

On the number of positive solutions of some semilinear elliptic problems

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Dedicated to the memory of Giovanni Prodi

1 Introduction and main results

In this Note we deal with semilinear elliptic Dirichlet boundary value problems like

$$\begin{cases} -\Delta u = \lambda u - f(u) + h(x) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \quad (D_h)$$

Here Ω is a bounded domain in \mathbb{R}^n with $C^{0,\nu}$ boundary $\partial\Omega$, $h \in C^{0,\nu}(\Omega)$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$(f1) \quad f \in C^2(\mathbb{R}), f(0) = f'(0) = 0 \text{ and } f''(s)s > 0 \quad \forall s \neq 0$$

$$(f2) \quad \lim_{s \rightarrow \pm\infty} \frac{f(s)}{s} = +\infty.$$

If $h = 0$ the problem (D_h) will be denoted by (D) . By a solution of (D_h) we mean a $C^{2,\nu}(\Omega)$ classical solution.

In order to state our main results some notation is in order. If $m \in L^\infty(\Omega)$, $\lambda_k[m]$ denotes the k -th eigenvalue of

$$\begin{cases} -\Delta u = \lambda m(x)u & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases} \quad (1)$$

If $m(x) \equiv 1$, we set $\lambda_k[m] = \lambda_k$.

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Theorem 1. *Suppose that f satisfies (f1–2) and let $\lambda > \lambda_1$. Then (D) has exactly one positive u_1 and one negative solution u_2 . Moreover, if $\lambda \neq \lambda_k$, there exists $\varepsilon_\lambda > 0$ such that if $\|h\|_{L^2} \leq \varepsilon_\lambda$, then (D_h) has one solution near each of $u = 0, u_1, u_2$.*

Remarks 2. (i) In [2] it has been proved that if (f1–2) hold and $\lambda_1 < \lambda < \lambda_2$ then (D) has exactly one positive and one negative solution and (D) has no other solution. As a byproduct of our arguments we will give a simple proof of this fact, see Remark 9. Moreover, in [2] it is shown that if λ_2 is simple and $\lambda_2 < \lambda \leq \lambda_2 + \delta$, with $\delta > 0$ sufficiently small, (D) has exactly 4 non-trivial solutions. In addition, there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that (D_h) with $\|h\|_{L^2} \leq \varepsilon$ has precisely 3 solutions, resp. 5 solutions, provided $\lambda_1 < \lambda < \lambda_2$, resp. $\lambda_2 < \lambda \leq \lambda_2 + \delta$.

(ii) Problem (D_h) with $f(u) = u^3$ has been also studied in [6]. Using the theory of singularities, it is evaluated the exact number of solutions of (D_h) (with no restriction on the norm of h) provided $\lambda_1 < \lambda < \lambda_1 + \Lambda$ for some $\Lambda > 0$. In general, the result cannot be extended to cover all $\lambda_1 < \lambda < \lambda_2$, see [7].

(iii) The first paper concerning with the precise number of solutions of semilinear elliptic problems is [4]. It deals with the so called *jumping nonlinearities*. See also [5] and references therein. \square

2 Preliminaries

We set $H = L^2(\Omega)$ with scalar product $(\cdot | \cdot)$ and norm $\|\cdot\|$, and define $K \in L(H, H)$ by setting

$$Kv = u \iff -\Delta u = v, \quad v|_{\partial\Omega} = 0.$$

Let us consider $F_\lambda \in C^2(H, H)$,

$$F_\lambda(u) = u - \lambda Ku - Kf(u).$$

With this notation, $u \in H$ is a weak (and, by regularity, classical) solution of (D_h) , resp. (D), whenever $F_\lambda(u) = Kh$, resp. $F_\lambda(u) = 0$.

We denote by Σ_λ the set of $u \in H$ such that $\text{Ker}[F'_\lambda(u)] \neq \{0\}$. The set Σ_λ is called the *singular set* of F_λ . See e.g. [3, Section 3]

Remark 3. If $u \in \Sigma_\lambda$ there exists an integer $i \geq 1$ such that $\lambda_i[\lambda - f'(u)] = 1$. \square

We denote by \mathcal{S}_λ the set of nontrivial solutions of (D) . It is well known that, if $\lambda > \lambda_1$ then (D) has at least 2 nontrivial solutions $u_1 > 0$ and $u_2 < 0$. This result can be proved in several manner: by variational methods, by sub- and super-solutions or by degree, cfr. e.g. [1].

Finally, $F_\lambda = Id - Compact$ and the solutions of (D) are bounded: there exists $C > 0$ such that

$$\|u\|_{L^2} \leq C, \quad \forall u \in \mathcal{S}_\lambda. \quad (2)$$

As a consequence, for all $r > C$ there holds

$$deg(F_\lambda, B_r, 0) = 1, \quad (3)$$

where B_r denotes the ball of radius r in H and deg denotes the Leray-Schauder (LS for short) degree.

3 Some lemmas

In this section we prove some lemmas. It is always understood that $(f1 - 2)$ hold.

Lemma 4. *Let $u \in \mathcal{S}_\lambda$. Then for all $x \in \Omega$ such that $u(x) \neq 0$ there exists a unique $t_u = t_u(x) \in]0, 1[$ such that $t_u u \in H$ and $F'_\lambda(t_u u)[u] = 0$. Hence $t_u u \in \Sigma_\lambda$ and $\Sigma_\lambda \neq \emptyset$.*

Proof. By the assumptions on f it follows that $\forall x \in \Omega$ such that $u(x) \neq 0$, there exists a unique $t_u = t_u(x) \in]0, 1[$ such that

$$f(u(x)) = f'(t_u(x)u(x))u(x). \quad (4)$$

Since the set $\{x \in \Omega : u(x) = 0\}$ has zero Lebesgue measure, then $0 < t_u(x) < 1$ for almost every $x \in \Omega$ and $t_u u \in H$. We claim that $t_u u \in \Sigma_\lambda$. Since u is a solution of (D) one has

$$u = \lambda K u - K f(u). \quad (5)$$

Using (4) and (5), we get

$$F'_\lambda(t_u u)[u] = u - \lambda K u + K f'(t_u u)u = K f'(t_u u)u - K f(u) = 0.$$

Then $F'_\lambda(t_u u)[u] = 0$ has the nontrivial solution $u \neq 0$ and hence $t_u u \in \Sigma_\lambda$, as claimed. \square

Example. If $f(s) = |s|^{p-1}s$ then (4) yields $t_u(x) \equiv p^{-1/(p-1)}$. \square

Remark 5. The preceding proof highlights that $u \in \text{Ker}[F'_\lambda(t_u u)]$. \square

If $\lambda \geq \lambda_1$ the singular set Σ_λ is not empty. Actually, if $\lambda > \lambda_1$ and $\Sigma_\lambda = \emptyset$ (if $\lambda = \lambda_1$ then $\Sigma_\lambda = \{0\}$), one could apply the Global Inversion Theorem (see [3, Theorem 3.1.8]) and (D) should have the trivial solution, only.

Next, we set

$$\Sigma_{\lambda,1} = \{u \in \Sigma_\lambda : \lambda_1[\lambda - f'(u)] = 1\} \quad \text{and} \quad \Sigma_{\lambda,2} = \Sigma_\lambda \setminus \Sigma_{\lambda,1}.$$

Although only $\Sigma_{\lambda,1}$ plays a role to prove Theorem 1, for completeness we will make in the sequel some remarks on $\Sigma_{\lambda,2}$, also.

We let $\mathcal{S}_{\lambda,1}$ be the set of solutions which do not change sign in Ω .

Lemma 6. *If $\lambda > \lambda_1$ then $\Sigma_{\lambda,1} \neq \emptyset$. Moreover, if $\lambda > \lambda_2$ then $\Sigma_{\lambda,2}$ is also not empty.*

Proof. As remarked in Section 2, if $\lambda > \lambda_1$ then $\mathcal{S}_{\lambda,1} \neq \emptyset$. Let $u \in \mathcal{S}_{\lambda,1}$. By Lemma 4 and Remark 5, $t_u u \in \Sigma_\lambda$ and a corresponding eigenfunction is u which does not change sign in Ω . Since the only eigenvalue with an eigenfunction that does not change sign in Ω is the first one, then $\lambda_1[\lambda - f'(t_u u)] = 1$. Hence $t_u u \in \Sigma_{\lambda,1}$, which is therefore not empty. Next, let $z \in \mathcal{S}_\lambda \setminus \mathcal{S}_{\lambda,1}$. By Lemma 4 we infer there exists $t_z(x)$ such that $t_z z \in \Sigma_\lambda$. By Remark 5 an eigenfunction corresponding to $t_z z$ is z which changes sign in Ω . Then $\lambda_1[\lambda - f'(t_z z)] \neq 1$ because the Kernel corresponding to the first eigenvalue is one dimensional and spanned by a positive function. Then $\lambda_i[\lambda - f'(t_z z)] = 1$ for some integer $i > 1$. Therefore $\Sigma_{\lambda,2} \neq \emptyset$ and the proof is complete. \square

Remarks 7. (i) If $\lambda < \lambda_1$, resp. $\lambda \leq \lambda_1$ one has that $\Sigma_\lambda = \emptyset$, resp. $\Sigma_{\lambda,1} = \emptyset$. Moreover, if $\lambda < \lambda_2$ one has that $\lambda - f'(u) < \lambda_2$ and hence $\lambda_i[\lambda - f'(u)] > 1$ for all $i \geq 2$. Thus $\Sigma_\lambda = \Sigma_{\lambda,1}$.

(ii) If $\lambda < \lambda_2$ then $\mathcal{S}_\lambda = \mathcal{S}_{\lambda,1}$. Otherwise, let z be a solution of (D) which changes sign. Then $t_z z \in \Sigma_\lambda$. Precisely, by (i) one has that $z \in \Sigma_{\lambda,1}$, namely $\lambda_1[\lambda - f'(t_z z)] = 1$. Moreover $z \in \text{Ker}[\lambda - f'(t_z z)]$, a contradiction because z changes sign.

(iii) $\Sigma_{\lambda,1}$ is a smooth manifold of codimension 1 in H . To see this, let $v \in \Sigma_{\lambda,1}$, namely $\lambda_1[\lambda - f'(v)] = 1$. Obviously $\lambda_1[\lambda - f'(v)]$ is simple,

$\text{Ker}[F'_\lambda(v)]$ is one dimensional and spanned by some $\phi_\lambda \in H$. By $f''(v)v > 0$ it follows

$$(F''_\lambda(v)[v, \phi_\lambda] | \phi_\lambda) = - \int_{\Omega} f''(v)v\phi_\lambda^2 < 0.$$

This suffices to apply [3, Lemma 3.2.1] and the result follows. \square

We now focus our attention to $\mathcal{S}_{\lambda,1}$ and $\Sigma_{\lambda,1}$. If $u \in H$ is a non degenerate (i.e. non singular) solution of $F_\lambda(u) = 0$, we denote by $\text{ind}(F_\lambda, u)$ its LS index.

Lemma 8. *Every $u \in \mathcal{S}_{\lambda,1}$ is non degenerate and $\text{ind}(F_\lambda, u) = 1$.*

Proof. By the preceding arguments, if $u \in \mathcal{S}_{\lambda,1}$ then $\lambda_1[\lambda - f'(t_u u)] = 1$, with $0 < t_u(x) < 1, \forall x \in \Omega$. Then $f'(t_u(x)u(x)) < f'(u(x)), \forall x \in \Omega$ and the monotonicity property of eigenvalues implies $\lambda_1[\lambda - f'(u)] > 1$, proving the Lemma. \square

Remark 9. Lemma 8 allows us to give a simple proof of the result of [2] for $\lambda \in]\lambda_1, \lambda_2[$. Actually, for such λ , $u = 0$ is non-degenerate with $\text{ind}(F_\lambda, 0) = -1$ while the total degree on a ball of radius $r \gg 1$ is 1, see (3). The $u \in \mathcal{S}_{\lambda,1}$ are also non-degenerate with index 1 and hence, F_λ being proper, their number is finite, say k . Moreover, if $\lambda \in]\lambda_1, \lambda_2[$, Remark 7-(ii) yields $\mathcal{S}_\lambda = \mathcal{S}_{\lambda,1}$. Then using the additivity property of the degree we get

$$1 = \text{ind}(F_\lambda, 0) + \sum_{u \in \mathcal{S}_{\lambda,1}} \text{ind}(F_\lambda, u) = -1 + k,$$

and thus $k = 2$. \square

4 Proof of Theorem 1

Theorem 1 cannot be proved by using the degree arguments outlined in Remark 9, because for $\lambda > \lambda_2$ problem (D) has other solutions than the ones in $\mathcal{S}_{\lambda,1}$, and we do not know if they are degenerate or not, see e.g. Remark 10.

To overcome this difficulty we consider the bifurcation problem $F_\lambda(u) = 0$. It is well known that from λ_1 emanates a continuous \mathcal{C} of solutions of $F_\lambda(u) = 0$. Moreover, near $(0, \lambda_1)$, \mathcal{C} is a uniquely determined curve and if $(\lambda, u) \in \mathcal{C}$, u does not change sign in Ω . For the bifurcation from a simple eigenvalue we refer, e.g. to [3, Sec. 5.4]. Let $\mathcal{C}^+ = \{(\lambda, u) : \lambda \in \mathbb{R}, u > 0, F_\lambda(u) = 0\}$. By the previous remark, near $(0, \lambda_1)$ one has that $\mathcal{C}^+ \subset \mathcal{C}$. If $(\lambda, u_\lambda) \in \mathcal{C}^+$ then by

Lemma 8 u_λ is non-degenerate and hence \mathcal{C}^+ is a C^1 curve. In particular, there are no secondary bifurcations on \mathcal{C}^+ . From (2) it follows that \mathcal{C}^+ is bounded in $[0, \Lambda] \times H$ for each $\Lambda > 0$. Since $F_\lambda(u) = 0$ has only the trivial solution provided $\lambda \leq \lambda_1$, then $\{u \in H : (\lambda, u) \in \mathcal{C}^+\} = \emptyset$ for these λ . Moreover $\overline{\mathcal{C}^+}$ (the closure of \mathcal{C}^+) cannot contain $(0, \lambda_k)$, with $k \geq 2$, because λ_1 is the unique eigenvalue from which bifurcate positive solutions. Next, suppose that there exists a solutions $z_\lambda > 0$ of $F_\lambda(u) = 0$ such that $(\lambda, z_\lambda) \notin \mathcal{C}^+$. Lemma 8 implies that z_λ is non-degenerate. By the continuation property of the topological degree, there is a branch (actually a C^1 curve) \mathcal{C}^* containing (λ, z_λ) . Repeating the preceding arguments, we deduce that \mathcal{C}^* shares the same properties of \mathcal{C}^+ . In particular, $\mathcal{C}^* \cap \mathcal{C}^+ = \emptyset$ because otherwise \mathcal{C}^+ (or \mathcal{C}^*) would have a secondary bifurcation. In addition $(0, \lambda_1) \in \overline{\mathcal{C}^*}$. Since the branch \mathcal{C} bifurcating from $(0, \lambda_1)$ is (locally) unique, we find a contradiction, proving that the only positive solutions of $F_\lambda(u) = 0$ belong to \mathcal{C}^+ . In a quite similar way one shows that $\mathcal{C}^- = \{(\lambda, u) : \lambda \in \mathbb{R}, u < 0, F_\lambda(u) = 0\}$ is a curve bifurcating from $(0, \lambda_1)$ which contains all the negative solutions of $F_\lambda = 0$. This proves that (D) has precisely one solutions $u_1 > 0$ and one solutions $u_2 < 0$. Both u_1, u_2 are non-degenerate. Moreover, if $\lambda \neq \lambda_k$, also $u = 0$ is non-degenerate. Thus the Local Inversion Theorem applies yielding a unique solution of (D_h) near $0, u_1, u_2$ provided $\|h\|_H \ll 1$. This completes the proof of Theorem 1. \square

Remark 10. It is known that u_1, u_2 are local minima of the energy functional

$$J_\lambda(u) = \frac{1}{2} \int_\Omega [|\nabla u|^2 - \lambda u^2] dx + \int_\Omega \left[\int_0^{u(x)} \bar{f}(s) ds \right] dx,$$

where

$$\bar{f}(s) = \begin{cases} \lambda s - f(s) & \text{if } -s^* \geq s \leq s^*, \\ -\lambda s^* + f(-s^*), & \text{if } s < -s^*, \\ \lambda s^* - f(s^*), & \text{if } s > s^*, \end{cases}$$

and $s^* > 0$ is such that $\lambda s^* - f(s^*) < 0$ and $-\lambda s^* - f(-s^*) > 0$. Moreover, if $\lambda > \lambda_2$ then (D) has at least a Mountain Pass solution $z \neq 0$. Theorem 1 implies that z changes sign in Ω . Unfortunately we are not able to estimate the precise number of changing sign solutions, the main difficulty being that we do not know if these solutions are non-degenerate. \square

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