The geometry emerging from the symmetries of a quantum system

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Abstract

We investigate the relation between the symmetries of a quantum system and its topological quantum numbers, in a general C^*-algebraic framework. We prove that, under suitable assumptions on the symmetry algebra, there exists a generalization of the Bloch-Floquet transform which induces a direct-integral decomposition of the algebra of observables. Such generalized transform selects uniquely the set of "continuous sections" in the direct integral, thus yielding a Hilbert bundle. The emerging geometric structure provides some topological invariants of the quantum system. Two running examples provide an Ariadne's thread through the paper. For the sake of completeness, we review two related theorems by von Neumann and Maurin and compare them with our result.

Key words: Topological quantum numbers, spectral decomposition, Bloch-Floquet transform, C^*-module, Hilbert bundle. **MSC 2010:** 81Q70; 46L08; 46L45; 57R22.

Contents

1 Introduction

Topological quantum numbers (see [Tho98] for a complete overview) play a prominent role in quantum physics: they appear, just to mention few examples, in the theory of the *Quantum Hall Effect* (QHE) [TKNN82] [BSE94] (see [Gra07] for a recent review), in the context of macroscopic polarization [Tho83] [KSV93] [Res92] [SPT09] and in the topological approach to quantum pumps [AEGS04] [GO08].

A typical way to compute a topological quantum number in presence of symmetries is to fibrate the C^*-algebra of physical observables, and the Hilbert space where it is represented, with respect to the action of an abelian symmetry group. The prototypical example is provided by periodic systems and the usual Bloch-Floquet transform.

Example 1.1 (*Periodic systems, intro*)**.** The electron dynamics in a periodic crystal is generated by

$$H_{\rm per} := -\frac{1}{2}\triangle + V_\Gamma \tag{1}$$

defined on a suitable domain (of essential self-adjointness) in the Hilbert space $L^2(\mathbb{R}^d)$. The periodicity of the crystal is described by the lattice $\Gamma := \{\gamma \in \mathbb{R}^d \ : \gamma = \sum_{j=1}^d n_j\ \gamma_j,\ n_j \in \mathbb{Z}\} \simeq \mathbb{Z}^d$ where $\{\gamma_1, \ldots, \gamma_d\}$ is a linear basis of $\mathbb{R}^d$. The potential V_Γ is Γ-periodic, i.e. $V_\Gamma(\ \cdot\ - \gamma) = V_\Gamma(\ \cdot\ )$ for all $\gamma \in \Gamma$. The $\mathbb{Z}^d$-symmetry is implemented by the translation operators $\{T_1, \ldots, T_d\}$, $(T_j\psi)(x) := \psi(x - \gamma_j)$. One defines the *Bloch-Floquet transform*, initially for $\psi \in \mathcal{S}(\mathbb{R}^d)$, by posing

$$(\mathcal{U}_{\rm BF}\psi)(k, y) := \sum_{\gamma\in\Gamma} e^{-ik\cdot\gamma}\big(T_j^{n_j}\psi\big)(y), \qquad k, y \in \mathbb{R}^d, \tag{2}$$

where $\gamma = \sum_j n_j\gamma_j$ (see [Kuc93] or [Pan07] for details). Definition (2) extends to a unitary operator

$$\mathcal{U}_{\rm BF} : L^2(\mathbb{R}^d) \longrightarrow \int_{\mathbb{B}}^{\oplus} \mathcal{H}(k)\, dk \tag{3}$$

where $\mathbb{B} \simeq \mathbb{T}^d$ is the fundamental cell of the dual lattice Γ^* (Brillouin zone) and

$$\mathcal{H}(k) := \big\{\varphi \in L^2_{\rm loc}(\mathbb{R}^d) : \varphi(y + \gamma) = e^{ik\cdot\gamma}\varphi(y) \quad \forall\gamma \in \Gamma\big\}.$$

In this representation, the Fermi projector $P_\mu = E_{(-\infty,\mu)}(H_{\rm per})$ is a decomposable operator, in the sense that $\mathcal{U}_{\rm BF} P_\mu \mathcal{U}_{\rm BF}{}^{-1} = \int_{\mathbb{B}}^{\oplus} P(k)\, dk$. Thus, under the assumption that μ lies in a spectral gap, the Fermi projector defines (canonically) a complex vector bundle over $\mathbb{B}$, whose fiber at $k \in \mathbb{B}$ is $\mathrm{Ran}P(k) \subset \mathcal{H}(k)$ (Bloch bundle). Some geometric properties of this vector bundle are physically measurable: for example, for $d = 2$, the Chern number corresponds to the transverse conductance measured in multiples of $\frac{1}{2\pi}$ (in natural units). As far as the time-reversal symmetric Hamiltonian (1) is concerned, such Chern number is zero; however, the generalization of this procedure to the case of magnetic translations is relevant in the understanding of the QHE. ◀▷

This paper addresses the following questions:

(I) to which extent is it possible to generalize the Bloch-Floquet transform? how general is the decomposition procedure outlined above?

(II) how does the topology (geometry) of the decomposition emerges?

(III) to which extent is this topological information unique? More precisely, does it depend on the Hilbert space representation of the algebra of observables?

As for question (III), we notice in [DFP10] that the datum of a C^*-algebra and a symmetry group does *not* characterize the topological information. Indeed the rational rotation C^*-algebra (alias the algebra of the *noncommutative torus* [Con94] [Boc01] [GVF01]) admits two inequivalent representations, respectively in $L^2(\mathbb{R})$ (*Harper representation*) and in $L^2(\mathbb{T}^2)$ or $\ell(\mathbb{Z}^2)$ (*Hofstadter representation*). Both representations can be fibered with respect to a $\mathbb{Z}^2$-symmetry, but the corresponding Chern numbers are different. This example is described in detail in [DFP10].

Here we investigate more generally how and under which conditions the symmetries of a physical system are related to observable effects whose origin is geometric (e.g. topological quantum numbers). Our approach is based on a general framework: $\mathcal{H}$ is a separable Hilbert space which corresponds to the physical states; $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ is a C^*-algebra of bounded operators on $\mathcal{H}$ which contains the relevant physical models (the self-adjoint elements of $\mathfrak{A}$ can be thought of as Hamiltonians); the commutant $\mathfrak{A}'$ (the set of all the elements in $\mathscr{B}(\mathcal{H})$ which commute with $\mathfrak{A}$) can be thought of as the *set of all the physical symmetries* with respect to the physics described by $\mathfrak{A}$; any commutative unital C^*-algebra $\mathfrak{S} \subset \mathfrak{A}'$ describes a set of simultaneously implementable physical symmetries.

Definition 1.2 (Physical frame). *A* physical frame *is a triple* $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ *where* $\mathcal{H}$ *is a separable Hilbert space,* $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ *is a* C^**-algebra and* $\mathfrak{S} \subset \mathfrak{A}'$ *is a commutative unital* C^**-algebra. The physical frame* $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ *is called* irreducible *if* $\mathfrak{S}$ *is* maximal commutative [(1)]. *Two physical frames* $\{\mathcal{H}_1, \mathfrak{A}_1, \mathfrak{S}_1\}$ *and* $\{\mathcal{H}_2, \mathfrak{A}_2, \mathfrak{S}_2\}$ *are said* (unitarily) equivalent *if there exists a unitary map* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *such that* $\mathfrak{A}_2 = U\mathfrak{A}_1U^{-1}$ *and* $\mathfrak{S}_2 = U\mathfrak{S}_1U^{-1}$.

One deals often with triples $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ whose C^*-algebra $\mathfrak{S}$ describes symmetries with an intrinsic group structure. In these cases $\mathfrak{S}$ is related to a representation of the group in $\mathcal{H}$, as stated in the following definition.

Definition 1.3 ($\mathbb{G}$-algebra). *Let* $\mathbb{G}$ *be a topological group and* $\mathbb{G} \ni g \mapsto U_g \in \mathscr{U}(\mathcal{H})$ *a strongly continuous unitary representation of* $\mathbb{G}$ *in the group* $\mathscr{U}(\mathcal{H})$ *of the unitary operators on* $\mathcal{H}$*. The representation is* faithful *if* $U_g = \mathbb{1}$ *implies* $g = e$ *(*e *is the identity of the group) and is* algebraically compatible *if the operators* $\{U_g \ : \ g \in \mathbb{G}\}$ *are linearly independent in* $\mathscr{B}(\mathcal{H})$*. Let* $\mathfrak{S}(\mathbb{G})$ *be the unital* C^**-algebra generated algebraically by* $\{U_g \ : \ g \in \mathbb{G}\}$ *and closed with respect to the operator norm of* $\mathscr{B}(\mathcal{H})$*. When the representation of* $\mathbb{G}$ *is faithful and algebraically compatible we say that* $\mathfrak{S}(\mathbb{G})$ *is a* $\mathbb{G}$-algebra *in* $\mathcal{H}$*.*

The answer to question (I) is provided in Section 6, with a completely satisfactory answer in the case in which $\mathfrak{S}$ is a $\mathbb{Z}^d$-algebra (Theorem 6.4). Questions (II) and (III) are addressed in Section 7, in particular by Theorem 7.9. During the preparation of the paper, we realized that this subject is closely related to von Neumann's complete spectral theorem and to Maurin's nuclear spectral theorem. For the sake of a self-consistent exposition we decided to review them in Section 3 and Section 4 respectively.

(1) We recall that a commutative C^*-algebra $\mathfrak{S} \subseteq \mathscr{B}(\mathcal{H})$ is *maximal commutative* if there is no other commutative C^*-algebra in $\mathscr{B}(\mathcal{H})$ which contains properly $\mathfrak{S}$. Clearly the condition of maximal commutativity implies the existence of a unit.

In greater detail, the content of the paper is the following:

- **Section 3** is devoted to review the *von Neumann's complete spectral theorem* (Theorem 3.1). This is a general result which provides an abstract decomposition of the type (3) for any physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$. The theorem provides a characterization of the base space for the decomposition (which coincides with the Gel'fand spectrum of the C^*-algebra $\mathfrak{S}$) and of the measure (basic measure) which glues the fibers in such a way that the Hilbert space structure is preserved. However, the construction of the fiber spaces is abstract and the fibration is unique only in a measure theoretic sense, so unsuitable to investigate topological properties.

- **Section 4** contains a review of the *Maurin's nuclear spectral theorem* (Theorem 4.1), a completion of the von Neumann's theorem. Maurin exhibits a characterization of the fiber spaces of the decomposition as spaces of *common generalized eigenvectors* for the operators $\mathfrak{S}$. Although the Maurin's result does not provide any algorithm to build the decomposition, it suggests how to generalize formula (2) to more general cases.

- **Section 5** concerns the notion of *wandering property* for a commutative C^*-algebra generated by a finite family of independent operators. This notion is of particular relevance when the generators of the C^*-algebra are unitary operators. In this case the wandering property assures that the C^*-algebra is a $\mathbb{Z}^N$-algebra with N the number of generators. Moreover the Gel'fand spectrum of the C^*-algebra is forced to be the dual group of $\mathbb{Z}^N$, i.e. the N-dimensional torus $\mathbb{T}^N$.

- **Section 6** contains our first novel contribution: the extension of the formula (17) to the case of a $\mathbb{Z}^N$-algebra which satisfies the wandering property. This *generalized Bloch-Floquet transform* provides a concrete recipe to decompose the Hilbert space according to the von Neumann and Maurin theorems. The general (algebraic) context in which the Bloch-Floquet transform makes sense allows us to generalize (to "algebrize") some customary notions like that of *localized Wannier functions*.

- **Section 7** is devoted to shows how a non trivial topology (and geometry) emerges in a canonical way from the decomposition induced by the generalized Bloch-Floquet transform. This geometric structure is essentially unique, so the emerging geometrical information is a fingerprint of the given physical frame.

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2 Some guiding examples

We support the theory with some explicit examples. A more difficult and physically relevant example is discussed in [DFP10].

A simple prototypical example

Example 2.1 (*Symmetries induced by a finite group*)**.** It is well known that every finite commutative group is isomorphic to a product group $\mathbb{F} = \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_N}$, where $\mathbb{Z}_{p_j} := \{[0], \ldots, [p_j - 1]\}$ is the *cyclic group* of order $p_j \in \mathbb{N}$. For every $t := (t_1, \ldots, t_N) \in \widehat{\mathbb{F}}$, with $\widehat{\mathbb{F}} := \prod_{j=1}^{N}\{0, \ldots, p_j - 1\}$, let $g_t := ([t_1], \ldots, [t_N])$ be a generic element in $\mathbb{F}$. The set of indices $\widehat{\mathbb{F}}$ coincides with the dual group of $\mathbb{F}$. The order of the group is equals to the order of its dual, $|\mathbb{F}| = |\widehat{\mathbb{F}}| = p_1 \ldots p_N$. Let $U : \mathbb{F} \to \mathscr{U}(\mathcal{H})$ be a faithful and algebraically compatible unitary representation on a separable Hilbert space $\mathcal{H}$. In particular $U_1 := U_{([1],[0],\ldots,[0])}, \ldots, U_N := U_{([0],[0],\ldots,[1])}$ is a minimal family of generators for the $\mathbb{F}$-algebra $\mathfrak{S}(\mathbb{F})$. Using a multiindex notation we can write $U_{g_t} = U_1^{t_1} \ldots U_N^{t_N} =: U^t$ for all $g_t \in \mathbb{F}$. The condition $U_j^{p_j} = \mathbb{1}$ implies that if U_j has an eigenvalue then it should be a root of the unity of order p_j, i.e. a suitable integer power of $z_j := \exp i(2\pi/p_j)$. Some relevant questions arise in a natural way: is it possible to compute algorithmically the eigenvalues and the eigenspaces of the generators U_j? Is it possible to diagonalize simultaneously the C^*-algebra $\mathfrak{S}(\mathbb{F})$ and to compute its Gel'fand spectrum (the set of the simultaneous eigenvalues)? The answers to these questions are implicit in the following formula:

$$P_t := \frac{1}{|\mathbb{F}|} \sum_{g_n \in \mathbb{F}} z^{-it\cdot n}\, U_{g_n} := \frac{1}{p_1 \ldots p_N} \sum_{n \in \widehat{\mathbb{F}}} (\mathrm{e}^{-i\frac{2\pi}{p_1} t_1})^{n_1} \ldots (\mathrm{e}^{-i\frac{2\pi}{p_N} t_N})^{n_N}\, U_1^{n_1} \ldots U_N^{n_N}. \tag{4}$$

For all $t \in \widehat{\mathbb{F}}$ equation (4) defines an orthogonal projection; indeed it is immediate to check that: 1) $P_t^\dagger = P_t$ (the adjoint produces a permutation of the indices in the sum); 2) $P_t P_{t'} = \delta_{t,t'} P_t$ (since $\sum_{0 \leqslant n \leqslant p_j - 1} z_j^{t_j n_j} = p_j\ \delta_{t_j,0}$); 3) from the property of algebraic compatibility it follows that $P_t \neq 0$ for all $t \in \widehat{\mathbb{F}}$; 4) $\bigoplus_{t \in \widehat{\mathbb{F}}} P_t = P_0 = \mathbb{1}$; 5) $U_j P_t = z_j^{\,t_j}\ P_t$ for all $j = 1, \ldots, N$. We will refer to P_t as the t-th *Bloch-Floquet projection*. The family of the projections $\{P_t\}_{t \in \widehat{\mathbb{F}}}$ induces an orthogonal decomposition of the Hilbert space $\mathcal{H}$ labelled by the set $\widehat{\mathbb{F}}$. Let $\mathcal{H}(t) := \mathrm{Ran}(P_t)$, then the map

$$\mathcal{H} \overset{\mathcal{U}_{\mathfrak{S}(\mathbb{F})}}{\longrightarrow} \bigoplus_{t \in \widehat{\mathbb{F}}} \mathcal{H}(t) \tag{5}$$

defined by $(\mathcal{U}_{\mathfrak{S}(\mathbb{F})}\varphi)(t) := P_t \varphi =: \varphi(t)$ is called *(discrete) Bloch-Floquet transform.*The transform $\mathcal{U}_{\mathfrak{S}(\mathbb{F})}$ is unitary since $\|\varphi\|^2_{\mathcal{H}} = \sum_{t \in \widehat{\mathbb{F}}} \|P_t \varphi\|^2_{\mathcal{H}}$. Every Hilbert space $\mathcal{H}(t)$ is a space of simultaneous eigenvectors for the C^*-algebra $\mathfrak{S}(\mathbb{F})$, and the corresponding eigenvalues are generated as functions of $z_1^{t_1}, \ldots, z_N^{t_N}$. In particular the Gel'fand spectrum of $\mathfrak{S}(\mathbb{F})$ (which coincides with the joint spectrum of the generating family $U_1, \ldots, U_N$) is (homeomorphic to) the dual group $\widehat{\mathbb{F}}$. Finally the transform $\mathcal{U}_{\mathfrak{S}(\mathbb{F})}$ maps the Hilbert space $\mathcal{H}$ into a "fibered" space over the discrete set $\widehat{\mathbb{F}}$. The Hilbert structure is obtained "gluing" the fiber spaces $\mathcal{H}(t)$ by the counting measure defined on $\widehat{\mathbb{F}}$ (direct integral, Appendix B). The natural projection $\bigoplus_{t \in \widehat{\mathbb{F}}} \mathcal{H}(t) \overset{\pi}{\to} \widehat{\mathbb{F}}$ induces on the fibered space also the structure of vector bundle (with 0-dimensional basis). ◀▷

In the rest of this work we will generalize the previous decomposition to cases in which the C^*-algebra of the symmetries is more complicated than the one generated by a finite group. However, this simple example encodes already many relevant aspects which appear in the general cases.

Two examples of physical frame

Example 2.2 (*Periodic systems, part one*)**.** The Gel'fand-Naĭmark Theorem shows that there exists a isomorphism between the commutative C^*-algebra $C_0(\sigma(H_{\mathrm{per}}))$ and a commutative non-unital C^*-algebra $\mathfrak{A}_0(H_{\mathrm{per}})$ of bounded operators in $\mathcal{H}$. The elements of $\mathfrak{A}_0(H_{\mathrm{per}})$ are the operators $f(H_{\mathrm{per}}) \in \mathscr{B}(\mathcal{H})$, for $f \in C_0(\sigma(H_{\mathrm{per}}))$, obtained via the spectral theorem. Let $\mathfrak{A}(H_{\mathrm{per}})$ be the multiplier algebra of $\mathfrak{A}_0(H_{\mathrm{per}})$ in $\mathscr{B}(\mathcal{H})$. This is a unital commutative C^*-algebra which contains $\mathfrak{A}_0(H_{\mathrm{per}})$ (as an essential ideal), its Gel'fand spectrum is a (Stone-Čech) compactification of $\sigma(H_{\mathrm{per}})$ and the Gel'fand isomorphism maps $\mathfrak{A}(H_{\mathrm{per}})$

into the unital C^*-algebra of the continuous and bounded functions on $\sigma(H_{\text{per}})$ denoted by $C_{\text{b}}(\sigma(H_{\text{per}}))$ (see Appendix A for details). We assume that $\mathfrak{A}(H_{\text{per}})$ is the C^*-algebra of physical models.

Since $[T_i; T_j] = 0$ for any i, j, it follows that the unital C^*-algebra $\mathfrak{S}_T$ generated by the translations, their adjoints and the identity operator is commutative. Moreover, since $[H_{\text{per}}; T_j] = 0$ it follows that $\mathfrak{S}_T \subset \mathfrak{A}(H_{\text{per}})'$. Then the translations $\mathfrak{S}_T$ are simultaneously implementable physical symmetries for the physics described by the Hamiltonian (1). Thus $\{L^2(\mathbb{R}^d), \mathfrak{A}(H_{\text{per}}), \mathfrak{S}_T\}$ is a physical frame. It is a convenient model to study the properties of an electron in a periodic medium. ◀▷

Example 2.3 (*Mathieu-like Hamiltonians, part one*)**.** Let $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ be the one-dimensional torus. In the Hilbert space $L^2(\mathbb{T})$ consider the Fourier orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ defined by $e_n(\theta) := (2\pi)^{-\frac{1}{2}}\mathrm{e}^{in\theta}$. Let $\mathfrak{u}$ and $\mathfrak{v}$ be the unitary operators defined, for $g \in L^2(\mathbb{T})$, by

$$(\mathfrak{u}g)(\theta) := \mathrm{e}^{i\theta}\, g(\theta), \qquad (\mathfrak{v}g)(\theta) := g(\theta - 2\pi\beta), \qquad \mathfrak{u}\mathfrak{v} = \mathrm{e}^{i2\pi\beta}\, \mathfrak{v}\mathfrak{u} \tag{6}$$

with $\beta \in \mathbb{R}$. The last equation in (6) shows that the unitaries $\mathfrak{u}$ and $\mathfrak{v}$ satisfy the commutation relation of a *noncommutative torus* with deformation parameter β (see [Boc01] Chapter 1 or [GVF01] Chapter 12 for more details). We will denote by $\mathfrak{A}_{\text{M}}^{\beta} \subset \mathscr{B}(L^2(\mathbb{T}))$ the unital C^*-algebra generated by $\mathfrak{u}$, $\mathfrak{v}$ ($\mathfrak{u}^0 = \mathfrak{v}^0 := \mathbb{1}$) and their adjoints. We will call $\mathfrak{A}_{\text{M}}^{\beta}$ the *Mathieu C^*-algebra* [(2)] and its elements will be called Mathieu-like operators. This name is due to the fact that the Hamiltonian $\mathfrak{h} := \mathfrak{u}+\mathfrak{u}^{\dagger}+\mathfrak{v}+\mathfrak{v}^{\dagger} \in \mathfrak{A}_{\text{M}}^{\beta}$ appears in the well know *(almost-)Mathieu eigenvalue equation*

$$(\mathfrak{h}g)(\theta) = g(\theta - 2\pi\beta) + g(\theta + 2\pi\beta) + 2\cos(\theta)g(\theta) = \varepsilon g(\theta). \tag{7}$$

The action of $\mathfrak{u}$ and $\mathfrak{v}$ on the Fourier basis is given explicitly by $\mathfrak{u}e_n = e_{n+1}$ and $\mathfrak{v}e_n = \mathrm{e}^{-i2\pi n\beta}e_n$ for all $n \in \mathbb{Z}$.

We focus now on the commutant ${\mathfrak{A}_{\text{M}}^{\beta}}'$ of the Mathieu C^*-algebra. Let $\mathfrak{s} \in \mathscr{B}(L^2(\mathbb{T}))$ be a bounded operator such that $[\mathfrak{s}; \mathfrak{u}] = 0 = [\mathfrak{s}; \mathfrak{v}_\beta]$ and let $\mathfrak{s}e_n = \sum_{m\in\mathbb{Z}} s_{n,m}\, e_m$ be the action of $\mathfrak{s}$ on the basis vectors. The relation $[\mathfrak{s}; \mathfrak{u}] = 0$ implies $s_{n+1,m+1} = s_{n,m}$ and the relation $[\mathfrak{s}; \mathfrak{v}_\beta] = 0$ implies $\mathrm{e}^{-i2\pi(m-n)\beta}s_{n,m} = s_{n,m}$ for all $n, m \in \mathbb{Z}$. If $\beta \notin \mathbb{Q}$ then $\mathrm{e}^{-i2\pi(m-n)\beta} \neq 1$ unless $n = m$, hence $s_{n,m} = 0$ if $n \neq m$ and the condition $s_{n+1,n+1} = s_{n,n}$ implies that $\mathfrak{s} = s\mathbb{1}$ with $s \in \mathbb{C}$. This shows that in the irrational case $\beta \notin \mathbb{Q}$ the commutant of the Mathieu C^*-algebra is trivial. To have a non trivial commutant we need to assume that $\beta := p/q$ with p, q non zero integers such that $\gcd(q,p) = 1$. In this case the condition $\mathfrak{s} \in {\mathfrak{A}_{\text{M}}^{p/q}}'$ implies that $s_{n,m} \neq 0$ if and only if $m - n = kq$ for some $k \in \mathbb{Z}$, moreover $s_{n,n+kq} = s_{0,kq} =: s'_k$ for all $n \in \mathbb{Z}$. Let $\mathfrak{w}$ be the unitary operator defined on the basis by $\mathfrak{w}e_n := e_{n+q}$, namely $\mathfrak{w} = (\mathfrak{u})^q$. The relations for the commutant imply that $\mathfrak{s} \in \mathfrak{A}_{\text{M}}^{p/q}$ if and only if $\mathfrak{s} = \sum_{k\in\mathbb{Z}} s'_k\, \mathfrak{w}^k$. Then in the rational case the commutant of the Mathieu C^*-algebra is the von Neumann algebra generated in $\mathscr{B}(L^2(\mathbb{T}))$ as the strong closure of the family of finite polynomials in $\mathfrak{w}$. We will denote by $\mathfrak{S}_{\text{M}}^{q}$ the unital commutative C^*-algebra generated by $\mathfrak{w}$. Observe that it does not depend on p. The triple $\{L^2(\mathbb{T}), \mathfrak{A}_{\text{M}}^{q/p}, \mathfrak{S}_{\text{M}}^{q}\}$ is an example of physical frame. ◀▷

[(2)] Such a algebra is a representation of the rotation C^*-algebra and in particular is faithful when $\beta \notin \mathbb{Q}$ [Boc01]. Since in this paper we focus on properties which *do* depend on the representation, we will adopt different names for images of the same abstract C^*-algebra under unitarily inequivalent representations.

Finally, we introduce some notation which will be useful in the following.

Remark 2.4 (*Notation*)**.** The N-dimensional torus $\mathbb{T}^N := \mathbb{R}^N/(2\pi\mathbb{Z})^N$ is parametrized by the cube $[0, 2\pi)^N$: for every $t = (t_1, \ldots, t_N)$ in the cube, $z(t) := (z_1(t), \ldots, z_N(t))$, with $z_j(t) := \mathrm{e}^{it_j}$, is a point of $\mathbb{T}^N$. The normalized Haar measure is $dz(t) = {dt_1 \ldots dt_N}/{(2\pi)^N}$. ♦◊

3 The complete spectral theorem by von Neumann

The complete spectral theorem is a useful generalization of the usual spectral decomposition of a normal operator on a Hilbert space. It shows that symmetries reduces the description of the full algebra $\mathfrak{A}$ to a family of simpler representations. The main tool used in the theorem is the notion of the direct integral of Hilbert spaces (Appendix B). The "spectral" content of the theorem amounts to the characterization of the base space for the decomposition (the "set of labels") and of the measure which glues together the spaces so that the Hilbert space structure is preserved. These information emerges essentially from the Gel'fand theory (Appendix A). The definitions of decomposable and continuously diagonal operator are reviewed in Appendix B.

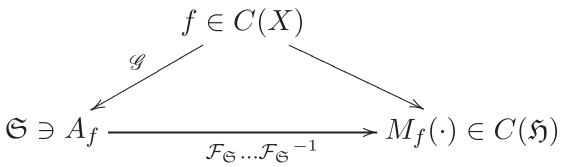
Theorem 3.1 (von Neumann's complete spectral theorem)**.** *Let $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ a physical frame and μ the basic measure carried by the spectrum X of $\mathfrak{S}$ (see Appendix A). Then there exist*

a) *a direct integral $\mathfrak{H} := \int_X^{\oplus} \mathcal{H}(x)\ d\mu(x)$ with $\mathcal{H}(x) \neq \{0\}$ for all $x \in X$,*

b) *a unitary map $\mathcal{F}_{\mathfrak{S}} : \mathcal{H} \to \mathfrak{H}$, called* $\mathfrak{S}$-Fourier transform [(3)],

such that:

(i) *the unitary map $\mathcal{F}_{\mathfrak{S}}$ intertwines the Gel'fand isomorphism $C(X) \ni f \overset{\mathscr{G}}{\longmapsto} A_f \in \mathfrak{S}$ and the canonical isomorphism of $C(X)$ onto the continuously diagonal operators $C(\mathfrak{H})$, i.e. the following diagram commutes*



(ii) *the unitary conjugation $\mathcal{F}_{\mathfrak{S}} \ldots \mathcal{F}_{\mathfrak{S}}^{-1}$ maps the elements of $\mathfrak{A}$ in decomposable operators on $\mathfrak{H}$; more precisely there is a measurable family $x \mapsto \pi_x$ of representations of $\mathfrak{A}$ on $\mathcal{H}(x)$ such that $\mathcal{F}_{\mathfrak{S}}\ \mathfrak{A}\ {\mathcal{F}_{\mathfrak{S}}}^{-1} = \int_X^{\oplus} \pi_x(\mathfrak{A})\ d\mu(x)$;*

(iii) *the representations π_x are irreducible if and only if the physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ is irreducible.*

Remark 3.2. For a complete proof of the above theorem one can see [Mau68] (Theorem 25 in Chapter I and Theorem 2 in Chapter V) or [Dix81] (Theorem 1 in Part II, Chapter 6). For our purposes it is interesting to recall how the fiber Hilbert spaces $\mathcal{H}(x)$ are constructed. For $\psi, \varphi \in \mathcal{H}$ let $\mu_{\psi,\varphi} = h_{\psi,\varphi}\ \mu$ the relation which links the spectral measure $\mu_{\psi,\varphi}$ with the basic measure μ. For μ-almost every $x \in X$ the value of the Radon-Nikodym derivative $h_{\psi,\varphi}$ in x defines a semi-definite sesquilinear form on $\mathcal{H}$, i.e. $(\psi; \varphi)_x := h_{\psi,\varphi}(x)$. Let $\mathcal{I}_x :=$

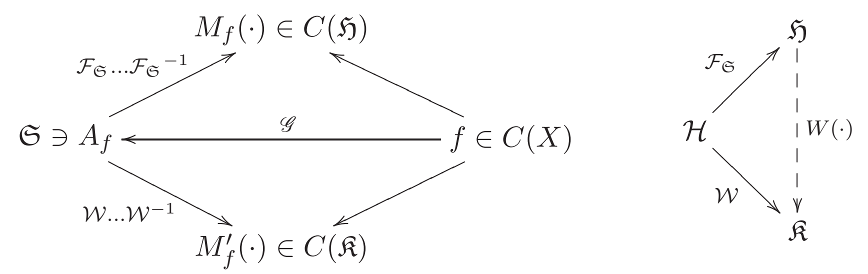
(3) According to the terminology used in [Mau68].

$\{\psi \in \mathcal{H} \ : \ h_{\psi,\psi}(x) = 0\}$. Then the quotient space $\mathcal{H}/\mathcal{I}_x$ is a pre-Hilbert space and $\mathcal{H}'(x)$ is defined to be the its completion. By construction $\mathcal{H}'(x) \neq \{0\}$ for μ-almost every $x \in X$. Let $N \subset X$ be the μ- negligible set on which $\mathcal{H}'(x)$ is trivial or not well defined. Then $\mathfrak{H} := \int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$ with $\mathcal{H}(x) := \mathcal{H}'(x)$ if $x \in X \setminus N$ and $\mathcal{H}(x) := H$ if $x \in N$ where H is an arbitrary non trivial Hilbert space. ♦◊

The previous theorem provides only a partial answer to our motivating questions. Firstly, it provides only a partial answer to question (I), since no explicit and computable "recipe" to construct the fiber Hilbert spaces is given. More importantly, Theorem 3.1 concerns a measure-theoretic decomposition of the Hilbert space, but it does not select a topological structure, yielding no answer to question (II). In more geometric terms, the elements of $\int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$ can be regarded as L^2-*sections* of a fibration over X, while the topological structure is encoded by the (still not defined) space of *continuous sections*. We will show in Section 7 that the Bloch-Floquet transform provides a natural choice of a subspace of $\int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$ which can be interpreted as the subspace of continuous sections, thus yielding a topological structure.

Given the triple $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$, the direct integral decomposition invoked in the statement of Theorem 3.1 is essentially unique in measure theoretic sense. The space X is unique up to homeomorphism: it agrees with the spectrum of $C(\mathfrak{H})$ in such a way that the canonical isomorphism of $C(X)$ onto $C(\mathfrak{H})$ may be identified with the Gel'fand isomorphism. As for the uniqueness of the direct integral decomposition, the following result holds true (see [Dix81] Theorem 3 in Part II Chapter 6).

Theorem 3.3 (Uniqueness). *With the notation of Theorem 3.1, let ν be a positive measure with support X, $\prod_{x\in X} \mathcal{K}(x)$ a field of non-zero Hilbert spaces over X, $\mathfrak{K} := \int_X^{\oplus} \mathcal{K}(x)\, d\nu(x)$, $C(\mathfrak{K})$ the commutative unital C^*-algebra of continuously diagonal operators on $\mathfrak{K}$ and $C(X) \to C(\mathfrak{K})$ the canonical isomorphism. Let $\mathcal{W}$ be a unitary (antiunitary) map from $\mathcal{H}$ onto $\mathfrak{K}$ transforming $A_f \in \mathfrak{S}$ into $M'_f(\cdot) \in C(\mathfrak{K})$ for all $f \in C(X)$, i.e. such that the first diagram commutes.*



Then, μ and ν are equivalent measures (so one can assume that $\mu = \nu$ up to a rescaling isomorphism). Moreover there exists a decomposable unitary (antiunitary) $W(\cdot)$ from $\mathfrak{H}$ onto $\mathfrak{K}$, such that $W(x) : \mathcal{H}(x) \to \mathcal{K}(x)$ is a unitary (antiunitary) operator μ-almost everywhere and $\mathcal{W} = W(\cdot) \circ \mathcal{F}_{\mathfrak{S}}$, i.e. the second diagram commutes.

Corollary 3.4 (Unitary equivalent triples). *Let $\{\mathcal{H}_1, \mathfrak{A}_1, \mathfrak{S}_1\}$ and $\{\mathcal{H}_2, \mathfrak{A}_2, \mathfrak{S}_2\}$ be two equivalent physical frames and U the unitary map which intertwines between them. Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ denote the direct integral decomposition of the two triples and let $\mathcal{F}_{\mathfrak{S}_1}$ and $\mathcal{F}_{\mathfrak{S}_2}$ be the two $\mathfrak{S}$-Fourier transforms. The unitary map $\mathcal{F}_{\mathfrak{S}_2} \circ U : \mathcal{H} \to \mathfrak{H}_2$ transforms $A_f \in \mathfrak{S}$ into $M'_f(\cdot) \in C(\mathfrak{H}_2)$ for all $f \in C(X)$. Then there exists a decomposable unitary operator $W(\cdot) : \mathfrak{H}_1 \to \mathfrak{H}_2$ such that $W(x) : \mathcal{H}_1(x) \to \mathcal{H}_2(x)$ is a unitary map μ-almost everywhere and $W(\cdot) = \mathcal{F}_{\mathfrak{S}_2} \circ U \circ {\mathcal{F}_{\mathfrak{S}_1}}^{-1}$.*

For sake of completeness, we also point out that two C^*-algebras $\mathfrak{S}_1$ and $\mathfrak{S}_2$ in $\mathfrak{A}'$ which are unitarily or anti-unitarily equivalent corresponds to decompositions which are equivalent (in the sense below).

Corollary 3.5 (Unitary equivalent symmetries algebras)**.** *Let $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}_1\}$ and $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}_2\}$ be physical frames such that $\mathfrak{S}_2 = U\mathfrak{S}_1 U^{-1}$ with U a unitary (anti-unitary) operator in $\mathcal{H}$. Let $\mathfrak{H}_1$ and $\mathfrak{H}_1$ denote the direct integral decomposition of the two triples and let $\mathcal{F}_{\mathfrak{S}_1}$ and $\mathcal{F}_{\mathfrak{S}_2}$ be the two $\mathfrak{S}$-Fourier transforms. The unitary (anti-unitary) map $\mathcal{F}_{\mathfrak{S}_2} \circ U : \mathcal{H} \to \mathfrak{H}_2$ transforms $A_f \in \mathfrak{S}$ into $M'_f(\cdot) \in C(\mathfrak{H}_2)$ for all $f \in C(X)$. Then there exists a unitary (anti-unitary) decomposable operator $W(\cdot) : \mathfrak{H}_1 \to \mathfrak{H}_2$ such that $W(x) : \mathcal{H}_1(x) \to \mathcal{H}_2(x)$ is a unitary (anti-unitary) map μ-almost everywhere and $W(\cdot) = \mathcal{F}_{\mathfrak{S}_2} \circ U \circ {\mathcal{F}_{\mathfrak{S}_1}}^{-1}$.*

4 The nuclear spectral theorem by Maurin

The complete spectral theorem by von Neumann shows that any physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ admits a representation in which the Hilbert space is decomposed (in a measure-theoretically unique way) in a direct integral $\mathcal{F}_{\mathfrak{S}} : \mathcal{H} \to \int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$, the elements of $\mathfrak{S}$ are diagonalized simultaneously and the C^*-algebra $\mathfrak{A}$ is decomposed on the fibers. The contribution of Maurin is a characterization of the fiber spaces $\mathcal{H}(x)$ as common generalized eigenspaces for $\mathfrak{S}$. In this sense the map $\mathcal{F}_{\mathfrak{S}}|_x$ restricted to the point $x \in X$ is a sort of generalization of the projection (4). The key ingredient for the Maurin's theorem is the notion of *(nuclear) Gel'fand triple.* A Gel'fand triple $\{\Phi, \mathcal{H}, \Phi^*\}$ consists of a separable Hilbert space $\mathcal{H}$, a norm-dense subspace $\Phi \subset \mathcal{H}$ such that Φ has a topology for which it is a nuclear space and the inclusion map $\imath : \Phi \hookrightarrow \mathcal{H}$ is continuous. Identifying $\mathcal{H}$ with its dual space $\mathcal{H}^*$ one has the the inclusion $\imath^* : \mathcal{H} \hookrightarrow \Phi^*$ where Φ^* is the topological dual of Φ and $\imath^*$ is the adjoint of the map $\imath$. The duality pairing between Φ and Φ^* has to be compatible with the scalar product on $\mathcal{H}$, namely $\langle \psi_1, \psi_2 \rangle = (\psi_1, \psi_2)_{\mathcal{H}}$ whenever $\psi_1 \in \Phi^* \cap \mathcal{H}$ and $\psi_2 \in \Phi$. In some sense the structure $\Phi \hookrightarrow \mathcal{H} \hookrightarrow \Phi^*$ provides an "enlargement" (*rigging*) of the space $\mathcal{H}$. In Appendix C the reader find a short review about the theory of nuclear spaces and the notion of Gel'fand triple.

Assume the notation of Theorem 3.1. Let $\{\xi_k(\cdot) \ : \ k \in \mathbb{N}\}$ be a *fundamental family* of orthonormal measurable vector fields (see Appendix B) for the direct integral $\mathfrak{H}$ defined by the $\mathfrak{S}$-Fourier transform $\mathcal{F}_{\mathfrak{S}}$. Any square integrable vector field $\varphi(\cdot)$ can be written in a unique way as $\varphi(\cdot) = \sum_{k \in \mathbb{N}} \widehat{\varphi}_k(\cdot)\ \xi_k(\cdot)$ where $\widehat{\varphi}_k \in L^2(X, d\mu)$ for all $k \in \mathbb{N}$. Equipped with this notation, the scalar product in $\mathfrak{H}$ reads

$$\langle \varphi(\cdot); \psi(\cdot) \rangle_{\mathfrak{H}} = \int_X \sum_{k=1}^{\dim\, \mathcal{H}(x)} \overline{\widehat{\varphi}_k(x)}\ \widehat{\psi}_k(x)\ d\mu(x).$$

For any $\varphi \in \mathcal{H}$ let $\varphi(\cdot) := \mathcal{F}_{\mathfrak{S}} \varphi$ be the square integrable vector field obtained from φ by the $\mathfrak{S}$-Fourier transform. Denote with $A_f \in \mathfrak{S}$ the operator associated with $f \in C(X)$ through the Gel'fand isomorphism. One checks that

$$(\widehat{\mathcal{F}_{\mathfrak{S}} A_f \varphi})_k(x) = (\xi_k(x); f(x)\varphi(x))_x = f(x)\ \widehat{\varphi}_k(x) \qquad k = 1, 2, \ldots, \dim\, \mathcal{H}(x). \tag{8}$$

Suppose that $\{\Phi, \mathcal{H}, \Phi^*\}$ is a Gel'fand triple for the space $\mathcal{H}$. If $\varphi \in \Phi$ then the map $\Phi \ni \varphi \mapsto \widehat{\varphi}_k(x) := (\xi_k(x); \varphi(x))_x \in \mathbb{C}$ is linear and moreover it is possible to show that it is continuous with respect to the nuclear topology of Φ. This means that there exists a functional $\eta_k(x) \in \Phi^*$ such that

$$\langle \eta_k(x); \varphi \rangle := \widehat{\varphi}_k(x) = (\xi_k(x); \varphi(x))_x \qquad k = 1, 2, \ldots, \dim\, \mathcal{H}(x). \tag{9}$$

Suppose that $A_f : \Phi \to \Phi$ is continuous with respect to the nuclear topology. Then from equations (8) and (9) and the sesquilinearity of the dual pairing between Φ and Φ^* one has that

$$\langle A_f\ \eta_k(x);\varphi\rangle = \langle \eta_k(x); A_{\overline{f}}\varphi\rangle = \overline{f}(x)\ \widehat{\varphi}_k(x) = \langle f(x)\eta_k(x);\varphi\rangle \qquad k=1,2,\ldots,\dim\ \mathcal{H}(x) \tag{10}$$

for all $\varphi \in \Phi$. Hence,

$$A_f\ \eta_k(x) = f(x)\ \eta_k(x)$$

as a linear functional. In this sense $\eta_k(x)$ is a *generalized eigenvector* for A_f. These arguments justify the following statement:

Theorem 4.1 (Maurin's nuclear spectral theorem)**.** *With the notation and the assumptions of Theorem 3.1 let $\{\Phi,\mathcal{H},\Phi^*\}$ be a nuclear Gel'fand triple for the space $\mathcal{H}$ such that Φ is $\mathfrak{S}$-invariant, i.e. each $A \in \mathfrak{S}$ is a continuous linear map $A : \Phi \to \Phi$. Then:*

(i) *for all $x \in X$ the $\mathfrak{S}$-Fourier transform $\mathcal{F}_{\mathfrak{S}}|_x : \Phi \to \mathcal{H}(x)$ such that $\Phi \ni \varphi \mapsto \varphi(x) \in \mathcal{H}(x)$ is continuous with respect the nuclear topology for μ-almost all $x \in X$;*

(ii) *there is a family of linear functionals $\{\eta_k(x)\ :\ k=1,2,\ldots,\dim\ \mathcal{H}(x)\} \subset \Phi^*$ such that equations* (9) *and* (10) *hold true for μ-almost all $x \in X$;*

(iii) *with the identification $\eta_k(x) \leftrightarrow \xi_k(x)$ the Hilbert space $\mathcal{H}(x)$ is (isomorphic to) a vector subspace of Φ^*; with this identification the $\mathcal{F}_{\mathfrak{S}}$-Fourier transform is defined on the dense set Φ by*

$$\Phi \ni \varphi \overset{\mathcal{F}_{\mathfrak{S}}|_x}{\longmapsto} \sum_{k=1}^{\dim\ \mathcal{H}(x)} \langle \eta_k(x);\varphi\rangle\ \eta_k(x) \in \Phi^* \tag{11}$$

and the scalar product in $\mathcal{H}(x)$ is formally defined by posing $(\eta_k(x);\eta_j(x))_x := \delta_{k,j}$;

(iv) *under the identification in* (iii) *the spaces $\mathcal{H}(x)$ become the* generalized common eigenspaces *of the operators in $\mathfrak{S}$ in the sense that if $A_f \in \mathfrak{S}$ then $A_f\ \eta_k(x) = f(x)\ \eta_k(x)$ for μ-almost every $x \in X$ and all $k=1,2,\ldots,\dim\ \mathcal{H}(x)$.*

For a proof we refer to [Mau68] (Chapter II). The identification at point (iii) of the Theorem 4.1 depends on the choice of a fundamental family of orthonormal measurable vector fields $\{\xi_k(\cdot)\ :\ k \in \mathbb{N}\}$ for the direct integral $\mathfrak{H}$, which is clearly not unique. If $\{\zeta_k(\cdot)\ :\ k \in \mathbb{N}\}$ is a second fundamental family of orthonormal measurable fields for $\mathfrak{H}$ then there exists a decomposable unitary map $W(\cdot)$ such that $W(x)\xi_k(x) = \zeta_k(x)$ for μ-almost all $x \in X$ and all $k \in \mathbb{N}$. The composition $U := {\mathcal{F}_{\mathfrak{S}}}^{-1} \circ W(\cdot) \circ \mathcal{F}_{\mathfrak{S}}$ is a unitary isomorphism of the Hilbert space $\mathcal{H}$ which induces a linear isomorphism between the Gel'fand triples $\{\Phi,\mathcal{H},\Phi^*\}$ and $\{\Psi,\mathcal{H},\Psi^*\}$ where $\Psi := U\Phi$ is a nuclear space in $\mathcal{H}$ with respect the induced topology of Φ by the map U (i.e. defined by the family of seminorms $p'_\alpha := p_\alpha \circ U^{-1}$) and Ψ^*, the topological dual of Ψ, is ${}^tU\Phi^*$, in view of the continuity of $U : \Phi \to \Psi$. The isomorphism of the Gel'fand triples is compatible with the direct integral decomposition. Indeed if $\vartheta_k(x) \leftrightarrow \zeta_k(x)$ is the identification between the new basis $\{\zeta_k(x)\ :\ k=1,2,\ldots,\dim\ \mathcal{H}(x)\}$ of $\mathcal{H}(x)$ and a family of linear functionals $\{\vartheta_k(x)\ :\ k=1,2,\ldots,\dim\ \mathcal{H}(x)\} \subset \Psi^*$ then equation (9) implies that for any $\varphi \in \Psi$

$$\langle \vartheta_k(x);\varphi\rangle := (\zeta_k(x);\varphi(x))_x = \big(\xi_k(x);W(x)^{-1}\varphi(x)\big)_x = \langle \eta_k(x);U^{-1}\varphi\rangle = \langle U\eta_k(x);\varphi\rangle. \tag{12}$$

Proposition 4.2. *Up to a canonical identification of isomorphic Gel'fand triples (global change of basis) the realization* (11) *of the fiber spaces* $\mathcal{H}(x)$ *as generalized common eigenspaces is canonical in the sense that it does not depend on the choice of fundamental family of orthonormal measurable fields.*

From Proposition 4.2 and Corollaries 3.4 and 3.5 it follows that:

Corollary 4.3. *Up to a canonical identification of isomorphic Gel'fand triples (global change of basis) the realization* (11) *of the fiber spaces* $\mathcal{H}(x)$ *as generalized common eigenspaces is preserved by a unitary transform of the triple* $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ *or by a unitary equivalent choice of the commutative* C^**-algebra* $\mathfrak{S}$.

Theorem 4.1 assumes the existence of a $\mathfrak{S}$-invariant nuclear space and the related Gel'fand triple. If $\mathfrak{S}$ is generated by a countable family, such nuclear space does exist and there is an algorithmic procedure to construct it.

Theorem 4.4 (Existence of the nuclear space [Mau68])**.** *Let* $\{A_1, A_2, \ldots\}$ *a countable family of commuting bounded normal operators on the separable Hilbert space* $\mathcal{H}$ *which generate (together with their adjoints and the identity) the commutative* C^**-algebra* $\mathfrak{S}$. *Then there exists a countable* $\mathfrak{S}$*-cyclic system* $\{\psi_1, \psi_2, \ldots\}$ *which generates a nuclear space* $\Phi \subset \mathcal{H}$ *such that:* a) Φ *is dense in* $\mathcal{H}$*;* b) *the embedding* $\imath : \Phi \hookrightarrow \mathcal{H}$ *is continuous;* c) *the maps* $A_j^m : \Phi \to \Phi$ *are continuous for all* $j, m \in \mathbb{N}$.

Remark 4.5. For the proof of Theorem 4.4 see [Mau68] (Chapter II, Theorem 6). We recall that a countable (or finite) family $\{\psi_1, \psi_2, \ldots\}$ of orthonormal vectors in $\mathcal{H}$ is a $\mathfrak{S}$-*cyclic system* for $\mathfrak{S}$ if the set $\{{A^{\dagger}}^b A^a \psi_k \ : \ k \in \mathbb{N}, \quad a, b \in \mathbb{N}^{\infty}_{\text{fin}}\}$ is dense in $\mathcal{H}$, where $\mathbb{N}^{\infty}_{\text{fin}}$ is the space of sequences in $\mathbb{N}$ which are definitely zero and $A^a := A_1^{a_1} A_2^{a_2} \ldots A_N^{a_N}$ for some integer N. Any C^*-algebra $\mathfrak{S}$ (not necessarily commutative) has many $\mathfrak{S}$-cyclic systems. Indeed one can start from a generic orthonormal vector $\psi_1 \in \mathcal{H}$ to build the closed subspace $\mathcal{H}_1$ spanned by the action of $\mathfrak{S}$ on ψ_1. If $\mathcal{H}_1 \neq \mathcal{H}$ one can choose a second orthonormal vector ψ_2 in the orthogonal complement of $\mathcal{H}_1$ to build the closed subspace $\mathcal{H}_2$. Since $\mathcal{H}$ is separable, this procedure produces a countable (or finite) family $\{\psi_1, \psi_2, \ldots\}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots$. Obviously this construction is not unique. The nuclear space Φ claimed in Theorem 4.4 depends on the choice of a $\mathfrak{S}$-cyclic system and generically many inequivalent choices are possible. ♦◊

5 The wandering property

An interesting and generally unsolved problem is the construction of the invariant subspaces of an operator or of a family of operators. Let $\mathfrak{S}$ be a C^*-algebra contained in $\mathscr{B}(\mathcal{H})$. If $\psi \in \mathcal{H}$ then the subspace $\mathfrak{S}[\psi]$ generated by the action of $\mathfrak{S}$ on the vector ψ is an invariant subspace for the C^*-algebra. The existence of a particular decomposition of the Hilbert space in invariant subspaces depends on the nature of the C^*-algebra. The problem is reasonably simple to solve for the C^*-algebras which satisfy the wandering property.

Definition 5.1 (wandering property)**.** *Let* $\mathfrak{S}$ *be a commutative unital* C^**-algebra generated by the countable family* $\{A_1, A_2, \ldots\}$ *of commuting bounded normal operators and their adjoints (with the convention* $A_j^0 := \mathbb{1}$*) in a separable Hilbert space* $\mathcal{H}$. *We will say that* $\mathfrak{S}$ *has the* wandering property *if there exists a (at most) countable family* $\{\psi_1, \psi_2, \ldots\} \subset \mathcal{H}$ *of orthonormal vectors which is* $\mathfrak{S}$*-cyclic (according to Remark 4.5) and such that*

$$(\psi_k; {A^{\dagger}}^b A^a \psi_h)_{\mathcal{H}} = \|A^a \psi_k\|^2_{\mathcal{H}} \ \delta_{k,h} \ \delta_{a,b} \qquad \forall \ h, k \in \mathbb{N}, \ \ \forall \ a, b \in \mathbb{N}^{\infty}_{\text{fin}}, \tag{13}$$

where $A^a := A_1^{a_1} A_2^{a_2} \dots A_N^{a_N}$, $\delta_{k,h}$ *is the usual Kronecker delta and* $\delta_{a,b}$ *is the Kronecker delta for the multiindices* a *and* b.

Let $\mathcal{H}_k := \mathfrak{S}[\psi_k]$ be the Hilbert subspace generated by the action of $\mathfrak{S}$ on the vector ψ_k. If $\mathfrak{S}$ has the wandering property then the Hilbert space decomposes as $\mathcal{H} = \bigoplus_{k\in\mathbb{N}} \mathcal{H}_k$ and each $\mathcal{H}_k$ is an $\mathfrak{S}$-invariant subspace. We will refer to $\mathcal{H}_k$ as *wandering subspaces* and to $\{\psi_1, \psi_2, \dots\}$ as the *wandering system.* In these subspaces each operator A_j acts as a unilateral weighted shift and this justifies the use of the adjective "wandering" (see [NF70] Chapter 1, Sections 1 and 2). The wandering property implies many interesting consequences.

Proposition 5.2. *Let* $\mathfrak{S}$ *be a commutative unital* C^**-algebra generated by the (at most) countable family* $\{A_1, A_2, \dots\}$ *of commuting bounded normal operators and their adjoints in a separable Hilbert space* $\mathcal{H}$. *Suppose that* $\mathfrak{S}$ *has the wandering property with respect the family of vectors* $\{\psi_1, \psi_2, \dots\}$, *then:*

(i) *the generators can not be selfadjoint nor nilpotent;*

(ii) *if the generators are invertible then* $\mathfrak{S}$ *must be finitely generated;*

(iii) *every generator which is unitary has no eigenvectors;*

(iv) *if the generators are unitary then* $\mathfrak{S}$ *is a* $\mathbb{Z}^N$*-algebra for some positive integers* N.

Proof. To prove (i) observe that equation (13) implies that $A_j^a = \mathbb{1}$ if an only if $a = 0$; then A_j can not be nilpotent. Moreover the condition $A_j = A_j^\dagger$ implies that $A_j\psi_k = 0$ for all ψ_k in the system and the $\mathfrak{S}$-cyclicity imposes $A_j = 0$. To prove (ii) suppose that the generators $A_1, A_2, \dots$ are invertible. Then $\varphi^{(k)}_{a_1} := A_1^{a_1}\psi_k$, $\varphi^{(k)}_{a_1,a_2} := A_1^{a_1}A_2^{a_2}\psi_k$, $\varphi^{(k)}_{a_1,a_2,\dots} := A_1^{a_1}A_2^{a_2}\dots\psi_k$ are all independent non zero vectors in $\mathcal{H}$. Then the cardinality of a basis of $\mathcal{H}_k$ is the cardinality of $\mathbb{N} + \mathbb{N}^2 + \mathbb{N}^3 + \cdots$. Since $\mathcal{H}$ is separable any basis of $\mathcal{H}_k$ must be at most countable, hence one must stop after a finite number of steps. This shows that the number of invertible generators is necessarily finite. If the generators of $\mathfrak{S}$ are unitary then from (ii) it follows that they must be in a finite number, i.e. $\{U_1, U_2, \dots U_N\}$, and (i) implies that they are not nilpotent. The map $\mathbb{Z}^N \ni a := (a_1, \dots, a_N) \mapsto U^a := U_1^{a_1}\dots U_N^{a_N} \in \mathscr{U}(\mathcal{H})$ is a faithful unitary representation of $\mathbb{Z}^N$ on $\mathcal{H}$. Since $\mathbb{Z}^N$ is discrete the representation is also (strongly) continuous. To prove (iii) observe that if $\{U, A_1, A_2, \dots\}$ is a set of commuting generators for $\mathfrak{S}$ with U unitary, then each vector $\varphi \in \mathcal{H}$ can be written as $\varphi = \sum_{n\in\mathbb{Z}} U^n\chi_n$ where $\chi_n = \sum_{k\in\mathbb{N}} \alpha_k A^{a(k)}\psi_k$. Evidently $U\varphi = \sum_{n\in\mathbb{Z}} U^n\chi_{n-1}$ and equation (13) implies that $\|\varphi\|^2_{\mathcal{H}} = \sum_{n\in\mathbb{Z}} \|\chi_n\|^2_{\mathcal{H}}$. If $U\varphi = \lambda\varphi$, with $\lambda \in \mathbb{S}^1$, then a comparison between the components provides $\chi_{n-1} = \lambda\chi_n$, i.e. $\chi_n = \lambda^{-n}\chi_0$ for all $n \in \mathbb{Z}$. This contradicts the convergence of the series expressing the norm of φ. To prove (iv) we need only to show that the representation is algebraically compatible. Suppose that $\sum_{a\in\mathbb{Z}^N} \alpha_a U^a = 0$; then from equation (13) it follows that $0 = (U^b\psi_k; \sum_{a\in\mathbb{Z}^N} \alpha_a U^a\psi_k)_{\mathcal{H}} = \alpha_b$ for all $b \in \mathbb{Z}^N$, and this concludes the proof. ■

Proposition 5.2 shows that the wandering property forces a commutative C^*-algebra generated by unitary operators to be a $\mathbb{Z}^N$-algebra. This is exactly what happens in the cases in which we are mostly interested.

Example 5.3 (*Periodic systems, part two*)**.** The commutative unital C^*-algebra $\mathfrak{S}_T$ defined in Example 2.2 is generated by a unitary faithful representation of $\mathbb{Z}^d$ on $L^2(\mathbb{R}^d)$, given by $\mathbb{Z}^d \ni m \mapsto T^m \in \mathscr{U}(L^2(\mathbb{R}^d))$ where $m := (m_1, \dots, m_d)$ and $T^m := T_1^{m_1}\dots T_d^{m_d}$. The C^*-algebra $\mathfrak{S}_T$ has the wandering property. Indeed let $\mathcal{Q}_0 := \{x = \sum_{j=1}^d x_j\gamma_j \ : \ -1/2 \leqslant x_j \leqslant$

$1/2,\ \ j=1,\ldots,d\}$ the *fundamental unit cell* of the lattice Γ and $\mathcal{Q}_m := \mathcal{Q}_0 + m$ its translated by the lattice vector $m := \sum_{j=1}^{d} m_j \gamma_j$. Let $\{\psi_k\}_{k\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$ be a family of functions with support in $\mathcal{Q}_0$ providing an orthonormal basis of $L^2(\mathcal{Q}_0)$ up to the natural inclusion $L^2(\mathcal{Q}_0) \hookrightarrow L^2(\mathbb{R}^d)$. This system is $\mathfrak{S}_T$-cyclic since $L^2(\mathbb{R}^d) = \bigoplus_{m\in\mathbb{Z}^d} L^2(\mathcal{Q}_m)$. Moreover, it is wandering under the action of $\mathfrak{S}_T$ since the intersection $\mathcal{Q}_0 \cap \mathcal{Q}_m$ has zero measure for every $m \neq 0$. The cardinality of the wandering system is $\aleph_0$. Proposition 5.2 assures that $\mathfrak{S}_T$ is a $\mathbb{Z}^d$-algebra. Moreover, as a consequence of Proposition 5.8, the Gel'fand spectrum of $\mathfrak{S}_T$ is homeomorphic to the d-dimensional torus $\mathbb{T}^d$ and the normalized basic measure is the Haar measure dz on $\mathbb{T}^d$. ◀▷

Example 5.4 (*Mathieu-like Hamiltonians, part two*). The unital commutative C^*-algebra $\mathfrak{S}^q_{\mathrm{M}} \subset \mathscr{B}(L^2(\mathbb{T}))$ defined in Example 2.3 is generated by a unitary faithful representation of the group $\mathbb{Z}$ on the Hilbert space $L^2(\mathbb{T})$. Indeed, the map $\mathbb{Z} \ni k \mapsto \mathfrak{w}^k \in \mathscr{U}(L^2(\mathbb{T}))$ is an injective group homomorphism. The set of vectors $\{e_0,\ldots,e_{q-1}\} \subset L^2(\mathbb{T})$ shows that the C^*-algebra $\mathfrak{S}^q_{\mathrm{M}}$ has the wandering property. In this case the cardinality of the wandering system is q. Proposition 5.2 assures that $\mathfrak{S}^q_{\mathrm{M}}$ is a $\mathbb{Z}$-algebra. Moreover, Proposition 5.8 will show that the Gel'fand spectrum of $\mathfrak{S}^q_{\mathrm{M}}$ is homeomorphic to the 1-dimensional torus $\mathbb{T}$ and the normalized basic measure on the spectrum coincide with the Haar measure dz on $\mathbb{T}$. The first claim agrees with the fact that the Gel'fand spectrum of $\mathfrak{S}^q_{\mathrm{M}}$ coincides with the (Hilbert space) spectrum of $\mathfrak{w}$, the generator of the C^*-algebra, and $\sigma(\mathfrak{w}) = \mathbb{T}$. The claim about the basic measure agrees with the fact that the vector e_0 is cyclic for the commutant of $\mathfrak{S}^q_{\mathrm{M}}$ (which is the von Neumann algebra generated by $\mathfrak{A}^{p/q}_{\mathrm{M}}$). Indeed, a general result (see Appendix A) assures that the spectral measure μ_{e_0,e_0} provides the basic measure. To determine μ_{e_0,e_0} let $\mathcal{P}(\mathfrak{w}) := \sum_{k\in\mathbb{Z}} \alpha_k \mathfrak{w}^k$ be a generic element of $\mathfrak{S}^q_{\mathrm{M}}$. From the definition of spectral measure it follows

$$\alpha_0 = (e_0; \mathcal{P}(\mathfrak{w}) e_0) = \int_{\mathbb{T}} \mathcal{P}(z)\ d\mu_{e_0,e_0}(z) = \sum_{k\in\mathbb{Z}} \alpha_k \int_0^{2\pi} \mathrm{e}^{ikt}\ d\widetilde{\mu}_{e_0,e_0}(t). \tag{14}$$

where the measure $\widetilde{\mu}_{e_0,e_0}$ is related to μ_{e_0,e_0} by the change of variables $\mathbb{T} \ni z \mapsto t \in [0, 2\pi)$ defined by $z := \mathrm{e}^{it}$ according to Remark 2.4. Equation (14) implies that $\widetilde{\mu}_{e_0,e_0}$ agrees with $dt/2\pi$ on $C(\mathbb{T})$, namely the basic measure μ_{e_0,e_0} is the Haar normalized measure. ◀▷

It is easy to yield examples of commutative unital C^*-algebras which have the wandering property but which are generated by a family of non unitary or non invertible operators.

Example 5.5. With the notations of Example 2.3 let $\widetilde{\mathfrak{w}}$ the operator defined on the Fourier basis $\{e_n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{T})$ by $\widetilde{\mathfrak{w}}\ e_n = w_{[n]}\ e_{n+q}$ where $[n]$ means n modulo q and $w_{[n]} \in \mathbb{C}$ are complex coefficients. The operator $\widetilde{\mathfrak{w}}$ is a bilateral weighted shift completely characterized by the fundamental weights $w_0,\ldots,w_{q-1}$. The adjoint of $\widetilde{\mathfrak{w}}$ is defined by $\widetilde{\mathfrak{w}}^\dagger\ e_n = \overline{w}_{[n-q]}\ e_{n-q} = \overline{w}_{[n]}\ e_{n-q}$ and an easy computation shows that $\widetilde{\mathfrak{w}}$ is normal, indeed $\widetilde{\mathfrak{w}}^\dagger \widetilde{\mathfrak{w}}\ e_n = \widetilde{\mathfrak{w}} \widetilde{\mathfrak{w}}^\dagger\ e_n = |w_{[n]}|^2 e_n$. If $|w_{[j]}| \neq 1$ for some $j = 0,\ldots,q-1$ then $\widetilde{\mathfrak{w}}$ is not unitary. However, the commutative unital C^*-algebra $\widetilde{\mathfrak{S}}^q_T$ generated by $\widetilde{\mathfrak{w}}$ has the wandering property with respect to the finite system of vectors $\{e_0,\ldots,e_{q-1}\}$. ◀▷

Example 5.6. Let $\mathcal{H} := L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$, $e^{(1)}_n := e_n \oplus 0$ and $e^{(2)}_n := 0 \oplus e_n$ where $\{e_n\}_{n\in\mathbb{Z}}$ is the Fourier basis of $L^2(\mathbb{T})$ according to the notations of Example 2.3. Obviously $\{e^{(1)}_n, e^{(2)}_n\}_{n\in\mathbb{Z}}$ is a basis for $\mathcal{H}$. The operators $\mathfrak{w}^{(1)} := \mathfrak{w} \oplus 0$ and $\mathfrak{w}^{(2)} := 0 \oplus \mathfrak{w}$ are not invertible, are normal and commute. Let $\mathfrak{W}^q$ be the commutative C^*-algebra generated by $\mathfrak{w}^{(1)}$, $\mathfrak{w}^{(2)}$ and their adjoints. It is immediate to check that $\mathfrak{W}^q$ has the wandering property with respect the the finite system of vectors $\{e^{(1)}_n, e^{(2)}_n\}_{n=0,\ldots,q-1}$. ◀▷

In the relevant cases of commutative unital C^*-algebras generated by unitary operators the wandering property provides a useful characterization of the Gel'fand spectrum and the basic measure. We state first the following useful result.

Lemma 5.7. *Let $\mathfrak{S}(\mathbb{G}) \subset \mathscr{B}(\mathcal{H})$ be a $\mathbb{G}$-algebra generated by the unitary representation $\mathbb{G} \ni g \mapsto U_g \in \mathscr{U}(\mathcal{H})$ of the commutative discrete group $\mathbb{G}$ on the separable Hilbert space $\mathcal{H}$. Then the Gel'fand spectrum X of $\mathfrak{S}(\mathbb{G})$ is homeomorphic to the dual group $\widehat{\mathbb{G}}$.*

Proof. Let $\ell^1(\mathbb{G})$ be the set of sequences $c := \{c_g\}_{g\in\mathbb{G}}$ such that $\|c\|_{\ell^1} := \sum_{g\in\mathbb{G}} |c_g| < +\infty$. Equipped with the convolution product $(c * d)_g := \sum_{h\in\mathbb{G}} c_h d_{g-h}$, it becomes a commutative Banach $*$-algebra. The algebraic compatibility of the representation of $\mathbb{G}$ on $\mathcal{H}$ implies that the map $\ell^1(\mathbb{G}) \ni c \overset{F}{\longmapsto} F(c) \in \mathscr{B}(\mathcal{H})$, defined by $F(c) := \sum_{g\in\mathbb{G}} c_g\ {U_g}^{-1}$, is injective. Let $\mathfrak{L}^1(\mathbb{G})$ be the image of $\ell^1(\mathbb{G})$ in $\mathscr{B}(\mathcal{H})$ via F. It is a dense unital $*$-subalgebra of $\mathfrak{S}(\mathbb{G})$. Moreover $F(c)F(d) = F(c * d)$ for all $c, d \in \ell^1(\mathbb{G})$ and $\mathfrak{L}^1(\mathbb{G})$ becomes a Banach unital $*$-algebra isomorphic to $\ell^1(\mathbb{G})$ when is endowed whit the norm $|||F(c)||| := \|c\|_{\ell^1}$. This implies that the Gel'fand spectrum of $\mathfrak{L}^1(\mathbb{G})$ coincides with the Gel'fand spectrum of $\ell^1(\mathbb{G})$ which is exactly the dual group $\widehat{\mathbb{G}}$ (see Remark A.1). Notice that $\|F(c)\|_{\mathscr{B}(\mathcal{H})} \leqslant |||F(c)|||$ for all $c \in \ell^1(\mathbb{C})$. The injection $\mathfrak{L}^1(\mathbb{G}) \hookrightarrow \mathscr{B}(\mathcal{H})$ is a faithful representation of the Banach $*$-algebra $\mathfrak{L}^1(\mathbb{G})$ on the Hilbert space $\mathcal{H}$. Then $\|\cdot\|_{\mathscr{B}(\mathcal{H})}$ provides the *universal C^*-norm* and $\mathfrak{S}(\mathbb{G})$ is the (unique up to isomorphism) *enveloping C^*-algebra* of $\mathfrak{L}^1(\mathbb{G})$ (see [Dix82] Section 2.7). The equality between the Gel'fand spectra of $\mathfrak{S}(\mathbb{G})$ and $\mathfrak{L}^1(\mathbb{G})$ (see [Dix82] Proposition 2.7.5) concludes the proof. ■

Proposition 5.8. *Let $\mathfrak{S}$ be a unital commutative C^*-algebra in a separable Hilbert space $\mathcal{H}$ generated by a family $\{U_1, \ldots, U_N\}$ of unitary operators. Assume the wandering property. Then:*

(i) *the Gel'fand spectrum of $\mathfrak{S}$ is homeomorphic to the N-dimensional torus $\mathbb{T}^N$;*

(ii) *the basic measure of $\mathfrak{S}$ is the normalized Haar measure dz on $\mathbb{T}^N$.*

Proof. The claim (i) follows from Proposition 5.2 which assures that $\mathfrak{S}$ is a $\mathbb{Z}^N$-algebra and Lemma 5.7 which implies that the Gel'fand spectrum of $\mathfrak{S}$ is $\widehat{\mathbb{Z}^N} \simeq \widehat{\mathbb{Z}}^N = \mathbb{T}^N$ (see [Rud62] Examples 1.2.7). To prove (ii) let $\{\psi_k\}_{k\in\mathbb{N}}$ be the wandering system of vectors for $\mathfrak{S}$ and $\mu_k := \mu_{\psi_k,\psi_k}$ the spectral measure defined by ψ_k. The Gel'fand isomorphism agrees with the identification of the generator $U_j \in \mathfrak{S}$ with $z_j \in C(\mathbb{T}^N)$. For all $U^a := U_1^{a_1} \ldots U_N^{a_N}$ it follows that

$$\delta_{a,0} = (\psi_k; U^a \psi_k) = \int_{\mathbb{T}^N} z^a\ d\mu_k(z) := \int_0^{2\pi} \ldots \int_0^{2\pi} z_1^{a_1}(t) \ldots z_N^{a_N}(t)\ d\widetilde{\mu}_k(t), \qquad (15)$$

where the measure $\widetilde{\mu}_k$ is related to μ_k by the change of variables $\mathbb{T}^N \ni z \mapsto t \in [0, 2\pi)^N$ defined by $z := \mathrm{e}^{it}$ according to Remark 2.4. Equation (15) shows that for all k the spectral measure $\widetilde{\mu}_k$ agrees with $dz(t) := {dt_1 \ldots dt_N}/{(2\pi)^N}$. Let A_f be the element of $\mathfrak{S}$ whose image via the Gel'fand isomorphism is the function $f \in C(\mathbb{T}^N)$. Then

$$(U^b\psi_j; A_f U^a \psi_k)_{\mathcal{H}} = \delta_{j,k} (\psi_k; A_f U^{a-b} \psi_k)_{\mathcal{H}} = \int_{\mathbb{T}^N} f(z)\ \delta_{j,k}\ z^{a-b}\ dz.$$

So the spectral measure $\mu_{U^b\psi_j, U^a\psi_k}$ is related to the Haar measure dz by the function $\delta_{j,k}\ z^{a-b}$. Let $\varphi := \sum_{k\in\mathbb{N}, a\in\mathbb{N}^N} \alpha_{a,k}\ U^a\psi_k$ be a generic vector in $\mathcal{H}$ then a direct computation shows that $\mu_{\varphi,\varphi}(z) = h_{\varphi,\varphi}(z)\ dz$, where $h_{\varphi,\varphi}(z) := \sum_{k\in\mathbb{N}} |F_\varphi^{(k)}(z)|^2$ with $F_\varphi^{(k)}(z) := \sum_{a\in\mathbb{N}^N} \alpha_{k,a} z^a$.

Since $F_{\varphi}^{(k)} \in L^2(\mathbb{T}^N)$ it follows that $|F_{\varphi}^{(k)}|^2 \in L^1(\mathbb{T}^N)$. Let $h_{\varphi,\varphi}^{(M)}(z) := \sum_{k=0}^{M} |F_{\varphi}^{(k)}(z)|^2$. Since $h_{\varphi,\varphi}^{(M+1)} \geqslant h_{\varphi,\varphi}^{(M)} \geqslant 0$ and $\int_{\mathbb{T}^N} h_{\varphi,\varphi}^{(M)}(z)\, dz = \sum_{k=0}^{M} \sum_{a\in\mathbb{N}^N} |\alpha_{k,a}|^2 \leqslant \|\varphi\|_{\mathcal{H}}$ for all M, it follows from the monotone convergence theorem (see [RS73] Theorem I.10) that $h_{\varphi,\varphi} \in L^1(\mathbb{T}^N)$. This concludes the proof. ■

Not every commutative C^*-algebra generated by a faithful unitary representation of $\mathbb{Z}^N$ has a wandering system. In this situation, even if the spectrum is still a torus, the basic measure can be inequivalent to the Haar measure.

Example 5.9. Let R_α the unitary operator on $L^2(\mathbb{R}^2)$ which implements a rotation around the origin of the angle α, with $\alpha \notin 2\pi\mathbb{Q}$. Clearly $R_\alpha^N = R_{N\alpha} \neq \mathbb{1}$ for every integer N, hence the commutative unital C^*-algebra $\mathfrak{R}_\alpha$ generated by R_α is a unitary faithful representation of $\mathbb{Z}$. The Gel'fand spectrum of $\mathfrak{R}_\alpha$, which coincides with the spectrum of R_α, is $\mathbb{T}$. Indeed, the vector $\psi_N(\rho,\phi) := \mathrm{e}^{iN\phi} f(\rho)$ (in polar coordinates) is an eigenvector corresponding to the eigenvalue $\mathrm{e}^{iN\alpha}$. The spectrum of R_α is the closure of $\{\mathrm{e}^{iN\alpha} \ : \ N \in \mathbb{Z}\}$, which is $\mathbb{T}$ in view of the irrationality of α. The existence of eigenvectors excludes the existence of a wandering system (see Proposition 5.2). Moreover, since R_α has a point spectrum it follows that the basic measure can not coincide with the Haar measure. Indeed, the spectral measure μ_{ψ_N,ψ_N} corresponding to the eigenvector ψ_N is the *Dirac measure* concentrated in $\mathrm{e}^{iN\alpha}$. ◀▷

6 The generalized Bloch-Floquet transform

The purpose of this section is to provide a general algorithm to construct the direct integral decomposition of a commutative C^*-algebra which appears in the von Neumann's complete spectral theorem. The idea is to generalize the construction of the Bloch-Floquet projections (4) but the problem is to find a consistent reinterpretation of the infinite sum. Maurin's nuclear spectral theorem shows the way to circumvent this obstacle. Bloch-Floquet projections should be reinterpreted as "projectors on an appropriate distributional space". In this approach a relevant role will be played by the wandering property. We will consider a commutative unital C^*-algebra $\mathfrak{S}$ on a separable Hilbert space $\mathcal{H}$ generated by the finite family $\{U_1, U_2, \ldots, U_N\}$ of unitary operators and which admits a wandering system $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$. According to the results of Section 5, $\mathfrak{S}$ is a $\mathbb{Z}^N$-algebra with Gel'fand spectrum $\mathbb{T}^N$ and with the Haar measure dz as basic measure. Further generalizations will be discussed at the end of this section.

Construction of the nuclear space

Consider the orthonormal basis $\{U^a\psi_k\}_{k\in\mathbb{N},a\in\mathbb{Z}^N}$, where $\{\psi_k\}_{k\in\mathbb{N}}$ is the wandering system, and denote as $\mathcal{L} \subset \mathcal{H}$ the family of all finite linear combinations of the vectors of such basis. For every integer $m \geqslant 0$ denote as $\mathcal{H}_m$ the finite dimensional Hilbert space generated by the finite set of vectors $\{U^a\psi_k \ : \ 0 \leqslant k, |a| \leqslant m\}$, where $|a| := |a_1| + \ldots + |a_N|$. Obviously $\mathcal{H}_m \subset \mathcal{L}$. Let denote with D_m the dimension of the subspace $\mathcal{H}_m$. If $\varphi = \sum_{k\in\mathbb{N},a\in\mathbb{Z}^N} \alpha_{k,a} U^a\psi_k$ is a generic element of $\mathcal{H}$ then the formula

$$p_m^2(\varphi) := D_m \sum_{0\leqslant k,|a|\leqslant m} |(U^a\psi_k;\varphi)_{\mathcal{H}}|^2 = D_m \sum_{0\leqslant k,|a|\leqslant m} |\alpha_{k,a}|^2 \qquad (16)$$

defines a of seminorm on $\mathcal{L}$ for every $m \geqslant 0$. From (16) it follows that $p_m \leqslant p_{m+1}$ for all m. The countable family of seminorms $\{p_m\}_{m\in\mathbb{N}}$ provides a locally convex topology for the vector space $\mathcal{L}$. Let denote with Σ the pair $\{\mathcal{L}, \{p_m\}_{m\in\mathbb{N}}\}$, i.e. the vector space $\mathcal{L}$

endowed with the locally convex topology induced by the seminorms (16). Σ is a complete and metrizable (i.e. Fréchet) space (see [Tre67] Chapter 10 Example III). However, for our purposes, we need a topology on $\mathcal{L}$ which is strictly stronger than the metrizable topology induced by the seminorms (16).

The quotient space $\Phi_m := \mathcal{L}/\mathcal{N}_m$, with $\mathcal{N}_m := \{\varphi \in \mathcal{L} \ : \ p_m(\varphi) = 0\}$, is isomorphic to the finite dimensional vector space $\mathcal{H}_m$, hence it is nuclear and Fréchet. This follows immediately observing that the norm p_m on Φ_m coincides, up to the positive constant $\sqrt{D_m}$, with the usual Hilbert norm. Obviously $\Phi_m \subset \Phi_{m+1}$ for all $m \geqslant 0$ and the topology of Φ_m agrees with the topology inherited from Φ_{m+1}, indeed $p_{m+1}|_{\Phi_m} = \sqrt{\frac{D_{m+1}}{D_m}}\, p_m$. We define Φ to be $\bigcup_{m\in\mathbb{N}} \Phi_m$ (which is $\mathcal{L}$ as a set) endowed with the *strict inductive limit topology* which is the stronger topology which makes continuous all the injections $\imath_m : \Phi_m \hookrightarrow \Phi$. The space Φ is called a *LF-space* (according to the definition of [Tre67] Chapter 13) and it is a nuclear space since it is the strict inductive limit of nuclear spaces (see [Tre67] Proposition 50.1). We will say that Φ is the *wandering nuclear space* defined by the $\mathbb{Z}^N$-algebra $\mathfrak{S}$ on the wandering system $\{\psi_k\}_{k\in\mathbb{N}}$.

Proposition 6.1. *The wandering nuclear space Φ defined by the $\mathbb{Z}^N$-algebra $\mathfrak{S}$ on the wandering system $\{\psi_k\}_{k\in\mathbb{N}}$ verifies all the properties stated in Theorem 4.4.*

Proof. A linear map $\jmath : \Phi \to \Psi$, with Ψ is an arbitrary locally convex topological vector space, is continuous if and only if the restriction $\jmath|_{\Phi_m}$ of $\jmath$ to Φ_m is continuous for each $m \geqslant 0$ (see [Tre67] Proposition 13.1). This implies that the canonical embedding $\imath : \Phi \hookrightarrow \mathcal{H}$ is continuous, since its restrictions are linear operators defined on finite dimensional spaces. The linear maps $U^a : \Phi \to \Phi$ for all $a \in \mathbb{N}^N$ are also continuous for the same reason. Finally Φ is norm-dense in $\mathcal{H}$ since as a set it is the dense domain $\mathcal{L}$. ■

The seminorms (16) are continuous with respect the strict inductive limit topology which defines Φ. This means that the strict inductive limit topology which defines Φ is stronger than the topology induced by the seminorms (16) which define the Fréchet space Σ. The space Φ is complete (see [Tre67] Theorem 13.1) but not metrizable (since every Φ_m is closed in the topology of Φ_{m+1}).

The transform

We are now in position to define the *generalized Bloch-Floquet transform* $\mathcal{U}_{\mathfrak{S}}$ for the C^*-algebra $\mathfrak{S}$. The Gel'fand spectrum of $\mathfrak{S}$ is $\mathbb{T}^N$ and the Gel'fand isomorphism associates to the generator U_j the function $z_j \in C(\mathbb{T}^N)$ according to the notation of Remark 2.4. For all $t \in [0, 2\pi)^N$ and for all $\varphi \in \Phi$ we define (formally for the moment) the Bloch-Floquet transform of φ at point t of the torus:

$$\boxed{\Phi \ni \varphi \overset{\mathcal{U}_{\mathfrak{S}}|_t}{\longmapsto} (\mathcal{U}_{\mathfrak{S}}\varphi)(t) := \sum_{a\in\mathbb{Z}^N} z^{-a}(t)\ U^a\varphi} \tag{17}$$

where we posed $z^a(t) := z_1^{a_1}(t)\ldots z_N^{a_N}(t)$ and $U^a := U_1^{a_1}\ldots U_N^{a_N}$. The linear combination which appears in equation (17) suggests that $(\mathcal{U}_{\mathfrak{S}}\varphi)(t)$ is a common generalized eigenvector for the elements of $\mathfrak{S}$, indeed a formal computation shows that

$$U_j(\mathcal{U}_{\mathfrak{S}}\varphi)(t) = z_j(t) \sum_{a\in\mathbb{Z}^N} z_j^{-1}(t) z^{-a}(t) U_j U^a \varphi = \mathrm{e}^{it_j}\ (\mathcal{U}_{\mathfrak{S}}\varphi)(t). \tag{18}$$

Theorem 6.2 (Generalized Bloch-Floquet transform). *Let $\mathfrak{S}$ be a $\mathbb{Z}^N$-algebra in the separable Hilbert space $\mathcal{H}$ with generators $\{U_1, \ldots, U_N\}$ and wandering system $\{\psi_k\}_{k\in\mathbb{N}}$, and let Φ the corresponding nuclear space. Under these assumptions the* generalized Bloch-Floquet transform (17) *defines an injective linear map from the nuclear space Φ into its topological dual Φ^* for every $t \in [0, 2\pi)^N$. More precisely the transform $\mathcal{U}_{\mathfrak{S}}|_t$ maps Φ onto a subspace $\Phi^*(t) \subset \Phi^*$ which is a common generalized eigenspace for the commutative C^*-algebra $\mathfrak{S}$; i.e. equation* (18) *holds true in Φ^*. The map $\mathcal{U}_{\mathfrak{S}}|_t : \Phi \to \Phi^*(t) \subset \Phi^*$ is a continuous linear isomorphism, provided Φ^* is endowed with the $*$-weak topology.*

Proof. We need to verify that the right-hand side of (17) is well defined as linear functional on Φ. A generic vector $\varphi \in \Phi$ is a finite linear combination $\varphi = \sum_{k\in\mathbb{N}}^{\text{fin}} \sum_{b\in\mathbb{Z}^N}^{\text{fin}} \alpha_{k,b}\, U^b \psi_k$ (the complex numbers $\alpha_{k,b}$ are different from zero only for a finite set of the values of the index k and the multiindex b). Let $\phi = \sum_{h\in\mathbb{N}}^{\text{fin}} \sum_{c\in\mathbb{N}^N}^{\text{fin}} \beta_{h,c}\, U^c \psi_h$ be another element in Φ. The linearity of the dual pairing between Φ^* and Φ and the compatibility of the pairing with the Hermitian structure of $\mathcal{H}$ imply

$$\langle (\mathcal{U}_{\mathfrak{S}}\varphi)(t); \phi\rangle := \sum_{k\in\mathbb{N}}^{\text{fin}} \sum_{b,c\in\mathbb{Z}^N}^{\text{fin}} \overline{\alpha}_{k,b}\beta_{k,c} \left(\sum_{a\in\mathbb{Z}^N} z^a(t)\ (U^{a+b}\psi_k; U^c\psi_k)_{\mathcal{H}} \right) \tag{19}$$

where in the right-hand side we used the orthogonality between the spaces generated by ψ_k and ψ_h if $k \neq h$. Without further conditions equation (19) is a finite sum in k, b, c (this is simply a consequence of the fact that φ and ϕ are "test functions") but it is an infinite sum in a which generally does not converge. However, if we use the wandering property of the system $\{\psi_k\}_{k\in\mathbb{N}}$ one has that $(U^{a+b}\psi_k; U^c\psi_k)_{\mathcal{H}} = \delta_{a+b,c}$ and the sum (19) becomes the finite sum

$$\langle (\mathcal{U}_{\mathfrak{S}}\varphi)(t); \phi\rangle := \sum_{k\in\mathbb{N}}^{\text{fin}} \sum_{b,c\in\mathbb{Z}^N}^{\text{fin}} \overline{\alpha}_{k,b}\beta_{k,c}\ z^c(t) z^{-b}(t). \tag{20}$$

Let $C_{\varphi;k} := \sum_{b\in\mathbb{Z}^N}^{\text{fin}} |\alpha_{k,b}|$ and $C_\varphi := \max_{k\in\mathbb{N}}\{C_{\varphi;k}\}$ (which is well defined since the set contains only a finite numbers of non-zero elements). An easy computation shows that

$$|\langle (\mathcal{U}_{\mathfrak{S}}\varphi)(t); \phi\rangle| \leqslant \sum_{k\in\mathbb{N}}^{\text{fin}} C_{\varphi,k} \left(\sum_{c\in\mathbb{Z}^N}^{\text{fin}} |\beta_{k,c}| \right) \leqslant C_\varphi \sum_{k\in\mathbb{N}}^{\text{fin}} \sum_{c\in\mathbb{Z}^N}^{\text{fin}} |\beta_{k,c}|\,.$$

Let $m \geqslant 0$ be the smallest integer such that $\phi \in \Phi_m$. The number of the coefficients $\beta_{k,c}$ different from zero (the number of the non-zero components of ϕ) is smaller than the dimension D_m of Φ_m. Using the Cauchy-Schwarz inequality one has

$$|\langle (\mathcal{U}_{\mathfrak{S}}\varphi)(t); \phi\rangle| \leqslant C_\varphi \sqrt{D_m} \left(\sum_{k\in\mathbb{N}}^{\text{fin}} \sum_{c\in\mathbb{N}^N}^{\text{fin}} |\beta_{k,c}|^2 \right)^{\frac{1}{2}} = C_\varphi\ p_m(\phi). \tag{21}$$

The inequality (21) shows that the linear map $(\mathcal{U}_{\mathfrak{S}}\varphi)(t) : \Phi \to \mathbb{C}$ is continuous when it is restricted to each finite dimensional space Φ_m. Since Φ is endowed with the strict inductive limit topology, this is enough to assure that $(\mathcal{U}_{\mathfrak{S}}\varphi)(t)$ is a continuous linear functional on Φ. Since the (21) does not depend on t it follows that $(\mathcal{U}_{\mathfrak{S}}\varphi)(t) \in \Phi^*$ for all $t \in [0, 2\pi)^N$ and for all $\varphi \in \Phi$. The linearity of the map $\mathcal{U}_{\mathfrak{S}}|_t : \Phi \to \Phi^*$ is immediate and from equation (20) it follows that $(\mathcal{U}_{\mathfrak{S}}\varphi)(t) = 0$ (as functional) implies that $\alpha_{k,b} = 0$ for all k and b, hence $\varphi = 0$. This prove the injectivity. To prove the continuity of the map $\mathcal{U}_{\mathfrak{S}}|_t : \Phi \to \Phi^*$, because the strict inductive topology on Φ, we need only to check the continuity of the maps $\mathcal{U}_{\mathfrak{S}}|_t : \Phi_m \to \Phi^*$ for all $m \geqslant 0$. Since Φ_m is a finite dimensional vector space with norm p_m we need to prove that the norm-convergence of the sequence $\varphi_n \to 0$ in Φ_m implies the $*$-weak convergence $(\mathcal{U}_{\mathfrak{S}}\varphi_n)(t) \to 0$ in Φ^*, i.e. $|\langle(\mathcal{U}_{\mathfrak{S}}\varphi_n)(t);\phi\rangle| \to 0$ for all $\phi \in \Phi$. As inequality (21) suggests, it is enough to show that $C_{\varphi_n} \to 0$. This is true since $\varphi_n := \sum_{0<k,|b|\leqslant m} \alpha_{k,b}^{(n)}\ U^b\psi_k \to 0$ in Φ_m implies $\alpha_{k,b}^{(n)} \to 0$. Finally, since the map $U^{-a} = (U^a)^\dagger$ is continuous on Φ for all $a \in \mathbb{Z}^N$ then $U^a : \Phi^* \to \Phi^*$ defines a continuous map which extends the operator U^a originally defined on $\mathcal{H}$. In this context the equation (18) is meaningful and shows that $\Phi^*(t) := \mathcal{U}_{\mathfrak{S}}|_t(\Phi) \subset \Phi^*$ is a space of common generalized eigenvectors for $\mathfrak{S}$. ■

The decomposition

The wandering system $\{\psi_k\}_{k\in\mathbb{N}}$ generates under the Bloch-Floquet transform a special family of elements of Φ^*, denoted as

$$\boxed{\zeta_k(t) := (\mathcal{U}_{\mathfrak{S}}\psi_k)(t) = \sum_{a\in\mathbb{Z}^N} z^{-a}(t)\ U^a\psi_k \qquad \forall\ k \in \mathbb{N}.} \tag{22}$$

The injectivity of the map $\mathcal{U}_{\mathfrak{S}}$ implies that the functionals $\{\zeta_k(t)\}_{k\in\mathbb{N}}$ are linearly independent for any t. If $\varphi = \sum^{\text{fin}}_{k\in\mathbb{N}}\sum^{\text{fin}}_{b\in\mathbb{Z}^N} \alpha_{k,b}\ U^b\psi_k$ is a generic element in Φ then a simple computation shows that

$$(\mathcal{U}_{\mathfrak{S}}\varphi)(t) = \sum^{\text{fin}}_{k\in\mathbb{N}}\sum^{\text{fin}}_{b\in\mathbb{Z}^N} \alpha_{k,b} \sum_{a\in\mathbb{N}^N} z^{-a}(t)\ U^{a+b}\psi_k = \sum^{\text{fin}}_{k\in\mathbb{N}} f_{\varphi;k}(t)\ \zeta_k(t) \tag{23}$$

where $f_{\varphi;k}(t) := \sum^{\text{fin}}_{b\in\mathbb{Z}^N} \alpha_{k,b}\ z^b(t)$. The equalities in (23) should be interpreted in the sense of "distributions", *i.e.* elements of Φ^*. The functions $f_{\varphi;k} : \mathbb{T}^N \to \mathbb{C}$, for all $k \in \mathbb{N}$, are finite linear combination of continuous functions, hence continuous. Equation (23) shows that any subspace $\Phi^*(t)$ is generated by finite linear combinations of the functionals (22). For all $t \in [0,2\pi)^N$ we denote by $\mathcal{K}(t)$ the space of the elements of the form $\sum_{k\in\mathbb{N}} \alpha_k\ \zeta_k(t)$ with $\{\alpha_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$. This is a Hilbert space with the inner product induced by the isomorphism with $\ell^2(\mathbb{N})$. In other words the inner product is induced by the "formal" conditions $(\zeta_k(t);\zeta_h(t))_t := \delta_{k,h}$. All the Hilbert spaces $\mathcal{K}(t)$ have the same dimension which is the cardinality of the system $\{\psi_k\}_{k\in\mathbb{N}}$.

Proposition 6.3. *For all $t \in [0,2\pi)^N$ the inclusions $\Phi^*(t) \subset \mathcal{K}(t) \subset \Phi^*$ holds true. Moreover the generalized Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}}|_t$ extends to a unitary isomorphism between the Hilbert space $\mathbb{H} \subset \mathcal{H}$ spanned by the orthonormal system $\{\psi_k\}_{k\in\mathbb{N}}$ and the Hilbert space $\mathcal{K}(t) \subset \Phi^*$ spanned by $\{\zeta_k(t)\}_{k\in\mathbb{N}}$ (assumed as orthonormal basis).*

Proof. The first inclusion $\Phi^*(t) \subset \mathcal{K}(t)$ follows from the definition. For the second inclusion we need to prove that $\omega(t) := \sum_{k\in\mathbb{N}} \alpha_k\ \zeta_k(t)$ is a continuous functional if $\{\alpha_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$. Let $\phi = \sum_{0\leqslant h,|c|\leqslant m} \beta_{h,c}\ U^c\psi_h$ be an element of $\Phi_m \subset \Phi$ then, from the sesquilinearity of the dual pairing and the Cauchy-Schwarz inequality it follows that

$$|\langle\omega(t);\phi\rangle|^2 \leqslant \left(\sum_{k\in\mathbb{N}} |\alpha_k|\ |\langle(\mathcal{U}_{\mathfrak{S}}\psi_k)(t);\phi\rangle|\right)^2 \leqslant \|\alpha\|^2_{\ell^2} \sum_{k\in\mathbb{N}} |\langle(\mathcal{U}_{\mathfrak{S}}\psi_k)(t);\phi\rangle|^2 \tag{24}$$

where $\|\alpha\|^2_{\ell^2} := \sum_{k\in\mathbb{N}} |\alpha_k|^2 < \infty$. From equation (19) it is evident that $\langle(\mathcal{U}_{\mathfrak{S}}\psi_k)(t);\phi\rangle = 0$ if $\psi_k \notin \Phi_m$, then equation (21) and $C_{\psi_k} = 1$ imply $|\langle\omega(t);\phi\rangle| \leqslant \|\alpha\|_{\ell^2}\sqrt{m}\ p_m(\phi)$. This inequality shows that $\omega(t)$ is a continuous functional when it is restricted to each subspace Φ_m and, because the strict inductive limit topology, this proves that $\omega(t)$ lies in Φ^*. Let $\omega_n(t) := \sum_{0\leqslant k\leqslant n}\alpha_k\ \zeta_k(t)$. Obviously $\omega_n(t) = (\mathcal{U}_{\mathfrak{S}}\varphi_n)(t) \in \Phi^*(t)$ since $\varphi_n := \sum_{0\leqslant k\leqslant n}\alpha_k\ \psi_k \in \Phi$. Moreover the inequality (24) can be used to show that $(\mathcal{U}_{\mathfrak{S}}\varphi_n)(t) \to \omega(t)$ when $n \to \infty$ with respect the $*$-weak topology of Φ^*. This enables us to define $\omega(t) := (\mathcal{U}_{\mathfrak{S}}\varphi)(t)$ for all $\varphi := \sum_{k\in\mathbb{N}}\alpha_k\ \psi_k \in \mathbb{H}$. Obviously the generalized Bloch-Floquet transform acts as a unitary isomorphism between $\mathbb{H}$ and $\mathcal{K}(t)$ with respect to the Hilbert structure induced in $\mathcal{K}(t)$ by the orthonormal basis $\{\zeta_k(t)\}_{k\in\mathbb{N}}$. ■

Theorem 6.4 (Bloch-Floquet spectral decomposition). *Let $\mathfrak{S}$ be a $\mathbb{Z}^N$-algebra in the separable Hilbert space $\mathcal{H}$ with generators $\{U_1,\ldots,U_N\}$, wandering system $\{\psi_k\}_{k\in\mathbb{N}}$ and wandering nuclear space Φ. The generalized Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}}$, defined on Φ by equation* (17)*, induces a direct integral decomposition of the Hilbert space $\mathcal{H}$ which is equivalent (in the sense of Theorem 3.3) to the decomposition of the von Neumann's theorem 3.1. Moreover the spaces $\mathcal{K}(t)$ spanned in Φ^* by the functionals* (22) *provide an explicit realization for the family of common eigenspaces of $\mathfrak{S}$ appearing in Maurin's theorem 4.1.*

Proof. Proposition 5.8 assures that the Gel'fand spectrum of $\mathfrak{S}$ is the N-dimensional torus $\mathbb{T}^N$ and the basic measure agrees with the normalized Haar measure dz. On the field of Hilbert spaces $\prod_{t\in\mathbb{T}^N}\mathcal{K}(t)$ we can introduce a measurable structure by the fundamental family of orthonormal vector fields $\{\zeta_k(\cdot)\}_{k\in\mathbb{N}}$ defined by (22). For all $\varphi\in\Phi$ the generalized Bloch-Floquet transform defines a square integrable vector field $(\mathcal{U}_{\mathfrak{S}}\varphi)(\cdot) \in \mathfrak{K} := \int^{\oplus}_{\mathbb{T}^N}\mathcal{K}(t)\ dz(t)$. Indeed equation (23) shows that $(\mathcal{U}_{\mathfrak{S}}\varphi)(t) \in \mathcal{K}(t)$ for any t and $\|(\mathcal{U}_{\mathfrak{S}}\varphi)(t)\|^2_t = \sum^{\text{fin}}_{k\in\mathbb{N}}|f_{\varphi;k}(t)|^2$ is a continuous function (finite sum of continuous functions) hence integrable on $\mathbb{T}^N$. In particular

$$\|(\mathcal{U}_{\mathfrak{S}}\varphi)(\cdot)\|^2_{\mathfrak{K}} = \int_{\mathbb{T}^N}\|(\mathcal{U}_{\mathfrak{S}}\varphi)(t)\|^2_t\ dz(t) = \sum_{k\in\mathbb{N}}\int_{\mathbb{T}^N}\underbrace{\left(\sum^{\text{fin}}_{b,c\in\mathbb{Z}^N}\overline{\alpha}_{k,b}\alpha_{k,c}\ z^{c-b}(t)\right)}_{=|f_{\varphi;k}(x)|^2}\ dz(t) = \|\varphi\|^2_{\mathcal{H}}.$$

In view of the density of Φ, $\mathcal{U}_{\mathfrak{S}}$ can be extended to an isometry from $\mathcal{H}$ to $\mathfrak{K}$. To prove that $\mathcal{U}_{\mathfrak{S}}$ is unitary we need surjectivity. Any square integrable vector field $\varphi(\cdot)\in\mathfrak{K}$ is uniquely characterized by its expansion on the basis $\{\zeta_k(\cdot)\}_{k\in\mathbb{N}}$, i.e. $\varphi(\cdot) = \sum_{k\in N}\widehat{\varphi}_k(\cdot)\ \zeta_k(\cdot)$ where $\{\widehat{\varphi}_k(t)\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$ for all $t\in[0,2\pi)^N$. The condition $\|\varphi(\cdot)\|^2_{\mathfrak{K}} = \int_{\mathbb{T}^N}\sum_{k\in\mathbb{N}}|\widehat{\varphi}_k(t)|^2\ dz(t) < +\infty$ shows that $\widehat{\varphi}_k \in L^2(\mathbb{T}^N)$ for all $k\in\mathbb{N}$. Let $\widehat{\varphi}_k(t) := \sum_{b\in\mathbb{Z}^N}\alpha_{k,b}z^b(t)$ be the Fourier expansion of $\widehat{\varphi}_k$. Since $\sum_{k\in\mathbb{N}}\sum_{b\in\mathbb{Z}^N}|\alpha_{k,b}|^2 = \sum_{k\in\mathbb{N}}\|\widehat{\varphi}_k\|^2_{L^2(\mathbb{T}^N)} = \|\varphi(\cdot)\|^2_{\mathfrak{K}} < +\infty$ it follows that $\{\alpha_{k,b}\}_{k\in\mathbb{N},b\in\mathbb{Z}^N}$ is an ℓ^2-sequences and the mapping

$$\varphi(\cdot) = \sum_{k\in N}\sum_{b\in\mathbb{Z}^N}\alpha_{k,b}\ z^b(\cdot)\ \zeta_k(\cdot) \quad \overset{\mathcal{U}_{\mathfrak{S}}{}^{-1}}{\longmapsto} \quad \varphi := \sum_{k\in\mathbb{N}}\sum_{b\in\mathbb{Z}^N}\alpha_{k,b}\ U^b\psi_k \tag{25}$$

defines an element $\varphi\in\mathcal{H}$ starting from the vector field $\varphi(\cdot)\in\mathfrak{K}$. It is immediate to check that $\mathcal{U}_{\mathfrak{S}}$ maps φ in $\varphi(\cdot)$, hence $\mathcal{U}_{\mathfrak{S}}$ is injective. If $A_f\in\mathfrak{S}$ is an operator associated with the continuous function $f\in C(\mathbb{T}^N)$ via Gel'fand isomorphism, then $\mathcal{U}_{\mathfrak{S}}A_f\mathcal{U}_{\mathfrak{S}}{}^{-1}\varphi(\cdot) = f(\cdot)\varphi(\cdot)$, i.e. $\mathcal{U}_{\mathfrak{S}}$ maps $A_f\in\mathfrak{S}$ in $M_f(\cdot)\in C(\mathfrak{K})$. This allows us to apply the Theorem 3.3 which assures that the direct integral $\mathfrak{K}$ coincides, up to a decomposable unitary transform, with the spectral decomposition of $\mathfrak{S}$ established in Theorem 3.1. ■

The generalized Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}}$ can be seen as a "computable" realization of the abstract $\mathfrak{S}$-Fourier transform $\mathcal{F}_{\mathfrak{S}}$. From Proposition 6.3 and from some general results on the theory of the direct integral (see [Dix81] Part II, Chapter 1, Section 8) one obtains the following identifications:

$$\mathcal{H} \xrightarrow{\mathcal{U}_{\mathfrak{S}} \ldots \mathcal{U}_{\mathfrak{S}}{}^{-1}} \int_{\mathbb{T}^N}^{\oplus} \mathcal{K}(t)\ dz(t) \simeq \int_{\mathbb{T}^N}^{\oplus} \mathbb{H}\ dz(t) \simeq L^2(\mathbb{T}^N, \mathbb{H}). \tag{26}$$

Since the dimension of $\mathbb{H}$ is the cardinality of the wandering system chosen to define the Bloch-Floquet transform and considering that the Theorem 3.3 assures that the direct integral decomposition is essentially unique (in measure theoretic sense) it follows:

Corollary 6.5. *Any two wandering systems associated to a $\mathbb{Z}^N$-algebra $\mathfrak{S}$ have the same cardinality. Any two wandering systems for $\mathfrak{S}$ are intertwined by a unitary operator which commutes with $\mathfrak{S}$.*

Example 6.6 (*Periodic systems, part three*)**.** In the case of Example 2.2 the Bloch-Floquet transform is the usual one (see [Kuc93], [Pan07])

$$(\mathcal{U}_{\mathfrak{S}_T}\varphi)(y,t) := \sum_{m\in\Gamma} z^{-m}(t) T^m \varphi(y) = \sum_{m\in\Gamma} \mathrm{e}^{-im_1 t_1} \ldots \mathrm{e}^{-im_d t_d} \varphi(y-m),$$

where $m := \sum_{j=1}^{d} m_j \gamma_j$, for all φ in the wandering nuclear space $\Phi \subset L^2(\mathbb{R}^d)$, built according to Proposition 6.1 from any orthonormal basis of $L^2(\mathcal{Q}_0)$. The fiber spaces in the direct integral decomposition are all unitarily equivalent to $L^2(\mathcal{Q}_0)$ hence the Hilbert space decomposition is

$$L^2(\mathbb{R}^d) \xrightarrow{\mathcal{U}_{\mathfrak{S}_T} \ldots \mathcal{U}_{\mathfrak{S}_T}{}^{-1}} \int_{\mathbb{T}^d}^{\oplus} L^2(\mathcal{Q}_0)\ dz(t).$$

◀▷

Example 6.7 (*Mathieu-like Hamiltonians, part three*)**.** In this case the wandering nuclear space Φ is the set of the finite linear combinations of the Fourier basis $\{e_n\}_{n\in\mathbb{Z}}$ and for all $g(\theta) := \sum_{n\in\mathbb{Z}}^{\mathrm{fin}} \alpha_n \mathrm{e}^{in\theta}$ in Φ the Bloch-Floquet transform is

$$(\mathcal{U}_{\mathfrak{S}_{\mathrm{M}}^q}\ g)(\theta,t) := \sum_{m\in\mathbb{Z}} \mathrm{e}^{-imt}\ \mathfrak{w}^m g(\theta) = \sum_{n\in\mathbb{Z}}^{\mathrm{fin}} \alpha_n \left(\sum_{m\in\mathbb{Z}} \mathrm{e}^{i[n\theta + m(q\theta - t)]} \right).$$

The collection $\zeta_k(\cdot;t) \in \Phi^*$, given by $\zeta_k(\theta;t) := \mathrm{e}^{ik\theta} \sum_{m\in\mathbb{Z}} \mathrm{e}^{im(q\theta - t)}$ with $k = 0, \ldots, q-1$, defines a fundamental family of orthonormal fields. The fiber spaces in the direct integral decomposition are all unitarily equivalent to $\mathbb{C}^q$ hence the Hilbert space decomposition is

$$L^2(\mathbb{T}) \xrightarrow{\mathcal{U}_{\mathfrak{S}_{\mathrm{M}}^q} \ldots \mathcal{U}_{\mathfrak{S}_{\mathrm{M}}^q}{}^{-1}} \int_{\mathbb{T}}^{\oplus} \mathbb{C}^q\ dz(t).$$

The images of the generators $\mathfrak{u}$ and $\mathfrak{v}$ under the map $\mathcal{U}_{\mathfrak{S}_{\mathrm{M}}^q} \ldots \mathcal{U}_{\mathfrak{S}_{\mathrm{M}}^q}^{-1}$ are the two t dependent $q \times q$ matrices

$$\mathfrak{u}(t) := \begin{pmatrix} 0 & & & \mathrm{e}^{it} \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix} \qquad \mathfrak{v}(t) := \begin{pmatrix} 1 & & & \\ & \mathrm{e}^{-i2\pi\frac{p}{q}} & & \\ & & \ddots & \\ & & & \mathrm{e}^{-i2\pi\frac{p}{q}(q-1)} \end{pmatrix}.$$

For all $t \in \mathbb{T}$ the matrices $\mathfrak{u}(t)$ and $\mathfrak{v}(t)$ generates a faithful irreducible representation of the C^*-algebra $\mathfrak{A}_{\mathrm{M}}^{p/q}$ on the Hilbert space $\mathbb{C}^q$ (see [Boc01] Theorem 1.9). ◀▷

Remark 6.8 (*Wannier vectors, part one*)**.** Equation (25) in the proof of Theorem 6.4 provides a recipe to invert the unitary map $\mathcal{U}_{\mathfrak{S}} : \mathcal{H} \to \mathfrak{K}$. According to (25) ${\mathcal{U}_{\mathfrak{S}}}^{-1}$ maps the fundamental vector fields $\zeta_k(\cdot)$ into the wandering vectors ψ_k, and it intertwines multiplication by the exponentials $z^a(\cdot)$ with the unitary operators U^a. We will say that ${\mathcal{U}_{\mathfrak{S}}}^{-1}(\varphi(\cdot))$ is the *Wannier vector* associated to the vector field $\varphi(\cdot)$. We will denote by the symbol $\mathfrak{F}$ the set of all the vector fields $\prod_{t\in\mathbb{T}^N}\mathcal{K}(t)$. Let $\Gamma^\infty(\mathfrak{F})$ be the set of square integrable vector fields $\varphi(\cdot)\in\mathfrak{K}$ whose component functions $\widehat{\varphi}_k(\cdot)$ are of class $C^\infty(\mathbb{T}^N)$. Similarly let $\Gamma^\omega(\mathfrak{F})$ be the set of square integrable vector fields which component functions are analytic, i.e. of class $C^\omega(\mathbb{T}^N)$. Obviously $\mathcal{U}_{\mathfrak{S}}(\Phi)\subset\Gamma^\omega(\mathfrak{F})\subset\Gamma^\infty(\mathfrak{F})\subset\mathfrak{K}\subset\mathfrak{F}$. By ordinary Fourier theory, one sees that if $\varphi(\cdot)\in\Gamma^\infty(\mathfrak{F})$ then the sequence of coefficients $\{\alpha_{k,a}\}_{k\in\mathbb{N},a\in\mathbb{Z}^N}$ which defines the component functions $\widehat{\varphi}_k(\cdot)$ decays faster than any polynomial. Similarly if $\varphi(\cdot)\in\Gamma^\omega(\mathfrak{F})$ then the sequence $\{\alpha_{k,a}\}_{k\in\mathbb{N},a\in\mathbb{Z}^N}$ has an exponential decay. In analogy with the ordinary Bloch-Floquet theory [Kuc93], these consideration suggest the name of super-polynomially localized Wannier vectors for the elements of $\Gamma^\infty_{\mathfrak{S}} := {\mathcal{U}_{\mathfrak{S}}}^{-1}(\Gamma^\infty(\mathfrak{F}))$ and *exponentially localized Wannier vectors* for the elements of $\Gamma^\omega_{\mathfrak{S}} := {\mathcal{U}_{\mathfrak{S}}}^{-1}(\Gamma^\omega(\mathfrak{F}))$. ♦◊

Remark 6.9 (*Further generalizations*)**.** It is natural to try to generalize the results of this section to more general situations, like a commutative unital C^*-algebra $\mathfrak{S}$ generated by a finite family $\{A_1,\ldots,A_N\}$ of commuting bounded normal operators and their adjoints. We can also suppose that the generators A_j are invertible and that $\mathfrak{S}$ has a wandering system $\{\psi_k\}_{k\in\mathbb{N}}$. Under these assumptions Proposition 6.1 and Theorem 6.2 hold true with some small changes. In other words one can show that there exists a nuclear space Φ obtained as inductive limit of the finite dimensional Hilbert spaces $\mathcal{H}_m$ spanned by $\{(A^\dagger)^a A^b\psi_k \ : \ 0\leqslant k,|a|,|b|\leqslant m\}$ on which the generalized Bloch-Floquet transform is well defined by

$$\Phi\ni\varphi \xrightarrow{\mathcal{U}_{\mathfrak{S}}|_t} (\mathcal{U}_{\mathfrak{S}}\varphi)(t) := \sum_{a,b\in\mathbb{N}^N}\frac{1}{\overline{f}^a(t)f^b(t)}\ (A^\dagger)^a A^b\varphi \tag{27}$$

where $f_j\in C(X)$ is the continuous function associated to the generator $A_j\in\mathfrak{S}$ by the Gel'fand isomorphism. The invertibility of the generators implies $0<|f_j(x)|\leqslant\|A_j\|_{\mathscr{B}(\mathcal{H})}$. However, to extend also the Bloch-Floquet spectral decomposition (i.e. the content of Theorem 6.4) to this case we need a positive answer to the following, to our knowledge open, question.

Question. Let $\mathfrak{S}$ be a commutative unital C^*-algebra on the separable Hilbert space $\mathcal{H}$ generated by the finite family $\{A_1,\ldots,A_N\}$ of commuting bounded normal operators (and their adjoints). Let $f_j\in C(X)$ the function which represents $A_j\in\mathfrak{S}$ via the Gel'fand isomorphism. Suppose that $\mathfrak{S}$ has a wandering system $\{\psi_1,\psi_2\ldots\}$. Under which conditions there exists a positive regular Borel measure μ on X such that

$$\int_X \overline{f}^b(x)f^a(x)\ d\mu(x) := \int_X \overline{f}_1^{b_1}(x)\ldots\overline{f}_N^{b_N}(x)f_1^{a_1}(x)\ldots f_N^{a_N}(x)\ d\mu(x) = \delta_{a,b} \tag{28}$$

for all $a,b\in\mathbb{N}^N$? Under which conditions the measure μ is also basic for $\mathfrak{S}$? ♦◊

7 Emergent geometry

From a geometric viewpoint, the field of Hilbert spaces $\mathfrak{F} := \prod_{x\in X}\mathcal{H}(x)$ can be regarded as a pseudo vector-bundle $\mathscr{E}\xrightarrow{\pi}X$, where

$$\mathscr{E} := \bigsqcup_{x\in X}\mathcal{H}(x) \tag{29}$$

is the disjoint union of the Hilbert spaces $\mathcal{H}(x)$. The use of the prefix "pseudo" refers to the fact that more ingredients are needed to turn $\mathscr{E} \xrightarrow{\pi} X$ into a vector bundle. First of all, the map π must be continuous, which requires a topology on $\mathscr{E}$. As a first attempt, one might consider $\mathscr{E} \xrightarrow{\pi} X$ as a sub-bundle of the trivial vector bundle $X \times \Phi^* \xrightarrow{\pi} X$, equipped with the topology induced by the inclusion, so that $\mathscr{E} \xrightarrow{\pi} X$ becomes a topological bundle whose fibers are Hilbert spaces. However, nothing ensures that the Hilbert space topology defined fiberwise is compatible with the topology of $\mathscr{E}$, a necessary condition to have a meaningful topological theory.

Geometric vs. analytic viewpoint

We begin our analysis with the definition of topological fibration of Hilbert spaces. Following [FD88] (Chapter II, Section 13) we have

Definition 7.1 (Geometric viewpoint: Hilbert bundle). *A* Hilbert bundle *is the datum of a topological Hausdorff spaces $\mathscr{E}$ (the* total space*) a compact Hausdorff space X (the* base space*) and a map $\mathscr{E} \xrightarrow{\pi} X$ (the* canonical projection*) which is a continuous* open *surjection such that:*

a) *for all $x \in X$ the fiber $\pi^{-1}(x) \subset \mathscr{E}$ is a Hilbert space;*

b) *the application $\mathscr{E} \ni p \longmapsto \|p\| \in \mathbb{C}$ is continuous;*

c) *the operation $+$ is continuous as a function on $\mathscr{S} := \{(p,s) \in \mathscr{E} \times \mathscr{E} \ : \ \pi(p) = \pi(s)\}$ to $\mathscr{E}$;*

d) *for each $\lambda \in \mathbb{C}$ the map $\mathscr{E} \ni p \longmapsto \lambda\, p \in \mathscr{E}$ is continuous;*

e) *for each $x \in X$, the collection of all subset of $\mathscr{E}$ of the form $\mathscr{U}(O, x, \varepsilon) := \{p \in \mathscr{E} \ : \ \pi(p) \in O, \quad \|p\| < \varepsilon\}$, where O is a neighborhood of x and $\varepsilon > 0$, is a basis of neighborhoods of $0_x \in \pi^{-1}(x)$ in $\mathscr{E}$.*

We will denote with the short symbol $\mathscr{E}_{\pi,X}$ the Hilbert bundle $\mathscr{E} \xrightarrow{\pi} X$. A *section* of $\mathscr{E}_{\pi,X}$ is a function $\psi : X \to \mathscr{E}$ such that $\pi \circ \psi = \mathrm{id}_X$. We denote by $\Gamma(\mathscr{E}_{\pi,X})$ the set of all *continuous sections* of $\mathscr{E}_{\pi,X}$. As showed in [FD88], from Definition 7.1 it follows that: (i) the scalar multiplication $\mathbb{C} \times \mathscr{E} \ni (\lambda, p) \mapsto \lambda\, p \in \mathscr{E}$ is continuous; (ii) the topology of $\mathscr{E}$ relativized to a fiber $\pi^{-1}(x)$ coincides with the norm topology of $\pi^{-1}(x)$; (iii) the set $\Gamma(\mathscr{E}_{\pi,X})$ has the structure of a (left) $C(X)$-module. The definition of Hilbert bundle includes all the requests which a "formal" fibration as (29) needs to fulfill to be a topological fibration with a topology compatible with the Hilbert structure of the fibers. In this sense the Hilbert bundle is the "geometric object" of our interest.

However, the structure that emerges in a natural way from the Bloch-Floquet decomposition (Theorem 6.4) is more easily understood from the analytic viewpoint. Switching the focus from the total space $\mathscr{E}$ to the space of sections $\mathfrak{F}$, the relevant notion is that of *continuous field of Hilbert spaces*, according to [Dix82] (Section 10.1) or [DD63] (Section 1).

Definition 7.2 (Analytic viewpoint: continuous field of Hilbert spaces). *Let X be a compact Hausdorff space X and $\mathfrak{F} := \prod_{x \in X} \mathcal{H}(x)$ a field of Hilbert spaces. A* continuous structure *on $\mathfrak{F}$ is the datum of a linear subspace $\Gamma \subset \mathfrak{F}$ such that:*

a) for each $x \in X$ the set $\{\sigma(x) \ : \ \sigma(\cdot) \in \Gamma\}$ is dense in $\mathcal{H}(x)$;

b) for any $\sigma(\cdot) \in \Gamma$ the map $X \ni x \mapsto \|\sigma(x)\|_x \in \mathbb{C}$ is continuous;

c) if $\psi(\cdot) \in \mathfrak{F}$ and if for each $\varepsilon > 0$ and each $x_0 \in X$, there is some $\sigma(\cdot) \in \Gamma$ such that $\|\sigma(x) - \psi(x)\|_x < \varepsilon$ on a neighborhood of x_0, then $\psi(\cdot) \in \Gamma$.

We will denote with the short symbol $\mathfrak{F}_{\Gamma,X}$ the field of Hilbert spaces $\mathfrak{F}$ endowed with the continuous structure Γ. The elements of Γ are called *continuous vector fields*. The condition b) may be replaced by the requirement that for any $\sigma(\cdot), \varrho(\cdot) \in \Gamma$, the function $X \ni x \mapsto (\sigma(x); \varrho(x))_x \in \mathbb{C}$ is continuous. Condition c) is called *locally uniform closure*. Locally uniform closure is needed in order that the linear space Γ is stable under multiplication by continuous functions on X. This condition implies that Γ is a *(left) $C(X)$-module*. A *total set* of continuous vector fields for $\mathfrak{F}_{\Gamma,X}$ is a subset $\Lambda \subset \Gamma$ such that $\Lambda(x) := \{\sigma(x) \ : \ \sigma(\cdot) \in \Lambda\}$ is dense in $\mathcal{H}(x)$ for all $x \in X$. The continuous field of Hilbert spaces is said to be *separable* if it has a countable total set of continuous vector fields.

The link between the notion of continuous field of Hilbert spaces and that of Hilbert bundle is very strong, as showed by the following result.

Proposition 7.3 (Equivalence between geometric and analytic viewpoint [DD63] [FD88]). *Let $\mathfrak{F}_{\Gamma,X}$ be a continuous field of Hilbert spaces over the compact Hausdorff space X. Let $\mathscr{E}(\mathfrak{F}) := \bigsqcup_{x\in X} \mathcal{H}(x)$ be the disjoint union of the Hilbert spaces $\mathcal{H}(x)$ and π the canonical surjection of $\mathscr{E}(\mathfrak{F})$ in X. Then there exists a unique topology $\mathfrak{T}$ on $\mathscr{E}(\mathfrak{F})$ making $\mathscr{E}(\mathfrak{F}) \xrightarrow{\pi} X$ a Hilbert bundle over X such that all the continuous vector fields in $\mathfrak{F}$ are continuous sections of $\mathscr{E}(\mathfrak{F})$. Moreover, assumed the compactness of X, it follows that every Hilbert bundle comes from a continuous field of Hilbert spaces.*

Proof. For the details of the proof, see [DD63] (Section 2) or [FD88] (Chapter II, Theorem 13.18). One has to equip the set $\mathscr{E}(\mathfrak{F}) := \bigsqcup_{x\in X} \mathcal{H}(x)$ with a topology which satisfies the axioms of Definition 7.1. Such a topology $\mathfrak{T}$ is generated by the basis of open sets whose elements are the tubular neighborhoods $\mathscr{U}(O, \sigma, \varepsilon) := \{p \in \mathscr{E}(\mathfrak{F}) \ : \ \pi(p) \in O, \ \ \|p - \sigma(\pi(p))\| < \varepsilon\}$ for all open sets $O \subseteq X$, all continuous vector fields $\sigma(\cdot) \in \Gamma$ and all positive number $\varepsilon > 0$. Since $\pi(\mathscr{U}(O, \sigma, \varepsilon)) = O$ it is clear that with respect to the topology $\mathfrak{T}$ the map $\pi : \mathscr{E}(\mathfrak{F}) \to X$ is a continuous open surjection. The topology induced by $\mathfrak{T}$ on $\mathcal{H}(x)$ is equivalent to the norm-topology of $\mathcal{H}(x)$. Any vector field $\sigma(\cdot) \in \mathfrak{F}$ can be seen as a map $\sigma : X \to \mathscr{E}(\mathfrak{F})$ such that $\pi \circ \sigma = \mathrm{id}_X$, i.e. it is a section of $\mathscr{E}(\mathfrak{F})$. It follows that $\sigma(\cdot) \in \Gamma$ if and only if σ is a continuous section.

Conversely, let $\mathscr{E}_{\pi,X}$ be a Hilbert bundle over the compact Hausdorff space X and let $\Gamma(\mathscr{E}_{\pi,X})$ the set of its continuous section. Let $\mathfrak{F}(\mathscr{E}) := \prod_{x\in X} \pi^{-1}(x)$ be the field of Hilbert spaces associated to the bundle $\mathscr{E}_{\pi,X}$. The compactness of X assures that $\mathscr{E}_{\pi,X}$ has *enough continuous sections*, i.e. $\{\sigma(x) \ : \ \sigma \in \Gamma(\mathscr{E}_{\pi,X})\} = \pi^{-1}(x) =: \mathcal{H}(x)$ (Douady-Dal Soglio-Hérault theorem, see [FD88] Appendix C). For all $\sigma \in \Gamma(\mathscr{E}_{\pi,X})$ the map $X \ni x \mapsto \|\sigma(x)\|_x$ is continuous since it is composition of continuous functions. Finally the the family $\Gamma(\mathscr{E}_{\pi,X})$ fulfiels the locally uniform closure property (see [FD88] Chapter II, Corollary 13.13). This proves that the set of continuous sections $\Gamma(\mathscr{E}_{\pi,X})$ defines a continuous structure on the field of Hilbert spaces $\mathfrak{F}(\mathscr{E})$. ■

We will say that the set $\mathscr{E}(\mathfrak{F})$ endowed with the topology $\mathfrak{T}$ and the canonical surjection π is the Hilbert bundle *associated* to the continuous structure Γ of $\mathfrak{F}$.

Triviality, local triviality and vector bundle structure

A Hilbert bundle is a generalization of a (infinite dimensional) vector bundle, in the sense that some other extra conditions are needed in order to turn it into a genuine vector bundle.

For the axioms of vector bundle we refer to [Lan85]. The most relevant missing condition, is the *local triviality property.*

Two Hilbert bundles $\mathscr{E}_{\pi,X}$ and $\mathscr{F}_{\tau,X}$ over the same base space X are said to be (isometrically) isomorphic if there exists a homeomorphism $\Theta : \mathscr{E} \to \mathscr{F}$ such that a) $\tau \circ \Theta = \pi$, b) $\Theta_x := \Theta|_{\pi^{-1}(x)}$ is a unitary map from the Hilbert space $\pi^{-1}(x)$ to the Hilbert space $\tau^{-1}(x)$. From the definition it follows that if the Hilbert bundles $\mathscr{E}_{\pi,X}$ and $\mathscr{F}_{\tau,X}$ are isomorphic then the map $\Gamma(\mathscr{E}_{\pi,X}) \ni \sigma \mapsto \Theta \circ \sigma \in \Gamma(\mathscr{F}_{\tau,X})$ is one to one. A Hilbert bundle is said to be *trivial* if it is isomorphic to the *constant* Hilbert bundle $X \times \mathbb{H} \to X$ where $\mathbb{H}$ is a fixed Hilbert space. It is called *locally trivial* if for every $x \in X$ there is a neighborhood O of x such that the *reduced Hilbert bundles* $\mathscr{E}|_O := \{(x,p) \in O \times \mathscr{E} \;:\; \pi(p) = x\} = \pi^{-1}(O)$ is isomorphic to the constant Hilbert bundle $O \times \mathbb{H} \to O$. Two continuous fields of Hilbert spaces $\mathfrak{F}_{\Gamma,X}$ and $\mathfrak{G}_{\Delta,X}$ over the same space X are said to be (isometrically) isomorphic if the associated Hilbert bundles $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ and $\mathscr{E}(\mathfrak{G}_{\Delta,X})$ are isomorphic. A continuous field of Hilbert spaces $\mathfrak{F}_{\Gamma,X}$ is said to be trivial (resp. locally trivial) if $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ is trivial (resp. locally trivial).

Proposition 7.4 ([FD88] [DD63]). *Let $\mathfrak{F}_{\Gamma,X}$ be a continuous field of Hilbert spaces over the compact Hausdorff space X and $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ the associated Hilbert bundle. Then:*

(i) *if $\mathfrak{F}_{\Gamma,X}$ is separable and X is second-countable (or equivalently metrizable) then the topology defined on the total space $\mathscr{E}(\mathfrak{F})$ is second-countable;*

(ii) *if* dim $\mathcal{H}(x) = \aleph_0$ *for all $x \in X$ and if X is a finite dimensional manifold then the Hilbert bundle $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ is trivial;*

(iii) *if* dim $\mathcal{H}(x) = q < +\infty$ *for all $x \in X$ then the Hilbert bundle $\mathscr{E}(\mathfrak{F}_{\Gamma,X})$ is a Hermitian* vector bundle *with* typical fiber $\mathbb{C}^q$.

Proof. For the proof of (i) one can see [FD88] (Chapter II, Proposition 13.21). The proof of (ii) is in [DD63] (Theorem 5). We will give a sketch of the proof of (iii). First of all, we remember that to prove that a Hilbert bundle is a vector bundle we need to prove the local triviality and the continuity of the transition functions (see [Lan85] Chapter III). However, if the fibers are finite dimensional then the continuity of the transition functions follows from the existence of the local trivializations (see [Lan85] Chapter III, Proposition 1). Let $\mathfrak{F} := \prod_{x\in X} \mathcal{H}(x)$ with dim $\mathcal{H}(x) = q$ for all $x \in X$ and $\{\sigma_1(\cdot), \ldots, \sigma_q(\cdot)\} \subset \Gamma$ such that for a fixed $x_0 \in X$ the collection $\{\sigma_1(x_0), \ldots, \sigma_q(x_0)\}$ is a basis for $\mathcal{H}(x_0)$. We show that $\{\sigma_1(x), \ldots, \sigma_q(x)\}$ is a basis for $\mathcal{H}(x)$ for all x in a suitable neighborhood of x_0. The function $\wp : X \times \mathbb{C}^q \to [0, +\infty)$ defined by $\wp(x, \alpha_1, \ldots, \alpha_q) := |\alpha| \; \| \sum_{j=1}^q \frac{\alpha_j}{|\alpha|} \sigma_j(x) \|_x$, with $|\alpha|^2 := \sum_{j=1}^q |\alpha_j|^2$, is continuous (composition of continuous function). Then the function $\kappa : X \to [0, +\infty)$ defined by $\kappa(x) := \inf_{|\alpha|=1} \wp(x, \alpha)$ is also continuous since the unit sphere in $\mathbb{C}^q$ is compact. Moreover $\kappa(x_0) > 0$. Since $\{\sigma_1(x), \ldots, \sigma_q(x)\}$ are linearly independent if and only if $\kappa(x) > 0$ it follows that the vectors are linearly independent in a suitable neighborhood O_{x_0} of x_0. In O_{x_0} we can use the Gram-Schmidt formula to obtain a local set of orthonormal continuous vector field $\{\widetilde{\sigma}_1(\cdot), \ldots, \widetilde{\sigma}_q(\cdot)\}$. This local frame enables us to define a map $h_{x_0} : \pi^{-1}(O_{x_0}) \to O_{x_0} \times \mathbb{C}^q$ by $h_{x_0}(p) := (\pi(p), \alpha_1, \ldots, \alpha_q)$ with $\pi(p) = x \in O_{x_0}$ and $p = \sum_{j=1}^q \alpha_j \widetilde{\sigma}_j(x)$. The map h_{x_0} is an homomorphism and for each $x \in O_{x_0}$ is a linear isomorphism between $\mathcal{H}(x)$ and $\mathbb{C}^q$. The collection $\{O_{x_0}\}_{x_0 \in X}$ is a open covering, so we can select by the compactness of X a finite covering $\{O_1, \ldots, O_\ell\}$. The family $\{(O_j, h_j)\}_{j=1,\ldots,\ell}$ is a finite trivializing atlas for the bundle. ■

Algebraic viewpoint

Roughly speaking a continuous field of Hilbert spaces is an "analytic object" while a Hilbert bundle is a "geometric object". There is also a third point of view which is of algebraic nature. We introduce an "algebraic object" which encodes all the relevant properties of the set of continuous vector fields (or continuous sections).

Definition 7.5 (Algebraic viewpoint: Hilbert module)**.** *A* (left) pre-C^*-module *over a commutative unital C^*-algebra $\mathcal{A}$ is a complex vector space Ω that is also a (left) $\mathcal{A}$-module with a pairing $\{\cdot;\cdot\} : \Omega \times \Omega \to \mathcal{A}$ satisfying, for $\sigma, \varrho, \varsigma \in \Omega$ and for $a \in \mathcal{A}$ the following requirements:*

a) $\{\sigma; \varrho + \varsigma\} = \{\sigma; \varrho\} + \{\sigma; \varsigma\}$;

b) $\{\sigma; a\varrho\} = a\{\sigma; \varrho\}$;

c) $\{\sigma; \varrho\}^* = \{\varrho; \sigma\}$;

d) $\{\sigma; \sigma\} > 0$ *if* $\sigma \neq 0$.

The map $|||\cdot||| : \Omega \to [0, +\infty)$ defined by $|||\sigma||| := \sqrt{\|\{\sigma;\sigma\}\|_{\mathcal{A}}}$ is a norm in Ω. The completion $\Gamma(\Omega)$ of Ω with respect the norm $|||\cdot|||$ is called (left) C^*-module *or* Hilbert module *over $\mathcal{A}$.*

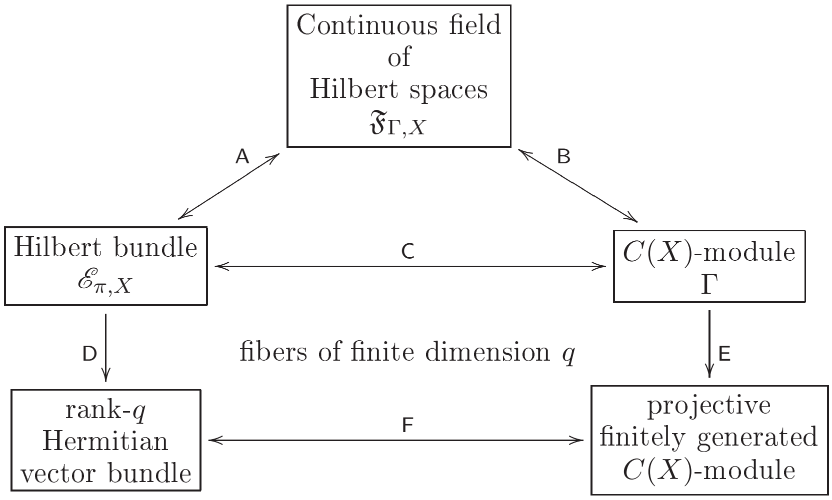
Proposition 7.6 (Equivalence between algebraic and analytic viewpoint [DD63])**.** *Let $\mathfrak{F}_{\Gamma,X}$ be a continuous field of Hilbert spaces over the compact Hausdorff space X. The set of continuous vector fields Γ has the structure of a Hilbert module over $C(X)$. Conversely any Hilbert module over $C(X)$ defines a continuous field of Hilbert spaces. This correspondence is one-to-one.*

Proof. For the details of the proof, see [DD63] (Section 3). To prove the first part of the statement we observe that for all pairs of continuous vector fields $\sigma(\cdot), \varrho(\cdot) \in \Gamma$ the pairing $\{\cdot;\cdot\} : \Gamma \times \Gamma \to C(X)$ defined fiberwise by the inner product, i.e. by posing $\{\sigma; \varrho\}(x) := (\sigma(x); \varrho(x))_x$, satisfies Definition 7.5. The norm is defined by $|||\sigma||| := \sup_{x\in X} \|\sigma(x)\|_x$ and Γ is closed with respect to this norm in view of the property of locally uniform closure.

Conversely let Ω a C^*-module over $C(X)$. For all $x \in X$ define a pre-Hilbert structure on Ω by $(\sigma; \varrho)_x := \{\sigma; \varrho\}(x)$. The set $\mathcal{I}_x := \{\sigma \in \Omega \ : \ \{\sigma; \sigma\}(x) = 0\}$ is a linear subspace of Ξ. On the quotient space $\Omega/\mathcal{I}_x$ the inner product $(\ ;\ )_x$ is a positive definite sesqulinear form and we denote by $\mathcal{H}(x)$ the related Hilbert space. The collection $\mathcal{H}(x)$ defines a field of Hilbert spaces $\mathfrak{F}(\Omega)$. For all $\sigma \in \Omega$ the canonical projection $\Omega \ni \sigma \xmapsto{\jmath_x} \sigma(x) \ni \Omega/\mathcal{I}_x$ defines a vector field $\sigma(\cdot) \in \mathfrak{F}(\Omega)$. It is easy to check that the map $\Omega \ni \sigma \xmapsto{\Gamma} \sigma(\cdot) \ni \mathfrak{F}(\Omega)$ is injective. We denote by $\Gamma(\Omega)$ the image of Ω in $\mathfrak{F}(\Omega)$. The family $\Gamma(\Omega)$ defines a continuous structure on $\mathfrak{F}(\Omega)$. Indeed $\{\sigma(x) \ : \ \sigma(\cdot) \in \Gamma(\Omega)\} = \Omega/\mathcal{I}_x$ which is dense in $\mathcal{H}(x)$ and $\|\sigma(x)\|_x^2 = \{\sigma; \sigma\}(x)$ is continuous. Finally locally uniform closure of $\Gamma(\Omega)$ follows from the closure of Ω with respect the norm $|||\sigma||| := \sup_{x\in X} \sqrt{\{\sigma;\sigma\}(x)}$ and the existence of a partition of the unit subordinate to a finite cover of X (since X is compact). ■

The Hilbert bundle emerging from the Bloch-Floquet decomposition

We are now in position to provide a complete answer to questions (II) and (III) in Section 1. Before proceeding with our analysis, it is useful to summarize in the following diagram the relations between the algebraic, the analytic and the geometric descriptions.



(30)

Arrows A and B summarize the content of Propositions 7.3 and 7.6 respectively, arrow D corresponds to point (iii) of Proposition 7.4, and arrow E follows by Proposition 53 in [Lan97]. Arrow F corresponds to the remarkable Serre-Swan Theorem (see [Lan97] Proposition 21), so arrow C can be interpreted as a generalization of the Serre-Swan Theorem.

Coming back to our original problem, let $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ be a physical frame with $\mathcal{H}$ a separable Hilbert space and $\mathfrak{S}$ a $\mathbb{Z}^N$-algebra with generators $\{U_1, \dots, U_N\}$ and wandering system $\{\psi_k\}_{k\in\mathbb{N}}$. The Bloch-Floquet decomposition (Theorem 6.4) ensures the existence of a unitary map $\mathcal{U}_\mathfrak{S}$, which maps $\mathcal{H}$ into the direct integral $\mathfrak{K} := \int_{\mathbb{T}^N}^{\oplus} \mathcal{K}(t)\ dz(t)$. Let $\mathfrak{F} := \prod_{t\in\mathbb{T}^N} \mathcal{K}(t)$ be the field of Hilbert spaces obtained from the collection of the spaces $\mathcal{K}(t)$ appearing in the Bloch-Floquet decomposition. The collection of vector fields $\mathfrak{K}$ is a subset of $\mathfrak{F}$ which has the structure of a Hilbert space and whose elements can be seen as L^2-sections of a "pseudo-Hilbert bundle" $\mathscr{E}(\mathfrak{F}) := \bigsqcup_{t\in\mathbb{T}^N} \mathcal{H}(t)$. This justifies the use of the notation $\mathfrak{K} = \Gamma_{L^2}(\mathscr{E})$. To answer question II in Section 1 we need to know how to select a priori a continuous structure $\Gamma \subset \mathfrak{K}$ for the field of Hilbert spaces $\mathfrak{F}$. In view of Proposition 7.3, this procedure is equivalent to select a priori the family of the continuous section $\Gamma(\mathscr{E})$ of the Hilbert bundle $\mathscr{E}$ inside the Hilbert space of the L^2-sections $\Gamma_{L^2}(\mathscr{E})$. We can use the generalized Bloch-Floquet transform to push back this problem at the level of the original Hilbert space $\mathcal{H}$ and to adopt the algebraic viewpoint. With this change of perspective the new, but equivalent, question which we need to answer is: does the physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$ select a Hilbert module over $C(\mathbb{T}^N)$ inside the Hilbert space $\mathcal{H}$? Generalizing an idea of [Gru01], we can use the transform $\mathcal{U}_\mathfrak{S}$ and the notion of wandering nuclear space Φ to provide a positive answer. The core of our analysis is the following result.

Proposition 7.7. *Let $\mathfrak{S}$ be a $\mathbb{Z}^N$-algebra in the separable Hilbert space $\mathcal{H}$ with generators $\{U_1, \dots, U_N\}$, wandering system $\{\psi_k\}_{k\in\mathbb{N}}$ and wandering nuclear space Φ. Let $\mathfrak{K}$ be the direct integral defined by the Bloch-Floquet transform $\mathcal{U}_\mathfrak{S} : \mathcal{H} \to \mathfrak{K}$. Then the Bloch-Floquet transform endowes Φ with the structure of a (left) pre-C^*-module over $C(\mathbb{T}^N)$. Let $\Gamma_\mathfrak{S}$ be the completion of Φ with respect the C^*-module norm. Then $\Gamma_\mathfrak{S}$ is a Hilbert module over $C(\mathbb{T}^N)$ such that $\Gamma_\mathfrak{S} \subset \mathcal{H}$.*

Proof. For any pair $\varphi, \phi \in \Phi$ we define the pairing $\{\varphi, \phi\}(t) := ((\mathcal{U}_\mathfrak{S}\varphi)(t); (\mathcal{U}_\mathfrak{S}\phi)(t))_t$. For any fixed t the pairing $\{\cdot;\cdot\}$ is a bilinear map from $\Phi \times \Phi$ to $\mathbb{C}$. Moreover it is easy to check that $\{\varphi, \phi\}(t)$ is a continuous function of t. Indeed $\varphi, \phi \in \Phi$ means that φ and ϕ are finite linear combinations of the vectors $U^a\psi_k$ and from equation (25) and the orthonormality of the fundamental vector fields $\zeta_k(\cdot)$ it follows that $\{\varphi,\phi\}(t)$ consists of a finite linear combination of the exponentials $\mathrm{e}^{it_1},\ldots,\mathrm{e}^{it_N}$. The set Φ is a complex vector space with the structure of a $C(\mathbb{T}^N)$-module defined by $C(\mathbb{T}^N)\times\Phi\ni(f,\varphi)\mapsto A_f\varphi\in\Phi$ where $A_f\in\mathfrak{S}$ is the operator associated to $f\in C(\mathbb{T}^N)$ by the Gel'fand isomorphism. With the module structure and the map $\{\cdot;\cdot\}:\Phi\times\Phi\to C(\mathbb{T}^N)$ the space Φ becomes a (left) pre-C^*-module over $C(\mathbb{T}^N)$.

The Hilbert module $\Gamma_{\mathfrak{S}}$ is defined to be the completion of Φ with respect the norm

$$|||\varphi|||^2 := \sup_{t\in\mathbb{T}^N}\|(\mathcal{U}_{\mathfrak{S}}\varphi)(t)\|_t^2 = \sup_{t\in\mathbb{T}^N}\left(\sum_{k\in\mathbb{N}}^{\text{fin}}|f_{\varphi;k}(t)|^2\right) \tag{31}$$

according to the notation in the proof of Theorem 6.4. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in Φ which is Cauchy with respect the norm $|||\cdot|||$. From (31), the unitarity of $\mathcal{U}_{\mathfrak{S}}$ and the normalization of the Haar measure dz on $\mathbb{T}^N$ it follows that $\|\varphi_n-\varphi_m\|_{\mathcal{H}}\leqslant|||\varphi_n-\varphi_m|||$, hence $\{\varphi_n\}_{n\in\mathbb{N}}$ is also Cauchy with respect the norm $\|\cdot\|_{\mathcal{H}}$, i.e. the limit $\varphi_n\to\varphi$ is an element of $\mathcal{H}$. ■

Remark 7.8 (*Wannier vectors, part two*). The proof of Proposition 7.7 provides more information about the relation of Γ with respect the relevant families of Wannier vectors. According to the notation introduced in Remark 6.8, we can prove that $\Phi\subset\Gamma_{\mathfrak{S}}^{\omega}\subset\Gamma_{\mathfrak{S}}^{\infty}\subset\Gamma_{\mathfrak{S}}\subset\mathcal{H}$. Indeed equation (31) implies the inequality $\sup_{t\in\mathbb{T}^N}|f_{\varphi_n;k}(t)-f_{\varphi_m;k}(t)|\leqslant|||\varphi_n-\varphi_m|||$ which assures that $f_{\varphi_n;k}\to f_{\varphi;k}$ uniformly with $f_{\varphi;k}\in C(\mathbb{T}^N)$ for all $k\in\mathbb{N}$. Then the components of $(\mathcal{U}_{\mathfrak{S}}\varphi)(\cdot)$ with respect the fundamental orthonormal frame $\{\zeta_k(\cdot)\}_{k\in\mathbb{N}}$ are continuous functions and this justifies the chain of inclusions claimed above. ♦◊

Once selected the Hilbert module $\Gamma_{\mathfrak{S}}$, we can use it to define a continuous field of Hilbert spaces as explained in Proposition 7.6. However, it is easy to convince themselves that the abstract construction proposed in Proposition 7.6 is concretely implemented by the generalized Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}}$. Then the set of vector fields $\Gamma(\mathfrak{F}):=\mathcal{U}_{\mathfrak{S}}(\Gamma_{\mathfrak{S}})$ defines a continuous structure on the field of Hilbert spaces $\mathfrak{F}:=\prod_{t\in\mathbb{T}^N}\mathcal{K}(t)$ and, in view of Proposition 7.3, a Hilbert bundle over the base manifold $\mathbb{T}^N$. This Hilbert bundle, which we will denote by $\mathscr{E}_{\mathfrak{S}}$, is the set $\bigsqcup_{t\in\mathbb{T}^N}\mathcal{K}(t)$ equipped by the topology prescribed by the set of the continuous sections $\Gamma(\mathfrak{F})$. The structure of $\mathscr{E}_{\mathfrak{S}}$ depends only on the equivalence class of the physical frame $\{\mathcal{H},\mathfrak{A},\mathfrak{S}\}$ and we will refer to it as the *Bloch-Floquet-Hilbert bundle*.

Theorem 7.9 (Emerging geometric structure). *Let $\mathfrak{S}$ be a $\mathbb{Z}^N$-algebra in the separable Hilbert space $\mathcal{H}$ with generators $\{U_1,\ldots,U_N\}$, wandering system $\{\psi_k\}_{k\in\mathbb{N}}$ and wandering nuclear space Φ. Let $\mathfrak{K}$ be the direct integral defined by the Bloch-Floquet transform $\mathcal{U}_{\mathfrak{S}}:\mathcal{H}\to\mathfrak{K}$ and $\Gamma_{\mathfrak{S}}\subset\mathcal{H}$ the Hilbert module over $C(\mathbb{T}^N)$ defined in Proposition 7.7. Then:*

(i) *the family of vector fields $\mathcal{U}_{\mathfrak{S}}(\Gamma_{\mathfrak{S}})=:\Gamma(\mathfrak{F})$ defines a continuous structure on $\mathfrak{F}$ which realizes the correspondence stated in Proposition 7.6;*

(ii) *the Bloch-Floquet-Hilbert bundle $\mathscr{E}_{\mathfrak{S}}$, defined by $\Gamma(\mathfrak{F})$ according to Proposition 7.3, depends only on the equivalence class of the physical frame $\{\mathcal{H},\mathfrak{A},\mathfrak{S}\}$.*

Proof. To prove (i) let $\mathcal{I}_t:=\{\varphi\in\Phi\ :\ ((\mathcal{U}_{\mathfrak{S}}\varphi)(t);(\mathcal{U}_{\mathfrak{S}}\varphi)(t))_t=0\}$. The space $\Phi/\mathcal{I}_t$ is a pre-Hilbert space with respect the scalar product induced by $\mathcal{U}_{\mathfrak{S}}|_t$. The map $\mathcal{U}_{\mathfrak{S}}|_t:\Phi/\mathcal{I}_t\to\mathcal{K}(t)$ is obviously isometric and so can be extended to a linear isometry from the norm-closure of $\Phi/\mathcal{I}_t$ into $\mathcal{K}(t)$. The map $\mathcal{U}_{\mathfrak{S}}|_t$ is also surjective, indeed $\mathcal{K}(t)$ is generated by the orthonormal basis $\{\zeta_k(t)\}_{k\in\mathbb{N}}$ and $\mathcal{U}_{\mathfrak{S}}|_t^{-1}\zeta_k(t)=\psi_k\in\Phi/\mathcal{I}_t$. Then the fiber Hilbert spaces appearing in the proof of Proposition 7.6 coincide, up to a unitary equivalence, with the fiber Hilbert spaces $\mathcal{K}(t)$ obtained through the Bloch-Floquet decomposition. Moreover the Bloch-Floquet transform acts as the map defined in the proof of Proposition ̊efvecmod, which sends the element of the Hilbert module Φ in the continuous section of $\mathfrak{F}$.

To prove (ii) let $\{\mathcal{H}_1, \mathfrak{A}_1, \mathfrak{S}_1\}$ and $\{\mathcal{H}_2, \mathfrak{A}_2, \mathfrak{S}_2\}$ be two physical frames related by a unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$. If $\mathfrak{S}_1$ is a $\mathbb{Z}^N$-algebra in $\mathcal{H}_1$ then also $\mathfrak{S}_2 = U\mathfrak{S}_1 U^{-1}$ is a $\mathbb{Z}^N$-algebra in $\mathcal{H}_2 = U\mathcal{H}_1$ and if $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_1$ is a wandering system for $\mathfrak{S}_1$ then $\{\widetilde{\psi}_k := U\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{H}_2$ is a wandering system for $\mathfrak{S}_2$ (with the same cardinality). The two wandering nuclear spaces $\Phi_1 \subset \mathcal{H}_1$ and $\Phi_2 \subset \mathcal{H}_2$ are related by $\Phi_2 = U\Phi_1$. Let $\mathcal{U}_{\mathfrak{S}_1} : \mathcal{H}_1 \to \mathfrak{H}_1$ and $\mathcal{U}_{\mathfrak{S}_2} : \mathcal{H}_2 \to \mathfrak{H}_2$ be the two generalized Bloch-Floquet transforms defined by the two equivalent physical frames. From the explicit expression of $\mathcal{U}_{\mathfrak{S}_2}$ and ${\mathcal{U}_{\mathfrak{S}_1}}^{-1}$, and in accordance with Corollary 3.4, one argues that $\mathcal{U}_{\mathfrak{S}_2} \circ U \circ {\mathcal{U}_{\mathfrak{S}_1}}^{-1} =: W(\cdot)$ is a decomposable unitary which is well defined for all t (it is essentially a change of basis on each fiber). Let $\varphi, \phi \in \Phi_1$ then

$$\begin{aligned}\{\varphi;\phi\}_1(t) :&= ((\mathcal{U}_{\mathfrak{S}_1}\varphi)(t);(\mathcal{U}_{\mathfrak{S}_1}\phi)(t))_t = (W(t)(\mathcal{U}_{\mathfrak{S}_1}\varphi)(t);W(t)(\mathcal{U}_{\mathfrak{S}_1}\phi)(t))_t \\ &= ((\mathcal{U}_{\mathfrak{S}_2}U\varphi)(t);(\mathcal{U}_{\mathfrak{S}_2}U\phi)(t))_t = \Big((\mathcal{U}_{\mathfrak{S}_2}\widetilde{\varphi})(t);(\mathcal{U}_{\mathfrak{S}_2}\widetilde{\phi})(t)\Big)_t =: \{\widetilde{\varphi};\widetilde{\phi}\}_2(t)\end{aligned}$$

where $\widetilde{\varphi} := U\varphi$ and $\widetilde{\phi} := U\phi$ are elements of Φ_2. This equation shows that Φ_1 and Φ_2 have the same $C(\mathbb{T}^N)$-module structure and so define the same abstract Hilbert module over $C(\mathbb{T}^N)$. The claim follows from the generalization of the Serre-Swan Theorem summarized by arrow C in (30). ■

Remark 7.10. With a proof similar to that of point (ii) of Theorem 7.9, one deduces also that the Bloch-Floquet-Hilbert bundle $\mathscr{E}_{\mathfrak{S}}$ does not depend on the choice of two unitarily (or antiunitarily) equivalent commutative C^*-algebras $\mathfrak{S}_1$ and $\mathfrak{S}_2$ inside $\mathfrak{A}$. Indeed also in this case the abstract Hilbert module structure induced by the two Bloch-Floquet transforms $\mathcal{U}_{\mathfrak{S}_1}$ and $\mathcal{U}_{\mathfrak{S}_2}$ is the same. ♦◊

Theorem 7.9 provides a complete and satisfactory answer to questions (II) and (III) in Section 1 for the interesting case of a $\mathbb{Z}^N$-algebras $\mathfrak{S}$. At this point, it is natural to deduce more information about the topology of the Bloch-Floquet-Hilbert bundle from the properties of the physical frame $\{\mathcal{H}, \mathfrak{A}, \mathfrak{S}\}$. An interesting property arises from the cardinality of the wandering system which depends only on the physical frame (see Corollary 6.5).

Corollary 7.11. *The Hilbert bundle $\mathscr{E}_{\mathfrak{S}}$ over the torus $\mathbb{T}^N$ defined by the continuous structure $\Gamma(\mathfrak{F})$ is trivial if the cardinality of the wandering system is $\aleph_0$, and is a rank-q Hermitian vector bundle if the cardinality of the wandering system is q. In the latter case the transition functions of the vector bundle can be expressed in terms of the fundamental orthonormal frame $\{\zeta_k(\cdot) := (\mathcal{U}_{\mathfrak{S}}\psi_k)\}_{k=1,\ldots,q}$.*

Proof. The claim follows from Propositions 7.4 and 7.3 jointly with the fact that the dimension of the fiber spaces $\mathcal{K}(t)$ is the cardinality of the wandering system as proved in Proposition 6.3. In the finite dimensional case the fundamental orthonormal frame $\{\zeta_k(\cdot)\}_{k\in\mathbb{N}}$, defined by (22), selects locally a family of frames and so provides the local trivializations for the vector bundle. ■

A Gel'fand theory, joint spectrum and basic measures

Let $\mathfrak{A}$ be a unital (not necessarily commutative) C^*-algebra and $\mathfrak{A}^\times$ the *group of the invertible elements* of $\mathfrak{A}$. The *algebraic spectrum* of $A \in \mathfrak{A}$ is defined to be $\sigma_{\mathfrak{A}}(A) := \{\lambda \in \mathbb{C} : (A - \lambda\mathbb{1}) \notin \mathfrak{A}^\times\}$. If $\mathfrak{A}_0$ is a non unital C^*-algebra and $\imath : \mathfrak{A}_0 \hookrightarrow \mathfrak{A}$ is the canonical embedding of $\mathfrak{A}_0$

in the unital C^*-algebra $\mathfrak{A}$ (see [BR87] Proposition 2.1.5) then one defines $\sigma_{\mathfrak{A}_0}(A) := \sigma_{\mathfrak{A}}(\imath(A))$ for all $A \in \mathfrak{A}_0$. This shows that the notion of spectrum is strongly linked to the existence of the unit. If $\mathfrak{A}$ is unital and $C^*(A) \subset \mathfrak{A}$ is the unital C^*-subalgebra generated algebraically by A, its adjoint $A^\dagger$ and $\mathbb{1}$ (=: A^0 for definition) then $\sigma_{\mathfrak{A}}(A) = \sigma_{C^*(A)}(A)$ (see [BR87] Proposition 2.2.7). As a consequence we have that if $\mathfrak{A} \subset \mathscr{B}(\mathcal{H})$ is a concrete C^*-algebra of operators on the Hilbert space $\mathcal{H}$ and $A \in \mathfrak{A}$ then the algebraic spectrum $\sigma_{\mathfrak{A}}(A)$ agrees with the *Hilbert space spectrum* $\sigma(A) := \{\lambda \in \mathbb{C} \ : \ (A - \lambda\mathbb{1}) \notin \mathrm{GL}(\mathcal{H})\}$ where $\mathrm{GL}(\mathcal{H}) := \mathscr{B}(\mathcal{H})^\times$ is the group of the invertible bounded linear operators on the Hilbert space $\mathcal{H}$.

Let denote with $\mathfrak{S}$ a commutative C^*-algebra. A *character* of $\mathfrak{S}$ is a nonzero homomorphism $x : \mathfrak{S} \to \mathbb{C}$ (also called pure state). The *Gel'fand spectrum* of $\mathfrak{S}$, denoted by $X(\mathfrak{S})$ or simply by X, is the set of all characters of $\mathfrak{S}$. The space X, endowed with the $*$-weak topology (topology of the pointwise convergence on $\mathfrak{S}$) becomes a topological Hausdorff space, which is compact if $\mathfrak{S}$ is unital and only locally compact otherwise (see [BR87] Theorem 2.1.11A). If $\mathfrak{S}$ is *separable* (namely it is generated algebraically by a countable family of commuting elements) then the $*$-weak topology in X is metrizable (see [Bré87] Theorem III.25) and if, in addition, $\mathfrak{S}$ is also unital then X is compact and metrizable which implies (see [Cho66] Proposition 18.3 and Theorem 20.9) that X is second-countable (has a countable basis), separable (has a countable everywhere dense subset) and complete. If $\mathcal{H}$ is a separable Hilbert space then also the C^*-algebra $\mathscr{B}(\mathcal{H})$ is separable. Summarizing, the Gel'fand spectrum of a commutative separable unital C^*-algebra has the structure of a *Polish space* (separable complete metric space).

The *Gel'fand-Naĭmark Theorem* (see [BR87] Section 2.3.5 or [GVF01] Section 1.2 or [Lan97] Section 2.2) states that there is a canonical isomorphism between any commutative unital C^*-algebra $\mathfrak{S}$ and the commutative C^*-algebra $C(X)$ of the continuous complex valued functions on its spectrum endowed with the norm of the uniform convergence. The *Gel'fand isomorphism* $C(X) \ni f \overset{\mathscr{G}}{\mapsto} A_f \in \mathfrak{S}$ maps any continuous f in the unique element A_f which satisfies the relation $f(x) = x(A_f)$ for all $x \in X$. Then we can use the continuous functions on X to "label" the elements of $\mathfrak{S}$. If $\mathfrak{S}_0$ is a non-unital commutative C^*-algebra then the Gel'fand-Naĭmark Theorem proves the isomorphism between $\mathfrak{S}_0$ and the commutative C^*-algebra $C_0(X_0)$ of the continuous complex valued functions vanishing at infinity on the locally compact space X_0 which is the spectrum of $\mathfrak{S}_0$. If $\mathfrak{S}_0 \subset \mathscr{B}(\mathcal{H})$ we define the *multiplier algebra* of $\mathfrak{S}_0$ to be $\mathfrak{S} := \{B \in \mathscr{B}(\mathcal{H}) \ : \ BA = AB \in \mathfrak{S}_0 \quad \forall A \in \mathfrak{S}_0\}$. The multiplier algebra is a unital commutative C^*-subalgebra of $\mathscr{B}(\mathcal{H})$ which contains $\mathfrak{S}_0$ as an *essential ideal*. The Gel'fand spectrum X of $\mathfrak{S}$ corresponds to the *Stone-Čech compactification* of the spectrum X_0 and one can prove that $C(X) \simeq C_{\mathrm{b}}(X_0)$. Thus, via Gel'fand isomorphism, the multiplier algebra $\mathfrak{S}$ can be described as the unital commutative C^*-algebra of the bounded continuous functions on the locally compact space X_0 (see [GVF01] pp. 13-15).

For every $A_f \in \mathfrak{S}$ one has that $\sigma_{\mathfrak{S}}(A_f) = \{f(x) \ : \ x \in X\}$ (se [Hör90] Theorem 3.1.6) then A_f is invertible if and only if $0 < |f(x)| \leqslant \|A_f\|_{\mathfrak{S}}$ for all $x \in X$. We often consider the relevant case when the unital commutative C^*-algebra is *finitely generated*, i.e. when $\mathfrak{S}$ is algebraically generated by a finite family $\{A_1, \ldots, A_N\}$ of commuting normal elements, their adjoints and $\mathbb{1}$ (=: A_j^0 for definition). We will say that the family of generators $\{A_1, \ldots, A_N\}$ is *minimal* if the any C^*-algebra generated by a proper subset of generators is strictly contained in $\mathfrak{S}$.

Let $f_1, \ldots, f_N$ be the continuous functions which label the elements of the generating system. The map $X \ni x \overset{\varpi}{\mapsto} (f_1(x), \ldots, f_N(x)) \in \mathbb{C}^N$ is a homeomorphism of the Gel'fand spectrum X on a compact subset of $\mathbb{C}^N$ called the *joint spectrum* of the generating system $\{A_1, \ldots, A_N\}$ (see [Hör90] Theorem 3.1.15). Then, when $\mathfrak{S}$ is finitely generated, we can identify the Gel'fand spectrum X with its homeomorphic image $\varpi(X)$ (the joint spectrum) which is a compact, generally proper, subset of $\sigma_{\mathfrak{S}}(A_1)\times\ldots\times\sigma_{\mathfrak{S}}(A_N)$. When $\{A_1,\ldots,A_N\}\subset\mathscr{B}(\mathcal{H})$ a necessary and sufficient condition for $\lambda:=(\lambda_1,\ldots,\lambda_N)$ to be in $\varpi(X)$ is that there exists a sequence of normalized vectors $\{\psi_n\}_{n\in\mathbb{N}}$ such that $\|(A_j-\lambda_j)\psi_n\|\to 0$ if $n\to\infty$ for all $j=1,\ldots,N$ (see [Sam91] Proposition 2).

Remark A.1 (*Dual group*)**.** The Gel'fand theory has an interesting application to abelian locally compact groups $\mathbb{G}$. Usually the *dual group* (or character group) $\widehat{\mathbb{G}}$ is defined to be the set of all continuous characters of $\mathbb{G}$, namely the set of all the continuous homomorphism of $\mathbb{G}$ into the circle group $\mathbb{S}^1:=\{z\in\mathbb{C}\ :\ |z|=1\}$. However, to endow $\widehat{\mathbb{G}}$ with a natural topology it is useful to give an equivalent definition of dual group. Since $\mathbb{G}$ is locally compact and abelian there exists a unique (up to a multiplicative constant) *invariant Haar measure* on $\mathbb{G}$ denoted by dg (see [Rud62] Section 1.1). The family of functions $L^1(\mathbb{G})$ (with respect the Haar measure) becomes a commutative Banach $*$-algebra, if multiplication is defined by convolution; it is called the *group algebra* of $\mathbb{G}$. If $\mathbb{G}$ is discrete then $L^1(\mathbb{G})$ is unital otherwise $L^1(\mathbb{G})$ has always an *approximate unit* (see [Rud62] Theorems 1.1.7 and 1.1.8). Every $\chi\in\widehat{\mathbb{G}}$ defines a linear multiplicative functional $\widehat{\chi}$ on $L^1(\mathbb{G})$ by $\widehat{\chi}(f):=\int_{\mathbb{G}}f(g)\chi(-g)\ d\mu(g)$ for all $f\in L^1(\mathbb{G})$ (the Fourier transform). This map defines a one to one correspondence between $\widehat{\mathbb{G}}$ and the Gel'fand spectrum of the algebra $L^1(\mathbb{G})$ (see [Rud62] Theorem 1.2.2). This enables us to consider $\widehat{\mathbb{G}}$ as the Gel'fand spectrum of $L^1(\mathbb{G})$ and when $\widehat{\mathbb{G}}$ is endowed with the usual $*$-weak topology with respect to $L^1(\mathbb{G})$ then it becomes a Hausdorff locally compact space. Moreover $\widehat{\mathbb{G}}$ is compact if $\mathbb{G}$ is discrete and it is discrete when $\mathbb{G}$ is compact (see [Rud62] Theorem 1.2.5). ♦◊

Let X be a compact Polish space and $\mathcal{B}(X)$ the Borel σ-algebra generated by the topology of X. The pair $\{X,\mathcal{B}(X)\}$ is called *standard Borel space.* A mapping $\mu:\mathcal{B}(X)\to[0,+\infty]$ such that: $\mu(\emptyset)=0$, $\mu(X)\leqslant\infty$ which is additive with respect the union of countable families of pairwise disjoint subsets of X is called a *(finite) Borel measure.* If $\mu(X)=1$ then we will said that μ is a *probability* Borel measure. Any Borel measure on a standard Borel space $\{X,\mathcal{B}(X)\}$ is *regular*, i.e. if for all $Y\in\mathcal{B}(X)$ one has that $\mu(Y)=\sup\{\mu(K)\ :\ K\subset Y,\ K\text{ compact}\}$ or $\mu(Y)=\inf\{\mu(O)\ :\ Y\subset O,\ O\text{ open}\}$. Let N be the union of all the open sets $O_\alpha\subset X$ such that $\mu(O)=0$. The closed set $X\setminus N$ is called the *support* of μ. If μ is a regular Borel measure then $\mu(N)=0$ and μ is concentrated on its support. For more details about the measure theory the reader can see [Rud87] (Chapters 1-2) or [RS73] (Section IV.4).

Let $\mathfrak{S}$ be a unital commutative C^*-algebra on the separable Hilbert space $\mathcal{H}$ with Gel'fand spectrum X. For all pairs $\psi,\varphi\in\mathcal{H}$ the mapping $C(X)\ni f\mapsto(\psi;A_f\varphi)_{\mathcal{H}}\in\mathbb{C}$ is a continuous linear functional on $C(X)$ hence the *Riesz-Markov Theorem* (see [Rud87] Theorem 2.14 or [RS73] Theorem IV.18) implies the existence of a unique regular (complex) Borel measure $\mu_{\psi,\varphi}$, with finite total variation, such that $(\psi;A_f\varphi)_{\mathcal{H}}=\int_X f(x)\ d\mu_{\psi,\varphi}(x)$ for all $f\in C(X)$. We will refer to $\mu_{\psi,\varphi}$ as a *spectral measure.* The union of the supports of the (positive) spectral measures $\mu_{\psi,\psi}$ is dense, namely for all open sets $O\subset X$ there exists a $\psi\in\mathcal{H}$ such that $\mu_{\psi,\psi}(O)>0$. A positive measures μ on X is said to be *basic* for the C^*-algebra $\mathfrak{S}$ if: for every $Y\subset X$, Y is locally μ-negligible if and only if Y is locally $\mu_{\psi,\psi}$-negligible for any $\psi\in\mathcal{H}$. From the definition if follows that: (i) if there exists a basic measure μ on X, then every other basic measure is *equivalent* (has the same null sets) to μ; (ii) for all $\psi,\varphi\in\mathcal{H}$ the spectral measure $\mu_{\psi,\varphi}$ is *absolutely continuous* with respect to μ, moreover there exists a unique element $h_{\psi,\varphi}\in L^1(X)$ (the *Radon-Nikodym derivative*) such that $\mu_{\psi,\varphi}=h_{\psi,\varphi}\mu$; (iii) since the union of the supports of the measures $\mu_{\psi,\psi}$ is dense in X, then the support of a basic measure μ is the whole X (see [Dix81] Part I, Chapter 7). The existence of a basic measure for a commutative C^*-algebra $\mathfrak{S} \subset \mathscr{B}(\mathcal{H})$ is generic. Indeed the existence of a basic measure is equivalent to the existence of a cyclic vector ϕ for the commutant $\mathfrak{S}'$ and the basic measure can be chosen to be the spectral measure $\mu_{\phi,\phi}$ (see [Dix81] Part I, Chapter 7, Proposition 3). Since a vector ϕ is cyclic for $\mathfrak{S}'$ if and only if it is separating for the commutative von Neumann algebra $\mathfrak{S}'' \supset \mathfrak{S}$, and since any commutative von Neumann algebra of operators on a separable Hilbert space has a separating vector, it follows that any commutative unital C^*-algebra $\mathfrak{S}$ of operators which acts on a separable Hilbert space has a basic measure carried on its spectrum (see [Dix81] Part I, Chapter 7, Propositions 4).

B Direct integral of Hilbert spaces

General references about the notion of a direct integral of Hilbert spaces can be found in [Dix81] (part II from chapter 1 to chapter 5) or in [Mau68] (chapter I, section 6). In what follows we assume that the pair $\{X, \mathcal{B}(X)\}$ is a standard Borel space and μ a (regular) Borel measure on X. For every $x \in X$ let $\mathcal{H}(x)$ be a Hilbert space with scalar product $(\ ;\ )_x$. The set $\mathfrak{F} := \prod_{x\in X} \mathcal{H}(x)$ (Cartesian product) is called a *field of Hilbert spaces* over X. A *vector field* $\varphi(\cdot)$ is an element of $\mathfrak{F}$, namely a map $X \ni x \mapsto \varphi(x) \in \mathcal{H}(x)$. A countable family $\{\xi_j(\cdot)\ :\ j \in \mathbb{N}\}$ of vector fields is called a *fundamental family* of measurable vector fields if:

a) for all $i, j \in \mathbb{N}$ the functions $X \ni x \mapsto (\xi_i(x); \xi_j(x))_x \in \mathbb{C}$ are measurable;

b) for each $x \in X$ the set $\{\xi_j(x)\ :\ j \in \mathbb{N}\}$ spans the space $\mathcal{H}(x)$.

The field $\mathfrak{F}$ has a *measurable structure* if it has a fundamental family of measurable vector fields. A vector field $\varphi(\cdot) \in \mathfrak{F}$ is said to be *measurable* if all the functions $X \ni x \mapsto (\xi_j(x); \varphi(x))_x \in \mathbb{C}$ are measurable for all $j \in \mathbb{N}$. The set of all measurable vector fields is a linear subspace of $\mathfrak{F}$. By the Gram-Schmidt orthonormalization we can always build a fundamental family of orthonormal measurable fields (see [Dix81] Part II, Chapter 1, Propositions 1 and 4). Such a family will be call a *measurable field of orthonormal frames*. Two fields are said to be equivalent if they are equal μ-almost everywhere on X. The *direct integral* $\mathfrak{H}$ of the Hilbert spaces $\mathcal{H}(x)$ (subordinate to the measurable structure of $\mathfrak{F}$), is the Hilbert space of the equivalence classes of measurable vector fields $\varphi(\cdot) \in \mathfrak{F}$ satisfying

$$\|\varphi(\cdot)\|^2_{\mathfrak{H}} := \int_X \|\varphi(x)\|^2_x \, d\mu(x) < \infty. \tag{32}$$

The scalar product on $\mathfrak{H}$ is defined by

$$\langle \varphi_1(\cdot); \varphi_2(\cdot)\rangle_{\mathfrak{H}} := \int_X (\varphi_1(x); \varphi_2(x))_x \, d\mu(x) < \infty. \tag{33}$$

The Hilbert space $\mathfrak{H}$ is often denoted by the symbol $\int_X^{\oplus} \mathcal{H}(x)\, d\mu(x)$. It is separable if X is separable.

Let ν be a positive measure equivalent to μ. The Radon-Nikodym theorem ensures the existence of a positive $\rho \in L^1(X, \mu)$ with $\frac{1}{\rho} \in L^1(X, \nu)$ such that $\nu = \rho\mu$. Let $\mathfrak{H}$ be the direct integral with respect μ, $\mathfrak{K}$ the direct integral with respect ν and $\varphi(\cdot) \in \mathfrak{H}$. The mapping $\mathfrak{H} \in \varphi(\cdot) \mapsto \varphi'(\cdot) \in \mathfrak{K}$ defined by $\varphi'(x) = \frac{1}{\sqrt{\rho(x)}}\varphi(x)$ for all $x \in X$ is an unitary map of $\mathfrak{H}$ onto $\mathfrak{K}$ and for fixed μ and ν. This isomorphism does not depend on the choice of the representative for ρ and it is called the *canonical rescaling isomorphism*.

A *(bounded) operator field* $A(\cdot)$ is a map $X \ni x \mapsto A(x) \in \mathscr{B}(\mathcal{H}(x))$. It is called measurable if the function $X \ni x \mapsto (\xi_i(x); A(x)\xi_j(x))_x \in \mathbb{C}$ is measurable for all $i, j \in \mathbb{N}$. A measurable operator field is called a *decomposable operator* in the Hilbert space $\mathfrak{H}$. Let $f \in L^\infty(X)$ (with respect the measure μ); then the map $X \ni x \mapsto M_f(x) := f(x)\mathbb{1}_x \in \mathscr{B}(\mathcal{H}(x))$ (with $\mathbb{1}_x$ the identity in $\mathcal{H}(x)$) defines a simple example of decomposable operator called *diagonal operator*. When $f \in C(X)$, the diagonal operator $M_f(\cdot)$ is called a *continuously diagonal operator*. Denote by $C(\mathfrak{H})$ (resp. $L^\infty(\mathfrak{H})$) the set of the continuously diagonal operators (resp. the set of diagonal operators on $\mathfrak{H}$). Suppose that $\mathcal{H}(x) \neq 0$ μ-almost everywhere on X, then the following facts hold true (see [Dix81] Part II, Chapter 2, Section 4): (i) $L^\infty(\mathfrak{H})$ is a commutative von Neumann algebra and the mapping $L^\infty(X) \ni f \to M_f(\cdot) \in L^\infty(\mathfrak{H})$ is a (canonical) isomorphism of von Neumann algebras; (ii) the commutant $L^\infty(\mathfrak{H})'$ is the von Neumann algebra of decomposable operators on $\mathfrak{H}$; (iii) the mapping $C(X) \ni f \to M_f(\cdot) \in C(\mathfrak{H})$ is a (canonical) homomorphism of C^*-algebras which becomes an isomorphism if the support of μ is all X; in this case X is the Gel'fand spectrum of $C(\mathfrak{H})$ and μ is a basic measure.

C Nuclear Gel'fand triples

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces with orthonormal basis $\{\psi_j^{(1)}\}_{j\in\mathbb{N}}$ and $\{\psi_j^{(2)}\}_{j\in\mathbb{N}}$. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear mapping and $A^\dagger : \mathcal{H}_2 \to \mathcal{H}_1$ its adjoint. The numbers $|A|^2 := \sum_{j\in\mathbb{N}} \|A\psi_j^{(2)}\|^2_{\mathcal{H}_2}$ and $|A^\dagger|^2 := \sum_{j\in\mathbb{N}} \|A^\dagger\psi_j^{(1)}\|^2_{\mathcal{H}_1}$ are independent on the choice of the orthonormal basis, $|A| = |A^\dagger|$ and if $|A| < \infty$ then A is bounded since $\|A\|_{\mathscr{B}(\mathcal{H}_1,\mathcal{H}_2)} < |A|$. A linear mapping $A \in \mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ for which $|A| < \infty$ is called *Hilbert-Schmidt operator* and the number $|A|$ is called the *Hilbert-Schmidt norm* of A. If A is a linear map between two Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$ at least one of which with finite dimension then A is Hilbert-Schmidt. The identity operator in $\mathcal{H}$ is Hilbert-Schmidt if and only if the dimension of $\mathcal{H}$ is finite. For more details on the theory of the Hilbert-Schmidt operators see [RS73] (Section VI.6)

General references for the theory of locally compact spaces and nuclear spaces are [Tre67] or [RS73] (Chapter V). A *locally convex vector space* Φ is a complex vector space with a family of seminorms $\{p_\alpha\}_{\alpha\in\mathfrak{J}}$ endowed with the natural (coarsest) topology which makes all the seminorms continuous. Moreover the family of seminorms need to be separating, i.e. $p_\alpha(\varphi) = 0$ for all $\alpha \in \mathfrak{J}$ implies that $\varphi = 0$ (this assures that Φ is Hausdorff). A locally convex vector space Φ is metrizable if and only if its topology is determined by a countable family of seminorms (countably normed space). A metrizable and complete locally convex space is called a *Fréchet space*. A locally convex vector space Φ is said to be *nuclear* if the system of seminorms $\{p_\alpha\}_{\alpha\in\mathfrak{J}}$ (or an equivalent system) satisfies the following conditions: a) for every $\alpha \in \mathfrak{J}$ the normed quotient space $\Phi_\alpha := \Phi/\mathfrak{N}_\alpha$, where $\mathfrak{N}_\alpha := \{\varphi \in \Phi \ : \ p_\alpha(\varphi) = 0\}$, is a pre-Hilbert space whose closure is denoted by $\mathcal{H}_\alpha$; b) for each $\alpha_1 \in \mathfrak{J}$ there exists such $\alpha_2 \in \mathfrak{J}$ such that $p_{\alpha_1} \leqslant p_{\alpha_2}$ (which implies $\Phi_{\alpha_2} \subset \Phi_{\alpha_1}$) and the canonical embedding $\imath_{\alpha_2,\alpha_1} : \mathcal{H}_{\alpha_2} \to \mathcal{H}_{\alpha_1}$ is a Hilbert-Schmidt operator. If Φ is a locally-convex vector space then Φ^* denotes the *topological dual space* of Φ, namely Φ^* is the set of the linear continuous functional on Φ. We will denote by $\langle \ ; \ \rangle : \Phi^* \times \Phi \to \mathbb{C}$ the dual (sesquilinear) pairing between Φ^* and Φ. The space Φ^* is a topological Hausdorff space with the $*$-weak topology (the topology of pointwise convergence). If $A : \Phi \to \Phi$ is a continuous linear map on Φ we can define pointwise the *transpose* ${}^tA$ of A by $\langle A^\dagger\eta; \varphi\rangle := \langle\eta; A\varphi\rangle$ for all $\eta \in \Phi^*$ and $\varphi \in \Phi$; it is a continuous map ${}^tA : \Phi^* \to \Phi^*$ with respect the $*$-weak topology.

A *rigged Hilbert space* is a triple $\Phi \hookrightarrow \mathcal{H} \hookrightarrow \Phi^*$ with $\mathcal{H}$ a separable Hilbert space, $\Phi \subset \mathcal{H}$ a norm-dense subspace such that Φ has a topology for which it is a nuclear space and the inclusion map $\imath : \Phi \hookrightarrow \mathcal{H}$ is continuous. Identifying $\mathcal{H}$ with its dual space $\mathcal{H}^*$ one has that

$\mathcal{H} \subset \Phi^*$ and the inclusion $\imath^* : \mathcal{H} \hookrightarrow \Phi^*$ is the adjoint of the map $\imath$. The duality pairing between Φ and Φ^* has to be compatible with the scalar product on $\mathcal{H}$, namely $\langle \psi_1, \psi_2 \rangle = (\psi_1, \psi_2)_{\mathcal{H}}$ whenever $\psi_1 \in \Phi^* \cap \mathcal{H}$ and $\psi_2 \in \Phi$. The specific triple $\{\Phi, \mathcal{H}, \Phi^*\}$ is often named the *(nuclear) Gel'fand triple*. If A is a bounded operator on $\mathcal{H}$ which leaves unchanged Φ and $A : \Phi \to \Phi$ is continuous with respect the nuclear topology of Φ then ${}^tA : \Phi^* \to \Phi^*$ is continuous and is an extension of the adjoint $A^\dagger$ defined in $\mathcal{H}$ to all the dual space Φ^*. Many references about the theory of the Gel'fand triple can be found in [dlM05].

References

[AEGS04] J. E. Avron, A. Elgart, G. M. Graf, and L. Sadun. Transport and dissipation in quantum pumps. *J. Stat. Phys.*, **116**: 425–473, 2004.

[Boc01] F. P. Boca. *Rotations C^*-algebras and almost Mathieu operators.* Theta Foundation, 2001.

[BR87] O. Bratteli and D. W. Robinson. *C^*- and W^*-Algebras, Symmetry Groups, Decomposition of States*, volume I of *Operator Algebras and Quantum Statistical Mechanics*. Springer-Verlag, 1987.

[Bré87] H. Brézis. *Analyse fonctionnelle, Théorie et Application.* Masson, 1987.

[BSE94] J. V. Bellissard, H. Schulz-Baldes, and A. van Elst. The Non Commutative Geometry of the Quantum Hall Effect. *J. Math. Phys.*, **35**: 5373–5471, 1994.

[Cho66] G. Choquet. *Topology.* Academic Press, 1966.

[Con94] A. Connes. *Noncommutative Geometry.* Academic Press, 1994.

[DD63] J. Dixmier and A. Douady. Champs continus d'espaces hilbertiens et de C^*-algèbres. *Bull. Soc. math. France*, **91**: 227–284, 1963.

[DFP10] G. De Nittis, F. Faure, and G. Panati. Topological duality between the Harper and Hofstadter models. In preparation, 2010.

[Dix81] J. Dixmier. *von Neumann Algebras.* North-Holland, 1981.

[Dix82] J. Dixmier. *C^*-Algebras.* North-Holland, 1982.

[dlM05] R. de la Madrid. The role of the rigged Hilbert space in Quantum Mechanics. *Eur. J. Phys.*, **26**, 2005.

[FD88] J. M. G. Fell and R. S. Doran. *Basic Representation Theory of Groups and Algebras*, volume 1 of *Representation of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles.* Academic Press Inc., 1988.

[GO08] G. M. Graf and G. Ortelli. Comparison of quantization of charge transport in periodic and open pumps. *Phys. Rev. B*, **77**: 033304, 2008.

[Gra07] G. M. Graf. Aspects of the Integer Quantum Hall Effect. *Proceedings of Symposia in Pure Mathematics*, **76.1**: 429–442, 2007.

[Gru01] M. J. Gruber. Non-commutative Bloch theory. *J. Math. Phys.*, **42**: 2438–2465, 2001.

[GVF01] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa. *Elements of Noncommutative Geometry*. Birkhäuser, 2001.

[Hör90] L. Hörmander. *Complex Analysis in Several Variables*. North-Holland, 1990.

[KSV93] R. D. King-Smith and D. Vanderbilt. Theory of polarization of crystalline solids. *Phys. Rev. B*, **47**: 1651–1654, 1993.

[Kuc93] P. Kuchment. *Floquet Theory for Partial Differential Equations*. Operator Theory: Advances and Applications. Birkhäuser Verlag, 1993.

[Lan85] S. Lang. *Differential Manifolds*. Springer-Verlag, 1985.

[Lan97] G. Landi. *An Introduction to Noncommutative Spaces and their Geometries*. Lecture Notes in Physics. Springer-Verlag, 1997.

[Mau68] K. Maurin. *General Eigenfunction Expansions and Unitary Representations of Topological Groups*. PWN, 1968.

[NF70] B. Sz. Nagy and C. Foias. *Harmonic Analysis of Operators on Hilbert Space*. American Elsevier. North-Holland, 1970.

[Pan07] G. Panati. Triviality of Bloch and Bloch-Dirac Bundles. *Ann. Henri Poincaré*, **8**: 995–1011, 2007.

[Res92] R. Resta. Theory of the electric polarization in crystals. *Ferroelectrics*, **136**: 51–75, 1992.

[RS73] M. Reed and B. Simon. *Functional Analysis*, volume I of *Methods of Modern Mathematical Phisics*. Academic Press, 1973.

[Rud62] W. Rudin. *Fourier Analysis on Groups*. Number 12 in Interscience Tracts in Pure and Applied Mathematics. Interscience Publishers, 1962.

[Rud87] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, 1987.

[Sam91] Y. S. Samoilenko. *Spectral Theory of Families of Self-Adjoint Operators*. Mathematics and Its Applications. Kluwer Academic Publishers, 1991.

[SPT09] C. Sparber, G. Panati, and S. Teufel. Geometric currents in piezoelectricity. *Arch. Rat. Mech. Anal.*, **91**: 387–422, 2009.

[Tho83] D. J. Thouless. Quantization of particle transport. *Phys. Rev. B*, **27**: 6083–6087, 1983.

[Tho98] D. J. Thouless. *Topological Quantum Numbers in Nonrelativistic Physics*. World Scientific Publishing, 1998.

[TKNN82] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. Nijs. Quantized Hall conductance in a two-dimensional periodic potential. *Phys. Rev. Lett.*, **49**: 405–408, 1982.

[Tre67] F. Treves. *Topological vector spaces, distributions and kernels*. Academic Press, 1967.