Bubbles with prescribed mean curvature: the variational approach

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Abstract. Let $H\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^1 mapping such that $H(p) \to H_\infty > 0$ as $|p| \to \infty$. We show that when H satisfies some global conditions then there exists an H-bubble, namely a sphere S in $\mathbb{R}^3$ such that the mean curvature of S at any regular point $p \in S$ equals $H(p)$.

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1 Introduction

In this paper we give some contribution to the following problem, raised by S.T. Yau in [25]: *"Let H be a real-valued function on $\mathbb{R}^3$. Find (reasonable) conditions on H to ensure that one can find a closed surface with prescribed genus in $\mathbb{R}^3$ whose mean curvature is given by H"*. In particular we are interested in the existence of $\mathbb{S}^2$-type surfaces in $\mathbb{R}^3$ with prescribed mean curvature H.

Spheres in $\mathbb{R}^3$ with mean curvature H can be characterized as parametric surfaces, more precisely as nonconstant solutions of the problem

$$\begin{cases} \Delta u = 2H(u)u_x \wedge u_y & \text{on } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} |\nabla u|^2 < \infty\,. \end{cases} \tag{1.1}$$

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If H satisfies suitable smoothness and growth assumptions (see [1], [2], [15], [19]), then any weak solution to (1.1) is actually a classical solution. Moreover it is conformal and it parametrizes a closed surface S with area $\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2$. Furthermore the surface S has mean curvature $H(p)$ at any point $p \in S$, with the exception of a finite number of singular points. Problem (1.1) has also some relevance with regard to the Plateau problem for disc-type surfaces with prescribed mean curvature. We quote for instance [9], [21] and [22] for a discussion of this feature.

Following [7], we will call *H-bubble* any nonconstant solution of (1.1). We are interested in finding sufficient global conditions for the existence of H-bubbles when H approaches a positive constant at infinity, that is,

$$H(p) \to H_\infty \quad \text{as } |p| \to \infty\,, \quad \text{for some } H_\infty \in (0,\infty)\,.$$

If $H \equiv H_\infty \in (0,\infty)$ is constant, then spheres of radius H_∞^{-1} everywhere placed are the only H-bubbles (i.e., admit parametrizations solving (1.1), see Lemma 0.1 in [4]).

Problem (1.1) is much more involved when H is variable. For example, one can exhibit a curvature H which is a small perturbation of a positive constant and such that no embedded H-bubble exists (see [8]). On the other hand, some existence results on H-bubbles are known, in different settings: we quote, for instance, [5], [7], [8], [13], [18], and [23].

In the present paper we study the H-bubble problem (1.1) by exploiting its variational nature. Indeed weak solutions to (1.1) correspond to critical points of an energy functional $\mathcal{E}$ suitably defined on a Sobolev space (see Section 2). As the energy $\mathcal{E}$ is unbounded, we are lead to look for mountain pass-type or even for higher unstable critical points.

Our main result is stated in the next theorem.

Theorem 1.1 *Let $H \in C^1(\mathbb{R}^3)$ be a mapping satisfying*

(H_1) $M_H := \sup_{p\in\mathbb{R}^3} |\nabla H(p)\cdot p\, p| < 1$*;*

(H_2) *there exists a constant $H_\infty > 0$ such that $H(p) = H_\infty + o(|p|^{-1})$ as $|p| \to \infty$.*

Then an H-bubble exists, provided that

$$c^* := \sup_{\substack{s>0\\ p\in\mathbb{R}^3}} \left(4\pi s^2 - 2\int_{B_s(p)} H(q)\,dq\right) < \frac{8\pi}{3H_\infty^2}\,. \tag{1.2}$$

Assumption (H_1) deeply affects the topological properties of the energy functional associated to (1.1). It is indeed a condition on the radial derivative of H. In fact (H_1) regards just the difference $H - H_\infty$, and it provides a restriction on the growth and on the decay at infinity along radial directions. More precisely, (H_1) implies that

$$|H(p) - H_\infty| \le \frac{M_H}{|p|} \quad \text{for all } p \in \mathbb{R}^3.$$

Assumption (H_2) enters into one of the main ingredients in our proof, namely, in the blow-up analysis of sequences of approximate solutions to (1.1) (compare with Theorem A.1).
We also exhibit some sufficient conditions for (1.2) that lead to the following existence results.

Theorem 1.2 *Let $H \in C^1(\mathbb{R}^3)$ be a mapping satisfying* (H_1) *and* (H_2). *Then an H-bubble exists, provided that one of the following conditions holds true:*

$$\inf_{p\in\mathbb{R}^3} \frac{H(p)}{H_\infty} > \frac{1}{\sqrt{2}} \tag{1.3}$$

$$(1 + M_H)^3 < 2\,. \tag{1.4}$$

Other sufficient conditions involve the negative part of the difference $K(p) := H(p) - H_\infty$.

Theorem 1.3 *Let $H \in C^1(\mathbb{R}^3)$ be a mapping satisfying* (H_1) *and* (H_2). *Then an H-bubble exists, provided that one of the following conditions holds true:*

$$\|K^-\|_\infty < H_\infty \left(1 - \frac{1}{\sqrt{2}}\right) \tag{1.5}$$

$$K^- \in L^1(\mathbb{R}^3) \quad \textit{and} \quad \|K^-\|_{L^1(\mathbb{R}^3)} < \frac{2\pi}{3H_\infty^2} \tag{1.6}$$

$$K^- \in L^3(\mathbb{R}^3) \quad \textit{and} \quad \left(1 + \frac{2\|K^-\|_{L^3(\mathbb{R}^3)}}{\sqrt[3]{36\pi}}\right)^3 < 2\,. \tag{1.7}$$

We can also state some existence results for

$$\begin{cases} \Delta u = 2\,(H_\infty + K(u))\, u_x \wedge u_y \quad \text{on } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} |\nabla u|^2 < \infty\,, \end{cases} \tag{1.8}$$

where $H_\infty > 0$ and $K : \mathbb{R}^3 \to \mathbb{R}$ are given. The following corollary is an immediate consequence of Theorem 1.3.

Corollary 1.4 *Let $K \in C^1(\mathbb{R}^3)$ be a mapping satisfying*

$$M_K := \sup_{p\in\mathbb{R}^3} |\nabla K(p) \cdot p\; p| < 1 \quad \textit{and} \quad K(p) = o(|p|^{-1}) \quad \textit{as } |p| \to \infty.$$

1. *Problem (1.8) has a nonconstant solution for any $H_\infty > 0$ large enough.*
2. *If $K^- \in L^1(\mathbb{R}^3)$ then problem (1.8) has a nonconstant solution for any $H_\infty > 0$ small enough.*
3. *If (1.7) is satisfied then problem (1.8) has a nonconstant solution for any $H_\infty > 0$.*

By using the functional change $u(x,y) \to u(y,x)$, one can easily derive similar results in case $H_\infty < 0$. To complete the picture we recall that no H-bubble exists if $H \in C^1(\mathbb{R}^3)$ satisfies (H_1) and $H(p) \to 0$ as $|p| \to \infty$ (see [9]).
Our approach leads to the existence of a parametric surface S which is a possibly branched immersion. By a result in [14], the surface S has at most a finite number of branch points.

The paper is organized as follows.
In Section 2 we first notice that it is not restrictive to assume $H_\infty = 1$. Then we recall some known facts about the volume functional and the variational setting. In Section 3 we study the topology of the sublevels of the energy functional associated to (1.1) and we prove Theorems 1.1, 1.2 and 1.3. In Section 4 we prove Proposition 3.4, which constitutes the main step in the proof of Theorem 1.1. In Theorem A.1 of Appendix A we describe the behavior of Palais-Smale sequences for (1.1).

Notation

We denote by $\omega\colon \mathbb{R}^2 \to \mathbb{S}^2$ the inverse of the stereographic projection from the north pole of the 2-dimensional sphere $\mathbb{S}^2$. More precisely

$$\omega(z) := (x\mu, y\mu, 1-\mu) \quad \text{where } \mu(z) = \frac{2}{1+|z|^2} \quad (z = (x,y) \in \mathbb{R}^2)\,.$$

By direct computation one can check that

$$\int_{\mathbb{R}^2} |\nabla\omega|^2 = 8\pi\,, \qquad \int_{\mathbb{R}^2} \omega \cdot \omega_x \wedge \omega_y = -4\pi\,. \tag{1.9}$$

Let $\alpha \in (1,\infty)$. We denote by $W^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)$ the Sobolev space of vector-valued maps $v \in L^\alpha(\mathbb{S}^2,\mathbb{R}^3)$ such that $\int_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2} v|^\alpha \, d\sigma < \infty$. It is well known that $W^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)$ is continuously embedded into L^∞ if and only if $\alpha > 2$.
We set $\widehat{W}^{1,\alpha} := \{v \circ \omega \mid v \in W^{1,\alpha}(\mathbb{S}^2,\mathbb{R}^3)\}$. If $\alpha = 2$ we simply write $H^1(\mathbb{S}^2,\mathbb{R}^3)$ and $\widehat{H}^1$ instead of $W^{1,2}(\mathbb{S}^2,\mathbb{R}^3)$ and $\widehat{W}^{1,2}$, respectively. We remark that constant maps belong to $\widehat{H}^1$, that is $\mathbb{R}^3 \hookrightarrow \widehat{H}^1$. We notice also that $\widehat{H}^1$ is a Hilbert space, isomorphic to $H^1(\mathbb{S}^2,\mathbb{R}^3)$. A Hilbertian norm in $\widehat{H}^1$ is:

$$\|u\|^2 = \|\nabla u\|_2^2 + \left| \fint_{\mathbb{S}^2} u \right|^2$$

where

$$\|\nabla u\|_2^2 := \int_{\mathbb{R}^2} |\nabla u|^2\,, \qquad \fint_{\mathbb{S}^2} u := \frac{1}{4\pi}\int_{\mathbb{R}^2} u\mu^2\,.$$

2 Functional setting and Palais-Smale sequences

In this preliminary section we recall some known facts about the Steffen volume functional and we point out some results in connection with the variational nature of problem (1.1). We start with a simple remark that will be quite useful in the sequel.

Remark 2.1 *For every $H \in C^1(\mathbb{R}^3)$ and $t \in \mathbb{R} \setminus \{0\}$ let $H_t \in C^1(\mathbb{R}^3)$ be the map defined by $H_t(p) = tH(tp)$ for any $p \in \mathbb{R}^3$. By easy computation one can check that a mapping $u \in \widehat{H}^1$ is an H-bubble, i.e., is a nonconstant weak solution of (1.1), if and only if $t^{-1}u$ is an H_t-bubble. Notice also that $\sup_{p\in\mathbb{R}^3} |(\nabla H_t(p) \cdot p)p| = \sup_{p\in\mathbb{R}^3} |(\nabla H(p) \cdot p)p|$.*

Let $H \in C^1(\mathbb{R}^3)$ be a function satisfying (H_1) and (H_2). According to the previous remark, it is not restrictive to assume $H_\infty = 1$. Then the map $K(p) := H(p) - 1$ satisfies

(K_1) $M_K = M_H = \sup_{p\in\mathbb{R}^3} |\nabla K(p) \cdot p\, p| < 1$;

(K_2) $K(p)p \to 0$ as $|p| \to \infty$.

The H-*volume functional* can be defined in $\widehat{H}^1$ as follows:

$$\mathcal{V}_H(u) := \mathcal{V}(u) + \mathcal{V}_K(u)\ , \quad \mathcal{V}_H : \widehat{H}^1 \to \mathbb{R}\,.$$

Here

$$\mathcal{V}_K(u) := \int_{\mathbb{R}^2} m_K(u)u \cdot u_x \wedge u_y\ , \qquad m_K(u) := \int_0^1 K(su)s^2\, ds\,,$$

and $\mathcal{V}(u)$ is the classical Bononcini-Wente volume of $u \in \widehat{H}^1$. More precisely, $\mathcal{V}$ is the unique continuous extension on $\widehat{H}^1$ of the functional

$$\mathcal{V}(u) = \frac{1}{3}\int_{\mathbb{R}^2} u \cdot u_x \wedge u_y$$

defined in $\widehat{H}^1 \cap L^\infty$. Notice that $\mathcal{V}$ is cubic, namely $\mathcal{V}(su) = s^3\mathcal{V}(u)$ for all $u \in \widehat{H}^1$ and $s \in \mathbb{R}$. Moreover, since $2u_x \wedge u_y = \mathrm{div}(u \wedge u_y, u_x \wedge u)$, the functional $\mathcal{V}$ is invariant with respect to translations in $\mathbb{R}^3$, that is,

$$\mathcal{V}(u + p) = \mathcal{V}(u) \quad \text{for any } u \in \widehat{H},\ p \in \mathbb{R}^3\,.$$

We also recall the classical Bononcini-Wente isoperimetric inequality (see [3], [24]):

$$|\mathcal{V}(u)| \le \frac{1}{12\sqrt{2\pi}}\|\nabla u\|_2^3 \quad \forall u \in \widehat{H}^1\,. \tag{2.1}$$

Remark 2.2 *Conditions* (K_1)–(K_2) *imply that $K(u) = -\int_1^\infty \nabla K(su) \cdot u\, ds$. Thus*

$$|K(u)u| \le M_K\ , \quad |m_K(u)u| \le \frac{M_K}{2} \quad \textit{for all } u \in \mathbb{R}^3\,. \tag{2.2}$$

This ensures that the K-volume $\mathcal{V}_K$ is well defined on $\widehat{H}^1$. Moreover it is continuous by Lebesgue's Theorem and

$$|\mathcal{V}_K(u)| \le \frac{M_K}{4}\int_{\mathbb{R}^2} |\nabla u|^2 \quad \forall u \in \widehat{H}^1\,. \tag{2.3}$$

Notice also that the vector-field $u \mapsto m_K(u)u$ is of class C^1 on $\mathbb{R}^3$ and satisfies

$$\mathrm{div}(m_K(u)u) = K(u) = 3m_K(u) + \nabla m_K(u) \cdot u \tag{2.4}$$

$$|\nabla m_K(u) \cdot u\ u| = |(3m_K(u) - K(u))u| \le \frac{M_K}{2} \quad \textit{for all } u \in \mathbb{R}^3. \tag{2.5}$$

Remark 2.3 *Let $s > 0$, $p \in \mathbb{R}^3$ and let ω be the parametrization of $\mathbb{S}^2$ defined in (1.9). Then the map $s\omega + p$ parameterizes the sphere of radius s centered at p. It turns out that*

$$\mathcal{V}_K(s\omega + p) = -\int_{B_s(p)} K(q)\, dq \quad \text{and} \quad \mathcal{V}_H(s\omega + p) = -\int_{B_s(p)} H(q)\, dq\,.$$

For the proof see [20], or compute directly $\omega_x \wedge \omega_y = -\mu^2 \omega$ and

$$\mathcal{V}_K(s\omega + p) = -s^2 \int_{\mathbb{R}^2} m_K(s\omega + p)(s\omega + p) \cdot \omega \mu^2 = -s^2 \int_{\mathbb{S}^2} m_K(s\omega + p)(s\omega + p) \cdot \omega\ d\omega\,.$$

The conclusion follows from (2.4), by the divergence Theorem, and from (1.9). More generally one can see that if $u \in \widehat{H}^1$ is a smooth conformal parametrization of the boundary of some region A of $\mathbb{R}^3$, and if $\nabla u(z) \neq 0$ for all $z \in \mathbb{R}^2$, then $\mathcal{V}_H(u)$ equals the signed H-volume enclosed by A.

Let us introduce the *energy "at infinity"*

$$\mathcal{E}_\infty(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + 2\mathcal{V}(u)\ , \quad \mathcal{E}_\infty : \widehat{H}^1 \to \mathbb{R}\,,$$

and the *H-energy* or just *energy* of $u \in \widehat{H}^1$:

$$\mathcal{E}(u) := \mathcal{E}_\infty(u) + 2\mathcal{V}_K(u)\ , \quad \mathcal{E} : \widehat{H}^1 \to \mathbb{R}\,.$$

Notice that, by Remark 2.3 and by (1.9),

$$\mathcal{E}(s\omega + p) = 4\pi s^2 - 2 \int_{B_s(p)} H(q)\ dq\ , \quad \forall s > 0,\ p \in \mathbb{R}^3. \tag{2.6}$$

The next Lemma is essentially contained in [20] (see also [9]).

Lemma 2.4 (i) *The functionals $\mathcal{V}$ and $\mathcal{E}_\infty$ are of class C^∞ in $\widehat{H}^1$.*

(ii) *The functionals $\mathcal{V}_K$ and $\mathcal{E}$ are continuous on $\widehat{H}^1$.*

(iii) *For every $u \in \widehat{H}^1$ and for every $\varphi \in C^\infty_c(\mathbb{R}^3)$ there exists the directional derivative*

$$\partial_\varphi \mathcal{E}(u) = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla \varphi + 2H(u)\varphi \cdot u_x \wedge u_y)$$

and the mapping $u \mapsto \partial_\varphi \mathcal{E}(u)$ is continuous on $\widehat{H}^1$.

In general the functional $\mathcal{E}$ is not Gateaux-differentiable (see [20]). If $\mathcal{E}$ is Fréchet differentiable at some point $u \in \widehat{H}$ we will use also the notation $\mathcal{E}'(u)[\varphi] = \partial_\varphi \mathcal{E}(u)$.

From (iii) in Lemma 2.4 we infer that an H-bubble is a nonconstant mapping $u \in \widehat{H}^1$ such that $\partial_\varphi \mathcal{E}(u) = 0$ for all $\varphi \in C^\infty_c(\mathbb{R}^2, \mathbb{R}^3)$.

Remark 2.5 *If (2.2) holds, then every H-bubble is a bounded mapping (see, e.g., [11] or [19]). Thus, the regularity theory for H-systems applies (see, for instance, [15] or [2]) to ensure that any H-bubble u is of class $C^{2,\alpha}$ as a map on $\mathbb{S}^2$. It also solves the conformality conditions, hence it describes a parametric surface $S = u(\mathbb{R}^2 \cup \{\infty\})$ such that $H(p)$ equals the mean curvature of S at any regular point $p \in S$ (see for instance [7]). In addition S has finite area $\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2$ and it has at most a finite number of branch points (see [14]).*

Remark 2.6 The case H constant. *Consider the case $K \equiv 0$, i.e. $H \equiv 1$. As proved in [4], solutions to (1.1) are explicitly known. In particular, a 1-bubble $U \in \widehat{H}^1$ parametrizes a sphere with constant mean curvature 1 and $\mathcal{E}_\infty(U) = \frac{4\pi k}{3}$ for some $k \in \mathbb{N}$.*
We can provide a variational characterization of 1-bubbles corresponding to embedded spheres. The functional $\mathcal{E}_\infty$ has a nice mountain pass geometry. Indeed, constant maps are local minima of $\mathcal{E}_\infty$ by the isoperimetric inequality (2.1). Moreover the energy $\mathcal{E}_\infty$ is unbounded from below since the functional $\mathcal{V}$ is cubic. Thus we can introduce the mountain-pass level

$$\underline{c}_\infty := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{E}_\infty(\gamma(t)) \quad \text{where} \quad \Gamma = \{\gamma \in C([0,1], \widehat{H}^1) \mid \gamma(0) = 0,\ \mathcal{E}_\infty(\gamma(1)) < 0\}.$$

Due to the invariance of $\mathcal{E}_\infty$ with respect to the conformal group on $\mathbb{R}^2$ and with respect to translations in $\mathbb{R}^3$ the Palais-Smale condition fails. However, nowadays standard rescaling arguments easily lead to the conclusion that $\underline{c}_\infty$ is a critical level for $\mathcal{E}$. In fact one can check that

$$\underline{c}_\infty = \inf_{\substack{u \in \widehat{H}^1 \\ \mathcal{V}(u) < 0}} \sup_{s > 0} \mathcal{E}_\infty(su) = \frac{4\pi}{3}.$$

Thus, we can conclude that embedded spheres of radius 1 are the mountain-pass critical points for $\mathcal{E}$ in $\widehat{H}^1$. Their Morse index was computed in [8] and [18].

One of the main steps in the proof of Theorem 1.1 concerns the analysis of the behavior of approximate solutions to (1.1). Let us recall the following definition.

Definition. *A Palais-Smale (shortly, PS) sequence for $\mathcal{E}$ at level $c \in \mathbb{R}$ is a sequence (u^n) in H^1 such that $\mathcal{E}$ is differentiable at u^n for any n, and*

$$\mathcal{E}(u^n) \to c, \qquad \|\mathcal{E}'(u^n)\|_{\widehat{H}^{-1}} \to 0,$$

where $\widehat{H}^{-1}$ denotes the dual space of $\widehat{H}^1$.

Condition (K_1) has two important consequences, that are pointed out in the next lemma.

Lemma 2.7 (i) *If (u^n) is a PS sequence for $\mathcal{E}$ in $\widehat{H}^1$, then $\sup_n \|\nabla u^n\|_2 < \infty$.*

(ii) *If $U \in \widehat{H}^1$ is an H-bubble, then $\mathcal{E}(U) > 0$ and $\mathcal{V}(U) < 0$.*

Proof. From (2.2) and from Lebesgue's Theorem it easily follows that for every $u \in \widehat{H}^1$ there exists

$$\partial_u \mathcal{E}(u) = \lim_{s\to 1} \frac{\mathcal{E}(su) - \mathcal{E}(u)}{s-1} = \int_{\mathbb{R}^2} |\nabla u|^2 + 2\int_{\mathbb{R}^2} K(u)u \cdot u_x \wedge u_y + 6\mathcal{V}(u) =: \mathcal{G}(u) \tag{2.7}$$

(see also Lemma 4.1 in Section 4). In particular, by (2.2),

$$\mathcal{G}(u) = 0 \quad \Rightarrow \quad 6\mathcal{V}(u) \le -(1 - M_K) \int_{\mathbb{R}^2} |\nabla u|^2 . \tag{2.8}$$

Using (2.5) we infer that for any $u \in \widehat{H}^1$ the following equality holds:

$$3\mathcal{E}(u) - \mathcal{G}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + 2\int_{\mathbb{R}^2} (3m_K(u) - K(u))u \cdot u_x \wedge u_y \ge \frac{1-M_K}{2}\int_{\mathbb{R}^2} |\nabla u|^2 . \tag{2.9}$$

Let us prove (i). If (u^n) is a PS sequence for $\mathcal{E}$ in $\widehat{H}^1$, then $|\mathcal{G}(u^n)| \le \|\mathcal{E}'(u^n)\|_{\widehat{H}^{-1}} \|\nabla u^n\|_2$ and consequently $\sup_n \|\nabla u^n\|_2 < \infty$ by (2.9), since $M_K < 1$ by assumption. Now we prove (ii). If $U \in \widehat{H}^1$ is an H-bubble, then multiplying $\Delta U = 2(1 + K(U))U_x \wedge U_y$ by U and integrating we obtain that $\mathcal{G}(U) = 0$. Thus (ii) follows from (2.8), (2.9) and from $M_K < 1$. □

We are able to furnish a complete description of a PS sequence. This is discussed in Theorem A.1 in Appendix A. The next proposition is an important consequence of Theorem A.1 and contains the essential information we need for our purposes.

Proposition 2.8 *Let $(u^n) \subset \widehat{H}^1$ be a PS sequence for $\mathcal{E}$ at the level $c \ne 0$.*

(i) *If $c \ne 4k\pi/3$ for any integer k then there exists at least an H-bubble.*

(ii) *If $c = 4\pi/3$ then there exists an H-bubble U with energy $\mathcal{E}(U) \le 4\pi/3$, or (u^n) blows exactly one round sphere placed at infinity, that is, up to a subsequence, there exists a sequence (g_n) of Möbius transformations such that*

$$\nabla(u^n \circ g_n) \to \nabla\omega \quad \text{in } L^2(\mathbb{R}^2).$$

Proof. The sequence $(\|\nabla u^n\|_2)$ is bounded by Lemma 2.7. Since $\mathcal{G}(u^n) = o(1)$, from (2.9) and from $c \ne 0$ we infer that $c > 0$. By the isoperimetric inequality (2.1) and by (2.3) we easily get that

$$\mathcal{E}(u) \le \frac{1 + M_K}{2}\|\nabla u\|_2^2 + 2\mathcal{V}(u) \quad \text{for any } u \in \widehat{H}^1. \tag{2.10}$$

Therefore the sequence $(\|\nabla u^n\|_2)$ is bounded away from zero. The first statement follows from Theorem A.1, since $\mathcal{E}_\infty(U)$ is a multiple of $4\pi/3$, for any 1-bubble U (see the appendix of [4]). To prove (ii) one has to use again Lemma 2.7, Theorem A.1 in the appendix and the results in [4]. □

3 Proof of Theorems 1.1, 1.2 and 1.3

In this section we take a mapping $H = 1 + K$, with $K \in C^1(\mathbb{R}^3)$ satisfying (K_1)–(K_2). Our aim is to construct a PS sequence at some positive energy level. We start by noticing that the functional $\mathcal{E}$ has a mountain pass structure, similarly to what observed for the functional $\mathcal{E}_\infty$ in Remark 2.6.

Lemma 3.1 *The energy $\mathcal{E}$ admits a mountain pass geometry in $\widehat{H}^1$. More precisely:*

(i) $\mathcal{E}(u) \geq \frac{1-M_K}{2}\|\nabla u\|_2^2 + O(\|\nabla u\|_2^3)$ *as* $\|\nabla u\|_2 \to 0$ *in* $\widehat{H}^1$*;*

(ii) *there exists* $u \in \widehat{H}^1$ *such that* $\mathcal{E}(u) < 0$.

Proof. (i) It follows from the estimate (2.3) and from the isoperimetric inequality (2.1).
(ii) Take $u \in \widehat{H}^1$ such that $\mathcal{V}(u) < 0$ and use (2.10) to estimate

$$\mathcal{E}(su) \leq (1 + M_K)\frac{s^2}{2}\|\nabla u\|_2^2 + 2s^3\mathcal{V}(u).$$

Then (ii) follows by taking $s > 0$ large enough. □

Accordingly to Lemma 3.1 we can define the mountain pass level as

$$\underline{c} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{E}(\gamma(t)) \ , \quad \text{where} \quad \Gamma := \{\gamma \in C([0,1], \widehat{H}^1) \mid \gamma(0) = 0 \ , \ \mathcal{E}(\gamma(1)) < 0\} \, .$$

In the next Lemma we give an estimate from above on $\underline{c}$ that will be useful also in Section 4.

Lemma 3.2 *For every $p \in \mathbb{R}^3$ one has that $\sup_{s>0}\mathcal{E}(s(\omega + p)) \geq \underline{c}$. Moreover $\underline{c} \leq \frac{4\pi}{3}$.*

Proof. From (1.9) we get $\mathcal{E}_\infty(\omega) = \frac{4\pi}{3}$. Thanks to Remark 2.3 and to (2.2), for any $|p| > 1$ one has that

$$|\mathcal{V}_K(s(\omega + p))| = \left|\int_{B_s(sp)} K(q)\, dq\right| \leq M_K \int_{B_s(sp)} \frac{dq}{|q|} = \frac{4\pi s^2 M_K}{3|p|} \, ,$$

since the map $q \to |q|^{-1}$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$. Then

$$\mathcal{E}(s(\omega + p)) = \mathcal{E}_\infty(s(\omega + p)) + 2\mathcal{V}_K(s(\omega + p)) \leq \frac{4\pi s^2}{3}\left(3 + \frac{2M_K}{|p|} - 2s\right) \tag{3.1}$$

by (1.9) and since the volume functional $\mathcal{V}$ is invariant with respect to translations in $\mathbb{R}^3$. In particular $\mathcal{E}(s(\omega + p)) < 0$ if $s > 0$ is large enough. Thus the map $s \mapsto s(\omega + p)$ is a mountain-pass path, up to a reparametrization. Hence $\sup_{s>0}\mathcal{E}(s(\omega + p)) \geq \underline{c}$. From (3.1) we get also

$$\mathcal{E}(s(\omega + p)) \leq \frac{4\pi}{3}\left(1 + \frac{2M_K}{3|p|}\right)^3 \quad \forall s > 0 \ , \ |p| > 1 \, ,$$

which in particular implies

$$\sup_{s>0}\mathcal{E}(s(\omega + p)) \leq \frac{4\pi}{3} + O(|p|^{-1}) \quad \text{as } |p| \to \infty \, . \tag{3.2}$$

Hence $\underline{c} \leq \frac{4\pi}{3}$. □

We are in position to state our first existence result.

Proposition 3.3 *Let $H = 1 + K$, with $K \in C^1(\mathbb{R}^3)$ satisfying* (K$_1$) *and* (K$_2$). *If*

$$\underline{c} < \frac{4\pi}{3} \tag{3.3}$$

then there exists an H-bubble $U \in \widehat{H}^1$ with $\mathcal{E}(U) = \underline{c}$.

Proof. By Lemma 3.1, the class of mountain-pass paths Γ is nonempty and $\underline{c} > 0$. The regularity properties of $\mathcal{E}$ given by Lemma 2.4 are sufficient to infer the existence of a PS sequence $(u^n) \subset \widehat{H}^1$ for $\mathcal{E}$ at level $\underline{c}$ (see, e.g., Theorem 4.3 in [17]). By Lemma 2.7, it turns out that $\sup_n \|\nabla u^n\|_2 < \infty$. In view of Proposition 2.8 we have that an H-bubble $U \in \widehat{H}^1$ exists and $\mathcal{E}(U) \le \underline{c}$. In fact, from the results in [7] it holds that $\mathcal{E}(U) = \underline{c}$. $\square$

We point out that at this step the assumption (K$_2$) can be weakened by asking that $K(p) \to 0$ as $|p| \to \infty$. This can be performed by a suitable approximation argument, as in [7]. In fact the existence of a minimal $(1+K)$-bubble with $K \in C^1(\mathbb{R}^3)$ vanishing at infinity and satisfying (K$_1$) and (3.3) has been already proved in [7]. In the next section we push on the investigation, considering also the case $\underline{c} \ge \frac{4\pi}{3}$, when it may happen that no $(1+K)$-bubble at level $\underline{c}$ exists.
The core of the proof of Theorem 1.1 is the following result.

Proposition 3.4 *Let $H = 1 + K$, with $K \in C^1(\mathbb{R}^3)$ satisfying* (K$_1$) *and* (K$_2$). *Assume in addition that*

$$\text{there exists no } (1+K)\text{-bubble with energy} \le \frac{4\pi}{3}. \tag{3.4}$$

Then there exist $\lambda, \rho > 0$ such that, setting

$$Z := [0,\lambda] \times B_\rho = \{(s,p) \in \mathbb{R} \times \mathbb{R}^3 \mid 0 \le s \le \lambda\ ,\ |p| \le \rho\}\ ,$$
$$\Phi := \{\phi \in C(Z, \widehat{H}^1) \mid \phi(s,p) = s(\omega + p)\ \forall (s,p) \in \partial Z\}\ ,$$

one has that

$$\frac{4\pi}{3} \le \sup_{(s,p)\in\partial Z} \mathcal{E}(s(\omega+p)) < \overline{c} := \inf_{\phi\in\Phi} \max_{(s,p)\in Z} \mathcal{E}(\phi(s,p))\ . \tag{3.5}$$

The proof of Proposition 3.4 will be accomplished in Section 4.

Proof of Theorem 1.1. By Remark 2.1 we can assume that $H_\infty = 1$. Hence in particular

$$c^* = \sup_{\substack{s>0\\ p\in\mathbb{R}^3}} \left(4\pi s^2 - 2\int_{B_s(p)} H(q)\,dq\right) = \sup_{\substack{s>0\\ p\in\mathbb{R}^3}} \mathcal{E}(s\omega + p) < \frac{8\pi}{3}\,. \tag{3.6}$$

by (1.2) and (2.6). If there exists an $(1+K)$-bubble U with energy $\mathcal{E}(U) \le 4\pi/3$ then we are done. If not, then $\underline{c} \ge 4\pi/3$ by Proposition 3.3, hence $\underline{c} = 4\pi/3$ by Lemma 3.2. Moreover

Proposition 3.4 applies. In particular we can define the higher minmax level $\overline{c}$ as in (3.5). By (3.6), (3.5) and by the definition of the class Φ we have that

$$\frac{4\pi}{3} < \overline{c} < \frac{8\pi}{3}\,. \tag{3.7}$$

Thanks to Proposition 3.4 and to Lemma 2.4, there exists a PS sequence $(u^n) \subset \widehat{H}^1$ for $\mathcal{E}$ at level $\overline{c}$ (use for example Theorem 4.3 in [17]). Therefore $\sup_n \|\nabla u^n\|_2 < \infty$ by Lemma 2.7, and thus Proposition 2.8 applies. The proof is complete. □

Remark 3.5 *Let us compare the existence results stated in Proposition 3.3 and in Theorem 1.1 with $H_\infty = 1$. The striking difference concerns the additional conditions (3.3) in Proposition 3.3 and (1.2) in Theorem 1.1. One has that (3.3) is fulfilled if*

$$\exists p \in \mathbb{R}^3 \text{ such that } \sup_{s>0} \mathcal{E}(s\omega + p) < \frac{4\pi}{3}\,,$$

whereas (1.2) holds when

$$\sup_{\substack{s>0 \\ p\in\mathbb{R}^3}} \mathcal{E}(s\omega + p) < \frac{8\pi}{3}\,.$$

In general Theorem 1.1 covers a larger class of curvature functions than Proposition 3.3 does. For example if $K < 0$ on $\mathbb{R}^3$ then (3.3) cannot hold.

Proof of Theorem 1.2. Put $H_0 = \inf_{p\in\mathbb{R}^3} H(p)$ and observe that

$$4\pi s^2 - 2\int_{B_s(p)} H(q)\,dq \le 4\pi s^2 - \frac{8\pi}{3} H_0 s^3 \le \frac{4\pi}{3H_0^2}$$

for any $p \in \mathbb{R}^3$, $s > 0$. Therefore (1.2) is fulfilled if (1.3) is satisfied. Now we assume that (1.4) holds. We compute

$$\int_{B_s(p)} H(q)\,dq = \frac{4\pi}{3} H_\infty s^3 - \mathcal{V}_K(s\omega + p)$$

and we use (2.3) to evaluate

$$4\pi s^2 - 2\int_{B_s(p)} H(q)\,dq \le 8\pi \left((1 + M_K)\frac{s^2}{2} - H_\infty \frac{s^3}{3} \right) \le \frac{4\pi}{3H_\infty^2}(1 + M_H)^3 < \frac{8\pi}{3}$$

for any $p \in \mathbb{R}^3$, $s > 0$. This implies that (1.2) holds. □

Proof of Theorem 1.3. Condition (1.5) is trivially equivalent to (1.3). Assume that (1.6) holds. We use $K^- = -\min\{H - H_\infty, 0\}$, (1.9) and the inequality

$$-\int_{B_s(p)} H(q)\,dq \le -\frac{4\pi}{3} H_\infty s^3 + \int_{\mathbb{R}^3} K^-(q)\,dq$$

to get

$$4\pi s^2 - 2\int_{B_s(p)} H(q)\,dq \le \frac{4\pi}{3H_\infty^2} + \int_{\mathbb{R}^3} K^-(q)\,dq$$

for any $p \in \mathbb{R}^3$, $s > 0$. Thus (1.2) is satisfied. Finally, if $K^- \in L^3$ we use Hölder inequality to estimate

$$-\int_{B_s(p)} H(q)\,dq \le -\frac{4\pi}{3} H_\infty s^3 + \int_{B_s(p)} K^-(q)\,dq \le -\frac{4\pi}{3} H_\infty s^3 + \left(\frac{4\pi}{3}\right)^{2/3} s^2 \|K^-\|_{L^3}\,.$$

Hence

$$4\pi s^2 - 2\int_{B_s(p)} H(q)\,dq \le \frac{4\pi}{3H_\infty^2}\left(1 + \frac{2\|K^-\|_{L^3}}{\sqrt[3]{36\pi}}\right)^3,$$

for any $p \in \mathbb{R}^3$, $s > 0$. Thus (1.7) implies (1.2). □

4 Proof of Proposition 3.4

The proof of Proposition 3.4 needs a deeper understanding of the topology of the energy sublevels. To this extent we will use some devices, that we introduce hereafter.

Main technical tools

- The open set

$$\Omega := \{u \in \widehat{H}^1 \mid \mathcal{V}(u) < 0\}\,.$$

  Observe that Ω contains the mappings $\omega + p$ for all $p \in \mathbb{R}^3$ and if $u \in \Omega$ then, up to a reparametrization, the radial path $s \mapsto su$ is a mountain pass path.

- The functional

$$\mathcal{G}(u) = \int_{\mathbb{R}^2} |\nabla u|^2 + 6\mathcal{V}(u) + 2\int_{\mathbb{R}^2} K(u) u \cdot u_x \wedge u_y\,, \quad \mathcal{G}\colon \widehat{H}^1 \to \mathbb{R}\,,$$

  already introduced in the proof of Lemma 2.7. Recall that $\mathcal{G}(u) = \partial_u \mathcal{E}(u)$ for any $u \in \widehat{H}^1$ (see (2.7)). In particular, $\mathcal{G}(u) = 0$ for any H-bubble u.

- The Nehari topological manifold

$$\Sigma := \{u \in \widehat{H}^1 \mid \mathcal{G}(u) = 0\,,\ \|\nabla u\|_2 > 0\}.$$

  We remark that in general Σ is not a differentiable manifold, since the functional $\mathcal{G}$ is not sufficiently regular. Notice also that $\Sigma \subset \Omega$ by (2.8).

- The vector-valued map

$$\mathcal{B}(u) = \frac{1}{8\pi}\int_{\mathbb{R}^2} \Pi(u)|\nabla u|^2 \ , \quad \mathcal{B}\colon \widehat{H}^1 \to \mathbb{R}^3, \tag{4.1}$$

where Π is the minimal distance projection of $\mathbb{R}^3$ onto the closed unit ball:

$$\Pi(p) = \begin{cases} p & \text{if } |p| < 1 \\ p/|p| & \text{if } |p| \geq 1. \end{cases}$$

Since $\Pi \circ u$ is bounded for any $u \in \widehat{H}^1$, the mapping $\mathcal{B}$ is well defined and continuous on $\widehat{H}^1$. Moreover $\mathcal{B}$ is conformally invariant. The knowledge of $\mathcal{B}(u)$ furnishes some information on the barycenter of the surface parametrized by the map $u \in \widehat{H}^1$.

In Lemma 4.2 we will also introduce a continuous, conformally invariant map $\mathcal{S}\colon \Omega \to (0,\infty)$, such that $\Sigma = \{\mathcal{S}(u)u \mid u \in \Omega\} = \{u \in \Omega \mid \mathcal{S}(u) = 1\}$. Hence Σ turns out to be a retract of Ω, with retraction $u \mapsto \mathcal{S}(u)u$.
We start by studying the behavior of the energy functional along radial paths.

Lemma 4.1 *Let $H = 1 + K$, with $K \in C^1(\mathbb{R}^3)$ satisfying* (K_1) *and* (K_2)*. For every $u \in \widehat{H}^1$ the mapping $s \mapsto \mathcal{E}(su)$ is of class C^2 in $(0,\infty)$. Moreover*

$$s\frac{d}{ds}\mathcal{E}(su) = \mathcal{G}(su) \tag{4.2}$$

$$\frac{d^2}{ds^2}\mathcal{E}(su) \leq 2\mathcal{G}(su) - (1 - M_K)\int_{\mathbb{R}^2} |\nabla u|^2 \quad \forall s > 0 \ , \ u \in \widehat{H}^1. \tag{4.3}$$

Proof. Fix $u \in \widehat{H}^1$ and set

$$f(s) := \int_{\mathbb{R}^2} m_K(su)u \cdot u_x \wedge u_y \, .$$

Since $\mathcal{E}(su) = \frac{s^2}{2}\|\nabla u\|_2^2 + 2s^3\mathcal{V}(u) + s^3 f(s)$ then the map $s \mapsto \mathcal{E}(su)$ enjoys the same regularity as f. Using (2.4), (2.5) and standard techniques one can show that f admits derivative at any $s > 0$. More precisely

$$f'(s) = \int_{\mathbb{R}^2} \frac{K(su) - 3m_K(su)}{s}\, u \cdot u_x \wedge u_y = \frac{1}{s}\left(\tilde{f}(s) - 3f(s)\right) \, ,$$

where

$$\tilde{f}(s) := \int_{\mathbb{R}^2} K(su)u \cdot u_x \wedge u_y \, .$$

Similarly, using (K_1), one can check that also $\tilde{f}$ admits derivative at any $s > 0$. Moreover, by standard arguments, the functions f' and $\tilde{f}'$ turn out to be continuous on $(0,\infty)$. Thus one gets that the mapping $s \mapsto \mathcal{E}(su)$ is of class C^2 in $(0,\infty)$ and

$$\begin{aligned}\frac{d}{ds}\mathcal{E}(su) &= s\int_{\mathbb{R}^2}|\nabla u|^2 + 2s^2\int_{\mathbb{R}^2}K(su)u\cdot u_x\wedge u_y + 6s^2\mathcal{V}(u)\\ \frac{d^2}{ds^2}\mathcal{E}(su) &= \int_{\mathbb{R}^2}|\nabla u|^2 + 4s\int_{\mathbb{R}^2}K(su)u\cdot u_x\wedge u_y\\ &\qquad + 2s^2\int_{\mathbb{R}^2}(\nabla K(su)\cdot u)u\cdot u_x\wedge u_y + 12s\mathcal{V}(u)\,.\end{aligned}$$

Then (4.2) is proved, and (4.3) follows from (K_1). □

Lemma 4.2 *Let $H = 1+K$, with $K \in C^1(\mathbb{R}^3)$ satisfying* (K_1) *and* (K_2)*. For every $u \in \Omega$ there exists a unique number $\mathcal{S}(u) > 0$ such that*

$$\sup_{s>0}\mathcal{E}(su) = \mathcal{E}(\mathcal{S}(u)u)\,.$$

Moreover the following facts hold:

(i) *the map $u \mapsto \mathcal{S}(u)$ is conformally invariant and continuous on Ω;*

(ii) $\Sigma = \{\mathcal{S}(u)u \mid u \in \Omega\} = \{u \in \Omega \mid \mathcal{S}(u) = 1\} = \{u \in \Omega \mid \mathcal{E}(u) = \sup_{s>0}\mathcal{E}(su)\}$*;*

(iii) *if $u \in \Omega$ then $\mathcal{G}(su) > 0$ if and only if $0 < s < \mathcal{S}(u)$.*

Proof. Fix $u \in \Omega$, and notice that $\mathcal{E}(su) \ge \frac{1-M_K}{2}s^2 + o(s^2)$ as $s \to 0$, by Lemma 3.1. Thus $\sup_{s>0}\mathcal{E}(su)$ is positive and finite. If $\frac{d}{ds}\mathcal{E}(\overline{s}u) = 0$ at some $\overline{s} > 0$, then $\frac{d^2}{ds^2}\mathcal{E}(\overline{s}u) < 0$ by (4.3). Thus the function $s \mapsto \mathcal{E}(su)$ has a unique critical point, which is a global maximum, and which will be denoted by $\mathcal{S}(u)$. In particular the map $u \mapsto \mathcal{S}(u)$ is well defined on Ω. Notice that for every $u \in \Omega$ the value $\mathcal{S}(u)$ is the unique positive solution of

$$\mathcal{F}(u,s) := \int_{\mathbb{R}^2}|\nabla u|^2 + 2s\int_{\mathbb{R}^2}K(su)u\cdot u_x\wedge u_y + 6s\mathcal{V}(u) = 0\,. \tag{4.4}$$

The map $\mathcal{F}$ is continuous on $\Omega \times (0,\infty)$, it admits continuous derivative

$$\partial_s\mathcal{F}(u,s) = 2\int_{\mathbb{R}^2}K(su)u\cdot u_x\wedge u_y + 2s\int_{\mathbb{R}^2}(\nabla K(su)\cdot u)u\cdot u_x\wedge u_y + 6\mathcal{V}(u)$$

and if $\mathcal{F}(u,s) = 0$ then, by (K_1),

$$\partial_s\mathcal{F}(u,s) = -\frac{1}{s}\int_{\mathbb{R}^2}|\nabla u|^2 + 2s\int_{\mathbb{R}^2}(\nabla K(su)\cdot u)u\cdot u_x\wedge u_y \le -\frac{1-M_K}{s}\int_{\mathbb{R}^2}|\nabla u|^2 < 0\,.$$

Hence the continuity of $\mathcal{S}$ follows from the Implicit Function Theorem (see, e.g., Theorem 15.1 in [12]). The conformal invariance of $\mathcal{S}(u)$ follows from (4.4), since $\mathcal{F}(u\circ g, s) = \mathcal{F}(u,s)$ for every conformal transformation g of $\mathbb{R}^2 \cup \{\infty\}$.
To prove (ii) we notice that $\Sigma \subset \Omega$ by (2.8). If $u \in \Sigma$ then $\mathcal{S}(u) = 1$ by (4.2), since the map $s \to \mathcal{E}(su)$ has a unique critical point in $(0,\infty)$. The converse again follows from (4.2). Statement (iii) is immediate. □

Remark 4.3 *The mapping $\mathcal{S}$ is implicitly defined by the equation $\mathcal{F}(u,s)=0$ in $\Omega\times(0,\infty)$, with $\mathcal{F}$ defined in (4.4). Since in general $\mathcal{F}$ is just continuous with respect to $u\in\widehat{H}^1$, one cannot expect more regularity than continuity for the mapping $\mathcal{S}$. On the other hand, the functional $\mathcal{F}$ turns out to be of class C^1 in $\widehat{W}^{1,\alpha}\times\mathbb{R}$, for any $\alpha>2$ (see, e.g., [8]). Hence the mapping $\mathcal{S}$ is of class C^1 in $\Omega\cap\widehat{W}^{1,\alpha}$, for any $\alpha>2$.*

Next we state a result that will be a key point in the proof of Proposition 3.4.

Proposition 4.4 *Let $H=1+K$, with $K\in C^1(\mathbb{R}^3)$ satisfying* (K_1) *and* (K_2). *Then*

$$\inf_{u\in\Sigma}\mathcal{E}(u)=\underline{c}\,.$$

In addition, if $U\in\Sigma$ is such that $\mathcal{E}(U)=\underline{c}$ and $U\in\widehat{W}^{1,\alpha}$ for some $\alpha>2$, then U is an H-bubble.

Proof. By Lemma 4.2 it holds that $\underline{c}\le\inf_\Sigma\mathcal{E}$. To prove the converse, set $s_p:=\mathcal{S}(\omega+p)$ for every $p\in\mathbb{R}^3$. From the definition of $\mathcal{S}$ we have that $s_p(\omega+p)\in\Sigma$ and

$$\inf_{u\in\Sigma}\mathcal{E}(u)\le\mathcal{E}(s_p(\omega+p))=\sup_{s>0}\mathcal{E}(s(\omega+p))\,.$$

Using (3.2) we obtain that $\inf_\Sigma\mathcal{E}\le 4\pi/3$. If $\underline{c}=4\pi/3$ then we are done. If $\underline{c}<4\pi/3$ then by Proposition 3.3 there exists an H-bubble U at the energy level $\underline{c}$. But then $U\in\Sigma$, and therefore $\underline{c}=\mathcal{E}(U)\ge\inf_\Sigma\mathcal{E}$. Thus the first statement is proved.
Next, let $U\in\Sigma$ be such that $\mathcal{E}(U)=\underline{c}$ and $U\in\widehat{W}^{1,\alpha}$ for some $\alpha>2$. Fix $\varphi\in C_c^\infty(\mathbb{R}^2,\mathbb{R}^3)$. For $t\in\mathbb{R}$ close to 0 it results that $\mathcal{V}(U+t\varphi)<0$, since $\mathcal{V}(U)<0$ by Lemma 4.2. Hence $\mathcal{S}(U+t\varphi)$ is well defined for $|t|$ small enough. Put $\xi(t)=\mathcal{S}(U+t\varphi)(U+t\varphi)\in\Sigma$ and notice that $\xi(0)=U$, as $U\in\Sigma$ (use Lemma 4.2). Since $U+t\varphi\in\widehat{W}^{1,\alpha}$, then by Remark 4.3 the map $t\mapsto\xi(t)$ is of class C^1. Taking into account that $\mathcal{S}(U)=1$ we get

$$\xi'(0)=\big(\mathcal{S}'(U)[\varphi]\big)\,U+\varphi\,.$$

The map $t\mapsto\mathcal{E}(\xi(t))$ is of class C^1 as well, since $\mathcal{E}\in C^1(\widehat{W}^{1,\alpha},\mathbb{R})$. In particular one can compute

$$\frac{d}{dt}\mathcal{E}(\xi(t))\bigg|_{t=0}=\mathcal{E}'(U)[\xi'(0)]=\mathcal{E}'(U)[\varphi]=\partial_\varphi\mathcal{E}(U)$$

since $\mathcal{E}'(U)[U]=\mathcal{G}(U)=0$. As $\mathcal{E}(U)=\inf_\Sigma\mathcal{E}$, one has that $\mathcal{E}(\xi(t))\ge\mathcal{E}(\xi(0))$ for $|t|$ small. Then $\partial_\varphi\mathcal{E}(U)=0$. This holds for any $\varphi\in C_c^\infty(\mathbb{R}^2,\mathbb{R}^3)$, that is, U is an H-bubble. $\square$

Remark 4.5 *Let $u \in \Sigma$ and $s > 0$. Then by (2.3) and (2.8) we have that*

$$\mathcal{E}(su) \le \left(\frac{s^2}{2}(1 + M_K) - \frac{s^3}{3}(1 - M_K) \right) \int_{\mathbb{R}^2} |\nabla u|^2 .$$

Hence, fixing $\lambda > \frac{3(1+M_K)}{2(1-M_K)}$, one has that $\mathcal{E}(\lambda u) < 0$ for any u in Σ.

From now on we assume that (3.4) holds. In particular from Lemma 3.2, Proposition 3.3 and Proposition 4.4 we get that

$$\inf_{u \in \Sigma} \mathcal{E}(u) = \underline{c} = \frac{4\pi}{3} . \tag{4.5}$$

In the next result we use the map $\mathcal{B} : \widehat{H}^1 \to \mathbb{R}^3$ defined in (4.1) to prove that the images of maps in Σ with small energy lie far from the origin.

Proposition 4.6 *Let $H = 1 + K$, with $K \in C^1(\mathbb{R}^3)$ satisfying* (K_1), (K_2) *and (3.4). Fix a large parameter λ as in Remark 4.5, and let*

$$r_K \in \left(0, \lambda^{-2}\right) . \tag{4.6}$$

Then there exists $\varepsilon \in (0, 1)$ such that

$$u \in \Sigma \, , \quad \mathcal{E}(u) \le \frac{4\pi}{3} + \varepsilon \quad \Rightarrow \quad |\mathcal{B}(u)| > r_K.$$

Proof. We argue by contradiction. Assume that there exists a sequence $(v^n) \subset \Sigma$ such that $\mathcal{E}(v^n) \to \frac{4\pi}{3}$ and $|\mathcal{B}(v^n)| \le r_K$ for every n. Up to a reparametrization, the map $\gamma^n : s \mapsto sv^n$, $\gamma^n : [0, \lambda] \to \widehat{H}^1$ is a mountain-pass path by Remark 4.5. Thus $\max_{s \in [0,\lambda]} \mathcal{E}(sv^n) = \mathcal{E}(v^n) \to \underline{c}$ by (4.5). According to the Mountain Pass Theorem (see again Theorem 4.3 in [17]), there exists a PS sequence $(u^n) \subset \widehat{H}^1$ for $\mathcal{E}$ at level $\underline{c}$ with $\mathrm{dist}(u^n, \mathrm{range}\, \gamma^n) \to 0$. Hence there is a sequence $(s_n) \subset [0, \lambda]$ such that $\nabla(u^n - s_n v^n) \to 0$ in L^2. Up to a subsequence, we can assume that $s_n \to \tilde{s} \in [0, \lambda]$ and, since $\sup_n \|\nabla v^n\|_2 < \infty$ by (2.9), then

$$\nabla(u^n - \tilde{s} v^n) \to 0 \quad \text{in } L^2(\mathbb{R}^2). \tag{4.7}$$

Notice that $\tilde{s} \neq 0$, otherwise $\|\nabla u^n\|_2 \to 0$ and consequently $\mathcal{E}(u^n) \to 0$, which is false. By (3.4) and (4.5), in view of Proposition 2.8, there exists a sequence (g_n) of Möbius transformations such that, for a subsequence,

$$\nabla(u^n \circ g_n) \to \nabla \omega \quad \text{in } L^2(\mathbb{R}^2). \tag{4.8}$$

Setting

$$\tilde{v}^n = v^n \circ g_n \, , \quad U = \frac{\omega}{\tilde{s}} \, ,$$

by (4.7) and (4.8) we infer that $\nabla \tilde{v}^n \to \nabla U$ strongly in L^2. Hence

$$\mathcal{B}(v^n) = \mathcal{B}(\tilde{v}^n) = \frac{1}{8\pi} \int_{\mathbb{R}^2} \Pi(\tilde{v}^n) |\nabla U|^2 + o(1) .$$

Let

$$q^n = \fint_{\mathbb{S}^2} \tilde{v}^n \,.$$

If (for a subsequence) $|q^n| \to \infty$, then $\tilde{v}^n - q^n \to U$ strongly in $\widehat{H}^1$, since $\fint_{\mathbb{S}^2} U = 0$. In particular $|\tilde{v}^n| \to \infty$ a.e. in $\mathbb{R}^2$. If $\frac{q^n}{|q^n|} \to \tilde{q}$, then $\Pi(\tilde{v}^n) \to \tilde{q}$ a.e. in $\mathbb{R}^2$. As $|\tilde{q}| = 1$ we obtain

$$|\mathcal{B}(v^n)| = \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla U|^2 + o(1) = \frac{1}{\tilde{s}^2} + o(1) \geq \frac{1}{\lambda^2} + o(1)\,,$$

by (1.9), since $\omega = \tilde{s}U$. But $|\mathcal{B}(v^n)| \leq r_K < \lambda^{-2}$ by (4.6). Hence we get a contradiction. Suppose now that (for a subsequence) $q^n \to q \in \mathbb{R}^3$. Then $\tilde{v}^n \to U + q$ strongly in $\widehat{H}^1$. Therefore, by continuity, $\mathcal{S}(U+q) = 1$ and $\mathcal{E}(U+q) = \frac{4\pi}{3}$. Hence $U + q \in \Sigma$ and, by Lemma 4.4 and by (4.5), $\mathcal{E}(U+q) = \inf_\Sigma \mathcal{E}$. Since $U+q$ is smooth as a map on $\mathbb{S}^2$, then Proposition 4.4 applies and we obtain a contradiction with (3.4). □

Lemma 4.7 *Let* $H = 1 + K$, *with* $K \in C^1(\mathbb{R}^3)$ *satisfying* (K$_1$), (K$_2$) *and (3.4). For every* $p \in \mathbb{R}^3$ *set* $s_p := \mathcal{S}(\omega + p)$. *As* $|p| \to \infty$ *it holds that*

$$s_p = 1 + o(1) \tag{4.9}$$

$$\mathcal{E}(s_p(\omega+p)) = \frac{4\pi}{3} + o(1) \tag{4.10}$$

$$\mathcal{B}(s_p(\omega+p)) = \frac{p}{|p|} + o(1)\,. \tag{4.11}$$

Proof. Since $\mathcal{G}(s_p(\omega+p)) = 0$, from (1.9) we obtain

$$s_p = 1 + \frac{1}{4\pi} \int_{\mathbb{R}^2} K(s_p(\omega+p)) s_p(\omega+p) \cdot \omega_x \wedge \omega_y\,, \tag{4.12}$$

as $\mathcal{V}(\omega+p) = \mathcal{V}(\omega) = -4\pi/3$ for any $p \in \mathbb{R}^3$. By (2.2) we evaluate

$$\left| \int_{\mathbb{R}^2} K(s_p(\omega+p)) s_p(\omega+p) \cdot \omega_x \wedge \omega_y \right| \leq \frac{M_K}{2} \int_{\mathbb{R}^2} |\nabla\omega|^2 = 4\pi M_K\,.$$

Thus $s_p \geq 1 - M_K$ for all $p \in \mathbb{R}^3$. Consequently $|s_p(\omega+p)| \to \infty$ as $|p| \to \infty$ pointwise on $\mathbb{R}^2$, and hence $\int_{\mathbb{R}^2} K(s_p(\omega+p)) s_p(\omega+p) \cdot \omega_x \wedge \omega_y \to 0$ as $|p| \to \infty$, by (2.2), (K$_2$) and by Lebesgue's Theorem. Thus (4.9) follows from (4.12). The limit in (4.10) follows from (3.2) and (4.5). Now we prove (4.11). Thanks to (4.9), it turns out that $|s_p(\omega+p)| \geq 1$ on $\mathbb{R}^2$ for $|p|$ large enough. This yields $\mathcal{B}(s_p(\omega+p)) = s_p^2 \mathcal{B}(\omega+p)$. Then

$$|\mathcal{B}(s_p(\omega+p)) - \mathcal{B}(\omega+p)| \leq |1 - s_p^2|\;\; |\mathcal{B}(\omega+p)| \leq |1 - s_p^2|$$

because $|\mathcal{B}(\omega+p)| \leq \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla\omega|^2 = 1$. Thus $\mathcal{B}(s_p(\omega+p)) = \mathcal{B}(\omega+p) + o(1)$. To conclude, notice that

$$\mathcal{B}(\omega+p) - \frac{p}{|p|} = \frac{1}{8\pi} \int_{\mathbb{R}^2} \left(\frac{\omega+p}{|\omega+p|} - \frac{p}{|p|} \right) |\nabla\omega|^2 = o(1)$$

by the dominated convergence theorem. □

Proof of Proposition 3.4 completed. We have to choose suitable parameters ρ and λ. For every $p \in \mathbb{R}^3$ set

$$\omega^p = s_p(\omega + p)$$

with $s_p = \mathcal{S}(\omega + p)$, as in Lemma 4.7. Notice that the mapping $p \mapsto \omega^p$ is continuous and that $\omega^p \in \Sigma$ by Lemma 4.2.

• **Choice of $\boldsymbol{\lambda}$**. Fix $\lambda > 1$ large as in Remark 4.5 and such that

$$\lambda > \frac{3s_p(1 + M_K)}{2(1 - M_K)} \quad \forall p \in \mathbb{R}^3.$$

This is possible since the mapping $p \mapsto s_p$ is bounded, thanks to (4.9). Since $\omega^p \in \Sigma$, with this choice of λ one has that

$$\mathcal{E}(\lambda(\omega + p)) < 0 \quad \text{and} \quad \max_{s\in[0,\lambda]} \mathcal{E}(s(\omega + p)) = \mathcal{E}(\omega^p) \quad \forall p \in \mathbb{R}^3. \tag{4.13}$$

• **Choice of $\boldsymbol{\rho}$**. Fix r_K as in (4.6) and let $\varepsilon \in (0, 1)$ be given by Proposition 4.6. According to Lemma 4.7 there exists $\rho > 0$ such that for all $p \in \mathbb{R}^3$ with $|p| = \rho$ one has that

$$\mathcal{E}(\omega^p) < \frac{4\pi}{3} + \varepsilon \tag{4.14}$$

$$|\mathcal{B}(\omega^p)| > 1 - \varepsilon\,. \tag{4.15}$$

Let $Z = [0, \lambda] \times B_\rho$ and let Φ be as in the statement of Proposition 3.4. From (4.13) it follows that

$$\sup_{(s,p)\in\partial Z} \mathcal{E}(s(\omega + p)) = \sup_{(s,p)\in[0,\lambda]\times\partial B_\rho} \mathcal{E}(s(\omega + p)) = \sup_{p\in\partial B_\rho} \mathcal{E}(\omega^p)\,.$$

Then

$$\frac{4\pi}{3} \le \sup_{(s,p)\in\partial Z} \mathcal{E}(s(\omega + p)) < \frac{4\pi}{3} + \varepsilon$$

by Lemma 3.2, (4.5) and (4.14), since $\omega^p \in \Sigma$. In order to conclude the proof we will show that $\bar{c} \ge \frac{4\pi}{3} + \varepsilon$. The argument was inspired by the proof of Theorem 2.1 in [16]. Assume by contradiction that there exists a map $\phi \in \Phi$ such that

$$\sup_{(s,p)\in Z} \mathcal{E}(\phi(s, p)) < \frac{4\pi}{3} + \varepsilon\,.$$

We set

$$\tilde{\phi}(t, p) := \phi(s_p t, p)\ , \quad (t, p) \in \widetilde{Z} := \left\{(t, p) \in \mathbb{R} \times \mathbb{R}^3 \mid (s_p t, p) \in Z\ \right\}\,.$$

Notice that $\tilde{\phi}(t, p) = s_p t(\omega + p)$ on $\partial\widetilde{Z}$ and

$$\sup_{(t,p)\in\widetilde{Z}} \mathcal{E}(\tilde{\phi}(t, p)) < \frac{4\pi}{3} + \varepsilon\,. \tag{4.16}$$

We split $\widetilde{Z}$ into:

$$\begin{aligned}\widetilde{Z}_+ &= \{(t,p) \in \widetilde{Z} \mid \mathcal{G}(\tilde{\phi}(t,p)) > 0 \ \text{ or } \ t = 0\}\\ \widetilde{Z}_- &= \{(t,p) \in \widetilde{Z} \mid \mathcal{G}(\tilde{\phi}(t,p)) < 0\}\\ \widetilde{Z}_0 &= \{(t,p) \in \widetilde{Z} \mid \mathcal{G}(\tilde{\phi}(t,p)) = 0\,, \ \ t > 0\}.\end{aligned}$$

Notice that $\widetilde{Z}_0 = \tilde{\phi}^{-1}(\mathbb{R}^3 \cup \Sigma)$. By Proposition 4.6 and by (4.16) we have that

$$\Big|\mathcal{B}(\tilde{\phi}(t,p))\Big| > r_K \quad \text{if } (t,p) \in \widetilde{Z}_0 \text{ and } \tilde{\phi}(t,p) \text{ is not constant.} \tag{4.17}$$

Consider the map $g\colon \widetilde{Z} \to \mathbb{R}^4$ defined by

$$g(t,p) = \Big(t, \mathcal{B}(\tilde{\phi}(t,p))\Big)\,,$$

and fix a point $p_0 \in \mathbb{R}^3$ with

$$0 < |p_0| < \min\{r_K, 1-\varepsilon\}.$$

We claim that the topological degree $\deg(g, \widetilde{Z}, (1,p_0))$ is well defined and

$$\deg(g, \widetilde{Z}, (1,p_0)) = 1. \tag{4.18}$$

Consider the homotopy $h\colon [0,1] \times \widetilde{Z} \to \mathbb{R}^4$ defined as follows:

$$h(\tau; (t,p)) = \Big(t, \tau p + (1-\tau)\mathcal{B}(\tilde{\phi}(t,p))\Big)\,.$$

Notice that $\deg\left(h(1;\cdot), (1,p_0)\right) = 1$, since $(1,p_0) \in \widetilde{Z}$. To prove that h is an admissible homotopy assume by contradiction that there exist $\tau \in [0,1]$ and $(t,p) \in \partial\widetilde{Z}$ such that $h(\tau;(t,p)) = (1,p_0)$. Then $t = 1$ and $(s_p, p) \in \partial Z$. On the other hand, $s_p < \lambda$ and therefore $|p| = \rho$. From $\tilde{\phi}(1,p) = \omega^p$ and from (4.15) we infer

$$|p_0| = |\tau p + (1-\tau)\mathcal{B}(\omega^p)| \geq \tau\rho + (1-\tau)(1-\varepsilon)$$

which is impossible, since $\tau \in [0,1]$, $\rho > 1$, $\varepsilon < 1$ and $|p_0| < 1 - \varepsilon$. Therefore h is admissible, and (4.18) is proved.
Next we notice that by (4.17) and by the excision property one has

$$\deg(g, \widetilde{Z}, (1,p_0)) = \deg(g, \widetilde{Z}_+, (1,p_0)) + \deg(g, \widetilde{Z}_-, (1,p_0))\,. \tag{4.19}$$

We will conclude the proof by showing that

$$\deg(g, \widetilde{Z}_+, (1,p_0)) = 0 \tag{4.20}$$

$$\deg(g, \widetilde{Z}_-, (1,p_0)) = 0\,, \tag{4.21}$$

that contradict (4.19), because of (4.18).

Proof of (4.20). The cylinder $\widetilde{Z}$ is compact and g is continuous. Thus there exists $\bar{R} > 1$ large, such that $(\bar{R}, p_0) \notin g(\widetilde{Z})$. Use the path $\tau \in [0,1] \mapsto \xi(\tau) = \big(\tau\bar{R} + (1-\tau), p_0\big)$ joining $(\bar{R}, p_0)$ with $(1, p_0)$. We claim that the triple $(g, \widetilde{Z}_+, \xi(\tau))$ is admissible, that is, $g(t,p) \neq \xi(\tau)$ for every $\tau \in [0,1]$ and $(t,p) \in \partial\widetilde{Z}_+$. Assume by contradiction that there exists $\tau \in [0,1]$ and $(t,p) \in \partial\widetilde{Z}_+$ such that $(t, \mathcal{B}(\tilde{\phi}(t,p))) = \big(\tau\bar{R} + (1-\tau), p_0\big)$. First of all we infer $t \geq 1$. Next, notice that $\partial\widetilde{Z}_+ \subset \widetilde{Z}_0 \cup \partial\widetilde{Z}$. But $(t,p) \notin \widetilde{Z}_0$ by (4.17), since $0 < |\mathcal{B}(\tilde{\phi}(t,p))| = |p_0| < r_K$. Thus $(t,p) \in \partial\widetilde{Z}$ and in particular $\tilde{\phi}(t,p) = t\omega^p$. Moreover $(t,p) \in \partial\widetilde{Z}_+ \setminus \widetilde{Z}_0$ imply $\mathcal{G}(\tilde{\phi}(t,p)) = \mathcal{G}(t\omega^p) > 0$. But then $t < 1$ by Lemma 4.2 (iii), a contradiction. Since ξ is admissible and $(\bar{R}, p_0) \notin g(\widetilde{Z})$, we conclude that $\deg(g, \widetilde{Z}_+, (1, p_0)) = \deg(g, \widetilde{Z}_+, (\bar{R}, p_0)) = 0$.

Proof of (4.21). Use the path $\tau \in [0,1] \mapsto \zeta(\tau) = (1 - 2\tau, p_0)$ joining $(-1, p_0) \notin g(\widetilde{Z})$ with $(1, p_0)$. We prove that the triple $(g, \widetilde{Z}_-, \zeta(\tau))$ is admissible, that is, $g(t,p) \neq \zeta(\tau)$ for every $\tau \in [0,1]$ and $(t,p) \in \partial\widetilde{Z}_-$. By contradiction, assume that there exists $(t,p) \in \partial\widetilde{Z}_-$ such that $g(t,p) = \zeta(\tau)$. Then $t = 1 - 2\tau \leq 1$ and $0 < |\mathcal{B}(\tilde{\phi}(t,p))| = |p_0| < r_K$. Thus $(t,p) \notin \widetilde{Z}_0$ by (4.17). On the other hand, $\partial\widetilde{Z}_- \subset \widetilde{Z}_0 \cup \partial\widetilde{Z}$, and thus $(t,p) \in \partial\widetilde{Z} \setminus \widetilde{Z}_0$. In particular $\tilde{\phi}(t,p) = t\omega^p$, and $\mathcal{G}(\tilde{\phi}(t,p)) = \mathcal{G}(t\omega^p) < 0$. But then $t > 1$ by Lemma 4.2 (iii), a contradiction. Since ζ is an admissible path and $(-1, p_0) \notin g(\widetilde{Z})$, then $\deg(g, \widetilde{Z}_-, (1, p_0)) = \deg(g, \widetilde{Z}_-, (-1, p_0)) = 0$. Thus the proof is complete. □

A Palais-Smale sequences

In this appendix we prove the following result.

Theorem A.1 *Let $H = 1 + K$, with $K \in C^1(\mathbb{R}^3)$ satisfying* (K_1) *and* (K_2)*. Let $(u^n) \subset \widehat{H}^1$ be a PS sequence for $\mathcal{E}$ with $\sup_n \|\nabla u^n\|_2 < \infty$. Then either $\nabla u^n \to 0$ strongly in L^2, or there exist a subsequence (u^n), an integer $k > 0$ and, for every index $i \in \{1, \ldots, k\}$, a mapping $U^i \in \widehat{H}^1$ and a sequence $(g_{n,i})$ of Möbius transformations such that, setting*

$$u^{n,i} = u^n \circ g_{n,i} \quad \text{and} \quad p^{n,i} = \fint_{\mathbb{S}^2} u^{n,i},$$

one has:

(i) *$u^{n,i} - p^{n,i} \to U^i$ weakly in $\widehat{H}^1$ and strongly in $H^1_{loc}(\mathbb{R}^2, \mathbb{R}^3)$;*

(ii) *U^i is an H_{p^i}-bubble, where $H_{p^i} = H(\cdot + p^i)$ if $p^{n,i} \to p^i \in \mathbb{R}^3$, or $H_{p^i} \equiv 1$ if $|p^{n,i}| \to \infty$;*

(iii) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}^2} |\nabla u^n|^2 = \sum_{i=1}^{k} \int_{\mathbb{R}^2} |\nabla U^i|^2$ *and* $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}^2} \nabla(U^i \circ g_{n,i}^{-1}) \cdot \nabla(U^j \circ g_{n,j}^{-1}) \to 0$ *if $i \neq j$;*

(iv) $\displaystyle\lim_{n\to\infty} \mathcal{E}(u^n) = \sum_{i\in I} \mathcal{E}(U^i + p^i) + \sum_{i \in I_\infty} \mathcal{E}_\infty(U^i)$ *, where I is the set of indices i such that $p^{n,i} \to p^i \in \mathbb{R}^3$, and I_∞ is the set of indices i such that $|p^{n,i}| \to \infty$.*

Notice that $U^i + p^i$ is an H-bubble for any $i \in I$.

Proof. Statements (i)–(iii) have already been proved in [6]. It remains only to check part (iv). To this purpose we first denote by I the set of indexes $i \in \{1, ..., k\}$ such that the sequence $(p^{n,i})$ is bounded. We can assume that for any $i \in I$ there exists a point $p^i \in \mathbb{R}^3$ such that $p^{n,i} \to p^i$ as $n \to \infty$. We set also $I_\infty = \{1, ..., k\} \setminus I$ and

$$\theta^n = u^n - \sum_{i=1}^{k} U^i \circ g_{n,i}^{-1} .$$

Notice that

$$\int_{\mathbb{R}^2} |\nabla \theta^n|^2 \to 0 \tag{A.1}$$

by statement (iii). The energy decomposition of $\mathcal{E}(u^n)$, as stated at point (iv) of Theorem A.1 will be proved in two steps.

Step 1: $\mathcal{V}(u^n) = \sum_{i=1}^{k} \mathcal{V}(U^i) + o(1)$.

Denote $\theta^{n,0} = u^n$ and, for $j = 1, \ldots, k$,

$$\theta^{n,j} = u^n - \sum_{i=1}^{j} U^i \circ g_{n,i}^{-1} \quad \text{and} \quad v^{n,j} = \theta^{n,j} \circ g_{n,j} - p^{n,j} + U^j .$$

Notice that for every $j = 1, \ldots, k$

$$v^{n,j} = \theta^{n,j-1} \circ g_{n,j} - p^{n,j} \tag{A.2}$$

$$v^{n,j} \to U^j \quad \text{weakly in } \widehat{H}^1. \tag{A.3}$$

By the invariance of $\mathcal{V}$ with respect to conformality and translations in $\mathbb{R}^3$, one has that

$$\begin{aligned} \mathcal{V}(\theta^{n,j}) = &\ \mathcal{V}(\theta^{n,j} \circ g_{n,j}) = \mathcal{V}(v^{n,j} - U^j) \\ = &\ \mathcal{V}(v^{n,j}) - \underbrace{\int_{\mathbb{R}^2} \left(v^{n,j} \cdot v_x^{n,j} \wedge U_y^j + v^{n,j} \cdot U_x^j \wedge v_y^{n,j} + U^j \cdot v_x^{n,j} \wedge v_y^{n,j} \right)}_{A_{n,j}} \\ & + \underbrace{\int_{\mathbb{R}^2} \left(v^{n,j} \cdot U_x^j \wedge U_y^j + U^j \cdot U_x^j \wedge v_y^{n,j} + U^j \cdot v_x^{n,j} \wedge U_y^j \right)}_{B_{n,j}} - \mathcal{V}(U^j) . \end{aligned}$$

By (A.3) one has that $B_{n,j} \to 3\mathcal{V}(U^j)$. Integration by parts and (A.3) lead to $A_{n,j} = \frac{3}{2} \int_{\mathbb{R}^2} \left(v^{n,j} \cdot v_x^{n,j} \wedge U_y^j + v^{n,j} \cdot U_x^j \wedge v_y^{n,j} \right) = 3\mathcal{V}(U^j) + o(1)$. Hence, by (A.2) and thanks to the invariances of $\mathcal{V}$, one obtains

$$\mathcal{V}(\theta^{n,j}) = \mathcal{V}(v^{n,j}) - \mathcal{V}(U^j) + o(1) = \mathcal{V}(\theta^{n,j-1}) - \mathcal{V}(U^j) + o(1) .$$

Applying this equality for $j = 1, \dots, k$, one infers that

$$\mathcal{V}(\theta^{n,k}) = \mathcal{V}(\theta^{n,0}) - \sum_{i=1}^{k} \mathcal{V}(U^i) + o(1)\,. \tag{A.4}$$

By (A.1) and by the isoperimetric inequality $\mathcal{V}(\theta^n) \to 0$. Since $\theta^{n,0} = u^n$ and $\theta^{n,k} = \theta^n$, the conclusion follows from (A.4).

Step 2: $\mathcal{V}_K(u^n) \to \sum_{i \in I} \mathcal{V}_K(U^i + p^i)$.

For every $i, j = 1, \dots, k$ define

$$U^{n,i} = U^i \circ g_{n,i}^{-1} \quad \text{and} \quad U^{n,i,j} = U^{n,i} \circ g_{n,j}\,.$$

Setting $Q_K(p) = m_K(p)p$, one has that

$$\begin{aligned}\mathcal{V}_K(u^n) = &\int_{\mathbb{R}^2} Q_K(u^n) \cdot \theta^n_x \wedge \theta^n_y + \sum_{i=1}^{k} \int_{\mathbb{R}^2} Q_K(u^n) \cdot \left(\theta^n_x \wedge U^{n,i}_y + U^{n,i}_x \wedge \theta^n_y\right) \\ &+ \sum_{i,j=1}^{k} \int_{\mathbb{R}^2} Q_K(u^n) \cdot U^{n,i}_x \wedge U^{n,j}_y\,.\end{aligned}$$

Since Q_K is bounded by (2.2), one has that

$$\left| \int_{\mathbb{R}^2} Q_K(u^n) \cdot \theta^n_x \wedge \theta^n_y \right| \le \frac{M_K}{4} \int_{\mathbb{R}^2} |\nabla \theta^n|^2 = o(1)$$

$$\left| \int_{\mathbb{R}^2} Q_K(u^n) \cdot \left(\theta^n_x \wedge U^{n,i}_y + U^{n,i}_x \wedge \theta^n_y\right) \right| \le \frac{M_K}{2} \|\nabla U^i\|_2 \|\nabla \theta^n\|_2 = o(1)\,.$$

Moreover

$$\int_{\mathbb{R}^2} Q_K(u^n) \cdot U^{n,i}_x \wedge U^{n,i}_y = \int_{\mathbb{R}^2} Q_K(u^{n,i}) \cdot U^i_x \wedge U^i_y \to \begin{cases} 0 & \text{if } |p^{n,i}| \to \infty \\ \mathcal{V}_K(U^i + p^i) & \text{if } p^{n,i} \to p^i \in \mathbb{R}^3 \end{cases}$$

because $Q_K(p) \to 0$ as $|p| \to \infty$ (by (K_2)) and $|u^{n,i}| \to \infty$ pointwise a.e. if $|p^{n,i}| \to \infty$, whereas $u^{n,i} \to U^i + p^i$ if $p^{n,i} \to p^i$. In addition, if $i \neq j$ then

$$\int_{\mathbb{R}^2} Q_K(u^n) \cdot U^{n,i}_x \wedge U^{n,j}_y = \int_{\mathbb{R}^2} Q_K(u^{n,i}) \cdot U^i_x \wedge U^{n,j,i}_y = \int_{\mathbb{R}^2} \left(Q_K(u^{n,i}) \wedge U^i_x\right) \cdot U^{n,j,i}_y \to 0$$

since the sequence $\left(Q_K(u^{n,i}) \wedge U^i_x\right)$ converges strongly in $L^2(\mathbb{R}^2)$, whereas $U^{n,j,i}_y \to 0$ weakly in $L^2(\mathbb{R}^2)$. Hence the conclusion follows. □

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