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Orbifold Vortex and Super Liouville Theory

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ABSTRACT

We study the nonabelian vortex counting problem on $\mathbb{C}/\mathbb{Z}_p$. At first we calculate vortex partition functions on the orbifold space using localization techniques, then we find how to extract orbifold vortex partitions function from orbifold linear quiver instanton partition functions. Finally, we study the AGT like relation between orbifold $SU(2)$ vortices and $\mathcal{N} = 1$ super Liouville theory in the mixed R/NS sector by fixing the dictionary among parameters in the common hypergeometric functions system.

1 Introduction

New connections between two completely different theories will generate interesting discoveries on both sides. One fair example in recent years is the discovery of the duality between $\mathcal{N} = 2$ quiver gauge theory and Liouville conformal field theory [1]. In [2], [5]and [6] the relations among surface operators of $\mathcal{N} = 2$ four dimensional gauge theories, degenerate fields of Liouville theory and two dimensional vortex theories are studied in detail. Recently, the AGT correspondence related to ALE instanton counting has been studied in [3], [4] and [12].

After [11], non-abelian vortices became a hot area of research. The moduli space of vortices on a Riemann surface was studied in [8] and the moduli space of vortices on $\mathbb{C}/\mathbb{Z}_p$ was studied in [13]. We use analogous localization techniques used in [5] to calculate vortex partition functions on $\mathbb{C}/\mathbb{Z}_p$, which is a similar extension of instanton partition functions on $\mathbb{C}^2$ to $\mathbb{C}^2/\mathbb{Z}_p$ calculated in [9]. This similarity is expectable from the string theory point of view.

In the context of string theory, linear quiver gauge theories have a geometrical realization as the low energy effective theories of $D4$-branes intersecting with $NS5$-branes [15], and instantons can be considered as $D0$-branes inside the $D4$-branes. By localization techniques, integrations over instanton moduli space turn into combinatoric problems associated with an arrow of two dimensional Young-tableaus and each $D0$-brane is associated with a box in the Young-tableaus [14]. When mass parameters are in special values, the instanton partition functions will degenerate into simpler forms characterized by one dimensional Young-tableaus [6]. We study the degeneration phenomenon of orbifold quiver instanton partition functions which not only tells us how to extract orbifold vortex partition functions from that of instantons but also gives information about surface operators of orbifold gauge theory.

After studying the relation between orbifold vortices and orbifold instantons, one is urged to study the AGT dual of orbifold vortices. It is difficult to find the conformal field theory dual of vortex partition functions directly. The trick here is that we can use four dimensional gauge theory as a bridge connecting conformal field theories and vortex theories [6]. The AGT dual of correlation functions of pure NS primary fields was studied in [3] and that of Whittaker vectors in the Ramond sector was studied in [12]. However, our analysis shows that in order to find the AGT dual of orbifold vortices, it is necessary to have a complete knowledge of AGT duality of super Liouville theory with both NS and Ramond sectors, which worths a single paper by itself. We study the super Liouville theory dual of orbifold vortices based on known results about correlation functions of degenerate fields in Ramond sector [10], [7] and show that orbifold vortex partition functions can be identified with correlation functions of lowest degenerate states in Ramond sector.

The organization of this paper is as follows. In section 2 we review necessary knowledge about instanton counting on $\mathbb{C}^2/\mathbb{Z}_p$. In section 3 we calculate vortex partition function on $\mathbb{C}/\mathbb{Z}_p$. In section 4 we give the relation between the two classes of partition functions obtained in section 2 and section 3. Then in section 5 we study the CFT dual of vortex partition function on $\mathbb{C}/\mathbb{Z}_p$. Section 6 contains discussions.

2 Instantons on $\mathbb{C}^2/\mathbb{Z}_p$

Our conventions and notations will mainly follow [9]. The gauge group is U(N) and the instanton number is k . Parameters for pure instanton counting are Coulomb branch parameters a_α where α runs from 1 to N and Ω-deformation parameters are ϵ_1, ϵ_2. Due to orbifold action $q_\alpha, \epsilon_1, \epsilon_2$ have respectively discrete charges $q_\alpha, 1, -1$. Notice that charges take value in $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ commutates with gauge group, under this assignment of charges, gauge group will break in the following way:

$$U(N) \longrightarrow \prod_q U(n_q)$$
$$n_q = \sum_\alpha \delta_{q,q_\alpha}$$

It seems that the discrete group will change the fixed point structure of instanton counting drastically, but due to the fact that $\Gamma \in U(1)^2$ of the localization torus action, fixed points are still characterized by N Young tableaus of total number of boxes equals to k. Similarly the auxiliary $U(k)$ group will also break as :

$$U(k) \longrightarrow \prod_q U(k_q)$$

As we know each box in a Young diagram represents an instanton, and the corresponding discrete charge is just $q_\alpha + i - j$ for a box at position (i, j) of the α-th Young diagram. So $k_q = \dim V_q$ =number of instantons with discrete charge q. Here V and W are complex linear spaces of dimension k and N. Then we have following linear decomposition of the Euler character of tangent bundle of instanton moduli space :

$$\begin{aligned} \chi_\Gamma &= V^* \otimes V\left(T_1 + T_2 - 1 - T_1T_2\right) + W^* \otimes V + V^* \otimes W T_1 T_2 \\ &= \sum_q \left(V_q^* V_{q+1} + V_{q+1}^* V_q - V_q^* V_q T_1 T_2 - V_q^* V_q + W_q^* V_q + V_q^* W_q T_1 T_2\right) \\ V_q &= \sum_{\alpha=1}^N \sum_{s \in Y_\alpha} T_{a_\alpha} T_1^{-j_s+1} T_2^{-i_s+1} \delta_{q_\alpha + i_s - j_s, q} \\ W_q &= \sum_{\alpha=1}^N T_{a_\alpha} \delta_{q_\alpha, q} \end{aligned} \tag{1}$$

After some algebra:

$$\begin{aligned}
\chi_\Gamma^{\text{vector}} &= -\sum_{\alpha,\beta}^{N}\sum_{s\in Y_\alpha}\left(T_{a_{\alpha,\beta}}T_1^{-L_\beta(s)}T_2^{A_\alpha(s)+1}+T_{a_{\beta,\alpha}}T_1^{L_\beta(s)+1}T_2^{-A_\alpha(s)}\right)\delta_{L_\beta(s)+A_\alpha(s)+1,q_{\alpha,\beta}}\\
&= -\sum_{\alpha,\beta}^{N}\sum_{s\in Y_\alpha}T_{a_{\alpha,\beta}}T_1^{-L_\beta(s)}T_2^{A_\alpha(s)+1}\delta_{L_\beta(s)+A_\alpha(s)+1,q_{\alpha,\beta}} \qquad (2)\\
&\quad -\sum_{\alpha,\beta}^{N}\sum_{t\in Y_\beta}T_{a_{\alpha,\beta}}T_1^{L_\alpha(s)+1}T_2^{-A_\beta(s)}\delta_{L_\alpha(s)+A_\beta(s)+1,q_{\beta,\alpha}}
\end{aligned}$$

To obtain 4d instanton partition functions, we need to set $T_1=e^{\epsilon_1}, T_2=e^{\epsilon_2}, T_{a_\alpha}=e^{a_\alpha}$. As it is known from [5] and [6], vortex partition functions lie in $\epsilon_+=\epsilon_1+\epsilon_2=0$ limit of degenerate instanton partition functions, we will take this limit in the following:

$$\begin{aligned}
\left(Z_\Gamma^{\text{vector}}(a,Y,q_\alpha)\right)^{-1} &= \prod_{\alpha,\beta}^{N}\prod_{s\in Y_\alpha}\left(a_{\alpha,\beta}+\epsilon_2\left(A_{Y_\alpha}(s)+1+L_{Y_\beta}(s)\right)\right)\delta_{A_{Y_\alpha}(s)+1+L_{Y_\beta}(s),q_{\alpha,\beta}}\\
&\quad \prod_{t\in W_\beta}\left(a_{\alpha,\beta}-\epsilon_2\left(A_{Y_\beta}(t)+1+L_{Y_\alpha}(s)\right)\right)\delta_{A_{Y_\beta}(t)+1+L_{Y_\alpha}(t),q_{\beta,\alpha}} \qquad (3)
\end{aligned}$$

Vector field contributions are in denominators of instanton partition functions, and numerators of instanton partition function will come from hypermultiplets. For our interest lies in linear quiver gauge theories, we will only consider hypermultiplets in (anti)fundamental and bifundamental representations. Since latter we will study N-node quiver gauge theory, we will take following notations similar to these in [6].

$$\left\{Y_\alpha^{(L)}\right\}_{\alpha=1}^{N} \quad : \quad \text{the Young tableaus of the } L\text{-th gauge factor.} \qquad (4)$$

$$\begin{aligned}
\left\{a_\alpha^{(L)}\right\}_{\alpha=1}^{N} &\quad : \quad \text{the Coulomb branch parameters of the L-th gauge factor.}\\
m_i &\quad = \quad \text{the i-th mass of bifundamental hypermultiplet} \qquad (5)\\
\mu_i &\quad = \quad \begin{cases}\text{masses of antifundamental hypermultiplets} & i\in[1,N]\\ \text{masses of fundamental hypermultiplets} & i\in[N+1,2N]\end{cases}\\
m_{\alpha,\beta}^{(L)} &\quad := \quad a_\alpha^{(L)}-a_\beta^{(L+1)}-m_L
\end{aligned}$$

$$\begin{aligned}
\left\{q_\alpha^{(L)}\right\}_{\alpha=1}^{N} &\quad : \quad \text{the discrete charges of Coulomb branch parameters of the L-th gauge factor.}\\
q_m^{(L)} &\quad : \quad \text{the discrete charge of the } L\text{-th bifundamental hypermultiplet.}\\
q_\alpha^{f} &\quad : \quad \text{the discrete charge of the } \alpha\text{-th fundamental hypermultiplet.} \qquad (6)\\
q_\alpha^{af} &\quad : \quad \text{the discrete charge of the } \alpha\text{-th antifundamental hypermultiplet.}\\
Q_{\alpha,\beta}^{(L)} &\quad = \quad q_\alpha^{(L)}-q_\beta^{(L+1)}+q_m^{(L)}
\end{aligned}$$

2.1 With bifundamental matter fields

From the vector field contribution, we can easily obtain the contribution from bifundamental hypermultiples:

$$\chi_{\Gamma}^{\mathrm{bifund},L} = \sum_{\alpha,\beta}^{N} T_{m_L} \left(\sum_{s\in Y_{\alpha}^{(L)}} T_{a_{\alpha,\beta}^{(L,L+1)}} T_1^{-L_{Y_{\beta}^{(L+1)}}(s)} T_2^{A_{Y_{\alpha}^{(L)}}(s)+1} \delta_{L_{Y_{\beta}^{(L+1)}}(s)+A_{Y_{\alpha}^{(L)}}(s)+1, Q_{\alpha,\beta}^{(L,L+1)}} + \right.$$

$$\left. \sum_{t\in Y_{\beta}^{(L+1)}} T_{a_{\alpha,\beta}^{(L,L+1)}} T_1^{L_{Y_{\alpha}^{(L)}}(t)+1} T_2^{-A_{Y_{\beta}^{(L+1)}}(t)} \delta_{L_{Y_{\alpha}^{(L)}}(t)+A_{Y_{\beta}^{(L+1)}}(t)+1, Q_{\beta,\alpha}^{(L,L+1)}} \right)$$

In $\epsilon_+ = 0$ limit, the contribution to instanton partition function from the L-th bifundamental hypermultiplet is:

$$\begin{aligned} Z_{\Gamma}^{\mathrm{bifund},L}(a,m,Y) &= \prod_{\alpha,\beta}^{N} \prod_{s\in Y_{\alpha}^{(L)}} \left(m_{\alpha,\beta}^{(L)} + \epsilon_2 \left(L_{Y_{\beta}^{(L+1)}}(s) + A_{Y_{\alpha}^{(L)}}(s) + 1 \right) \right) \delta_{L_{Y_{\beta}^{(L+1)}}(s)+A_{Y_{\alpha}^{(L)}}(s)+1, Q_{\alpha,\beta}^{(L)}} \\ & \prod_{t\in Y_{\beta}^{(L+1)}} \left(m_{\alpha,\beta}^{(L)} - \epsilon_2 \left(L_{Y_{\alpha}^{(L)}}(t) + A_{Y_{\beta}^{(L+1)}}(t) + 1 \right) \right) \delta_{L_{Y_{\alpha}^{(L)}}(t)+A_{Y_{\beta}^{(L+1)}}(t)+1, Q_{\beta,\alpha}^{(L)}} \end{aligned}$$

2.2 With fundamental matter fields

It is easy to obtain contributions from fundamental hypermultiplets by either direct calculation or reduction from that of bifundamental hypermultiplets'. The results are:

$$\begin{aligned} Z_{\Gamma}^{\mathrm{fund},q_{\beta}^{f}}(a,m,Y) &= \prod_{\alpha=1}^{N} \prod_{\beta=1}^{F} \prod_{s\in Y_{\alpha}} \left(a_{\alpha} - m_{\beta} + \epsilon_1 (i_s - 1) + \epsilon_2 (j_s - 1) + \epsilon_+ \right) \delta_{j-i, q_{\alpha}-q_{\beta}^{f}} \\ Z_{\Gamma}^{\mathrm{antifund},q_{\beta}^{af}}(a,m,Y) &= \prod_{\alpha=1}^{N} \prod_{f=1}^{F} \prod_{s\in Y_{\alpha}} \left(a_{\alpha} + m_{\beta} + \epsilon_1 (i_s - 1) + \epsilon_2 (j_s - 1) \right) \delta_{j-i, q_{\alpha}-q_{\beta}^{af}} \end{aligned}$$

2.3 Different sectors

For the N node $SU(N)$ linear quiver theory on $\mathbb{C}/\mathbb{Z}_p$, we have different branches determined by discrete charges. The generic formula for a specific branch of orbifold instanton partition function is:

$$\begin{aligned} Z_{Quiver}\left(a,m,\left\{q_{\alpha}^{(L)}\right\};\left\{q_{\alpha}^{\mathrm{af}}\right\};\left\{q_{\alpha}^{f}\right\}\right) &= \sum_{Y} \prod_{\beta=1}^{N} q_{\beta}^{|Y^{(\beta)}|} Z_{\Gamma}^{\mathrm{antifund},q_{\beta}^{\mathrm{af}}}\left(a,m,Y^{(1)}\right) \\ & Z_{\Gamma}^{\mathrm{fund},q_{\beta}^{f}}\left(a,m,Y^{(N)}\right) Z_{N}^{\Gamma}\left(\left\{q_{\alpha}^{(N)}\right\},Y^{(N)}\right) \\ & \prod_{L=1}^{N-1} Z_{L}^{\Gamma}\left(\left\{q_{\alpha}^{(L)}\right\},Y^{(L)}\right) Z_{L,L+1}^{\Gamma}\left(Y^{L},Y^{L+1}\right) \end{aligned} \tag{7}$$

Where

$$
\begin{aligned}
Z_L^\Gamma\left(\left\{q_\alpha^{(L)}\right\}, Y^{(L)}\right) &:= Z_\Gamma^{\text{vec}}\left(a^{(L)}, Y^{(L)}, \left\{q_\alpha^{(L)}\right\}\right);\\
Z_{L,L+1}^\Gamma\left(Y^L, Y^{L+1}\right) &:= Z_\Gamma^{\text{bifund},L}\left(a^{(L)}, m_L, \left\{q_\alpha^{(L)}\right\}, \left\{q_m^{(L)}\right\}, Y^L, Y^{L+1}\right)
\end{aligned}
$$

In general, orbifold instanton counting has two counting parameters if the first Chern class, c_1, of orbifold instanton moduli space is nontrivial. For simplicity we will only consider the case when $c_1 = 0$.

We will see later, in order to extract vortex partition function from instanton, up-to Weyl symmetry, we need to choose discrete charges in following way: $q_\alpha^{(1)} - q_\alpha^f = \delta_{1,\alpha} \bmod p$ and $q_\alpha^{(L)} - q_\alpha^{(L+1)} + q_m = \delta_{\alpha,L+1} \bmod p$.

3 Vortices on $\mathbb{C}/\mathbb{Z}_p$

The moduli space of orbifold vortex was studied in [13] using moduli matrix method. In the following we will studying the orbifold vortex counting problem. As we know from [5], [11] the moduli space of vortices can be considered as a Lagrangian submanifold of that of instantons. Similar mechanism can be used for the orbifold case. Recall that the moduli space of vortex partition function on $\mathbb{C}$ is given by following ADHM like data:

$$
\mathcal{M}_{N,k} = \left\{(B,I)\,\middle|\,\left[B,B^\dagger\right] + II^\dagger = c\mathbb{I}_k\right\}\big/U(k)
$$

Where $B \in \text{End}(V,V), I \in \text{Hom}(V,W)$. V and W are complex linear spaces of dimension k and N . When there is an extra $\mathbb{Z}_n$ action, V and W have further decomposition:

$$
\begin{aligned}
V_q &= \sum_{\alpha=1}^{N}\sum_{j=1}^{k_\alpha} T_{a_\alpha} T_\hbar^{-i+1}\delta_{q_\alpha+i-1,q}\\
W_q &= \sum_{\alpha=1}^{N} T_{a_\alpha}\delta_{q_\alpha,q}\\
\chi_\Gamma &= V^*\otimes V\left(T_1 - 1\right) + W^*\otimes V = \sum_q \left(V_q^* V_{q+1} - V_q^* V_q + W_q^* V_q\right)
\end{aligned}
\tag{8}
$$

A short calculation shows:

$$
\chi_\Gamma = \sum_{\alpha,\beta=1}^{N} T_{a_{\alpha,\beta}} \sum_{i=1}^{k_\alpha} T_\hbar^{-i+1+k_\beta}\delta_{-i+1+k_\beta,q_{\alpha,\beta}} \tag{9}
$$

So the vector field contribution is:

$$
\left(Z_{\Gamma,\text{vortex}}^{\text{vector}}(a,\hbar;k;q)\right)^{-1} = \prod_{\alpha,\beta=1}^{N}\prod_{i=1}^{k_\alpha}\left(a_{\alpha,\beta} + \hbar\left(k_\beta + 1 - i\right)\right)\delta_{-i+1+k_\beta,q_{\alpha,\beta}} \tag{10}
$$

Similarly, we get contributions from matter fields in fundamental representation:

$$
\begin{aligned}
Z^{\text{fund},q^f_\beta}_{\Gamma,\text{vortex}}(a,m,\hbar;k) &= \prod_{\alpha=1}^{N}\prod_{\beta=1}^{F}\prod_{i=1}^{k_\alpha}(a_\alpha - m_\beta + \hbar(i-1))\,\delta_{1-i,q_\alpha - q^f_\beta} \\
Z^{\text{antifund},q^{af}_\beta}_{\Gamma,\text{vortex}}(a,m,\hbar;k) &= \prod_{\alpha=1}^{N}\prod_{\beta=1}^{F}\prod_{s\in Y_\alpha}(a_\alpha + m_\beta + \hbar(i-1))\,\delta_{1-i,q_\alpha - q^{af}_\beta}
\end{aligned}
\tag{11}
$$

Orbifold vortex partition functions also have many sectors determined by discrete charges:

$$
Z_{\text{vortex}}\left(\left\{a,m,q^{(L)}_\alpha\right\};\left\{q^f_\alpha\right\}\right) = \sum_k \prod_{\beta=1}^{N} z_\beta^{k_\beta}\, Z^{\text{fund},q^f_\beta}_{\Gamma,\text{vortex}}(a,m,k) Z^{\text{vector}}_{\Gamma,\text{vortex}}\left(a,k;\{q_\alpha\}\right) \tag{12}
$$

Where z_β are N counting parameters.

4 Vortex From Instantons

We can extract orbifold vortex partition functions from orbifold instanton partition functions following the same strategy for non-orbifold case [6]. For $SU(N)$ vortex, we need to consider $SU(N)$ N-node linear quiver gauge theory. The instanton partition function of this gauge theory is characterized by N N-dimensional arrows of Young-tableaus, which in noted by $Y^{(L)}_\alpha$ in (4). Then by setting masses of antifundamental hypermultiplets and bifundamental hypermultiplets to special values, the Young-tableaus are forced to have following simple form:

$$
Y^{(L)}_\alpha = \begin{cases} k_L & \alpha = L \\ \emptyset & \text{otherwise} \end{cases} \tag{13}
$$

Through direct calculation, we will show how to get this constraint naturally. Then we prove the equality between this orbifold instanton partition function and the $SU(N)$ orbifold vortex partition function. A necessary tool to achieve these goals is the following proposition.

Proposition 4.1 *For generic orbifold space, when $m^{(L)}_{\alpha,\beta} = 0$, $Y^{(L)}_\alpha$ should equal to $Y^{(L+1)}_\beta$ and when $m^{(L)}_{\alpha,\beta} = \epsilon_2$, $Y^{(L+1)}_\beta$ should have one more row than that of $Y^{(L)}_\alpha$. In this latter situation, if we further suppose the orbifold space is $\mathbb{C}/\mathbb{Z}_2$, $Y^{(L)}_\alpha$ has M rows with lengths $k_1 \le k_2 \le \ldots \le k_M$ and $Y^{(L+1)}_\beta$ had $M+1$ rows with lengths $l_0 \le l_1 \le \ldots \le l_M$, then for $1 \le i \le M$ either $k_i = l_{i-1}$ or $k_i = l_i + 1$.*

One important observation is that in the self-dual limit $\epsilon_+ = 0$, the boxes contribute to orbifold instanton partition function are picked out by relative hook length. So, upto minor modifications the proof of the degeneracy phenomenon in [6] is valid in orbifold case and the above proposition can be proved analogously.

4.1 Constraint from fundamental hypermultiplets

From the formula (10), we know for antifundamental hypermultiplets, if we want to get $Y_\alpha = \emptyset$, it is necessary that $a_\alpha + m_f = 0$ and the box $(1,1)$ satisfy the δ-function, that is $q_\alpha - q_\beta^{af} = 0$ for some β. On the other hand , if we want to reduce Y_α to be one row, then $a_\alpha + m_f = -\epsilon_2$ and the box $(1,2)$ should satisfy the δ-function, that is $q_\alpha - q_\beta^{af} = 1$ for some β. In order to satisfy (13), we should take:

$$\left\{ \begin{array}{c} a_\alpha^{(1)} + m_\alpha = -\epsilon_2 \; \delta_{\alpha,1} \\ q_\alpha^{(1)} - q_\alpha^{\text{af}} = \delta_{\alpha,1} \bmod p \end{array} \right. \tag{14}$$

4.2 Constraint from bifundamental hypermultiplets

Using (4.1), It is easy to find that in order to satisfy (13), following identities should be satisfied:

$$\left\{ \begin{array}{c} m_{\alpha,\alpha}^{(L)} = \epsilon_2 \delta_{\alpha,L+1} \\ Q_{\alpha,\alpha}^{(L)} = -\delta_{\alpha,L+1} \bmod p \end{array} \right. \tag{15}$$

Which means:

$$Q_{\alpha,\beta}^{(L)} = \left\{ \begin{array}{cc} q_{\alpha,\beta}^{(L)} = q_{\alpha,\beta}^{(L+1)} & \alpha \in [1,L]; \beta = [1,L] \\ q_{\alpha,\beta}^{(L+1)} & a \in [1,L]; \beta \in [L+1,N] \\ q_{\alpha,\beta}^{(L)} & \alpha \in [L+1,N]; \beta = [1,L] \end{array} \right.$$

We see that the pattern of $Q_{\alpha,\beta}^{(L)}$ is the same as that of $m_{\alpha,\beta}^{(L)}$. So the degeneration of ordinary instanton partition functions also happens for instanton partition functions on orbifolds. With almost the same calculation as in [6], we extract orbifold vortex partition function from orbifold instanton partition function.

4.3 Reshuffling the partition function

In order to make formulas lighter, we will make the δ-functions of discrete charges implicit and use following notations:

$$(x)_k^+ := (x)_k = \prod_{i=0}^{k-1} (x + i\epsilon_2) \qquad (x)_k^- := \prod_{i=1}^{k} (x - \epsilon_2 i)$$

Now let's input (15) into (8) and find the contribution from the L-th vector-multiplet is:

$$\begin{aligned}
\left(Z_L^\Gamma\right)^{-1} &= A\cdot B\cdot C \\
A &= \prod_{\alpha,\beta=1}^{L}\left(a_{\alpha,\beta}^{(L)}\right)_{k_\alpha,k_\beta} \\
B &= \prod_{\alpha=1}^{L}\prod_{\beta=L+1}^{N}(-1)^{k_\alpha}\left(a_{\beta,\alpha}^{(L)}\right)_{k_\alpha} = \prod_{\alpha=1}^{L}\prod_{\beta=L+1}^{N}\left(a_{\alpha,\beta}^{(L)}\right)^{-}_{k_\alpha} \qquad (16)\\
C &= \prod_{\beta=1}^{L}\prod_{\alpha=L+1}^{N}\left(a_{\alpha,\beta}^{(L)}\right)^{+}_{k_\beta}
\end{aligned}$$

After suitable reshuffling we also get the contribution from the L-th bifundamental hypermultiplet as:

$$\begin{aligned}
Z_{L,L+1}^\Gamma &= \mathrm{I}\cdot\mathrm{II}\cdot\mathrm{III} \\
\mathrm{I} &= \left\{\prod_{\alpha=1}^{L}\prod_{\beta=1}^{L}\left(m_{\alpha,\beta}^{(L)}\right)_{k_\alpha,k_\beta}\right\} \\
\mathrm{II} &= \left\{\prod_{\alpha=1}^{L}\prod_{\beta=L+2}^{N}\left(m_{\alpha,\beta}^{(L)}\right)^{-}_{k_\alpha}\right\}\left\{\prod_{\alpha=1}^{L}\left(m_{\alpha,L+1}^{(L)}\right)_{k_\alpha,k_{L+1}}\right\} \qquad (17)\\
\mathrm{III} &= \left\{\prod_{\alpha=L+1}^{N}\prod_{\beta=1}^{L}\left(m_{\alpha,\beta}^{(L)}\right)^{+}_{k_\beta}\right\}\left\{\prod_{\alpha=L+1}^{N}\left(m_{\alpha,L+1}^{(L)}\right)^{+}_{k_{L+1}}\right\}
\end{aligned}$$

So:

$$Z_L^\Gamma Z_{L,L+1}^\Gamma = \frac{\left\{\prod_{\alpha=1}^{L}\left(a_{\alpha,L+1}^{(L+1)}\right)_{k_\alpha,k_{L+1}}\right\}}{\left\{\prod_{\alpha=1}^{L}\left(a_{\alpha,L+1}^{(L)}\right)^{-}_{k_\alpha}\right\}}\left\{\prod_{\alpha=L+2}^{N}\left(a_{\alpha,L+1}^{(L+1)}\right)^{+}_{k_{L+1}}\right\}(\epsilon_2)^{+}_{k_{L+1}} \qquad (18)$$

Other factors are:

$$\begin{aligned}
Z_\Gamma^{\text{fund}} &= \prod_{f=1}^{N}\prod_{i=1}^{k_1}\left(a_1^{(1)}+m_f-\epsilon_2(i-1)\right) = (-\epsilon_2)^{-}_{k_1}\prod_{f=2}^{N}\left(a_{1,f}^{(1)}\right)^{-}_{k_1} \qquad (19)\\
Z_N^\Gamma &= \prod_{\alpha=1}^{N}(\epsilon_2)^{+}_{k_\alpha}(\epsilon_2)^{-}_{k_\alpha}\prod_{\alpha<\beta}^{N}\left(a_{\alpha,\beta}^{(N)}\right)_{k_\alpha,k_\beta}\left(a_{\beta,\alpha}^{(N)}\right)_{k_\beta,k_\alpha} \qquad (20)
\end{aligned}$$

Parameters in above formulas are not independent, since from the explicit form of $m_{\alpha,\alpha}^{(L)}$, we know:

$$a_{\alpha,\beta}^{(L+1)} - a_{\alpha,\beta}^{(L)} = -\epsilon_2\left(\delta_{\alpha,L+1}-\delta_{\beta,L+1}\right)$$

It follows that:

$$\begin{array}{rr} a^{(L)}_{K,L} = a^{(N)}_{K,L} & L \in [2,N], K < L \\ a^{(L)}_{K,L+1} = a^{(K)}_{K,L+1} & L \in [K,N-1], K \in [1,N-1] \end{array} \tag{21}$$

Similar relations are found for discrete charges:

$$\begin{array}{rr} Q^{(L)}_{K,L} = Q^{(N)}_{K,L} & L \in [2,N], K < L \\ Q^{(L)}_{K,L+1} = Q^{(K)}_{K,L+1} & L \in [K,N-1], K \in [1,N-1] \end{array} \tag{22}$$

This induce the identification of following factors:

$$\prod_{L=1}^{N-1} \left\{ \prod_{\alpha=1}^{L} \left(a^{(L+1)}_{\alpha,L+1} \right)_{k_\alpha, k_{L+1}} \right\} = \prod_{\alpha<\beta}^{N} \left(a^{(N)}_{\beta,\alpha} \right)_{k_\beta,k_\alpha} \left(a^{(N)}_{\alpha,\beta} \right)_{k_\alpha,k_\beta}$$

$$\left\{ \prod_{f=2}^{N} \left(a^{(1)}_{1,f} \right)_{k_1} \right\} \left\{ \prod_{L=1}^{N-1} \prod_{\alpha=L+2}^{N} \left(a^{(L+1)}_{L+1,\alpha} \right)_{k_{L+1}} \right\} = \prod_{L=1}^{N-1} \left\{ \prod_{\alpha=1}^{L} \left(a^{(L)}_{\alpha,L+1} \right)_{k_\alpha} \right\}$$

With these identities we have :

$$Z_{\text{Quiver}}(k) = \frac{\prod_{\beta=1}^{N} Z_{\Gamma}^{\text{fund},q_\beta^f} \left(Y^{(N)} \right)}{\prod_{\alpha=1}^{N} (\epsilon_2)_{k_\alpha} \prod_{\alpha<\beta}^{N} \left(a^{(N)}_{\beta,\alpha} \right)_{k_\beta,k_\alpha}} \tag{23}$$

The equality in above formula is exact upto an overall sign factor which will disappear after redefine counting parameters. We recognize above formula is the same as orbifold vortex partition function, if we identify $a^{(N)}_\alpha$ in (8) with a_α in (12). A comment here is that the moduli space of orbifold instanton may have nontrivial first Chern class. We will concentrate on the case when the first Chern class is trivial which will give extra constraints on Young-tableaus. But this does not affect all the arguments in this section.

5 Vortex on $\mathbb{C}/\mathbb{Z}_2$ and $\mathcal{N}=1$ Super Liouville Theory

In [3] and [4] people discussed about AGT like relation between instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathcal{N}=1$ super Liouville theory. In the following, we will study the relation between SU(2) vortex on $\mathbb{C}/\mathbb{Z}_2$ and degenerate states in $\mathcal{N}=1$ super Liouville theory.

5.1 SU(2) Vortex on $\mathbb{C}/\mathbb{Z}_2$

Vector field contribution

$$\begin{aligned} \left(Z^{\text{vector}}_{\Gamma,\text{vortex}}\left(a,\hbar;k;q_{1,2}\right)\right)^{-1} &= U^{\text{vector}}_{\Gamma,\text{vortex}}(\hbar,k)O^{\text{vector}}_{\Gamma,\text{vortex}}\left(a,\hbar;k;q_{1,2}\right) \\ U^{\text{vector}}_{\Gamma,\text{vortex}}(\hbar,k) &= \prod_{\alpha=1}^{2}(2\hbar)^{\lfloor\frac{k_\alpha}{2}\rfloor}\left\lfloor\frac{k_\alpha}{2}\right\rfloor! \\ O^{\text{vector}}_{\Gamma,\text{vortex}}\left(a,\hbar;k;q_{1,2}\right) &= \prod_{i=1}^{k_1}\left(a_{1,2}+\hbar\left(k_2+1-i\right)\right)\delta_{-i+1+k_2,q_{1,2}} \\ & \prod_{j=1}^{k_2}\left(a_{2,1}+\hbar\left(k_1+1-j\right)\right)\delta_{-j+1+k_1,q_{2,1}} \end{aligned} \tag{24}$$

The contributions to the partition function from vector field are classified by $q_{1,2}$. Since $q_{1,2}$ takes value in $\mathbb{Z}_2$, there are two different branches. In the following, we will set $a_1 = a; a_2 = -a$ and $\lfloor x\rfloor$ is the floor function that is the largest integer not greater than x .

$$O^{\text{vector}}_{\Gamma,\text{vortex}}(a,\hbar;k;0) = D^0_{k_1,k_2}\prod_{i=1}^{\lfloor\frac{k_1}{2}\rfloor}(-2a+2\hbar i)(2\hbar i)\prod_{i=1}^{\lfloor\frac{k_2}{2}\rfloor}(2a+2\hbar i)(2\hbar i) \tag{25}$$

$$O^{\text{vector}}_{\Gamma,\text{vortex}}(a,\hbar;k;1) = D^1_{k_1,k_2}\prod_{i=1}^{\lceil\frac{k_1}{2}\rceil}(-2a+\hbar(2i-1))\prod_{i=1}^{\lceil\frac{k_2}{2}\rceil}(2a+\hbar(2i-1)) \tag{26}$$

The pre-factors are defined as:

$$D^0_{k_1,k_2} := \begin{cases} \frac{2a(-1)^{\frac{k_1+k_2}{2}}}{2a+\hbar(k_2-k_1)}(-1)^{k_1} & k_1+k_2\text{even} \\ (-1)^{\frac{k_2+1+k_1}{2}}2a & k_1+k_2\text{odd} \end{cases} \tag{27}$$

$$D^1_{k_1,k_2} = \begin{cases} (-1)^{\frac{k_1+k_2}{2}}(-1)^{k_1} & k_1+k_2\text{even} \\ \frac{(-1)^{\frac{k_2-1+k_1}{2}}}{2a+\hbar(k_2-k_1)}(-1)^{k_1} & k_1+k_2\text{odd} \end{cases} \tag{28}$$

Fundamental hypermultiplets contribution

Since $q_\alpha - q_f$ only can take values of 0 and 1, there are four types contributions from fundamental hypermultiplets. When $q_{1,2} = 0$, we have:

$$
\begin{aligned}
Z_{\Gamma,\text{vortex}}^{\text{fund},0,0}(a, m_f, \hbar; k) &= \prod_{\alpha=1}^{2} \prod_{i=1}^{\lceil \frac{k_\alpha}{2} \rceil} (m_{\alpha,f} + 2\hbar(i-1)) \\
Z_{\Gamma,\text{vortex}}^{\text{fund},0,1}(a, m, \hbar; k) &= \prod_{\alpha=1}^{2} \prod_{i=1}^{\lfloor \frac{k_\alpha}{2} \rfloor} (m_{\alpha,f} + \hbar(2i-1))
\end{aligned}
\tag{29}
$$

Where $m_{\alpha,f} = a_\alpha - m_f$ and $\lceil x \rceil$ is the ceiling function that is the smallest integer not less than x. When $q_{1,2} = 1$, we have:

$$
\begin{aligned}
Z_{\Gamma,\text{vortex}}^{\text{fund},1,0}(a, m_f, \hbar; k) &= \prod_{i=1}^{\lceil \frac{k_1}{2} \rceil} (m_{1,f} + 2\hbar(i-1)) \prod_{i=1}^{\lfloor \frac{k_2}{2} \rfloor} (m_{2,f} + \hbar(2i-1)) \\
Z_{\Gamma,\text{vortex}}^{\text{fund},1,1}(a, m, \hbar; k) &= \prod_{i=1}^{\lfloor \frac{k_1}{2} \rfloor} (m_{1,f} + \hbar(2i-1)) \prod_{i=1}^{\lceil \frac{k_2}{2} \rceil} (m_{2,f} + 2\hbar(i-1))
\end{aligned}
\tag{30}
$$

Notice that we use two integers in the superscript to denote the types of fundamental hypermultiplet contributions.

Vortex partition functions

Unlike non-orbifold case, where there is only one vortex partition function, orbifold vortex partition function has many sectors characterized by discrete charges.

$$
\begin{aligned}
Z_\Gamma^{\text{vortex}}(q_{1,2}, p_1, p_2; k) := \; & Z_{\Gamma,\text{vortex}}^{\text{vector}}(a, \hbar; k; q_{1,2}) \, Z_{\Gamma,\text{vortex}}^{\text{fund},q_{1,2},p_1}(a, m_1, \hbar; k) \\
& Z_{\Gamma,\text{vortex}}^{\text{fund},q_{1,2},p_2}(a, m_2, \hbar; k)
\end{aligned}
\tag{31}
$$

In the LHS of above formula we make a and the mass parameters implicit to make the formula shorter. In general there are eight different types, since the integers of the LHS can only take values in 0 and 1. Four examples related to our discussion are:

$$
Z_\Gamma^{\text{vortex}}(0,0,0;k) = \frac{1}{D^0_{k_1,k_2}} \frac{\prod_{\alpha=1}^{2} \prod_{i=1}^{\lceil \frac{k_\alpha}{2} \rceil} (m_{\alpha,1} + 2\hbar(i-1)) \prod_{i=1}^{\lceil \frac{k_\alpha}{2} \rceil} (m_{\alpha,2} + 2\hbar(i-1))}{\prod_{i=1}^{\lfloor \frac{k_1}{2} \rfloor} (-2a + 2\hbar i)(2\hbar i) \prod_{i=1}^{\lfloor \frac{k_2}{2} \rfloor} (2a + 2\hbar i)(2\hbar i)} \tag{32}
$$

$$
Z_\Gamma^{\text{vortex}}(0,0,1;k) = \frac{1}{D^0_{k_1,k_2}} \frac{\prod_{\alpha=1}^{2} \prod_{i=1}^{\lceil \frac{k_\alpha}{2} \rceil} (m_{\alpha,1} + 2\hbar(i-1)) \prod_{i=1}^{\lfloor \frac{k_\alpha}{2} \rfloor} (m_{\alpha,2} + 2\hbar(i-1))}{\prod_{i=1}^{\lfloor \frac{k_1}{2} \rfloor} (-2a + 2\hbar i)(2\hbar i) \prod_{i=1}^{\lfloor \frac{k_2}{2} \rfloor} (2a + 2\hbar i)(2\hbar i)} \tag{33}
$$

$$Z^{\text{vortex}}_{\Gamma}(1,0,0;k) = \frac{1}{D^1_{k_1,k_2}} \frac{\prod_f^2 \prod_{i=1}^{\lceil \frac{k_1}{2} \rceil} (m_{1,f} + 2\hbar(i-1)) \prod_{i=1}^{\lfloor \frac{k_2}{2} \rfloor} (m_{2,f} + \hbar(2i-1))}{\prod_{\alpha=1}^2 (2\hbar)^{\lfloor \frac{k_\alpha}{2} \rfloor} \lfloor \frac{k_\alpha}{2} \rfloor ! \prod_{i=1}^{\lceil \frac{k_1}{2} \rceil} (-2a + \hbar(2i-1)) \prod_{i=1}^{\lceil \frac{k_2}{2} \rceil} (2a + \hbar(2i-1))} \tag{34}$$

$$Z^{\text{vortex}}_{\Gamma}(1,0,1;k) = \frac{1}{D^0_{k_1,k_2}} \frac{\prod_f^2 \prod_{i=1}^{\lfloor \frac{k_1}{2} \rfloor} (m_{1,f} + \hbar(2i-1)) \prod_{i=1}^{\lceil \frac{k_2}{2} \rceil} (m_{2,f} + 2\hbar(i-1))}{\prod_{\alpha=1}^2 (2\hbar)^{\lfloor \frac{k_\alpha}{2} \rfloor} \lfloor \frac{k_\alpha}{2} \rfloor ! \prod_{i=1}^{\lceil \frac{k_1}{2} \rceil} (-2a + \hbar(2i-1)) \prod_{i=1}^{\lceil \frac{k_2}{2} \rceil} (2a + \hbar(2i-1))} \tag{35}$$

Since there are more branches of orbifold instanton partition functions than the types of four point correlation functions, it is reasonable that not all kinds of orbifold instanton partition function has a super Liouville theory explanation. Correspondingly not all of above vortex partition functions will correspond to correlation functions with degenerate states in super Liouville theory. Considering the symmetry between fundamental and antifundamental hypermultiplets of linear quiver gauge theories, we will show in following subsection only (33), (34), and (35) may have conformal filed theory explanations. Let's first concentrate on (33):

$$\begin{aligned}
&Z^{\text{vortex}}_{\Gamma}(0,0,1) := \sum_k z_1^{k_1} z_2^{k_2} Z^{\text{vortex}}_{\Gamma}(0,0,1;k) := \\
&\sum_l \big(z_1^{2l_1} z_2^{2l_2} Z^{\text{vortex}}_{\Gamma}(0,0,1;\{2l_1,2l_2\}) + z_1^{2l_1} z_2^{2l_2+1} Z^{\text{vortex}}_{\Gamma}(0,0,1;\{2l_1,2l_2+1\}) \\
&+ z_1^{2l_1+1} z_2^{2l_2} Z^{\text{vortex}}_{\Gamma}(0,0,1;\{2l_1+1,2l_2\}) + z_1^{2l_1+1} z_2^{2l_2+1} Z^{\text{vortex}}_{\Gamma}(0,0,1;\{2l_1+1,2l_2+1\}) \big)
\end{aligned}$$

Where l_1 and l_2 are non-negative integers.

For l_1 and l_2 even:

$$\begin{aligned}
&\sum_l z_1^{2l_1} z_2^{2l_2} Z^{\text{vortex}}_{\Gamma}(0,0,1;\{2l_1,2l_2\}) = \\
&\left(1 + \frac{\hbar}{2a}\left(z_2\partial_{z_2} - z_1\partial_{z_1}\right)\right) F\left(\frac{m_{1,1}}{2\hbar}, \frac{m_{1,2}}{2\hbar}, \frac{-2a}{2\hbar}+1, -z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar}, \frac{m_{2,2}}{2\hbar}, \frac{2a}{2\hbar}+1, -z_2^2\right)
\end{aligned} \tag{36}$$

For l_1 even and l_2 odd:

$$\begin{aligned}
&\sum_l z_1^{2l_1} z_2^{2l_2+1} Z^{\text{vortex}}_{\Gamma}(0,0,1;\{2l_1,2l_2+1\}) = \\
&\frac{-z_2}{2a} F\left(\frac{m_{1,1}}{2\hbar}, \frac{m_{1,2}}{2\hbar}, \frac{-2a}{2\hbar}+1, -z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar}, \frac{m_{2,2}}{2\hbar}, \frac{2a}{2\hbar}+1, -z_2^2\right)
\end{aligned} \tag{37}$$

For l_1 odd and l_2 even:

$$\begin{aligned}
&\sum_l z_1^{2l_1+1} z_2^{2l_2} Z^{\text{vortex}}_{\Gamma}(0,0,1;\{2l_1+1,2l_2\}) = \\
&-\frac{m_{1,1}m_{1,2}z_1}{2a} F\left(\frac{m_{1,1}}{2\hbar}+1, \frac{m_{1,2}}{2\hbar}+1, \frac{-2a}{2\hbar}+1, -z_1^2\right) F\left(\frac{m_{1,1}}{2\hbar}, \frac{m_{2,1}}{2\hbar}, \frac{2a}{2\hbar}+1, -z_2^2\right)
\end{aligned} \tag{38}$$

For l_1 odd and l_2 odd:

$$\sum_l z_1^{2l_1+1} z_2^{2l_2+1} Z_\Gamma^{\text{vortex}}\left(0,0,1;\{2l_1+1,2l_2+1\}\right) = \tag{39}$$

$$z_1 z_2 m_{1,1} m_{1,2}\left(1+\frac{\hbar}{2a}\left(z_2\partial_{z_2}-z_1\partial_{z_1}\right)\right) F\left(\frac{m_{1,1}}{2\hbar}+1,\frac{m_{1,2}}{2\hbar}+1,\frac{-2a}{2\hbar}+1,-z_1^2\right)$$
$$F\left(\frac{m_{2,1}}{2\hbar},\frac{m_{2,2}}{2\hbar},\frac{2a}{2\hbar}+1,-z_2^2\right)$$

Separately, each of them can be considered as some intertwine differential operators acting on products of two hypergeometric functions.

Another type of vortex partition function which we want to calculate explicitly is (35):

$$Z_\Gamma^{\text{vortex}}\left(1,0,1\right) := \sum_k z_1^{k_1} z_2^{k_2} Z_\Gamma^{\text{vortex}}\left(q_{1,2}=1,0,1;k\right) := \tag{40}$$
$$\sum_l \left(z_1^{2l_1} z_2^{2l_2} Z_\Gamma^{\text{vortex}}\left(1,0,1;\{2l_1,2l_2\}\right) + z_1^{2l_1} z_2^{2l_2+1} Z_\Gamma^{\text{vortex}}\left(1,0,1;\{2l_1,2l_2+1\}\right)\right.$$
$$\left.+ z_1^{2l_1+1} z_2^{2l_2} Z_\Gamma^{\text{vortex}}\left(1,0,1;\{2l_1+1,2l_2\}\right) + z_1^{2l_1+1} z_2^{2l_2+1} Z_\Gamma^{\text{vortex}}\left(1,0,1;\{2l_1+1,2l_2+1\}\right)\right)$$

For l_1 even and l_2 even:

$$\sum_l z_1^{2l_1} z_2^{2l_2} Z_\Gamma^{\text{vortex}}\left(1,0,1;\{2l_1,2l_2\}\right) =$$
$$F\left(\frac{m_{1,1}}{2\hbar},\frac{m_{1,2}}{2\hbar},\frac{-2a}{2\hbar}-\frac{3}{2},-z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar},\frac{m_{2,2}}{2\hbar},\frac{2a}{2\hbar}-\frac{3}{2},-z_2^2\right) \tag{41}$$

For l_1 even and l_2 odd:

$$\sum_l z_1^{2l_1} z_2^{2l_2+1} Z_\Gamma^{\text{vortex}}\left(1,0,1;\{2l_1,2l_2+1\}\right) = z_2 \frac{2a+2\hbar+\hbar\left(z_2\partial_{z_2}-z_1\partial_{z_1}\right)}{2a+\hbar}$$
$$F\left(\frac{m_{1,1}}{2\hbar},\frac{m_{1,2}}{2\hbar},\frac{-2a}{2\hbar}-\frac{3}{2},-z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar},\frac{m_{2,2}}{2\hbar},\frac{2a}{2\hbar}+\frac{1}{2},-z_1^2\right) \tag{42}$$

For l_1 odd and l_2 even:

$$\sum_l z_1^{2l_1+1} z_2^{2l_2} Z_\Gamma^{\text{vortex}}\left(1,0,1;\{2l_1+1,2l_2\}\right) = z_1 m_{1,1} m_{1,2} \frac{2a-2\hbar+\hbar\left(z_2\partial_{z_2}-z_1\partial_{z_1}\right)}{2a-\hbar}$$
$$F\left(\frac{m_{1,1}}{2\hbar}+1,\frac{m_{1,2}}{2\hbar}+1,\frac{-2a}{2\hbar}+\frac{3}{2},-z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar},\frac{m_{2,2}}{2\hbar},\frac{2a}{2\hbar}+\frac{1}{2},-z_2^2\right) \tag{43}$$

For l_1 odd and l_2 odd:

$$\sum_l z_1^{2l_1+1} z_2^{2l_2+1} Z_\Gamma^{\text{vortex}}\left(1,0,1;\{2l_1+1,2l_2+1\}\right) = \frac{z_1 z_2 m_{1,1} m_{1,2}}{(-2a+\hbar)(2a+\hbar)}$$
$$F\left(\frac{m_{1,1}}{2\hbar}+1,\frac{m_{1,2}}{2\hbar}+1,\frac{-2a}{2\hbar}+\frac{3}{2},-z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar},\frac{m_{2,2}}{2\hbar},\frac{2a}{2\hbar}+\frac{3}{2},-z_2^2\right) \tag{44}$$

A universal property of SU(2) $\mathbb{Z}_2$ orbifold vortex partition functions is that they are quadratic forms of Gaussian hypergeometric functions. This is the same for non-orbifold case and one big difference is the effective counting parameter is $2\hbar$ for orbifold case while $\hbar$ for non-orbifold case. We will see the CFT correspondence of these properties.

5.2 Relation to super Liouville theory

Since we know the relation between orbifold vortex partition function and orbifold instanton partition function, we can find the relation between orbifold and vortex through degeneration procedure on super Liouville theory side. There are in principle two ways. (1) Calculate directly the correlation function between two lowest degenerate states and three non-degenerate primary states in $\mathcal{N}=1$ super Liouville theory. (2) If we know the complete AGT relation between partition functions of SU(2) instantons on $\mathbb{C}^2/\mathbb{Z}_2$ and correlation functions of $\mathcal{N}=1$ super Liouville theory with both Ramond and NS primary fields, we get the relation between orbifold vortex and $\mathcal{N}=1$ super Liouville theory almost for free. However, technically both ways are blocked. There are no results concerning (1) and (2) in literature. In the following we will use existing results to analysis the AGT dual of orbifold vortex.

Correlation functions with degenerate fields

As it is clear from previous calculation, in order to extract vortex partition function from instanton partition function, the parameters $m^{(L)}_{\alpha,\alpha}$ are restrict to be 0 or ϵ_2, this means on the CFT side the fusion rule is that from lowest degenerate states, i.e. those with momentum equals $\frac{-b}{2}$. It is known that the lowest degenerate states in NS- and R-sector have momentum equal to $\frac{-3b}{2}$ and $\frac{-b}{2}$ respectively. So the CFT dual of $SU(2)$ orbifold vortex should come from five point correlation functions with two lowest degenerate states in the R-sector. Possible configurations are show in Fig 1. In the following we will use V^R_α and V^{NS}_α to denote primary fields with momentum α in Ramond- and NS-sector.

In Fig 1, V^R_α and V^{NS}_α denote primary fields with momentum α in Ramond- and NS-sector and I_R, I_{NS} are identity operators in Ramond- and NS-sector. To exactly check our proposal, we need to know the AGT correspondence of the following correlation functions in super Liouville theory:

$$\left\langle V^R_{\alpha_1}V^R_{\alpha_2}V^R_{\alpha_3}V^R_{\alpha_4}\right\rangle_{\text{NS}} \quad \text{and} \quad \left\langle V^{\text{NS}}_{\alpha_1}V^R_{\alpha_2}V^R_{\alpha_3}V^{\text{NS}}_{\alpha_4}\right\rangle_R \tag{45}$$

$$\left\langle V^{\text{NS}}_{\alpha_1}V^R_{\alpha_2}V^{\text{NS}}_{\alpha_3}V^R_{\alpha_4}\right\rangle_R \quad \text{and} \quad \left\langle V^R_{\alpha_1}V^R_{\alpha_2}V^{\text{NS}}_{\alpha_3}V^{\text{NS}}_{\alpha_4}\right\rangle_{\text{NS}} \tag{46}$$

The subscript in above correlation functions is used to emphasize the types of internal states. Notice that except the first correlation function in (45), the other three are four point correlation function with two Ramond and two NS primary fields. The latter three are not trivially related, since they have different internal states.

The first internal state of the correlation function in Fig 1.a is in NS sector and correspondingly the Kac determinant which gives denominators of conformal blocks is also in NS

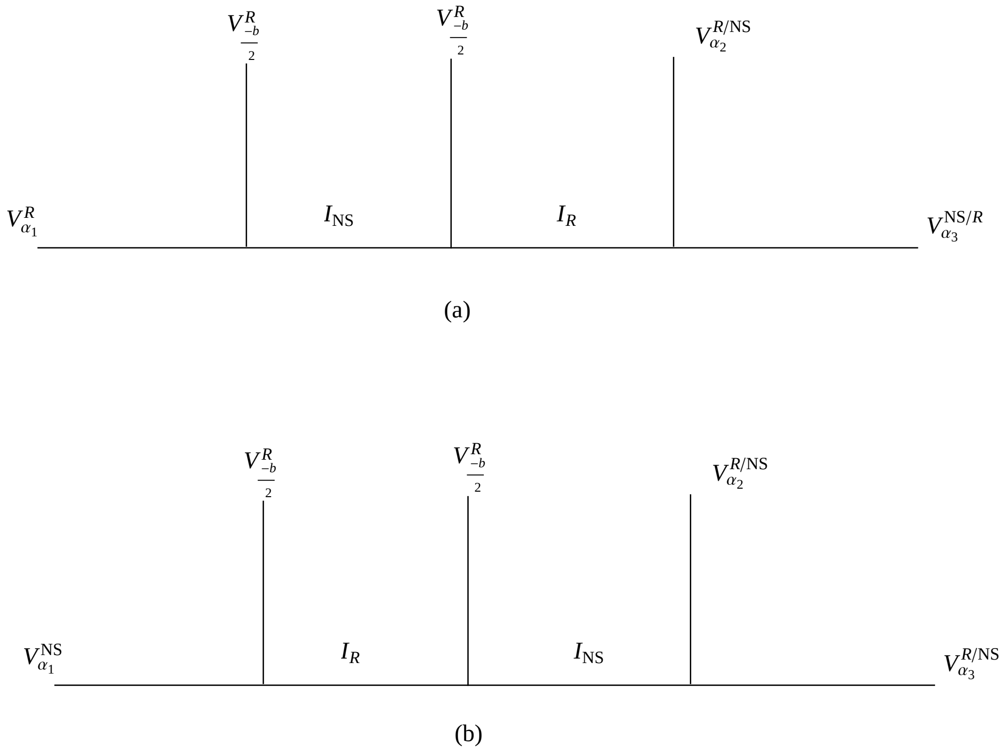

Figure 1: five point correlation functions corresponding to $SU(2)$ $\mathbb{Z}_2$ orbifold vertices.

sector. From [3], we can expect that $q_1^{(1)} = q_2^{(1)} \bmod 2$, since they determine the form of denominators of instanton partition functions (3). Similarly, from [12], we will conjecture that $q_1^{(2)} = q_2^{(2)} + 1 \bmod 2$. According to [6], the fusion rule of the first $V^R_{\frac{-b}{2}}$ corresponds to the choice (14), this means that when $q_1^{(1)} = q_2^{(1)} \bmod 2$, $q_1^{\mathrm{af}} = q_2^{\mathrm{af}} + 1 \bmod 2$ and when $q_1^{(1)} = q_2^{(1)} + 1 \bmod 2$, $q_1^{\mathrm{af}} = q_2^{\mathrm{af}} \bmod 2$. Our choice of the discrete charges is different from that of [3], which in our language is $q_1^{(1)} = q_2^{(1)} \bmod 2$ and $q_1^{\mathrm{af}} = q_2^{\mathrm{af}} \bmod 2$. If we further consider the symmetry between fundamental and antifundamental hypermultiplets, $q_\alpha^f = q_\alpha^{\mathrm{af}}$, we find that only (33), (34), (35) can be identified as correlation function in Fig 1.a

However, presently there are no results in literature that we can use to give a direct check of our claim. What we know are the following four point correlation functions in Fig.2, which are calculated in [10].

For $\left\langle V^{NS}_{\alpha_1} V^R_{\frac{-b}{2}} V^R_{\alpha_2} V^{NS}_{\alpha_3} \right\rangle$, the hypergeometric function factors are:

$$F\left(\frac{1}{2b^{-1}}\left(\alpha_1+\alpha_3+\alpha_4\right)+\frac{3}{4}, \frac{1}{2b^{-1}}\left(a_1+\alpha_3-\alpha_4\right)+\frac{3}{4}, \frac{2\alpha_1}{2b^{-1}}+\frac{3}{2}\right) \tag{47}$$

$$F\left(\frac{1}{2b^{-1}}\left(a_1+\alpha_3+\alpha_4\right)+\frac{1}{4}, \frac{1}{2b^{-1}}\left(a_1+\alpha_3-\alpha_4\right)+\frac{1}{4}, \frac{2\alpha_1}{2b^{-1}}+\frac{1}{2}\right) \tag{48}$$

For $\left\langle V^R_{\alpha_1} V^R_{\frac{-b}{2}} V^R_{\alpha_2} V^R_{\alpha_3} \right\rangle$, the hypergeometric function factors are:

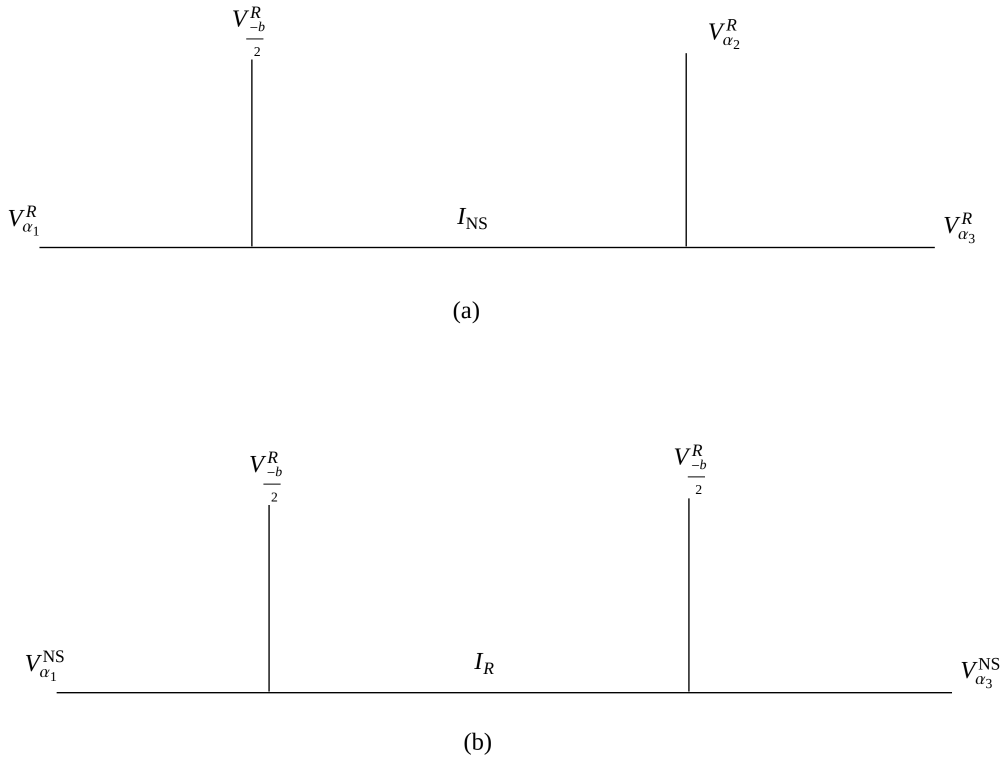

Figure 2: four point correlation functions in Super Liouville theory

$$F\left(\frac{1}{2b^{-1}}\left(a_1+\alpha_3+\alpha_4\right)+\frac{3}{4},\frac{1}{2b^{-1}}\left(a_1+\alpha_2-\alpha_3\right)+\frac{3}{2},\frac{2\alpha_1}{2b^{-1}}+\frac{3}{2}\right) \tag{49}$$

$$F\left(\frac{1}{2b^{-1}}\left(a_1+\alpha_3+\alpha_4\right)+\frac{3}{4},\frac{1}{2b^{-1}}\left(a_1+\alpha_2-\alpha_3\right)+\frac{3}{2},\frac{2\alpha_1}{2b^{-1}}+\frac{1}{2}\right) \tag{50}$$

The one for Fig 2.b is also calculated in [7] with a different convention.

$$F\left(\frac{1}{2b^{-1}}\left(\alpha_1+\alpha_2-\alpha_3\right)-\frac{1}{4},\frac{1}{2b^{-1}}\left(\alpha_1+\alpha_2+\alpha_3\right)-\frac{1}{4},\frac{2\alpha_1}{2b^{-1}}+1\right) \tag{51}$$

We can see that after a linear map between parameters of orbifold vortices and degenerate four point correlation functions in super Liouville theory, we can identify the hypergeometric function factors of both sides.

$$\begin{aligned} b^{-1} &= \hbar \\ \alpha_1 &= a+\text{const} \\ \alpha_2+\alpha_3 &= m_1+\text{const} \\ \alpha_2+\alpha_3 &= m_2+\text{const} \end{aligned}$$

The constants depends on which pair of hypergeometric functions we are comparing. This is an evidence that orbifold vortex partition functions should correspond to correlation functions of lowest degenerate Ramond fields as show in Fig.1. It also tells us that the identification of parameters of orbifold instanton partition functions and that of correlation functions of the super Liouville theory in mixed sectors is the same–upto a constant shit– as in original AGT paper [1]. It is important to notice that as in non-orbifold case the four point correlation functions in Fig.2 can not be identified with Abelian vortex partition function, since the former has three parameters– the three momentums, while the latter has only two parameters–the two masses of fundamental hypermultiplets. So a direct check of our proposal should start from a direct clear calculation of the correlation functions in Fig.1, which is a hard problem due to the subtleties coming from the multi-branch of super conformal generator in R-sector and also the double vacuua in R-sector. We leave this problem in future study.

If we consider four point correlation functions with one degenerate fields as the "partition" function of surface operators, we will have two types of simple surface operators in the gauge theory dual of $\mathcal{N}=1$ super Liouville theory, since super Liouville theory has two types of lowest degenerate states. Exactly, for $\mathbb{Z}_2$ orbifold $SU(2)$ gauge theory with flavor number equals 2, the instanton partition functions only have two types of lowest degeneration.

6 Discussions

We consider some functions which are the four dimensional limit of strip amplitudes satisfying the same boxes selection rule as orbifold instanton partition functions and denote them by $\mathcal{A}(a, m, Y)$, where a and m are parameters associated with Coulomb branch parameters and masses, and Y are N-dimensional arrow of Young-tableaus. Then a natural question is whether we can reduce orbifold instanton partition function of a quiver gauge theory to these functions with general Young-tableaus as we did for non-orbifold case [6]. By the proposition 4.1, we can show that it is doable for two situations. (1)If discrete charges take value in $\mathbb{Z}_p$ for general p, Y should be an arrow of N rows, which is just the vortex case. (2) If discrete charges take value in $\mathbb{Z}_2$, Y can be arbitrary. This makes the $\mathbb{Z}_2$ case especially simple and it is expected to interpret simple surface operators in $\mathbb{Z}_2$ orbifold gauge theory as degenerate fields in $\mathcal{N}=1$ super Liouville theory.

Using degenerate fields as a probe, we should be able to get a full AGT correspondence between instanton partition functions on $\mathbb{C}/\mathbb{Z}_2$ and $\mathcal{N}=1$ super Liouville theory. Exactly, we get a relation between a certain branch of instanton partition function and correlation function with four primary Ramond fields and check this relation up-to three instanton contributions. Further checks to higher order instanton contributions and other types of correlation functions are left for future work [16].

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References

[1] L. F. Alday, D. Gaiotto and Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories," arXiv:0906.3219 [hep-th].

[2] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, "Loop and surface operators in N=2 gauge theory and Liouville modular geometry," arXiv:0909.0945 [hep-th]. C. Kozcaz, S. Pasquetti and N. Wyllard, "A and B model approaches to surface operators and Toda theories" JHEP **1008** (2010) 042 [arXiv:1004.2025 [hep-th]]. T. Dimofte, S. Gukov and L. Hollands, "Vortex Counting and Lagrangian 3-manifolds," arXiv:1006.0977 [hep-th]. A. Marshakov, A. Mironov and A. Morozov, "On AGT Relations with Surface Operator Insertion and Stationary Limit of Teor. Mat. Fiz. **164** (2010) 1:3 [arXiv:1011.4491 [hep-th]]; H. Kanno and Y. Tachikawa, "Instanton counting with a surface operator and the chain-saw quiver," JHEP **1106**, 119 (2011); M. Taki, "Surface Operator, Bubbling Calabi-Yau and AGT Relation," arXiv:1007.2524 [hep-th]. D. Gaiotto, "Surface Operators in N=2 4d Gauge Theories," arXiv:0911.1316 [hep-th]. H. Awata, H. Fuji, H. Kanno, M. Manabe and Y. Yamada, "Localization with a Surface Operator, Irregular Conformal Blocks and Open Topological String," arXiv:1008.0574 [hep-th]. K. Maruyoshi and M. Taki, "Deformed Prepotential, Quantum Integrable System and Liouville Field Nucl. Phys. B **841**, 388 (2010) [arXiv:1006.4505 [hep-th]]. U. Bruzzo, W. y. Chuang, D. E. Diaconescu, M. Jardim, G. Pan and Y. Zhang, "D-branes, surface operators, and ADHM quiver representations," arXiv:1012.1826 [hep-th]. A. Mironov and A. Morozov, "Nekrasov Functions from Exact BS Periods: the Case of SU(N)," J. Phys. A **43** (2010) 195401 [arXiv:0911.2396 [hep-th]]. A. Mironov and A. Morozov, "Nekrasov Functions and Exact Bohr-Sommerfeld Integrals," JHEP **1004** (2010) 040 [arXiv:0910.5670 [hep-th]]. S. Shadchin, "On F-term contribution to effective action," JHEP **0708**, 052 (2007) [arXiv:hep-th/0611278]. S. Pasquetti, "Factorisation of N = 2 theories on the squashed 3-sphere," arXiv:1111.6905 [hep-th]

[3] V. Belavin and B. Feigin, "Super Liouville conformal blocks from N=2 SU(2) quiver gauge theories" arXiv:1105.5800 [hep-th]. A. Belavin, V. Belavin and M. Bershtein, "Instantons and 2d Superconformal field theory," arXiv:1106.4001 [hep-th]. G. Bonelli, K. Maruyoshi and A. Tanzini, "Instantons on ALE spaces and Super Liouville Conformal Field Theories," arXiv:1106.2505 [hep-th]. G. Bonelli, K. Maruyoshi and A. Tanzini, "Gauge Theories on ALE Space and Super Liouville Correlation Functions," arXiv:1107.4609 [hep-th].

[4] T. Nishioka, Y. Tachikawa, "Para-Liouville/Toda central charges from M5-branes" arXiv:1106.1172 [hep-th]. N. Wyllard, " Coset conformal blocks and N=2 gauge theo-

ries" arXiv:1109.4264 [hep-th]. M. N. Alfimov, G. M. Tarnopolsky, "Parafermionic Liouville field theory and instantons on ALE spaces " arXiv:1110.5628 [hep-th]. A.A. Belavin, M.A. Bershtein, B.L. Feigin, A.V. Litvinov, G.M. Tarnopolsky, "Instanton moduli spaces and bases in coset conformal field theory " arXiv:1111.2803 [hep-th].

[5] G. Bonelli, A. Tanzini and J. Zhao, "Vertices, Vortices and Interacting Surface Operators," arXiv: 1102.0184v1 [hep-th].

[6] G. Bonelli, A. Tanzini and J. Zhao, "The Liouville side of the Vortex" arXiv: 1107.2787v1 [hep-th]

[7] D. Chorazkiewicz, L. Hadasz and Z. Jaskolski, "Braiding properties of the N=1 superconformal blocks (Ramond sector)" arXiv:1108.2355 [hep-th]

[8] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, "Moduli space of non-Abelian vortices", Phys. Rev. Lett. **96** (2006) 161601. M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, "Solitons in the Higgs phase: The moduli matrix approach", J. Phys **A39** (2006) R315–R392. T. Fujimori, G. Marmorini, M. Nitta, K. Ohashi, N. Sakai, "The Moduli Space Metric for Well-Separated Non-Abelian Vortices," Phys. Rev. **D82**, 065005 (2010). M. Eto, T. Fujimori, M. Nitta, K. Ohashi, N. Sakai, "Dynamics of Non-Abelian Vortices," arXiv:1105.1547. Y. Yoshida, "Localization of Vortex Partition Functions in $\mathcal{N} = (2,2)$ Super Yang-Mills theory", arXiv:1101.0872, [hep-th]. A. Miyake, K. Ohta, and N. Sakai, "Volume of Moduli Space of Vortex Equations and Localization", arXiv:1105.2087, [hep-th]. M. Eto, T. Fujimori, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai, "Non-Abelian vortices on cylinder: Duality between vortices and walls," Phys. Rev. **D73**, 085008 (2006). G. S. Lozano, D. Marques, F. A. Schaposnik, "Non-Abelian Vortices on the Torus," JHEP **0709**, 095 (2007). A. D. Popov, "Non-Abelian Vortices on Riemann Surfaces: An Integrable Case," Lett. Math. Phys. **84**, 139–148 (2008). J. M. Baptista, "Non-Abelian vortices on compact Riemann surfaces," Commun. Math. Phys. **291**, 799–812 (2009). J. M. Baptista, "On the L^2-metric of vortex moduli spaces," Nucl. Phys. **B844**, 308–333 (2011). N. S. Manton, N. A. Rink, "Geometry and Energy of Non-abelian Vortices," J. Math. Phys. **52**, 043511 (2011). N. S. Manton, N. Sakai, "Maximally Non-Abelian Vortices from Self-dual Yang–Mills Fields," Phys. Lett. **B687**, 395-399 (2010).

[9] F. Fucito and J. F. Morales and R. Poghossian, "Multi instanton calculus on ALE spaces," arXiv:hep-th/0406243

[10] T. Fukuda and K. Hosomichi, "Super Liouville Theory with Boundary," arXiv: hep-th/0202032

[11] A. Hanany and D. Tong, "Vortices, instantons and branes", JHEP **0307** (2003) 037, hep-th/0306150.
R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, "Nonabelian superconductors: Vortices and confinement in $\mathcal{N} = 2$ SQCD", Nucl. Phys. **B673** (2003) 187–216, hep-th/0307287.

[12] Y. Ito, "Ramond sector of super Liouville theory from instantons on an ALE space" arXiv:1110.2176 [hep-th].

[13] T. Kimura, M. Nitta, "Vortices on Orbifolds" arXiv:1108.3563 [hep-th]

[14] N. A. Nekrasov, "Seiberg-Witten Prepotential From Instanton Counting," Adv. Theor. Math. Phys. **7**, 831 (2004) [arXiv:hep-th/0206161]
R. Flume and R. Poghossian, Int. J. Mod. Phys. A **18** (2003) 2541, arXiv:hep-th/0208176.
U. Bruzzo, F. Fucito, J. F. Morales and A. Tanzini, JHEP **0305** (2003) 054, arXiv:hep-th/0211108.

[15] E. Witten, "Solutions of four-dimensional field theories via M-theory," Nucl. Phys. B **500** (1997) 3 [arXiv:hep-th/9703166].

[16] J. Zhao, etc, to appear