SOME REMARKS ON THE CONVERGENCE OF SOLUTIONS TO ELLIPTIC EQUATIONS UNDER WEAK HYPOTHESES ON THE DATA

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Abstract. We study the asymptotic behavior of solutions to elliptic equations of the form

$$\begin{cases} -\operatorname{div}(A_k\nabla u_k) = f_k & \text{in } \Omega, \\ u_k = w_k & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, w_k is weakly converging in $H^1(\Omega)$, f_k is weakly converging in $H^{-1}(\Omega)$, and A_k is a sequence square matrices satisfying some uniform ellipticity and boundedness conditions, and H-converging in Ω. In particular, we characterize the weak limits of the solutions u_k and of their momenta $A_k\nabla u_k$. When A_k is symmetric and $w_k = w = 0$, we characterize the limits of the energies for the solutions.

Keywords: Elliptic equations, G-convergence, H-convergence, Γ-convergence.

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1. Introduction

In this paper, we study the asymptotic behavior, as k tends to $+\infty$, of the solutions u_k to elliptic problems of the form

$$\begin{cases} -\operatorname{div}(A_k\nabla u_k) = f_k & \text{in } \Omega, \\ u_k = w_k & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, A_k are $n\times n$ matrices of $L^\infty(\Omega)$ functions satisfying the usual ellipticity and boundedness conditions, uniformly with respect to k, f_k are elements of the dual $H^{-1}(\Omega)$ of the Sobolev space $H^1_0(\Omega)$, while w_k is a sequence of elements of $H^1(\Omega)$ weakly converging to w in $H^1(\Omega)$, and the equality $u_k = w_k$ is intended in the usual sense $u_k - w_k \in H^1_0(\Omega)$.

These problems have been extensively studied in the 70s. Since in many applications the coefficients of the matrices A_k are strongly oscillating and the weak convergence is not useful for these problems, two new notions of convergence for the coefficients were introduced: G-convergence for the symmetric case (see [7, 14, 15]) and H-convergence for the general case (see [10, 11]). These notions have been widely used in homogenization problems, where, in the periodic case, the limit matrix can be obtained by solving some auxiliary problems in the periodicity cell (see [1, 2, 4, 8, 9, 12, 13]).

The main result concerning H-convergence is the following (see [11, Theorem 2]): if A_k H-converges to A in Ω and f_k strongly converges to f in $H^{-1}(\Omega)$ then

$$u_k \rightharpoonup u \quad \text{weakly in } H^1(\Omega),$$
$$A_k\nabla u_k \rightharpoonup A\nabla u \quad \text{weakly in } [L^2(\Omega)]^n,$$

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where u is the solution to the problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega, \\ u = w & \text{on } \partial\Omega. \end{cases}$$

The novelty of our paper is that we assume only that f_k converges to f weakly in $H^{-1}(\Omega)$. We prove (see Theorem 3.2) that, up to a subsequence, there exist $g \in H^{-1}(\Omega)$ and $\psi \in [L^2(\Omega)]^n$ such that for every open set $U \subset \Omega$ and every sequence $w_k \in H^1(U)$ weakly converging in $H^1(U)$ to some function w we have that the solutions v_k of the problems

$$\begin{cases} -\mathrm{div}(A\nabla v_k) = f_k & \text{in } U, \\ v_k = w_k & \text{on } \partial\Omega, \end{cases}$$

satisfy the following two conditions

$$v_k \rightharpoonup v \quad \text{weakly in } H^1(U),$$
$$A_k\nabla v_k \rightharpoonup A\nabla v + \psi \quad \text{weakly in } [L^2(U)]^n,$$

where v is the solution to the problem

$$\begin{cases} -\mathrm{div}(A\nabla v) = g & \text{in } U, \\ v = w & \text{on } \partial U. \end{cases}$$

We note that both g and ψ depend on the sequences A_k and f_k, but not on w_k nor on U. When A_k does not converge pointwise and f_k does not converge strongly, it is easy to construct simple examples, even in dimension one, showing that we might have $g \neq f$ and $\psi \neq 0$.

We first study the problems in Ω when $w = 0$. Since the functions u_k are bounded in $H^1_0(\Omega)$, we extract a non-relabeled subsequence converging weakly in $H^1_0(\Omega)$ to some function u, and we define $g = -\mathrm{div}(A\nabla u)$. Since the vector functions $A_k\nabla u_k$ are bounded in $[L^2(\Omega)]^n$, we extract a non-relabeled subsequence such that $A_k\nabla u_k$ converges weakly in $[L^2(\Omega)]^n$ to a limit, that we express as $A\nabla u + \psi$. To prove that g and ψ do not depend on w_k nor on U we use known results of the theory of H-convergence.

If the matrices A_k are symmetric and $w = 0$, then for every open set $U \subset \Omega$ the solutions u^U_k, u^U to the problems

$$\begin{cases} -\mathrm{div}(A_k\nabla u^U_k) = f_k & \text{in } U, \\ u^U_k \in H^1_0(U), \end{cases} \quad \text{and} \quad \begin{cases} -\mathrm{div}(A_k\nabla u^U) = g & \text{in } U, \\ u^U \in H^1_0(U), \end{cases} \tag{1.2}$$

can be characterized as the minimizers of the functionals defined for every $v \in H^1_0(U)$ as

$$F_k(v, U) = \frac{1}{2}\int_U A_k\nabla v \cdot \nabla v dx - \langle f_k, \tilde{v}\rangle,$$
$$F_0(v, U) = \frac{1}{2}\int_U A\nabla v \cdot \nabla v dx - \langle g, \tilde{v}\rangle,$$

where $\tilde{v}$ is the extension to Ω of v obtained by setting $\tilde{v} = 0$ outside U.

Therefore, in the symmetric case we can obtain some useful information on the behavior of the sequence u^U_k using Γ-convergence techniques, for which we refer to [5] and [3]. Although in general the sequence $F_k(\cdot, U)$ does not Γ-converge to $F_0(\cdot, U)$, we can prove the following result (see Theorem 4.2): there exist a subsequence, non-relabeled, and a bounded non-negative Radon measure ν on Ω such that

$$F_k(\cdot, U) \quad \Gamma\text{-converges to} \quad F_0(\cdot, U) - \nu(U) \quad \text{in the weak topology of } H^1_0(U), \tag{1.3}$$

for "almost every" open set $U \subset \Omega$ in the sense of Definition 4.1. This implies that the energies $F_k(u^U_k, U)$ converge to $F_0(u^U, U) - \nu(U)$, hence $\nu(U)$ can be interpreted as the asymptotic energy gap between u^U and the sequence u^U_k. Simple examples, even in dimension one, show that if A_k does not converge pointwise to A and f_k does not converge strongly to f, then the measure ν might be non-trivial.

2. Notation and Preliminary results

Throughout the paper Ω will always denote an open bounded subset of $\mathbb{R}^n$, $n \geq 1$. The space $H^1(\Omega)$ is the space of $L^2(\Omega)$ functions whose first order distributional derivatives are in $L^2(\Omega)$. The space $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$, the closure in $H^1(\Omega)$ of $C_c^\infty(\Omega)$. We will endow $H_0^1(\Omega)$ with the norm

$$||u||_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u(x)|^2 dx \right)^{\frac{1}{2}},$$

which by Poincaré inequality is equivalent on $H_0^1(\Omega)$ to the usual norm of $H^1(\Omega)$. We denote the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ with $\langle \cdot, \cdot \rangle$. If $U \subset \Omega$ is an open set, $f \in H^{-1}(\Omega)$, and $u \in H_0^1(U)$ we denote by $\langle f, u \rangle$ the real number $\langle f, \tilde{u} \rangle$, where $\tilde{u}$ is the extension to zero of u outside of U.

From now on α, β will be two positive constants satisfying $0 < \alpha \leq \beta < +\infty$. We denote by $\mathcal{M}_\alpha^\beta(\Omega)$ the space of matrices $A \in [L^\infty(\Omega)]^{n\times n}$ such that

$$\alpha|\xi|^2 \leq A(x)\xi \cdot \xi, \quad |A(x)\xi|^2 \leq \beta A(x)\xi \cdot \xi, \qquad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^n. \tag{2.1}$$

For every $n \times n$ matrix B the operatorial norm $|B|$ is defined as

$$|B| = \sup\{|B\xi| : \xi \in \mathbb{R}^n, |\xi| \leq 1\}.$$

Notice that the second inequality in (2.1) implies

$$|A(x)| \leq \beta \quad \text{ for a.e. } x \in \Omega. \tag{2.2}$$

We endow the space $\mathcal{M}_\alpha^\beta(\Omega)$ with the H-convergence, introduced by Murat and Tartar in the 70s (see for instance [11]).

Definition 2.1 (H-convergence)**.** Let A_k, $A \in \mathcal{M}_\alpha^\beta(\Omega)$. The sequence A_k is said to H-converge to A in Ω if and only if for every $f \in H^{-1}(\Omega)$ the solutions u_k, u to the problems

$$\begin{cases} -\text{div}(A_k \nabla u_k) = f & \text{in } \Omega, \\ u_k \in H_0^1(\Omega), \end{cases} \qquad \begin{cases} -\text{div}(A \nabla u) = f & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

satisfy the following conditions:

(1) (*convergence of solutions*) $u_k \rightharpoonup u$ weakly in $H_0^1(\Omega)$;
(2) (*convergence of momenta*) $A_k \nabla u_k \rightharpoonup A\nabla u$ weakly in $[L^2(\Omega)]^n$.

On the subspace of $\mathcal{M}_\alpha^\beta(\Omega)$ of symmetric matrices Spagnolo introduced in 1968 (see [15]) the similar notion of G-convergence.

Definition 2.2. Let A_k, $A \in \mathcal{M}_\alpha^\beta(\Omega)$ be symmetric. The sequence A_k is said to G-converge to A in Ω if and only if for every $f \in H^{-1}(\Omega)$ condition (1) of Definition 2.1 is satisfied.

It turns out that for matrices that are symmetric the notion of H-convergence and of G-convergence are equivalent. For a proof of this result, we refer the reader to [7] and to [4, Proposition 13.6].

Proposition 2.3. *Let A_k, $A \in \mathcal{M}_\alpha^\beta(\Omega)$ be symmetric. Then A_k G-converges to A in Ω if and only if A_k H-converges to A in Ω.*

It is important to notice that $\mathcal{M}_\alpha^\beta(\Omega)$ is sequentially compact with respect to the H-convergence.

Theorem 2.4 ([11, Theorem 2])**.** *Let $A_k \in \mathcal{M}_\alpha^\beta(\Omega)$. Then there exists an $A \in \mathcal{M}(\Omega)$ and a subsequence A_{h_k} H-converging to A in Ω. In particular, if A_k is symmetric, A is symmetric as well.*

Given A_k, $A \in \mathcal{M}_\alpha^\beta(\Omega)$ symmetric and $U \subset \Omega$ open, we denote by $Q_k(\cdot, U)$, $Q(\cdot, U)$ the functionals defined for every $v \in H_0^1(U)$ bas

$$Q_k(v, U) = \frac{1}{2}\int_U A_k \nabla v \cdot \nabla v dx, \tag{2.3}$$

$$Q(v, U) = \frac{1}{2}\int_U A \nabla v \cdot \nabla v dx. \tag{2.4}$$

The H-convergence of a sequence of symmetric matrices can be characterized in terms of the Γ-convergence of the associated quadratic functionals (2.3). For the main properties of the Γ-convergence, we refer the reader to [3, 5].

Theorem 2.5 ([5, Theorem 24.5])**.** *Let A_k, $A \in \mathcal{M}_\alpha^\beta(\Omega)$ be symmetric. Then A_k H-converges to A in Ω if and only if $Q_k(\cdot, \Omega)$ Γ-converges to $Q(\cdot, \Omega)$ in the weak topology of $H_0^1(\Omega)$.*

3. Asymptotic behavior of solutions to Elliptic problems

Let $A_k \in \mathcal{M}_\alpha^\beta(\Omega)$ be H-convergent to $A \in \mathcal{M}_\alpha^\beta(\Omega)$ in Ω and let $f_k \in H^{-1}(\Omega)$ be convergent to f weakly in $H^{-1}(\Omega)$. It is easy to see that in general the sequence of solutions u_k to

$$\begin{cases} -\mathrm{div}(A_k \nabla u_k) = f_k & \text{in } \Omega, \\ u_k \in H_0^1(\Omega), \end{cases} \tag{3.1}$$

does not converge to u^f the solution to

$$\begin{cases} -\mathrm{div}(A \nabla u^f) = f & \text{in } \Omega, \\ u^f \in H_0^1(\Omega). \end{cases}$$

From now on, u_k will always denote the sequence of solutions to (3.1). Since by Lax-Milgram Theorem the $H_0^1(\Omega)$ norm of u_k is bounded by the $H^{-1}(\Omega)$ norm of f_k and since A_k satisfies (2.2), there exists a suitable non-relabeled subsequence such that u_k converges weakly in $H_0^1(\Omega)$ and $A_k \nabla u_k$ converges weakly in $[L^2(\Omega)]^n$. Without loss of generality, we will always suppose that for a certain function $u \in H_0^1(\Omega)$,

$$u_k \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \tag{3.2}$$

$$A_k \nabla u_k \rightharpoonup A \nabla u + \psi \quad \text{weakly in } [L^2(\Omega)]^n, \tag{3.3}$$

where ψ is a $[L^2(\Omega)]^n$ vector field. We now introduce the linear functional $g \in H^{-1}(\Omega)$ defined as

$$g = -\mathrm{div}(A \nabla u). \tag{3.4}$$

It is obvious that u is the solution to

$$\begin{cases} -\mathrm{div}(A \nabla u) = g & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{3.5}$$

The term g and the vector field ψ, which depend on the sequences A_k, f_k and on the open set Ω, can be used to study the asymptotic behavior of solutions to elliptic problems on open subsets $U \subset \Omega$, with possibly non-zero boundary conditions. To prove these results we need the following lemma.

Lemma 3.1 ([11, Theorem 1])**.** *Let $A_k \in \mathcal{M}_\alpha^\beta(\Omega)$ be H-converging to A in Ω and let $U \subset \Omega$ be an open subset. Suppose that the following conditions are all satisfied:*

(1) $v_k \in H^1(U)$,
(2) $f_k \in H^{-1}(U)$,
(3) $-\mathrm{div}(A_k \nabla v_k) = f_k$ *in* U,
(4) v_k *converges to* v *weakly in* $H^1(U)$,
(5) f_k *converges to* f *strongly* $H^{-1}(U)$.

Then $A_k\nabla v_k$ *converges to* $A\nabla v$ *weakly in* $[L^2(U)]^n$ *and* $A_k\nabla v_k\cdot\nabla v_k$ *converges to* $A\nabla v\cdot\nabla v$ *in the sense of distributions on* U.

We are now ready to study the problems on an open subset of Ω.

Theorem 3.2. *Let* w_k, $w \in H^1(U)$ *and let* v_k *be the solution to the non-homogeneous boundary value problem*

$$\begin{cases} -\mathrm{div}(A_k\nabla v_k) = f_k & \text{in } U,\\ v_k - w_k \in H^1_0(U). \end{cases}$$

Assume that w_k *converges to* w *weakly in* $H^1(U)$. *Then* v_k *converges weakly in* $H^1(U)$ *to the solution* v *to the problem*

$$\begin{cases} -\mathrm{div}(A\nabla v) = g & \text{in } U,\\ v - w \in H^1_0(U), \end{cases} \tag{3.6}$$

and $A_k\nabla v_k$ *converges to* $A\nabla v + \psi$ *weakly in* $[L^2(U)]^n$.

Proof. From the uniform ellipticity of the matrices A_k and from the boundedness of the two sequences $||w_k||_{H^1(U)}$ and $||f_k||_{H^{-1}(\Omega)}$, we obtain that the sequence v_k is bounded in $H^1(U)$. Therefore passing to a non-relabeled subsequence we may assume that v_k converges weakly in $H^1(U)$ to a function v^*, with $v^* - w \in H^1_0(\Omega)$, and $A_k\nabla w$ converges weakly in $[L^2(U)]^n$ to a vector function Φ.

We want to prove that v^* and Φ are independent of the chosen subsequence and that $v^* = v$ and $\Phi = A\nabla v + \psi$. Consider the sequence $z_k = v_k - u_k \in H^1(U)$, which converges to $z = v^* - u$ weakly in $H^1(U)$.

Since

$$-\mathrm{div}(A_k\nabla z_k) = 0 \quad \text{in } U,$$

Lemma 3.1 implies that

$$-\mathrm{div}(A\nabla(v^* - u)) = -\mathrm{div}(A\nabla z) = 0 \quad \text{in } U,$$

and in addition that $A_k\nabla z_k$ converges to $A\nabla z$ weakly in $[L^2(U)]^n$. Since $v^* - w \in H^1_0(\Omega)$, recalling the definition of g (see (3.4)), we obtain that v^* is a solution of the problem

$$\begin{cases} -\mathrm{div}(A\nabla v^*) = g & \text{in } U,\\ v^* - w \in H^1_0(U). \end{cases}$$

By uniqueness of the solution to (3.6), we have proved that $v^* = v$. Since the limit does not depend on the chosen subsequence, the whole sequence v_k converges to v weakly in $H^1(U)$.

Since

$$A_k\nabla v_k - A_k\nabla u_k = A_k\nabla z_k \rightharpoonup A\nabla z = A\nabla v - A\nabla u \quad \text{weakly in } [L^2(U)]^n,$$

(3.3) implies that

$$A\nabla v_k \rightharpoonup A\nabla v + \psi \quad \text{weakly in } [L^2(U)]^n,$$

hence $\Phi = A\nabla v + \psi$ and the whole sequence $A_k\nabla v_k$ converges to $A\nabla v + \psi$ weakly in $[L^2(U)]^n$. □

4. The symmetric case

Suppose now that the sequence $A_k \in \mathcal{M}^\beta_\alpha(\Omega)$ and its H-limit A are symmetric and that $U \subset \Omega$ is an open subset. From now on, u^U_k will denote the sequence of solutions to the elliptic problems

$$\begin{cases} -\mathrm{div}(A_k\nabla u^U_k) = f_k & \text{in } U,\\ u^U_k \in H^1_0(U). \end{cases} \tag{4.1}$$

We will denote by u^U the weak limit in $H^1_0(\Omega)$ of u^U_k (which exists by Theorem 3.2), and by g the linear term defined in (3.4). Under the additional hypothesis of symmetry, Theorem

2.5 and Theorem 3.2 allow us to compute the Γ-limit $F(\cdot,U)$ in the weak topology of $H^1_0(U)$ of the functionals $F_k(\cdot,U)$ defined for $v\in H^1_0(\Omega)$ as

$$F_k(v,U)=\frac{1}{2}\int_U A_k\nabla v\cdot\nabla v dx-\langle f_k,v\rangle=Q_k(v,U)-\langle g,v\rangle, \tag{4.2}$$

where $Q_k(\cdot,U)$ is the quadratic functional defined by (2.3).

As we will see, the Γ-limit $F(\cdot,U)$ is closely related to the functional $F_0(\cdot,U)$, defined for $v\in H^1_0(U)$ as

$$F_0(v,U)=\frac{1}{2}\int_U A\nabla v\cdot\nabla v dx-\langle g,v\rangle=Q(v,U)-\langle g,v\rangle, \tag{4.3}$$

where $Q(\cdot,U)$ is the quadratic functional defined by (2.4).

Before stating the main result of this section, we briefly recall a definition in set theory (see [5, Definition 14.10]).

Definition 4.1. Let $V\subset\mathbb{R}^n$ be an open set. We say that a collection $\mathcal{R}$ of open subsets of V is rich if for every family $(U_t)_{t\in(0,1)}$ of open subsets of V satisfying the property that if $t_1,t_2\in(0,1)$ and $t_1<t_2$ then $U_{t_1}\subset\subset U_{t_2}$, the set $\{t\in(0,1):\ U_t\notin\mathcal{R}\}$ is at most countable.

Theorem 4.2. *Let $A_k\in\mathcal{M}^\beta_\alpha(\Omega)$ be a sequence of symmetric matrices H-converging to $A\in\mathcal{M}^\beta_\alpha(\Omega)$ in Ω. Consider the sequence of functionals $F_k(\cdot,U)$ defined by (4.2) for $U\subset\Omega$ open. Then there exist a non-relabeled subsequence, a non-negative bounded Radon measure ν on Ω, and a rich collection $\mathcal{R}$ of open subsets of Ω such that for every $U\in\mathcal{R}$ the sequence $F_k(\cdot,U)$ Γ-converges in the weak topology of $H^1_0(U)$ to the functional defined as*

$$F(v,U)=F_0(v,U)-\nu(U)\quad\text{for every } v\in H^1_0(U). \tag{4.4}$$

To prove Theorem 4.2 we proceed in several steps. The first one is to compute the Γ-$\liminf$ and the Γ-$\limsup$ of $F_k(\cdot,U)$. To deal with this problem, we introduce two set functions ν', ν'' defined for every open set $U\subset\Omega$ by

$$-\nu'(U)=\inf_{\substack{v_k\in H^1_0(U)\\ v_k\rightharpoonup 0}}\ \liminf_{k\to+\infty}F_k(v_k,U), \tag{4.5}$$

$$-\nu''(U)=\inf_{\substack{v_k\in H^1_0(U)\\ v_k\rightharpoonup 0}}\ \limsup_{k\to+\infty}F_k(v_k,U). \tag{4.6}$$

It is easy to see that the infimum is attained. It is immediate to check that the set functions ν' and ν'' are both increasing and bounded, and that $0\le\nu''\le\nu'$.

Notice that, being the functionals $F_k(\cdot,U)$ equi-coercive in the weak topology of $H^1_0(U)$, the Γ-convergence of the sequence $F_k(\cdot,U)$ can be characterized sequentially (see for instance [5, Theorem 8.17]), so that $-\nu'(U)$ and $-\nu''(U)$ are precisely the value of the Γ-$\liminf$ and of the Γ-$\limsup$ of $F_k(\cdot,U)$ at $v=0$. In what follows the Γ-$\liminf$ and the Γ-$\limsup$ of $F_k(\cdot,U)$ in the weak topology of $H^1_0(U)$ will always be denoted by $F'(\cdot,U)$ and by $F''(\cdot,U)$ and, when they coincide, the Γ-limit will be denoted by $F(\cdot,U)$.

Proposition 4.3. *Let $U\subset\Omega$ be open and let $F_k(\cdot,U)$ be the functionals defined in (4.2). Then:*

(1) *For every $v\in H^1_0(U)$*

$$F'(v,U)=F_0(v,U)-\nu'(U); \tag{4.7}$$

(2) *For every $v\in H^1_0(U)$*

$$F''(v,U)=F_0(v,U)-\nu''(U). \tag{4.8}$$

Proof. We prove only (4.7), the proof of (4.8) being analogous. Let $Q_k(\cdot, U)$ and $Q(\cdot, U)$ the quadratic functionals defined in (2.3) and (2.4). We introduce the non-negative constants α_k, α', α'', and α defined as

$$\alpha_k = Q_k(u_k^U, U), \quad \alpha' = \liminf_{k\to+\infty} \alpha_k, \quad \alpha'' = \limsup_{k\to+\infty} \alpha_k, \quad \alpha = Q(u^U, U).$$

Since u_k^U is the solution to (4.1) it follows that

$$F_k(v, U) + \alpha_k = Q_k(v - u_k^U, U) \qquad \text{for every } v \in H_0^1(U). \tag{4.9}$$

Fix $v \in H_0^1(U)$. Theorem 2.5 then implies that $Q_k(\cdot, U)$ Γ-converges to $Q(\cdot, U)$ in the weak topology of $H_0^1(U)$, so that if $v_k \rightharpoonup v$ weakly in $H_0^1(U)$ the liminf inequality and (4.9) yield

$$\begin{aligned}\liminf_{k\to+\infty} F_k(v_k, U) + \alpha'' \geq \liminf_{k\to+\infty}(F_k(v_k, U) + \alpha_k) &= \liminf_{k\to+\infty} Q_k(v_k - u_k^U, U)\\ &\geq Q(v - u^U, U) = F_0(v, U) + \alpha, \end{aligned} \tag{4.10}$$

whence, taking the infimum with respect to all the sequences v_k converging to v weakly in $H_0^1(U)$,

$$F'(v, U) \geq F_0(v, U) + \alpha - \alpha''. \tag{4.11}$$

To prove the converse inequality, we argue as follows. Fix $v \in H_0^1(U)$ and consider a sequence $z_k \in H_0^1(U)$ converging to $v - u^U$ weakly in $H_0^1(U)$, so that $v_k = z_k + u_k^U$ converges weakly to v. Then (4.9) yields

$$F'(v, U) + \alpha'' \leq \limsup_{k\to+\infty}(F_k(v_k, U) + \alpha_k) = \limsup_{k\to+\infty} Q_k(z_k, U).$$

Since this last inequality holds for any $z_k \rightharpoonup z$ weakly in $H_0^1(U)$ and $Q_k(\cdot, U)$ Γ-converges to $Q(\cdot, U)$, we obtain that $F'(v, U) \leq Q(z, U) - \alpha''$. Hence (4.10) and (4.11) finally imply that

$$F'(v, U) = F_0(v, U) + \alpha - \alpha''.$$

Evaluating this last expression at $v = 0$, one gets $\nu'(U) = \alpha'' - \alpha$, concluding the proof. □

Corollary 4.4. *The sequence $F_k(\cdot, U)$ Γ-converges in the weak topology of $H_0^1(U)$ if and only if $\nu'(U) = \nu''(U)$, and in that case the Γ-limit is given by*

$$F(v, U) = F_0(v, U) - \nu'(U) = F_0(v, U) - \nu''(U) \quad \text{for every } v \in H_0^1(U).$$

The second step of our analysis is to make sure that the set $\mathcal{U} := \{U \subset \Omega : U \text{ open}, \nu'(U) = \nu''(U)\}$ is rich and to construct the measure ν appearing in (4.4). To this aim, we fix a countable dense collection $\mathcal{D}$ of open subsets of Ω (i.e., we assume that for all open sets U, W such that $U \subset\subset W \subset \Omega$ there exists $V \in \mathcal{D}$ such that $U \subset\subset V \subset\subset W$). By a diagonal argument and by the compactness of Γ-convergence (see [5, Corollary 8.12]), we can pass to a non-relabeled subsequence such that $F_k(\cdot, U)$ Γ-converges in the weak topology of $H_0^1(U)$ for every $U \in \mathcal{D}$. Therefore from now on, we will always suppose that $\mathcal{U}$ itself is dense.

Finally, we denote by ν the common inner regular envelope of ν' and ν'', i.e., the increasing set function defined for $U \subset \Omega$ open by

$$\nu(U) = \sup\{\nu'(V) : V \text{ open}, V \subset\subset U\} = \sup\{\nu''(V) : V \text{ open}, V \subset\subset U\}.$$

Remark 4.5. Since ν' and ν'' are increasing and the set $\mathcal{U}$ is dense, by [5, Proposition 14.14] $\mathcal{U}$ is actually rich.

We now prove that ν extends to a Borel measure on Ω.

Proposition 4.6. *The set function ν is the restriction to the class of open subsets of Ω of a Borel measure μ.*

Proof. The proof is based on De Giorgi-Letta Theorem (see [6, Theorem 5.6] for the original result and see [5, Theorem 14.23] for the particular statement used in this proof). Since ν is clearly increasing and inner regular, we only need to prove that it is subadditive and superadditive to conclude that it extends to a Borel measure on Ω.

To prove that ν is superadditive it is enough to show that ν'' is superadditive (see [5, Proposition 14.18]). Let V, $W \subset \Omega$ be open sets with $V \cap W = \emptyset$. Consider two sequences $v_k \in H_0^1(V)$ and $w_k \in H_0^1(W)$ such that $v_k \rightharpoonup 0$ weakly in $H_0^1(V)$, $w_k \rightharpoonup 0$ weakly in $H_0^1(W)$, and

$$-\nu''(V) = \limsup_{k\to+\infty} F_k(v_k, V), \quad -\nu''(W) = \limsup_{k\to+\infty} F_k(w_k, W).$$

Let $z_k \in H_0^1(V \cup W)$ be equal to v_k on V and to w_k on W. Then

$$-\nu''(V \cup W) \le \limsup_{k\to\infty} F_k(z_k, V \cup W) \le -\nu''(V) - \nu''(W),$$

which concludes the proof of superadditivity.

Arguing as in the proof of [5, Proposition 18.4], we see that, the subadditivity of ν follows from the following property: if V', V, $W \subset \Omega$ are open sets with $V' \subset\subset V \subset \Omega$, then

$$\nu'(V' \cup W) \le \nu'(V) + \nu'(W). \tag{4.12}$$

To prove this inequality we argue as follows. Fix a cut off function φ between V' and V (i.e., $\varphi \in C_c^\infty(\Omega)$, $\operatorname{supp}(\varphi) \subset V$, $\varphi = 1$ in a neighborhood of $\overline{V'}$, and $0 \le \varphi \le 1$ in Ω) and consider a sequence $z_k \in H_0^1(V' \cup W)$ such that $z_k \rightharpoonup 0$ weakly in $H_0^1(V' \cup W)$ and

$$-\nu'(V' \cup W) = \liminf_{k\to+\infty} F_k(z_k, V' \cup W).$$

We define $v_k = \varphi z_k$ and $w_k = (1-\varphi) z_k$, and we observe that $v_k \in H_0^1(V)$, $w_k \in H_0^1(W)$, that $v_k \rightharpoonup 0$ weakly in $H_0^1(V)$ and $w_k \rightharpoonup 0$ weakly in $H_0^1(W)$. Therefore

$$\begin{aligned} -\nu'(V) - \nu'(W) &\le \liminf_{k\to+\infty} F_k(v_k, V) + \liminf_{k\to+\infty} F_k(w_k, W) \le \liminf_{k\to+\infty} (F_k(v_k, V) \\ &\quad + F_k(w_k, W)) = \liminf_{k\to+\infty} (Q_k(v_k, V) + Q_k(w_k, W) - \langle f_k, z_k\rangle). \end{aligned} \tag{4.13}$$

We note that, since $\varphi^2 \le \varphi$, we have

$$Q_k(v_k, V) \le \frac{1}{2}\int_V \varphi A_k \nabla z_k \cdot \nabla z_k dx + \frac{1}{2}\int_V z_k^2 A_k \nabla\varphi \cdot \nabla\varphi dx + \int_V \varphi z_k A_k \nabla z_k \cdot \nabla \varphi dx,$$

and a similar inequality holds for $Q_k(w_k, W)$, with V and φ replaced by W and $1-\varphi$. Since the sequence z_k converges to zero strongly in $L^2(V' \cup W)$ by Rellich Compactness Theorem, we have

$$Q_k(v_k, V) + Q_k(w_k, W) \le \frac{1}{2}\int_{V' \cup W} A_k \nabla z_k \cdot \nabla z_k dx + \varepsilon_k,$$

with $\varepsilon_k \to 0$ as k tends to $+\infty$. Therefore (4.13) implies

$$-\nu'(V) - \nu'(W) \le \liminf_{k\to+\infty} F_k(z_k, V' \cup W) = -\nu'(V' \cup W).$$

This proves (4.12) and concludes the proof of the theorem. □

Proof of Theorem 4.2. The result follows from Corollary 4.4, Remark 4.5, and Proposition 4.6. □

The following proposition shows that there may exist an open set $U_0 \subset \Omega$ such that ν' and ν'' are not regular at U_0, hence $\nu(U_0) < \nu''(U_0) < \nu'(U_0)$. As a consequence of this, Corollary 4.4 implies that (4.4) can not hold for $U = U_0$.

Proposition 4.7. *Assume $n \geq 2$ and that $A_k = A = I$, where I is the identity matrix. Let $U_0 \subset\subset \Omega$ be open and $x_0 \in \partial U_0$. Then there exists a sequence $f_k \in L^\infty(\Omega)$ converging to zero weakly in $H^{-1}(\Omega)$ and such that if ν', ν'' are the set functions defined by* (4.5) *and* (4.6) *then*

(1) *$\nu'(U) = \nu''(U) = 0$ for every $U \subset \Omega$ open such that $x_0 \notin \overline{U}$;*
(2) *$\nu''(U_0) > 0$.*

Proof. We begin by constructing an auxiliary sequence of functions g_k supported on balls centered at 0. The sequence f_k will be obtained as a suitable translation of these g_k. Fix $0 < R < \operatorname{dist}(U_0, \Omega^c)$ and two sequences $0 < r_k < R_k < R$ monotonically converging to 0. Consider the sequence $g_k = c_k \chi_{B(0,r_k)}$, where $\chi_{B(0,r_k)}$ is the characteristic function of $B(0, r_k)$, the open ball of center 0 and radius r_k, and $c_k = 8||\chi_{B(0,r_k)}||^{-1}_{H^{-1}(B(0,R))}$. Since $||g_k||_{H^{-1}(B(0,R))} = 8$, there exists a non-relabeled subsequence converging weakly in $H^{-1}(B(0,R))$ to some g_0. Since $r_k \to 0$, if $\varphi \in C_c^\infty(B(0,R))$ vanishes in a neighborhood of 0 we have that $\langle g_0, \varphi\rangle = 0$. On the other hand, every function in $H_0^1(\Omega)$ can be approximated strongly in $H_0^1(\Omega)$ by a sequence of functions in $C_c^\infty(\Omega)$ vanishing near 0, so that by the previous remark we conclude that $\langle g_0, v\rangle = 0$ for every $v \in H_0^1(\Omega)$, hence $g_0 = 0$. Since this result is not dependent on the subsequence, we conclude that the whole sequence g_k converges to 0 weakly in $H^{-1}(B(0,R))$.

Since $||g_k||_{H^{-1}(B(0,R))} = 8$, for every k there exists $\zeta_k \in H_0^1(B(0,R))$ with $||\zeta_k||_{H_0^1(B(0,R))} \leq 1$ such that

$$\int_{B(0,R)} g_k \zeta_k dx \geq 8 - \frac{1}{k}.$$

Passing to a non-relabeled subsequence we might assume that ζ_k converges to ζ_0 weakly in $H_0^1(B(0,R))$, with $||\zeta_0||_{H_0^1(B(0,R))} \leq 1$. Let $v_k = \zeta_k - \zeta_0$. Then $||v_k||_{H_0^1(B(0,R))} \leq 2$ and

$$\int_{B(0,R)} g_k v_k \geq 8 - \varepsilon_k, \tag{4.14}$$

where $\varepsilon_k = \frac{1}{k} - \langle g_k, \zeta_0\rangle$. Notice that ε_k converges to zero since g_k converges to 0 weakly in $H^{-1}(B(0,R))$.

Consider a sequence $\varphi_k \in C_c^\infty(\Omega)$ with $\operatorname{supp}(\varphi_k) \subset B(0,R_k)$, $\varphi_k = 1$ on $B(0,r_k)$, $0 \leq \varphi_k \leq 1$ on $B(0,R)$, and such that $|\nabla\varphi_k| \leq \frac{2}{R_k - r_k}$. Consider the sequence $w_k = \varphi_k v_k$, which obviously still satisfies (4.14). We claim that there exists a choice of r_k and R_k such that w_k converges to 0 weakly in $H_0^1(B(0,R))$. To prove this, it is enough to show that $||w_k||_{H_0^1(B(0,R))}$ is bounded, since by Rellich compactness Theorem and dominated convergence Theorem the sequence w_k converges to 0 strongly in $L^2(B(0,R))$. By direct computations

$$\begin{aligned}
\int_{B(0,R)} |\nabla w_k|^2 dx &= \int_{B(0,R)} (|\nabla v_k|^2 \varphi_k^2 + |\nabla\varphi_k|^2 v_k^2 + 2 v_k \varphi_k \nabla\varphi_k \cdot \nabla v_k) dx \\
&\leq 2 + \frac{4}{(R_k - r_k)^2} \int_{B(0,R)} v_k^2 + \frac{4}{R_k - r_k} \Big(\int_{B(0,R)} v_k^2 dx\Big)^{1/2} \Big(\int_{B(0,R)} |\nabla v_k|^2 dx\Big)^{1/2} \\
&\leq 2 + \frac{4}{R_k - r_k} \Big(\int_{B(0,R)} v_k^2 dx\Big)^{1/2} \Big(\frac{1}{R_k - r_k} \Big(\int_{B(0,R)} v_k^2 dx\Big)^{1/2} + 2\Big).
\end{aligned}$$

Recalling that v_k converges weakly to 0 in $H_0^1(B(0,R))$ and hence strongly in $L^2(B(0,R))$ by Rellich Compactness Theorem, we can choose R_k, r_k so that $\frac{1}{R_k - r_k}(\int_{B(0,R)} v_k^2 dx)^{\frac{1}{2}} \leq 1$. It follows that $||w_k||^2_{H_0^1(B(0,R))} \leq 14$, concluding the proof of the fact that w_k converges to 0 weakly in $H_0^1(B(0,R))$.

Fix now a sequence $x_k \in U_0$ converging to x_0 and such that $B(x_k, R_k) \subset U_0$ for $k \in \mathbb{N}$ large enough. We set $f_k(x) = g_k(x - x_k)$ and $z_k(x) = w_k(x - x_k)$. Clearly, f_k converges to 0 weakly in $H^{-1}(\Omega)$ and z_k converges to 0 weakly in $H_0^1(U_0)$. Since w_k coincides with v_k on $B(0, r_k)$ and v_k satisfies (4.14), it follows that

$$\int_\Omega f_k z_k dx \geq 8 - \varepsilon_k.$$

Since $||w_k||_{H_0^1(B(0,R))} \leq 14$, the previous inequality gives

$$F_k(z_k, U_0) = \frac{1}{2} \int_{U_0} |\nabla z_k|^2 dx - \int_{U_0} f_k z_k dx \leq 7 - 8 + \varepsilon_k.$$

Recalling the definition of ν'', we obtain

$$-\nu''(U_0) \leq \limsup_{k \to +\infty} F_k(z_k, U_0) \leq -1.$$

This concludes the proof of (2). Condition (1) is trivially satisfied and the proof of the proposition is thus concluded. □

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