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Lump solutions in SFT. Complements

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Abstract: Recently a possible violation of the equation of motion for the recently proposed lump solutions in open SFT has been pointed out in the literature. In this paper we argue that, when the issue is considered in the appropriate mathematical setting of distribution theory, no violations to the equation of motion occur.

Keywords: String Field Theory, Tachyon Condensation, Lump.

Contents

1. Introduction

This paper is a comment on [2, 3, 4]. Recently, following an earlier suggestion of [38], a general method has been proposed, [1], to obtain new exact analytic solutions in Witten's cubic open string field theory (OSFT) [5], and in particular solutions that describe inhomogeneous tachyon condensation. On general grounds it is expected that an OSFT defined on a particular boundary conformal field theory (BCFT) has classical solutions describing other boundary conformal field theories [6, 7]. Analytic solutions have actually been constructed describing the tachyon vacuum [8, 39, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] and others describing general marginal boundary deformations of the initial BCFT [22, 23, 25, 26, 27, 28, 29, 30, 31, 32], see also the reviews [33, 34]. In this panorama solutions describing inhomogeneous and relevant boundary deformations of the initial BCFT were not known until recently, though their existence was predicted [6, 7, 35]. This absence was filled up in [38, 1], and in [2, 3] the energy of a D24-brane solution was calculated for the first time. In [4] these results were extended to analytic SFT solutions corresponding to D(25-p)-branes, for any p, and their energy was calculated.

Notwithstanding these successes, some formal problems has remained behind. Such problems were pointed out especially in [3] and a solution was proposed in [2] in one appendix. In this note we would like to return to these issuess and discuss them in full detail. They concern a possible violation of the SFT equation of motion for the string field candidates considered in [1, 2, 4, 3], which originates from the use of a Schwinger parametrization of inverse elements. We argue in this paper that, when the issue is considered in the proper mathematical setting, no violations to the equation of motion occur for the solutions considered in [2, 4]. The appropriate setting is that of distribution theory. We show below that these solutions must be considered as distributions in the appropriate mathematical framework. Once this is done any ambiguity, which may give rise to spurious terms in the equation of motion, disappears.

The paper is organized as follows. After a review of [1] and [2], in section 2 we outline the main problem, which arises when we represent $\frac{1}{K+\phi_u}$ by means of a Schwinger parametrization. In section 3 we introduce a new formal representation of $\frac{1}{K+\phi_u}$, which helps formulating the problem in a clearer way. In section 4 we show that with this new representation we get the same results as in [2]. In section 5 we compute the would be offending term in the SFT equation of motion and show that it is in fact related to an ambiguity in the formalism. In section 6 we argue that this term can be removed by considering the problem in the appropriate mathematical framework of distribution theory.

1.1 Review of the previous results

In [1], to start with, the well-known K, B, c algebra defined by

$$K = \frac{\pi}{2} K_1^L |I\rangle, \qquad B = \frac{\pi}{2} B_1^L |I\rangle, \qquad c = c\left(\frac{1}{2}\right)|I\rangle, \tag{1.1}$$

was enlarged as follows. In the sliver frame (obtained by mapping the UHP to an infinite cylinder C_2 of circumference 2, by the sliver map $\tilde{z} = \frac{2}{\pi}\arctan z$), by adding a (relevant) matter operator

$$\phi = \phi\left(\frac{1}{2}\right)|I\rangle \tag{1.2}$$

with the properties

$$[c, \phi] = 0, \qquad [B, \phi] = 0, \qquad [K, \phi] = \partial\phi, \tag{1.3}$$

In this new algebra Q has the following action:

$$Q\phi = c\partial\phi + \partial c\delta\phi. \tag{1.4}$$

It can be easily proven that

$$\psi_\phi = c\phi - \frac{1}{K+\phi}(\phi - \delta\phi)Bc\partial c \tag{1.5}$$

does indeed satisfy the OSFT equation of motion

$$Q\psi_\phi + \psi_\phi\psi_\phi = 0. \tag{1.6}$$

It is clear that (1.5) is a deformation of the Erler–Schnabl solution, see [10], which can be recovered for $\phi = 1$.

In order to prove that (1.5) is a solution, one demands that $(c\phi)^2 = 0$, which requires the OPE of ϕ at nearby points to be not too singular.

Using the K, B, c, ϕ algebra one can show that

$$\mathcal{Q}_{\psi_\phi} \frac{B}{K+\phi} = Q\frac{B}{K+\phi} + \left\{\psi_\phi, \frac{B}{K+\phi}\right\} = 1.$$

So, unless the homotopy–field $\frac{B}{K+\phi}$ is singular, the solution has trivial cohomology, which is the defining property of the tachyon vacuum [38, 39]. On the other hand, in order for the solution to be well defined, the quantity $\frac{1}{K+\phi}(\phi - \delta\phi)$ should be well defined too. Finally, in order to be able to show that (1.5) satisfies the equation of motion, one needs $K + \phi$ to be invertible.

In full generality we thus have a new nontrivial solution if

1. $\frac{1}{K+\phi}$ is singular, but
2. $\frac{1}{K+\phi}(\phi - \delta\phi)$ is regular and
3. $\frac{1}{K+\phi}(K + \phi) = 1$.

In [1] sufficient conditions for ϕ to comply with the firts two requirements were determined. Let us parametrize the worldsheet RG flow, referred to above, by a parameter u, where $u = 0$ represents the UV and $u = \infty$ the IR, and rewrite ϕ as ϕ_u, with $\phi_{u=0} = 0$. Then we require for ϕ_u the following properties under the coordinate rescaling $f_t(z) = \frac{z}{t}$

$$f_t \circ \phi_u(z) = \frac{1}{t}\,\phi_{tu}\left(\frac{z}{t}\right) \tag{1.7}$$

and, most important, that the partition function

$$g(u) \equiv Tr[e^{-(K+\phi_u)}] = \left\langle e^{-\int_0^1 ds\, \phi_u(s)}\right\rangle_{C_1}, \tag{1.8}$$

satisfies the asymptotic finiteness condition

$$\lim_{u\to\infty} \left\langle e^{-\int_0^1 ds\, \phi_u(s)}\right\rangle_{C_1} = \texttt{finite}. \tag{1.9}$$

It was pointed out in [1] that this satisfies the first two conditions above i.e. guarantees not only the regularity of the solution but also its 'non-triviality', in the sense that if this condition is satisfied, it cannot fall in the same class as the ES tachyon vacuum solution. It would seem that the last condition above cannot be satisfied in view of the first. But this is not the case.

We will consider in the sequel a specific relevant operator ϕ_u and the corresponding SFT solution. This operator generates an exact RG flow studied by Witten in [36], see also [37], and is based on the operator (defined in the cylinder C_T of width T in the arctan frame)

$$\phi_u(s) = u(X^2(s) + 2\ln u + 2A), \tag{1.10}$$

where A is a constant first introduced in [38]. In C_1 we have

$$\phi_u(s) = u(X^2(s) + 2\ln Tu + 2A) \tag{1.11}$$

and on the unit disk D,

$$\phi_u(\theta) = u(X^2(\theta) + 2\ln\frac{Tu}{2\pi} + 2A). \tag{1.12}$$

If we set

$$g_A(u) = \langle e^{-\int_0^1 ds\, \phi_u(s)}\rangle_{C_1} \tag{1.13}$$

we have

$$g_A(u) = \langle e^{-\frac{1}{2\pi}\int_0^{2\pi} d\theta\, u\left(X^2(\theta)+2\ln\frac{u}{2\pi}+2A\right)}\rangle_D.$$

According to [36],

$$g_A(u) = Z(2u)e^{-2u(\ln\frac{u}{2\pi}+A)}, \tag{1.14}$$

where

$$Z(u) = \frac{1}{\sqrt{2\pi}}\sqrt{u}\Gamma(u)e^{\gamma u} \tag{1.15}$$

Requiring finiteness for $u \to \infty$ we get $A = \gamma - 1 + \ln 4\pi$, which implies

$$g_A(u) \equiv g(u) = \frac{1}{\sqrt{2\pi}}\sqrt{2u}\Gamma(2u)e^{2u(1-\ln(2u))} \tag{1.16}$$

and

$$\lim_{u\to\infty} g(u) = 1. \tag{1.17}$$

Moreover, as it turns out, $\delta\phi_u = -2u$, and so:

$$\phi_u - \delta\phi_u = u\partial_u\phi_u(s). \tag{1.18}$$

Therefore the ϕ_u just introduced satisfies all the required properties and consequently $\psi_u \equiv \psi_{\phi_u}$ must represent a D24 brane solution.

In I the expression for the energy of the lump solution was determined by evaluating a three–point function on the cylinder C_T of circumference T in the arctan frame. It is given by

$$\begin{aligned} E[\psi_u] &= -\frac{1}{6}\langle\psi_u\psi_u\psi_u\rangle \\ &= \frac{1}{6}\int_0^\infty d(2uT)\;(2uT)^2\int_0^1 dy\int_0^y dx\,\frac{4}{\pi}\sin\pi x\sin\pi y\sin\pi(x-y) \\ &\cdot g(uT)\Bigg\{-\Big(\frac{\partial_{2uT}g(uT)}{g(uT)}\Big)^3 + G_{2uT}(2\pi x)G_{2uT}(2\pi(x-y))G_{2uT}(2\pi y) \\ &-\frac{1}{2}\Big(\frac{\partial_{2uT}g(uT)}{g(uT)}\Big)\Big(G^2_{2uT}(2\pi x) + G^2_{2uT}(2\pi(x-y)) + G^2_{2uT}(2\pi y)\Big)\Bigg\}. \end{aligned} \tag{1.19}$$

where $G_u(\theta)$ represents the correlator on the boundary, first determined by Witten, [36]:

$$G_u(\theta) = \frac{1}{u} + 2\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k+u} \tag{1.20}$$

Moreover $\mathcal{E}_0(t_1, t_2, t_3)$ represents the ghost three–point function in C_T.

$$\mathcal{E}_0(t_1, t_2, t_3) = \langle Bc\partial c(t_1+t_2)\partial c(t_1)\partial c(0)\rangle_{C_T} = -\frac{4}{\pi}\sin\frac{\pi t_1}{T}\sin\frac{\pi(t_1+t_2)}{T}\sin\frac{\pi t_2}{T}. \tag{1.21}$$

Finally, to get (1.19) a change of variables $(t_1, t_2, t_3) \to (T, x, y)$, where

$$x = \frac{t_2}{T}, \qquad y = 1 - \frac{t_1}{T}.$$

is needed.

The expression (1.19) has been evaluated in [2]. As it turns out, this expression has a UV ($s \approx 0$, setting $s = 2uT$) singularity, which must be subtracted away. Therefore the result one obtains in general will depend on this subtraction. In [2] it has been pointed out that a physical significance can be assigned only to a subtraction-independent quantity, and it has been shown how to define and evaluate such a quantity. First a new solution to the EOM, depending on a parameter ε, has been introduced

$$\psi_U^\varepsilon = c(\phi_u + \varepsilon) - \frac{1}{K + \phi_u + \varepsilon}(\phi_u + \varepsilon - \delta\phi_u)Bc\partial c. \tag{1.22}$$

and it has been shown that it is gauge equivalent to the tachyon vacuum solution, its energy being 0 (after the same UV subtraction as in the previous case). Then, using it, a solution to the EOM at the tachyon condensation vacuum has been obtained. The equation of motion at the tachyon vacuum is

$$\mathcal{Q}\Phi + \Phi\Phi = 0, \quad \text{where} \;\; \mathcal{Q}\Phi = Q\Phi + \psi_u^\varepsilon\Phi + \Phi\psi_u^\varepsilon. \tag{1.23}$$

One can easily show that

$$\Phi_0 = \psi_u - \psi_U^\varepsilon \tag{1.24}$$

is a solution to (1.23). The action at the tachyon vacuum is $-\frac{1}{2}\langle\mathcal{Q}\Phi, \Phi\rangle - \frac{1}{3}\langle\Phi, \Phi\Phi\rangle$. Thus the energy of Φ_0 is

$$E[\Phi_0] = -\frac{1}{6}\langle\Phi_0, \Phi_0\Phi_0\rangle = -\frac{1}{6}\big[\langle\psi_u, \psi_u\psi_u\rangle - \langle\psi_u^\varepsilon, \psi_u^\varepsilon\psi_u^\varepsilon\rangle - 3\langle\psi_u^\varepsilon, \psi_u\psi_u\rangle + 3\langle\psi_u, \psi_u^\varepsilon\psi_u^\varepsilon\rangle\big]. \tag{1.25}$$

The UV subtractions necessary for each correlator at the RHS of this equation are always the same, therefore they cancel out. $E[\Phi_0]$ turns out to be precisely the D24-brane energy. In [4] the same result was extended to any Dp-brane lump.

2. The problem with the Schwinger representation

In order to obtain (1.19) one has to use the following Schwinger representation

$$\frac{1}{K+\phi_u} = \int_0^\infty dt\, e^{-t(K+\phi_u)} \tag{2.1}$$

of the inverse of $K+\phi_u$. When using such a Schwinger representation, however, the identity

$$\frac{1}{K+\phi_u}(K+\phi_u) = I, \tag{2.2}$$

would seem not to be satisfied. To illustrate the problem, let us calculate the overlap of both the left and the right hand sides of (2.2) with $Y = \frac{1}{2}\partial^2 c\partial cc$. The right hand side is trivial and, in our normalization, it is

$$\mathrm{Tr}(Y \cdot I) = \lim_{t\to 0}\langle Y(t)\rangle_{C_t}\langle 1\rangle_{C_t} = \frac{V}{2\pi}. \tag{2.3}$$

To calculate the left hand side we need the Schwinger representation

$$\mathrm{Tr}\big[Y \cdot \frac{1}{K+\phi_u}(K+\phi_u)\big] = \int_0^\infty dt \mathrm{Tr}\big[Y \cdot e^{-t(K+\phi_u)}(K+\phi_u)\big] \tag{2.4}$$

Making the replacement

$$e^{-t(K+\phi_u)}(K+\phi_u) \to -\frac{d}{dt}e^{-t(K+\phi_u)} \tag{2.5}$$

one obtains

$$\mathrm{Tr}\big[Y \cdot \frac{1}{K+\phi_u}(K+\phi_u)\big] = g(0) - g(\infty) = \frac{V}{2\pi} - g(\infty), \tag{2.6}$$

which is different form (2.3) because $g(\infty)$ is nonvanishing. The latter relation is often written in a stronger form

$$\int_0^\infty dt\, e^{-t(K+\phi_u)}(K+\phi_u) = 1 - \Omega_u^\infty, \qquad \Omega_u^\infty = \lim_{\Lambda\to\infty} e^{-\Lambda(K+\phi_u)} \tag{2.7}$$

This (strong) equality, however, has to be handled with great care. If the latter is taken literally, we could also write

$$\frac{1}{K+\phi_u} = \int_0^\infty dt\, e^{-t(K+\phi_u)} + \frac{1}{K+\phi_u}\Omega_u^\infty \tag{2.8}$$

instead of (2.1). This would imply that eq.(2.2) is not satisfied, and, consequently, the equation of motion is not satisfied by ψ_u.

3. A new (formal) representation for $\frac{1}{K+\phi_u}$

Let us introduce a small parameter ϵ, and remark that we can formally write

$$\frac{1}{K+\phi_u} = \frac{1}{K+\phi_u+\epsilon-\epsilon} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{(K+\phi_u+\epsilon)^{n+1}} \tag{3.1}$$

We can also rewrite it as

$$\frac{1}{K+\phi_u} = \sum_{n=0}^{\infty} \frac{(-\epsilon)^n}{n!} \partial_\epsilon^n \frac{1}{(K+\phi_u+\epsilon)} = e^{-\epsilon\partial_\epsilon} \frac{1}{(K+\phi_u+\epsilon)} \tag{3.2}$$

where $e^{-\epsilon\partial_\epsilon}$ means

$$e^{-\epsilon\partial_\epsilon} = e^{-a\partial_\epsilon}|_{a=\epsilon}$$

This expansion has the advantage that it expresses $\frac{1}{K+\phi_u}$ in terms of $\frac{1}{(K+\phi_u+\epsilon)}$. The latter, as was shown in [2], does not suffer from the same ambiguity as $\frac{1}{K+\phi_u}$. We can write in general

$$\frac{1}{K+\phi_u+\epsilon} = \int_0^\infty dt\, e^{-t(K+\phi_u+\epsilon)} \tag{3.3}$$

So we will use (3.1) or (3.2) as our definition of $\frac{1}{K+\phi_u}$. Of course now we have to take care that the series in (3.1) and (3.2) converge, or that the shift operator $e^{-\epsilon\partial_\epsilon}$ acts on objects whose dependence on ϵ is regular. We recall that if $f(\epsilon)$ and $g(\epsilon)$ are two regular functions of ϵ, then

$$e^{-\epsilon\partial_\epsilon}(f(\epsilon)g(\epsilon)) = \left(e^{-\epsilon\partial_\epsilon} f(\epsilon)\right)\left(e^{-\epsilon\partial_\epsilon} g(\epsilon)\right) \tag{3.4}$$

but, of course, this may not be true if the functions are not regular and the equation may become ambiguous. It should also be noticed that (3.2) may be interpreted as

$$\text{either} \qquad e^{-\epsilon\partial_\epsilon}\int_0^\infty dt\, e^{-t(K+\phi_u+\epsilon)}, \qquad \text{or} \qquad \int_0^\infty dt\, e^{-\epsilon\partial_\epsilon} e^{-t(K+\phi_u+\epsilon)} \tag{3.5}$$

These are the same in case of regularity, but may give rise to an ambiguity otherwise. The just mentioned ambiguities will be the main focus of this paper.

Let us see an example straightaway. According to this new representation the lump solution can be written as

$$\psi_u = c\phi_u - \sum_{n=0}^{\infty} \frac{\epsilon^n}{(K+\phi_u+\epsilon)^{n+1}}(\phi_u - \delta\phi_u)Bc\partial c \tag{3.6}$$

One of the basic expressions considered in the previous section is $\frac{1}{K+\phi_u}(\phi_u - \delta\phi_u)$, which must be nonsingular. Let us check it on the basis of the new prescription

$$
\begin{aligned}
\mathrm{Tr}\left[\frac{1}{K+\phi_u}(\phi_u - \delta\phi_u)\right] &= \sum_{n=0}^{\infty} \mathrm{Tr}\big[\frac{\epsilon^n}{(K+\phi_u+\epsilon)^{n+1}}(\phi_u - \delta\phi_u)\big] \\
&= \sum_{n=0}^{\infty} \frac{(-\epsilon)^n}{n!}\,\partial_\epsilon^n \mathrm{Tr}\big[\frac{1}{K+\phi_u+\epsilon}(\phi_u - \delta\phi_u)\big] \\
&= \sum_{n=0}^{\infty} \frac{(-\epsilon)^n}{n!}\,\partial_\epsilon^n \int_0^\infty dt\; e^{-\epsilon t}\mathrm{Tr}\big[u\partial_u\phi_u e^{-t(K+\phi_u)}\big] \\
&= -\sum_{n=0}^{\infty} \frac{(-\epsilon)^n}{n!}\,\partial_\epsilon^n \int_0^\infty dt\; e^{-\epsilon t}\frac{u}{t}\partial_u \mathrm{Tr}\big[e^{-t(K+\phi_u)}\big] \\
&= -e^{-\epsilon\partial_\epsilon} \int_0^\infty dt\; e^{-\epsilon t}\frac{u}{t}\partial_u g(ut)
\end{aligned} \tag{3.7}
$$

where $g(u)$ has been defined above. Setting $y = ut$ we can write

$$
\begin{aligned}
\mathrm{Tr}\left[\frac{1}{K+\phi}(\phi - \delta\phi_u)\right] &= -e^{-\epsilon\partial_\epsilon} \int_0^\infty dy\; e^{-\epsilon y/u}\,\partial_y g(y) \\
&= -\int_0^\infty dy\; e^{-\epsilon\partial_\epsilon} e^{-\epsilon y/u}\partial_y g(y) = -\int_0^\infty dy\partial_y g(y)
\end{aligned} \tag{3.8}
$$

This is the result we would obtain by using directly (2.1).

Two remarks are in order. First, in the passage from the first to the second line of (3.8) we exchange integration and summation: this is allowed if the integral is convergent without the $e^{-\epsilon y/u}$ factor[1]. The integrand we are considering, $\partial_y g(y)$, behaves like $1/y^2$ for large y, so this condition is satisfied in the IR. Alternatively one can analyse the applicability of the shift operator $e^{-\epsilon\partial_\epsilon}$ to the integral $\int_0^\infty dy\; e^{-\epsilon y/u}\,\partial_y g(y)$. This is correct as long as the integral is differentiable as a function of ϵ, which is the case when $\int_0^\infty dy\;\; \partial_y g(y)$ is convergent.

As for the UV, $y \approx 0$, (and this is the second remark) we know that a subtraction is needed, but it involves only the 0-th order term of the summation and it does not depend on ϵ. The 0-th order term can be easily treated separately and the relevant subtraction is precisely the same as the one needed in the RHS of the second line of (3.8).

The convergence of the integral in the IR in the previous example is crucial. As a counterexample let us consider

$$
\begin{aligned}
\mathrm{Tr}\left[\frac{1}{K+\phi_u}\right] &= \sum_{n=0}^{\infty} \mathrm{Tr}\big[\frac{\epsilon^n}{(K+\phi_u+\epsilon)^{n+1}} = \sum_{n=0}^{\infty} \frac{(-\epsilon)^n}{n!}\,\partial_\epsilon^n \mathrm{Tr}\big[\frac{1}{K+\phi_u+\epsilon}\big] \\
&= \sum_{n=0}^{\infty} \frac{(-\epsilon)^n}{n!}\,\partial_\epsilon^n \int_0^\infty dt\; e^{-\epsilon t}\mathrm{Tr}\big[e^{-t(K+\phi_u)}\big] \\
&= -\sum_{n=0}^{\infty} \frac{(-\epsilon)^n}{n!}\,\partial_\epsilon^n \int_0^\infty dt\; e^{-\epsilon t}\, g(ut)
\end{aligned} \tag{3.9}
$$

[1]Actually the integral could be less than convergent, even be logarithmically divergent in the IR, but in the sequel we will not meet such an occurrence.

The integral diverges because $g(t) \to 1$ as $t \to \infty$, as a consequence we cannot exchange summation and integration. It is easy to show that the summation diverges even in the presence of the $e^{-\epsilon t}$ in the integrand. Alternatively one can argue that $e^{-\epsilon \frac{\partial}{\partial \epsilon}}$ cannot be exchanged with the integral because the latter is discontinuous at $\epsilon = 0$.

The new representation agrees with the old one (see [1]) on the fact that $\frac{1}{K+\phi_u}$ is singular.

3.1 The energy

Of course it is very important that with the new representation we are able to show that we obtain for the energy the same result as in [2]. We recall that the energy expression for ψ_u was obtained by means of the replacement (1.19) and takes the following form

$$E[\psi_u] = -\frac{1}{6}\langle \psi_u \psi_u \psi_u \rangle = \int_0^\infty ds\, F(s) \tag{3.10}$$

where $F(s)$ behaves like $\sim 1/s^2$ for large s and needs a subtraction in the UV. With the new prescription we get

$$\begin{aligned} E[\psi_u] =& -\frac{1}{6}\langle \psi_u \psi_u \psi_u \rangle = \Big(\sum_{n=0}^{\infty} (-1)^n \frac{\epsilon^n}{n!}\partial_\epsilon^n\Big) \mathrm{Tr}\Big[\Big(\frac{1}{K+\phi_u+\epsilon}(\phi_u - \delta\phi_u)Bc\partial c\Big)^3\Big] \\ =& \sum_{n=0}^{\infty} (-1)^n \frac{\epsilon^n}{n!}\partial_\epsilon^n \int_0^\infty ds\ e^{-\epsilon s/2u} F(s) = \int_0^\infty ds\ e^{-\epsilon s/2u} \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{\epsilon s}{2u}\Big)^n F(s) \\ =& \int_0^\infty ds F(s). \end{aligned} \tag{3.11}$$

that is, the same result as (3.10).

In the previous derivation there are a few nontrivial passages. The first is the passage from the second to the third term. We justify it in Appendix A. Another nontrivial step is the one in the second line (i.e. exchanging summation and integration). This is possible only because the integrand is convergent in the IR and the singularity near $s \approx 0$ is present only in the zeroth order term. The latter can be easily dealt with separately from the others, it can be isolated and subtracted, and this subtraction is independent of ϵ and is the same mentioned above for $F(s)$.

In conclusion the new prescription gives for $E[\psi_u]$ the same result obtained in [2] by using the Schwinger representation (2.1) for $\frac{1}{K+\phi_u}$.

In order to complete our derivation we have to consider the last two terms in (1.25). Proceeding as in Appendix A we find (we stress the different role of ϵ and ε)

$$
\begin{aligned}
&-3\langle\psi_u\psi_u\psi_\varepsilon\rangle\\
&\quad = 3\sum_{n=0}^{\infty}\frac{(-\epsilon)^n}{n!}\partial_\epsilon^n \mathrm{Tr}\Big[\Big(\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)Bc\partial c\Big)^2\frac{1}{K+\phi_u+\varepsilon}(\phi_u+\varepsilon-\delta\phi_u)Bc\partial c\Big]\\
&\quad = 3\sum_{n=0}^{\infty}\frac{(-\epsilon)^n}{n!}\partial_\epsilon^n\int ds\,dx\,dy\,e^{-\epsilon s\frac{x+y}{2u}}F_1(\varepsilon,s,x,y)\\
&\quad = 3\int ds\,dx\,dy\,e^{-\epsilon s\frac{x+y}{2u}}\sum_{n=0}^{\infty}\frac{1}{n!}\big(s\frac{x+y}{2u}\big)^nF_1(\varepsilon,s,x,y)\\
&\quad = 3\int ds\,dx\,dy\,F_1(\varepsilon,s,x,y) \qquad (3.12)
\end{aligned}
$$

where $F_1(\varepsilon,s,x,y)$ is related to the integrand of the functional $\langle\psi_u\psi_u\psi_\varepsilon\rangle$ calculated in [2] by using Schwinger representation (2.1) for $\frac{1}{K+\phi_u}$. We have introduced the usual variables

$$
T=t_1+t_2+t_3,\quad x=\frac{t_1}{T},\quad y=\frac{t_2}{T},\quad s=2uT \qquad (3.13)
$$

and one can identify $F_1(\varepsilon,s,x,y)$ with the integrand of eq.(9.5) in [2] one must make the change of variables $x\leftrightarrow y$ followed by $y\to 1-y$.

Similarly

$$
\begin{aligned}
&3\langle\psi_u\psi_\varepsilon\psi_\varepsilon\rangle\\
&\quad = -3\sum_{n=0}^{\infty}\frac{(-\epsilon)^n}{n!}\partial_\epsilon^n)\mathrm{Tr}\Big[\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)Bc\partial c\Big(\frac{1}{K+\phi_u+\varepsilon}(\phi_u+\varepsilon-\delta\phi_u)Bc\partial c\Big)^2\Big]\\
&\quad = -3\sum_{n=0}^{\infty}\frac{(-\epsilon)^n}{n!}\partial_\epsilon^n\int ds\,dx\,dy\,e^{-\epsilon s\frac{x}{2u}}F_2(\varepsilon,s,x,y)\\
&\quad = -3\int ds\,dx\,dy\,e^{-\epsilon s\frac{x}{2u}}\sum_{n=0}^{\infty}\frac{1}{n!}\big(s\frac{x}{2u}\big)^nF_2(\varepsilon,s,x,y)\\
&\quad = -3\int ds\,dx\,dy\,F_2(\varepsilon,s,x,y) \qquad (3.14)
\end{aligned}
$$

Once again $F_2(\varepsilon,s,x,y)$ is related to the integrand of the functional $\langle\psi_u\psi_\varepsilon\psi_\varepsilon\rangle$ calculated in [2] by using Schwinger representation (2.1) for $\frac{1}{K+\phi_u}$. Identification of $F_1(\varepsilon,s,x,y)$ with the integrand of eq.(9.6) in [2] requires the same change of variables as above.

As for the UV, the same holds as for (3.10): the subtraction is ϵ-independent, it involves only 0-th order terms in each series and can be treated separately.

Summarizing, the results obtained with the new prescription (3.1,3.2) for eq.(1.25) are the same as the results we obtained by straightforwardly using the Schwinger representation (2.1) for $\frac{1}{K+\phi_u}$ in [2].

4. About the closed string overlap

In [1] it was shown that the ψ_u solution can satisfy the closed string overlap condition. Now we are in a position to clarify some aspects of this problem. The closed string overlap (CSO) is closely related to the traces we have considered above. Since the contribution from the identity piece of the solution is zero the CSO is given by

$$\begin{aligned}
\mathrm{Tr}[V_c\psi_u] &= -\sum_{n=0}^{\infty}\mathrm{Tr}\big[V_c\frac{\epsilon^n}{(K+\phi_u+\epsilon)^{n+1}}(\phi_u-\delta\phi_u)Bc\partial c\big]\\
&= -\sum_{n=0}^{\infty}\frac{(-\epsilon)^n}{n!}\partial_\epsilon^n\mathrm{Tr}\big[V_c\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)Bc\partial c\big]\\
&= -\sum_{n=0}^{\infty}\frac{(-\epsilon)^n}{n!}\partial_\epsilon^n\int_0^\infty dt\ e^{-\epsilon t}\mathrm{Tr}\big[V_c e^{-t(K+\phi_u)}(\phi_u-\delta\phi_u)Bc\partial c\big]\\
&= -\int_0^\infty dt\ e^{-\epsilon t}\sum_{n=0}^{\infty}\frac{(\epsilon t)^n}{n!}\mathrm{Tr}\big[V_c e^{-t(K+\phi_u)}(\phi_u-\delta\phi_u)Bc\partial c\big]\\
&= -e^{-\epsilon\partial_\epsilon}\int_0^\infty dt\ e^{-\epsilon t}\mathrm{Tr}\big[V_c e^{-t(K+\phi_u)}(\phi_u-\delta\phi_u)Bc\partial c\big]\\
&= -\int_0^\infty dt\ \mathrm{Tr}\big[V_c e^{-t(K+\phi_u)}(\phi_u-\delta\phi_u)Bc\partial c\big]
\end{aligned}\tag{4.1}$$

which is the result we would obtain by using the Schwinger representation directly for $\frac{1}{K+\phi_u}$, as was done in [1]. Once again all this is correct if, in the above expressions, the integrand, when $e^{-\epsilon t}$ is replaced by 1, is convergent, i.e. if the integral in the last line of (4.1) is well defined.

5. Concerning the identity $\frac{1}{K+\phi_u}(K+\phi_u)=I$

Let us return to section 2 and eqs.(2.2), (2.1) and (2.8). Applying our new prescription we get

$$\begin{aligned}
\frac{1}{K+\phi_u}(K+\phi_u) &= \sum_{n=0}^{\infty}\frac{(-\epsilon)^n}{n!}\,\partial_\epsilon^n\frac{1}{K+\phi_u+\epsilon}(K+\phi_u)\\
&= e^{-\epsilon\partial_\epsilon}\left(1-\frac{\epsilon}{K+\phi_u+\epsilon}\right)\\
&= 1-e^{-\epsilon\partial_\epsilon}\frac{\epsilon}{K+\phi_u+\epsilon}
\end{aligned}\tag{5.1}$$

The expression $e^{-\epsilon\partial_\epsilon}\frac{\epsilon}{K+\phi_u+\epsilon}$ is a more appropriate way to write Ω_u^∞. It is of course formally vanishing, but, as we have explained many times, to make any sense of such expressions one has to evaluate it in correlators. For instance, taking the trace, as in section 2, we are led to evaluate

$$\mathrm{Tr}\left[\frac{\epsilon}{K+\phi_u+\epsilon}\right]=\epsilon\int_0^\infty dt\,e^{-\epsilon t}\,g(ut)\tag{5.2}$$

Since, once again, $g(\infty)=1$, the limit $\epsilon \to 0$ is not continuous, and this depend on the fact, as we have seen many times, that the integral in the RHS of (5.2) is (linearly) divergent when the factor $e^{-\epsilon t}$ is replaced by 1. As a consequence the shift operator $e^{-\epsilon\partial_\epsilon}$ cannot be applied in a consistent way in (5.2). In fact it is not clear what value one should assign to the expression

$$e^{-\epsilon\partial_\epsilon}\left(\epsilon\int_0^\infty dt\, e^{-\epsilon t}\, g(ut)\right) \tag{5.3}$$

On the other hand, if (5.1) is inserted in a correlator (like the energy one) where the integrand without the exponential factor decreases fast enough, then the result of the application of $e^{-\epsilon\partial_\epsilon}$ to $\frac{\epsilon}{K+\phi_u+\epsilon}$ is unambiguously 0. For instance, suppose we are allowed to write

$$e^{-\epsilon\partial_\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right)=\big(e^{-\epsilon\partial_\epsilon}\epsilon\big)e^{-\epsilon\partial_\epsilon}\left(\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right) \tag{5.4}$$

The first piece on the RHS is obviously vanishing. Therefore, the whole term is vanishing as long as the second piece is finite. It is obvious that singling out the term $\big(e^{-\epsilon\partial_\epsilon}\epsilon\big)$ in the RHS of (5.4) makes sense only in this case. This can also be seen by contracting with a string field which has a regular overlap with (5.4), for instance $\partial^2 c\; e^{-(K+\phi)}$

$$\begin{aligned}
&\mathrm{Tr}\Big[\partial^2 c\; e^{-(K+\phi)}e^{-\epsilon\partial_\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\Big] \qquad (5.5)\\
&= e^{-\epsilon\partial_\epsilon}\epsilon\int_0^\infty dt\; e^{-\epsilon t}\mathrm{Tr}\big[(\phi_u-\delta\phi_u)e^{-(t+1)(K+\phi)}\big]\langle\partial^2 c(t+1)c\partial c(0)\rangle_{C_{t+1}}\\
&= e^{-\epsilon\partial_\epsilon}\epsilon\int_0^\infty dt\; e^{-\epsilon t}\langle(\phi_u(0)-\delta\phi_u(0))e^{-\int_0^{t+1}ds\phi(s)}\rangle_{C_{t+1}}\langle\partial^2 c(t+1)c\partial c(0)\rangle_{C_{t+1}}\\
&= -e^{-\epsilon\partial_\epsilon}\epsilon\int_0^\infty dt\; e^{-\epsilon t}G(t)\frac{u}{t+1}\partial_u g\big(u(t+1)\big)=2e^{-\epsilon\partial_\epsilon}\epsilon\int_0^\infty dt\; e^{-\epsilon t}\frac{u}{t+1}\partial_u g\big(u(t+1)\big)
\end{aligned}$$

where the ghost contribution is given by

$$G(t)=\langle\partial^2 c(t+1)(c\partial c)(0)\rangle_{C_{t+1}}=-2.$$

Using (3.4) we can write eq.(5.5) as

$$\begin{aligned}
&2\big(e^{-\epsilon\partial_\epsilon}\epsilon\big)e^{-\epsilon\partial_\epsilon}\int_0^\infty dt\; e^{-\epsilon t}\frac{u}{t+1}\partial_u g\big(u(t+1)\big)\\
&\qquad=2\big(e^{-\epsilon\partial_\epsilon}\epsilon\big)\int_0^\infty dt\;\frac{u}{t+1}\partial_u g\big(u(t+1)\big)=0.
\end{aligned} \tag{5.6}$$

We note that this last result does not need any UV subtraction.

Now let us examine an (apparent) violation of the equation of motion due to the second term in the RHS of (5.1). To this end we rewrite

$$\psi_u \to \psi_{u,\epsilon}=c\phi_u - e^{-\epsilon\partial_\epsilon}\tfrac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c \tag{5.7}$$

and apply Q to it. Using in particular

$$Q\left(e^{-\epsilon\partial_\epsilon}\frac{1}{(K+\phi_u+\epsilon)}\right)=-e^{-\epsilon\partial_\epsilon}\frac{1}{(K+\phi_u+\epsilon)}(Q\phi_u)\frac{1}{(K+\phi_u+\epsilon)} \tag{5.8}$$

and proceeding as in section 3.2 of [1], we find

$$\begin{aligned} Q\psi_{u,\epsilon}&=Q\left(c\phi_u-e^{-\epsilon\partial_\epsilon}\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c\right) \\ &=e^{-\epsilon\partial_\epsilon}\left[1+\frac{1}{(K+\phi_u+\epsilon)}(c\partial\phi_u+\partial c\delta\phi_u)\frac{1}{(K+\phi_u+\epsilon)}B-\frac{1}{(K+\phi_u+\epsilon)}K\right](\phi_u-\delta\phi_u)c\partial c \\ &=e^{-\epsilon\partial_\epsilon}\left[\left(c\phi_u-\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)\partial c\right)\frac{1}{(K+\phi_u+\epsilon)}+\frac{\epsilon}{(K+\phi_u+\epsilon)}c\right](\phi_u-\delta\phi_u)Bc\partial c \\ &=-\psi_{u,\epsilon}\psi_{u,\epsilon}+e^{-\epsilon\partial_\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right) \end{aligned} \tag{5.9}$$

In a regular setting, that is when inserted in a correlator regular in ϵ, this boils down to the usual eom $Q\psi_u=-\psi_u\psi_u$, and in particular the second piece in the RHS of the last line vanishes. Let us see what happens when (5.9) is inserted in the expression of the energy. We have

$$\begin{aligned} -\langle\psi_u Q\psi_u\rangle &\to -\langle\psi_{u,\epsilon}Q\psi_{u,\epsilon}\rangle \\ &= \langle\psi_{u,\epsilon}\psi_{u,\epsilon}\psi_{u,\epsilon}\rangle+\langle\psi_{u,\epsilon}e^{-\epsilon\partial_\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\rangle \end{aligned} \tag{5.10}$$

The second term in the RHS equals

$$e^{-\epsilon\partial_\epsilon}\langle\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c\ \frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle$$

With the usual procedure we can write this as

$$e^{-\epsilon\partial_\epsilon}\left(\epsilon\int_0^\infty dt_1dt_2\ e^{-\epsilon T}\mathcal{G}(t_1,t_2)u^2g(uT)\Big\{\Big(-\frac{\partial_{uT}g(uT)}{g(uT)}\Big)^2+2G^2_{2uT}(\frac{2\pi t_1}{T})\Big\}\right), \tag{5.11}$$

where the ghost part is given by

$$\mathcal{G}(t_1,t_2)=\langle(Bc\partial c)(t_1)(c\partial c)(0)\rangle_{C_T}=\frac{t_1}{\pi}\sin(\frac{2\pi t_1}{T})-\frac{2T}{\pi^2}\sin^2(\frac{\pi t_1}{T}). \tag{5.12}$$

In Appendix B we show that (5.11) reduces to the form

$$e^{-\epsilon\partial_\epsilon}\left(\epsilon\int_0^\infty ds\, e^{-\epsilon s}\mathcal{F}(s)\right) \tag{5.13}$$

where $\mathcal{F}(s)\to$ const for large s. Therefore once again $\int_0^\infty ds\, e^{-\epsilon s}\mathcal{F}(s)$ is discontinuous at $\epsilon=0$, and we do not know what value must be assigned to (5.13). As a consequence the additional piece in RHS of (5.10) cannot be assigned an unambiguous value without a suitable prescription[2].

In what follows we would like to argue that when an appropriate regularization is introduced the nature of this term becomes clear: it is a spurious term and must not be taken into account.

[2]Remembering (3.5) the expression (5.13) could mean an indefinite number if the integration is carried out last, or a constant if $e^{-\epsilon\partial_\epsilon}$ operates last, see also Appendix B.

6. Spurious terms: comments and conclusions

We have seen that the term $e^{-\epsilon\partial_\epsilon}\frac{\epsilon}{K+\phi_u+\epsilon}$ in (5.1), when inserted in correlators, is either identically vanishing or ambiguous. The first case occurs when it is inserted in a regular correlator, i.e. in a correlator which is convergent when the factor $e^{-\epsilon t}$ coming from the Schwinger representation of $\frac{1}{K+\phi_u+\epsilon}$ is replaced by 1. The second case is when the correlator is at least linearly divergent in the IR. In the latter case we need a definite prescription. It is almost evident that, a regularization being defined on regular elements, it will lead to the vanishing of the $e^{-\epsilon\partial_\epsilon}\frac{\epsilon}{K+\phi_u+\epsilon}$ contribution also in the second case.

It is however opportune to come to the same conclusion in a well defined setting. We propose two ways. One was outlined in Appendix D of [2]. It is based on viewing $K+\phi_u$ as the operator $K_1^L+\phi_u(\frac{1}{2})$ applied to the identity state $|I\rangle$. This operator being self-adjoint we can introduce its resolvent, which can be regarded as its regularized inverse, and prove eq.(2.2) in general. In this framework no spurious piece can ever arise. In this way, however, we are obliged to use two different formalisms, an operator formalism to verify the equation of motion and a Schwinger representation for $\frac{1}{K+\phi_u}$ in order to compute the lump energy. It would be desirable to have a unified treatment. From the above remarks it is evident that the best candidate is the formalism of distributions. Let us approach the problem first with the help of some heuristics. We interpret the appearance of the Schwinger representation of $\frac{1}{K+\phi_u}$ in the various correlators as the evaluation of such a vector on dual test elements characterized by the presence of the function $g(s)$ (all correlators we consider contain $g(s)$ or derivatives thereof). We attribute to $g(s)$ the nature of test function. It is true that in this paper we have used a precise $g(s)$, see (1.16), but varying the constant A we can obtain infinite different behaviours and, in any case, in its capacity of test function it is not important for $g(s)$ to have a physical meaning. Thus we reconsider all the correlators we have obtained by replacing our $g(s)$ with a true test function, by which, heuristically, we mean that it must decrease at least like $\frac{1}{s^2}$ for large s. It is basic in distribution theory that properties of distributions are defined by evaluating them on regular test functions. We have noticed that the distribution $e^{-\epsilon\partial_\epsilon}\frac{\epsilon}{K+\phi_u+\epsilon}$ is zero when evaluated on regular correlators. Therefore in distribution theory this expression is identically vanishing. It follows that as a distribution $\frac{1}{K+\phi_u}$ identically satisfies (2.2). It follows also that, interpreted in distribution theory, the second term in the last line of (5.9) should simply not be there, and, as a consequence, any offending terms in the equation of motion disappear.

It is perhaps useful, in order to appreciate the importance of the regularization implicit in distribution theory, to consider a counterexample in ordinary distribution theory. Let us take the distribution $\mathbf{x}_+^\lambda$ described in ch.1 of [48]. One starts from the function

$$x_+^\lambda = \begin{cases} x^\lambda & x>0 \\ 0 & x\leq 0 \end{cases} \tag{6.1}$$

This defines a distribution, which in the case of $\lambda=-1$ takes the form

$$\mathbf{x}_+^{-1} = \delta(x) + \mathrm{x}_+^{-1} \tag{6.2}$$

The distribution x_+^{-1} in turn is defined by evaluating it on the generic test function $\varphi(x)$ as follows:

$$(\mathrm{x}_+^{-1}, \varphi) = \int_0^\infty dx\, x^{-1}\, (\varphi(x) - \varphi(0)) \tag{6.3}$$

It is immediate to show that $x \cdot \mathbf{x}_+^{-1}$ is the distribution 1 in the relevant semi-axis (and, by continuity, even in 0). We can arrive at the same conclusion mimicking our procedure in the previous section. We can write

$$x \cdot \mathbf{x}_+^{-1} = 1 - e^{-\epsilon\partial_\epsilon} \frac{\epsilon}{x_+ + \epsilon} = 1 - e^{-\epsilon\partial_\epsilon} \left(\epsilon \int_0^\infty dt\, e^{-t(x_+ + \epsilon)} \right) \tag{6.4}$$

It is easy to show that the second term in the RHS vanishes identically on any (regular) test function. But, not too surprisingly, it might not vanish if we evaluate it on a non-test function. For instance let us evaluate it on $\delta(x)$, which is a limit of test functions, but not itself a test function. Integrating over x we obtain[3]

$$e^{-\epsilon\partial_\epsilon} \left(\epsilon \int_0^\infty dx\, \delta(x) \int_0^\infty dt\, e^{-t(x_+ + \epsilon)} \right) = e^{-\epsilon\partial_\epsilon} \left(\epsilon \int_0^\infty dt\, e^{-t\epsilon} \right) = e^{-\epsilon\partial_\epsilon} \left(\epsilon \frac{1}{\epsilon} \right) \tag{6.5}$$

which is indeterminate, due to the discontinuity of the t integral at $\epsilon = 0$. A similar discussion applies also to the distribution $\mathbf{r}^\lambda$, where r is a radial distance in a flat space.

From this example we understand that the appearance of the second term in the last line of (5.9) is a consequence of evaluating the $\frac{1}{K+\phi_u}$ distribution outside the realm of validity of distribution theory, therefore it is definitely a spurious term.

Invoking distribution theory in order to get rid of the spurious terms in the equation of motion (and elsewhere) may seem *ad hoc* at first sight, but the interpretation in terms of distribution theory provides a consistent regularization we need in order to make sense of the ambiguities pointed out in the previous sections. Anyhow this is a familiar procedure in theoretical physics in order to carefully define various physical solutions. For instance, brane solutions in supergravity are often characterized by a metric that explodes when we approach the brane location in the transverse direction, as it depends on some negative power of r, r being the transverse distance. However the relevant physical quantities, like the energy density, are finite. There is only one way to give an unambiguous meaning to such solutions, which is to interpret them in the framework of distribution theory.

Let us summarize our conclusions. The formal presence of the term Ω_u^∞ in the RHS of (2.8) or of $e^{-\epsilon\partial_\epsilon}\frac{\epsilon}{K+\phi_u+\epsilon}$ on the RHS of (5.1) is simply the spy of the fact that we are evaluating the identity (2.2) on a divergent correlator. If the correlator's integrand is convergent enough any such addition as $\frac{1}{K+\phi_u}\Omega_u^\infty$ is absent and $\frac{1}{K+\phi_u}$ is correctly represented by (2.1). The appearance of Ω_u^∞ or $e^{-\epsilon\partial_\epsilon}\frac{\epsilon}{K+\phi_u+\epsilon}$ is a pathology of the Schwinger representation which may show up if the problem is not formulated in the proper setting. The appropriate setting is that of distribution theory. In this framework the spurious terms are identically vanishing and there are no violations of the equation of motion.

[3]The choice of $\delta(x)$ is not accidental. It mimics the evaluation on the zero eigenvalue of $K + \phi_u$.

We would like to add a final comment on the possibility of a more rigorous treatment of our distribution theory interpretation. An ordinary distribution is just a linear continuous functional on a space of test functions. We can heuristically extend this definition to string fields. A string field distribution is a linear functional on the space of test string fields. For instance in our previous equations $\frac{1}{K+\phi_u}$ is evaluated (ignoring for simplicity the ghost part) on dual vectors represented by $\langle I|$, $\langle(\phi_u - \delta\phi_u)|$, etc.[4]. Some are test string fields other are not, depending on their behaviour in the IR (and in the UV), which is recognizable when the distribution is evaluated on them. A formalization of the idea of string field distribution is possible, but to our best knowledge the relevant formalism has not been developed so far. Perhaps the right mathematical setting is offered by the vector distribution theory. The theory of vector distributions was developed by Laurent Schwartz, [47]. The basic objects are a topological vector space and the space of test functions. A distributions is a linear continuous map from the latter to the former. More practically we can think of test vector functions as tensor products of ordinary scalar test functions by vectors and a vector distribution as a space dependent vector, while the evaluation on a vector test function is the ordinary scalar product followed by an ordinary integration. In our case the expression $\frac{1}{K+\phi_u}$ should be regarded as a vector distribution. It goes without saying that much work has to be done in order to clarify definitions and show applicability of such formalism in the context of SFT.

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Appendix

A. Some calculations

Let us justify the first line of (3.10) order by order in ϵ. Let us start from from the lowest order

$$\psi_u = c\phi_u - \frac{1}{K+\phi_u+\epsilon}\big(1+\frac{\epsilon}{K+\phi_u+\epsilon}\big)(\phi_u - \delta\phi_u)Bc\partial c + \mathcal{O}(\epsilon^2) \tag{A.1}$$

We can write

$$\begin{aligned}\langle\psi_u\psi_u\psi_u\rangle &= -\mathrm{Tr}\Big[\Big(\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c\Big)^3\Big]\\ &- 3\epsilon\mathrm{Tr}\Big[\Big(\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c\Big)^2\frac{1}{(K+\phi_u+\epsilon)^2}(\phi_u-\delta\phi_u)Bc\partial c\Big] + \mathcal{O}(\epsilon^2)\end{aligned} \tag{A.2}$$

[4]In the energy expression we have also triple products of the distribution in question. So we should be able to define products of distributions. This can usually be done in distribution theory, but since, as we have seen, such triple products in this paper do not have any convergence problem, they can be dealt with with ordinary analysis.

As for the first order term we have

$$
\begin{aligned}
&-3\epsilon\mathrm{Tr}\Big[\Big(\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c\Big)^2\frac{1}{(K+\phi_u+\epsilon)^2}(\phi_u-\delta\phi_u)Bc\partial c\Big]\\
&=\epsilon\partial_\epsilon\mathrm{Tr}\Big[\Big(\frac{1}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)Bc\partial c\Big)^3\Big]=\epsilon\partial_\epsilon\int_0^\infty e^{-\epsilon s/2u}F(s)\\
&=-\frac{\epsilon}{2u}\int_0^\infty e^{-\epsilon s/2u}sF(s) \qquad\qquad (A.3)
\end{aligned}
$$

where $F(s)$ is the integrand of the lump energy functional after the angular integrations are carried out (see above).

Up to second order we can write

$$
\psi_u=c\phi_u-\frac{1}{K+\phi_u+\epsilon}\big(1+\frac{\epsilon}{K+\phi_u+\epsilon}+\frac{\epsilon^2}{(K+\phi_u+\epsilon)^2}\big)(\phi_u-\delta\phi_u)Bc\partial c+\mathcal{O}(\epsilon^3) \qquad (A.4)
$$

and

$$
\begin{aligned}
&\langle\psi_u\psi_u\psi_u\rangle=-\mathrm{Tr}\Big[\Big(\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)Bc\partial c\Big)^3\Big]\\
&-3\epsilon\mathrm{Tr}\Big[\Big(\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)Bc\partial c\Big)^2\frac{1}{(K+\phi_u+\epsilon)^2}(\phi_u-\delta\phi_u)Bc\partial c\Big]\\
&-3\epsilon^2\mathrm{Tr}\Big[\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)Bc\partial c\Big(\frac{1}{(K+\phi_u+\epsilon)^2}(\phi_u-\delta\phi_u)Bc\partial c\Big)^2\Big]\\
&-3\epsilon^2\mathrm{Tr}\Big[\Big(\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)Bc\partial c\Big)^2\frac{1}{(K+\phi_u+\epsilon)^3}(\phi_u-\delta\phi_u)Bc\partial c\Big]+\mathcal{O}(\epsilon^3), \quad (A.5)
\end{aligned}
$$

or

$$
\langle\psi_u\psi_u\psi_u\rangle=-\big(1-\epsilon\partial_\epsilon+\frac{\epsilon^2}{2}\partial_\epsilon^2\big)\mathrm{Tr}\Big[\Big(\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)Bc\partial c\Big)^3\Big]+\mathcal{O}(\epsilon^3) \qquad (A.6)
$$

Including the third order one gets

$$
\langle\psi_u\psi_u\psi_u\rangle=-\big(1-\epsilon\partial_\epsilon+\frac{\epsilon^2}{2!}\partial_\epsilon^2-\frac{\epsilon^3}{3!}\partial_\epsilon^3\big)\mathrm{Tr}\Big[\Big(\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)Bc\partial c\Big)^3\Big]+\mathcal{O}(\epsilon^4) \qquad (A.7)
$$

and so on. Finally, by induction,

$$
E[\psi_u]=-\frac{1}{6}\langle\psi_u\psi_u\psi_u\rangle=\big(\sum_{n=0}^{\infty}(-1)^n\frac{\epsilon^n}{n!}\,\partial_\epsilon^n\big)\mathrm{Tr}\Big[\Big(\frac{1}{K+\phi_u+\epsilon}(\phi_u-\delta\phi_u)Bc\partial c\Big)^3\Big] \qquad (A.8)
$$

B. Spurious terms

The purpose of this appendix is to deduce eq.(5.13). Denoting $x=\frac{t_1}{T}$, Eq.(5.12) can be rewritten as

$$
e^{-\eta\partial_\eta}\eta\int_0^\infty dss^2\int_0^1 dx\mathcal{E}(x)\;e^{-\eta s}\mathrm{g}(s)\Big\{\Big(-\frac{\partial_s\mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2+\frac{1}{2}G_s^2(2\pi x)\Big\}, \qquad (B.1)
$$

where

$$\mathcal{E}(x) = \langle (Bc\partial c)(x)(c\partial c)(0)\rangle_{C_1} = \frac{-1+\cos(2\pi x)+\pi x \sin(2\pi x)}{\pi^2} \tag{B.2}$$

Since $\int_0^1 dx \mathcal{E}(x) = -\frac{3}{2\pi^2}$, the term with no G_s is given by

$$-\frac{3}{2\pi^2}\eta \int_0^\infty dss^2\, e^{-\eta s} \mathrm{g}(s)\Big(-\frac{\partial_s \mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2 \tag{B.3}$$

As $\mathrm{g}(s) \approx \frac{1}{\sqrt{s}}$ in the UV we are in the case of eq.(8.13) of [2] and so the UV contribution vanishes for $\eta \to 0$. In the IR we are in the case of eq.(8.17) [2] in the paper and so the IR contribution vanishes too. It can be easily proven that

$$\begin{aligned} 3\int_0^1 dx \mathcal{E}(x) G_s^2(2\pi x) &= \frac{4}{\pi}\int_0^1 dy \int_0^y dx \sin\pi x \sin\pi y \sin\pi(x-y) \\ &\quad \cdot \Big(G_s^2(2\pi x) + G_s^2(2\pi(x-y)) + G_s^2(2\pi y)\Big) \end{aligned} \tag{B.4}$$

where the expression in the RHS is the same as eq.(3.7) of [2]. Therefore we have

$$\begin{aligned} & e^{-\eta\partial_\eta}\left(\frac{1}{2}\eta\int_0^\infty dss^2\, e^{-\eta s}\mathrm{g}(s)\int_0^1 dx \mathcal{E}(x,0) G_s^2(2\pi x)\right) \\ &= e^{-\eta\partial_\eta}\left(\frac{1}{6}\eta\int_0^\infty dss^2\, e^{-\eta s}\mathrm{g}(s)\frac{4}{\pi}\int_0^1 dy\int_0^y dx \sin\pi x \sin\pi y \sin\pi(x-y)\right. \\ &\quad \left.\cdot \Big(G_s^2(2\pi x) + G_s^2(2\pi(x-y)) + G_s^2(2\pi y)\Big)\right) \end{aligned} \tag{B.5}$$

We can now avail ourselves of the results in [2]. The integration over x and y leads to an integrand in s that behaves like a constant for large s, if one abstracts from the factor $e^{-\eta s}$. Under these conditions the limit for $\eta \to 0$ of the s integral is discontinuous and we are not allowed to exchange $e^{-\eta\partial_\eta}$ with the integration. We can still obtain a finite result if we follow a definite prescription. If we first multiply η by the result of the integration and subsequently apply $e^{-\eta\partial_\eta}$ we obtain -2β, where β is the number introduced in [2]. This result is the same as the one obtained by [3]. *But one should not forget that it is prescription-dependent.*

It is interesting to see how to produce a prescription-independent result. Let us take $\Phi(\epsilon,\varepsilon) = \psi_{u,\epsilon} - \psi_u^\varepsilon$, where ψ_u^ε is the tachyon vacuum solution defined in [2] where ε is meant as a gauge parameter. We get

$$\begin{aligned} Q\psi_{u,\epsilon} &= -\psi_{u,\epsilon}\psi_{u,\epsilon} + e^{-\epsilon\partial_\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u - \delta\phi_u)c\partial c\right) \\ Q\psi_u^\varepsilon &= -\psi_u^\varepsilon\psi_u^\varepsilon \\ \mathcal{Q}\Phi(\epsilon,\varepsilon) &= -\Phi(\epsilon,\varepsilon)\Phi(\epsilon,\varepsilon) + e^{-\epsilon\partial_\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right) \end{aligned} \tag{B.6}$$

where $\mathcal{Q}\Phi = Q\Phi + \psi_u^\varepsilon\Phi + \Phi\psi_u^\varepsilon$. Moreover

$$\begin{aligned} -\langle\Phi(\epsilon,\varepsilon)\mathcal{Q}\Phi(\epsilon,\varepsilon)\rangle &= \langle\Phi(\epsilon,\varepsilon)\Phi(\epsilon,\varepsilon)\Phi(\epsilon,\varepsilon)\rangle \\ &\quad + \langle\Phi(\epsilon,\varepsilon)e^{-\epsilon\partial_\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\rangle \end{aligned} \tag{B.7}$$

If we use the just defined prescription, the second term in the RHS equals

$$
\begin{aligned}
& e^{-\epsilon\partial_\epsilon}\langle\psi_{u,\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\rangle - e^{-\epsilon\partial_\epsilon}\langle\psi_u^\varepsilon\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\rangle \\
= & -2\beta - e^{-\epsilon\partial_\epsilon}\langle\frac{1}{(K+\phi_u+\varepsilon)}(\phi_u+\varepsilon-\delta\phi_u)Bc\partial c\ \frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle \\
= & -2\beta - e^{-\epsilon\partial_\epsilon}\langle\frac{1}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)Bc\partial c\ \frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle \\
& -e^{-\epsilon\partial_\epsilon}\langle\frac{\varepsilon}{(K+\phi_u+\varepsilon)}Bc\partial c\ \frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle
\end{aligned}
\tag{B.8}
$$

The last two terms in the RHS equal, respectively,

$$
\begin{aligned}
& e^{-\epsilon\partial_\epsilon}\langle\frac{1}{(K+\phi_u+\varepsilon)}(\phi_u-\delta\phi_u)Bc\partial c\ \frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle \\
= & e^{-\epsilon\partial_\epsilon}\left(\epsilon\int_0^\infty dt_1dt_2\ e^{-\varepsilon t_2-\epsilon t_1}\mathcal{G}(t_1,t_2)u^2g(uT)\Big\{\Big(-\frac{\partial_{uT}g(uT)}{g(uT)}\Big)^2+2G^2_{2uT}(\frac{2\pi t_1}{T})\Big\}\right) \\
= & e^{-\eta\partial_\eta}\left(\eta\int_0^\infty dss^2\int_0^1 dx\mathcal{E}(x)\ e^{-T(\varepsilon(1-x)+\epsilon x)}\mathrm{g}(s)\Big\{\Big(-\frac{\partial_s\mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2+\frac{1}{2}G_s^2(2\pi x)\Big\}\right) \\
= & e^{-\eta\partial_\eta}\left(\eta\int_0^\infty dss^2\int_0^1\ e^{-\eta s}dx\mathcal{E}(1-x)\ e^{s\frac{\epsilon-\varepsilon}{2u}x}\mathrm{g}(s)\Big\{\Big(-\frac{\partial_s\mathrm{g}(s)}{\mathrm{g}(s)}\Big)^2+\frac{1}{2}G_s^2(2\pi x)\Big\}\right)
\end{aligned}
\tag{B.9}
$$

and

$$
\begin{aligned}
& e^{-\epsilon\partial_\epsilon}\langle\frac{\varepsilon}{(K+\phi_u+\varepsilon)}Bc\partial c\ \frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\rangle \\
= & e^{-\epsilon\partial_\epsilon}\left(\epsilon\varepsilon\int_0^\infty dt_1dt_2\ e^{-\varepsilon t_2-\epsilon t_1}\mathcal{G}(t_1,t_2)\frac{u}{t_1+t_2}\partial_u g(uT)\right) \\
= & e^{-\epsilon\partial_\epsilon}\left(\epsilon\varepsilon\int_0^\infty dT\,T\int_0^1 dx\ e^{-T(\varepsilon(1-x)+\epsilon x)}\mathcal{E}(x)u\partial_u g(uT)\right) \\
= & e^{-\eta\partial_\eta}\left(\eta\frac{\varepsilon}{2u}\int_0^\infty dss\ e^{-\eta s}\int_0^1 dx\ \mathcal{E}(1-x)\ e^{s\frac{\epsilon-\varepsilon}{2u}x}\partial_s\mathrm{g}(s)\right)
\end{aligned}
\tag{B.10}
$$

For the sake of simplicity, we evaluate these quantities in the limit $\varepsilon\to 0$. Now, one can either use gauge freedom to fix $\varepsilon=\epsilon$, or one can argue that both angular integrations are finite even without the $e^{s\frac{\epsilon-\varepsilon}{2u}x}$ factors so that in the limit $\epsilon,\varepsilon\to 0$ the integration is continuous in ϵ,ε and the factors can be dropped. Thus, using always the same prescription, the former integral is just -2β. The latter is the same as eq.(4.21) of [3]. It is convergent both in the UV and the IR.

So we find

$$
\langle\Phi(\epsilon,\varepsilon)e^{-\epsilon\partial_\epsilon}\left(\frac{\epsilon}{(K+\phi_u+\epsilon)}(\phi_u-\delta\phi_u)c\partial c\right)\rangle = -2\beta+2\beta-0
$$

This is a prescription-independent result, the reason being that the overall s integrand has the right convergent behaviour for large s in order to guarantee continuity in ϵ also at $\epsilon=0$.

In the language of distribution theory we could say that $\Phi(\epsilon,\varepsilon)$ is a good dual test string field for the EOM whereas $\psi_{u,\epsilon}$ is not.

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