.8})$$

These agree with the U 's computed from $\langle G_1|G_4 \rangle$ and $\langle G_2|G_3 \rangle$ respectively.

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