Critical points of the Moser-Trudinger functional

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August 30, 2011

Abstract

On a smooth bounded domain $\Omega \subset \mathbb{R}^2$ we study the heat flow $u_t = \Delta u + \lambda(t) u e^{u^2}$ ($\lambda(t)$ is such that $\frac{d}{dt}\|u(t,\cdot)\|_{H_0^1} = 0$) introduced by T. Lamm, F. Robert and M. Struwe in [23] to investigate the Moser-Trudinger functional

$$E(v) = \int_\Omega \left(e^{v^2} - 1\right) dx, \quad v \in H_0^1(\Omega).$$

We prove that if u blows-up as $t \to \infty$ and if $E(u(t,\cdot))$ remains bounded, then for a sequence $t_k \to \infty$ we have $u(t_k,\cdot) \rightharpoonup 0$ in H_0^1 and $\|u(t_k,\cdot)\|^2_{H_0^1} \to 4\pi L$ for an integer $L \geq 1$.

We couple these results with a topological technique to prove that if Ω is not contractible, then for every $0 < \Lambda \in \mathbb{R} \setminus 4\pi\mathbb{N}$ the functional E constrained to $M_\Lambda = \{v \in H_0^1(\Omega) : \|v\|^2_{H_0^1} = \Lambda\}$ has a positive critical point. We prove that when Ω is the unit ball and Λ is large enough, then $E|_{M_\Lambda}$ has no positive critical points, hence showing that the topological assumption on Ω is natural.

Key Words: Moser-Trudinger inequality, critical points, blow-up analysis, variational methods

AMS subject classification: 35B33, 35K55, 35B44, 35A15, 35A01

1 Introduction

Let $\Omega \Subset \mathbb{R}^2$ be a smooth bounded and connected open set. It is well-known that there is a Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{2p}{2-p}}(\Omega)$ for $p \in [1,2)$, but $H_0^1(\Omega) := W_0^{1,2}(\Omega) \not\hookrightarrow L^\infty(\Omega)$. Indeed it was proven by N. Trudinger [36] that $e^{u^2} \in L^1(\Omega)$ whenever $u \in H_0^1(\Omega)$. This embedding was sharpened by J. Moser [29] who showed that

$$\sup_{u \in H_0^1(\Omega),\, \|u\|^2_{H_0^1} \leq 4\pi} \int_\Omega \left(e^{u^2} - 1\right) dx \leq C|\Omega|, \qquad \|u\|_{H_0^1} := \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}, \tag{1}$$

and

$$\sup_{u\in H_0^1(\Omega),\ \|u\|^2_{H_0^1}\le 4\pi+\delta}\int_\Omega\left(e^{u^2}-1\right)dx=+\infty,\qquad \text{for every } \delta>0. \tag{2}$$

Since then, a formidable amount of work has been devoted to the study of the functional

$$E(u):=\int_\Omega\left(e^{u^2}-1\right)dx,\qquad u\in H_0^1(\Omega)$$

and in particular of its critical points. Clearly $u\equiv 0$ is the only global minimum of E, but because of (2) we cannot look for a global maximizer of E in H_0^1. Instead one might hope to find a maximizer of $E|_{M_\Lambda}$, i.e. of E constrained to the manifold

$$M_\Lambda:=\left\{u\in H_0^1(\Omega):\|u\|^2_{H_0^1}=\Lambda\right\}$$

for $\Lambda\in(0,4\pi]$, or to find other kinds of critical points (local maxima or minima, saddle points, etc.) when $\Lambda>4\pi$. As long as $\Lambda<4\pi$ the embedding (1) is in fact compact, so that the existence of a maximizer is elementary, but when $\Lambda\ge 4\pi$, compactness is lost and in fact also the Palais-Smale condition does not hold anymore, see [3].

In spite of these difficulties Carleson and Chang [9] proved that when $\Omega=B_1(0)$ (the unit ball) E has a maximizer in $M_{4\pi}$. This result was extended by Struwe [32] who proved the existence of a maximizer in $M_{4\pi}$ when Ω is close to a ball, and finally by Flucher [21] for any bounded smooth Ω (see also [13] for a related result in higher dimension).

The existence of critical points on M_Λ in the supercritical regime, i.e. for $\Lambda>4\pi$, is even more challenging, and to the fundamental question of the existence of critical points of $E|_{M_\Lambda}$ for Λ large only few answers have been given. Monahan [28] gave numerical evidence that when $\Omega=B_1(0)$ then for some $\Lambda^*>4\pi$ the functional $E|_{M_\Lambda}$ has a local maximum and a mountain pass critical point for every $\Lambda\in(4\pi,\Lambda^*)$. Assuming that a local maximum of $E|_{M_\Lambda}$ exists (which is always guaranteed by the result in [21]) Struwe proved in [32] that for some $\Lambda^*=\Lambda^*(\Omega)>4\pi$ and for a.e. $\Lambda\in(4\pi,\Lambda^*)$ a second critical point exists. The result was then extended in [23] to all values of $\Lambda\in(4\pi,\Lambda^*)$ through the more precise information given by the parabolic flow, compared to the one given by the Palais-Smale condition.

Further, using implicit function methods, Del Pino, Musso and Ruf [15] were able to characterize some of these critical points as one-peaked bubbling functions which blow-up as $\Lambda\searrow 4\pi$. In the same paper they showed that if Ω is not contractible, then for some $\Lambda^\dagger>8\pi$ the functional $E|_{M_\Lambda}$ has a critical point for $\Lambda\in(8\pi,\Lambda^\dagger)$. When Ω is a radially symmetric annulus they also proved for any $1\le\ell\in\mathbb{N}$ the existence of some $\Lambda^*_\ell>4\pi\ell$ such that $E|_{M_\Lambda}$ has a critical points when $\Lambda\in(4\pi\ell,\Lambda^*_\ell)$. We also refer to [33] and [23] for related results on domains with *small holes*, in the spirit of [12] (where the Yamabe equation was treated).

The previous results, in particular those in [15], suggest that at least when Ω is not contractible $E|_{M_\Lambda}$ might have critical points even when Λ is much larger that 4π. As we shall now describe, in this work we prove that this is indeed often the case: we sharpen the blow-up analysis of the heat flow introduced by Lamm, Robert and Struwe (Problem (1) below) and use this improvement to show that whenever Ω is not contractible and $0 < \Lambda \in \mathbb{R} \setminus 4\pi\mathbb{N}$, $E|_{M_\Lambda}$ has a critical point. Our result can be viewed as a counterpart of the main theorem in [5], where the Yamabe equation with Dirichlet boundary conditions was treated, and where solutions were found with variational and topological methods.

Given $u_0 \in C^\infty(\overline{\Omega})$ with $u_0 \geq 0$ and $u_0|_{\partial\Omega} = 0$, following [23] we consider the non-linear parabolic problem

$$\begin{cases} u_t e^{u^2} = \Delta u + \lambda u e^{u^2} & \text{in } [0,\infty) \times \Omega; \\ u = 0 & \text{on } [0,\infty) \times \partial\Omega; \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{3}$$

where $\lambda > 0$ depends on t and is chosen as in [23, Section 1.2] so that

$$\|u(t,\cdot)\|^2_{H^1_0(\Omega)} = \Lambda := \|u_0\|^2_{H^1_0(\Omega)}, \quad \text{for all } t \geq 0.$$

Our main result concerning the study of (3) is the following:

Theorem 1 *For any $0 \leq u_0 \in C^\infty(\Omega)$ with $u_0|_{\partial\Omega} = 0$ Problem (3) has a unique solution u provided that $E(u(t,\cdot))$ remains bounded in t. Moreover there exists a sequence $t_k \to \infty$ such that $\lambda(t_k) \to \lambda_\infty \geq 0$ and, setting $u_k := u(t_k,\cdot)$, one of the following is true:*

(i) $u_k \to u_\infty \geq 0$ in $H^1_0(\Omega)$ and in $C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$, where u_∞ solves

$$-\Delta u_\infty = \lambda_\infty u_\infty e^{u_\infty^2} \quad \text{in } \Omega; \tag{4}$$

(ii) $u_k \rightharpoonup 0$ in $H^1_0(\Omega)$ and there exists a non-empty finite set $S = \{x^{(1)}, \dots, x^{(J)}\} \subset \Omega$ such that $u_k \to 0$ in $C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega} \setminus S)$ and

$$|\nabla u_k|^2 dx \rightharpoonup \sum_{j=1}^J 4\pi L_j \delta_{x^{(j)}}, \quad \lambda_k u_k^2 e^{u_k^2} dx \rightharpoonup \sum_{j=1}^J 4\pi L_j \delta_{x^{(j)}} \tag{5}$$

weakly in the sense of measures, where $L_j \in \mathbb{N}$. In particular

$$\Lambda = \|u_0\|^2_{H^1_0} = 4\pi L, \quad L = \sum_{j=1}^J L_j \in \mathbb{N}. \tag{6}$$

Let us remark that a critical point u of $E|_{M_\Lambda}$ satisfies

$$\begin{cases} -\Delta u = \lambda u e^{u^2} & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega; \\ \|u\|^2_{H^1_0} = \Lambda, \end{cases} \tag{7}$$

for some $\lambda > 0$. Adimurthi [1] proved that for every $\lambda \in (0, \lambda_1)$ Problem (7) has a solution for some $\Lambda \leq 4\pi$, and no solution for $\lambda \geq \lambda_1$, where λ_1 is the first eigenvalue of the Laplacian in Ω.

Anyway, since λ is a Lagrange multiplier, we are more interested in prescribing Λ rather than λ.
As a consequence of our proof we also deduce the following compactness theorem for critical points of $E|_{M_\Lambda}$, i.e. solutions of (7) with equibounded energies, improving the result in Druet [19] for the nonlinearity in (4) (see also [4] and [2] for previous related results, where the blow-up profile was identified).

Theorem 2 *Let u_k be a sequence of positive solutions to*

$$\begin{cases} -\Delta u_k = \lambda_k u_k e^{u_k^2} & \text{in } \Omega; \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

such that $\lambda_k \to \lambda_\infty \geq 0$ and such that $\|u_k\|^2_{H^1_0(\Omega)} \to \Lambda < \infty$. Then up to a subsequence, either (i) or (ii) of Theorem 1 holds.

In [19] and [23] it was proven that in case (ii) above $u_k \to u_\infty$ weakly in $H^1_0(\Omega)$ and strongly in $C^{1,\alpha}_{\text{loc}}(\overline{\Omega} \setminus S)$, where $u_\infty \geq 0$ solves (4) and a priori might be non-zero. Then the quantization of the energy in (6) is of the form $\Lambda = 4\pi L + \|u_\infty\|^2_{H^1_0(\Omega)}$, for some integer L, which has limited use without a control on $\|u_\infty\|^2_{H^1_0(\Omega)}$. Indeed with regard to the applications, the most crucial improvement of our Theorem 1 (and also of Theorem 2) is that $u_\infty \equiv 0$. To prove this we consider first the radially symmetric situation, as in [23], and we analyze different cases concerning the monotonicity of some function φ_k, see Proposition 8, whose definition is inspired by the concept of *isolated simple* blow-up point introduced in [30]. We treat the general non-radial case using a Harnack type inequality, see Proposition 9, and controlling suitable radial averages of the solutions, together with an inductive procedure. The fact that blow-up sequences do not accumulate at $\partial\Omega$ follows from rather standard moving plane techniques.

The final picture that we get is then much closer to the geometric situation of the Liouville equation $-\Delta u = Ve^{2u}$ in $\Omega \subset \mathbb{R}^2$, as studied by Brezis-Merle [7, Theorem 3] (where in particular the counterpart of $u_\infty \equiv 0$ was proven) and Li-Shafrir [25] (who showed quantization for the conformal volume) than so far known. It would be interesting to see whether each blow-up point $x^{(j)}$ in Theorem 1 carries a quantization value equal *exactly* to 4π, as shown in [24] for the Liouville equation on compact surfaces with no boundary.

Employing Theorem 1, we derive a rather general existence result for (7).

Theorem 3 *Let $\Omega \subseteq \mathbb{R}^2$ be smooth, connected, bounded and non contractible, and let $\Lambda > 0, \Lambda \in \mathbb{R} \setminus 4\pi\mathbb{N}$. Then $E|_{M_\Lambda}$ has a positive critical point, i.e. (7) has a positive solution for some $\lambda > 0$.*

The strategy of our proof is to produce a trajectory of (3) with prescribed norm $\Lambda \in \mathbb{R} \setminus 4\pi\mathbb{N}$ for which $E(u(t,\cdot))$ stays bounded, by properly choosing the initial data. Then case (ii) of Theorem 1 cannot occur because of (6) and necessarily $u(t_k,\cdot)$ strongly converges to a solution of (7) for a sequence $t_k \to \infty$.

In order to produce such a trajectory we use a topological argument inspired from [6] and [18], which helps to characterize the functions u with given H_0^1 norm and for which $E(u)$ is large. First, see Lemma 14, we extend the Moser-Trudinger inequality (1) showing that we get a uniform control on $\int_\Omega e^{u^2} dx$ also for $\|u\|^2_{H_0^1(\Omega)} > 4\pi$, provided the function e^{u^2} is *spread enough* over Ω. Naively, we can increase $\|u\|^2_{H_0^1}$ by 4π each time e^{u^2} *separates* into two distinct parts: this was done initially by W.Chen and C.Li for the Liouville equation in [10].

This naïve description is useful then to characterize the high levels of the functional E. In fact, if $\|u\|^2_{H_0^1(\Omega)} \in (4\pi\ell, 4\pi(\ell+1))$ and if $E(u)$ is large enough, then the probability measure $\frac{e^{u^2}dx}{\int_\Omega e^{u^2}dx}$ has to *concentrate* near at most ℓ points of Ω, see Lemma 16. As in [18], this suggests to consider the set of probability measures

$$\Omega_\ell = \left\{ \sum_{i=1}^{\ell} t_i \delta_{x_i} \ : \ x_i \in \Omega, t_i \geq 0, \sum_{i=1}^{\ell} t_i = 1 \right\},$$

to describe such functions u. In Subsection 3.3 we show indeed that this description is somehow optimal, in the sense that we can embed Ω_ℓ (more precisely, a retraction $\mathcal{B}_\ell$ of it, see [6]) into arbitrarily high sublevels of E, by considering suitable test functions. This part turns out to be technically quite different from [18].

We exploit then the non contractibility of Ω (which also implies that of $\mathcal{B}_\ell$) in order to find the flow trajectory described above, see Proposition 25 and Subsection 3.4.

Although solutions to (7) with norm close to 4π exist for all domains, our non contractibility assumption in Theorem 3 is somehow natural, as it is shown by our next theorem.

Theorem 4 *For $\Omega = B_1(0)$ there exists $\Lambda^\sharp \geq 4\pi$ such that for $\Lambda > \Lambda^\sharp$ the functional E has no* positive *critical points on M_Λ.*

The proof of this result rests on the fact that a positive solution to (7) with $\Omega = B_1(0)$ is radially symmetric by [22], and on our decay estimate of the energy near the blow-up points, see Subsection 2.2. In fact we will show that every solution u_μ to (7) in $B_1(0)$ is uniquely determined by $\mu := u_\mu(0)$, and that $\lim_{\mu\to\infty} \|u_\mu\|^2_{H_0^1} = 4\pi$, $\lim_{\mu\to 0} \|u_\mu\|^2_{H_0^1} = 0$. In particular, an immediate consequence of the simple proof of Theorem 4 is the existence of blowing-up solutions to (7) for $\Lambda \searrow 4\pi$, see Remark 27 for more details.

It would be interesting to prove multiplicity of solutions to (7) using category arguments (e.g. in the spirit of [16], where the result in [35] for Liouville equations on tori was extended). Besides, we expect that applying Morse inequalities (see [17]) one could obtain that, for a generic choice of $\Lambda \in (8\ell\pi, 8(\ell+1)\pi)$, problem (7) admits at least $\binom{\ell+\mathfrak{g}-1}{\mathfrak{g}-1}$ solutions, where $\mathfrak{g} = \dim H_1(\Omega; \mathbb{Z})$ (in case $\mathfrak{g} \geq 1$).

In the following the letter C denotes a large constant which may change from line to line and even within the same line.

Acknowledgements

The authors are supported by the project FIRB-Ideas *Analysis and Beyond*, and L.M. was partially supported by the Swiss National Fond Grant no. PBEZP2-129520. During the preparation of this work F.D.M. and L.M. were hosted by SISSA, while A.M. by Centro de Giorgi in Pisa: they are grateful to these institutions for the kind hospitality.

2 Proof of Theorems 1 and 2

In this section we shall first prove Theorem 1 and clarify how its proof applies to Theorem 2 as well.

2.1 The analysis of Lamm-Robert-Struwe

We will use and improve upon Lemma 4.1 and Theorem 4.2 from [23], which we recall for convenience.

Lemma 5 *([23]) Under the hypothesis of Theorem 1, suppose that* $\limsup_{t\to\infty}\sup_{x\in\Omega} u(t,x) = \infty$ *and that* $E(u(t,\cdot)) \le C$ *for some constant* C*. Then there exists a sequence* $t_k \to \infty$ *such that as* $k \to \infty$ *we have* $\lambda_k := \lambda(t_k) \to \lambda_\infty \in [0,\infty)$, $0 \le u_k := u(t_k,\cdot) \rightharpoonup u_\infty$ *in* $H_0^1(\Omega)$ *and*

$$\lim_{k\to\infty}\sup_{\Omega} u_k = \infty, \quad \lim_{k\to\infty}\frac{1}{\lambda_k}\int_\Omega |g_k|^2 e^{u_k^2}dx \to 0, \quad g_k := u_t(t_k,\cdot). \tag{9}$$

Theorem 6 *([23]) Let* t_k *and* $u_k := u(t_k,\cdot)$ *be as in Lemma 5. Then, up to extracting a subsequence, there exist* I *sequences* $x_{i,k} \to x^{(i)} \in \overline{\Omega}$ *such that* $u_k(x_{i,k}) \to \infty$ *and, setting* $r_{i,k} \downarrow 0$ *with*

$$r_{i,k}^2 f_k(x_{i,k}) = 4, \quad f_k := \lambda_k u_k^2 e^{u_k^2}, \tag{10}$$

the following holds true:

$$\eta_{i,k}(x) := u_k(x_{i,k})(u_k(r_{i,k}x + x_{i,k}) - u_k(x_{i,k})) + \log 2 \to \eta_\infty(x) = \log\frac{2}{1+|x|^2} \tag{11}$$

in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ *and*

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{Rr_{i,k}(x_{i,k})}} f_k(x)dx = \int_{\mathbb{R}^2} e^{2\eta_\infty}dx = 4\pi. \tag{12}$$

Moreover for $1 \le i,j \le I$ *and* $i \ne j$

$$\lim_{k\to\infty}\frac{|x_{i,k}-x_{j,k}|}{r_{i,k}} = \infty, \quad \lim_{k\to\infty}\frac{\mathrm{dist}(x_{i,k},\partial\Omega)}{r_{i,k}} = \infty. \tag{13}$$

Finally

$$\sup_{x\in\Omega}\min_{1\le i\le I}|x-x_{i,k}|^2 f_k(x)\le C,$$

and $u_k\to u_\infty\ge 0$ *weakly in* $H^1_0(\Omega)$ *and strongly in* $C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\setminus\{x^{(i)}:1\le i\le I\})$*, where* u_∞ *solves* (4).

Another important tool in the proof of Theorem 1 is the fact the blow-up points stay away from the boundary, as we shall prove in Subsection 2.4.

Lemma 7 *In Theorem 6 we have* $x^{(i)}\in\Omega$ *for* $1\le i\le I$.

The proof of Theorem 1 is not based on a Pohozaev-type identity as in [34] and [23], but on a simpler decay estimate of u_k away from the blow-up points $x_{i,k}$, which has some partial analogies with Lemma 3 of [25] (originating in [31], see also [8]) and a decay estimate from [20] in the context of the Liouville equation. Moreover we will prove that each blow-up sequence is *simple*, i.e. it carries an energy of 4π and $\Lambda=4\pi I$, where I is as in Theorem 6. More precisely, we will prove (14) below, which is reminiscent of Lemma 5 of [25]. In all previous works [19, 23, 27, 34] on the subject this problem was left open and it was only proven that each blowing-up sequence $(x_{i,k})$ brings an energy of $4\pi L_i$ for some integer $L_i\ge 1$, partly because the nonlinearity $\lambda u e^{u^2}$ is more difficult to deal with than the geometric counterpart e^{2u}. We still do not know whether two blowing-up sequences can converge to the same blow-up point ($x^{(i)}=x^{(j)}$ for some $1\le i<j\le I$), see also the comments in the introduction.

Global existence of solutions to (1) when E stays bounded was proven in [23, Section 2], and if $\sup_{\Omega\times[0,\infty)}u<\infty$ we have that case (i) occurs, see [23, Section 3.2]. Considering Theorem 6 it remains to prove that if

$$\limsup_{t\to\infty}\sup_{x\in\Omega}u(t,x)=\infty,$$

then we can complement the claim of Theorem 6 by showing that $u_\infty\equiv 0$ and

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{\Omega\setminus\bigcup_{1\le i\le I}B_{Rr_{i,k}}(x_{i,k})} f_k dx=0, \tag{14}$$

which together with (12), yields the convergence of $f_k dx$ in (5). To see that this also implies the convergence of $|\nabla u_k|^2dx$, observe first that by Hölder's inequality and (9)

$$\int_\Omega |g_k|u_k e^{u_k^2}dx\le\left(\frac{1}{\lambda_k}\int_\Omega g_k^2 e^{u_k^2}dx\right)^{\frac12}\left(\int_\Omega f_k dx\right)^{\frac12}=o(1)\left(\int_\Omega f_k dx\right)^{\frac12}, \tag{15}$$

with $o(1)\to 0$ as $k\to\infty$, hence by integration by parts

$$\Lambda=\int_\Omega u_k(-\Delta u_k)dx=(1+o(1))\int_\Omega f_k dx. \tag{16}$$

Coming back to (5), using (15) and $u_k \to 0$ in $L^2(\Omega)$, we have for any $\varphi \in C_c^2(\Omega)$

$$\int_\Omega |\nabla u_k|^2 \varphi dx = \int_\Omega \frac{\Delta(u_k^2)}{2} \varphi dx + \int_\Omega f_k \varphi dx + o(1) \sup_\Omega |\varphi|$$
$$= \int_\Omega \frac{u_k^2}{2} \Delta\varphi dx + \int_\Omega f_k \varphi dx + o(1) = \int_\Omega f_k \varphi dx + o(1),$$

hence $f_k dx$ and $|\nabla u_k|^2 dx$ have the same weak limit in the sense of measures.

2.2 Proof of Theorem 1 in the radially symmetric case

As in [23] we first consider the case when $\Omega = B_\delta$ and u_k is radially symmetric. We shall write $u_k(x) = \overline{u}_k(|x - x_k|)$, where $x_k = 0$ in this subsection (with this trick, we will be able to "recycle" several formulas in the next subsection) for a non-negative function $\overline{u}_k : [0, \delta] \to \mathbb{R}$. Then in Theorem 6 we have $I = 1$, $x_{1,k} = x_k = 0$ and we infer $u_k \to u_\infty$ in $C^1_{\mathrm{loc}}(\overline{B}_\delta \setminus \{0\})$ (but one could also use Lemma 26 below). It remains to prove the following proposition.

Proposition 8 *We have $u_\infty \equiv 0$ and $\Lambda = 4\pi$.*

Proof. Set for a given $\nu \in (1, 2)$

$$\varphi_k(r) = r^{2\nu} \overline{f}_k(r) := r^{2\nu} \lambda_k \overline{u}_k^2(r) e^{\overline{u}_k^2(r)}. \tag{17}$$

We compute

$$\varphi_k'(r) = r^{2\nu} \lambda_k e^{\overline{u}_k^2} [2\overline{u}_k \overline{u}_k' + 2\overline{u}_k^3 \overline{u}_k' + 2\nu r^{-1} \overline{u}_k^2] = 2r^{2\nu-1} \overline{f}_k [r\overline{u}_k'/\overline{u}_k + r\overline{u}_k \overline{u}_k' + \nu], \tag{18}$$

hence, using that $\lim_{k\to\infty} \overline{u}_k(Rr_k) = \infty$ for $R > 0$, which follows from (11), we infer

$$\lim_{k\to\infty} \varphi_k'(Rr_k) < 0 \Leftrightarrow \lim_{k\to\infty} Rr_k \overline{u}_k(Rr_k) \overline{u}_k'(Rr_k) < -\nu, \qquad \text{for any } R > 0.$$

For fixed $R_0 = R_0(\nu) > 0$ large enough (11) implies

$$\lim_{k\to\infty} R_0 r_k \overline{u}_k(R_0 r_k) \overline{u}_k'(R_0 r_k) = \lim_{k\to\infty} R_0 r_k u_k(x_k) \overline{u}_k'(R_0 r_k) = \lim_{k\to\infty} R_0 \eta_k'(R_0) = -\frac{2R_0^2}{1 + R_0^2} < -\nu. \tag{19}$$

Then there exists $k_0(R_0) \in \mathbb{N}$ such that $\varphi'(R_0 r_k) < 0$ for $k \geq k_0(R_0)$. Set

$$\rho_k := \sup\{r \in [R_0 r_k, \delta] : \varphi_k' < 0 \text{ in } [R_0 r_k, r)\}. \tag{20}$$

Clearly, again resting on (11),

$$\lim_{k\to\infty} \frac{\rho_k}{r_k} = \infty, \quad \frac{\overline{u}_k(\rho_k)}{\overline{u}_k(r_k \cdot)} \to 0 \text{ in } L^\infty_{\mathrm{loc}}([0, +\infty)), \quad \varphi_k'(\rho_k) = 0 \text{ if } \rho_k < \delta, \tag{21}$$

and up to taking $k_0(R_0)$ even larger we have

$$\varphi_k(rr_k) \leq C_\nu r_k^{2(\nu-1)}, \quad \text{for } r \in [0, R_0],\ k \geq k_0(R_0).$$

Then by the definition of ρ_k we immediately infer

$$r^{2\nu}\overline{f}_k(r) \le C_\nu r_k^{2(\nu-1)}, \quad \text{for } 0 \le r \le \rho_k. \tag{22}$$

An important consequence of (22) is the following estimate on the *neck* $B_{\rho_k}(x_k)\backslash B_{Rr_k}(x_k)$ (recall that $x_k = 0$):

$$\begin{aligned}\int_{B_{\rho_k}(x_k)\backslash B_{Rr_k}(x_k)} \overline{f}_k(|x-x_k|)dx &\le C_\nu r_k^{2(\nu-1)} \int_{B_{\rho_k}(x_k)\backslash B_{Rr_k}(x_k)} |x-x_k|^{-2\nu}dx \\ &\le Cr_k^{2(\nu-1)}\int_{Rr_k}^{\rho_k} r^{1-2\nu}dr \\ &= CR^{2(1-\nu)} - C\left(\frac{r_k}{\rho_k}\right)^{2(\nu-1)},\end{aligned}$$

which, together with (21) implies

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{\rho_k}(x_k)\backslash B_{Rr_k}(x_k)} \overline{f}_k(|x-x_k|)dx = 0. \tag{23}$$

We now consider 3 cases.

Case 1 Up to a subsequence $\lim_{k\to\infty}\rho_k = \delta' \in (0,\delta]$. Then (22) for $r = \delta'/2$ implies at once that (setting $u_\infty(x) = \overline{u}_\infty(|x|)$)

$$0 = \lim_{k\to\infty}\overline{f}_k(\delta'/2) = \lambda_\infty \overline{u}_\infty^2(\delta'/2)e^{\overline{u}_\infty^2(\delta'/2)}. \tag{24}$$

If $\lambda_\infty > 0$ we conclude that $\overline{u}_\infty(\delta'/2) = 0$, hence $u_\infty \equiv 0$ by the maximum principle. If $\lambda_\infty = 0$, then $\Delta u_\infty = 0$, hence $u_\infty \equiv 0$. In both cases (12) and (23) imply that $\Lambda = 4\pi$ and we are done.

Case 2 Up to a subsequence $\lim_{k\to\infty}\rho_k = 0$ and $\lim_{k\to\infty}\overline{u}_k(\rho_k) = 0$. Observe that $\overline{u}_k(r)$ is monotone decreasing for $R > 0$, $r \ge Rr_k$, and $k \ge k_0(R)$. Indeed by the divergence theorem, (11) and (15) we can write

$$\begin{aligned}2\pi r u_k(x_k)\overline{u}_k'(r) &= u_k(x_k)\int_{B_r(x_k)}\Delta u_k dx = -\int_{B_r(x_k)}\lambda_k u_k(x_k)u_k e^{u_k^2}dx + \int_{B_r(x_k)} u_k(x_k)g_k e^{u_k^2}dx \\ &\le -\int_{B_R(x_k)} e^{2\eta_\infty}dx + o(1) < 0, \quad \text{for } r \ge Rr_k.\end{aligned} \tag{25}$$

Then

$$\lim_{k\to\infty}\overline{u}_k(r) = \overline{u}_\infty(r) = 0 \quad \text{for } r > 0,$$

i.e. $u_\infty \equiv 0$. Still using (12) and (23) we infer that $\Lambda = 4\pi$.

Case 3 Up to a subsequence $\lim_{k\to\infty}\rho_k=0$ and $\lim_{k\to\infty}\overline{u}_k(\rho_k)\in(0,\infty]$. We claim that this is not possible. Indeed we will prove that

$$\lim_{k\to\infty}\rho_k\overline{u}_k(\rho_k)\overline{u}_k'(\rho_k)=0, \tag{26}$$

which clearly contradicts (18) and $\varphi_k'(\rho_k)=0$ in (21). In order to prove (26) we write

$$\begin{aligned}-2\pi\rho_k\overline{u}_k(\rho_k)\overline{u}_k'(\rho_k) &= \overline{u}_k(\rho_k)\int_{B_{\rho_k}(x_k)}(-\Delta u_k)dx\\ &=\int_{B_{Rr_k}(x_k)}\frac{\overline{u}_k(\rho_k)}{u_k}f_kdx+\int_{B_{\rho_k}(x_k)\setminus B_{Rr_k}(x_k)}\frac{\overline{u}_k(\rho_k)}{u_k}f_kdx\\ &\quad-\int_{B_{\rho_k}(x_k)}\overline{u}_k(\rho_k)g_ke^{u_k^2}dx\\ &=:I_{R,k}+II_{R,k}+III_k.\end{aligned} \tag{27}$$

According to (21), $\lim_{k\to\infty}\frac{\overline{u}_k(\rho_k)}{\overline{u}_k(rr_k)}=0$ uniformly for $r\in[0,R]$ for every $R\geq 0$ and with (12) we infer at once $\lim_{k\to\infty}I_{R,k}=0$. As for $II_{R,k}$ we use (23) and (25) to obtain

$$\lim_{R\to\infty}\lim_{k\to\infty}II_{R,k}\leq\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{\rho_k}(x_k)\setminus B_{Rr_k}(x_k)}\overline{f}_k(|x-x_k|)dx=0.$$

Finally, since by (21) and (25) we have $\overline{u}_k(\rho_k)\leq\overline{u}_k(r)$ for $r\in[0,\rho_k]$ and k large enough, with (15) and (16) we get $\lim_{k\to\infty}III_k=0$. This proves (26). □

2.3 Proof of Theorem 1 in the general case

It remains to prove the following proposition.

Proposition 9 *Using the notation of Theorem 6 we have $u_\infty\equiv 0$ and (14) holds. In particular $\Lambda=4\pi I$.*

In order to apply the ideas of the rotationally symmetric case to the general case, we shall need the following estimate, taken from [23, Lemma 5.14], which extends to the parabolic case a similar statement from [19, Proposition 2] and which allows us to compare u_k with its average on circles

$$\sup_{B_{R_k(y)/2}(y)}|u_k^2-u_k^2(y)|\leq C,\quad R_k(x):=\min_{1\leq i\leq I}|x-x_{i,k}|, \tag{28}$$

where C depends on Λ and Ω only.

One of the main difficulties in the proof of Proposition 9 is that different blow-up sequences $(x_{i,k})$ may converge to the same point. In order to deal with that we proceed by induction, proving the following lemma.

Lemma 10 *(a) For $1 \leq \ell \leq I-1$ the following holds: given $1 \leq i \leq I$, $\mathcal{J} \subset \{1,\dots,I\}\setminus\{i\}$ with $\#\mathcal{J} = \ell$ such that $x_{j,k} \to x^{(i)}$ for $j \in \mathcal{J}$, and numbers $s_k \in (0, \mathrm{dist}(x_{i,k}, \partial\Omega)]$ such that $r_{i,k} = o(s_k)$, $d_{i,j,k} = o(s_k)$ for $j \in \mathcal{J}$, and $d_{i,j,k} \geq 2s_k$ for $j \in \{1,\dots,I\}\setminus(\mathcal{J}\cup\{i\})$, where $d_{i,j,k} := |x_{i,k} - x_{j,k}|$, we have*

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{s_k}(x_{i,k})\setminus\bigcup_{j\in\mathcal{J}\cup\{i\}} B_{Rr_{j,k}}(x_{j,k})} f_k dx = 0. \tag{29}$$

(b) $u_\infty \equiv 0$.

Proof. We first prove part (a) inductively, starting with the case $\ell = 0$. Let $1 \leq i \leq I$ be fixed, $\mathcal{J} = \emptyset$ and s_k be as in the lemma. To simplify the notation write $x_k := x_{i,k} \to x^{(i)}$ and $r_k := r_{i,k}$. Set

$$\overline{u}_k(r) := \fint_{\partial B_r(x_k)} u_k d\sigma, \quad 0 < r \leq \delta := \mathrm{dist}(x^{(i)}, \partial\Omega)/2,$$

and set $\varphi_k(r)$, $\overline{f}_k(r)$ as in (17). Notice that $\delta > 0$ by Lemma 7. As before for $R_0 = R_0(\nu)$ and $k \geq k_0(R_0)$ we have $\varphi_k'(R_0 r_k) < 0$ and we let ρ_k be as in (20). Then (21) and (22) hold true. A simple covering argument shows that (28) and $s_k \leq 2d_{i,j,k}$ for $j \neq i$ imply

$$\sup_{\partial B_r(x_k)} u_k^2 - \inf_{\partial B_r(x_k)} u_k^2 \leq C, \quad \text{for } r \leq s_k, \tag{30}$$

whence

$$\sup_{\partial B_r(x_k)} e^{u_k^2} \leq C e^{\overline{u}_k^2(r)}, \quad \text{for } r \leq s_k \tag{31}$$

and

$$\sup_{\partial B_r(x_k)} u_k^2 e^{u_k^2} \leq C_1(1+\overline{u}_k^2(r))e^{\overline{u}_k^2(r)} \leq C_2 \inf_{\partial B_r(x_k)}(1+u_k^2)e^{u_k^2}, \quad \text{for } r \leq s_k. \tag{32}$$

As before we distinguish 3 cases.

Case 1 Up to extracting a subsequence, $\lim_{k\to\infty}\rho_k = \delta' \in (0,\delta]$. Then as before (22) implies (24). If $\lambda_\infty > 0$, then $\overline{u}_\infty(\delta'/2) = 0$. Then, since $u_\infty \geq 0$, the maximum principle implies $u_\infty \equiv 0$. If $\lambda_\infty = 0$, then $\Delta u_\infty \equiv 0$, hence $u_\infty \equiv 0$. Assume also $\lim_{k\to\infty} s_k = s > 0$ (up to a subsequence). Then (23) and (32) yield for $\varepsilon \in (0,s)$

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_\varepsilon(x_k)\setminus B_{Rr_k}(x_k)} f_k dx \leq C \lim_{R\to\infty}\lim_{k\to\infty}\int_{B_\varepsilon(x_k)\setminus B_{Rr_k}(x_k)} \overline{f}_k(|x-x_k|)dx + C\varepsilon^2 = C\varepsilon^2, \tag{33}$$

while $u_k \to u_\infty \equiv 0$ in $C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\setminus\{x_j : 1 \leq j \leq I\})$ implies

$$\lim_{k\to\infty}\int_{B_{s_k}(x_k)\setminus B_\varepsilon(x_k)} f_k dx = 0. \tag{34}$$

Letting $\varepsilon \to 0$ we finally obtain (29). If $\lim_{k\to\infty} s_k = 0$, we simply take $\varepsilon = s_k$ in (33) and we are done.

Case 2 Up to extracting a subsequence, $\lim_{k\to\infty}\rho_k = 0$ and $\lim_{k\to\infty}\overline{u}_k(\rho_k) = 0$. As in (25), $\overline{u}_k'(r) \le 0$ for $r \ge \rho_k > r_k$ and k large, and we infer $u_\infty \equiv 0$ as in Case 1. Assume now that $s_k \le \rho_k$. Then, in analogy to (33), (23) and (32) yield

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{s_k}(x_k)\setminus B_{Rr_k}(x_k)} f_k dx = 0. \tag{35}$$

If instead $s_k > \rho_k$ and $s_k \to 0$ by monotonicity $\overline{u}_k(r) \to 0$ uniformly for $r \in [\rho_k, s_k]$; together with $\lim_{k\to\infty}\lambda_k < \infty$ and (30) this gives

$$\lim_{k\to\infty}\int_{B_{s_k}(x_k)\setminus B_{\rho_k}(x_k)} f_k dx \le C s_k^2 = o(1),$$

and we are done. If $s_k \to \delta' > 0$ then we split the above integral into an integral over $B_\varepsilon(x_k)\setminus B_{\rho_k}(x_k)$ and an integral over $B_{s_k}(x_k)\setminus B_\varepsilon(x_k)$ and let ε tend to 0 as in (33) and (34).

Case 3 Up to extracting a subsequence, $\lim_{k\to\infty}\rho_k = 0$ and $\lim_{k\to\infty}\overline{u}_k(\rho_k) \in (0,\infty]$. If up to extracting a subsequence $s_k \le \rho_k$, then we see again with (23) and (32) that

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{s_k}(x_k)\setminus B_{Rr_k}(x_k)} f_k dx \le \lim_{k\to\infty} C s_k^2 + C\int_{B_{s_k}(x_k)\setminus B_{Rr_k}(x_k)} \overline{f}_k(|x-x_k|)dx = 0,$$

and we are done. Assume instead that $s_k \ge \rho_k$. We want to find a contradiction by proving that (26) holds. As in (27) write $-2\pi\rho_k\overline{u}_k(\rho_k)\overline{u}_k'(\rho_k) = I_{R,k} + II_{R.k} + III_k$. By Theorem 6 and (21)

$$\lim_{R\to\infty}\lim_{k\to\infty} I_{R,k} \le \lim_{R\to\infty}\lim_{k\to\infty}\frac{\overline{u}_k(\rho_k)}{\overline{u}_k(Rr_k)}\int_{B_{Rr_k}(x_k)} f_k dx = 0.$$

As for III_k, first observe that by Hölder's inequality, (15) and (16) we have

$$\begin{aligned} III_k &\le \left(\int_{B_{\rho_k}(x_k)} \lambda_k \overline{u}_k^2(\rho_k) e^{u_k^2} dx\right)^{\frac{1}{2}} \left(\int_{B_{\rho_k}(x_k)} \frac{1}{\lambda_k} g_k^2 e^{u_k^2} dx\right)^{\frac{1}{2}} \\ &= o(1)\left(\int_{B_{\rho_k}(x_k)} \lambda_k \overline{u}_k^2(\rho_k) e^{u_k^2} dx\right)^{\frac{1}{2}}. \end{aligned} \tag{36}$$

Using (32), (25) and (21) we see that for $R > 0$ fixed and k large

$$\int_{B_{\rho_k}(x_k)} \lambda_k \overline{u}_k^2(\rho_k) e^{u_k^2} dx \le C\int_{B_{\rho_k}(x_k)\setminus B_{Rr_k}(x)} \overline{f}_k dx + \frac{\overline{u}_k^2(\rho_k)(1+o(1))}{\overline{u}_k^2(Rr_k)}\int_{B_{Rr_k}(x_k)} f_k dx = o(1) \tag{37}$$

where we used (23). This proves that $\lim_{k\to\infty} III_k = 0$.

To bound $II_{R,k}$ recall that $\overline{u}_k(\rho_k) \ge \frac{1}{C}$. By (25) we also have $\overline{u}_k \ge \frac{1}{C}$ on $[Rr_k, \rho_k]$ and (11) implies that $\overline{u}_k \to \infty$ uniformly on $[0, Rr_k]$ as $k \to \infty$. Then we bound with (30) and (31)

$$u_k(x) \le C\overline{u}_k(|x-x_k|) \quad \text{for } 0 < |x-x_k| \le \rho_k,$$

and

$$\lambda_k \overline{u}_k(\rho_k) u_k(x) e^{u_k^2(x)} \leq C\overline{f}_k(|x-x_k|), \text{ for } 0<|x-x_k|\leq \rho_k,$$

hence by (23)

$$\lim_{R\to\infty}\lim_{k\to\infty} II_{R,k} \leq \lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{\rho_k}(x_k)\setminus B_{Rr_k}(x_k)} C\overline{f}_k(|x-x_k|)dx = 0.$$

Going back to (27), we see that (26) is proven. As in the symmetric case, we obtain a contradiction from (18) and (21). This completes the proof of the lemma when $\ell = 0$. Notice that for $\ell = 0$ and s_k as above, we have proven that $u_\infty \equiv 0$ unless we are in Case 3 and $s_k < \rho_k$.

We will now prove that the validity of Lemma 10 with $\#\mathcal{J} = \ell$ follows from its validity with $\#\mathcal{J} \in \{0,\dots,\ell-1\}$. Define $\overline{u}_k$ and ρ_k as before with $x_k = x_{1,k}$, $r_k = r_{1,k}$, and let s_k and $\mathcal{J}$ be as in the statement of the lemma with $i = 1$ (up to a relabeling) and $\#\mathcal{J} = \ell$. Again we split the proof into three cases, but this time we shall start with Case 3, which is the most complex.

Case 3 Assume that up to a subsequence

$$\lim_{k\to\infty}\rho_k = 0, \quad \lim_{k\to\infty}\overline{u}_k(\rho_k) \in (0,\infty], \quad s_k \geq \rho_k. \tag{38}$$

In particular (21) and (22) still hold true. The terms $I_{R,k}$, $II_{R,k}$ and III_k are as in (27) and set also $\mathcal{I} := \{i \in \mathcal{J} : d_{1,i,k} \leq 2\rho_k\}$. To bound III_k as in (36) we just need to prove that

$$\int_{B_{\rho_k}(x_k)} \lambda_k \overline{u}_k^2(\rho_k) e^{u_k^2} dx \leq C.$$

We split $B_{\rho_k}(x_k)$ as $B_{\rho_k}(x_k) = A_{1,k} \cup A_{2,k}$, where for some $L \geq 2$ to be fixed

$$A_{1,k} := B_{\rho_k}(x_k) \setminus \bigcup_{i\in\mathcal{I}} B_{d_{1,i,k}/L}(x_{i,k}), \quad A_{2,k} := B_{\rho_k}(x_k) \cap \bigcup_{i\in\mathcal{I}} B_{d_{1,i,k}/L}(x_{i,k}). \tag{39}$$

The point of this splitting is that (30) can be easily replaced by

$$\sup_{\partial B_r(x_k)\setminus\bigcup_{i=2}^I B_{d_{1,i,k}/L}(x_{i,k})} u_k^2 - \inf_{\partial B_r(x_k)\setminus\bigcup_{i=2}^I B_{d_{1,i,k}/L}(x_{i,k})} u_k^2 \leq C(L), \tag{40}$$

which follows from (28). Also (31) and (32) are subject to an analogous change.

To simplify the notations, we shall suppress the index k in the sets $A_{1,k}$, $A_{2,k}$. The integral over A_1 can be dealt with as in (37), this time using (40). Now split A_2 as $A_2 = A_3 \cup A_4$, with

$$A_3 := A_2 \cap \{x : u_k(x) \leq \overline{u}_k(\rho_k)\}, \quad A_4 := A_2 \cap \{x : u_k(x) > \overline{u}_k(\rho_k)\}.$$

Then

$$\int_{A_4} \lambda_k \overline{u}_k^2(\rho_k) e^{u_k^2} dx \leq \int_{A_4} f_k dx \leq \Lambda,$$

while using

$$\lim_{k\to\infty}\rho_k^2\overline{f}_k(\rho_k)\le C_\nu\lim_{k\to\infty}\left(\frac{r_k}{\rho_k}\right)^{2(\nu-1)}=0,$$

which follows from (21) and (22), we conclude

$$\int_{A_3}\lambda_k\overline{u}_k^2(\rho_k)e^{u_k^2}dx\le\int_{A_3}\overline{f}_k(\rho_k)dx\le C\sum_{i\in\mathcal{I}}d_{1,i,k}^2\overline{f}_k(\rho_k)\le C\rho_k^2\overline{f}_k(\rho_k)\to 0.$$

In particular $\lim_{k\to\infty}III_k=0$. The proof of $\lim_{R\to\infty}\lim_{k\to\infty}I_{R,k}=0$ is as before.

We now want to bound $II_{R,k}$. Since the integral over A_1 is infinitesimal as before, we can focus on the integrals over A_3 and A_4. We have

$$\int_{A_3}\lambda_k\overline{u}_k(\rho_k)u_ke^{u_k^2}dx\le\int_{A_3}\overline{f}_k(|x-x_k|)dx=o(1)$$

by (23). In order to bound the analogous integral over A_4 fix $i\in\mathcal{I}$ and set

$$A_{4,i}:=A_4\cap B_{d_{1,i,k}/L}(x_{i,k}).$$

We first claim that

$$\lim_{k\to 0}\frac{\overline{u}_k(\rho_k)}{u_k(x_{j,k})}=0,\quad\text{for } j\in\mathcal{I}. \tag{41}$$

Indeed from (22) and (10) we have for $j\in\mathcal{I}$

$$\frac{\overline{f}_k(\rho_k)}{f_k(x_{j,k})}\le\frac{C_\nu r_k^{2(\nu-1)}r_{j,k}^2}{4\rho_k^{2\nu}}=o(1)\left(\frac{r_k}{\rho_k}\right)^{2(\nu-1)}=o(1),$$

where we also used that $r_{j,k}=o(1)d_{1,j,k}=o(1)\rho_k$ by (13). Then (41) follows at once. Consider

$$\mathcal{J}_i:=\{j\in\mathcal{I}:j\ne i,\ d_{i,j,k}=o(d_{1,i,k})\}.$$

Upon choosing L larger we can assume that up to a subsequence

$$B_{2d_{1,i,k}/L}(x_{i,k})\cap B_{d_{1,j,k}/L}(x_{j,k})=\emptyset\quad\text{ for } j\notin\mathcal{J}_i\cup\{i\}. \tag{42}$$

Now let us bound

$$\begin{aligned}\int_{A_{4,i}}\lambda_k\overline{u}_k(\rho_k)u_ke^{u_k^2}dx\le&\int_{\bigcup_{j\in\mathcal{J}_i\cup\{i\}}B_{Rr_{j,k}}(x_{j,k})}\lambda_k\overline{u}_k(\rho_k)u_ke^{u_k^2}dx+\int_{A_{4,i}\setminus\bigcup_{j\in\mathcal{J}_i\cup\{i\}}B_{Rr_{j,k}}(x_{j,k})}f_kdx\\ =:&IV_{i,R,k}+V_{i,R,k}.\end{aligned}$$

Using (11) and (41) we get

$$IV_{i,R,k}\le\sum_{j\in\mathcal{J}_i\cup\{i\}}\frac{\overline{u}_k(\rho_k)(1+o(1))}{u_k(x_{j,k})}\int_{B_{Rr_{j,k}}(x_{j,k})}f_kdx=o(1)4\pi. \tag{43}$$

Clearly $\#\mathcal{J}_i \le \ell-1$, hence assuming that Lemma 10 holds for $\#\mathcal{J} \in \{0,\dots,\ell-1\}$, and applying it with $s_k = d_{1,i,k}/L$ we infer

$$\lim_{R\to\infty}\lim_{k\to\infty} V_{i,R,k} = 0.$$

In conclusion, summing over $i \in \mathcal{J}$, we have proven (26) and get a contradiction from (18) and (21), hence (38) does not hold.

If in (38) we have $s_k < \rho_k$ instead of $s_k \ge \rho_k$, then similar to the case $\ell = 0$ we find

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{s_k}(x_k)\setminus\left(B_{Rr_k}(x_k)\cup\bigcup_{i\in\mathcal{J}} B_{d_{1,i,k}/L}(x_{i,k})\right)} f_k dx = 0,$$

and thanks to the inductive hypothesis applied for $i \in \mathcal{J}$ and $s_k = d_{1,i,k}/L$,

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_{d_{1,i,k}/L}(x_{i,k})\setminus\bigcup_{j\in\mathcal{J}_i\cup\{i\}} B_{Rr_{j,k}}(x_{j,k})} f_k dx = 0,$$

where L has been taken possibly larger, so that (42) holds.

Case 1 Up to a subsequence $\lim_{k\to\infty}\rho_k = \delta' \in (0,\delta]$. The proof of the fact that $u_\infty \equiv 0$ and of (23) remain unchanged. Assume that up to a subsequence $\lim_{k\to\infty} s_k = s > 0$. We replace (33) with

$$\lim_{R\to\infty}\lim_{k\to\infty}\int_{B_\varepsilon(x_k)\setminus\left(B_{Rr_k}\cup\bigcup_{i\in\mathcal{J}} B_{d_{1,i,k}/L}(x_{i,k})\right)} f_k dx \le C\varepsilon^2, \quad \varepsilon \in (0,s),$$

whose proof is similar thanks to (40), and we replace (34) with

$$\lim_{k\to\infty}\int_{B_{s_k}(x_k)\setminus\left(B_\varepsilon(x_k)\cup\bigcup_{i\in\mathcal{I}} B_{d_{1,i,k}/L}(x_{i,k})\right)} f_k dx = 0,$$

where we used that $u_k \to 0$ uniformly on $B_{s_k}(x_k)\setminus B_\varepsilon(x_k)$. The integral over $\bigcup_{i\in\mathcal{I}} B_{d_{1,i,k}/L}(x_{i,k})$ can be treated using the inductive hypothesis, hence we get (29) by letting $\varepsilon \to 0$. If $\lim_{k\to\infty} s_k = 0$ as in the case $\ell = 0$ we take $\varepsilon = s_k$ above and conclude in a similar way.

Case 2 Up to a subsequence $\lim_{k\to\infty}\rho_k = 0$ and $\lim_{k\to\infty}\overline{u}_k(\rho_k) = 0$. Again, we proceed as in the case $\ell = 0$, treating the integrals over $\bigcup_{i\in\mathcal{I}} B_{d_{1,i,k}/L}(x_{i,k})$ using the inductive hypothesis.

In conclusion, we have proven (29), i.e. part (a) of the lemma. Moreover we have shown that $u_\infty \equiv 0$, if we can find $1 \le i \le I$, $\mathcal{J}$ and numbers $s_k \in (0, \mathrm{dist}(x_{i,k},\partial\Omega)]$ as in the statement of the lemma such that we do not fall in the case when

$$\lim_{k\to\infty}\rho_k = 0, \quad \overline{u}_k(\rho_k) \in (0,\infty], \quad s_k < \rho_k. \tag{44}$$

Pick any i with $1 \le i \le I$ and set

$$\mathcal{J} := \{j : 1 \le j \le I,\ j \ne i,\ d_{i,j,k} = o(1)\}, \quad s_k = s_{i,k} := \min\left\{\mathrm{dist}(x_{i,k},\partial\Omega), \min_{j\notin\mathcal{J}\cup\{i\}} d_{i,j,k}/2\right\}. \tag{45}$$

Lemma 7 implies $\liminf_{k\to\infty} s_k > 0$ and (44) cannot hold, hence also part (b) is proven. $\square$

Proof of Proposition 9. For $1 \le i \le I$, apply Lemma 10 with $\mathcal{J}$ and $s_k = s_{i,k}$ defined as in (45). Since $u_k \to u_\infty \equiv 0$ in $C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega} \setminus \{x^{(1)}, \ldots, x^{(I)}\})$ and $\liminf_{k\to\infty} s_{i,k} > 0$, in addition to (29) we also have

$$\lim_{k\to\infty} \int_{\Omega \setminus \bigcup_{i=1}^{I} B_{s_{i,k}}(x_{i,k})} f_k dx = 0,$$

and (14) follows. $\square$

2.4 Proof of Lemma 7

Our approach closely follows [22], where the elliptic case is treated. For $x \in \partial\Omega$ let $\nu(x) = (\nu_1(x), \nu_2(x)) \in \mathbb{R}^2$ be the exterior unit normal to $\partial\Omega$ at x.

Lemma 11 *Let $x \in \partial\Omega$ with $\nu_1(x) > 0$ and fix $0 < t_0 < T < \infty$. Then there exists $\delta > 0$, such that $u_{x_1}(t,x) < 0$ for $(t,x) \in [t_0, T] \times (\Omega \cap B_\delta(x))$.*

Proof. Since $u_t = \Delta u + \lambda(t) f(u)$, with $f(0) = 0$, we can write (3) as

$$e^{u^2(t,x)} u_t(t,x) = \Delta u(t,x) + \lambda(t) c(x,t) u(t,x),$$

which is linear and to which we can apply the parabolic Hopf maximum principle (see [26], Lemma 2.8). $\square$

Consider a point $x_0 \in \partial\Omega$ and up to a rotation assume that $\nu(x_0) = (-1, 0)$. Up to a dilation independent of x_0 and a translation (which does depend on x_0) we can also assume that $x_0 = (1,0)$ and $\overline{B}_1(0) \cap \overline{\Omega} = \{x_0\}$. Then we invert Ω and u with respect to $\partial B_1(0)$, i.e. we define

$$\tilde{u}(t,x) := u(t, x/|x|^2) \quad \text{ for } x \in \overline{\tilde{\Omega}} := \{y \in \mathbb{R}^2 : y/|y|^2 \in \overline{\Omega}\} \subset B_1(0) \setminus \{0\}.$$

The conformal covariance of Δ in $\mathbb{R}^2$ implies that

$$e^{\tilde{u}^2(t,x)} \tilde{u}_t(t,x) = |x|^4 \Delta \tilde{u}(t,x) + \lambda(t) \tilde{u}(t,x) e^{\tilde{u}^2(t,x)}.$$

For $\tau > 0$ we set

$$\Sigma(\tau) := \{x \in \tilde{\Omega} : x_1 \ge \tau\}, \quad T_\tau := \{(\tau, x_2) \in \mathbb{R}^2 : x_2 \in \mathbb{R}\}.$$

Notice that $\sup\{\tau : \Sigma(\tau) \neq \emptyset\} = 1$. Set also

$$x^\tau := (-x_1 + 2\tau, x_2), \quad \Sigma'(\tau) := \{x^\tau : x \in \Sigma(\tau)\} = \text{reflection of } \Sigma(\tau) \text{ across } T_\tau$$

and let

$$\tau_1 := \inf\{\tau : \Sigma'(\tau) \subset \tilde{\Omega} \text{ and } \nu_1(x) > 0 \text{ for } x \in \partial\Sigma(\tau) \setminus T_\tau\} < 1,$$

where the last inequality follows from the strict convexity of $\tilde{\Omega}$ at $x_0 = (1,0)$.

By Lemma 11 we have that for some $t_0 > 0$ and $\delta > 0$ independent of x_0

$$\tilde{u}_{x_1}(t_0, x) < 0, \quad \text{for } x \in \Sigma(1 - 2\delta). \tag{46}$$

Up to translating the solution, we can and do assume that $t_0 = 0$.

Proposition 12 *With the above notation and assumptions, given any τ such that*

$$\max\{\tau_1, 1 - \delta\} < \tau < 1, \tag{47}$$

where δ is as in (46)*, we have*

$$\tilde{u}_{x_1}(t, x) < 0 \text{ and } \tilde{u}(t, x) < \tilde{u}(t, x^\tau) \quad \text{for } (t, x) \in [0, \infty) \times \Sigma(\tau). \tag{48}$$

Before proving the proposition we need to state the following lemma.

Lemma 13 *Assume that for some τ as in* (47) *we have*

$$\tilde{u}_{x_1}(t, x) \leq 0, \quad \tilde{u}(t, x) \leq \tilde{u}(t, x^\tau), \quad \tilde{u}(t, x) \not\equiv \tilde{u}(t, x^\tau), \quad \text{for } (t, x) \in [0, T] \times \Sigma(\tau). \tag{49}$$

Then

$$\tilde{u}(t, x) < \tilde{u}(t, x^\tau) \text{ in } (0, T] \times \Sigma(\tau) \quad \text{and} \quad \tilde{u}_{x_1} < 0 \text{ in } (0, T] \times (T_\tau \cap \overline{\tilde{\Omega}}). \tag{50}$$

Proof. Set

$$v(t, x) := \tilde{u}(t, x^\tau), \quad w(t, x) := v(t, x) - \tilde{u}(t, x) \quad \text{for } (t, x) \in [0, T] \times \Sigma'(\tau).$$

Then $w \leq 0$, $w \not\equiv 0$ in $[0, T] \times \Sigma'(\tau)$, $w \equiv 0$ on $T_\tau \cap \Omega$ and writing $v_t = e^{-v^2}|x|^4 \Delta v + \lambda v$, we obtain

$$w_t(t, x) = e^{-v^2(t,x)}|x|^4 \Delta w(t, x) + c(t, x)w(t, x) \quad \text{for } (t, x) \in [0, T] \times \Sigma'(\tau),$$

for some smooth function c. The equation is uniformly parabolic since $|x| \geq \varepsilon > 0$ for $x \in \tilde{\Omega}$ and v is smooth, hence bounded locally in time, therefore the maximum principle implies that $w < 0$ in $(0, T] \times \Sigma'(\tau)$ and $-2\tilde{u}_{x_1} = w_{x_1} > 0$ on $(0, T] \times (T_\tau \cap \overline{\tilde{\Omega}})$. □

Proof of Proposition 12. Given τ as in (47), consider a maximal $T \in (0, \infty]$ such that the inequalities in (48) hold for $(t, x) \in (0, T) \times \Sigma(\tau)$. Assume that $T < \infty$. By continuity, the two inequalities in (49) are satisfied, and clearly $\tilde{u}(t, x) \not\equiv \tilde{u}(t, x^\tau)$ for any $(t, x) \in (0, T) \times \Sigma(\tau)$. Then Lemma 13 implies that (50) holds. Moreover, letting τ increase to 1 we see that $\tilde{u}_{x_1} < 0$ on $(0, T] \times \overline{\Sigma}(\tau)$, hence on $(0, T'] \times \overline{\Sigma}(\tau)$ for some $T' > T$ by continuity. If also $\tilde{u}(t, x) < \tilde{u}(t, x^\tau)$ on $(0, T''] \times \Sigma(\tau)$ for some $T'' > T$ we contradict the maximality of T. Otherwise we can find a sequence (t_j, x_j) with

$$t_j \downarrow T, \quad \Sigma(\tau) \ni x_j \to \overline{x} \in \overline{\Sigma}(\tau), \quad u(t_j, x_j) \geq u(t_j, x_j^\tau).$$

By Lemma 13 we have $\overline{x}^\tau \in \partial\Sigma(\tau)$. If $\overline{x} \in \partial\Sigma(\tau) \setminus T_\tau$, we have $\overline{x}^\tau \in \tilde{\Omega}$, hence

$$0 = u(T, \overline{x}) = u(T, \overline{x}^\tau) > 0,$$

contradicting the maximum principle. Then $\overline{x} \in T_\tau \cap \overline{\tilde{\Omega}}$. By the mean value theorem we can find $\sigma_j \in (0,1)$ such that the points $\tilde{x}_j := \sigma_j x_j + (1-\sigma_j)x_j^\tau$ satisfy $\tilde{u}_{x_1}(t_j, \tilde{x}_j) \geq 0$. Since $\tilde{x}_j \to \overline{x}$ we infer $\tilde{u}_{x_1}(T, \overline{x}) \geq 0$, contradicting (50), hence $T = \infty$ and we are done. □

Proof of Lemma 7. Given $x_0 \in \partial\Omega$, perform the rotation and translation as above. By (48) we then have

$$u_{x_1}(t,x) > 0 \quad \text{for } t > 0,\ x = (x_1, 0),\ 1 \leq x_1 < \delta^* \leq \min\left\{\frac{1}{1-\delta}, \frac{1}{\tau_1}\right\}. \tag{51}$$

By the smoothness of Ω we can choose $\delta^* > 1$ independent of x_0 such that (51) holds. Then we have proven that $\nabla u(t,x) \neq 0$ for $t \geq t_0$ set as in Lemma 11 and x in a neighborhood of $\partial\Omega$. If $x_{i,k} \to x^{(i)} \in \partial\Omega$ for some $1 \leq i \leq I$, then (11) would imply that $u(t_k, \cdot)$ has a critical point in $B_{r_{i,k}}(x_{i,k})$ for k large enough, a contradiction. □

2.5 Proof of Theorem 2

The proof of Theorem 6 applies to sequences of non-negative solutions to

$$\Delta u_k + \lambda_k u_k e^{u_k^2} = g_k e^{u_k^2} \text{ in } \Omega \tag{52}$$

where $\lambda_k \to \lambda_\infty \geq 0$, $g_k : \Omega \to \mathbb{R}$ is measurable, $\lim_{k\to\infty} \|u_k\|^2_{H_0^1} = \Lambda$ (the condition $E(u_k) \leq C$ is not needed, see [23]) and

$$\lim_{k\to\infty} \frac{1}{\lambda_k} \int_\Omega g_k^2 e^{u_k^2} dx \to 0.$$

In our proof of Theorem 1 we started with Theorem 6, Lemma 5 and Lemma 7, therefore reducing essentially to solutions to (52) which blow-up, but not on the boundary ($x^{(i)} \in \Omega$). When $g_k \equiv 0$, which is the case of Theorem 2, this was proven in [22, p. 223], see also [2], hence the previous proof can be applied to Theorem 2.

3 Proof of Theorem 3

In this section we prove Theorem 3: we first show how the Moser-Trudinger inequality can be improved under suitable conditions on the function u, and then use this improvement to characterize the high levels of the functional $E|_{M_\Lambda}$. Finally, we use the construction of suitable test functions to build a min-max scheme for finding critical points of $E|_{M_\Lambda}$.

3.1 Improvement of the M-T inequality

In the next result we show uniform boundedness of $\int_\Omega e^{u^2} dx$ for values of $\|u\|^2_{H_0^1}$ greater than 4π, provided the measure $e^{u^2} dx$ is *spread* into different regions of the domain.

Lemma 14 *For a fixed integer ℓ, let $\Omega_1, \dots, \Omega_{\ell+1}$ be subsets of Ω satisfying $\mathrm{dist}(\Omega_i, \Omega_j) \geq \delta_0$ for $i \neq j$ and some $\delta_0 > 0$. Let also $\gamma_0 \in \left(0, \frac{1}{\ell+1}\right)$, $\delta_1 \in (0, 4\pi(\ell+1))$. Then there exists a constant $C = C(\ell, \delta_0, \delta_1, \gamma_0, \Omega)$ such that*

$$\int_\Omega e^{u^2} dx \leq C \tag{53}$$

for all the functions $u \in H_0^1(\Omega)$ satisfying

$$\frac{\int_{\Omega_i} e^{u^2} dx}{\int_\Omega e^{u^2} dx} \geq \gamma_0, \qquad \forall\, i \in \{1, \dots, \ell+1\}; \qquad \textit{and} \qquad \|u\|^2_{H_0^1} \leq 4\pi(\ell+1) - \delta_1. \tag{54}$$

Proof. We can find $\ell+1$ functions $g_1, \dots, g_{\ell+1}$ satisfying the following properties

$$\begin{cases} g_i(x) \in [0,1] & \text{for every } x \in \Omega; \\ g_i(x) = 1, & \text{for every } x \in \Omega_i; \\ g_i(x) = 0, & \text{if } \mathrm{dist}(x, \Omega_i) \geq \frac{\delta_0}{2}; \\ \|g_i\|_{C^4(\overline{\Omega})} \leq C_{\delta_0}, & \end{cases} \tag{55}$$

where $C_{\delta_0} > 0$ depends only on δ_0. For any $\varepsilon > 0$ using the inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ we can find a constant C_{ε,δ_0} depending only on ε and δ_0 such that, for any $i \in \{1, \dots, \ell+1\}$ there holds

$$\int_\Omega |\nabla(g_i v)|^2 dx \leq \int_\Omega g_i^2 |\nabla v|^2 dx + \varepsilon \int_\Omega |\nabla v|^2 dx + C_{\varepsilon,\delta_0} \int_\Omega v^2 dx, \quad \text{for } v \in H^1(\Omega). \tag{56}$$

From our assumptions and from (55) we deduce

$$\int_\Omega e^{u^2} dx \leq \frac{1}{\gamma_0} \int_{\Omega_i} e^{u^2} dx \leq \frac{1}{\gamma_0} \int_\Omega e^{(g_i u)^2} dx. \tag{57}$$

If we write u as $u = u_1 + u_2$ with $u_1 \in L^\infty(\Omega)$, from (57) and using again $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ we get for $\varepsilon > 0$

$$\int_\Omega e^{u^2} dx \leq \frac{1}{\gamma_0} e^{\left(1+\frac{1}{\varepsilon}\right)\|u_1\|^2_\infty} \int_\Omega e^{(1+\varepsilon)(g_i u_2)^2} dx. \tag{58}$$

We now choose i such that $\int_\Omega g_i^2 |\nabla u_2|^2 dx \leq \int_\Omega g_j^2 |\nabla u_2|^2 dx$ for every $j \in \{1, \dots, \ell+1\}$. Since the functions $g_1, \dots g_{\ell+1}$ have disjoint supports, from (56) applied with $v = u_2$ we then get

$$\int_\Omega |\nabla(g_i u_2)|^2 dx \leq \frac{1}{\ell+1} \int_\Omega |\nabla u_2|^2 dx + \varepsilon \int_\Omega |\nabla u_2|^2 dx + C_{\varepsilon,\delta_0} \int_\Omega u_2^2 dx. \tag{59}$$

Next we choose $\lambda_{\varepsilon,\delta_0}$ to be an eigenvalue of $-\Delta$ such that $\frac{C_{\varepsilon,\delta_0}}{\lambda_{\varepsilon,\delta_0}} < \varepsilon$, where C_{ε,δ_0} is given in the last formula, and we set

$$u_1 = P_{V_{\varepsilon,\delta_0}} u; \qquad\qquad u_2 = P_{V^\perp_{\varepsilon,\delta_0}} u.$$

Here $V_{\varepsilon,\delta_0} \subset H_0^1(\Omega)$ is the direct sum of the eigenspaces of $-\Delta$ with eigenvalues less than or equal to $\lambda_{\varepsilon,\delta_0}$, and $P_{V_{\varepsilon,\delta_0}}, P_{V^\perp_{\varepsilon,\delta_0}}$ denote the projections onto V_{ε,δ_0} and $V^\perp_{\varepsilon,\delta_0}$ respectively. Notice that for $v \in V_{\varepsilon,\delta_0}$ we have $\|v\|_{L^\infty} \leq \tilde{C}\|v\|_{L^2} \leq \hat{C}\|v\|_{H_0^1}$, where $\tilde{C}$ and $\hat{C}$ depend only on ε, δ_0 and Ω, hence by our choice of u_1 and u_2 there holds

$$\|u_1\|^2_{L^\infty(\Omega)} \leq \hat{C}\int_\Omega |\nabla u_1|^2 dx, \qquad C_{\varepsilon,\delta_0}\int_\Omega u_2^2 dx \leq \frac{C_{\varepsilon,\delta_0}}{\lambda_{\varepsilon,\delta_0}}\int_\Omega |\nabla u_2|^2 dx < \varepsilon \int_\Omega |\nabla u_2|^2 dx.$$

The second inequality in the last formula and (59) imply for ε small enough (depending on δ_1 and ℓ)

$$\int_\Omega |\nabla(g_i u_2)|^2 dx \leq \left(\frac{1}{\ell+1} + 2\varepsilon\right)\int_\Omega |\nabla u_2|^2 dx \leq \left(\frac{1}{\ell+1} + 2\varepsilon\right)\int_\Omega |\nabla u|^2 dx < 4\pi - \delta_1'$$

for some $\delta_1' > 0$ depending on δ_1, and $(1+\varepsilon)\|g_i u_2\|^2_{H_0^1} \leq 4\pi$. In particular from the Moser-Trudinger inequality (1) we get

$$\int_\Omega e^{(1+\varepsilon)(g_i u_2)^2} dx \leq C.$$

Notice also that

$$\|u_1\|^2_{L^\infty(\Omega)} \leq \hat{C}\int_\Omega |\nabla u_1|^2 dx \leq \hat{C}4\pi(\ell+1).$$

Hence the last formulas and (58) imply (53) for a constant C which depends only on ε (which in turn only depends on δ_1 and ℓ), δ_0, γ_0 and Ω. □

The next lemma, proven in [18] (Lemma 2.3), is a criterion which implies the situation described by the first condition in (54).

Lemma 15 *Let ℓ be a given positive integer, and consider $\varepsilon, r > 0$. Suppose that for a non-negative function $f \in L^1(\Omega)$ with $\|f\|_{L^1(\Omega)} = 1$ there holds*

$$\int_{\Omega\cap\bigcup_{i=1}^\ell B_r(p_i)} f_i dx < 1 - \varepsilon \qquad \text{for every } \ell\text{-tuple } p_1, \dots, p_l \in \Omega.$$

Then there exist $\bar{\varepsilon} > 0$ and $\bar{r} > 0$, depending only on ε, r, ℓ and Ω (but not on f), and $\ell+1$ points $\bar{p}_{1,f}, \dots, \bar{p}_{\ell+1,f} \in \Omega$ satisfying

$$\int_{B_{\bar{r}}(\bar{p}_{j,f})\cap\Omega} f dx \geq \bar{\varepsilon}, \text{ for } j = 1, \dots, \ell+1, \qquad \text{and} \qquad B_{2\bar{r}}(\bar{p}_{i,f}) \cap B_{2\bar{r}}(\bar{p}_{j,f}) \cap \Omega = \emptyset, \text{ for } i \neq j.$$

Finally, by the last two results, we can characterize the functions in $H_0^1(\Omega)$ for which the value of E is large.

Lemma 16 *If $\Lambda \in (4\pi\ell, 4\pi(\ell+1))$ with $\ell \geq 1$, the following property holds. For any $\varepsilon > 0$ and any $r > 0$ there exists a large positive $L = L(\varepsilon, r, \Lambda)$ such that, for every $u \in M_\Lambda^L := \{v \in M_\Lambda : E(v) \geq L\}$ there exist ℓ points $p_{1,u}, \dots, p_{\ell,u} \in \Omega$ such that*

$$\frac{\int_{\Omega\setminus\bigcup_{i=1}^\ell B_r(p_{i,u})} e^{u^2} dx}{\int_\Omega e^{u^2} dx} \leq \varepsilon.$$

Proof. Suppose by contradiction that the statement is not true, namely that there exist ε, $r > 0$ and $(u_n)_n \subset M_\Lambda$ with $E(u_n) \to \infty$ and such that for every ℓ-tuple $p_1, \dots, p_\ell$ in Ω there holds

$$\frac{\int_{\bigcup_{i=1}^{\ell} B_r(p_i)} e^{u_n^2} dx}{\int_\Omega e^{u_n^2} dx} < 1 - \varepsilon.$$

Then we can apply Lemma 15 with $f = \frac{e^{u_n^2}}{\int_\Omega e^{u_n^2} dx}$, and in turn Lemma 14 with $\delta_0 = 2\bar{r}$, $\delta_1 = \Lambda - 4\pi(\ell+1)$, $\Omega_i = B_{\bar{r}}(\bar{p}_{i,u_n}) \cap \Omega$ and $\gamma_0 = \bar{\varepsilon}$, where $\bar{\varepsilon}$, $\bar{r}$ and $\bar{p}_{i,u_n}$ are given by Lemma 15 and $i = 1, \dots, \ell+1$. Then $E(u_n) \le C$, a contradiction. $\square$

3.2 The topological argument

Let us consider the following set

$$\Omega_\ell = \left\{ \sum_{i=1}^{\ell} t_i \delta_{x_i} \; : \; x_i \in \Omega, t_i \ge 0, \sum_{i=1}^{\ell} t_i = 1 \right\}$$

endowed with the weak topology of distributions (δ_{x_i} is the Dirac mass at x_i). This is known in literature as the set of formal barycenters of Ω of order ℓ. It is convenient to consider on Ω_ℓ the metric of $C^1(\Omega)^*$ which induces the same topology and which we will denote by $\mathrm{dist}(\cdot,\cdot)$.

We first find a proper subset of Ω which is homotopically equivalent to the whole domain.

Lemma 17 *If Ω is a smooth, connected, bounded and non-contractible domain of $\mathbb{R}^2$, then there exists a subset $\mathcal{B} = \cup_{j=1}^{\mathfrak{g}} A_j \subset \Omega$, with $\mathfrak{g} = \dim H_1(\Omega;\mathbb{Z})$, with A_j homeomorphic to S^1 and $A_i \cap A_j = \{P\}$ for some $P \in \Omega$, and which is a deformation retract of Ω.*

We will refer to $\mathcal{B}$ in the above statement as a *bouquet of $\mathfrak{g}$ loops centered at P*. Notice that Ω being non-contractible implies $\mathfrak{g} = \dim H_1(\Omega;\mathbb{Z}) > 0$.

Proof. Fix $\bar{r} > 0$ and $\bar{x} \in \Omega$ such that $B_{2\bar{r}}(\bar{x}) \subset \Omega$, and for some $\rho \le \bar{r}$ consider $\mathfrak{g}$ balls, $B_\rho(x_i) \subset B_{\bar{r}}(\bar{x})$ with $x_i \in \partial B_{\bar{r}}(\bar{x})$, $i = 1, \dots, \mathfrak{g}$, where $\mathfrak{g} = \dim H_1(\Omega;\mathbb{Z})$ is the numbers of holes of Ω. By taking ρ sufficiently small we can assume that $\overline{B_\rho(x_i)} \cap \overline{B_\rho(x_j)} = \emptyset$ for any $i \ne j$. Now we can find a bouquet $\tilde{\mathcal{B}} = \cup_{j=1}^{\mathfrak{g}} \tilde{A}_j \subset \Omega$ of $\mathfrak{g}$ loops centered at $\bar{x}$ and such that any loop of the bouquet contains one of the balls $B_\rho(x_i)$.

Let us focus on the $\mathfrak{g}+1$ connected components, $\Gamma_0, \Gamma_1, \dots, \Gamma_{\mathfrak{g}}$, of $\partial\Omega$. Each Γ_i divides the plane $\mathbb{R}^2$ into two distinct regions: a bounded one and an unbounded one. Without loss of generality we can assume that Ω is contained in the bounded region enclosed by Γ_0, while the opposite occurs for any Γ_i with $i \in \{1, \dots, \mathfrak{g}\}$; roughly speaking Γ_0 is the exterior boundary of Ω, while the other Γ_i's surround the $\mathfrak{g}$ holes of Ω. In the following we will denote, for any i, by $\tilde{\Omega}_i$ the bounded set enclosed by Γ_i.

Now, there exists a homeomorphism η between $\big(\Omega \cup (\cup_{i=1}^{\mathfrak{g}} \overline{\tilde{\Omega}}_i)\big) \setminus \big(\cup_{i=1}^{\mathfrak{g}} B_\rho(x_i)\big)$ and Ω, and it is possible to retract $\big(\Omega \cup (\cup_{i=1}^{\mathfrak{g}} \overline{\tilde{\Omega}}_i)\big) \setminus \big(\cup_{i=1}^{\mathfrak{g}} B_\rho(x_i)\big)$ onto $\tilde{\mathcal{B}}$, namely there exists a map

$$\tilde{H} : \Big(\big(\Omega \cup (\cup_{i=1}^{\mathfrak{g}} \overline{\tilde{\Omega}}_i)\big) \setminus \big(\cup_{i=1}^{\mathfrak{g}} B_\rho(x_i)\big)\Big) \times [0,1] \longrightarrow \big(\Omega \cup (\cup_{i=1}^{\mathfrak{g}} \overline{\tilde{\Omega}}_i)\big) \setminus \big(\cup_{i=1}^{\mathfrak{g}} B_\rho(x_i)\big)$$

with $\tilde{H}(\cdot,0)=\mathrm{Id}$, $\mathrm{range}(\tilde{H}(\cdot,1))\subset\tilde{\mathcal{B}}$ for every $x\in$ and $\tilde{H}|_{\tilde{\mathcal{B}}}(\cdot,1)=\mathrm{Id}|_{\tilde{\mathcal{B}}}$.

Let us define $\mathcal{B}:=\eta^{-1}(\tilde{\mathcal{B}})$ and $H:\Omega\times[0,1]\to\Omega$ as $H(x,t):=\eta(\tilde{H}(\eta^{-1}(x),t))$. Then H is the deformation retraction of Ω onto $\mathcal{B}$. □

Next we focus on the set of formal barycenters of $\mathcal{B}$ of order ℓ, namely

$$\mathcal{B}_\ell=\left\{\sum_{i=1}^{\ell}t_i\delta_{x_i}\ :\ x_i\in\mathcal{B},t_i\geq 0,\sum_{i=1}^{\ell}t_i=1\right\}.$$

In [6] the homology groups of $\mathcal{B}_\ell$ have been computed and since its $(2k-1)$-th Betti number is positive, provided $k\geq 1$ and $\mathfrak{g}\geq 1$, we have immediately the following result.

Lemma 18 *Let Ω, $\mathcal{B}$ and $\mathfrak{g}$ be as above. If $\mathfrak{g}\geq 1$, then $\mathcal{B}_\ell$ is non-contractible for any $\ell\geq 1$.*

3.3 Some correspondence between superlevels of E and $\mathcal{B}_\ell$

In this section we show how to map $\mathcal{B}_\ell$ into $M_\Lambda^L=\{u\in M_\Lambda : E(u)\geq L\}$ if $\Lambda\in(4\pi\ell,4\pi(\ell+1))$ and if L is large, and then how to define a *natural inverse.*

Given $x\in\Omega$ and $\mu>0$ we define the function $\varphi_{\mu,x}:\Omega\to\mathbb{R}$ by

$$\varphi_{\mu,x}(y)=\mu-\frac{1}{\mu}\log\left(1+\frac{|y-x|^2}{r_\mu^2}\right),\qquad r_\mu^2:=4\mu^{-2}e^{-\mu^2}.$$

For a barycenter $\sigma=\sum_{i=1}^{\ell}t_i\delta_{x_i}\in\mathcal{B}_\ell$ we let

$$\tilde{\varphi}_{\mu,\sigma}=\sum_{i=1}^{k}t_i\varphi_{\mu,x_i},$$

and then consider the test functions

$$\varphi_{\mu,\sigma}=\left(\frac{\Lambda}{\int_\Omega|\nabla(\tilde{\varphi}_{\mu,\sigma}-f_{\mu,\sigma})|^2dx}\right)^{\frac{1}{2}}(\tilde{\varphi}_{\mu,\sigma}-f_{\mu,\sigma}),$$

where $f_{\mu,\sigma}$ solves

$$\begin{cases}\Delta f_{\mu,\sigma}=0 & \text{in } \Omega\\ f_{\mu,\sigma}=\tilde{\varphi}_{\mu,\sigma} & \text{on } \partial\Omega.\end{cases}$$

Clearly $\varphi_{\mu,\sigma}\in M_\Lambda$ and $\varphi_{\mu,\sigma}>0$ in Ω by the maximum principle, since each φ_{μ,x_i} is superharmonic and $f_{\mu,\sigma}$ is harmonic. We are now going to show that for $\Lambda\in(4\pi\ell,4\pi(\ell+1))$ the measures $\frac{e^{\varphi_{\mu,\sigma}^2}dx}{\int_\Omega e^{\varphi_{\mu,\sigma}^2}dx}$ are close to $\mathcal{B}_\ell$ in the sense of distributions for μ large.

Notation. Until the end of this section $o(1)$ will denote any quantity tending to 0 as $\mu\to\infty$, uniformly with respect to $\sigma\in\mathcal{B}_\ell$.

Lemma 19

$$\int_\Omega |\nabla\varphi_{\mu,x}|^2 dx = 4\pi + o(1).$$

Proof. We set $\delta = \mathrm{dist}(\mathcal{B}, \partial\Omega) > 0$. One obtains easily that

$$\nabla\varphi_{\mu,x}(y) = -\frac{2(y-x)\mu e^{\mu^2}}{4 + |y-x|^2\mu^2 e^{\mu^2}}, \tag{60}$$

hence

$$|\nabla\varphi_{\mu,x}(y)| \le \frac{2|y-x|\mu e^{\mu^2}}{4} \le \frac{1}{2}, \qquad \text{for } y \in B_{\mu^{-1}e^{-\mu^2}}(x),$$

and

$$|\nabla\varphi_{\mu,x}(y)| \le \frac{2|y-x|\mu e^{\mu^2}}{|y-x|^2\mu^2 e^{\mu^2}} \le \frac{2}{\mu\delta} \qquad \text{for } y \in \Omega\setminus B_\delta(x). \tag{61}$$

Therefore we are able to conclude thanks to the change of variable $s = r^2\mu^2 e^{\mu^2}$

$$\begin{aligned}
\int_\Omega |\nabla\varphi_{\mu,x}|^2 dx &= \int_{B_\delta(x)\setminus B_{\mu^{-1}e^{-\mu^2}}(x)} |\nabla\varphi_{\mu,x}|^2 dx + o(1)\\
&= 2\pi\int_{\mu^{-1}e^{-\mu^2}}^{\delta} \frac{4r^2\mu^2 e^{2\mu^2}}{(4 + r^2\mu^2 e^{\mu^2})^2} r\, dr + o(1)\\
&= \frac{8\pi}{\mu^2}\int_{e^{-\mu^2}}^{\delta^2\mu^2 e^{\mu^2}} \frac{s}{(4+s)^2} ds + o(1)\\
&= \frac{4\pi}{\mu^2}\left[\log(\delta^2\mu^2 e^{\mu^2} + 4) - \log(e^{-\mu^2} + 4) + \frac{4}{4 + \delta^2\mu^2 e^{\mu^2}} - \frac{4}{4 + e^{-\mu^2}}\right] + o(1)\\
&= 4\pi + o(1).
\end{aligned}$$

□

Lemma 20 *Let $\delta := \mathrm{dist}(\mathcal{B}, \partial\Omega)$. For $\sigma \in \mathcal{B}_\ell$ and $\mu \ge \mu_0(\delta)$, $f_{\mu,\sigma} \le 0$ and $\|f_{\mu,\sigma}\|_{C^1(\Omega)} \le C\frac{\log\mu}{\mu}$, where $C = C(\delta)$ does not depend on σ.*

Proof. We consider a compact subset $\tilde\Omega$ of Ω such that $\tilde\Omega \supset \left\{x \in \Omega \,|\, \mathrm{dist}(x, \mathcal{B}) < \frac{\delta}{2}\right\}$.
Let $\sigma = \sum_{i=1}^{\ell} t_i\delta_{x_i}$. Using that for $y \in \bar\Omega\setminus\tilde\Omega$ we have $\frac{\delta}{2} \le |y - x_i| \le \mathrm{diam}(\Omega)$ we estimate

$$\begin{aligned}
\tilde\varphi_{\mu,\sigma}(y) &= \sum_{i=1}^k t_i\left(\mu - \frac{1}{\mu}\log\left(1 + \frac{|y-x_i|^2}{4}\mu^2 e^{\mu^2}\right)\right) \le \sum_{i=1}^k t_i\left(\mu - \frac{1}{\mu}\log\left(\frac{\delta^2}{16}\mu^2 e^{\mu^2}\right)\right)\\
&= -\frac{2}{\mu}\log\mu - \frac{2}{\mu}\log\left(\frac{\delta}{4}\right) = -\frac{2(1+o(1))}{\mu}\log\mu
\end{aligned} \tag{62}$$

and

$$\tilde\varphi_{\mu,\sigma}(y) \ge \sum_{i=1}^k t_i\left(\mu - \frac{1}{\mu}\log\left(1 + \frac{\mathrm{diam}^2(\Omega)}{2}\mu^2 e^{\mu^2}\right)\right) = -\frac{2(1+o(1))}{\mu}\log\mu. \tag{63}$$

As a consequence of (62) and (63) and of the maximum principle for harmonic functions we get $f_{\mu,\sigma} \leq 0$ for μ sufficiently large and $\|f_{\mu,\sigma}\|_{C^0(\Omega)} \leq C\frac{\log\mu}{\mu}$.

From (61) it follows that

$$|\nabla\varphi_{\mu,x}(y)| \leq \frac{2|y-x|\mu e^{\mu^2}}{|y-x|^2\mu^2 e^{\mu^2}} \leq \frac{4}{\delta\mu}, \qquad \text{for } y\in\Omega\setminus B_{\delta/2}(x).$$

Then $|\nabla\tilde\varphi_{\mu,\sigma}| \leq \frac{4}{\delta\mu}$ in $\overline{\Omega}\backslash\tilde\Omega$ and similarly $|\nabla^2\tilde\varphi_{\mu,\sigma}| \leq \frac{4}{\delta^2\mu}$ in $\overline{\Omega}\backslash\tilde\Omega$, hence $\|\nabla\tilde\varphi_{\mu,\sigma}\|_{C^2(\overline{\Omega}\backslash\tilde\Omega)} \leq C\frac{\log\mu}{\mu}$, and from Schauder theory we get $\|f_{\mu,\sigma}\|_{C^1(\Omega)} \leq C\frac{\log\mu}{\mu^2}$. □

Lemma 21 *For $\sigma = \sum_{i=1}^{\ell} t_i\delta_{x_i} \in \mathcal{B}_\ell$ set $d_{j,h} := |x_j - x_h|$, $1 \leq j,h \leq \ell$. Then we have*

$$\frac{4\pi}{\ell} + o(1) \overset{I}{\leq} \int_\Omega |\nabla(\tilde\varphi_{\mu,\sigma} - f_{\mu,\sigma})|^2 dx \overset{II}{\leq} 4\pi \sum_{\substack{1\leq j,h\leq\ell \\ d_{j,h}\leq 2r_\mu}} t_j t_h + \frac{8\pi}{\mu^2}\sum_{\substack{1\leq j,h\leq\ell \\ d_{j,h}>2r_\mu}} t_j t_h \log\frac{1}{d_{j,h}} + o(1).$$

Proof. From Lemma 20 for $\mu \geq \mu_0(\delta)$ we have $\|\nabla f_{\mu,\sigma}\|_\infty \leq C\frac{\log\mu}{\mu}$ for any $\sigma\in\mathcal{B}_\ell$ and we easily infer

$$\int_\Omega |\nabla\tilde\varphi_{\mu,\sigma} - \nabla f_{\mu,\sigma}|^2 dx = \int_\Omega |\nabla\tilde\varphi_{\mu,\sigma}|^2 dx + o(1). \tag{64}$$

We have by definition

$$\int_\Omega |\nabla\tilde\varphi_{\mu,\sigma}|^2 dx = \sum_{1\leq j,h\leq\ell} t_j t_h \int_\Omega \nabla\varphi_{\mu,x_j}\cdot\nabla\varphi_{\mu,x_h} dx. \tag{65}$$

To prove inequality II we will split the above sum by distinguishing two cases:

(a) $d_{j,h} > 2r_\mu$,

(b) $d_{j,h} \leq 2r_\mu$ (possibly $j = h$).

From (60) we get

$$|\nabla\varphi_{\mu,x_j}(y)\cdot\nabla\varphi_{\mu,x_h}(y)| \leq \frac{4}{\mu^2|y-x_j|\,|y-x_h|}. \tag{66}$$

Now if j and h satisfy condition (a) we bound, using that $r_\mu^2\mu^2 e^{\mu^2} = 4$,

$$\frac{4}{\mu^2}\int_{B_{d_{j,h}/2}(x_j)} \frac{dy}{|y-x_j|\,|y-x_h|} \leq \frac{8\pi}{\mu^2}\int_0^{\frac{d_{j,h}}{2}} \frac{1}{r}\frac{1}{d_{j,h}-r} r\,dr = \frac{8\pi}{\mu^2}\log 2 = o(1), \tag{67}$$

and similarly on $B_{d_{j,h}/2}(x_h)$. Moreover

$$\frac{4}{\mu^2}\int_{B_{2d_{j,h}}(x_j)\backslash\left(B_{\frac{d_{j,h}}{2}}(x_j)\cup B_{\frac{d_{j,h}}{2}}(x_h)\right)} \frac{dy}{|y-x_j|\,|y-x_h|} \leq \frac{8\pi}{\mu^2}\int_{\frac{d_{j,h}}{2}}^{2d_{j,h}} \frac{1}{r}\frac{2}{d_{j,h}} r\,dr = \frac{24\pi}{\mu^2} = o(1). \tag{68}$$

The main contribution of the Dirichlet integral is outside $B_{2d_{j,h}}(x_j)$, and can be estimated by

$$\frac{4}{\mu^2}\int_{\Omega\setminus B_{2d_{j,h}}(x_j)}\frac{dy}{|y-x_j|\,|y-x_h|}\leq\frac{8\pi}{\mu^2}\int_{2d_{j,h}}^{\mathrm{diam}(\Omega)}\frac{dr}{r-d_{j,h}}=\frac{8\pi}{\mu^2}\log\frac{1}{d_{j,h}}+o(1).$$

Hence we have proven that when j and h satisfy (a) we have

$$\int_\Omega\nabla\varphi_{\mu,x_j}\cdot\nabla\varphi_{\mu,x_h}dy\leq\frac{8\pi}{\mu^2}\log\frac{1}{d_{j,h}}+o(1).$$

We now consider the case when j and h satisfy condition (b). From $d_{j,h}\leq 2r_\mu$ we infer, $B_{r_\mu}(x_j)\cup B_{r_\mu}(x_h)\subset B_{3r_\mu}(x_j)$, whence

$$\begin{aligned}\int_{\Omega\setminus B_{3r_\mu}(x_j)}\nabla\varphi_{\mu,x_j}\cdot\nabla\varphi_{\mu,x_h}dy &\leq \frac{4}{\mu^2}\int_{\Omega\setminus B_{3r_\mu}(x_j)}\frac{dy}{|y-x_j|\,|y-x_h|}\leq\frac{8\pi}{\mu^2}\int_{3r_\mu}^{\mathrm{diam}(\Omega)}\frac{1}{r}\frac{1}{r-d_{j,h}}rdr\\ &\leq \frac{8\pi}{\mu^2}|\log(3r_\mu-d_{j,h})|+o(1)\leq\frac{8\pi}{\mu^2}|\log r_\mu|+o(1)\\ &\leq 4\pi+o(1)\end{aligned}$$

where the second last inequality follows from $0\leq d_{j,h}\leq 2r_\mu$. Now bound with (60)

$$|\nabla\varphi_{\mu,x_i}(y)|=\frac{2}{\mu}\frac{|y-x_i|}{r_\mu^2+|y-x_i|^2}\leq\frac{2|y-x_i|}{\mu r_\mu^2}$$

and, since $|y-x_j|\leq 3r_\mu$ and $|y-x_h|\leq 5r_\mu$ for $y\in B_{3r_\mu}(x_j)$, we get

$$\begin{aligned}\int_{B_{3r_\mu}(x_j)}|\nabla\varphi_{\mu,x_j}\cdot\nabla\varphi_{\mu,x_\ell}|dy &\leq \frac{4}{\mu^2}\int_{B_{3r_\mu}(x_j)}\frac{|y-x_j|}{r_\mu^2}\frac{|y-x_h|}{r_\mu^2}dy\\ &\leq \frac{4}{\mu^2}\int_{B_{3r_\mu}(x_j)}\frac{3r_\mu}{r_\mu^2}\frac{5r_\mu}{r_\mu^2}dy=o(1).\end{aligned}\tag{69}$$

Summing over j and h we obtain II.

To prove inequality I we will use some facts derived above; in particular, by virtue of (65), (66), (67), (68) and (69), we obtain

$$\begin{aligned}\int_\Omega|\nabla\tilde\varphi_{\mu,\sigma}|^2dy &= \sum_{i=1}^{\ell}t_i^2\int_\Omega|\nabla\varphi_{\mu,x_i}|^2dy+\sum_{d_{j,h}>2r_\mu}\int_{\Omega\setminus B_{2d_{j,h}}(x_j)}\nabla\varphi_{\mu,x_j}\cdot\nabla\varphi_{\mu,x_h}dy\\ &\quad+\sum_{\substack{d_{j,h}\leq 2r_\mu\\ j\neq h}}\int_{\Omega\setminus B_{3r_\mu}(x_j)}\nabla\varphi_{\mu,x_j}\cdot\nabla\varphi_{\mu,x_h}dy+o(1).\end{aligned}$$

Now applying Lemma 19, noticing that $\sum_{i=1}^{\ell}t_i^2\geq\frac{1}{\ell}$, and checking that the second and the third term are non-negative, being $(y-x_j)\cdot(y-x_h)$ positive in both the domains of integration, we conclude that

$$\int_\Omega|\nabla\tilde\varphi_{\mu,\sigma}|^2dy\geq\frac{4\pi}{\ell}+o(1),$$

which immediately implies I in view of (64). $\square$

Lemma 22 *If $\Lambda \in (4\pi\ell, 4\pi(\ell+1))$, then*

$$\lim_{\mu\to\infty}\int_\Omega e^{\varphi^2_{\mu,\sigma}}dx = \infty, \qquad \textit{uniformly for } \sigma \in \mathcal{B}_\ell.$$

Proof. It is sufficient to show that

$$\lim_{\mu\to\infty}\int_{\bigcup_{i=1}^{\ell} B_{r_\mu}(x_i)} e^{\varphi^2_{\mu,\sigma}}dx = \infty, \qquad \text{uniformly for } \sigma \in \mathcal{B}_\ell. \tag{70}$$

We have $f_{\mu,\sigma} < 0$ for $\mu \geq \mu_0(\delta)$ and $\|f_{\mu,\sigma}\|_{C^0} = o(1) \to 0$ as $\mu \to \infty$ by Lemma 20, and since $\inf_{y\in\Omega}\varphi_{\mu,x_i}(y) = -|o(1)|$, we deduce

$$\varphi^2_{\mu,\sigma} \geq \Lambda\frac{\tilde\varphi^2_{\mu,\sigma} + o(1)}{\int_\Omega |\nabla(\tilde\varphi_{\mu,\sigma} - f_{\mu,\sigma})|^2dx}. \tag{71}$$

For $x \in B_{r_\mu}(x_i)$, using that if $d_{i,j} \geq 2r_\mu$ we have $\frac{d_{i,j}}{2} \leq |x - x_j| \leq 2d_{i,j}$, we infer

$$\tilde\varphi_{\mu,\sigma}(x) = \mu\sum_{j=1}^{\ell} t_j\left(1 - \log(r_\mu^2 + |x - x_j|) + o(1)\right) = \mu\left(\sum_{j\in J_i} t_j + \sum_{j\notin J_i}\frac{2t_j}{\mu^2}\log\frac{1}{d_{i,j}} + o_\mu(1)\right), \tag{72}$$

where $J_i := \{j \in \{1,\dots,\ell\} | d_{i,j} \leq 2r_\mu\}$.

We next show that there exist $\bar i \in \{1,\dots,k\}$ and $\varepsilon > 0$ depending only on Λ such that

$$\varphi_{\mu,\sigma}(x) \geq (1+\varepsilon)\mu, \qquad \text{for } x \in B_{r_\mu}(x_{\bar i}). \tag{73}$$

Indeed we have $\Lambda > 4\pi\ell(1+2\varepsilon)^2$ for some $\varepsilon > 0$. If the claim were false, for any $i \in \{1,\dots,k\}$ we could find $\tilde x_i \in B_{r_\mu}(x_i)$ such that $\varphi_{\mu,\sigma}(\tilde x_i) < (1+\varepsilon)\mu$. Squaring, summing and using (71), (72) and inequality II of Lemma 21 we would have

$$\begin{aligned}
\ell(1+\varepsilon)^2 >& \frac{1}{\mu^2}\sum_{i=1}^{\ell}\varphi^2_{\mu,\sigma}(\tilde x_i) \geq \frac{\Lambda}{\int_\Omega |\nabla(\tilde\varphi_{\mu,\sigma} - f_{\mu,\sigma})|^2dx}\\
&\times\sum_{i=1}^{\ell}\left(\sum_{\substack{j\in J_i\\ h\in J_i}} t_jt_h + \frac{2}{\mu^2}\sum_{\substack{j\in J_i\\ h\notin J_i}} t_jt_h\log\frac{1}{d_{i,h}} + \frac{4}{\mu^4}\sum_{\substack{j\notin J_i\\ h\notin J_i}} t_jt_h\log\frac{1}{d_{i,j}}\log\frac{1}{d_{i,h}} + o(1)\right)\\
>& \frac{\ell(1+2\varepsilon)^2\left[\sum_{\substack{1\leq j,h\leq\ell\\ d_{j,h}\leq 2r_\mu}} t_jt_h + \frac{2}{\mu^2}\sum_{\substack{1\leq j,h\leq\ell\\ d_{j,h}>2r_\mu}} t_jt_h\log\frac{1}{d_{j,h}} + \text{positive terms} + o(1)\right]}{\sum_{\substack{1\leq j,h\leq\ell\\ d_{j,h}\leq 2r_\mu}} t_jt_h + \frac{2}{\mu^2}\sum_{\substack{1\leq j,h\leq\ell\\ d_{j,h}>2r_\mu}} t_jt_h\log\frac{1}{d_{j,h}} + o(1)}\\
\geq& \ell(1+2\varepsilon)^2 + o(1),
\end{aligned}$$

where we used that $\frac{1}{\mu^2}\log\frac{1}{d_{i,j}} \geq o(1)$ in the third inequality, while in the last inequality we used that the denominator is greater than a positive constant independent of $\sigma \in \mathcal{B}_\ell$. We obtained a contradiction, hence (73) holds true. Then it follows

$$\int_{\bigcup_{i=1}^{\ell} B_{r_\mu}(x_i)} e^{\varphi^2_{\mu,\sigma}} dx \geq \int_{B_{r_\mu}(x_{\bar{i}})} e^{(1+\varepsilon)^2\mu^2} dx = \frac{4\pi e^{(1+\varepsilon)^2\mu^2}}{\mu^2 e^{\mu^2}} \to +\infty, \qquad \text{as } \mu \to \infty,$$

so we are done. □

Lemma 23 *We have*

$$\lim_{\mu\to\infty} \operatorname{dist}\left(\frac{e^{\varphi^2_{\mu,\sigma}} dx}{\int_\Omega e^{\varphi^2_{\mu,\sigma}} dx}, \Omega_\ell\right) = 0, \qquad \textit{uniformly for } \sigma \in \mathcal{B}_\ell.$$

Proof. By Lemma 22 it is sufficient to show that, given any $\varepsilon > 0$, there exists $\mu_0(\varepsilon, \Lambda)$ such that

$$\int_{\Omega\setminus\bigcup_{i=1}^{\ell} B_\varepsilon(x_i)} e^{\varphi^2_{\mu,\sigma}} dx \leq C, \qquad \text{for } \mu \geq \mu_0(\varepsilon, \Lambda), \tag{74}$$

where C does not depend on $\sigma = \sum_{i=1}^{\ell} t_i \delta_{x_i} \in \mathcal{B}_\ell$. Indeed, noticing that for $y \notin B_{2r_\mu}(x_i)$

$$\varphi_{\mu,x_i}(y) = \mu + \frac{1}{\mu}\log(4\mu^{-2}e^{-\mu^2}) - \frac{1}{\mu}\log(|y-x_i|^2 + r_\mu^2) = -\frac{2}{\mu}\log|y-x_i| + o(1),$$

that $\|f_{\mu,\sigma}\|_\infty = o(1)$ by Lemma 20, and using inequality I of Lemma 21 one has that $\varphi_{\mu,\sigma}(x) = o(1)$ for $x \in \Omega \setminus \cup_{i=1}^{\ell} B_\varepsilon(x_i)$, and (74) follows. □

From Lemma 16 we deduce immediately the following result.

Lemma 24 *Suppose $\Lambda \in (4\pi\ell, 4\pi(\ell+1))$, with $\ell \geq 1$. Then for any $\varepsilon > 0$ there exists a large positive $L = L(\varepsilon, \Lambda)$ such that for every $u \in M_\Lambda^L := \{u \in M_\Lambda : E(u) \geq L\}$ we have*

$$\operatorname{dist}\left(\frac{e^{u^2} dx}{\int_\Omega e^{u^2} dx}, \Omega_\ell\right) < \varepsilon.$$

Proposition 25 *Suppose $\Lambda \in (4\pi\ell, 4\pi(\ell+1))$ with $\ell \geq 1$. Then, there exists $L > 0$ and a continuous projection $\Psi : M_\Lambda^L \to \mathcal{B}_\ell$ such that for $\mu \geq \mu_0(L, \Lambda)$ the map $\Phi(\sigma) := \Psi(\varphi_{\mu,\sigma})$ is homotopically equivalent to the identity on $\mathcal{B}_\ell$.*

Proof. By the results in Section 3 of [18], there exists a continuous retraction $\hat{\Psi}$ from a neighborhood of Ω_ℓ (as a subset of $\mathcal{P}(\Omega) = \{\text{probability measures on } \Omega\}$ with respect to the distance denoted by dist) onto Ω_ℓ. By Lemma 24, the map

$$u \mapsto \hat{\Psi}\left(\frac{e^{u^2} dx}{\int_\Omega e^{u^2} dx}\right) \in \Omega_\ell$$

is well defined on M_Λ^L for L large and is continuous with respect to the H_0^1 topology. Let $\mathcal{R}$ be the retraction of Ω onto $\mathcal{B}$ given by Lemma 17, define the retraction $\tilde{\mathcal{R}} : \Omega_\ell \to \mathcal{B}_\ell$ by

$$\sigma = \sum_{i=1}^{\ell} t_i \delta_{x_i} \mapsto \tilde{\mathcal{R}}(\sigma) = \sum_{i=1}^{\ell} t_i \delta_{\mathcal{R}(x_i)},$$

and finally define a continuous map $\Psi : M_\Lambda^L \to \mathcal{B}_\ell$ by

$$\Psi(u) := \tilde{\mathcal{R}} \circ \hat{\Psi} \left(\frac{e^{u^2} dx}{\int_\Omega e^{u^2} dx} \right).$$

We need to show that $\Phi \simeq \mathrm{Id}_{\mathcal{B}_\ell}$. Consider the homotopy $H : [0,1] \times \mathcal{B}_\ell \to \mathcal{P}(\Omega)$ given by

$$H(s,\sigma) = s\sigma + (1-s) \frac{e^{\varphi_{\mu,\sigma}^2} dx}{\int_\Omega e^{\varphi_{\mu,\sigma}^2} dx}.$$

By Lemma 23, $\mathrm{dist}(H(s,\sigma), \mathcal{B}_\ell)$ is small for μ sufficiently large, $\sigma \in \mathcal{B}_\ell$ and $s \in [0,1]$ (here we also use that $\mathcal{B}_\ell$ is convex). Therefore $\hat{\Psi}$ is well defined on the image of H and we can define the homotopy $\mathcal{H} : [0,1] \times \mathcal{B}_\ell \to \mathcal{B}_\ell$

$$\mathcal{H}(s,\sigma) = \tilde{\mathcal{R}} \circ \hat{\Psi} \circ H(s,\sigma).$$

Clearly $\mathcal{H}(0,\cdot) = \Phi$ and $\mathcal{H}(1,\cdot) = \mathrm{Id}_{\mathcal{B}_\ell}$. □

3.4 Conclusion

If we are able to find $0 \le u_0 \in C^\infty(\overline{\Omega})$ with $u_0|_{\partial\Omega} = 0$ and $\|u_0\|_{H_0^1(\Omega)} = \Lambda \in (4\pi\ell, 4\pi(\ell+1))$ such that the functional E stays bounded along the trajectory $u(t,\cdot)$ of the flow (3) starting from u_0 at time $t = 0$, Theorem 1 implies that $u(t,\cdot)$ converges to a positive solution of Problem (7).

Therefore it remains to find such a u_0. Given $\mathcal{B}$ as above, fix L and $\mu_0(L,\Lambda)$ as in Proposition 25. Fix $\mu \ge \mu_0(L,\Lambda)$ so large that $E(\varphi_{\mu,\sigma}) \ge L$ for every $\sigma \in \mathcal{B}_\ell$, and define the *(contractible) topological cone* $\mathcal{C}_\ell$ over $\mathcal{B}_\ell$ as

$$\mathcal{C}_\ell = \left\{ \gamma_{\sigma,s} \;=\; \Lambda^{\frac{1}{2}} \frac{s\varphi_1 + (1-s)\varphi_{\mu,\sigma}}{\|s\varphi_1 + (1-s)\varphi_{\mu,\sigma}\|_{H_0^1(\Omega)}} \;:\; \sigma \in \mathcal{B}_\ell,\ s \in [0,1] \right\} \subset M_\Lambda,$$

where $\varphi_1 \in C^\infty(\overline{\Omega})$ is non-negative, $\varphi|_{\partial\Omega} = 0$ and $\varphi \not\equiv 0$.

We claim that there exists $(s,\sigma) \in [0,1] \times \mathcal{B}_\ell$ such that if $u_{\sigma,s}$ is the solution to (1) with $u_0 = \gamma_{\sigma,s}$, then $E(u_{\sigma,s}(t,\cdot)) \le 2L$ for every $t \in [0,\infty)$, and we are done.

In order to prove the claim let us consider a smooth truncation of the flow (3) satisfying:

$$\begin{cases} u_t = 0 & \text{in } \{t \in [0,\infty) \,|\, E(u(t,\cdot)) \ge 3L\} \times \Omega; \\ u_t e^{u^2} = \Delta u + \lambda u e^{u^2} & \text{in } \{t \in [0,\infty) \,|\, E(u(t,\cdot)) \le 2L\} \times \Omega; \\ u = 0 & \text{on } [0,\infty) \times \partial\Omega; \\ u(0,x) = u_0(x) & \text{for } x \in \Omega; \\ u > 0 & \text{in } (0,\infty) \times \Omega \end{cases} \tag{75}$$

and $\frac{d}{dt}E(u(t,\cdot)) > 0$ if $E(u(t,\cdot)) < 3L$. This can be done since for a solution to (1) we have

$$\frac{d}{dt}E(u(t,\cdot)) = \frac{1}{\lambda(t)}\int_{\{t\}\times\Omega} u_t^2 e^{u^2}dx \geq 0,$$

see [23]. We will denote by $\tilde{\gamma}_{\sigma,s}(t) = \tilde{\gamma}_{\sigma,s}(t,\cdot)$ the solution of (75) with $u_0 = \gamma_{\sigma,s}$.

Suppose by contradiction that for all $(\sigma, s) \in \mathcal{B}_\ell \times [0,1]$ there exists $\bar{t} > 0$ (which by compactness can be taken independent of (σ, s)) such that $E(\tilde{\gamma}_{\sigma,s}(\bar{t})) \geq 2L$. Then by Proposition 25 the map

$$(\sigma, s) \mapsto \Psi\left(\tilde{\gamma}_{\sigma,s}(\bar{t})\right)$$

would be a homotopy in $\mathcal{B}_\ell$ between Φ and a constant map. Indeed $\Phi(\sigma) = \Psi(\tilde{\gamma}_{\sigma,0}(0))$, and $\Phi \simeq \Phi^{\bar{t}} := \Psi(\tilde{\gamma}_{\cdot,0}(\bar{t}))$ in $\mathcal{B}_\ell$, since E increases along the flow, and $\gamma_{\sigma,0} = \varphi_{\mu,\sigma} \in M^L_\Lambda$, hence $\tilde{\gamma}_{\sigma,0}(t) \in M^L_\Lambda$ for $t \in [0,\bar{t}]$. Finally $\Phi^{\bar{t}} \simeq \Psi(\tilde{\gamma}_{\cdot,1}(\bar{t})) \equiv const$. This proves that $\Phi \simeq const$ in $\mathcal{B}_\ell$, but this is impossible since $\Phi \simeq \mathrm{Id}_{\mathcal{B}_\ell}$ and $\mathcal{B}_\ell$ is non-contractible by Lemma 18. □

4 Proof of Theorem 4

Given $\mu, \lambda > 0$ let $\overline{u}_{\mu,\lambda} \in C^\infty([0, T_{\mu,\lambda}))$ be the solution to the ODE

$$-\frac{\partial^2\overline{u}}{\partial r^2} - \frac{1}{r}\frac{\partial\overline{u}}{\partial r} = \lambda\overline{u}e^{\overline{u}^2}, \quad \overline{u}(0) = \mu, \quad \overline{u}'(0) = 0, \tag{76}$$

where $[0, T_{\mu,\lambda})$, $T_{\mu,\lambda} \in (0,\infty]$, is the maximal interval of existence for (76) (in fact $T_{\mu,\lambda} = \infty$, but we will not prove this). Then $u_{\mu,\lambda}(x) := \overline{u}_{\mu,\lambda}(|x|)$ satisfies

$$-\Delta u_{\mu,\lambda} = \lambda u_{\mu,\lambda} e^{u^2_{\mu,\lambda}} \quad \text{in } B_{T_{\mu,\lambda}}, \qquad u_{\mu,\lambda}(0) = \mu.$$

Set

$$\tau(\mu) := \inf\{r \in (0, T_{\mu,1}] : \overline{u}_{\mu,1}(r) = 0\}.$$

We claim that $\tau(\mu) < \infty$ for every $\mu > 0$. To see this, fix $r_0 \in (0, \tau(\mu))$. Then for $r \geq r_0$ the divergence theorem yields

$$\overline{u}'_{\mu,1}(r) = \frac{1}{2\pi r}\int_{B_r} \Delta u_{\mu,1}dx \leq \frac{1}{2\pi r}\int_{B_{r_0}} \Delta u_{\mu,1}dx < -\frac{\varepsilon}{r},$$

for some positive ε. The claim easily follows from standard comparison arguments.

Now notice that, for any $\lambda, \lambda' > 0$

$$\overline{u}_{\mu,\lambda}\left(\sqrt{\frac{\lambda'}{\lambda}}r\right) = \overline{u}_{\mu,\lambda'}(r)$$

and for every $\mu > 0$ set $\overline{u}_\mu := \overline{u}_{\mu,\tau^2(\mu)} = \overline{u}_{\mu,\lambda_\mu}$, where $\lambda_\mu := \tau^2(\mu)$. Then $\overline{u}_\mu$ is positive in $[0,1)$ and $\overline{u}_\mu(1) = 0$. By ODE theory the function $\Phi : \mu \mapsto \overline{u}_\mu|_{[0,1]}$ belongs to $C^\infty((0,\infty), C^\infty([0,1]))$.

Every non-negative critical point of $E(\cdot, B_1(0))$ on M_Λ is smooth and satisfies

$$-\Delta u = \lambda u e^{u^2} \text{ in } B_1, \quad u = 0 \text{ on } \partial B_1, \tag{77}$$

for some $\lambda > 0$. By [22, Theorem 1] we have that u is radially symmetric, i.e. we can write $u(x) = \overline{u}(|x|)$, where $\overline{u}$ satisfies (76) with $\mu = u(0)$ and the additional condition that $\overline{u} > 0$ on $[0, 1)$ and $\overline{u}(1) = 0$. This is possible only if $\overline{u} = \overline{u}_\mu$. Hence we have proven that every solution $0 \le u \not\equiv 0$ of (77) is of the form $u(x) = u_\mu(x) := \overline{u}_\mu(|x|)$ for some $\mu > 0$. Define

$$\mathcal{E}(\mu) := \|u_\mu\|^2_{H^1_0(B_1)}, \quad \Lambda^\sharp := \sup_{\mu \in (0,\infty)} \mathcal{E}(\mu).$$

We claim that $\Lambda^\sharp < \infty$. Indeed $\mathcal{E}$ is continuous and it is clear that $u_\mu \to 0$ smoothly as $\mu \downarrow 0$, hence $\lim_{\mu \downarrow 0} \mathcal{E}(\mu) = 0$. Moreover we claim that $\lim_{\mu \to \infty} \mathcal{E}(\mu) = 4\pi$, hence by continuity $\Lambda^\sharp < \infty$.

In order to prove the last claim, we observe that the proof of Proposition 8 applies to the present situation if we can show that (19) holds. For that we cannot use the work of Druet [19] or Theorem 6 because we do not have a bound on the energy of u_μ. On the other hand for what we need such a bound is not necessary, see Lemma 26 below. □

Lemma 26 *For every $\mu > 0$ let u_μ be as in the proof of Theorem 4 and let $r_\mu > 0$ be such that $r_\mu^2 \lambda_\mu \mu^2 e^{\mu^2} = 4$. Then as $\mu \to \infty$*

$$\eta_\mu(x) := \mu(u_\mu(r_\mu x) - \mu) + \log 2 \to \eta_\infty(x) = \log \frac{2}{1 + |x|^2} \quad \text{in } C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2). \tag{78}$$

Proof. We first prove that $\lim_{\mu \to \infty} r_\mu = 0$. Otherwise we have for a sequence $\mu_k \to \infty$ that $\lambda_{\mu_k} \mu_k^2 e^{\mu_k^2} \le C$. Then, using that $\overline{u}'_\mu \le 0$ in $[0, 1]$ we see that

$$-\mu_k \Delta u_{\mu_k} \le \lambda_{\mu_k} \mu_k^2 e^{\mu_k^2} \le C \quad \text{in } B_1,$$

hence as $k \to \infty$ we get $\Delta u_{\mu_k} \to 0$ and $u_{\mu_k} \to 0$ uniformly, contradicting $u_{\mu_k}(0) = \mu_k \to \infty$.

Set now $v_\mu(x) = u_\mu(r_\mu x) - \mu$. We claim that $v \to 0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ as $\mu \to \infty$. Indeed

$$-\Delta v_\mu(x) = \frac{4}{\mu} \frac{u_\mu(r_\mu x)}{\mu} e^{(u(r_\mu x) - \mu)(u(r_\mu x) + \mu)} \to 0 \quad \text{uniformly as } \mu \to \infty$$

since $4/\mu \to 0$, $0 \le u_\mu/\mu \le 1$ and $(u(r_\mu x) - \mu)(u(r_\mu x) + \mu) \le 0$. Now notice that $v_\mu \le 0$ and $v_\mu(0) = 0$. Then the Harnack inequality implies the claim.

Then we have

$$-\Delta \eta_\mu = V_\mu e^{2 m a_\mu \eta_\mu} \quad \text{in } B_{1/r_\mu},$$

where

$$V_\mu(x) = 2^{(1 - u_\mu(r_\mu x)/\mu)} \frac{u_\mu(r_\mu x)}{\mu} \to 1, \quad a_\mu = \frac{1}{2}\left(\frac{u_\mu(r_\mu x)}{\mu} + 1\right) \to 1 \quad \text{in } C^0_{\mathrm{loc}}(\mathbb{R}^2).$$

Considering that $\eta_\mu \le 0$, $\Delta \eta_\mu$ is locally bounded and $\eta_\mu(0) = \log 2$ we have $\eta_\mu \to \eta^*$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2)$ by the Harnack inequality, where $-\Delta \eta^* = e^{2\eta^*}$ and $\eta^*(0) = \log 2$. It then follows from the uniqueness of solutions to the Cauchy problem (recall we are in the radial case) that $\eta^* = \eta_\infty$. □

Remark 27 *As anticipated in the introduction, the arguments of the proof of Theorem 4 yield immediately existence of blowing-up sequences of solutions to* (7) *when* Λ *approaches* 4π *from above. This has long been an open problem: Adimurthi and Prashanth [3] were only able to prove the existence of blowing-up Palais-Smale sequences, while more recently Del Pino, Musso and Ruf, with an approach technically much richer, showed that blowing-up solutions exist for any domain* Ω*, see [15]. Our approach applies only to the unit ball, but it is on the other hand very simple and relatively explicit. We also remark that by the result of Struwe [32] in Theorem 4 we can take* $\Lambda^\sharp > 4\pi$.

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