

TOWARDS THE CLASSIFICATION OF EXACTLY SOLVABLE
FEYNMAN PATH INTEGRALS: δ -FUNCTION PERTURBATIONS
AND BOUNDARY-PROBLEMS AS MISCELLANEOUS SOLVABLE
MODELS

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ABSTRACT

In this contribution I present further results on steps towards a Table of Feynman Path Integrals. Whereas the usual path integral solutions of the harmonic oscillator (Gaussian path integrals), of the radial harmonic oscillator (Besselian path integrals), and the (modified) Pöschl-Teller potential(s) (Legendrian path integrals) are well known and can be performed explicitly by exploiting the convolution properties of the various types, a perturbative method opens other possibilities for calculating path integrals. Here I want to demonstrate the perturbation expansion method for point interactions and boundary problems in path integrals.

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1. Introduction. About 25 years ago Smorodinsky, Winternitz et al. [34] started to classify potential problems in quantum mechanics according to their separability in the Schrödinger equation. In particular, they were interested to find the dynamical symmetry groups of systems in three-dimensional space, where the Coulomb-potential with the symmetry group $O(4)$ is just but one, and listed numerous examples of separable systems, including the separating coordinate systems. It is, of course, desirable to consider an analogous program in the context of Feynman path integrals. Whereas no new systems will be found for such a consideration, the path integral approach is globally in comparison to the Schrödinger approach which is only locally. This opens the possibility to look at, say, the dynamical symmetries of a system from a path integral point of view, and to study the system in question by a group path integration. A comprehensive study along these lines does not exist yet, but in joint work with F. Steiner we are going to compile a list of exactly solvable Feynman path integrals [26], a Table of Feynman Path Integrals [27].

Whereas the usual path integral solutions of the harmonic oscillator and the general quadratic Lagrangian (Gaussian path integrals), of the radial harmonic oscillator (Besselian path integrals), and the (modified) Pöschl-Teller potential(s) (Legendrian path integrals) are well known and can be performed explicitly by exploiting the convolution properties of the various types (i.e. they can be seen as a particular group path integration, respectively a part of it), a perturbative method opens other possibilities for calculating path integrals and it is worthwhile to study old problems in this context (e.g. [33]). In this contribution I want to demonstrate this approach in a perturbation expansion for point interactions and boundary problems in path integrals. This will include one-, two- and three-dimensional δ -function perturbations, the one-dimensional δ' -function perturbation, and Dirichlet and Neumann boundary-conditions, respectively.

2. Formulation of the Path Integral. First of all, let us set up the definition of the Feynman path integral. We first consider the simple case of a classical Lagrangian $\mathcal{L}(\vec{x}, \dot{\vec{x}}) = m|\dot{\vec{x}}|^2/2 - V(\vec{x})$ in D dimensions. Then the integral kernel ($\vec{x} \in R^D$)

$$K(\vec{x}'', \vec{x}'; t'', t') = \langle \vec{x}'' | e^{-iH(t''-t')/\hbar} | \vec{x}' \rangle \Theta(t'' - t') , \quad (1)$$

(where H is the Hamiltonian of the system) of the time-evolution equation

$$\Psi(\vec{x}'', t'') = \int_{R^D} K(\vec{x}'', \vec{x}'; t'', t') \Psi(\vec{x}'', t') d\vec{x}' , \quad (2)$$

is represented in the form (Feynman path integral [12], [13])

$$\begin{aligned} K(\vec{x}'', \vec{x}'; t'', t') &= \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{ND/2} \prod_{j=1}^{N-1} \int_{R^D} dx_{(j)} \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} (\vec{x}_{(j)} - \vec{x}_{(j-1)})^2 - \epsilon V(\vec{x}_{(j)}) \right] \right\} \\ &\equiv \int_{\vec{x}(t')=\vec{x}'}^{\vec{x}(t'')=\vec{x}''} \mathcal{D}\vec{x}(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} |\dot{\vec{x}}|^2 - V(\vec{x}) \right] dt \right\} . \end{aligned} \quad (3)$$

Here we have used the abbreviations $\epsilon = (t'' - t')/N \equiv T/N$, $x_{(j)} = x(t_{(j)})$ ($t_{(j)} = t' + \epsilon j$, $j = 0, \dots, N$), and we interpret the limit $N \rightarrow \infty$ as equivalent to $\epsilon \rightarrow 0$, T fixed.

The next step is to consider a generic classical Lagrangian of the form $\mathcal{L}(\vec{q}, \dot{\vec{q}}) = mg_{ab}(\vec{q})\dot{q}^a\dot{q}^b/2 - V(\vec{q})$ in some D -dimensional Riemannian space M with line element $ds^2 = g_{ab}(\vec{q})dq^a dq^b$. This case, as first systematically discussed by DeWitt [10], requires a careful treatment. The Feynman path integral is most conveniently constructed by considering the Weyl-ordering prescription (e.g. [25] and references therein) in the corresponding quantum Hamiltonian. The result then is

$$\begin{aligned} K(\vec{q}'', \vec{q}'; t'', t') &= [g(\vec{q}')g(\vec{q}'')]^{-1/4} \lim_{N \rightarrow \infty} \left(\frac{m}{2\pi i \epsilon \hbar} \right)^{ND/2} \prod_{j=1}^{N-1} \int_M d\vec{q}_{(j)} \cdot \prod_{j=1}^N \sqrt{g(\vec{q}_{(j)})} \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[\frac{m}{2\epsilon} g_{ab}(\vec{q}_{(j)}) \Delta q_{(j)}^a \Delta q_{(j)}^b - \epsilon V(\vec{q}_{(j)}) - \epsilon \Delta V(\vec{q}_{(j)}) \right] \right\} . \end{aligned} \quad (4)$$

Here $\vec{q}_{(j)} = (\vec{q}_{(j)} + \vec{q}_{(j-1)})/2$ denotes the midpoint coordinate, $\Delta \vec{q}_{(j)} = (\vec{q}_{(j)} - \vec{q}_{(j-1)})$, and $\Delta V(\vec{q})$ is a well-defined “quantum potential” of order \hbar^2 having the form ($\Gamma_a = \partial_a \ln \sqrt{g}$, $g = \det(g_{ab})$)

$$\Delta V(\vec{q}) = \frac{\hbar^2}{8m} \left[g^{ab} \Gamma_a \Gamma_b + 2(g^{ab} \Gamma_a)_{,b} + g^{ab}{}_{,ab} \right] . \quad (5)$$

The midpoint prescription together with ΔV appears in a completely natural way as an unavoidable consequence of the Weyl-ordering prescription in the corresponding quantum Hamiltonian

$$H = -\frac{\hbar^2}{2m}g^{-1/2}\partial_a g^{1/2}g^{ab}\partial_b + V(\vec{q}) = \frac{1}{8m}\left(g^{ab}p_a p_b + 2p_a g^{ab}p_b + p_a p_b g^{ab}\right) + V(\vec{q}) + \Delta V(\vec{q}) , \quad (6)$$

with $p_a = -i\hbar(\partial_a + \Gamma_a/2)$, the momentum operator conjugate to the coordinate q_a in M . Of course, choosing another prescription leads to a different lattice definition in (4) and a different quantum potential $\widehat{\Delta V}$. However, every consistent lattice definition of (4) can be transformed into another one by carefully expanding the relevant metric terms (integration measure- and kinetic energy term).

Indispensable tools in path integral techniques are transformation rules. Here coordinate and time transformation must be mentioned which may be explicitly time-dependent or time-independent. The combination of these two kind of transformation as developed by Duru and Kleinert [11] in the context of the Coulomb problem, boosted enormously the further development of path integration techniques, for a recent review see e.g. [20] for Coulombian problems, and [27]. However, due lack of space, these transformation techniques will not be reviewed here, c.f. Refs. [11], [14], [20], [25], [27], and [30].

3. Basic Path Integrals. In this Section we present for completeness some path integrals which we consider as the Basic Path Integrals, c.f. also [26]. From these path integrals, respectively their Green functions, many other path integrals can be solved by applying the basic ones.

3.1. Path Integral for the Harmonic Oscillator. The first elementary example is the path integral for the harmonic oscillator. It has been first evaluated by Feynman [12]. We have the identity ($\vec{x} \in R^D$)

$$\begin{aligned} & \int_{\vec{x}(t')=\vec{x}'}^{\vec{x}(t'')=\vec{x}''} \mathcal{D}\vec{x}(t) \exp \left[\frac{im}{2\hbar} \int_{t'}^{t''} \left(|\dot{\vec{x}}|^2 - \omega^2 \vec{x}^2 \right) dt \right] \\ &= \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}} \exp \left\{ \frac{im\omega}{2\hbar} \left[(|\vec{x}'|^2 + |\vec{x}''|^2) \cot \omega T - \frac{2\vec{x}' \cdot \vec{x}''}{\sin \omega T} \right] \right\} . \end{aligned} \quad (7)$$

We do not state the expansion into wave-functions (\propto Hermite polynomials) which can be done by means of the Mehler formula, nor the corresponding Green's function.

The path integral for quadratic Lagrangians can also be stated exactly

$$\int_{\vec{x}(t')=\vec{x}'}^{\vec{x}(t'')=\vec{x}''} \mathcal{D}\vec{x}(t) \exp \left(\frac{i}{\hbar} \int_{t'}^{t''} \mathcal{L}(\vec{x}, \dot{\vec{x}}) dt \right) = \left(\frac{1}{2\pi i\hbar} \right)^{D/2} \sqrt{\det \left(-\frac{\partial^2 S_{Cl}[\vec{x}'', \vec{x}']}{\partial x'_a \partial x'_b} \right)} \exp \left(\frac{i}{\hbar} S_{Cl}[\vec{x}'', \vec{x}'] \right) . \quad (8)$$

Here $\mathcal{L}(\vec{x}, \dot{\vec{x}})$ denotes any classical Lagrangian at most quadratic in \vec{x} and $\dot{\vec{x}}$, and $S_{Cl}[\vec{x}'', \vec{x}'] = \int_{t'}^{t''} dt \mathcal{L}(\vec{x}_{Cl}, \dot{\vec{x}}_{Cl})$ the corresponding classical action evaluated along the classical solution \vec{x}_{Cl} satisfying the boundary conditions $\vec{x}_{Cl}(t') = \vec{x}'$, $\vec{x}_{Cl}(t'') = \vec{x}''$. The determinant appearing in (8) is known as the van Vleck-Pauli-Morette determinant (see e.g. [10] and references therein). The explicit evaluation of $S_{Cl}[\vec{x}'', \vec{x}']$ may have any degree of complexity due to complicated classical solutions of the Euler-Lagrange equations as the classical equations of motion. Of course, the harmonic oscillator is but a simple example of (8).

3.2. Path Integral for the Radial Harmonic Oscillator. In order to evaluate the path integral for the radial harmonic oscillator, one has to perform a separation of the angular variables, see Refs. [16], [35]. Path integrals related to the radial harmonic oscillator may be called of Besselian type [28]. Here we are not going into the subtleties of the Besselian functional measure due to the Bessel functions which appear in the lattice approach [14], [16], [25], [35], [38] which is actually necessary for the explicit evaluation of the radial harmonic oscillator path integral. One obtains (modulo the above mentioned subtleties) ($r > 0$)

$$\begin{aligned} & \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp \left[\frac{i}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{r}^2 - \hbar^2 \frac{\lambda^2 - \frac{1}{4}}{2mr^2} - \frac{m}{2} \omega^2 r^2 \right) dt \right] \\ &= \sqrt{r'r''} \frac{m\omega}{i\hbar \sin \omega T} \exp \left[-\frac{m\omega}{2i\hbar} (r'^2 + r''^2) \cot \omega T \right] I_\lambda \left(\frac{m\omega r'r''}{i\hbar \sin \omega T} \right) , \end{aligned} \quad (9)$$

where $I_\lambda(z)$ denotes the modified Bessel function.

3.3. *Path Integral for the Pöschl-Teller Potential.* There are two further basic path integral solutions based on the SU(2) [6], [29] and SU(1, 1) [6] group path integration, respectively. These path integrations may be called of Legendrian type [28]. The first yields the path integral identity for the solution of the Pöschl-Teller potential according to ($0 < x < \pi/2$)

$$\begin{aligned}
& \frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - \frac{\hbar^2}{2m} \left(\frac{\kappa^2 - \frac{1}{4}}{\sin^2 x} + \frac{\lambda^2 - \frac{1}{4}}{\cos^2 x} \right) \right] dt \right\} \\
&= \frac{m}{\hbar^2} \sqrt{\sin 2x' \sin 2x''} \frac{\Gamma(m_1 - L_E) \Gamma(L_E + m_1 + 1)}{\Gamma(m_1 + m_2 + 1) \Gamma(m_1 - m_2 + 1)} \\
&\quad \times \left(\frac{1 - \cos 2x_{<}}{2} \right)^{(m_1 - m_2)/2} \left(\frac{1 + \cos 2x_{<}}{2} \right)^{(m_1 + m_2)/2} \\
&\quad \times \left(\frac{1 - \cos 2x_{>}}{2} \right)^{(m_1 - m_2)/2} \left(\frac{1 + \cos 2x_{>}}{2} \right)^{(m_1 + m_2)/2} \\
&\quad \times {}_2F_1 \left(-L_E + m_1, L_E + m_1 + 1; m_1 - m_2 + 1; \frac{1 - \cos 2x_{<}}{2} \right) \\
&\quad \times {}_2F_1 \left(-L_E + m_1, L_E + m_1 + 1; m_1 + m_2 + 1; \frac{1 + \cos 2x_{>}}{2} \right) \tag{10}
\end{aligned}$$

with $m_{1/2} = \frac{1}{2}(\lambda \pm \kappa)$, $L_E = -\frac{1}{2} + \frac{1}{2}\sqrt{2mE}/\hbar$, and $x_{<,>}$ the larger, smaller of x', x'' , respectively. ${}_2F_1(a, b; c; z)$ denotes the hypergeometric function. Here we have used the fact that it is possible to state closed expressions for the (energy dependent) Green's functions for the Pöschl-Teller (and modified Pöschl-Teller potential, respectively), by summing up the spectral expansion, c.f. [31]. The case of the modified Pöschl-Teller potential will not be given here. It follows from a properly chosen coordinate transformation of the Pöschl-Teller potential case [31].

4. Point Interactions. The general method for the time-ordered perturbation expansion is simple. We assume that we have a potential $W(\vec{x}) = V(\vec{x}) + \tilde{V}(\vec{x})$ ($\vec{x} \in R^D$) in the path integral, where it is assumed that W is so complicated that a direct path integration is not possible. However, the path integral corresponding to $V(\vec{x})$ is assumed to be known, which we call $K^{(V)}(T)$. We expand the path integral containing $V(\vec{x})$ in a perturbation expansion about $\tilde{V}(\vec{x})$ in the following way. The initial kernel corresponding to V propagates in ϵ -time unperturbed, then it is interacting with \tilde{V} , propagates again in another ϵ -time unperturbed, a.s.o, up to the final state. Let us denote the path integral corresponding to the potential V by

$$K^{(V)}(\vec{x}'', \vec{x}'; T) = \int_{\vec{x}(t')=\vec{x}'}^{\vec{x}(t'')=\vec{x}''} \mathcal{D}\vec{x}(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} |\dot{\vec{x}}|^2 - V(\vec{x}) \right] dt \right\} . \tag{11}$$

We introduce the (energy-dependent) Green function (resolvent kernel)

$$G^{(V)}(\vec{x}'', \vec{x}'; E) = \frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} K^{(V)}(\vec{x}'', \vec{x}'; T) \tag{12}$$

$$K^{(V)}(\vec{x}'', \vec{x}'; T) = \frac{1}{2\pi i} \int_{-\infty}^\infty G^{(V)}(\vec{x}'', \vec{x}'; E) e^{-iET/\hbar} dE . \tag{13}$$

This gives the series expansion (see also e.g. [4], [13], [17], [19], [32], [36])

$$\begin{aligned}
& K(\vec{x}'', \vec{x}'; T) = \int_{\vec{x}(t')=\vec{x}'}^{\vec{x}(t'')=\vec{x}''} \mathcal{D}\vec{x}(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} |\dot{\vec{x}}|^2 - V(\vec{x}) - \tilde{V}(\vec{x}) \right] dt \right\} \\
&= K^{(V)}(\vec{x}'', \vec{x}'; T) + \sum_{n=1}^\infty \left(-\frac{i}{\hbar} \right)^n \prod_{j=1}^n \int_{t'}^{t_{j+1}} dt_j \int_{R^D} d\vec{x}_j \\
&\quad \times K^{(V)}(\vec{x}_1, \vec{x}'; t_1 - t') \tilde{V}(\vec{x}_1) K^{(V)}(\vec{x}_2, \vec{x}_1; t_2 - t_1) \times \dots
\end{aligned}$$

$$\dots \times \tilde{V}(x_{n-1})K^{(V)}(\vec{x}_n, \vec{x}_{n-1}; t_n - t_{n-1})\tilde{V}(\vec{x}_n)K^{(V)}(x'', \vec{x}_n; t'' - t_n) . \quad (14)$$

Here we have ordered the time as $t' = t_0 < t_1 < t_2 < \dots < t_{n+1} = t''$ and paid attention to the fact that $K^{(V)}(t_j - t_{j-1})$ is different from zero only if $t_j > t_{j-1}$.

4.1. δ -Function Perturbations in One Dimension. Let us consider first $D = 1$ and $\tilde{V}(x) = -\gamma\delta(x - a)$. Introducing the Green function $G^{(\delta)}(E)$ for the perturbed problem similarly as for $G^{(V)}(E)$, it is due to the convolution theorem possible to sum up the perturbation series and one obtains

$$\begin{aligned} & \frac{i}{\hbar} \int_0^\infty dT e^{iTE/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - V(x) + \gamma\delta(x - a) \right] dt \right\} \\ &= G^{(V)}(x'', x'; E) + \frac{G^{(V)}(x'', a; E)G^{(V)}(a, x'; E)}{1/\gamma - G^{(V)}(a, a; E)} . \end{aligned} \quad (15)$$

An implicit equation for the propagator has been stated by Gaveau and Schulman [15]. The simple example for a δ -function perturbation with $V \equiv 0$ has been discussed by several authors, e.g. [4], [17], [19], [32], and it is possible to state explicitly the corresponding propagator $K^{(\delta)}(T)$. In a similar way, one treats the case of a radial δ -function perturbation, so called shell-interactions [2], [19] ($\vec{x} \in R^D$):

$$\begin{aligned} & \frac{i}{\hbar} \int_0^\infty dT e^{iTE/\hbar} \int_{\vec{x}(t')=\vec{x}'}^{\vec{x}(t'')=\vec{x}''} \mathcal{D}\vec{x}(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} |\dot{\vec{x}}|^2 - V(r) + \gamma\delta(r - a) \right] dt \right\} \\ &= \sum_{l=0}^{\infty} S_l^\mu(\Omega'') S_l^{\mu*}(\Omega') \left[G_l^{(V)}(r'', r'; E) + \frac{G_l^{(V)}(r'', a; E)G_l^{(V)}(a, r'; E)}{a^{1-D}/\gamma - G_l^{(V)}(a, a; E)} \right] . \end{aligned} \quad (16)$$

Here the $S_l^\mu(\Omega)$ denote the real hyperspherical harmonics on the sphere $S^{(D-1)}$. It is obvious that more than one δ -function perturbation can be taken into account, in one dimension as well as in the radial case, c.f. [2], [21], [22] for details. In the limit of infinitely many singular perturbations a lattice is obtained, c.f. [2] and [18].

4.2. δ' -Function Perturbations in One Dimension. Due to the specific property of one-dimensional space, it is also possible to study δ' -interactions in one dimension. The δ' -function perturbation, also called dipole interaction, is a rather strange object and there seems to no way to generate this kind of interaction from the usual δ -function perturbations, e.g. by an appropriate limiting procedure in the definition of the derivative. However, if one makes the delay over the one-dimensional Dirac equation and its corresponding path integral representation [13] it is possible to incorporate δ' interactions [24]: One considers a δ -function perturbation in the path integral representation for the one-dimensional Dirac equation. If one chooses the “electron” component, one recovers in the non-relativistic limit the usual δ -function perturbation. If one chooses the “positron” component, a δ' -function perturbation is generated in the non-relativistic limit [2]. The result then has the form

$$\begin{aligned} & \frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - V(x) \right] dt \right\} \\ &= G^{(V)}(x'', x'; E) - \frac{G_{,x'}^{(V)}(x'', a; E)G_{,x''}^{(V)}(a, x'; E)}{1/\gamma + \widehat{G}_{,x'x''}^{(V)}(a, a; E)} , \end{aligned} \quad (17)$$

where $\widehat{G}_{,x'x''}^{(V)}(a, a; E)$ denotes a regularized Green function with all infinities subtracted. For the e.g. $V \equiv 0$ case one obtains

$$\frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 + \gamma\delta'(x - a) \right] dt \right\}$$

$$\begin{aligned}
&= \frac{1}{\hbar} \sqrt{\frac{m}{2E}} \exp\left(-\frac{\sqrt{-2mE}}{\hbar} |x'' - x'|\right) \\
&\quad - \frac{m^2 \exp\left[-\sqrt{-2mE} (|x'' - a| + |a - x'|)/\hbar\right]}{\hbar^4 \frac{1}{\gamma} - m\sqrt{-2mE}/\hbar^3} \text{sign}(x'' - a) \text{sign}(x' - a) . \quad (18)
\end{aligned}$$

The corresponding propagator is given by

$$\begin{aligned}
&\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 + \gamma \delta'(x - a)\right] dt\right\} \\
&= \sqrt{\frac{m}{2\pi i \hbar T}} \exp\left[-\frac{m}{2i\hbar T} (x'' - x')^2\right] \\
&\quad + \sqrt{\frac{m}{2\pi i \hbar T}} \exp\left[-\frac{m}{2i\hbar T} (|x'' - a| + |x' - a|)^2\right] \text{sign}(x'' - a) \text{sign}(x' - a) \\
&\quad + \frac{\hbar^2}{2m\gamma} \exp\left[-\frac{\hbar^2}{m\gamma} (|x'' - a| + |x' - a|) + \frac{i}{\hbar} \frac{\hbar^6 T}{2m^3 \gamma^2}\right] \\
&\quad \times \text{erfc}\left\{\sqrt{\frac{m}{2i\hbar T}} \left[(|x'' - a| + |x' - a|) - \frac{i\hbar^3 T}{m^2 \gamma} \right]\right\} \text{sign}(x'' - a) \text{sign}(x' - a) . \quad (19)
\end{aligned}$$

For $\gamma > 0$ there is one bound state with energy level and wave-function given by

$$E^{(\delta')} = -\frac{\hbar^6}{2m^3 \gamma^2}, \quad \Psi^{(\delta')}(x) = \frac{\hbar}{\sqrt{m\gamma}} \exp\left(-\frac{\hbar^2}{m\gamma} |x - a|\right) \text{sign}(x - a) . \quad (20)$$

4.3. δ -Function Perturbations in Two and Three Dimensions. We consider δ -function perturbations in two and three dimensions. A naïve generalization of the corresponding one-dimensional result gives a divergence. The origin of this divergence can be easily seen. One must evaluate the corresponding Green function of the unperturbed problem at both arguments being equal, an expression which diverges logarithmically in two dimensions, and in three dimensions there is a simple pole. Consequently, one must regularize the problem which basically consists of finding the proper Friedrich extension in order to make the corresponding Hamiltonian self-adjoint. This topic is addressed in a comprehensive way by Albeverio et al. [2] and has been put into the path integral language in [23]. The main idea is that in the regularization procedure a new (regularized) coupling α is introduced, while letting γ in the heuristic expression “ $\gamma\delta(\vec{x} - \vec{a})$ ” be zero in a suitable way, as first pointed out by Berezin and Faddeev [5]. We just cite the result: For the Green function of a δ -function perturbation in two and three dimensions one obtains:

$$\begin{aligned}
&\frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} \int_{\vec{x}(t')=\vec{x}'}^{\vec{x}(t'')=\vec{x}''} \mathcal{D}_{\Gamma_{\alpha,\vec{a}}^{(V)}} \vec{x}(t) \exp\left\{\frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} |\dot{\vec{x}}|^2 - V(\vec{x}) + \gamma \delta(\vec{x} - \vec{a})\right] dt\right\} \\
&= G^{(V)}(\vec{x}'', \vec{x}'; E) + (\Gamma_{\gamma,\vec{a}}^{(V)})^{-1} G^{(V)}(\vec{x}'', \vec{a}; E) G^{(V)}(\vec{a}, \vec{x}'; E) . \quad (21)
\end{aligned}$$

with $\Gamma_{\gamma,\vec{a}}^{(V)}$ given by $\Gamma_{\gamma,\vec{a}}^{(V)} = \alpha g_{0,\lambda} - g_{1,\lambda}$. Here denote ($\alpha \in (-\infty, \infty]$, $r = |\vec{x}|$)

$$g_{0,\lambda} = \lim_{r \rightarrow 0} \frac{g(r)}{G_\lambda^{(0)}(r)}, \quad g_{1,\lambda} = \lim_{r \rightarrow 0} \frac{[g(r) - g_{0,\lambda} G_\lambda^B(r)]}{F_\lambda^{(0)}(r)}, \quad (22)$$

where $F_\lambda^{(0)}(r) = r^\lambda$, $G_\lambda^{(0)}(r) = -(m/\pi\hbar^2)\sqrt{r} \ln r$ for $D = 2$, $G_\lambda^{(0)}(r) = -(m/2\pi\hbar^2)$ for $D = 3$; $G_\lambda^B(r)$ denotes the asymptotic expansion of the irregular solution of the corresponding Schrödinger problem up to order r^t , $t \leq 2\lambda - 1$; a convenient way for choosing $g(r)$ is to take the reduced Green function of the unperturbed problem, i.e. $g(r) = \Omega^{-1}(D) r^\lambda G^{(V)}(r, r; E)$, with $\Omega(D)$ the volume of the D -dimensional unit-sphere (for more details, c.f. [2] and [23] for a path integral discussion including some instructive

examples). In our notation the index $\Gamma_{\gamma, \vec{a}}^{(V)}$ in $\mathcal{D}_{\Gamma_{\gamma, \vec{a}}^{(V)}}(x)$ denotes the to-be-performed regularization in the path integral. The connection between λ and the dimension D is given by $\lambda = (D - 1)/2$. For a three-dimensional singular perturbation with $V \equiv 0$ it is possible to state the corresponding propagator in a straightforward way, c.f. [37], whereas in two dimensions only a rather complicated integral representation can be given, c.f. [1] and [23].

5. Boundary Conditions. The results from the incorporation of the δ - and δ' -function perturbation in one dimension make it possible to build in Dirichlet and Neumann boundary-conditions, respectively, in the path integral [22]. Some general, however not explicit, results have been archived in Refs. [7] and [8], and a particular example (the free particle) was discussed in [9]. Making now in our formalism the strength of the δ -function perturbations in (15) infinitely repulsive produces Dirichlet boundary-conditions (D) at the location of the δ -function, i.e., we obtain a path integral representation in a half-space with a boundary with Dirichlet boundary-conditions at $x = a$:

$$\begin{aligned} & \frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{(W_{all})}^{(D)} x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - V(x) \right] dt \right\} \\ &= G^{(V)}(x'', x'; E) - \frac{G^{(V)}(x'', a; E) G^{(V)}(a, x'; E)}{G^{(V)}(a, a; E)}. \end{aligned} \quad (23)$$

Note that for a symmetrical model, i.e. $V(x) = V(-x)$, a δ -function perturbation at $x = 0$ yields in the limit $\gamma \rightarrow \infty$ a doubly generated energy level spectrum with one ‘‘spurious ground state’’ $\tilde{\Psi}_0$ with infinite negative energy and a corresponding wave-function concentrated at $x = 0$, i.e. $|\tilde{\Psi}_0(x)|^2 = \delta(x)$ (e.g. [3] and references therein).

Similarly, making the strength of the δ' -function perturbations in (17) infinitely repulsive produces Neumann boundary-conditions (N) at the location of the δ' -function, i.e., we obtain a path integral representation in half-space with a boundary with Neumann boundary-conditions at $x = a$ [24]:

$$\begin{aligned} & \frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{(W_{all})}^{(N)} x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - V(x) \right] dt \right\} \\ &= G^{(V)}(x'', x'; E) - \frac{G_{,x'}^{(V)}(x'', a; E) G_{,x''}^{(V)}(a, x'; E)}{\widehat{G}_{,x'x''}^{(V)}(a, a; E)}. \end{aligned} \quad (24)$$

We can further consider motion in boxes, where any combination of Dirichlet and Neumann boundary-conditions is allowed, e.g.: For the motion in the box $a < x < b$ with Neumann boundary-conditions for $x = a$ and $x = b$ we obtain:

$$\begin{aligned} & \frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{(a < x < b)}^{(NN)} x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - V(x) \right] dt \right\} \\ &= \frac{\begin{vmatrix} G^{(V)}(x'', x'; E) & G_{,x'}^{(V)}(x'', b; E) & G_{,x'}^{(V)}(x'', a; E) \\ G_{,x''}^{(V)}(b, x'; E) & \widehat{G}_{,x'x''}^{(V)}(b, b; E) & \widehat{G}_{,x'x''}^{(V)}(b, a; E) \\ G_{,x''}^{(V)}(a, x'; E) & \widehat{G}_{,x'x''}^{(V)}(a, b; E) & \widehat{G}_{,x'x''}^{(V)}(a, a; E) \end{vmatrix}}{\begin{vmatrix} \widehat{G}_{,x'x''}^{(V)}(b, b; E) & \widehat{G}_{,x'x''}^{(V)}(b, a; E) \\ \widehat{G}_{,x'x''}^{(V)}(a, b; E) & \widehat{G}_{,x'x''}^{(V)}(a, a; E) \end{vmatrix}}. \end{aligned} \quad (25)$$

Finally, we can put the results of the motion in a half-space with Dirichlet *and* Neumann boundary-conditions together to find a path integral representation for potential problems with absolute value dependence:

$$\frac{i}{\hbar} \int_0^\infty dT e^{iET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} \dot{x}^2 - V(|x|) \right] dt \right\}$$

$$= G^{(V)}(x'', x'; E) - \frac{G^{(V)}(x'', 0; E)G^{(V)}(0, x'; E)}{2G^{(V)}(0, 0; E)} - \frac{G_{,x'}^{(V)}(x'', 0; E)G_{,x''}^{(V)}(0, x'; E)}{2\widehat{G}_{,x'x''}^{(V)}(0, 0; E)}. \quad (26)$$

Important applications of the latter equation are the so-called “double-oscillator” $V^{(DO)}(x) = (m/2) \omega^2 (|x| - a)^2$ and the one-dimensional Kepler problem $V^{(1-D)}(x) = k|x|$.

6. Summary. In this contribution I have presented some recent developments in path integral techniques, i.e. the incorporation of point interactions and boundary problems in the path integral. They were δ -function perturbations in one dimension, δ' -function perturbations in one dimension, and δ -function perturbations in two and three dimensions. The results for the δ - and δ' -function perturbations, respectively, made it possible to incorporate Dirichlet and Neumann boundary-conditions in the path integral. This could be achieved by making the strength of the singular perturbations infinitely repulsive. All the corresponding Green functions were stated, including some important combinations, for the quantum motion in boxes with Dirichlet and/or Neumann boundary conditions at the walls, and for potential problems with absolute value dependence.

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