

An example of a weakly mixing BV vector field which is not strongly mixing

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Abstract

We give an example of a weakly mixing vector field $b \in L^\infty([0, 1], \text{BV}(\mathbb{T}^2))$ which is not strongly mixing, in the setting first introduced in [3]. The example is based on a work of Chacon [4] who constructed a weakly mixing automorphism which is not strongly mixing on $([0, 1], \mathcal{B}([0, 1]), |\cdot|)$, where $\mathcal{B}([0, 1])$ are the Borel subsets of $[0, 1]$ and $|\cdot|$ is the one-dimensional Lebesgue measure.

Key words: weak mixing, automorphisms, divergence-free vector fields, Regular Lagrangian Flows.

MSC2020: 35Q35, 37A25.

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1 Introduction

We consider the measure space $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), |\cdot|)$ where the torus \mathbb{T}^2 is the unit square $[0, 1]^2$ with periodic boundary conditions, $\mathcal{B}(\mathbb{T}^2)$ denotes the Borel subsets of \mathbb{T}^2 and $|\cdot|$ represents the normalized Lebesgue measure. We consider the group of automorphisms on the torus, that is

$$G(\mathbb{T}^2) = \{T : \mathbb{T}^2 \rightarrow \mathbb{T}^2 \text{ invertible and measure-preserving}\},$$

made of all measurable and invertible maps that preserve the Lebesgue measure, in symbols:

$$|T^{-1}(A)| = |A|, \quad \forall A \in \mathcal{B}(\mathbb{T}^2).$$

Among automorphisms, we look for those that *mix* sets, that is

- T is *weakly mixing* if $\forall A, B \in \mathcal{B}(\mathbb{T}^2)$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} [|T^{-j}(A) \cap B| - |A||B|]^2 = 0;$$

- T is (*strongly*) *mixing* if $\forall A, B \in \mathcal{B}(\mathbb{T}^2)$

$$\lim_{n \rightarrow \infty} |T^{-n}(A) \cap B| = |A||B|.$$

One can easily observe that the strong mixing implies the weak mixing, while there are counterexamples of weakly mixing automorphisms that are not strongly mixing (see for instance [4]).

These objects are well understood from the point of view of Ergodic Theory, whereas their counterpart in the dynamics of incompressible fluids is a recent research topic that presents many challenging open questions. In Fluid Dynamics one focuses mostly on (divergence-free) vector fields $b = b(t, x)$ with $t \in [0, 1]$, $x \in \mathbb{T}^2$, whose flow $X(t)$, that is the solution of the following ODE system

$$\begin{cases} \dot{X}(t, x) = b(t, X(t, x)), \\ X(0, x) = x, \end{cases}$$

is an automorphism of $G(\mathbb{T}^2)$ when evaluated at time $t = 1$. Indeed, thanks to the incompressibility condition ($\operatorname{div} b = 0$), at any instant of time the flow $X(t)$ can be regarded as an invertible and measure-preserving map from \mathbb{T}^2 into itself. The theory of DiPerna and Lions [6] and Ambrosio [9] establishes existence, uniqueness and stability for the (Regular Lagrangian) flow of divergence-free vector fields living in $L^\infty([0, 1], W_{\operatorname{loc}}^{1,1}(\mathbb{T}^2))$ or $L^\infty([0, 1], \operatorname{BV}_{\operatorname{loc}}(\mathbb{T}^2))$, giving the possibility to exploit on the one side the rough regularity in space of the vector field, on the other side to establish deep connections with Ergodic Theory. To be more precise,

Definition 1.1. Let $b \in L^1([0, 1] \times \mathbb{T}^2; \mathbb{T}^2)$. A map $X : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a *Regular Lagrangian Flow* (RLF) for the vector field b if

1. for a.e. $x \in \mathbb{T}^2$ the map $t \rightarrow X(t, x)$ is an absolutely continuous integral solution of

$$\begin{cases} \frac{d}{dt}x(t) = b(t, x(t)); \\ x(0) = x. \end{cases}$$

2. there exists a positive constant C independent of t such that

$$|X(t)^{-1}(A)| \leq C|A|, \quad \forall A \in \mathcal{B}(\mathbb{T}^2).$$

The above discussion motivates the following

Definition 1.2. Let $b \in L^\infty([0, 1], \operatorname{BV}_{\operatorname{loc}}(\mathbb{T}^2))$ be a divergence-free vector field. We say that b is weakly mixing (resp. strongly mixing) if its unique RLF $X(t)$, when evaluated at $t = 1$, is a weakly mixing automorphism (resp. strongly mixing).

The definition of mixing vector fields was first given in [3], but there are examples of strongly mixing vector fields in previous literature. For instance, in [7] the authors give an explicit example of a strongly mixing vector field $b \in L^\infty([0, 1], W^{s,p}(\mathbb{T}^d))$ for $s < \frac{1+\sqrt{5}}{2}$ and $p \in [1, \frac{1}{2s+1-\sqrt{5}})$ whose RLF at time $t = 1$ is the *Folded Baker's map*. We should remark that the advantage of the rough regularity of the vector field, especially in the case of the BV regularity, is that it allows for rigid *cut and paste* motions, since the flows originated by these vector fields do not preserve the property of a set to be connected. These constructions would be hard to reproduce for vector fields with higher regularity in time and space. Nevertheless, a stochastic approach (see for example [2]) investigates mixing vector fields with higher regularity, but does not furnish deterministic examples of vector fields with the desired mixing properties.

The aim of this paper is the construction of a vector field $b \in L^\infty([0, 1], \operatorname{BV}(\mathbb{T}^2))$, which is weakly mixing but not strongly mixing. As proved in [12] for automorphisms and in [3] for vector fields, the weakly mixing behaviour is *typical*, while strongly mixing vector fields are *few* in the sense of Baire Category Theorem (see also [1],[11],[8],[10],[13],[15]). Nevertheless it is hard to give examples of weakly mixing automorphisms/vector fields that are not strongly mixing. As written above, Zlatoš and Elgindi construct a strongly mixing vector field in [7], while there are no examples in literature (up to our knowledge) of vector fields that are weakly mixing but not strongly mixing. We want especially to underline that the parallelism with Ergodic Theory, which has investigated different stages of the mixing behaviour (see for instance *mild mixing*, *light mixing*, *partial mixing*), can give new insights from the point of view of the dynamics of an incompressible fluid.

Our examples is based on a work of Chacon [4], who constructed a weakly mixing automorphism that is not strongly mixing, in the one-dimensional space $([0, 1], \mathcal{B}([0, 1]), |\cdot|)$, where $|\cdot|$ is the one-dimensional Lebesgue measure. The importance of his work is that a general procedure to build up weakly mixing automorphisms that are not strongly mixing is given. Indeed his example easily extends to multiple dimensions, but we will focus on the dimension $d = 2$ to avoid technicalities and to provide some *visual* expressions of the vector fields under consideration. In particular we will construct a divergence-free vector field $b \in L^\infty([0, 1], \text{BV}(\mathbb{T}^2))$ whose Regular Lagrangian Flow $X(t)$, when evaluated at time $t = 1$, is a weakly mixing automorphism that is not strongly mixing.

The idea of the paper is the definition of a setting (see *Configurations* in Section 3) that helps to relate an automorphism, call it U , to the vector field b^U whose RLF at time $t = 1$ is U .

A *configuration* γ is a $n \times n$ matrix that takes values in the set $\{1, 2, \dots, n^2\}$. Moreover we ask that $\gamma_{ij} = \gamma_{hk}$ iff $i = h, j = k$. Every configuration indeed represents an enumeration of the subsquares of the torus of the grid $\mathbb{N} \times \mathbb{N}^{\frac{1}{n}}$ (see Figures 1,2). A *movement* $T : \mathcal{C}(n) \rightarrow \mathcal{C}(n)$ is a one-to-one map from the space of configurations $\mathcal{C}(n)$ into itself, that is, it is a *permutations* of the subsquares of the grid $\mathbb{N} \times \mathbb{N}^{\frac{1}{n}}$. Among movements the following are relevant for Chacon's example: simple exchange, sort, rotation (Section 3), and each one of them can be realized as the flow, evaluated at time $t = 1$, of some BV divergence-free vector field (see Subsection 3.1 and Figure 3).

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Figure 1: The grid on the torus with $k = 4$ and an enumeration of subsquares.

15	9	1	2
11	12	5	6
3	4	10	13
7	8	14	16

Figure 2: The automorphism U_4 sends the starting configuration into this final one.

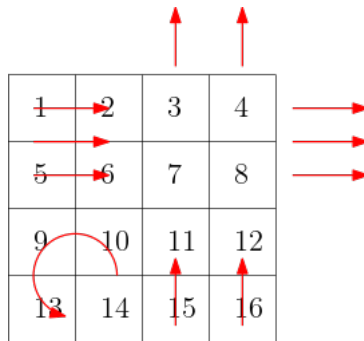


Figure 3: Vector fields on the torus that move rigidly the subsquares.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Figure 4: The 15 puzzle.

The key example that one has to keep in mind while approaching this problem is the *15 puzzle* where one performs rigid movements on subsquares in order to reach the desired configuration (see Figure 4).

Chacon's method provides a sequence of automorphisms $\{U_k\}$ that are permutations of the subsquares of the grid $\mathbb{N} \times \mathbb{N}^{\frac{1}{k}}$. To give an intuition, we fix for example $k = 4$ and we assume to enumerate squares as in Figure 1. Chacon's automorphism $U_4 \in G(\mathbb{T}^2)$ moves the previous enumeration of squares (the starting configuration) into the final configuration in Figure 2. Then he considers the limit map $U = \lim_{k \rightarrow \infty} U_k$, which is a weakly mixing automorphism that is not strongly mixing (see Sections 2.3, 4).

Using Chacon's construction and the setting of configurations, we prove in particular the following result (Section 5):

Theorem 1.3. *There exists a divergence-free vector field $b^U \in L_t^\infty \text{BV}_x$ whose RLF $X^U(t)$ when evaluated at time $t = 1$ is the Chacon's map U , that is b^U is a weakly mixing vector field that is not strongly mixing.*

We conclude remarking that all these constructions are possible assuming to work in $[0, 1]^d$ instead of \mathbb{T}^d .

Plan of the paper. After a short introduction on automorphisms in Ergodic Theory and on Regular Lagrangian Flows (Subsection 2.2), we introduce the Canonical Chacon's transformation in the one-dimensional space (Section 2.3), that will be a model for the extension to higher dimensions (see Section 4). In Section 3 we introduce the notions of *configurations* and *movements*, and in Section 3.1 we describe how to generate vector fields acting on the torus as the movements previously defined, giving some estimates on their total variation. In Section 4 we describe the two dimensional automorphism U in terms of movements, and finally in Section 5 we construct the RLF flow $X^U(t)$ with the property that $X^U(1) = U$.

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2 Preliminaries

For notations and preliminaries we refer mainly to [3]. Throughout the paper we will consider divergence-free vector fields $b : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$ in the space $L^\infty([0, 1], \text{BV}(\mathbb{T}^2))$ (or alternatively in $L^\infty([0, 1], \text{BV}(\mathbb{R}^2))$ with $\text{supp } b \subset\subset [0, 1]^2$), in short $b \in L_t^\infty \text{BV}_x$. The measure space under interest is $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), |\cdot|)$ where $|\cdot|$ is the two dimensional Lebesgue measure, and \mathbb{T}^2 is the unit square $[0, 1]^2$ with periodic boundary conditions. We remark that the analysis of the next sections can be easily extended to higher dimensions, but we will focus on the two dimensional case to avoid technical issues.

2.1 Mixing automorphisms

We call $G(\mathbb{T}^2)$ the group (w.r.t. the composition) of automorphisms of the torus, namely:

Definition 2.1. An *automorphism* of the measure space $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), |\cdot|)$ is an invertible map $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ measurable and measure-preserving, that is

$$|A| = |T(A)| = |T^{-1}(A)|, \quad \forall A \in \mathcal{B}(\mathbb{T}^2).$$

From now on we will say that T is an automorphism of the torus meaning that it is an automorphism of the measure space $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), |\cdot|)$. We remark that the definitions and propositions that we give in this section hold true for (Ω, Σ, μ) where Ω is locally compact and separable, while μ is a complete and normalized measure on Ω .

Definition 2.2. Let $T \in G(\mathbb{T}^2)$ be an automorphism. Then

- T is **ergodic** if $\forall A \in \mathcal{B}(\mathbb{T}^2)$ such that $T^{-1}(A) = A$ then $|A| = 0$ or 1 ;
- T is **weakly mixing** if $\forall A, B \in \mathcal{B}(\mathbb{T}^2)$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} [|T^{-j}(A) \cap B| - |A||B|]^2 = 0; \quad (2.1)$$

- T is **(strongly) mixing** if $\forall A, B \in \mathcal{B}(\mathbb{T}^2)$

$$\lim_{n \rightarrow \infty} |T^{-n}(A) \cap B| = |A||B|. \quad (2.2)$$

It is easy to see that strong mixing implies weak mixing which implies the ergodicity. Let now $T \in G(\mathbb{T}^2)$, then the Kolmogorov Koopman operator (see Chapter 1 in [5]) $U_T : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$ is defined as

$$U_T f(x) \doteq f(T(x)), \quad \forall f \in L^2(\mathbb{T}^2). \quad (2.3)$$

We observe that any operator U_T of the form considered has eigenfunctions $f = \text{const}$ corresponding to the eigenvalue 1. Then we have the following:

Theorem 2.3 (Mixing, Theorem 2 [5]). *T is weakly mixing iff U_T has no eigenfunctions which are not constants.*

2.2 Mixing vector fields

We are interested in divergence-free vector fields b whose flow $t \rightarrow X(t) \in C([0, 1]; G(\mathbb{T}^2))$ when evaluated at time $t = 1$ is a weakly/strongly mixing automorphism. When the velocity field $b : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is Lipschitz in space, uniformly in time, then its *flow* is well-defined in the classical sense, indeed it is the map $X : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ satisfying

$$\begin{cases} \frac{d}{dt} X(t, x) = b(t, X(t, x)), \\ X_0(x) = x. \end{cases}$$

In the presence of discontinuities of the vector field (as the BV regularity in space) we can still give a notion of a flow known as the *Regular Lagrangian Flow* associated with the vector field b .

Definition 2.4 (Regular Lagrangian Flow). Let $b \in L^1([0, 1] \times \mathbb{T}^2; \mathbb{T}^2)$. A map $X : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a *Regular Lagrangian Flow* (RLF) for the vector field b if

1. for a.e. $x \in \mathbb{T}^2$ the map $t \rightarrow X(t)(x)$ is an absolutely continuous integral solution of

$$\begin{cases} \frac{d}{dt} x(t) = b(t, x(t)), \\ x(0) = x. \end{cases}$$

2. there exists a positive constant C independent of t such that

$$|X(t)^{-1}(A)| \leq C|A|, \quad \forall A \in \mathcal{B}(\mathbb{T}^2).$$

Existence, uniqueness and stability for Regular Lagrangian Flows were established in [6] for the Sobolev case, and in [9] for the BV regularity of the vector field. In particular, if $b \in L_t^\infty \text{BV}_x$ is a divergence-free vector field (that is, the distributional divergence $D \cdot b(t) = 0$ \mathcal{L}^1 -a.e. t), then its unique RLF $t \rightarrow X(t)$ is a flow of automorphisms.

Definition 2.5. Let $b \in L^\infty([0, 1], \text{BV}_{\text{loc}}(\mathbb{T}^2))$ be a divergence-free vector field. We say that b is weakly mixing (strongly mixing) if its unique RLF $X(t)$ when evaluated at $t = 1$ is a weakly mixing (respectively strongly mixing) automorphism of \mathbb{T}^2 .

The above definition is motivated by the discussion in [3] (see page 13, Lemma 3.7). The main point of this work, as in [3],[7], is the construction of a vector field whose RLF, when evaluated at time $t = 1$, is an invertible and measure-preserving map which is weakly mixing but not strongly mixing. In the sequel we will build flows of measure-preserving maps originating from divergence-free vector fields; more precisely, if a flow $X : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is invertible, measure-preserving for \mathcal{L}^1 -a.e. t and the map $t \rightarrow X(t)$ is differentiable for \mathcal{L}^1 -a.e. t and $\dot{X}(t) \in L^1(\mathbb{T}^2)$, then the *vector field associated with $X(t)$* is the divergence-free vector field defined by

$$b(t)(x) = b(t, x) = \dot{X}(t)(X^{-1}(t, x)). \quad (2.4)$$

2.3 Canonical Chacon's Transformation

We present here the one dimensional canonical Chacon's Transformation [4], that can be easily extended to higher dimension (see for example Section 4 for the two dimensional): indeed, this construction is based on a general geometric approach which consists in mapping subintervals of the same length linearly onto each other (see also [14] for further reference). Let us consider $I = [0, 1]$ and let $|\cdot|$ be the Lebesgue measure. The aim is to construct a weakly mixing automorphism $T \in G(I)$ which is not strongly mixing.

Definition 2.6. A *column* C is a finite sequence of disjoint subintervals $J \subset I$ called *levels*. The number of levels in a column is its *height* h .

We define a family of automorphisms $\{T_k\}_k \subset G(I)$ by induction. Let C_0 be the column $C_0 \doteq I_{0,1} = [0, \frac{2}{3}]$, and let the *remaining set* be $R_0 = [\frac{2}{3}, 1]$. The height of the column C_0 is $h_0 = 1$, since it has a unique level $I_{0,1}$. We divide C_0 into three disjoint subintervals with same length: $I_{0,1}(1) = [0, \frac{2}{9}]$, $I_{0,1}(2) = [\frac{2}{9}, \frac{4}{9}]$, $I_{0,1}(3) = [\frac{4}{9}, \frac{2}{3}]$. We call *spacer* the interval $S_0 = [\frac{2}{3}, \frac{8}{9}]$ and $R_1 = R_0 \setminus S_0$. We observe that the spacer has the same length of $I_{0,1}(j)$ for $j = 1, 2, 3$. We put the spacer on the top of the middle interval $I_{0,1}(2)$ (see Figure 5). We define the piecewise linear map $T_1 : I \rightarrow I$ in the following way:

$$\begin{cases} T_1(I_{0,1}(1)) = I_{0,1}(2), \\ T_1(I_{0,1}(2)) = S_0, \\ T_1(S_0) = I_{0,1}(3). \end{cases}$$

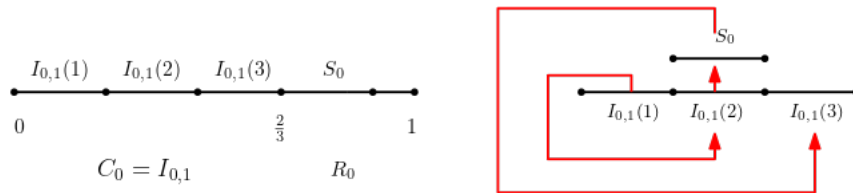


Figure 5: In the left figure the Column C_0 , in the right figure the geometric representation of the action of the automorphism T_1 .

In the set $I \setminus (I_{0,1}(1) \cup I_{0,1}(2) \cup S_0)$ the map T_1 is defined in such a way that T_1 is invertible and measure-preserving (for simplicity we can assume $T_1(I_{0,1}(3)) = I_{0,1}(1)$ and $T_1 = \text{id}$ otherwise). A useful notation to represent T_1 in a simpler way is using the language of permutations, that is

$$T_{1|_{[0,1] \setminus R_1}} = \begin{pmatrix} I_{0,1}(1) & I_{0,1}(2) & I_{0,1}(3) & S_0 \\ I_{0,1}(2) & S_0 & I_{0,1}(1) & I_{0,1}(3) \end{pmatrix}. \quad (2.5)$$

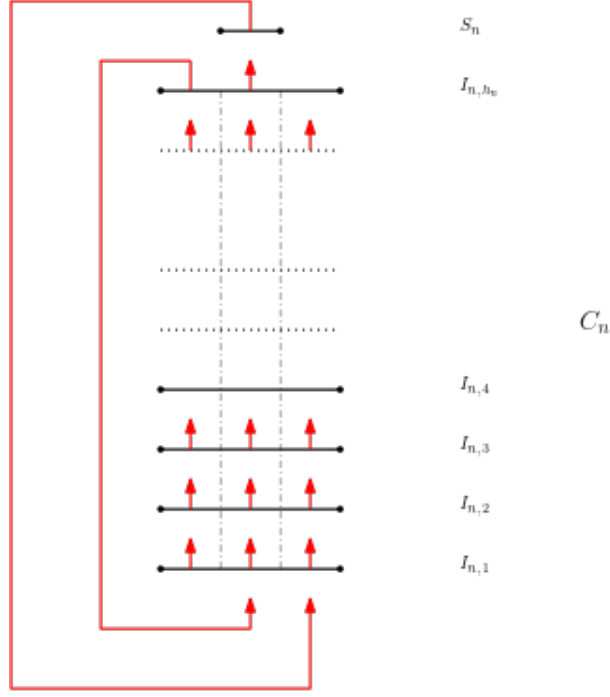


Figure 7: A column C_n and the action of the automorphism T_{n+1} .

We construct the column C_1 of height $h_1 = 3h_0 + 1 = 4$ putting one on the top of the other the intervals in the order $I_{0,1}(1), I_{0,1}(2), S_0, I_{0,1}(3)$: the intervals are arranged so that each point is located below its image (see Figure 6). We rename the levels $I_{1,1} = I_{0,1}(1), I_{1,2} = I_{0,1}(2), I_{1,3} = S_0, I_{1,4} = I_{0,1}(3)$.

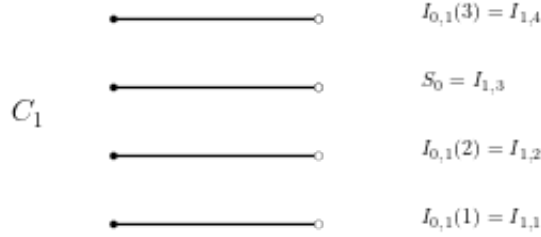


Figure 6: The column C_1 in which the levels are arranged one on top the other.

The inductive step is performed in the following way: we start with a column C_n made of $h_n = 3h_{n-1} + 1$ levels $I_{n,1}, I_{n,2}, \dots, I_{n,h_n}$. We divide each level into three disjoint consecutive subintervals of the same length, that is $I_{n,j} = I_{n,j}(1) \cup I_{n,j}(2) \cup I_{n,j}(3)$ for $j = 1, \dots, h_n$. We consider the set $R_n = \left[\frac{3^{n+1}-1}{3^{n+1}}, 1 \right]$ and we construct $S_n = \left[\frac{3^{n+1}-1}{3^{n+1}}, \frac{3^{n+2}-1}{3^{n+2}} \right]$, with $|S_n| = \frac{2}{3^{n+2}} = |I_{n,j}(i)|$ for $j = 1, \dots, h_n, i \in \{1, 2, 3\}$ and we put it on the top of the middle interval $I_{n,h_n}(2)$ of the bottom level I_{n,h_n} (see Figure 7). We finally define the piecewise linear map T_{n+1} in the following way:

$$\begin{cases} T_{n+1}(I_{n,j}(i)) = I_{n,j+1}(i) & \forall j = 1, \dots, h_n - 1, \quad \forall i \in \{1, 2, 3\}, \\ T_{n+1}(I_{n,h_n}(1)) = I_{n,1}(2), \\ T_{n+1}(I_{n,h_n}(2)) = S_n, \\ T_{n+1}(I_{n,h_n}(3)) = I_{n,1}(1), \\ T_{n+1}(S_n) = I_{n,1}(3), \\ T_{n+1} = \text{id} & \text{otherwise.} \end{cases}$$

The new column C_{n+1} of height $h_{n+1} = 3h_n + 1$ is obtained arranging the levels starting from the bottom $I_{n,1}(1), I_{n,2}(1), \dots, I_{n,h_n}(1), I_{n,1}(2), I_{n,2}(2), \dots, I_{n,h_n}(2), S_n, I_{n,1}(3), \dots, I_{n,h_n}(3)$. We rename them as $I_{n+1,j}$ $j = 1, \dots, h_{n+1}$ following the same order of the levels and we define $R_{n+1} = R_n \setminus S_n$.

Definition 2.7. The Canonical Chacon's map is the automorphism $T = \lim_{n \rightarrow \infty} T_n$.

Observe that the limit map T is well defined, invertible and measure-preserving, indeed one can easily check that $T_{n+1} = T_n$ on $\left(\bigcup_{j=1}^{h_n-1} I_{n,j}\right)^c$.

Theorem 2.8 (Chacon [4]). *T is weakly mixing but not strongly mixing.*

We will give a proof of this theorem in the case of two dimensional examples (see Proposition 4.2).

3 Configurations and movements

In this subsection we introduce the notion of rows, columns and configurations. The advantage of this abstract description is to simplify the action of two-dimensional automorphisms on the torus as the composition of a finite number of *movements*, where each one can be described as the flow (evaluated at time $t = 1$) of a divergence-free BV vector field.

Definition 3.1. A row $r(i)$ of length $n \in \mathbb{N}$ and index $i \in \mathbb{N}$ is a integer-valued $1 \times n$ matrix

$$r(i) = (i, j_1 \quad i, j_2 \quad \dots \quad i, j_n),$$

where $j_1, j_2, \dots, j_n \in \mathbb{N}$.

Definition 3.2. A column $c(j)$ of length $n \in \mathbb{N}$ and index $j \in \mathbb{N}$ is a integer-valued $n \times 1$ matrix

$$c(j) = (i_1, j \quad i_2, j \quad \dots \quad i_n, j)^T,$$

where $i_1, i_2, \dots, i_n \in \mathbb{N}$.

Definition 3.3. A configuration γ of size $n \in \mathbb{N}$ is a integer-valued $n \times n$ matrix, where the entries $\gamma_{i,j} \in \{1, 2, \dots, n^2\}$ and $\gamma_{i,j} = \gamma_{h,k}$ iff $i = h, j = k$. We can denote γ both by rows as $\gamma = (r(1), r(2), \dots, r(n))^T$ and by columns $\gamma = (c(1), c(2), \dots, c(n))$. We call $\mathcal{C}(n)$ the space of configurations of size n .

Remark 3.4. One can easily observe that $\#\mathcal{C}(n) = n^{2!}$, being γ all possible permutations of $\{1, 2, \dots, n^2\}$.

Definition 3.5. Let $\gamma \in \mathcal{C}(n)$ be a configuration. Two entries $\gamma_{i,j}, \gamma_{i',j'}$ are *adjacent* if $|i' - i| = 1$ and $j' = j$ or $i' = i$ and $|j' - j| = 1$.

Definition 3.6. A movement is a bijective map $S : \mathcal{C}(n) \rightarrow \mathcal{C}(n)$.

Definition 3.7 (Simple exchange). Let $\gamma \in \mathcal{C}(n)$ and let $\gamma_{i,j}, \gamma_{i',j'}$ be two adjacent entries. For simplicity we can assume $i' = i + 1, j' = j$ according to Definition 3.5. A *simple exchange* is a map $E_s(i, j; i + 1; j) : \mathcal{C}(n) \rightarrow \mathcal{C}(n)$ that exchanges the two entries, that is

$$E_s(i, j; i + 1; j) \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ i, 1 & \dots & i, j & \dots & i, n \\ i + 1, 1 & \dots & i + 1, j & \dots & i + 1, n \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ i, 1 & \dots & i + 1, j & \dots & i, n \\ i + 1, 1 & \dots & i, j & \dots & i + 1, n \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Definition 3.8 (Sort). Let $\gamma \in \mathcal{C}(n)$ and let $i \in \mathbb{N}$ be some fixed row index. Let $j, j' \in \mathbb{N}$ with $j < j'$. Then the *sort* operation on columns is $S_c(i; j, j') : \mathcal{C}(n) \rightarrow \mathcal{C}(n)$ defined by

$$S_c(i; j, j') \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & i, j & i, j + 1 & \dots & i, j' - 1 & i, j' & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & i, j' & i, j & i, j + 1 & \dots & i, j' - 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Similarly, if $i, i', j \in \mathbb{N}$ and $i < i'$ $S_r(i, i'; j) : \mathcal{C}(n) \rightarrow \mathcal{C}(n)$ is defined by

$$S_r(i, i'; j) \begin{pmatrix} \dots & \dots & \dots \\ \dots & i, j & \dots \\ \dots & i + 1, j & \dots \\ \dots & \dots & \dots \\ \dots & i', j & \dots \\ \dots & \dots & \dots \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots \\ \dots & i', j & \dots \\ \dots & i, j & \dots \\ \dots & \dots & \dots \\ \dots & i' - 1, j & \dots \\ \dots & \dots & \dots \end{pmatrix}. \quad (3.1)$$

Example 3.9. We give the example of some configuration γ and the new configuration $S_c(1; 2, 4)(\gamma)$ obtained by exchanging rows:

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \quad S_c(1; 2, 4)(\gamma) = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Remark 3.10. We observe that if $j = 1$ and $j' = n$ then the sort operation is simply a shift. Since we will work on the torus, the shifts are easier to perform than sort movements. In particular we will see that the cost of a shift, in terms of the total variation, is lower than the cost of a sort movement.

Definition 3.11 (Rotation). Let $\gamma \in \mathcal{C}(n)$ and let $i \geq 2$ and $j < n$. Then the counterclockwise rotation $R_{i,j}^- : \mathcal{C}(n) \rightarrow \mathcal{C}(n)$ is the following map

$$R_{i,j}^- \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & i-1, j & i-1, j+1 & \dots \\ \dots & i, j & i, j+1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & i-1, j+1 & i, j+1 & \dots \\ \dots & i-1, j & i, j & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad (3.2)$$

while the clockwise rotation $R_{i,j}^+ : \mathcal{C}(n) \rightarrow \mathcal{C}(n)$ is the following map

$$R_{i,j}^+ \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & i-1, j & i-1, j+1 & \dots \\ \dots & i, j & i, j+1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} = \begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & i, j & i-1, j & \dots \\ \dots & i, j+1 & i-1, j+1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \quad (3.3)$$

Example 3.12. We give the example of some configuration γ and the new configuration $R_{2,3}^-(\gamma)$ obtained by a counterclockwise rotation:

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \quad R_{2,3}^-(\gamma) = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 5 & 6 & 3 & 7 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

3.1 Flows of vector fields associated with movements

From now on we will consider the two dimensional torus $\mathbb{T}^2 = [0, 1]^2$ with periodic boundary conditions. The idea of this subsection is to apply the previous description via configurations, rows and columns to the two dimensional torus. In particular our aim is to find vector fields whose flow acts on the torus as the movements previously defined (Sort, Exchange, Rotation) and we want to give some estimates on their total variation. We remark here that the BV regularity of the vector fields make possible all these constructions, allowing for rigid *cut and paste motions*, in the spirit of [3] and [7].

Following the notation introduced in [3] we define a flow that rotates subrectangles of \mathbb{T}^2 and the vector field associated with it. More precisely we define the *rotation flow* $r_t : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ for $t \in [0, 1]$ in the following way: call

$$V(x) = \max \left\{ \left| x_1 - \frac{1}{2} \right|, \left| x_2 - \frac{1}{2} \right| \right\}^2, \quad (x_1, x_2) \in \mathbb{T}^2.$$

Then the *rotation field* is $r : \mathbb{T}^2 \rightarrow \mathbb{R}^2$

$$r(x) = \nabla^\perp V(x), \quad (3.4)$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ is the orthogonal gradient. Finally the rotation flow r_t is the flow of the vector field r , i.e. the unique solution to the following ODE system:

$$\begin{cases} \dot{r}(t, x) = r(r(t, x)), \\ r(0, x) = x. \end{cases} \quad (3.5)$$

This flow rotates the unit square $[0, 1]^2$ counterclockwise of an angle $\frac{\pi}{2}$ in a unit interval of time. We recall here Lemma 2.6 in [3] that gives the cost, in terms of the total variation, of the rotation of a rectangle:

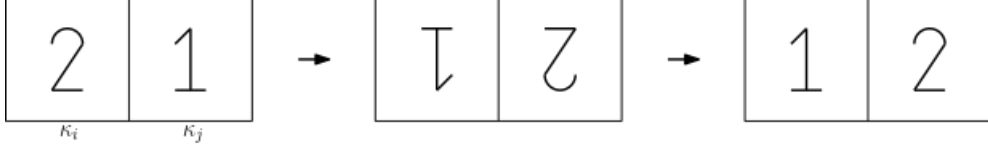


Figure 8: A transposition between two squares is an exchange.

Lemma 3.13. *Let $Q \subset [0, 1]^2$ be a rectangle of sides $a, b > 0$. Consider the rotating flow on the torus*

$$(R_Q)(t, x) = \chi_Q^{-1} \circ r(t) \circ \chi_Q(x), \quad \text{if } x \in Q.$$

where $\chi_Q : Q \rightarrow [0, 1]^2$ is the affine map sending Q into $[0, 1]^2$ and $r(t)$ is the rotation flow defined in (3.5). Let $b(t)^{R_Q}$ the divergence-free vector field associated with $(R_Q)(t)$ (extended to 0 outside the rectangle Q). Then

$$\text{Tot.Var.}(b^R(t))(\mathbb{T}^2) = 4a^2 + 4b^2, \quad \forall t \in [0, 1].$$

Let us fix now some $k \in \mathbb{N}$ giving the size of the partition, and let us consider the grid given by $\mathbb{N} \times \mathbb{N}^{\frac{1}{k}}$ made of squares of side $\frac{1}{k}$. We consider the subsquares $Q_{i,j} = [\frac{j-1}{k}, \frac{j}{k}] \times [\frac{k-i}{k}, \frac{k-i+1}{k}]$ and we identify each subsquare with an entry γ_{ij} of some configuration $\gamma \in \mathcal{C}(k)$. A horizontal stripe $H_i = [0, 1] \times [\frac{k-i}{k}, \frac{k-i+1}{k}]$ is a row, while a vertical stripe $V_j = [\frac{j-1}{k}, \frac{j}{k}] \times [0, 1]$ is a column. We observe that if $Q_{i,j}$ and $Q_{i',j'}$ are adjacent subsquares (that is, they share a common side), then $\gamma_{i,j}$ and $\gamma_{i',j'}$ are adjacent entries (Definition 3.5). From now on we will identify every $\gamma \in \mathcal{C}(k)$ with an enumeration of the subsquares of the torus, and any movement $T : \mathcal{C}(k) \rightarrow \mathcal{C}(k)$ with an automorphism $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ that sends rigidly every subsquare of the grid $\mathbb{N} \times \mathbb{N}^{\frac{1}{k}}$ into another one of the same grid. We start introducing the vector fields whose RLF, when evaluated at time $t = 1$, are movements.

We first recall the transposition vector field, first introduced in [3]. Let κ_i, κ_j be two adjacent squares of size $\frac{1}{k}$ and let $Q = \kappa_i \cup \kappa_j$, then the *transposition flow* between κ_i, κ_j is $T(t)(\kappa_i, \kappa_j) : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ defined as

$$T(t)(\kappa_i, \kappa_j) = \begin{cases} \chi^{-1} \circ r(4t) \circ \chi & x \in \overset{\circ}{Q}, t \in [0, \frac{1}{2}], \\ \chi_i^{-1} \circ r(4t) \circ \chi_i & x \in \overset{\circ}{\kappa}_i, t \in [\frac{1}{2}, 1], \\ \chi_j^{-1} \circ r(4t) \circ \chi_j & x \in \overset{\circ}{\kappa}_j, t \in [\frac{1}{2}, 1], \\ x & \text{otherwise,} \end{cases} \quad (3.6)$$

where the map $\chi : Q \rightarrow \mathbb{T}^2$ is the affine map sending the rectangle Q into the torus \mathbb{T}^2 , χ_i, χ_j are the affine maps sending κ_i, κ_j into the torus \mathbb{T}^2 and r is the rotation flow (3.5). This invertible measure-preserving flow has the property to exchange the two subsquares in the unit time interval (Figure 8). Moreover, by the computations done in Lemma 3.13, we can estimate the total variation of the vector field $b^T(t)(\kappa_i, \kappa_j)$ associated with $T(t)(\kappa_i, \kappa_j)$ (recall 2.4) as

$$\text{Tot.Var.}(b^T(t)(\kappa_i, \kappa_j))(\mathbb{T}^2) \leq 4 \frac{20}{k^2}. \quad (3.7)$$

We also observe that also L^∞ estimates on the vector field are easily available: indeed

$$\|b^T(\kappa_i, \kappa_j)\|_{L^\infty_x} \leq 4 \frac{2}{k}. \quad (3.8)$$

Definition 3.14 (Simple exchange vector field). Let us fix $i, i' \leq k$ with $|i - i'| = 1$ and $j \leq k$ (or alternatively $i \in \mathbb{N}$ and $j, j' \in \mathbb{N}$ with $|j - j'| = 1$). The simple exchange *vector field* is $b(t)(i, j; i', j) \in L_t^\infty \text{BV}_x$ such that, if $X(i, j; i', j)$ is its RLF evaluated at time $t = 1$, then

$$X(i, j; i', j)(\mathbb{T}^2) = E_s(i, j; i', j)(\gamma), \quad \forall \gamma \in \mathcal{C}(k). \quad (3.9)$$

The construction of this vector field is easy: fix for example $i' = i + 1$. Then take the two adjacent subsquares Q_{ij} and $Q_{i+1,j}$ and perform a transposition between them $T(t)(Q_{ij}, Q_{i+1,j})$. Then define

$$b(t)(i, j; i + 1, j) = b^T(t)(Q_{ij}, Q_{i+1,j}).$$

Clearly one has

$$\|b(i, j; i + 1, j)\|_{L^\infty_x} \leq \frac{8}{k}, \quad \text{Tot.Var.}(b(t)(i, j; i + 1, j))(\mathbb{T}^2) \leq 4 \frac{20}{k^2}. \quad (3.10)$$

Similarly we have

Definition 3.15 (Sort vector field). Let $i \leq k$ be some fixed row index. Let $j, j' \leq k$ with $j < j'$. Then the *sort vector field* (on columns) is $b^{S_c}(i; j, j')(t) \in L_t^\infty \text{BV}_x$ such that, if $X^{S_c}(i; j, j')$ is its RLF evaluated at time $t = 1$ then

$$X^{S_c}(i; j, j')(\mathbb{T}^2) = S_c(i; j, j')(\gamma), \quad \forall \gamma \in \mathcal{C}(k). \quad (3.11)$$

Remark 3.16. Similarly one has the sort vector field on rows $b^{S_r}(i, i'; j)(t)$ for some $i < i' \leq k$, $j \leq k$.

To construct a sort vector field (on columns, but the construction for rows is identical), we fix $\gamma \in \mathcal{C}(k)$, a row index i and two columns indices $j < j'$. The idea is to perform $j' - j$ exchanges between squares (recalling that an exchange is a transposition). For clarity see Figure 9 and compare with Example 3.9.

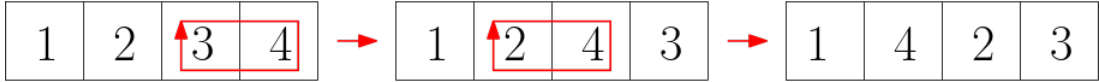


Figure 9: An example of a sort vector fields on a horizontal stripe of the torus. Here $j = 2, j' = 4$. The transpositions to be performed are $j' - j = 2$.

We define the flow in the following way:

$$X^{S_c}(i; j, j')(t) = \begin{cases} T_{(j'-j)t}(Q_{i(j'-1)}, Q_{ij'}) & \text{for } t \in \left[0, \frac{1}{j'-j}\right], \\ T_{(j'-j)t-1}(Q_{i(j'-2)}, Q_{i(j'-1)}) & \text{for } t \in \left[\frac{1}{j'-j}, \frac{2}{j'-j}\right], \\ \dots & \\ T_{(j'-j)t-(j'-j-2)}(Q_{i(j+1)}, Q_{i(j+2)}) & \text{for } t \in \left[\frac{j'-j-2}{j'-j}, \frac{j'-j-1}{j'-j}\right], \\ T_{(j'-j)t-(j'-j-1)}(Q_{ij}, Q_{i(j+1)}) & \text{for } t \in \left[\frac{j'-j-1}{j'-j}, 1\right], \\ x & \text{otherwise.} \end{cases} \quad (3.12)$$

By definition one has

$$b^{S_c}(i; j, j')(t) = \dot{X}^{S_c}(i; j, j')(t) \circ X^{S_c}(i; j, j')^{-1}(t), \quad (3.13)$$

so that

$$\sup_{t,x} |b^{S_c}(i; j, j')(t, x)| \leq \sup_{t,x} |\dot{X}^{S_c}(i; j, j')(t, x)| \leq (j - j') \frac{8}{k} \leq 8,$$

being the rotation vector field of the order of the side of subsquares. This implies that

$$\|b^{S_c}(i; j, j')\|_{L_t^\infty L_x^\infty} \leq 8. \quad (3.14)$$

Similarly one has, for every $t \in [0, 1]$

$$\text{Tot.Var.}(b^{S_c}(t)(i; j, j'))(\mathbb{T}^2) \leq (j' - j) \cdot 4 \cdot 4 \left(\frac{4}{k^2} + \frac{1}{k^2} \right) \leq \frac{80}{k}. \quad (3.15)$$

Remark 3.17. Recalling Remark 3.10 we observe that for the shift operation (that is $j' = k$ and $j = 1$) we can consider another vector field, exploiting the structure of the torus. We will still denote it as the sort vector field, but

$$b(t, x)(i; 1, k) = \begin{cases} \frac{1}{k}, & \text{if } x \in H_i, \\ 0 & \text{otherwise.} \end{cases} \quad (3.16)$$

So we have instead the following estimates:

$$\|b(i, 1, k)\|_{L_{t,x}^\infty} \leq \frac{1}{k}, \quad \|\text{Tot.Var.}(b(i, 1, k)(\mathbb{T}^2))\|_\infty \leq 2\frac{1}{k}. \quad (3.17)$$

This tells us that both the shift vector field and the sort vector field have total variation of the order of the side of the squares of the grid.

Definition 3.18 (Rotation vector field). Let $2 \leq i \leq k$ and $j \leq k - 1$. Then the counterclockwise rotation vector field is $b^-(i, j) \in L_t^\infty \text{BV}_x$ such that, if $X^-(i, j)$ is its RLF evaluated at time $t = 1$, then

$$X^-(i, j)(\mathbb{T}^2) = R_{ij}^-(\gamma), \quad \forall \gamma \in \mathcal{C}(k). \quad (3.18)$$

The clockwise rotation vector field is $b^+(i, j) \in L_t^\infty \text{BV}_x$ such that, if $X^+(i, j)$ is its RLF evaluated at time $t = 1$, then

$$X^+(i, j)(\mathbb{T}^2) = R_{ij}^+(\gamma), \quad \forall \gamma \in \mathcal{C}(k). \quad (3.19)$$

We write here just the counterclockwise case since the other one is identical. Here we have that, if we fix i, j as in the definition, we call $Q = Q_{(i-1)j} \cup Q_{ij} \cup Q_{i(j+1)} \cup Q_{(i-1)(j+1)}$, then

$$X^-(i, j)(t) = \begin{cases} \chi_Q^{-1} \circ r_{2t} \circ \chi_Q, & x \in \overset{\circ}{Q}, t \in [0, \frac{1}{2}], \\ \chi_{lm}^{-1} \circ r_{2t}^{-1} \circ \chi_{lm}, & x \in Q_{lm}, t \in [\frac{1}{2}, 1], l = i, m = j, j + 1, \text{ or } l = i - 1, m = j, j + 1, \\ x & \text{otherwise,} \end{cases} \quad (3.20)$$

where χ is the affine map sending Q into \mathbb{T}^2 and χ_{lm} is the affine map sending Q_{lm} into \mathbb{T}^2 . Again, one has

$$|\dot{X}^-(i, j)(t, x)| \leq \frac{2}{k}, \quad (3.21)$$

from which one gets

$$\|b^-(i, j)\|_{L_{t,x}^\infty} \leq \frac{2}{k}, \quad \text{Tot.Var.}(b^-(t)(i, j))(\mathbb{T}^2) \leq 2 \cdot 4 \left(\frac{4}{k^2} + \frac{4}{k^2} \right), \quad \forall t. \quad (3.22)$$

In the Figure 10 an example of the action of a clockwise rotation vector field.

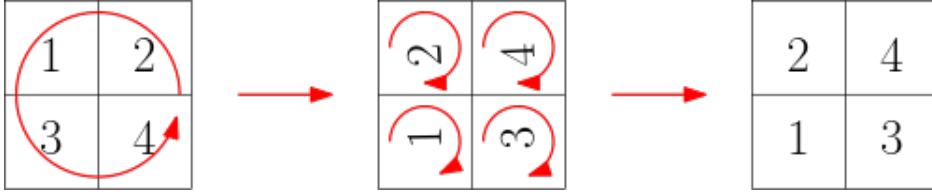


Figure 10: Action of the counterclockwise rotation vector field, where $k = 2$, $i = 2$, $j = 1$.

Remark 3.19. If $X : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is one of the flows previously considered (simple exchange, sort/shift, rotation), then, if k is the side of the squares in which the torus is tiled, one has

$$\|\dot{X}\|_{L_t^\infty L_x^\infty} \leq \max\left(\frac{8}{k}, \frac{(j-j')8}{k}, \frac{2}{k}\right) \quad (3.23)$$

We remark also that, if b is a vector field associated with a movement, then, $\forall t \in [0, 1]$, it holds

$$\|\text{Tot.Var.}(b)(\mathbb{T}^2)\|_\infty \leq \max\left(\frac{80}{k^2}, (j-j')\frac{80}{k^2}, \frac{64}{k^2}\right), \quad (3.24)$$

which follows by the second of (3.10),(3.15) and the second of (3.22).

4 Two dimensional construction

Our aim is the construction of a weakly mixing vector field which is not strongly mixing, that is a divergence-free vector field $b \in L^\infty([0, 1], \text{BV}(\mathbb{T}^2))$ whose RLF $X_{\lfloor t=1}$ when evaluated at time $t = 1$ is a weakly mixing automorphism of \mathbb{T}^2 but not a strongly mixing automorphism. The idea is to adapt Chacon's one-dimensional construction of Subsection 2.3 and decompose the map into simple movements (Section 3), described as the flow of some divergence-free BV vector field (Subsection 3.1). Also in this case the fundamental idea is to define U as the limit of a family of automorphisms $\{U_k\}_k$.

Let us consider the two dimensional torus \mathbb{T}^2 as the unit square $Q_1 = Q_{1,1} = [0, 1]^2$ with the canonical identification of boundaries. Using the notation via configurations (see Section 3) we say that Q_1 can be

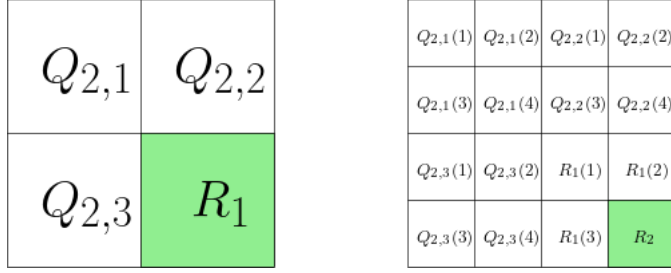


Figure 11: Subdivisions of the two dimensional torus.

identified with the configuration $\gamma_1 = (1) \in \mathcal{C}(1)$. We define $h_1 = 0$ and we divide the square $Q_1 = Q_{1,1}$ into four identical subsquares each one of side $\frac{1}{2}$, more precisely: $Q_{1,1} = Q_{1,1}(1) \cup Q_{1,1}(2) \cup Q_{1,1}(3) \cup R_1$ where $Q_{1,1}(1) = [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$, $Q_{1,1}(2) = [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ and $Q_{1,1}(3) = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$. Clearly $R_1 = [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$. We define U_1 on these subsquares in such a way that

$$U_1(Q_{1,1}(1)) = Q_{1,1}(2), \quad U_1(Q_{1,1}(2)) = Q_{1,1}(3) \quad (4.1)$$

and U_1 is measure-preserving and invertible on $Q_{1,1}$. More precisely, we define U_1 as

$$U_1(x) = \begin{cases} x + (\frac{1}{2}, 0) & \text{if } x \in \mathring{Q}_{1,1}(1), \\ x + (-\frac{1}{2}, -\frac{1}{2}) & \text{if } x \in \mathring{Q}_{1,1}(2), \\ x + (0, \frac{1}{2}) & \text{if } x \in \mathring{Q}_{1,1}(3), \\ x & \text{otherwise.} \end{cases} \quad (4.2)$$

We put $h_2 = 4h_1 + 3$ and we rename the subsquares $Q_{1,1}(i)$ as $Q_{2,1} = Q_{1,1}(1)$, $Q_{2,2} = Q_{1,1}(2)$, $Q_{2,3} = Q_{1,1}(3)$. We look at this via configurations: let us take $\gamma_2 \in \mathcal{C}(2)$, for example

$$\gamma_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}. \quad (4.3)$$

Then we can easily represent $U_1(\gamma_2)$ as

$$U_1(\gamma_2) = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}. \quad (4.4)$$

Here we see the advantage of the representation via configurations, which is more immediate. We continue our construction dividing each subsquare $Q_{2,i}$ with $i \in \{1, 2, 3\}$ into 4 subsquares of side equal to $\frac{1}{4}$: $Q_{2,i} = Q_{2,i}(1) \cup Q_{2,i}(2) \cup Q_{2,i}(3) \cup Q_{2,i}(4)$ and we divide also $R_1 = R_1(1) \cup R_1(2) \cup R_1(3) \cup R_2$ into 4 subsquares (see Figure 11). We define the map U_2 in the following way

$$U_2(x) = \begin{cases} U_1(x) & \text{if } x \in \mathring{Q}_{2,1} \cup \mathring{Q}_{2,2}, \\ x + (\frac{1}{4}, \frac{1}{2}) & \text{if } x \in \mathring{Q}_{2,3}(1), \\ x + (\frac{1}{4}, 0) & \text{if } x \in \mathring{Q}_{2,3}(2), \\ x + (\frac{3}{4}, \frac{1}{4}) & \text{if } x \in \mathring{Q}_{2,3}(3), \\ x + (\frac{1}{4}, 0) & \text{if } x \in \mathring{Q}_{2,3}(4), \\ x + (-\frac{1}{2}, \frac{1}{4}) & \text{if } x \in \mathring{R}_1(1), \\ x + (-\frac{1}{2}, \frac{1}{4}) & \text{if } x \in \mathring{R}_1(2), \\ x + (-\frac{1}{2}, \frac{3}{4}) & \text{if } x \in \mathring{R}_1(3), \\ x & \text{otherwise,} \end{cases} \quad (4.5)$$

or, via configurations, fixing some $\gamma_4 \in \mathcal{C}(4)$ we get

$$U_2 \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 9 & 10 & 13 & 14 \\ 11 & 12 & 15 & 16 \end{pmatrix} = \begin{pmatrix} 15 & 9 & 1 & 2 \\ 13 & 14 & 3 & 4 \\ 5 & 6 & 10 & 11 \\ 7 & 8 & 12 & 16 \end{pmatrix}. \quad (4.6)$$

We put $h_3 = 4h_2 + 3 = 15$ and we rename the squares as

$$\begin{aligned} Q_{3,1} \doteq Q_{2,1}(1) &\rightarrow Q_{3,2} \doteq Q_{2,2}(1) \rightarrow Q_{3,3} \doteq Q_{2,3}(1) \rightarrow \\ &\rightarrow Q_{3,4} = Q_{2,1}(2) \rightarrow \cdots \rightarrow Q_{3,14} \doteq Q_{2,3}(3) \rightarrow Q_{3,15} \doteq R_1(3), \end{aligned}$$

with the property that if $Q_{3,i} \rightarrow Q_{3,i+1}$ then $U_2(Q_{3,i}) = Q_{3,i+1}$.

The inductive step is the following: at the n -th step we have $Q_{n,1}, \dots, Q_{n,h_n}$ subsquares each one of area $\frac{1}{4^{n-1}}$ and side $l_n = \frac{1}{2^{n-1}}$. We divide each subsquare into 4 identical subsquares: $Q_{n,i} = Q_{n,i}(1) \cup Q_{n,i}(2) \cup Q_{n,i}(3) \cup Q_{n,i}(4)$ and we divide also R_{n-1} into 4 identical subsquares $R_{n-1} = R_{n-1}(1) \cup R_{n-1}(2) \cup R_{n-1}(4) \cup R_n$. We put $h_{n+1} = 4h_n + 3 = 4^n - 1$ and we define the map U_n in the following way:

$$U_n(x) = \begin{cases} U_{n-1}(x) & \text{if } x \in \overset{\circ}{Q}_{n,1} \cup \cdots \cup \overset{\circ}{Q}_{n,h_n-1}, \\ x + (\frac{5}{4^{n-1}} - 1, 1 - \frac{1}{2^{n-1}}) & \text{if } x \in \overset{\circ}{Q}_{n,h_n}(1), \\ x + (\frac{1}{4^{n-1}}, 0) & \text{if } x \in \overset{\circ}{Q}_{n,h_n}(2), \\ x + (\frac{3}{4^{n-1}}, \frac{1}{4^{n-1}}) & \text{if } x \in \overset{\circ}{Q}_{n,h_n}(3), \\ x + (\frac{1}{4^{n-1}}, 0) & \text{if } x \in \overset{\circ}{Q}_{n,h_n}(4), \\ x + (-1 + \frac{1}{2^{n-1}}, \frac{2^n-3}{4^{n-1}}) & \text{if } x \in \overset{\circ}{R}_{n-1}(1), \\ x + (\frac{1}{2^{n-1}} - 1, \frac{2^n-3}{4^{n-1}}) & \text{if } x \in \overset{\circ}{R}_{n-1}(2), \\ x + (\frac{1}{2^{n-1}} - 1, \frac{2^n-1}{4^{n-1}}) & \text{if } x \in \overset{\circ}{R}_{n-1}(3), \\ x & \text{otherwise.} \end{cases} \quad (4.7)$$

We underline that, by definition, we have that

$$U_n(x) = U_{n-1}(x) \text{ if } x \in Q_{n,1} \cup \cdots \cup Q_{n,h_n-1}. \quad (4.8)$$

Finally we rename the squares as follows:

$$\begin{aligned} Q_{n+1,1} \doteq Q_{n,1}(1) &\rightarrow Q_{n+1,2} \doteq Q_{n,2}(1) \rightarrow \cdots \rightarrow Q_{n+1,h_n} \doteq Q_{n,3}(1) \rightarrow \\ &\rightarrow \cdots \rightarrow Q_{n+1,4h_n+2} \doteq Q_{n,h_n}(3) \rightarrow Q_{n+1,h_n+1} \doteq R_n(3), \end{aligned}$$

with the property that if $Q_{n+1,i} \rightarrow Q_{n+1,i+1}$ then $U_n(Q_{n+1,i}) = Q_{n+1,i+1}$.

finally we define $U = \lim_{n \rightarrow \infty} U_n$ (well defined by condition 4.8). Then the following propositions hold:

Proposition 4.1. *The map U is measure-preserving and ergodic.*

The proof of the proposition relies on the notion of *sufficient semi-rings*, that can be taken from [14].

Proposition 4.2. *The map U is a weakly mixing automorphism of \mathbb{T}^2 that is not strongly mixing.*

Proof. The idea of the proof is the same of Chacon (see [4]). Indeed: take for example a subsquare $Q_n = Q_{n,i}$ with n sufficiently large. We will prove that

$$|U^{h_k}(Q_n) \cap Q_n| = \frac{1}{4}|Q_n| > |Q_n|^2, \quad \forall k \geq n. \quad (4.9)$$

We first recall that $U = \lim_{n \rightarrow \infty} U_n$ and that $U_{n+1} = U_n$ on $Q_{n+1,1} \cup Q_{n+1,2} \cup \cdots \cup Q_{n+1,h_{n+1}-1}$. We remember that $Q_n = Q_{n,i}(1) \cup Q_{n,i}(2) \cup Q_{n,i}(3) \cup Q_{n,i}(4)$, so

$$\begin{aligned} U_n^{h_n}(Q_{n,i}(1)) &= Q_{n,i}(2), \\ U_n^{h_n}(Q_{n,i}(2)) &= Q_{n,i-1}(3), \\ U_n^{h_n}(Q_{n,i}(3)) &= Q_{n,i-1}(4), \\ U_n^{h_n}(Q_{n,i}(4)) &= Q_{n,i-1}(1). \end{aligned}$$

This implies that

$$|U_n^{h_n}(Q_n) \cap Q_n| = \frac{1}{4}|Q_n|,$$

for any Q_n sublevel of the n -th column. Since (4.8) holds and also $U_n^{h_n}(Q_{n,i}(1)) = Q_{n,i}(2)$, one has

$$|U^{h_n}(Q) \cap Q| \geq \frac{1}{4}|Q|.$$

In particular, if one considers $k \geq n$ and takes a sublevel Q_k of the k -th column, one gets similarly

$$|U^{h_k}(Q_k) \cap Q_k| \geq \frac{1}{4}|Q_k|.$$

We observe now that any level Q_n of the n -th column has 4^{k-n} copies into the k -th column. For example, if $k = n + 1$ one gets $Q_n = Q_{n+1}^1 \cup Q_{n+1}^2 \cup Q_{n+1}^3 \cup Q_{n+1}^4$. But then

$$|U^{h_{n+1}}(Q_n) \cap Q_n| = \sum_{j=1}^4 |U^{h_{n+1}}(Q_{n+1}^j) \cap Q_n| \geq \sum_{j=1}^4 |U^{h_{n+1}}(Q_{n+1}^j) \cap Q_{n+1}^j| \geq \sum_{j=1}^4 \frac{1}{4}|Q_{n+1}^j| = \frac{1}{4}|Q|.$$

More in general

$$|U^{h_k}(Q_n) \cap Q_n| = \sum_{Q' \text{ copies}} |U^{h_k}(Q') \cap Q_n| \geq \sum_{Q' \text{ copies}} |U^{h_k}(Q') \cap Q'| \geq \sum_{Q' \text{ copies}} \frac{1}{4}|Q'| = \frac{1}{4}|Q|.$$

This gives a diverging sequence $\{k\}$ and a set Q_n for which the strongly mixing condition does not hold (if one simply requires that $|Q_n| < \frac{1}{4}$), therefore U cannot be strongly mixing (see Definition 2.2). Whereas U is weakly mixing. Indeed, by the previous proposition, U is measure-preserving and ergodic. By Theorem 2.3 we have to prove that if $f \in \mathcal{L}^2(\mathbb{T}^2)$ is an eigenfunction of \mathcal{U}_U of eigenvalue λ , with $|\lambda| = 1$, then $f = c$ for some constant c . By Lemma 2 of [5] (chapter 1, page 14), we need only to prove that $\lambda = 1$. Since f is non-zero on a set of positive measure, for every $\epsilon > 0$ there exists a constant k such that the set

$$A = \{x : |f(x) - k| < \epsilon\}$$

has positive measure. By the property of sufficient semi-rings (see again [14]) there exists a subsquare $Q = Q_{n,i}$ such that

$$\frac{|Q \cap A|}{|Q|} > \frac{7}{8}. \quad (4.10)$$

As in the previous computations, since $Q = Q_{n,i}(1) \cup Q_{n,i}(2) \cup Q_{n,i}(3) \cup Q_{n,i}(4)$, using that U is measure-preserving, $U^{h_n}(Q_{n,i}(1)) = Q_{n,i}(2)$, $U^{2h_n+1}(Q_{n,i}(1)) = Q_{n,i}(3)$ and that the inequality (4.10) holds, we obtain that there exists a point $x \in A \cap Q$ such that $U^{h_n}(x) \in A \cap Q$, and that there exists $y \in A \cap Q$ such that $U^{2h_n+1}(y) \in A \cap Q$. Indeed, by (4.10) we have that $|Q_{n,i}(2) \cap A| > \frac{1}{8}|Q|$ and for the same reason that

$$|Q_{n,i}(1) \cap A| > \frac{1}{8}|Q|,$$

therefore, by applying the map U^{h_n} one gets

$$|Q_{n,i}(2) \cap U^{h_n}(A)| > \frac{1}{8}|Q|,$$

which implies that

$$|(Q \cap A) \cap U^{h_n}(Q \cap A)| > 0.$$

An analogous argument proves that

$$|(Q \cap A) \cap U^{2h_n+1}(Q \cap A)| > 0.$$

Therefore

$$|f(x) - k| < \epsilon, \quad |\lambda^{h_n} f(x) - k| < \epsilon, \quad |f(y) - k| < \epsilon, \quad |\lambda^{2h_n+1} f(y) - k| < \epsilon. \quad (4.11)$$

So by the previous equations we have that

$$\lambda^{h_n} = \frac{k + \delta_2}{k + \delta_1}, \quad \lambda^{2h_n+1} = \frac{k + \delta_4}{k + \delta_3}$$

with $|\delta_i| < \epsilon$ for every i . That is

$$\lambda = \frac{(k + \delta_4)(k + \delta_1)^2}{(k + \delta_3)(k + \delta_2)^2}, \quad (4.12)$$

and since this hold for every $\epsilon > 0$, we conclude that $\lambda = 1$. \square

The maps U_n can be decomposed into simple movements, using the notations of configurations.

We have the following

Proposition 4.3. *For every $n \geq 2$ the map U_n can be expressed as*

$$U_n = U_{n-1} \circ V_n, \quad (4.13)$$

where $V_n : \mathcal{C}(2^n) \rightarrow \mathcal{C}(2^n)$ and

$$\begin{aligned} V_n = & E_s(2^n, 2^n - 3; 2^n, 2^n - 2) \circ R^+(2^n, 2^n - 3) \circ R^+(2^n, 2^n - 3) \circ \\ & \circ S_c^2(2^n - 1; 2^n - 3, 2^n) \circ R^-(2^n, 2^n - 3) \circ S_c(2^n; 2^n - 3, 2^n - 1). \end{aligned} \quad (4.14)$$

Proof. For understanding the situation we first fix $n = 1$, and the starting configuration

$$\gamma = \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 9 & 10 & 13 & 14 \\ 11 & 12 & 15 & 16 \end{pmatrix},$$

We first apply V_1 finding

$$\begin{aligned} & \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 9 & 10 & 13 & 14 \\ 11 & 12 & 15 & 16 \end{pmatrix} \xrightarrow{S_c(4;1,3)} \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 9 & 10 & 13 & 14 \\ 15 & 11 & 12 & 16 \end{pmatrix} \xrightarrow{R^-(4,1)} \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 10 & 11 & 13 & 14 \\ 9 & 15 & 12 & 16 \end{pmatrix} \xrightarrow{S_c^2(3;1,4)} \\ & \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 13 & 14 & 10 & 11 \\ 9 & 15 & 12 & 16 \end{pmatrix} \xrightarrow{(R^+(4,1))^2} \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 15 & 9 & 10 & 11 \\ 14 & 13 & 12 & 16 \end{pmatrix} \xrightarrow{E_s(4,1;1,2)} \begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 15 & 9 & 10 & 11 \\ 13 & 14 & 12 & 16 \end{pmatrix}. \end{aligned}$$

Then by applying U_1 to the last configuration

$$\begin{pmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ 15 & 9 & 10 & 11 \\ 13 & 14 & 12 & 16 \end{pmatrix} \xrightarrow{U_1} \begin{pmatrix} 15 & 9 & 1 & 2 \\ 13 & 14 & 3 & 4 \\ 5 & 6 & 10 & 11 \\ 7 & 8 & 12 & 16 \end{pmatrix},$$

to compare with (4.6). For a generic $n \in \mathbb{N}$, one has to observe that, recalling (4.8), one has

$$U_n(x) = U_{n-1}(x) \text{ if } x \in Q_{n,1} \cup \dots \cup Q_{n,h_n-1}. \quad (4.15)$$

with $h_n = 4^{n-1} - 1$. We list a series of useful observations:

- consider a configuration $\gamma \in \mathcal{C}(2^n)$, then the map V_n acts only on the subsquares

$$Q_{n+1,h_n}, Q_{n+1,2h_n}, Q_{n+1,2h_n+1}, Q_{n+1,3h_n+1}, Q_{n+1,3h_n+2}, Q_{n+1,4h_n+2}, Q_{n+1,4h_n+3}, R_n; \quad (4.16)$$

- $Q_{n,1} \cup \dots \cup Q_{n,h_n-1}$ in equation 4.15 correspond to the following subsquares of the refined grid

$$Q_{n+1,1}, Q_{n+1,2}, \dots, Q_{n+1,h_n-1}, Q_{n+1,h_n+1}, Q_{n+1,h_n+2}, \dots, Q_{n+1,2h_n-1},$$

$$Q_{n+1,2h_n+2}, Q_{n+1,2h_n+3}, \dots, Q_{n+1,3h_n-1}, Q_{n+1,3h_n}, \dots, Q_{n+1,3h_n+3},$$

$$Q_{n+1,3h_n+4}, \dots, Q_{n+1,4h_n+1};$$

- therefore we have to check $U_{n-1} \circ V_n$ only on the subsquares of (4.16);
- we recall that, by the enumeration chosen, $U_n(Q_{n+1,i}) = Q_{n+1,i+1}$ and $U_n(Q_{n+1,h_n+1}) = Q_{n+1,1}$;

- we recall also that

$$U_{n-1}(Q_{n,h_n}) = Q_{n,1} = Q_{n+1,1} \cup Q_{n+1,h_n+1} \cup Q_{n+1,2h_n+2} \cup Q_{n+1,3h_n+3}, \quad (4.17)$$

and

$$U_{n-1} \lrcorner R_{n-1} = Id. \quad (4.18)$$

That is, since $Q_n = Q_{n+1,h_n} \cup Q_{n+1,2h_n} \cup Q_{n+1,3h_n+1} \cup Q_{n+1,4h_n+2}$, then

$$\begin{aligned} Q_{n+1,h_n} &\xrightarrow{U_n^{-1}} Q_{n+1,1}, \\ Q_{n+1,2h_n} &\xrightarrow{U_n^{-1}} Q_{n+1,h_n+1}, \\ Q_{n+1,3h_n+1} &\xrightarrow{U_n^{-1}} Q_{n+1,2h_n+2}, \\ Q_{n+1,4h_n+2} &\xrightarrow{U_n^{-1}} Q_{n+1,3h_n+3}. \end{aligned} \quad (4.19)$$

Finally, we consider the adjacent subsquares Q_{n,h_n} and R_{n-1} in Figure 12 and their refinement (equation 4.16) on which V_n acts.

Q_{n+1,h_n}	$Q_{n+1,2h_n}$	$Q_{n+1,2h_n+1}$	$Q_{n+1,3h_n+2}$
$Q_{n+1,3h_n+1}$	$Q_{n+1,4h_n+2}$	$Q_{n+1,4h_n+3}$	R_n

Figure 12: The adjacent subsquares Q_{n,h_n} and R_{n-1} and their refinement.

Then the action of V_n is the following

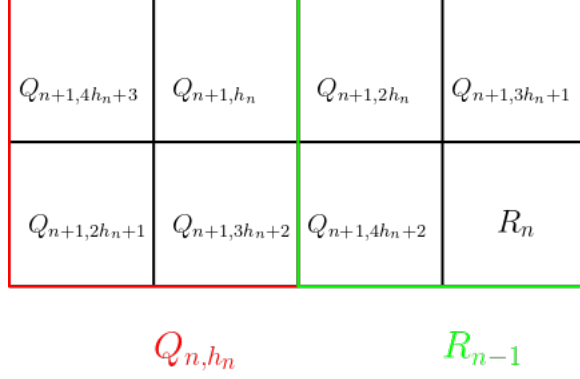
$$\begin{aligned} Q_{n+1,h_n} &\rightarrow Q_{n+1,2h_n}, \\ Q_{n+1,2h_n} &\rightarrow Q_{n+1,2h_n+1}, \\ Q_{n+1,2h_n+1} &\rightarrow Q_{n+1,3h_n+1}, \\ Q_{n+1,3h_n+1} &\rightarrow Q_{n+1,3h_n+2}, \\ Q_{n+1,3h_n+2} &\rightarrow Q_{n+1,4h_n+2}, \\ Q_{n+1,4h_n+2} &\rightarrow Q_{n+1,4h_n+3}, \\ Q_{n+1,4h_n+3} &\rightarrow Q_{n+1,h_n}, \\ R_n &\rightarrow R_n, \end{aligned}$$

as described in Figure 13.

$Q_{n+1,4h_n+3}$	Q_{n+1,h_n}	$Q_{n+1,2h_n}$	$Q_{n+1,3h_n+1}$
$Q_{n+1,2h_n+1}$	$Q_{n+1,3h_n+2}$	$Q_{n+1,4h_n+2}$	R_n

Figure 13: The action of V_n on the refinement of Q_{n,h_n} and R_n .

Thus the map V_n acts only on the sublevels $Q_{n+1,i}$ of the $n+1$ -column, not on the sublevels $Q_{n,i}$ of the n -th column.



Finally we recall that, by the enumeration chosen, by using the observation (4.17) and the definition of U_{n-1} in equations (4.19) one finally has

$$\begin{aligned}
Q_{n+1,h_n} &\xrightarrow{V_n} Q_{n+1,2h_n} \xrightarrow{U_{n-1}} Q_{n+1,h_n+1} = U_n(Q_{n+1,h_n}), \\
Q_{n+1,2h_n} &\xrightarrow{V_n} Q_{n+1,2h_n+1} \xrightarrow{U_{n-1}} Q_{n+1,2h_n+1} = U_n(Q_{n+1,2h_n}), \\
Q_{n+1,2h_n+1} &\xrightarrow{V_n} Q_{n+1,3h_n+1} \xrightarrow{U_{n-1}} Q_{n+1,2h_n+2} = U_n(Q_{n+1,2h_n+1}), \\
Q_{n+1,3h_n+1} &\xrightarrow{V_n} Q_{n+1,4h_n+2} \xrightarrow{U_{n-1}} Q_{n+1,3h_n+3} = U_n(Q_{n+1,3h_n+1}),
\end{aligned}$$

and so on. □

We will use similar ideas to recover the flow of the weakly mixing vector field in the next section.

Remark 4.4. The interested reader can observe that $\forall n$ the map U_n is a cyclic permutation of subsquares of the same area (see [3] for the definitions). This observation is in the spirit of the result proved by Halmos in [12] that states that weakly mixing automorphisms are a residual G_δ -set in $G(\mathbb{T}^2)$ with the L^1 -topology. A key ingredient for the proof is to show that cyclic permutations of subsquares are dense in $G(\mathbb{T}^2)$ with the L^1 -topology. In [3] similar ideas are used to recover that weakly mixing vector fields are dense with respect to the $L^1_t L^1_x$ -topology. In this paper we are doing a different thing: we are considering those automorphisms/vector fields that are weakly mixing but not strongly mixing, fixing a specific cyclic permutation. Therefore we cannot deduce from the previous computations the density of weakly mixing automorphisms.

5 The weakly mixing vector field

In this final section we provide a weakly mixing vector field $b^U \in L_t^\infty \text{BV}_x$ whose RLF $X^U(t)$ evaluated at time $t = 1$ is the map U constructed in the previous section.

Our aim is to construct a flow X^n with $n \geq 2$ that, evaluated at time $t = 1$, gives the map V_n introduced in (4.14). We first consider $\gamma_2 \in \mathcal{C}(2)$: we can take to fix the ideas the configuration (4.3). We start defining the following flow $X^1 : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ as

$$X^1(t) = \begin{cases} T(2t)(Q_{2,1}, Q_{2,2}), & t \in [0, \frac{1}{2}], \\ T(2t-1)(Q_{2,1}, Q_{2,3}) \circ T(1)(Q_{2,1}, Q_{2,2}), & t \in [\frac{1}{2}, 1]. \end{cases} \quad (5.1)$$

where the $Q_{2,i}$ $i = 1, 2, 3$ are the subsquares defined in the previous section. One can easily see that $X^1(1)(\gamma_2) = U_1(\gamma_2)$.

Building block flow. Let us consider the two adjacent subsquares Q_{n,h_n}, R_{n-1} of side $\frac{1}{2^n}$ for some $n \in \mathbb{N}$ and consider their subdivisions into the sublevels $Q_{n+1,i}$ as in Proposition 4.3. We look for a flow $X^n(Q_{n,h_n}, R_{n-1})[0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ that moves the subsquares within the time interval $[0, 1]$ with

the property that $X^n(Q_{n,h_n}, R_{n-1})(1) = V_n$. We define therefore $X^n(Q_{n,h_n}, R_{n-1})(t)$ as

$$X^n(Q_{n,h_n}, R_{n-1})(t) = \begin{cases} X^{S_c}(2^{n+1}, 2^{n+1} - 3, 2^{n+1} - 1)(7t) & t \in [0, \frac{1}{7}], \\ X^-(2^{n+1}, 2^{n+1} - 3)(7t - 1) \circ X^n(Q_{n,h_n}, R_{n-1})(\frac{1}{7}) & t \in [\frac{1}{7}, \frac{2}{7}], \\ X^{S_c}(2^{n+1} - 1; 2^{n+1} - 3, 2^{n+1})(7t - 2) \circ X^n(Q_{n,h_n}, R_{n-1})(\frac{2}{7}) & t \in [\frac{2}{7}, \frac{3}{7}], \\ X^{S_c}(2^{n+1} - 1; 2^{n+1} - 3, 2^{n+1})(7t - 3) \circ X^n(Q_{n,h_n}, R_{n-1})(\frac{3}{7}) & t \in [\frac{3}{7}, \frac{4}{7}], \\ X^+(2^{n+1}, 2^{n+1} - 3)(7t - 4) \circ X^n(Q_{n,h_n}, R_{n-1})(\frac{4}{7}) & t \in [\frac{4}{7}, \frac{5}{7}], \\ X^+(2^{n+1}, 2^{n+1} - 3)(7t - 5) \circ X^n(Q_{n,h_n}, R_{n-1})(\frac{5}{7}) & t \in [\frac{5}{7}, \frac{6}{7}], \\ X(2^{n+1}, 2^{n+1} - 3; 2^{n+1}, 2^{n+1} - 2)(7t - 6) \circ X^n(Q_{n,h_n}, R_{n-1})(\frac{6}{7}) & t \in [\frac{6}{7}, 1]. \end{cases}$$

Let us call

$$b^n(Q_{n,h_n}, R_{n-1})(t, x) \doteq \dot{X}^n(Q_{n,h_n}, R_{n-1})(t, (X^n)^{-1}(t, x)).$$

Then we have the following

Proposition 5.1. *There exist two positive constants $C_1, C_2 > 0$ such that the following estimates hold:*

$$\|\dot{X}^n\|_{L_t^\infty L_x^\infty} \leq \frac{C_1}{2^n}, \quad (5.2)$$

$$\|TV(b^n)(\mathbb{T}^2)\|_{L^\infty} \leq \frac{C_2}{2^{2n}}. \quad (5.3)$$

Proof. Let us fix $t \in [0, 1]$, then by (3.23), since the sort operation occurs for a maximum of 4 adjacent subsquares, that is $j - j' = 3$, we have that

$$\|\dot{X}^n\|_{L_t^\infty L_x^\infty} \leq \frac{7 \cdot 24}{2^n} = \frac{C_1}{2^n}.$$

To prove the second estimate we observe that, by (3.24), since again the sort operation occurs between at most 4 squares, one has

$$|\text{Tot.Var.}(b^n)(t)(\mathbb{T}^2)| \leq 7 \cdot \frac{240}{2^{2n}} = \frac{C_2}{2^{2n}}, \quad (5.4)$$

which concludes the proof. \square

Finally we can define the flow $X^U(t)$ as

$$X^U(t) = \begin{cases} X^1(2t - 1) \circ X^U(\frac{1}{2}) & t \in [\frac{1}{2}, 1], \\ X^2(Q_{2,3}, R_1)(4t - 2) \circ X^U(\frac{1}{4}) & t \in [\frac{1}{4}, \frac{1}{2}], \\ \dots & \dots \\ X^n(Q_{n,4h_n}, R_{n-1})(2^n t - 2^n + 2) \circ X^U(2^{-n}) & t \in [2^{-n}, 2^{-n+1}], \\ \dots & \dots \end{cases} \quad (5.5)$$

One can observe that the map $X^U : [0, 1] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is well-defined, $X^U \in C([0, 1]; L^1(\mathbb{T}^2))$ and differentiable \mathcal{L}^1 -a.e. t and for every $t \in [0, 1]$ it is an invertible and measure-preserving map from the torus into itself. We also remark that for every $x \in \mathbb{T}^2$ we have that $\lim_{t \rightarrow 0} X^U(t, x) = x$, which tells us that X^U is a flow.

Therefore, by (2.4) one can define the divergence-free vector field $b^U(t, x) \doteq \dot{X}^U(t, (X^U)^{-1}(t, x))$.

Proposition 5.2. *The divergence-free vector field b^U lives in the space $L_t^\infty \text{BV}_x$. Moreover, its RLF $X^U(1)$ when evaluated at time $t = 1$ is the map U , that is b^U is a weakly mixing vector field that is not strongly mixing.*

Proof. Let us fix $t \in [2^{-n}, 2^{-n+1}]$. Then

$$b^U(t, x) = 2^n \dot{X}^n(Q_{n,4h_{n-1}+3}, R_{n-1})(2^n t - 2^n + 2, (X^U)^{-1}(t, x)).$$

Therefore

$$\begin{aligned} \|b^U(t)\|_{L_x^1} &= 2^n \int_{\mathbb{T}^2} |\dot{X}^n(Q_{n,4h_{n-1}+3}, R_{n-1})(2^n t - 2^n + 2, (X^U)^{-1}(t, x))| dx \\ &= 2^n \int_{\mathbb{T}^2} |\dot{X}^n(Q_{n,4h_{n-1}+3}, R_{n-1})(2^n t - 2^n + 2, y)| dy \end{aligned} \quad (5.6)$$

$$\leq 2^n |Q_{n,4h_{n-1}+3} \cup R_{n-1}| \|\dot{X}^n(Q_{n,4h_{n-1}+3}, R_{n-1})\|_{L_t^\infty L_x^\infty} \quad (5.7)$$

$$\leq 2^n \cdot 2 \cdot 2^{-n} \frac{C_1}{2^n} \leq 2C_1, \quad (5.8)$$

where (5.6) follows by the fact that $X^U(t)$ is measure-preserving, (5.7) by the fact that X^n acts only on $Q_{n,4h_n} \cup R_{n-1}$ and (5.8) follows by (5.2). To compute the total variation of b^U , we observe that, for every $t \in [0, 1]$ one has

$$\dot{X}^U(t, (X^U)^{-1}(t, x)) = 2^n \dot{X}^n(2^n t - 2^n + 2, (X^n)^{-1}(2^n t - 2^n + 2, x)),$$

so that $|\text{Tot.Var.}(b^U(t)(\mathbb{T}^2))| = 2^n |\text{Tot.Var.}(b^n(t)(\mathbb{T}^2))|$. By using (5.3), one has that, if $t \in [2^{-n}, 2^{-n+1}]$

$$|\text{Tot.Var.}(b^U(t)(\mathbb{T}^2))| = 2^n |\text{Tot.Var.}(b^n(t)(\mathbb{T}^2))| \leq 2^n \frac{C_2}{2^{2n}},$$

which implies that

$$\|\text{Tot.Var.}(b^U)(\mathbb{T}^2)\|_{L^\infty} \leq C_2.$$

To conclude the proof we have to show that the map $X^U(1) = U$ is the Chacon's map introduced in Section 4. More precisely, we have to prove that

$$X^1(1) \circ X^2(Q_{2,3}, R_1)(1) \circ \dots \circ X^n(Q_{n,4h_{n-1}+3}, R_{n-1})(1) = U_n.$$

This easily follows by Proposition 4.3 observing that the map $X^1(1) = U_1$ and $X^n(Q_{n,4h_{n-1}+3}, R_{n-1})(1) = V_n$. \square

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