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Correlation functions of two–matrix models

L.Bonora

International School for Advanced Studies (SISSA/ISAS)
Via Beirut 2, 34014 Trieste, Italy
INFN, Sezione di Trieste.

C.S.Xiong

Physikalisches Institut der Universität Bonn
Nussallee 12, 53115 Bonn, Germany

Abstract

We show how to calculate correlation functions of two matrix models. Our method consists in making full use of the integrable hierarchies and their reductions, which were shown in previous papers to naturally appear in multi–matrix models. The second ingredient we use are the W –constraints. In fact an explicit solution of the relevant hierarchy, satisfying the W –constraints (string equation), underlies the explicit calculation of the correlation functions. In the course of our derivation we do not use any continuum limit technique. This allows us to find many solutions which are invisible to the latter technique.

1 Introduction

Matrix models are believed to provide a (discrete) description of two dimensional gravity coupled to matter. Multi-matrix models might also provide a new approach to non-perturbative QCD. An accurate study of these models is therefore strongly motivated. While one-matrix models have been widely investigated, our knowledge of multi-matrix models is not as satisfactory (see however [1], [4],[5],[2],[3],[6],[7],[8],[9],[10], [12],[13],[14]).

In this paper we bring to a conclusion an analysis started in previous papers [15],[16],[17],[18] (see also [19], [20]). For simplicity we concentrate on two-matrix models and show how to find *exact genus by genus solutions* in these models *. By exact solutions we mean that we calculate the correlation functions of these theories without any approximation technique (except for genus expansion). The distinctive features of our approach are both its effectiveness – we are able to compute explicit solutions – and the method itself – we do not use any continuum limit technique. This allows us to find many solutions which are invisible to the latter technique, as will be clear in a moment.

The essential tools used in this paper are the same introduced in the previous ones. In particular here we rely on the basic analysis carried out in [16], although the emphasis on particular aspects may have changed in the meantime.

For a model of $N \times N$ hermitean matrices with bilinear coupling, we proceed as follows. 1) We associate to the initial matrix model partition function Z_N a discrete linear system (there are several equivalent ones), we identify the coupling conditions (string equations) and finally we specify how we can reconstruct Z_N from the quantities appearing in the linear system; we obtain in this way an extended description of our matrix model in the sense that this description is equivalent to the original path integral whenever the latter exists, but it is valid also for values of the coupling parameters for which the path integral is not defined. 2) We derive the W_∞ constraints.

At this point we are faced with an integrable hierarchy of *differential-difference* equations; the solutions we are interested in are those that satisfy the above W_∞ constraints. To select them out we continue as follows.

3) We separate the dependence on N from the dependence on the coupling parameters via a procedure introduced in [16], which consists in substituting the first flow equations into the remaining ones. This leaves us with a hierarchy of *purely differential* equations. 4) This new hierarchy is also integrable and contains integrable subsystems (reduced hierarchies) which can be nicely classified. 5) We identify the critical points. 6) We integrate the differential equations and plug the solutions into the W_∞ constraints to fix possible integration constants. In this way we obtain the sought for correlation functions.

As one will see, all the above steps are of *algorithmic* nature.

The results of our approach are of two types. The first, preliminary to the second, consists in the identification and analysis of the integrable hierarchy of differential equations which appear in two-matrix models. Since this is a long and self-consistent analysis we preferred to publish it in a separate paper [21]. The second group of

* Our approach is not of the type initiated by M.Kontsevitch [22], although in a few cases the two approaches provide equivalent descriptions.

results is collected in this paper and concerns more specifically the calculus of correlation functions of two–matrix models. We are interested in the genus expansion of the solutions and concentrate in particular on genus 0 results. They can be compared with the 2D gravity theory in the continuum and with topological field theory models.

There are two broad families of models and correlation functions (CF’s). To understand this point one should remember that two–matrix models describe the interaction of the first matrix M_1 with itself, which is described by a (possibly infinite) polynomial in M_1 , of the second matrix M_2 with itself, which is described by a similar polynomial in M_2 , and finally by a bilinear interaction $M_1 M_2$ with a coupling c . The first family of models is the one in which $c \neq 0$, i.e. we consider the whole theory. The second family consists of models in which either $c = 0$ or c does not appear, and the M_2 sector is disregarded. We provide one example of the first family, but we mostly concentrate on the second. In the latter the matrix M_2 plays a spectator role; its self–interaction and interaction with M_1 are however essential. This goes as follows. *Every model is characterized by an integrable hierarchy, which is a reduction of the KP hierarchy.* The order of the second potential determine a subclass of these models: if the order is p then the model is characterized by either a $2p$ –field representation of the KP hierarchy or by one of its integrable reductions. The models are accordingly labeled \mathcal{M}_p^l with $l = 0, \dots, p - 1$, where \mathcal{M}_p^{p-1} corresponds to the just mentioned $2p$ –field representation of KP, while the others are reductions. In particular \mathcal{M}_p^0 corresponds to the p –th KdV hierarchy. The latter are characterized by the fact that the correlation functions do not depend on the size of the matrices, N . For all the other models the correlation functions do depend on N , therefore these models cannot be ‘seen’ with continuum limit techniques. The first potential has another role, the non–vanishing coupling parameters in it determine the *small phase space* and the critical points.

We remark that an explicit solution of the relevant hierarchy, satisfying the W –constraints (string equation), underlies the explicit calculation of the correlation functions of a given model. The solutions of the hierarchies relevant for CF’s are characterized by a genus by genus homogeneity, defined according to a degree which is determined by the particular critical point one is considering. This is a distinctive feature of enormous practical utility.

In elaborating our method we have been inspired by earlier works, in particular [23],[24],[25] and [26]. The models we obtain are very close to the topological field theory models coupled to gravity studied therein.

This paper is organized as follows. In section 2 we review the relevant results of [17]. In section 3 we show how one can compute CF’s directly from the W –constraints; we show an example of CF’s with $c \neq 0$. In section 4 we collect the relevant results of [18] and [21]. Section 5 is entirely devoted to the explicit calculation of genus 0 CF’s at the first critical point via integration of the relevant hierarchy in some specific models, precisely the three models specified by the 4–field representation of the KP–hierarchy and its integrable reductions. In section 6 we give a general prescription for calculating CF’s and a few more explicit examples. Section 7 is devoted to higher critical points: we show again how to calculate the corresponding correlation functions. The last section is devoted to a brief discussion of the connection of the previous models with topological field theories and 2D gravity coupled to matter.

The Appendices contain explicit expressions for W -constraints, flow equations and CF's for the various models.

2 Review of previous results

For simplicity we limit ourselves to two-matrix models. They are initially defined by the partition function

$$Z_N(t, c) = \int dM_1 dM_2 e^{Tr U}$$

where M_1 and M_2 are Hermitian $N \times N$ matrices and

$$U = V_1 + V_2 + cM_1 M_2$$

with potentials

$$V_\alpha = \sum_{r=1}^{p_\alpha} t_{\alpha,r} M_\alpha^r \quad \alpha = 1, 2. \quad (2.1)$$

The p_α 's are finite positive integers.

A clarification is in order concerning the coupling constants $t_{\alpha,r}$. Later on we will let $p_\alpha \rightarrow \infty$, thereby introducing an infinite number of couplings. Therefore the coupling constants split into two sets, those appearing in eq.(2.1) (which define the model), and the remaining ones (which are introduced only for computational purposes). In terms of ordinary field theory the former are analogous to the interaction couplings, while the latter correspond to external currents (coupled to composite operators). In the case of matrix models we will not make any formal distinction between them. This will allow us to obtain very symmetrical and powerful formulas – the W -constraints for example. Of course the distinction is substantial and will appear when calculating the correlation functions. These will depend only on the interaction couplings, while the external ones will be set to zero once they have done their job.

Let us return to the models defined at the beginning. Our purpose is to study their correlation functions (CF's), i.e. the correlation functions of the operators

$$\text{Tr} M_1^k, \quad \text{Tr} M_2^k, \quad \text{Tr}(M_1 M_2)$$

See the beginning of section 3 for more precise definitions.

The ordinary procedure to calculate the partition function consists of three steps [27],[28],[29]: (i) one integrates out the angular parts so that only the integrations over the eigenvalues are left; (ii) one introduces the orthogonal polynomials

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers}, \quad \eta_n(\lambda_2) = \lambda_2^n + \text{lower powers}$$

which satisfy the orthogonality relations

$$\int d\lambda_1 d\lambda_2 \xi_n(\lambda_1) e^{V_1(\lambda_1) + V_2(\lambda_2) + c\lambda_1 \lambda_2} \eta_m(\lambda_2) = h_n(t, c) \delta_{nm} \quad (2.2)$$

(iii), using the orthogonality relation (2.2) and the properties of the Vandermonde determinants, one can easily calculate the partition function

$$Z_N(t, c) = \text{const } N! \prod_{i=0}^{N-1} h_i \quad (2.3)$$

Knowing the partition function means knowing the coefficients $h_n(t, c)$'s.

The information concerning the latter can be encoded in a suitable linear system plus some coupling conditions, together with a relation that allows us to identify Z_N . But before we pass to that we need some convenient notations. For any matrix M , we define

$$(\mathcal{M})_{ij} = M_{ij} \frac{h_j}{h_i}, \quad \bar{M}_{ij} = M_{ji}, \quad M_l(j) \equiv M_{j, j-l}.$$

As usual we introduce the natural gradation

$$\text{deg}[E_{ij}] = j - i$$

and, for any given matrix M , if all its non-zero elements have degrees in the interval $[a, b]$, then we will simply write: $M \in [a, b]$. Moreover M_+ will denote the upper triangular part of M (including the main diagonal), while $M_- = M - M_+$. We will write

$$\text{Tr}(M) = \sum_{i=0}^{N-1} M_{ii}$$

Analogously, if L is a pseudodifferential operator, L_+ means the purely differential part of it, while $L_- = L - L_+$.

Let us come now to the step 1) mentioned in the introduction. First it is convenient to pass from the basis of orthogonal polynomials to the basis of orthogonal functions

$$\Psi_n(\lambda_1) = e^{V_1(\lambda_1)} \xi_n(\lambda_1), \quad \Phi_n(\lambda_2) = e^{V_2(\lambda_2)} \eta_n(\lambda_2).$$

The orthogonality relation (2.2) becomes

$$\int d\lambda_1 d\lambda_2 \Psi_n(\lambda_1) e^{c\lambda_1 \lambda_2} \Phi_m(\lambda_2) = \delta_{nm} h_n(t, c). \quad (2.4)$$

As usual we will denote the semi-infinite column vectors with components $\Psi_0, \Psi_1, \Psi_2, \dots$, and $\Phi_0, \Phi_1, \Phi_2, \dots$, by Ψ and Φ , respectively.

Next we introduce the following Q -type matrices

$$\int d\lambda_1 d\lambda_2 \Psi_n(\lambda_1) \lambda_\alpha e^{c\lambda_1 \lambda_2} \Phi_m(\lambda_2) \equiv Q_{nm}(\alpha) h_m = \bar{Q}_{mn}(\alpha) h_n, \quad \alpha = 1, 2. \quad (2.5)$$

Both $Q(1)$ and $\bar{Q}(2)$ are Jacobi matrices: their pure upper triangular part is $I_+ = \sum_i E_{i, i+1}$.

Beside the above Q matrices, we will need two P -type matrices, defined by

$$\int d\lambda_1 d\lambda_2 \left(\frac{\partial}{\partial \lambda_1} \Psi_n(\lambda_1) \right) e^{c\lambda_1 \lambda_2} \Phi_m(\lambda_2) \equiv P_{nm}(1) h_m \quad (2.6)$$

$$\int d\lambda_1 d\lambda_2 \Psi_n(\lambda_1) e^{c\lambda_1 \lambda_2} \left(\frac{\partial}{\partial \lambda_2} \Phi_m(\lambda_2) \right) \equiv P_{mn}(2) h_n \quad (2.7)$$

The two matrices (2.5) we introduced above are not completely independent. More precisely both $Q(\alpha)$'s can be expressed in terms of only one of them and one matrix P . Expressing the trivial fact that the integral of the total derivative of the integrand in eq.(2.4) with respect to λ_1 and λ_2 vanishes, we can easily derive the constraints or *coupling conditions*

$$P(1) + c_{12}Q(2) = 0, \quad c_{12}Q(1) + \bar{P}(2) = 0,$$

It is just these coupling conditions that lead to the famous $W_{1+\infty}$ -constraints on the partition function at the discrete level. We will also refer to them at times as *string equations*. From them it follows at once that

$$Q(\alpha) \in [-m_\alpha, n_\alpha], \quad \alpha = 1, 2$$

where

$$m_1 = p_2 - 1, \quad m_2 = 1$$

and

$$n_1 = 1, \quad n_2 = p_1 - 1$$

These equations show that a finite band structure for the $Q(\alpha)$ is only allowed if the number of perturbations is finite. Conversely, if we want, say, $Q(1)$ to possess the full discrete KP structure – we should let $p_2 \rightarrow \infty$.

The derivation of the linear systems associated to our matrix model is very simple. We take the derivatives of eqs.(2.4) with respect to the time parameters $t_{\alpha,r}$, and use eqs.(2.5). We get in this way the time evolution of Ψ and Φ , which can be represented in two different ways:

(*) Discrete Linear System I:

$$\left\{ \begin{array}{l} Q(1)\Psi(\lambda_1) = \lambda_1\Psi(\lambda_1), \\ \frac{\partial}{\partial t_{1,k}}\Psi(\lambda_1) = Q_+^k(1)\Psi(\lambda_1), \quad 1 \leq k \leq p_1, \\ \frac{\partial}{\partial t_{2,k}}\Psi(\lambda_1) = -Q_-^k(2)\Psi(\lambda_1), \quad 1 \leq k \leq p_2, \\ \frac{\partial}{\partial \lambda}\Psi(\lambda_1) = P(1)\Psi(\lambda_1). \end{array} \right. \quad (2.8)$$

The corresponding consistency conditions are

$$[Q(1), P(1)] = 1 \quad (2.9a)$$

$$\frac{\partial}{\partial t_{\alpha,k}}Q(1) = [Q(1), Q_-^k(\alpha)] \quad (2.9b)$$

$$\frac{\partial}{\partial t_{\alpha,k}}P(1) = [P(1), Q_-^k(\alpha)] \quad (2.9c)$$

(2.9b) and (2.9c) are discrete KP hierarchies, whose integrability and meaning were discussed in [17].

(**) Discrete Linear System II:

$$\left\{ \begin{array}{l} \bar{Q}(2)\Phi(\lambda_2) = \lambda_2\Phi(\lambda_2), \\ \frac{\partial}{\partial t_{2,k}}\Phi(\lambda_2) = \bar{Q}_+^k(2)\Phi(\lambda_2), \\ \frac{\partial}{\partial t_{1,k}}\Phi(\lambda_1) = -\bar{Q}_-^k(1)\Phi(\lambda_2), \quad 1 \leq k \leq p_1 \\ \frac{\partial}{\partial \lambda_2}\Phi(\lambda_2) = P(2)\Phi(\lambda_2). \end{array} \right. \quad (2.10)$$

with consistency conditions

$$[\bar{Q}(2), P(2)] = 1, \quad (2.11a)$$

$$\frac{\partial}{\partial t_{\alpha,k}}\bar{Q}(2) = [\bar{Q}(2), \bar{Q}_-^k(\alpha)] \quad (2.11b)$$

$$\frac{\partial}{\partial t_{\alpha,k}}P(2) = [P(2), \bar{Q}_-^k(\alpha)] \quad (2.11c)$$

The third element announced in the introduction is the link between the quantities that appear in the linear system and in the coupling conditions with the original partition function. We have

$$\frac{\partial}{\partial \alpha_r} \ln Z_N(t, c) = \text{Tr}(Q^r(\alpha)), \quad \alpha = 1, 2 \quad (2.12)$$

It is evident that, by using the flow equations above we can express all the derivatives of Z_N in terms of the elements of the Q matrices. For example

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{\alpha,r}} \ln Z_N(t, c) = \left(Q^r(\alpha) \right)_{N, N-1}, \quad (2.13)$$

$$1 \leq r \leq p_\alpha; \quad \alpha = 1, 2$$

We also recall the coupling dependence of the partition function

$$\frac{\partial}{\partial c} \ln Z_N(t, c) = \text{Tr}(Q(1)Q(2)) \quad (2.14)$$

Knowing all the derivatives with respect to the coupling parameters we can reconstruct the partition function up to an overall integration constant.

We also remark that, since the RHS of the above equations is always defined, they give us a definition of Z_N even in subsets of the parameter space where the path-integral is ill-defined.

We will be using the following coordinatization of the Jacobi matrices

$$Q(1) = I_+ + \sum_i \sum_{l=0}^{m_1} a_l(i) E_{i, i-l}, \quad \bar{Q}(2) = I_+ + \sum_i \sum_{l=0}^{m_2} b_l(i) E_{i, i-l} \quad (2.15)$$

One can immediately see that

$$\left(Q_+(1) \right)_{ij} = \delta_{j, i+1} + a_0(i) \delta_{i, j}, \quad \left(Q_-(2) \right)_{ij} = R_i \delta_{j, i-1}$$

As a consequence of this coordinatization, eq.(2.13) gives in particular the important relation

$$\frac{\partial^2}{\partial t_{1,1}^2} \ln Z_N(t, c) = a_1(N), \quad (2.16)$$

To end this subsection we write down explicitly the $t_{1,1}$ - and $t_{2,1}$ -flows, which will play a very important role in what follows

$$\frac{\partial}{\partial t_{1,1}} a_l(j) = a_{l+1}(j+1) - a_{l+1}(j) + a_l(j) (a_0(j) - a_0(j-l)) \quad (2.17a)$$

$$\frac{\partial}{\partial t_{q,1}} a_l(j) = R_{j-l+1} a_{l-1}(j) - R_j a_{l-1}(j-1) \quad (2.17b)$$

$$\frac{\partial}{\partial t_{1,1}} b_l(j) = R_{j-l+1} b_{l-1}(j) - R_j b_{l-1}(j-1) \quad (2.17c)$$

$$\frac{\partial}{\partial t_{q,1}} b_l(j) = b_{l+1}(j+1) - b_{l+1}(j) + b_l(j) (b_0(j) - b_0(j-l)) \quad (2.17d)$$

2.1 $W_{1+\infty}$ constraints

The $W_{1+\infty}$ constraints (or simply W -constraints) on the partition function for our two-matrix model were obtained in [17] by putting together both coupling conditions and consistency conditions (see above). In other words the $W_{1+\infty}$ constraints contain all the available information. They take the form

$$W_n^{[r]} Z_N(t, c) = 0, \quad \tilde{W}_n^{[r]} Z_N(t, c) = 0 \quad r \geq 0; \quad n \geq -r, \quad (2.18)$$

where

$$W_n^{[r]} \equiv (-c)^n \mathcal{L}_n^{[r]}(1) - \mathcal{L}_{-n}^{[r+n]}(2) \quad (2.19a)$$

$$\tilde{W}_n^{[r]} \equiv (-c)^n \mathcal{L}_n^{[r]}(2) - \mathcal{L}_{-n}^{[r+n]}(1) \quad (2.19b)$$

The generators $\mathcal{L}_n^{[r]}(1)$ are differential operators involving N and $t_{1,k}$, while $\mathcal{L}_n^{[r]}(2)$ have the same form with $t_{1,k}$ replaced by $t_{2,k}$. One of the remarkable aspects of (2.18) is that the dependence on the coupling c is nicely factorized. The $\mathcal{L}_n^{[r]}(1)$ satisfy the following $W_{1+\infty}$ algebra

$$[\mathcal{L}_n^{[1]}(1), \mathcal{L}_m^{[1]}(1)] = (n-m) \mathcal{L}_{n+m}^{[1]}(1) \quad (2.20a)$$

$$[\mathcal{L}_n^{[2]}(1), \mathcal{L}_m^{[1]}(1)] = (n-2m) \mathcal{L}_{n+m}^{[2]}(1) + m(m+1) \mathcal{L}_{n+m}^{[1]}(1) \quad (2.20b)$$

$$[\mathcal{L}_n^{[2]}(1), \mathcal{L}_m^{[2]}(1)] = 2(n-m) \mathcal{L}_{n+m}^{[3]}(1) - (n-m)(n+m+3) \mathcal{L}_{n+m}^{[2]}(1) \quad (2.20c)$$

and in general

$$[\mathcal{L}_n^{[r]}(1), \mathcal{L}_m^{[s]}(1)] = (sn - rm) \mathcal{L}_{n+m}^{[r+s-1]}(1) + \dots, \quad (2.21)$$

for $r, s \geq 1$; $n \geq -r, m \geq -s$. Here dots denote lower than $r + s - 1$ rank operators. We notice that this $W_{1+\infty}$ algebra is not simple, as it contains a Virasoro subalgebra spanned by the $\mathcal{L}_n^{[1]}(1)$'s. We see that once we know these generators and $\mathcal{L}_{-2}^{[2]}(1)$, the remaining ones are produced by the algebra itself. The explicit expression of the basic generators is given in Appendix A1.

The algebra of the $\mathcal{L}_n^{[r]}(2)$ is just a copy of the above one, and the algebra satisfied by the $W_n^{[r]}$ and by $\tilde{W}_n^{[r]}$ is isomorphic to both. We refer to this abstract algebra with the symbol \mathcal{W} .

3 How to compute correlation functions from the W -constraints

This section is a sort of intermezzo. It does not represent our final solution of the problem. But still it will teach us a lot. We said above that the W -constraints contain all the available information of our two-matrix model. Therefore, starting from them only, we should be able to calculate all the correlation functions of the model. This is indeed true and we are going to give in the following a sample calculation of this fact. To simplify the notation we set

$$\begin{aligned} t_{1,k} &\equiv t_k, & t_{2,k} &\equiv s_k \\ \text{Tr}M_1^k &\equiv \tau_k, & \text{Tr}M_2^k &\equiv \sigma_k \end{aligned}$$

From the definition of the partition function we inherit the definition of the correlation functions

$$\begin{aligned} \langle\langle \tau_{k_1} \dots \tau_{k_n} \rangle\rangle &= \frac{\partial}{\partial t_{k_1}} \dots \frac{\partial}{\partial t_{k_n}} \ln Z_N(t, c) \\ \langle\langle \sigma_{k_1} \dots \sigma_{k_n} \rangle\rangle &= \frac{\partial}{\partial s_{k_1}} \dots \frac{\partial}{\partial s_{k_n}} \ln Z_N(t, c) \end{aligned}$$

These formulae do not make any distinction between internal and external couplings (see the specification at the beginning of section 2). Eventually the external couplings will be set to zero.

Now we write the W -constraints in this new language. $W_{-1}^{[1]}Z_N = 0$ and $\tilde{W}_{-1}^{[1]}Z_N = 0$ become, respectively

$$\sum_{k=2}^{\infty} kt_k \langle\langle \tau_{k-1} \rangle\rangle + Nt_1 + c \langle\langle \tau_1 \rangle\rangle = 0 \quad (3.1a)$$

$$\sum_{k=2}^{\infty} ks_k \langle\langle \sigma_{k-1} \rangle\rangle + Ns_1 + c \langle\langle \sigma_1 \rangle\rangle = 0 \quad (3.1b)$$

Instead $W_0^{[1]}Z_N = 0$ and $\tilde{W}_0^{[1]}Z_N = 0$ give rise to the same equation

$$\sum_{k=1}^{\infty} kt_k \langle\langle \tau_k \rangle\rangle = \sum_{k=1}^{\infty} ks_k \langle\langle \sigma_k \rangle\rangle \quad (3.2)$$

The constraint $W_1^{[1]}Z_N = 0$ becomes

$$\begin{aligned}
& c\left(\sum_{l=1}^{\infty} lt_l \ll \tau_{l+1} \gg + (N+1) \ll \tau_1 \gg\right) + \sum_{l_1, l_2=1}^{\infty} l_1 l_2 s_{l_1} s_{l_2} \ll \sigma_{l_1+l_2-1} \gg \\
& + \sum_{l=3}^{\infty} l s_l \sum_{k=1}^{l-2} \left(\ll \sigma_k \sigma_{l-k-1} \gg + \ll \sigma_k \gg \ll \sigma_{l-k-1} \gg \right) \\
& + (2N+1) \sum_{l=2}^{\infty} l s_l \ll \sigma_{l-1} \gg + (N^2 + N) s_1 = 0
\end{aligned} \tag{3.3a}$$

It should be clear by now that $\tilde{W}_1^{[1]}Z_N = 0$ gives an exactly symmetrical equation where t_k and τ_k are interchanged with s_k and σ_k , respectively. In a similar way we can write all the other W -constraints.

Let us consider now the problem at issue in a simplified situation, i.e. in genus 0. This will be enough to shed light on the main features. In this case the W -constraints come out simplified, [27]. To see this assign a suitable degree to the couplings

$$\deg \equiv [\], \quad [t_k] = [s_k] = x - k, \quad [N] = x, \quad [c] = x - 2 \tag{3.4}$$

and the degree

$$[\mathcal{F}^{(0)}] = 2x \tag{3.5}$$

to the genus zero part of the “free energy” (let us set from now on $\ln Z = \mathcal{F}$). Here x is, for the time being, an unspecified positive real number. These assignments follow directly from the W -constraints and the first flow equations. We have only arbitrarily (but without loss of generality) fixed the scale. We will be looking for CF’s homogeneous (genus by genus) in the couplings in accordance with the above degree.

In the W -constraints the genus 0 contribution is the homogeneous part of highest degree. In any correlation function we should henceforth append a label $\ll \gg_0$ to indicate this contribution. However we will avoid this cumbersome operation by understanding that the correlation functions in the remaining part of this section refer only to the genus 0 contribution. With this convention the genus 0 version of (3.1a), (3.1b) and (3.2) remain (formally) the same, while (3.3a) takes the form

$$\begin{aligned}
& c\left(\sum_{l=1}^{\infty} lt_l \ll \tau_{l+1} \gg + N \ll \tau_1 \gg\right) + \sum_{l_1, l_2=1}^{\infty} l_1 l_2 s_{l_1} s_{l_2} \ll \sigma_{l_1+l_2-1} \gg \\
& + \sum_{l=3}^{\infty} l s_l \sum_{k=1}^{l-2} \ll \sigma_k \gg \ll \sigma_{l-k-1} \gg + 2N \sum_{l=2}^{\infty} l s_l \ll \sigma_{l-1} \gg + N^2 s_1 = 0
\end{aligned} \tag{3.6}$$

and so on.

The next step consists in specifying the critical point (or the model) we want to consider. This subject will be explained in detail later. For the time being we consider the first critical point of the simplest possible model:

$$2t_2 = -1 \quad t_k = 0, \quad k > 2, \quad 2s_2 = -1 \quad s_k = 0, \quad k > 2, \tag{3.7}$$

Therefore CF's will be functions of $N, c, t_1 \equiv t$ and $s_1 \equiv s$ only. To distinguish this from the general case we will denote the CF's with the symbol $\langle \cdot \rangle$ instead of $\ll \cdot \gg$. In order to preserve homogeneity, we set

$$x = 2$$

in the above degree assignments. The genus 0 W -constraints (3.1a), (3.1b), (3.2) and (3.6) become respectively

$$\begin{aligned} - \langle \tau_1 \rangle + Nt + c \langle \sigma_1 \rangle &= 0, \\ - \langle \sigma_1 \rangle + Ns + c \langle \tau_1 \rangle &= 0, \\ - \langle \tau_2 \rangle + t \langle \tau_1 \rangle &= - \langle \sigma_2 \rangle + s \langle \sigma_1 \rangle, \\ c(- \langle \tau_3 \rangle + t \langle \tau_2 \rangle + N \langle \tau_1 \rangle) \\ &+ \langle \sigma_3 \rangle - 2s \langle \sigma_2 \rangle + (s^2 - 2N) \langle \sigma_1 \rangle + N^2 s = 0 \end{aligned}$$

Writing down the appropriate formulas for the other constraints one quickly realizes that these formulas form a recursive and overdetermined system of algebraic equations for the one-point correlation functions. A simple computer program allow us to explicitly calculate them. For example we obtain

$$\langle \tau_1 \rangle = N \frac{t + cs}{1 - c^2}, \quad \langle \sigma_1 \rangle = N \frac{s + ct}{1 - c^2} \quad (3.9)$$

and so on (the first few are written down in Appendix C1). We immediately remark that, setting $c = 0$ and $s = 0$ in these formulas, we obtain the CF's of the NLS model already met in the context of one-matrix model, [16]. The NLS model correspond to \mathcal{M}_2^1 in the classification of the present paper.

One can also derive the multi-point correlation functions by simply differentiating an appropriate number of times the W -constraints with respect to the couplings and repeating the same procedure. However this method of calculating correlation functions is not very economical since it relies on our being able to solve systems of algebraic equations which become more and more complicated the higher the critical point is. There is also another reason why this method is insufficient: since we do not have at our disposal a theory of the reduction of the \mathcal{W} algebra and of the W -constraints, it is impossible to study the reductions of our system, i.e. to study consistent subsystems which, as we shall see, constitute a large part of the rich structure of two matrix models.

We need another method and this is what we are going to study in the following sections. However from the above example we have extracted some useful information:

1) integrability of the discrete hierarchy on which the W -constraints are based translates itself into a recursive set of equations for the CF's;

2) setting $c = 0$ (at the end of the computation) and disregarding s_k and σ_k , reduces the CF's to those of a simpler model, the NLS model studied in [16] – as will be seen, this is a general fact.

3) it follows from [16] that, in the same conditions as at the previous point, the constraints (2.18) can be replaced by the much simpler ones

$$L_n Z = 0 \quad (3.10)$$

$$\begin{aligned}
L_n = & \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+n}} + 2N \frac{\partial}{\partial t_n} \\
& + \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_k \partial t_{n-k}} + N^2 \delta_{n,0} + N t_1 \delta_{n,-1}, \quad n \geq -1
\end{aligned} \tag{3.11}$$

The L_n 's satisfies the Virasoro algebra and these Virasoro constraints are enough to completely determine the correlation functions. Effective W constraints of this type will appear also in other cases (other critical points, other models).

4 Differential Hierarchies of Two-Matrix Models.

The method alluded to at the end of the previous section is based on extracting differential hierarchies of flow equations from the discrete ones that characterize two-matrix models. Let us resume the summary of section 2. We recalled there that two-matrix models can be represented by means of coupled discrete linear systems, whose consistency conditions give rise to discrete KP hierarchies and string equations. Here we review the method introduced in [15] and used in [16] and [17], to transform the discrete linear systems into equivalent differential systems whose consistency conditions are purely differential hierarchies. This is tantamount to separating the N dependence from the dependence on the couplings.

The clue to the construction are the first flows, i.e. the $t_{1,1}$ and $t_{2,1}$ flows. For the sake of simplicity let us consider the *system I* and the flow (2.17a). Let us consider the generic situation in which $Q(1)$ has $m_1 = p_2 - 1$ lower diagonal lines (see the parametrization (2.15)). To begin with let us notice that

$$\frac{\partial}{\partial t_{1,1}} \Psi_j = \Psi_{n+1} + a_0(n) \Psi_n \tag{4.1}$$

and let us adopt for any function $f(t)$ the convention

$$f' \equiv \frac{\partial f}{\partial t_{1,1}} \equiv \partial f,$$

We can rewrite

$$\Psi_n = \hat{B}_n \Psi_{n+1} \tag{4.2}$$

where

$$\hat{B}_n \equiv \frac{1}{\partial - a_0(n)} = \partial^{-1} \sum_{l=0}^{\infty} (a_0(n) \partial^{-1})^l \tag{4.3}$$

Then it is an easy exercise to prove that the discrete spectral equation

$$Q(1) \Psi(\lambda_1) = \lambda_1 \Psi(\lambda_1)$$

is transformed into the pseudodifferential one

$$L_n(1) \Psi_n = \lambda_1 \Psi_n \tag{4.4}$$

where

$$\begin{aligned}
L_n(1) &= \partial + \sum_{l=1}^{m_1} a_l(n) \hat{B}_{n-l} \hat{B}_{n-l+1} \dots \hat{B}_{n-1} \\
&= \partial + \sum_{l=1}^{m_1} a_l(n) \frac{1}{\partial - a_0(n-l)} \cdot \frac{1}{\partial - a_0(n-l+1)} \dots \frac{1}{\partial - a_0(n-1)}
\end{aligned} \tag{4.5}$$

$L_n(1)$ is an operator of KP type *. Actually it is in general a reduction of the KP operator

$$L_{KP} = \partial + \sum_{l=1}^{\infty} w_l \partial^{-l}$$

where w_l are generic (i.e. unrestricted) coordinates.

Proceeding in the same way for the other equations of *system I* we obtain the new system in differential form

$$\left\{ \begin{array}{l}
L_n(1)\Psi_n = \lambda_1 \Psi_n \\
\frac{\partial}{\partial t_{1,r}} \Psi_n = \left(L_n^r(1)\right)_+ \Psi_n, \\
\frac{\partial}{\partial t_{2,r}} \Psi_n = -\left(L_n^r(2)\right)_- \Psi_n, \\
M_n(1)\Psi_n = \frac{\partial}{\partial \lambda_1} \Psi_n
\end{array} \right. \tag{4.6}$$

In particular we have the following consistency conditions

$$\frac{\partial}{\partial t_{1,r}} L_n(1) = [(L_n^r(1))_+, L_n(1)], \tag{4.7}$$

$$\frac{\partial}{\partial t_{2,r}} L_n(1) = [L_n(1), (L_n^r(2))_-], \tag{4.8}$$

They are purely differential equations.

Let us come now to the n dependence of the above equations. The operator $L_n(1)$ introduced above (4.5) depends on the coordinates of many sectors. We showed in [17] that, using the first flow equations, we can transform it into a pseudodifferential operator depending only on the n -th sector coordinates. This was implemented at the price of introducing a very complicated expression. For our purposes in this paper this is not very convenient and we had better proceed in another way. Precisely we introduce m_1 “fields” S_1, \dots, S_{m_1} , related to the “field” a_0 in the following way

$$S_i(n) \equiv a_0(n-i) \tag{4.9}$$

Then we can rewrite $L_n(1)$ in the following way

$$L_n(1) = \partial + \sum_{l=1}^{m_1} a_l(n) \frac{1}{\partial - S_l(n)} \cdot \frac{1}{\partial - S_{l-1}(n)} \dots \frac{1}{\partial - S_1(n)}$$

*In [17] we imposed the conditions $a_l(n) = 0$, $n < l$, but actually there is no reason to impose these conditions on the basis of the matrix model, therefore we drop them here. Consequently the discussion about non-universality in section 6 of [17] must be dropped too; in the language of that section, the only alternative present in multi-matrix model is the universal one.

So we have achieved the same result as in [17], except that, of course, the fields S_i are not independent. But following a long tradition in field theory, we will consider these fields as completely independent from one another in all the intermediate steps of our calculations and only eventually impose the condition (4.9).

To further simplify the notation we will consider henceforth the lattice label n on the same footing as the couplings and write

$$a_i(n, \dots) \equiv a_i(n)(\dots), \quad S_i(n, \dots) \equiv S_i(n)(\dots)$$

where dots denote the dependence on $t_{1,k}, t_{2,k}$ and c . So the expression of $L_1(n)$ get further simplified to

$$L = \partial + \sum_{l=1}^m a_l \frac{1}{\partial - S_l} \cdot \frac{1}{\partial - S_{l-1}} \cdots \frac{1}{\partial - S_1} \quad (4.10)$$

where, for simplicity, we have dropped the label (1) too. A similar simplification has to be understood also for the other equations of the *system I* above. This simplified form is the one we constantly refer to throughout the remaining part of the paper. In particular we recall the integrable hierarchy

$$\frac{\partial}{\partial t_r} L = [(L^r)_+, L], \quad (4.11)$$

and $t_r = t_{1,r}$.

We can of course extract an analogous pseudo-differential operator for *system II*, by using the first flows (2.17d). One can also reconstruct the discrete systems from the differential ones and the first flows, for an explicit example see [30].

This hierarchy and the relevant integrable reductions have been studied in [18] and in the companion of this paper [21]. There we showed that to each operator such as (4.10) there correspond m distinct integrable reductions which are obtained via Dirac reduction procedure by successively suppressing the S fields one by one (the order being irrelevant). Of each of them we gave a Lax operator representation, which not only constitutes a very quick proof of integrability, but also provides a very efficient way to explicitly calculate all flows. To each such Lax operator, as we shall see, we can associate a full set of correlation functions, i.e. a model. Therefore we can say that for any $p \equiv p_2 = m + 1$ (p_2 is the order of the potential V_2) there correspond p distinct integrable models \mathcal{M}_p^l , where l runs from p to 0 and counts the number of S fields. The original unreduced model \mathcal{M}_p^{p-1} is defined of course by the Lax operator (4.10) above. The most reduced model \mathcal{M}_p^0 is characterized by the p -th KdV Lax operator and the p -th KdV hierarchy.

To end this section let us comment a bit on the naturalness of the reduction procedure. Our reduction procedure consists in restricting the second Hamiltonian according to the chosen constraints and calculating the second Poisson brackets according to the Dirac recipe. The rest is automatic. In the framework of the Hamiltonian systems this reduction procedure is completely natural and allows us to select consistent subsystems of a given integrable system, or, in other words, to find solutions of the initial systems in which a part of the degrees of freedom are disregarded. We stressed in section 2 that our unreduced systems are equivalent to the initial path integral and that they even provide an enlarged definition of the latter. However we

do not know in general how to go back from a reduced system to the path integral formulation. We know that in the simplest case (one–matrix model giving rise to the NLS hierarchy, see [16]) the reduction is the KdV hierarchy and that the KdV hierarchy corresponds to the even potential case via double scaling limit. Perhaps a generalization of this fact can be found also for two–matrix models (see in this sense [13]). But we dare say this problem is still open.

5 The four–field KP hierarchy

In this section we show how to compute the CF’s of a definite model, the \mathcal{M}_3^2 , and of its reductions \mathcal{M}_3^1 and \mathcal{M}_3^0 , in genus 0 by explicit use of the integrable hierarchy. As we shall see, once one knows the hierarchy of flow equations the CF’s are almost completely known. The role of the W –constraints is very limited. Let us start with \mathcal{M}_3^2 .

5.1 The model \mathcal{M}_3^2

The model \mathcal{M}_3^2 is specified by the Lax operator

$$L = \partial + a_1 \frac{1}{\partial - S_1} + a_2 \frac{1}{\partial - S_2} \frac{1}{\partial - S_1} \quad (5.1)$$

which arises in the context of the previous section when we consider a potential V_2 of highest order 3. We have therefore altogether four fields a_1, a_2, S_1, S_2 . The Poisson brackets, the hamiltonians and some of the flow equations have been given in [18].

The calculus of CF’s via the hierarchy can be subdivided in several steps.

Step 1. Meaning of the fields. We specify the connection of the fields characterizing the model with the correlation functions. Due to equation (2.16) we have

$$a_1 = \ll \tau_1 \tau_1 \gg \quad (5.2)$$

In the following we will often denote τ_1 by P .

The interpretation for a_2 can be inferred from the second flow equation for a_1 , [18]:

$$\frac{\partial a_1}{\partial t_2} = (a_1' + 2a_2 + 2a_1 S_1)' \quad (5.3)$$

Therefore we have

$$a_2 = \frac{1}{2} \ll \tau_2 P \gg - \frac{1}{2} \ll PPP \gg - \ll PP \gg S_1 \quad (5.4)$$

via formal integration of eq.(5.3). Formal integration is an operation we will often use in the following. It is the formal inverse of ∂ (no integration constant involved) and is a proper operation in the context of integrable hierarchies whenever the pseudodifferential calculus applies. For example it is certainly correct when, as above, we apply it to expressions containing only the abstract symbols of the fields, it may be incorrect if applied to a particular solution. In the latter case one must be careful about the constants of integration.

Going back to eq.(5.4), we see that the meaning of a_2 in terms of correlation functions is clear once the meaning of S_1 is clear. An interpretation for the latter and for S_2 can be derived from the first flows. In \mathcal{M}_3^2 the first flows are

$$\begin{aligned} a'_0(N) &= a_1(N+1) - a_1(N) \\ a'_1(N) &= a_2(N+1) - a_2(N) + a_1(N)(a_0(N) - a_0(N-1)) \\ a'_2(N) &= a_2(N)(a_0(N) - a_0(N-2)) \end{aligned}$$

Let us introduce the operator $D_0 = e^{\partial_0} - 1$ defined by

$$D_0 f_N = f_{N+1} - f_N$$

for any function f depending on the discrete index N . Using the definition of the S fields, the first flows can be rewritten as

$$S'_1 + D_0 S'_1 = D_0 a_1 \tag{5.6a}$$

$$a'_1 = D_0 a_2 + a_1 D_0 S_1 \tag{5.6b}$$

$$a'_2 = a_2 (D_0 S_1 + D_0 S_2) \tag{5.6c}$$

It is very convenient at this point to consider $t_0 \equiv N$ on the same footing as the couplings t_k and to associate to it an “operator” Q , in the same way as t_1 is associated to P , etc.. The label $_0$ used above was motivated by the fact that $\partial_0 = \frac{\partial}{\partial t_0}$ mimics the derivative with respect to N . Keeping track of the operator Q is tantamount to studying the N dependence in the CF’s. Then (5.6a) implies

$$S_1 = \ll (1 - e^{-Q})P \gg, \quad S_2 = \ll e^{-Q}(1 - e^{-Q})P \gg \tag{5.7}$$

Similarly, eqs.(5.6c) and (5.6b) become

$$a_2 = \exp \left(\ll (e^Q - 1)(1 - e^{-2Q}) \gg \right) f \tag{5.8a}$$

$$D_0 a_2 = \ll PPP \gg - \ll PP \gg \ll (e^Q - 1)(1 - e^{-Q})P \gg \tag{5.8b}$$

where f is an integration constant which in general might depend on all the couplings except t_1 . These two are compatibility conditions which come from the request that ∂ and ∂_0 commute.

Eqs.(5.2, 5.4) and (5.7) provide the connection between fields and CF’s we were looking for.

Step 2. The genus 0 contribution. In this section we will be interested in the genus 0 part of the CF’s, which corresponds to considering the dispersionless limit of the hierarchy, i.e. the limit in which, according to the degree analysis in section 3, only the first derivative terms (in t_1 and t_0) are retained (see [35],[36]). This corresponds to assigning the following degrees

$$[t_k] = x - k \quad k \geq 0, \quad [\mathcal{F}^{(0)}] = 2x$$

as in section 3, and moreover

$$[a_1^{(0)}] = 2, \quad [a_2^{(0)}] = 3, \quad [S_1^{(0)}] = [S_2^{(0)}] = 1$$

and keeping the leading order terms in every equation. In this limit the two fields $S_1^{(0)}$ and $S_2^{(0)}$ collapse to a single field $S^{(0)}$ for

$$S_1^{(0)} = \ll PQ \gg = S_2^{(0)}$$

From now on in this subsection we drop the label (0) which indicates the genus 0 contribution since only the latter will be considered. With this understanding the first few dispersionless flow equations are collected in Appendix B1. The compatibility equations (5.8a) and (5.8b) become

$$a_2 = e^{2\ll QQ \gg} f_0(t_0) \quad (5.9a)$$

$$\ll PPP \gg = \partial_0 a_2 + \ll PP \gg \ll PQQ \gg \quad (5.9b)$$

Moreover, from (5.4),

$$a_2 = \frac{1}{2} \ll \tau_2 P \gg - \ll PP \gg \ll PQ \gg \quad (5.10)$$

Step 3. The first critical point. For the type of solutions we are looking for the first critical point is by definition the one for which the correlation functions contain only nonnegative integral powers of the couplings. In this model this circumstance is implemented if

$$3t_3 = -1, \quad t_k = 0 \quad k > 3 \quad (5.11)$$

We want to preserve homogeneity, therefore in the above degree assignment we set $x = 3$, i.e.

$$[t_k] = 3 - k \quad k \geq 0, \quad [a_1] = 2, \quad [a_2] = 3, \quad [S] = 1, \quad [\mathcal{F}^{(0)}] = 6 \quad (5.12)$$

The CF's will therefore be homogeneous functions of N, t_1, t_2 . At times we will refer to this subset of the parameter space as the ‘‘small phase space’’.

This choice of the critical point corresponds, in the path integral language, to considering the potential

$$V_1 = t_1 \tau_1 + t_2 \tau_2 - \frac{1}{3} \tau_3 \quad (5.13)$$

as the interacting potential, while the other τ_k are regarded as external fields.

Step 4. Integrating the flow equations. From the homogeneity ansatz we have

$$S = at_2, \quad a_1 = bt_1 + ct_2^2, \quad a_2 = dt_0 + et_1 t_2 + ft_2^3$$

where a, b, c, d, e, f are numerical constants to be determined. If we plug these expressions in the t_2 dispersionless flow equations

$$\frac{\partial a_1}{\partial t_2} = (2a_2 + 2a_1 S)' \quad (5.14)$$

$$\frac{\partial a_2}{\partial t_2} = 2a_2' S + 4a_2 S' \quad (5.15)$$

$$\frac{\partial S}{\partial t_2} = (S^2 + 2a_1)' \quad (5.16)$$

we find

$$a = 2b, \quad c = 2b^2, \quad e = 0, \quad f = 0$$

Then we can write

$$\begin{aligned} \langle PP \rangle &= \frac{1}{2}(at_1 + a^2t_2^2) \\ \langle \tau_2 P \rangle &= 2dt_0 + a^2t_1t_2 + a^3t_2^3 \\ \langle PQ \rangle &= at_2 \end{aligned}$$

where, as in section 3, the symbol $\langle \cdot \rangle$ denotes the correlation function evaluated in the small phase space. The second equation is obtained from (5.14) via formal integration. Integrating these equations with respect to t_1, t_2 and t_0 respectively, and comparing the results*, we find $2d = a$ and

$$\langle P \rangle = \frac{1}{4}at_1^2 + at_0t_2 + \frac{1}{2}a^2t_1t_2^2 + \frac{1}{4}a^3t_2^4 \quad (5.17)$$

After this example it should be clear how to proceed in order to calculate $\langle \tau_2 \rangle$. We need to know $\langle \tau_2 P \rangle$, $\langle \tau_2 Q \rangle$ and $\langle \tau_2 \tau_2 \rangle$. The first two are obtained by formally integrating the t_2 flows of a_1 and S . The third is obtained in the following way. We differentiate eq.(5.10) with respect to t_2 and get

$$\frac{\partial a_2}{\partial t_2} = \frac{1}{2} \ll \tau_2 \tau_2 P \gg - a_1 \frac{\partial S}{\partial t_2} - \frac{\partial a_1}{\partial t_2} S$$

Then we plug eqs.(5.14) and (5.16) into this equation and obtain

$$\frac{1}{2} \ll \tau_2 \tau_2 P \gg = \left(4a_2 S + 2a_1 S^2 + a_1^2 \right)' \quad (5.18)$$

This can be formally integrated. So altogether we obtain

$$\begin{aligned} \ll \tau_2 \tau_2 \gg &= 8a_2 S + 4a_1 S^2 + 2a_1^2 \\ \ll \tau_2 P \gg &= 2a_2 + 2a_1 S \\ \ll \tau_2 Q \gg &= S^2 + 2a_1 \end{aligned}$$

Evaluating the RHS's in the small phase space, integrating and comparing as before, we find

$$\langle \tau_2 \rangle = at_0t_1 + 2a^2t_0t_2^2 + a^3t_1t_2^3 + \frac{1}{2}a^2t_1^2t_2 + \frac{1}{2}a^4t_2^5 \quad (5.19)$$

We can continue in this way and calculate the other one-point CF's, the pattern for the derivation does not change. They all depend on the constant a . This constant cannot be determined on the basis of the flow equations alone. Before passing to the

*The reason why we have to compare the results of the different integrations, i.e. keep track of the integration constants, is that the correlation functions we find do not belong to the space in which ∂^{-1} is a formal integral (for example the space of rapidly decreasing functions). This remark applies to all the correlation functions calculated in this paper and means that integrability is effective in a far larger space than the one in which the formal rules of the pseudodifferential calculus strictly apply.

determination of a , let us remark that while the RHS of eq.(5.15) is not a derivative w.r.t. t_1 , the RHS of (5.18) is. This circumstance, crucial for us to be able to calculate the CF (otherwise we would be left with an undetermined integration constant), is typical of all the flows and is rooted in the integrability of the hierarchy.

Step 5. The W -constraints. To determine a we need the W -constraints. Let us consider the $W_{-1}^{[1]}Z_N = 0$ constraint at $c = 0$. It takes the form

$$- \langle \tau_2 \rangle + 2t_2 \langle \tau_1 \rangle + t_0 t_1 = 0$$

Inserting the above expressions for $\langle \tau_2 \rangle$ and $\langle \tau_1 \rangle$ we find that the constant a must be equal to 1. This is a general fact: provided $a = 1$, the correlation functions calculated by integrating the flow equations satisfy all the W -constraints if $c = 0$ (and the $t_{2,k}$ couplings are forgotten). This confirms what we have already found in section 3 concerning the NLS correlation functions.

A sample of the one-point CF's is collected in Appendix C2. We recall that multi-point correlation functions involving only the operators Q, P, τ_2 can be obtained from these results by simply differentiating a suitable number of times $\langle P \rangle$ and $\langle \tau_2 \rangle$ w.r.t. t_0, t_1 and t_2 . For the multi-point CF's involving τ_k with $k \geq 3$, we have to first differentiate suitably chosen flow equations w.r.t. suitably chosen flow parameters, insert the flow equations in the expressions so obtained and proceed as above.

A remark is in order concerning the constant a determined in this section. The value of this constant $a = 1$ is directly connected with the chosen fixed point $3t_3 = -1$. The correlation function with generic a are appropriate to the choice $3t_3 = -\frac{1}{a}$. In other words a is a normalization constant[†]. Nevertheless the fact remains that once the critical point is fixed the hierarchy is not enough to completely determine the CF's. We need also the W -constraints.

Step 6. The CF $\langle Q \rangle$. This is the only one-point function we have not calculated yet. From the above results we have

$$\langle PQ \rangle = t_2, \quad \langle \tau_2 Q \rangle = t_1 + 2t_2^2$$

Integrating the first equation w.r.t t_1 and the second w.r.t. t_2 and comparing, we find

$$\langle Q \rangle = t_1 t_2 + \frac{2}{3} t_2^3 + y t_0 \tag{5.20}$$

where y is an undetermined numerical constant. This of course implies in particular that $\langle QQ \rangle = y$. Further information could a priori come from the compatibility eqs.(5.9a, 5.9b). However, inserting $\langle PPP \rangle = 1/2$ and $\langle PQQ \rangle = 0$, they only give us

$$f_0 = \frac{1}{2} e^{-2y t_0}$$

Since we have used all the information at hand, the number y remains undetermined.

[†]A normalization problem is always present in this kind of models and, when comparing CF's obtained in different contexts or with different methods, one should be careful about the normalization of both the coupling constants and the critical point.

Finally, integrating $\langle Q \rangle$, $\langle P \rangle$ and $\langle \tau_2 \rangle$ w.r.t. t_0, t_1 and t_2 respectively, and comparing the results we find

$$\mathcal{F}^{(0)} \equiv \langle 1 \rangle = t_0 t_1 t_2 + \frac{2}{3} t_0 t_2^3 + \frac{1}{4} t_1^2 t_2^2 + \frac{1}{2} t_1 t_2^3 + \frac{1}{12} t_1^3 + \frac{1}{2} y t_0^2$$

This completes our genus 0 analysis of the model \mathcal{M}_3^2 .

5.2 The model \mathcal{M}_3^1 .

The model \mathcal{M}_3^1 is obtained from \mathcal{M}_3^2 via hamiltonian reduction – the constraint is $S_1 = 0$ –, and it is specified by the Lax operator

$$L = \partial^2 + a_1 + a_2 \frac{1}{\partial - S_2} \quad (5.21)$$

We have therefore three fields a_1, a_2 and $S_2 \equiv S$. The Poisson brackets, the hamiltonians and some of the flow equations have been given in [18].

The connection of the fields a_1 and S_2 with the correlation functions is the same as before. The interpretation for a_2 is slightly different from the previous one. In fact from the second flow equation for a_1 , [18]:

$$\frac{\partial a_1}{\partial t_2} = 2a_2'$$

we deduce via formal integration

$$a_2 = \frac{1}{2} \ll \tau_2 P \gg \quad (5.22)$$

The first critical point and the degree assignment are the same as in the previous model.

From now on we will be dealing with genus 0 and before we proceed let us summarize the basic relations. We have

$$3t_3 = -1, \quad t_k = 0 \quad k > 3$$

and

$$[t_k] = 3 - k \quad k \geq 0, \quad [a_1] = 2, \quad [a_2] = 3, \quad [S] = 1, \quad [\mathcal{F}^{(0)}] = 6$$

Moreover

$$a_1 = \ll PP \gg, \quad a_2 = \frac{1}{2} \ll \tau_2 P \gg, \quad S = \ll PQ \gg \quad (5.23)$$

and again Q is coupled to $t_0 \equiv N$. The CF's will therefore be homogeneous functions of N, t_1, t_2 . We have remarked above that this choice of the critical point corresponds to considering (5.13) as the interacting potential.

As before we now integrate the dispersionless flow equations (the first few are collected in Appendix B2). From the homogeneity ansatz we have as before

$$S = at_2, \quad a_1 = bt_1 + ct_2^2, \quad a_2 = dt_0 + et_1 t_2 + ft_2^3$$

where a, b, c, d, e, f are constants to be determined. If we plug these expressions in the t_2 dispersionless flow equations

$$\frac{\partial a_1}{\partial t_2} = 2a'_2, \quad \frac{\partial a_2}{\partial t_2} = 2(a_2 S)', \quad \frac{\partial S}{\partial t_2} = (S^2 + a_1)'$$

we find

$$a = b, \quad c = 0, \quad e = 0, \quad f = 0$$

Then we can write

$$\begin{aligned} \langle PP \rangle &= at_1 \\ \langle \tau_2 P \rangle &= 2dt_0 \\ \langle PQ \rangle &= at_2 \end{aligned}$$

Integrating these equations with respect to t_1, t_2 and t_0 respectively, and comparing the results, we find $2d = a$ and

$$\langle P \rangle = \frac{1}{2}at_1^2 + at_0t_2 \tag{5.24}$$

Proceeding now in the same way as before and, in particular, using the identification of a_2 contained in (5.23), we find

$$\langle \tau_2 \rangle = at_0t_1 + a^2t_0t_2^2$$

as well as all the one–point functions we wish. All of them depend on the numerical constant a .

It would be natural at this point to plug the one–point functions we found into the W –constraints of subsection 2.2, as before. However one quickly realizes that this gives inconsistent results. *Those W –constraints are incompatible with the reduced hierarchy.* This is not surprising. One should remember the origin of the W –constraints. They come both from the string equations and the flow equations. Since we have reduced the hierarchy it is evident that we have to change accordingly also the W –constraints. Unfortunately we do not have a general theory for reducing the \mathcal{W} algebra and consequently the W –constraints. However we can easily do without it. Remember that the only piece of information we need is the constant a . It is not difficult to guess the form of the constraints (at least in the case $c = 0$) which are compatible with the one–point CF's we have calculated. Once this form is determined it will tell us what the constant a is. In fact, if we consider the case $c = 0$ and disregard the $t_{2,k}$ dependence, it is enough to calculate the Virasoro constraints, $L_n Z = 0^\ddagger$.

The way to proceed is as follows. We try the following form for L_{-1}

$$L_{-1} = \frac{1}{2} \sum_{k=3}^{\infty} kt_k \frac{\partial}{\partial t_{k-2}} + C_{-1} \tag{5.25}$$

[‡]The CF's of the model \mathcal{M}_3^2 do not satisfy Virasoro constraints such as $L_n Z = 0$. Therefore some models satisfy Virasoro constraints, while others do not. This is a phenomenological observation we do not have an explanation for.

Here C_{-1} is a term not containing derivatives w.r.t. the parameters. Now we impose $L_{-1}Z = 0$, i.e.

$$- \langle P \rangle + 2C_{-1} = 0$$

i.e.

$$C_{-1} = \frac{1}{4}at_1^2 + \frac{1}{2}at_0t_2$$

In a similar way we proceed with

$$L_0 = \frac{1}{2} \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_k} + C_0$$

and with the other generators. We quickly realize that there exists a solution only if

$$a = 2$$

What we have done is legitimate if the new generators close over an algebra of the Virasoro type and if the constraints $L_n Z = 0$ are satisfied by the CF's for $a = 2$. These new constraints are then compatible with the reduced hierarchy. These requirements are actually true, to the extent we have been able to verify them. The new generators are written down in Appendix A2 and the first few correlation functions are collected in Appendix C3.

5.3 The model \mathcal{M}_3^0 .

The model \mathcal{M}_3^0 is obtained from \mathcal{M}_3^1 via hamiltonian reduction – the constraint being $S_2 = 0$ –, and it is specified by the Lax operator

$$L = \partial^3 + a_1 \partial + a_2 \tag{5.26}$$

This is the Lax operator which gives rise to the Boussinesq hierarchy. We have therefore two fields a_1, a_2 . The Poisson brackets, the hamiltonians and some of the flow equations have been given in [18]. In the Boussinesq hierarchy the t_{3k} flows with $k = 1, 2, 3, \dots$ do not appear[§]. It is therefore natural to ignore $t_0 \equiv N$. This is of course consistent only if we find a solution which does not depend on t_k with $k = 0 \pmod{3}$. This remarkable fact indeed comes true.

The connection of the fields a_1 and a_2 with the correlation functions is the same as in the model \mathcal{M}_3^0

$$a_1 = \ll PP \gg, \quad a_2 = \frac{1}{2} \ll \tau_2 P \gg, \tag{5.27}$$

But the first critical point and the degree assignment are different. Precisely we find a solution with the desired properties if we set

$$4t_4 = -1, \quad t_k = 0 \quad k > 4$$

[§]From the reduction procedure from \mathcal{M}_3^1 it is possible to define t_3, t_6, \dots flows in the reduced model, but they are not integrable.

and

$$[t_k] = 4 - k \quad k \geq 0, \quad [a_1] = 2, \quad [a_2] = 3, \quad [\mathcal{F}^{(0)}] = 8$$

We concentrate now on genus 0, as before. The CF's will therefore be homogeneous functions of t_1 and t_2 . As before we now integrate the dispersionless flow equations (the first few can be derived from Appendix B3). From the homogeneity ansatz we have

$$a_1 = at_2, \quad a_2 = bt_1$$

where a and b are numerical constants to be determined. Inserting this into the t_2 flow we find $a = 2b$. Using (5.27) we find

$$\langle P \rangle = at_1t_2$$

Similarly from

$$\langle \tau_2 P \rangle = at_1, \quad \langle \tau_2 \tau_2 \rangle = -\frac{2}{3}a^2t_2^2$$

we obtain

$$\langle \tau_2 \rangle = \frac{1}{2}at_1^2 - \frac{2}{9}a^2t_2^3$$

Moreover

$$\langle \tau_4 \rangle = \frac{1}{3}a^2t_1^2t_2 - \frac{2}{27}a^3t_2^4$$

and so on. As before we are left with an undetermined numerical constant a which only the appropriate W -constraints can fix.

We proceed as in the previous example to determine the W -constraints appropriate for this model. For example we guess the following form for $\mathcal{L}_{-1}^{[1]}$

$$\mathcal{L}_{-1}^{[1]} = \frac{1}{3} \sum_{k=4}^{\infty} 'kt_k \frac{\partial}{\partial t_{k-3}} + C_{-1} \quad (5.28a)$$

$$\mathcal{L}_0^{[1]} = \frac{1}{3} \sum_{k=1}^{\infty} 'kt_k \frac{\partial}{\partial t_k} + C_0 \quad (5.28b)$$

where the prime on the summation symbol means that the sum is over all $k > 0$ excluding the multiples of 3. Next we impose

$$\mathcal{L}_{-1}^{[1]}Z = 0 \quad \text{and} \quad \mathcal{L}_0^{[1]}Z = 0$$

We find that these conditions imply

$$a = 6, \quad C_{-1} = 2t_1t_2, \quad C_0 = \frac{1}{9}$$

A more thorough analysis confirms this and shows that the constraints compatible with the Boussinesq hierarchy are

$$\mathcal{L}_n^{[r]}Z = 0, \quad r = 1, 2, \quad n \geq -r \quad (5.29)$$

The generators $\mathcal{L}_n^{[r]}$ with $r = 1, 2$ form a closed quadratic algebra, the W_3 algebra, see Appendix A3. It corresponds to the version calculated in [33] with central charge equal to 2. The first few one-point correlation functions of this model are collected in Appendix C4.

Let us stress the distinctive features of the Boussinesq model:

- 1) the CF's do not depend on N ;
- 2) the appropriate \mathcal{W} algebra closes over $\mathcal{L}_n^{[1]}$ and $\mathcal{L}_n^{[2]}$;
- 3) the coupling constant c does not come into play.

6 Other models

6.1 General recipe

The three models discussed above describe essentially all the features of the \mathcal{M}_p^l . And, as far as the method is concerned, it remains the same for all the other models. For all the \mathcal{M}_p^l with $1 \leq l \leq p-1$ the first critical point is fixed by the condition

$$pt_p = -1, \quad t_k = 0 \quad k > p \quad (6.1)$$

and the degree assignment is

$$\begin{aligned} [t_k] &= p - k, & [\mathcal{F}] &= 2p \\ [S] &= 1, & [a_1] &= 2, \dots, [a_{p-1}] = p \end{aligned} \quad (6.2)$$

where S denote the genus 0 part of the fields S_i , which is common to all of them. The CF's will be homogeneous functions of t_0, t_1, \dots, t_{p-1} .

For the models \mathcal{M}_p^0 (the p -th KdV models) instead, one must first of all disregard all the t_k with k a multiple of p ; the first critical point is

$$(p+1)t_{p+1} = -1, \quad t_k = 0 \quad k > p+1 \quad (6.3)$$

and the degree assignment is

$$\begin{aligned} [t_k] &= p+1 - k, & [\mathcal{F}] &= 2p+2 \\ [S] &= 1, & [a_1] &= 2, \dots, [a_{p-1}] = p \end{aligned} \quad (6.4)$$

The CF's will be homogeneous functions of t_1, \dots, t_{p-1} .

In all the cases the method consists in fixing the form of the fields on the basis of homogeneity; this leaves a few undetermined numerical constants; inserting the fields into the flow equations determines them up to an overall constant; in turn this can be fixed via the appropriate W -constraints. The latter take the universal form (2.18) in all the unreduced \mathcal{M}_p^{p-1} models. In the other cases they can be constructed on the basis of the compatibility with the hierarchy characteristic of the model.

6.2 Other examples

For completeness we have explicitly worked out two more examples in genus 0, \mathcal{M}_4^3 and \mathcal{M}_4^0 . In the former the fields take the form

$$S = t_3, \quad a_1 = \frac{2}{3}t_2 + t_3^2, \quad a_2 = \frac{1}{3}t_1 + \frac{2}{3}t_2t_3 + \frac{2}{3}t_3^3, \quad a_3 = \frac{1}{3}t_0 \quad (6.5)$$

The first few dispersionless flow equations are collected in Appendix B4 and the first few one–point CF’s are in Appendix C5.

As for model \mathcal{M}_4^0 we have

$$a_1 = 12t_3, \quad a_2 = 8t_2, \quad a_3 = 4t_1 + 18t_3^2 \quad (6.6)$$

The first few dispersionless flows and one–point CF’s are in Appendix B5 and C6, respectively. The generators for the appropriate W –constraints are to be found in Appendix A4.

7 Higher genus contributions

Once the genus zero correlation functions of a given model are known, it is an easy task (at least in principle) to calculate the CF’s in higher genus. There are several methods one can adopt. The most immediate one is to use the W –constraints. These are either the original constraints (2.18) or the effective constraints we calculated in the previous sections. The important point is that, although the latter have been determined from the dispersionless versions of the relevant hierarchies, they are valid in general and provide the contributions to any genus. An example will suffice to illustrate this point.

Let us consider the NLS model, which we met at the end of section 3 and was studied in detail in [16]. Eq.(3.10) gives us the effective Virasoro constraints. We have seen that the genus zero contribution is represented by the highest order terms with respect to the degree assignment (3.4, 3.5). An inspection of eqs.(3.10, 3.11) tells us that the second nontrivial contribution will have degree $2x$ less than the leading order, the third $4x$ less, etc. Therefore the free energy has a genus expansion

$$\mathcal{F} = \sum_{h=0}^{\infty} \mathcal{F}^{(h)}, \quad [\mathcal{F}^{(h)}] = 2x(1-h) \quad (7.1)$$

Inserting this into (3.10) we find

$$\begin{aligned} & \sum_{k=2}^{\infty} kt_k \ll \tau_{k+n} \gg_h + 2N \ll \tau_n \gg_h + \sum_{k=1}^{n-1} \left(\ll \tau_k \tau_{n-k} \gg_{h-1} \right. \\ & \left. + \sum_{h'=0}^h \ll \tau_k \gg_{h'} \ll \tau_{n-k} \gg_{h-h'} \right) + (N^2 \delta_{n,0} + Nt_1 \delta_{n,-1}) \delta_{h,0} = 0 \quad (7.2) \end{aligned}$$

where $\ll \cdot \gg_h$ represents the genus h contribution to the $\ll \cdot \gg$ CF. Eqs.(7.2) are recursive, therefore it is easy to turn the crank and calculate the higher genus contributions

to the order we wish from those of genus 0. A few examples at the critical point $2t_2 = -1$ are given in Appendix D.

From the structure of the W –constraints appearing in two–matrix models it is easy to see that genus recursiveness is a general characteristic. Therefore computability of higher genus contributions is guaranteed. We recall that genus recursiveness is typical of topological field theories.

Clearly it is possible to calculate higher genus correlation functions starting from the relevant hierarchies and computing the genus expansion of them. This is the best method if one wishes compact formulas for the CF’s.

8 Higher critical points

The first critical point by definition implies a dependence of the basic fields on the couplings specified by homogeneous polynomials (i.e. with non-negative integer powers). Higher critical points are characterized still by a homogenous dependence, but with rational and/or negative powers of the couplings. For the models of type \mathcal{M}_p^l with $l = 1, \dots, p-1$, characterized by a small phase space with parameters t_0, t_1, \dots, t_{p-1} , higher critical points are specified by

$$kt_k = \pm 1, \quad k > p, \quad t_l = 0 \quad \text{for } l > k \quad (8.1)$$

and the degree assignment is *

$$\begin{aligned} [t_l] &= k - l, & [\mathcal{F}] &= 2k \\ [S] &= 1, & [a_1] &= 2, \dots, [a_{p-1}] = p \end{aligned} \quad (8.2)$$

For the models of type \mathcal{M}_p^0 (p-th KdV hierarchy) characterized by a small phase space with parameters t_1, \dots, t_{p-1} , higher critical points are specified by

$$kt_k = \pm 1, \quad k > p + 1, \quad t_l = 0 \quad \text{for } l > k, \quad k \neq np \quad (8.3)$$

and the degree assignment is †

$$\begin{aligned} [t_l] &= k - l, & [\mathcal{F}] &= 2k \\ [a_1] &= 2, \dots, [a_{p-1}] = p \end{aligned} \quad (8.4)$$

The \pm sign in eqs.(8.1, 8.3) has to be chosen in such a way as to avoid complex coefficients in the small phase space expressions of the basic fields.

A few examples will suffice. For the model \mathcal{M}_3^0 the second critical point is fixed by

$$5t_5 = 1, \quad t_l = 0 \quad l > 5$$

and, for simplicity, we set $t_4 = 0$. Consequently the degree assignment is

$$[t_l] = 5 - k, \quad [\mathcal{F}] = 10, \quad [a_1] = 2, \quad [a_2] = 3$$

It is convenient at this point to take immediately into account the constraint $\mathcal{L}_{-1}^{[1]}Z = 0$. It reads

$$\langle \tau_2 \rangle = -6t_1t_2 \quad (8.5)$$

Now eq.(5.27) implies $a_2 = -3t_2$. Using again the second flow equations (Appendix B3) we also find

$$a_1 = 3t_1^{\frac{1}{2}} \quad (8.6)$$

*The parameters t_0, t_1, \dots, t_{p-1} are the self-coupling parameters of the model, while t_p, \dots, t_{k-1} are to be regarded as external couplings. If one wants models in which all t_1, t_2, \dots, t_{k-1} are self-interacting parameters one should look at models of the type \mathcal{M}_k^l .

†The t_{p+1}, \dots, t_{k-1} are to be regarded as external couplings, see previous footnote.

At this point it is easy to calculate the correlation functions following the method of section 5. A few of them are collected in Appendix C4.

Another example is the second critical point of the model \mathcal{M}_4^0 :

$$6t_6 = -1, \quad t_5 = 0, \quad t_l = 0 \quad l > 6$$

Consequently the degree assignment is

$$[t_l] = 6 - k, \quad [\mathcal{F}] = 12, \quad [a_1] = 2, \quad [a_2] = 3, \quad [a_3] = 4$$

The constraint $\mathcal{L}_{-1}^{[1]}Z = 0$ tells us that

$$\langle \tau_2 \rangle = 8t_2^2 + 12t_1t_3 \quad (8.7)$$

From this and the second flow equations we can derive

$$a_1 = -\frac{4}{3}t_1t_3^{-1}, \quad a_2 = 6t_3, \quad a_3 = \frac{4}{9}t_1^2t_3^{-2} + 4t_2 \quad (8.8)$$

from which we can easily derive the CF's (see Appendix C6).

We conclude that we can at least in principle (i.e. modulo technical difficulties) solve the \mathcal{M}_p^l models at higher critical points as well.

8.1 Non-perturbative equations

This paper privileges genus expanded solutions. We do not want to get involved here into the problems of non-perturbative solutions. However it is useful to spend a few words at least on the way we can extract non-perturbative equations. We show a few examples. The first is the NLS model (\mathcal{M}_2^1 model) at the second critical point $3t_3 = -1, t_2 = 0$. Differentiating the $W_{-1}^{[1]}Z = 0$ at $c = 0$ w.r.t. t_0 and t_1 and using the second flow equations one finds

$$R' + 2RS = t_0, \quad -S' + 2R + S^2 = t_1 \quad (8.9)$$

where, as usual, we set $a_1 = R$. These two equations must be supplemented with the t_0 flows:

$$\left(e^{\partial_0} - 1\right)R = e^{\partial_0}S', \quad \left(e^{\partial_0} - 1\right)S = (\ln R)' \quad (8.10)$$

The above equations specify the t_0, t_1 dependence of R and S .

The second example is the second critical point of the \mathcal{M}_3^0 model, which has been specified above. Differentiating w.r.t. t_1 and t_2 the constraint $\mathcal{L}_{-1}^{[1]}$ and using the second flow equations we obtain

$$2a_2 - a_1' + 6t_2 = 0, \quad 4a_2' - \frac{7}{3}a_1'' - \frac{2}{3}a_1^2 + 6t_1 = 0 \quad (8.11)$$

These should be supplemented with the (complete) second flow equations

$$\frac{\partial a_1}{\partial t_2} = 2a_2' - a_1'', \quad \frac{\partial a_2}{\partial t_2} = a_2'' - \frac{2}{3}a_1''' - \frac{2}{3}a_1a_1' \quad (8.12)$$

The four equations above specify the non-perturbative dependence of a_1, a_2 on t_1 and t_2 .

An interesting example is the next critical point in the same model, \mathcal{M}_3^0 , i.e.

$$7t_7 = -1, \quad t_4 = 0, \quad t_l = 0, \quad l > 7 \quad (8.13)$$

and t_5 is an external coupling. Differentiating the constraint $\mathcal{L}_{-1}^{[1]}Z = 0$ with respect to t_1 and t_2 we obtain

$$\begin{aligned} - \ll \tau_4 P \gg + 5t_5 \ll \tau_2 P \gg + 6t_2 &= 0 \\ - \ll \tau_4 \tau_2 \gg + 5t_5 \ll \tau_2 \tau_2 \gg + 6t_1 &= 0 \end{aligned}$$

respectively. Now, using the t_2 and t_4 flows (see Appendix B3), we find

$$\begin{aligned} a_1''' - 2a_2'' + 2a_1a_1' - 4a_1a_2 + 15t_5(2a_2 - a_1') + 18t_2 &= 0 \\ \frac{1}{3}a_1'''' - 4a_2^2 + 2a_1a_1'' + a_1'a_1' + 4a_1'a_2 + \frac{8}{3}a_1^3 + 15t_5\left(-\frac{1}{2}a_1'' - \frac{2}{3}a_1^2\right) + 18t_1 &= 0 \end{aligned} \quad (8.14)$$

With the redefinition

$$a_1 \rightarrow a_1 - \frac{15}{2}t_5, \quad a_2 \rightarrow a_2 - \frac{1}{2}a_1'$$

and a shift of t_1 , we obtain the same equations derived in ref.[3] (see also [24]) for the Ising model on discretized Riemann surfaces (up to normalization of the quantities involved): t_1, t_2 and t_5 are related to the cosmological constant, the magnetic field and the parameter T of [3], respectively. Eqs.(8.14) together with (8.12) determine the t_1, t_2 dependence of a_1, a_2 .

9 Interpretation and comments

Let us briefly introduce in this last section two points that require a longer and more careful elaboration: a possible connection of the previous models with 2D gravity and topological field theories. We will then comment on some leftover problems.

On the basis of the continuum theory one would expect the chemical potential (to be assimilated to a_1) of 2D gravity coupled to conformal matter in genus 0, to behave like

$$a_1 \sim t^{\frac{2}{p_0+q_0-1}} \quad (9.1)$$

where t is proportional to the cosmological constant, and p_0, q_0 are relatively prime integers. From the above examples and general statements one can see that the behaviour of the models \mathcal{M}_p^l at the various critical points can abundantly account for (9.1), provided we interpret either t_1 or t_0 as t .

Let us come now to the interpretation of the models we have presented in this paper in terms of topological field theories coupled to topological gravity. The latter models are completely specified once one gives the primaries \mathcal{O}_α , the metric $\eta_{\alpha\beta}$ and the fusion coefficients $C_{\alpha\beta}^\gamma$. As primaries of our models we can take the fields coupled to the parameters of the small phase space, η and C can be derived from the CF's of

the primaries at the first critical point. For example, for the model \mathcal{M}_3^1 the primaries can be chosen to be: $\tau_0 \equiv Q$, $\tau_1 \equiv P$ and τ_2 and by inspection of Appendix C3 we find

$$\eta_{02} = \eta_{11} = \eta_{20} = 2$$

while the remaining $\eta_{\alpha\beta}$ vanish. For some models, like the \mathcal{M}_p^0 ones, a similar identification can be very easily carried out (see also [24]). There is therefore room for an interpretation in terms of topological field theories. However further elaboration is required. We defer a complete discussion on this point as well as a study of the connection with Landau–Ginsburg theories to a forthcoming paper.

It remains for us to point out a few problems concerning the completeness of the proofs and arguments presented in this paper. The first remark concerns the statements, made in section 5 and 6, about the agreement between the calculated correlation functions and the W -constraints. The agreement has been checked up to a remarkable order (seventh, eighth or even higher order CF) with a computer. In order to produce a rigorous proof one should be able to find compact formulae for the CF's, analogous to those we exhibited in [16]. These formulae are perhaps not beyond reach.

A second question is: did we study in this paper all the models contained in the two-matrix theory? We recall that for any small phase space of the type t_1, t_2, \dots, t_{p-1} or t_0, t_1, \dots, t_{p-1} , for any p , we presented one or more models. Does two-matrix theory describe other integrable models with a more peculiar small phase space (such as, for example, t_1, t_2, t_4, t_7)? To answer this question one has to ascertain whether there exist other reductions beside those considered in [18],[21].

An entirely new problem is posed by the calculation of the correlation functions with $c \neq 0$, as we did in section 3, but via the integrable hierarchies. In this case we have to consider two integrable hierarchies, the first (system I) and the second (system II). The novelty consists in the fact that we want to know the dependence of the solutions of the first on the parameters of the second and vice versa, plus the dependence on c . In [17] we showed that the corresponding flows commute.

Finally a few words on general multi-matrix models (with bilinear couplings). The results of [17] compared with those of [21] tell us that very little has to be expected in terms of new integrable hierarchies and new models. Of course if one wishes to study CF's dependent on all the couplings (see the previous paragraph) the problem presents new and perhaps interesting aspects.

Appendices

Appendix A1

The basic $\mathcal{L}_n^{[r]}(1)$ generators that appear in (2.18) are

$$\begin{aligned} \mathcal{L}_n^{[1]}(1) \equiv & \sum_{k=1}^{\infty} k t_{1,k} \frac{\partial}{\partial t_{1,k+n}} + (N + \frac{n+1}{2}) \frac{\partial}{\partial t_{1,n}} + \frac{1}{2} \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_{1,k} \partial t_{1,n-k}} \\ & + N t_{1,1} \delta_{n,-1} + \frac{1}{2} N(N+1) \delta_{n,0}, \quad n \geq -1. \end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_n^{[2]}(1) &\equiv \sum_{l_1, l_2=1}^{\infty} l_1 l_2 t_{1, l_1} t_{1, l_2} \frac{\partial}{\partial t_{1, l_1 + l_2 + n}} + \sum_{l=1}^{\infty} l t_{1, l} \sum_{k=1}^{l+n-1} \frac{\partial^2}{\partial t_{1, k} \partial t_{1, l+n-k}} \\
&+ \frac{1}{3} \sum_{l=1}^{n-2} \sum_{k=1}^{n-l-1} \frac{\partial^3}{\partial t_{1, l} \partial t_{1, k} \partial t_{1, n-l-k}} \\
&+ (N + \frac{n}{2} + 1) \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_k \partial t_{n-k}} + (2N + n + 2) \sum_{l=1}^{\infty} l t_l \frac{\partial}{\partial t_{n+l}} \\
&+ (N^2 + (n+2)N + \frac{1}{3}(n+1)(n+2)) \frac{\partial}{\partial t_{1, n}} \\
&+ (2N^2 t_{1, 2} + N t_{1, 1}^2) \delta_{n, -2} + N(N+1) t_{1, 1} \delta_{n, -1} \\
&+ \frac{1}{3} N(N+1)(N+2) \delta_{n, 0}, \quad n \geq -2.
\end{aligned}$$

In the above formulas it is understood that when $t_{1, k}$ appear with $k \leq 0$, the corresponding term is absent. As already noted the higher rank generators can be derived from the algebra \mathcal{W} itself.

Actually it remains for us to specify what the $\mathcal{L}_n^{[0]}(1)$'s are. These are given by

$$\mathcal{L}_n^{[0]} \equiv \frac{\partial}{\partial t_{1, n}} + N \delta_{n, 0}, \quad n \geq 0$$

These generators represent an abelian extension of the \mathcal{W} algebra, of which they form an abelian subalgebra. The other generators behave tensorially with respect to $\mathcal{L}_n^{[0]}$. However since the latter play a minor role in this paper, we do not insist on this point.

Appendix A2

Here are the generators of the Virasoro algebra for the reduced model \mathcal{M}_3^1 :

$$L_{-1} = \frac{1}{2} \sum_{k=3}^{\infty} k t_k \frac{\partial}{\partial t_{k-2}} + \frac{1}{2} t_1^2 + N t_2,$$

$$L_0 = \frac{1}{2} \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + \frac{3}{8} N^2 + \frac{1}{16},$$

$$L_n = \frac{1}{2} \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+2n}} + \frac{1}{4} N(n+2) \frac{\partial}{\partial t_{2n}} + \frac{1}{8} \sum_{k=1}^{2n-1} \frac{\partial^2}{\partial t_k \partial t_{2n-k}}, \quad n \geq 1$$

Appendix A3

The generators of the \mathcal{W} algebra appropriate for the \mathcal{M}_3^0 (Boussinesq) model are

$$\mathcal{L}_n^{[1]} = \frac{1}{3} \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+3n}} + \frac{1}{18} \sum_{\substack{k, l \\ k+l=3n}} \frac{\partial^2}{\partial t_k \partial t_l} + \frac{1}{2} \sum_{\substack{k, l \\ k+l=-3n}} k l t_k t_l + \frac{1}{9} \delta_{n, 0}, \quad \forall n$$

$$\begin{aligned}
\mathcal{L}_n^{[2]} &= \frac{1}{9} \sum_{l_1, l_2=1}^{\infty} l_1 l_2 t_{l_1} t_{l_2} \frac{\partial}{\partial t_{l_1+l_2+3n}} + \frac{1}{27} \sum_{\substack{l, k, j \\ l-k-j=-3n}} l t_l \frac{\partial^2}{\partial t_k \partial t_j} \\
&+ \frac{1}{243} \sum_{\substack{l, k, j \\ l+k+j=3n}} \frac{\partial^3}{\partial t_l \partial t_k \partial t_j} + \frac{1}{9} \sum_{\substack{l, k, j \\ l+k+j=-3n}} l k j t_l t_k t_j, \quad \forall n
\end{aligned}$$

In these expressions summations are limited to the terms such that no index involved is either negative or multiple of 3. Contrary to the previous cases we have introduced the generators for all n . In this way the above two sets of generators form a closed algebra, the W_3 algebra, [31],[32],[12]

$$\begin{aligned}
[\mathcal{L}_n^{[1]}, \mathcal{L}_m^{[1]}] &= (n-m)\mathcal{L}_{n+m}^{[1]} + \frac{1}{6}(n^3-n)\delta_{n+m,0} \\
[\mathcal{L}_n^{[1]}, \mathcal{L}_m^{[2]}] &= (2n-m)\mathcal{L}_{n+m}^{[2]} \\
[\mathcal{L}_n^{[2]}, \mathcal{L}_m^{[2]}] &= -\frac{1}{54}(n-m)\left((n^2+m^2+4nm)+3(n+m)+2\right)\mathcal{L}_{n+m}^{[1]} \\
&+ \frac{1}{9}(n-m)\Lambda_{n+m} + \frac{1}{810}n(n^2-1)(n^2-4)\delta_{n+m,0}
\end{aligned}$$

where

$$\Lambda_n = \sum_{k \leq -1} \mathcal{L}_k^{[1]} \mathcal{L}_{n-k}^{[1]} + \sum_{k \geq 0} \mathcal{L}_{n-k}^{[1]} \mathcal{L}_k^{[1]}$$

This corresponds to the quantum W_3 algebra calculated in [33] when the central charge is 2.

Appendix A4

The generators of the \mathcal{W} algebra appropriate for the \mathcal{M}_4^0 model are

$$\begin{aligned}
\mathcal{L}_n^{[1]} &= \frac{1}{4} \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+4n}} + \frac{1}{32} \sum_{k=1}^{4n+1} \frac{\partial^2}{\partial t_k \partial t_{4n-k}} + \frac{1}{2} \sum_{k+l=-4n} k l t_k t_l + \frac{5}{32} \delta_{n,0}, \quad \forall n \\
\mathcal{L}_2^{[2]} &= \frac{1}{16} \sum_{l_1, l_2=1}^{\infty} l_1 l_2 t_{l_1} t_{l_2} \frac{\partial}{\partial t_{l_1+l_2-8}} + \frac{1}{64} \sum_{\substack{l, j, k \\ l-j-k=8}} l t_l \sum \frac{\partial^2}{\partial t_k \partial t_j} \\
&+ \frac{1}{12} \sum_{\substack{l, k, j \\ l+k+j=8}} l k j t_l t_k t_j,
\end{aligned}$$

In these expressions summations are limited to the terms such that no index involved is either negative or multiple of 4.

From the above generators we can generate the W_4 algebra.

Appendix B1

Here are the first few equations of motion of the model \mathcal{M}_3^2 in the dispersionless limit ($t_k = t_{1,k}$):

$$\frac{\partial a_1}{\partial t_2} = (2a_2 + 2a_1 S)', \quad \frac{\partial a_2}{\partial t_2} = 2a_2' S + 4a_2 S', \quad \frac{\partial S}{\partial t_2} = (S^2 + 2a_1)'$$

$$\begin{aligned} \frac{\partial a_1}{\partial t_3} &= (3a_1 S^2 + 6a_2 S + 3a_1^2)' \\ \frac{\partial a_2}{\partial t_3} &= 3a_2' S^2 + 6a_2 (S^2)' + 3a_1 a_2' + 6a_1' a_2 \\ \frac{\partial S}{\partial t_3} &= (S^3 + 6a_1 S + 3a_2)' \end{aligned}$$

$$\begin{aligned} \frac{\partial a_1}{\partial t_4} &= (4a_1 S^3 + 12a_2 S^2 + 12a_1^2 S + 12a_1 a_2)' \\ \frac{\partial a_2}{\partial t_4} &= 4a_2' S^3 + 8a_2 (S^3)' + 12a_1 a_2' S + 24a_2 (a_1 S)' \\ \frac{\partial S}{\partial t_4} &= (S^4 + 12a_1 S^2 + 12a_2 S + 6a_1^2)' \end{aligned}$$

Appendix B2

Here are the first few equations of motion of the model \mathcal{M}_3^1 in the dispersionless limit and with the convention $t_k = t_{1,k}$:

$$\frac{\partial a_1}{\partial t_2} = 2a_2', \quad \frac{\partial a_2}{\partial t_2} = 2(a_2 S)', \quad \frac{\partial S}{\partial t_2} = (S^2 + a_1)'$$

$$\begin{aligned} \frac{\partial a_1}{\partial t_3} &= \left(\frac{3}{4}a_1^2 + 3a_2 S\right)' \\ \frac{\partial a_2}{\partial t_3} &= \left(\frac{3}{2}a_1 a_2 + 3a_2 S^2\right)' \\ \frac{\partial S}{\partial t_3} &= \left(S^3 + \frac{3}{2}a_1 S + \frac{3}{2}a_2\right)' \end{aligned}$$

$$\begin{aligned} \frac{\partial a_1}{\partial t_4} &= (4a_2 S^2 + 4a_1 a_2)' \\ \frac{\partial a_2}{\partial t_4} &= (4a_2 S^3 + 2a_2^2 + 4a_1 a_2 S)' \\ \frac{\partial S}{\partial t_4} &= (S^4 + 2a_1 S^2 + 4a_2 S + a_1^2)' \end{aligned}$$

Appendix B3

Here are the first few (complete) equations of motion of the model \mathcal{M}_3^0 with the convention $t_k = t_{1,k}$. We disregard the t_k flows with $k = 0 \pmod 3$:

$$\begin{aligned}\frac{\partial a_1}{\partial t_2} &= 2a_2' - a_1'', & \frac{\partial a_2}{\partial t_2} &= a_2'' - \frac{2}{3}a_1''' - \frac{2}{3}a_1a_1' \\ \frac{\partial a_1}{\partial t_4} &= \left(-\frac{1}{3}a_1''' + \frac{2}{3}a_2'' - \frac{2}{3}a_1a_1' + \frac{4}{3}a_1a_2\right)' \\ \frac{\partial a_2}{\partial t_4} &= \left(\frac{1}{3}a_2''' - \frac{2}{9}a_1'''' + \frac{2}{3}a_1a_2' + \frac{2}{3}a_2^2 - \frac{2}{3}a_1a_1'' - \frac{1}{2}a_1'a_1' - \frac{4}{27}a_1^3\right)'\end{aligned}$$

Appendix B4

Here are the first few equations of motion of the model \mathcal{M}_4^3 in the dispersionless limit ($t_k = t_{1,k}$):

$$\begin{aligned}\frac{\partial a_1}{\partial t_2} &= (2a_2 + 2a_1S)', & \frac{\partial a_2}{\partial t_2} &= 2a_3' + 2a_2'S + 4a_2S', \\ \frac{\partial a_3}{\partial t_2} &= 2a_3'S + 6a_3S', & \frac{\partial S}{\partial t_2} &= (S^2 + 2a_1)' \\ \frac{\partial a_1}{\partial t_3} &= (3a_1^2 + 3a_3 + 3a_1S^2 + 6a_2S)' \\ \frac{\partial a_2}{\partial t_3} &= 3a_2S^2 + 6a_2(S^2)' + 6a_3'S + 9a_3S' + 3a_1a_2' + 6a_1'a_2 \\ \frac{\partial a_3}{\partial t_3} &= 3a_1a_3' + 9a_1'a_3 + 3a_3'S^2 + 9a_3(S^2)' \\ \frac{\partial S}{\partial t_3} &= (S^3 + 6a_1S + 3a_2)'\end{aligned}\tag{9.2}$$

Appendix B5

Here are the first few equations of motion of the model \mathcal{M}_4^0 in the dispersionless limit and with the convention $t_k = t_{1,k}$. We disregard the t_k flows with $k = 0 \pmod 4$:

$$\begin{aligned}\frac{\partial a_1}{\partial t_2} &= 2a_2', & \frac{\partial a_2}{\partial t_2} &= \left(2a_3 - \frac{1}{2}a_1^2\right)', & \frac{\partial a_3}{\partial t_2} &= -\frac{1}{2}a_1'a_2 \\ \frac{\partial a_1}{\partial t_3} &= \left(-\frac{3}{8}a_1^2 + 3a_3\right)', & \frac{\partial a_2}{\partial t_3} &= -\frac{3}{4}(a_1a_2)', & \frac{\partial a_3}{\partial t_3} &= -\frac{3}{8}(a_2^2)' + \frac{3}{4}a_1a_3'\end{aligned}$$

$$\begin{aligned}\frac{\partial a_1}{\partial t_5} &= \left(\frac{5}{8}a_2^2 + \frac{5}{4}a_1a_3 - \frac{5}{32}a_1^3\right)' \\ \frac{\partial a_2}{\partial t_5} &= \left(\frac{5}{4}a_2a_3 - \frac{15}{32}a_1^2a_2\right)' \\ \frac{\partial a_3}{\partial t_5} &= \frac{5}{8}(a_3^2)' + \frac{5}{32}a_1^2a_3' - \frac{5}{16}(a_1a_2)'a_2\end{aligned}$$

Appendix C1

In this Appendix we collect the first few genus 0 one-point CF's calculated in section 3. Small phase space: N, t_1 . Critical point: $2t_2 = -1$.

$$\begin{aligned}
\langle \tau_1 \rangle &= N \frac{t + cs}{1 - c^2}, & \langle \sigma_1 \rangle &= N \frac{s + ct}{1 - c^2} \\
\langle \tau_2 \rangle &= N \frac{s^2 + t^2 + 2cst}{(1 - c^2)^2} + \frac{N^2 - Ns^2}{1 - c^2} \\
\langle \tau_3 \rangle &= N \frac{t + cs}{(1 - c^2)^3} (c^2 s^2 + 2cst + t^2 + 3N - 3Nc^2) \\
\langle \tau_4 \rangle &= \frac{N}{(1 - c^2)^4} (c^4 s^4 + 4c^3 s^3 t + 6c^2 s^2 t^2 + 4cst^3 + t^4 \\
&\quad + 6Nc^2 s^2 - 6Nc^4 s^2 + 12Ncst - 12Nc^3 st \\
&\quad + 6Nt^2 - 6Nc^2 t^2 + 2N^2 - 4N^2 c^2 + 2N^2 c^4)
\end{aligned}$$

and in general

$$\langle \sigma_k \rangle (t, s) = \langle \tau_k \rangle (s, t)$$

Appendix C2

One-point correlation functions of the model \mathcal{M}_3^2 in genus 0. Small phase space: $t_0 \equiv N, t_1, t_2$. Critical point: $3t_3 = -1$.

$$\begin{aligned}
\langle Q \rangle &= t_1 t_2 + \frac{2}{3} t_2^3 + y t_0, \\
\langle P \rangle &= \frac{1}{4} t_1^2 + t_0 t_2 + \frac{1}{2} t_1 t_2^2 + \frac{1}{4} t_2^4, \\
\langle \tau_2 \rangle &= t_0 t_1 + 2t_0 t_2^2 + t_1 t_2^3 + \frac{1}{2} t_1^2 t_2 + \frac{1}{2} t_2^5, \\
\langle \tau_3 \rangle &= \frac{1}{4} t_1^3 + 3t_0 t_1 t_2 + \frac{3}{2} t_1^2 t_2^2 + \frac{9}{4} t_1 t_2^4 + \frac{3}{4} t_0^2 + 4t_0 t_2^3 + t_2^6, \\
\langle \tau_4 \rangle &= \frac{3}{2} t_0 t_1^2 t_2^3 + 9t_0 t_1 t_2^2 + 4t_1^2 t_2^3 + 5t_1 t_2^5 + 3t_0^2 t_2 + \frac{17}{2} t_0 t_2^4 + 2t_2^7, \\
\langle \tau_5 \rangle &= \frac{5}{2} t_0^2 t_1 + \frac{15}{2} t_0 t_1^2 t_2 + 25t_0 t_1 t_2^3 + \frac{15}{4} t_1^3 t_2^2 + \frac{85}{8} t_1^2 t_2^4 \\
&\quad + \frac{5}{16} t_1^4 + \frac{45}{4} t_1 t_2^6 + 10t_0^2 t_2^2 + \frac{37}{2} t_0 t_2^5 + \frac{65}{16} t_2^8
\end{aligned}$$

In the first equation y represents an undetermined numerical constant.

Appendix C3

One-point correlation functions of the model \mathcal{M}_3^1 in genus 0. Small phase space: $t_0 \equiv N, t_1, t_2$. First critical point: $3t_3 = -1$.

$$\begin{aligned}
\langle Q \rangle &= 2t_1t_2 + \frac{4}{3}t_2^3 + yt_0, & \langle P \rangle &= t_1^2 + 2t_0t_2, & \langle \tau_2 \rangle &= 2t_0t_1 + 4t_0t_2^2 \\
\langle \tau_3 \rangle &= t_1^3 + 6t_0t_1t_2 + 8t_0t_2^3 + \frac{3}{4}t_0^2 \\
\langle \tau_4 \rangle &= 4t_0t_1^2 + 16t_0t_1t_2^2 + 4t_0^2t_2 + 16t_0t_2^4 \\
\langle \tau_5 \rangle &= \frac{5}{4}t_1^4 + \frac{15}{4}t_1t_0^2 + 15t_0t_1^2t_2 + 40t_0t_1t_2^3 + 15t_0^2t_2^2 + 32t_0t_2^5
\end{aligned}$$

Appendix C4

One-point correlation functions of the Boussinesq model \mathcal{M}_3^0 in genus 0. Small phase space: t_1, t_2 . First critical point: $4t_4 = -1$.

$$\begin{aligned}
\langle \tau_1 \rangle &= 6t_1t_2, & \langle \tau_2 \rangle &= 3t_1^2 - 8t_2^3 \\
\langle \tau_4 \rangle &= 12t_1^2t_2 - 16t_2^4, & \langle \tau_5 \rangle &= 5t_1^3 - 40t_1t_2^3 \\
\langle \tau_7 \rangle &= 28t_1^3t_2 - 112t_1t_2^4, & \langle \tau_8 \rangle &= 10t_1^4 - 160t_1^2t_2^3 + \frac{256}{3}t_2^6
\end{aligned} \tag{9.3}$$

Here follow the one-point correlation functions of the same model at the second critical point: $5t_5 = 1$.

$$\langle \tau_1 \rangle = 2t_1^{\frac{3}{2}} - 3t_2^2, \quad \langle \tau_2 \rangle = -6t_1t_2 \tag{9.4}$$

$$\langle \tau_4 \rangle = -8t_1^{\frac{3}{2}}t_2 + 4t_2^3, \quad \langle \tau_5 \rangle = -2t_1^{\frac{5}{2}} + 15t_1t_2^2$$

Appendix C5

Genus 0 one-point correlation functions of the model \mathcal{M}_4^3 . Small phase space: $t_0 \equiv N, t_1, t_2, t_3$. First critical point: $4t_4 = -1$.

$$\begin{aligned}
\langle Q \rangle &= t_1t_3 + \frac{2}{3}t_2^2 + 3t_2t_3^2 + \frac{9}{4}t_3^4 + yt_0 \\
\langle P \rangle &= \frac{2}{3}t_1t_2 + t_1t_3^2 + t_0t_3 + \frac{4}{3}t_2^2t_3 + \frac{10}{3}t_2t_3^3 + 2t_3^5
\end{aligned}$$

$$\begin{aligned}
\langle \tau_2 \rangle &= \frac{1}{3}t_1^2 + \frac{8}{3}t_1t_2t_3 + \frac{10}{3}t_1t_3^3 + \frac{4}{3}t_0t_2 + 3t_0t_3^2 + \frac{8}{27}t_2^3 \\
&\quad + \frac{16}{3}t_2^2t_3 + \frac{34}{3}t_2t_3^4 + \frac{19}{3}t_2^6 \\
\langle \tau_3 \rangle &= 10t_1t_2t_3^2 + 10t_1t_3^4 + t_1^2t_3 + t_0t_1 + \frac{4}{3}t_1t_2^2 + 6t_0t_2t_3 \\
&\quad + 9t_0t_3^3 + \frac{32}{9}t_2^3t_3 + \frac{68}{3}t_2^2t_3^3 + 38t_2t_3^5 + 19t_3^7
\end{aligned}$$

Appendix C6

Genus 0 one-point correlation functions of the model \mathcal{M}_4^0 . Small phase space: t_1, t_2, t_3 . First critical point: $5t_5 = -1$.

$$\begin{aligned}
\langle \tau_1 \rangle &= 8t_2^2 + 12t_1t_3, & \langle \tau_2 \rangle &= 16t_1t_2 - 72t_2t_3^2 \\
\langle \tau_3 \rangle &= 6t_1^2 - 72t_2^2t_3 + 81t_3^4 \\
\langle \tau_5 \rangle &= 40t_1t_2^2 + 30t_1^2t_3 - 360t_2^2t_3^2 + 243t_3^5
\end{aligned}$$

One-point correlation functions of the same model \mathcal{M}_4^0 at the second critical point: $6t_6 = -1$.

$$\begin{aligned}
\langle \tau_1 \rangle &= 12t_2t_3 - \frac{2}{3}t_1^2t_3^{-1}, & \langle \tau_2 \rangle &= 12t_1t_3 + 8t_2^2 \\
\langle \tau_3 \rangle &= 12t_1t_2 + \frac{2}{9}t_1^3t_3^{-2} - \frac{27}{2}t_3^3 \\
\langle \tau_5 \rangle &= -\frac{5}{54}t_1^4t_3^{-2} - \frac{10}{3}t_1^2t_2t_3^{-1} + \frac{45}{2}t_1t_3^2 + 30t_2^2t_3
\end{aligned}$$

9.1 Appendix D

In this Appendix we collect results on correlation functions of the NLS (\mathcal{M}_2^1) model. Small phase space: N, t_1 . First critical point: $2t_2 = -1$. The genus h contribution is denoted $\langle \cdot \rangle_h$.

Genus 0

$$\langle \tau_r \rangle_0 = \sum_{\substack{k \\ 2 \leq 2k \leq r+2}} \frac{r!}{(r-2k+2)!(k-1)!k!} N^k t_1^{r-2k+2}$$

Genus 1

$$\langle \tau_1 \rangle_1 = \langle \tau_2 \rangle_1 = \langle \tau_3 \rangle_1 = 0, \quad \langle \tau_4 \rangle_1 = N,$$

$$\begin{aligned}
\langle \tau_5 \rangle_1 &= 5Nt_1, & \langle \tau_6 \rangle_1 &= 15Nt_1^2 + 10N^2, \\
\langle \tau_7 \rangle_1 &= 35Nt_1^3 + 70N^2t_1, & \langle \tau_8 \rangle_1 &= 70Nt_1^4 + 280N^2t_1^2 + 70N^3 \\
\langle \tau_9 \rangle_1 &= 126Nt_1^5 + 840N^2t_1^3 + 630N^3t_1, \\
\langle \tau_{10} \rangle_1 &= 210Nt_1^6 + 2100N^2t_1^4 + 3150N^3t_1^2 + 420N^4
\end{aligned} \tag{9.5}$$

$$\langle \tau_1 \tau_i \rangle_1 = 0, \quad 1 \leq i \leq 4, \quad \langle \tau_1 \tau_5 \rangle_1 = 5N, \quad \langle \tau_1 \tau_6 \rangle_1 = 30Nt_1$$

$$\langle \tau_2 \tau_2 \rangle_1 = \langle \tau_2 \tau_3 \rangle_1 = 0, \quad \langle \tau_2 \tau_4 \rangle_1 = 4N$$

$$\langle \tau_2 \tau_5 \rangle_1 = 30Nt_1, \quad \langle \tau_3 \tau_3 \rangle_1 = 3N$$

Genus 2

$$\langle \tau_i \rangle_2 = 0 \quad 1 \leq i \leq 7, \quad \langle \tau_8 \rangle_2 = 21N, \quad \langle \tau_9 \rangle_2 = 189Nt_1$$

$$\langle \tau_{10} \rangle_2 = 945Nt_1^2 + 483N^2, \quad \langle \tau_{11} \rangle_2 = 3465Nt_1^3 + 5313N^2t_1$$

References

- [1] M. Douglas, Phys.Lett.**B238** (90) 176.
- [2] E.Brezin, M.Douglas, V.Kazakov and S.Shenker, Phys.Lett.**B237** (1990).
D.J.Gross and A.A.Migdal, Phys.Rev.Lett. **64** (1990) 717.
- [3] C.Crnkovic, P.Ginsparg, and G.Moore, Phys.Lett.**B237** (1990)196.
- [4] J.Goeree, Nucl.Phys.**B358**(1991)737.
- [5] E.Gava and K.Narain, Phys.Lett.**B263**(1991)213.
E.Martinec, Comm.Math.Phys. **138B**— (1991) 437.
- [6] T.Tada and T.Yamaguchi, Phys.Lett. **B250** (1990) 38.
T.Tada, Phys.Lett. **B259** (1991)442.
S.Odake, Phys.Lett. **B269** (1991)300.
- [7] Jan De Boer, Nucl.Phys. **336B** (1991) 602.
- [8] S.Ryang, Phys.Rev.**D46** (1992).
- [9] C.Ahn and K.Shigemoto, “Multi-matrix model and 2D Toda multi-component hierarchy” CLNS 91/1054.

- [10] S.Karchev, A.Marshakov, A.Mironov, A.Morozov and A.Zabrodin, Nucl.Phys. **366B** (1991) 569.
A.Marshakov, A.Mironov and A.Morozov, Mod.Phys.Lett.**A7**(1992) 1345.
- [11] I.K.Kostov, Phys.Lett.**B297** (1992) 74.
- [12] S.Panda and S.Roy, Phys.Lett. **B296** (1992) 23.
- [13] J.-M.Daul, V.A.Kazakov and I.K.Kostov *Rational Theories of 2d Gravity from the Two-Matrix Model* CERN-TH6834/93.
- [14] M.Staudacher *Combinatorial Solutions of the Two-Matrix model*, RU-92-64.
- [15] L.Bonora and C.S.Xiong, Phys.Lett.**B285**(1992)191.
- [16] L.Bonora and C.S.Xiong, Int.Jour.Mod.Phys.A **8**(1993)2973.
- [17] L.Bonora and C.S.Xiong, Nucl.Phys.**405B** (1993) 191.
- [18] L.Bonora and C.S.Xiong, Phys.Lett.**B317**(1993)329.
- [19] L. Bonora, M. Martellini and C. S. Xiong, Nucl.Phys. **B375**(1992)453.
- [20] C. S. Xiong, Phys.Lett.**B279**(1992)341.
- [21] L.Bonora and C.S.Xiong, *The (N, M) -th KdV hierarchy and the associated algebras*, SISSA 171/93/EP, BONN-HE-46/93, AS-ITP-49/93
- [22] M.Kontsevich, Commun. Math.Phys.**147**(1992)1.
- [23] E.Witten, Nucl.Phys.**340B** (1990) 281.
- [24] R.Dijkgraaf and E.Witten, Nucl.Phys. **342B** (1990) 486.
- [25] E.Witten, “Algebraic geometry associated with matrix models of two dimensional gravity”, IASSNS-HEP-91/74.
- [26] R. Dijkgraaf, Lecture notes at Cargèse Summer School, 1991.
- [27] D. Bessis, C. Itzykson, G.Parisi and J.-B. Zuber, Comm.Math.Phys. **59**(1978) 35.
D. Bessis, C. Itzykson, and J.-B. Zuber, Adv. Appl. Math.**1**(1980) 109.
- [28] C.Itzykson and B.Zuber, J.Math.Phys. **21** (1980) 411.
- [29] M.L.Metha, Comm.Math.Phys.**79** (1981) 327.
S. Chedha, G. Mahoux and M. Mehta, J. Phys. A: Math. Gen. **14**(1981)579.
- [30] H.Aratyn, L.A.Ferreira, J.F.Gomes and A.H.Zimerman, *Toda and Volterra lattice equations from discrete symmetries of KP hierarchies* IFT-P.041/93, hep-th/9307147.
- [31] M.Adler, Invent.Math. **50**(1979)219.
- [32] H.P.McKean, Adv.Math.Supp. **3** (1978) 217.
- [33] A.B.Zamolodchikov, Theor.Math.Phys. **65**(1986) 1205.
- [34] L. Dickey, *Soliton equations and Hamiltonian Systems*, World Scientific, 1991.
- [35] I.Krichever, Comm.Math.Phys.**143**(1992)415.
- [36] B.Dubrovin, Nucl.Phys.**379B** (1992) 627.