

Algebra of Operators in AdS-Rindler

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We discuss the algebra of operators in AdS-Rindler wedge, particularly in AdS₅/CFT₄. We explicitly construct the algebra at $N = \infty$ limit and discuss its Type III₁ nature. We will consider $1/N$ corrections to the theory and describe how several divergences can be renormalized and the algebra becomes Type II_∞. This will make it possible to associate a density matrix to any state in the Hilbert space and thus a von Neumann entropy.

I. INTRODUCTION

There has been a recent interest in studying algebra of operators in perturbative limits of quantum gravity in the context of AdS/CFT. Leutheusser and Liu studied the $N = \infty$ limit of the thermofield double state above Hawking-Page temperature [1, 2]. They noted that the algebra of operators exterior to the black hole horizon form what is called Type III₁ von Neumann algebra. From the boundary perspective, the algebra elements are the single trace operators whose thermal one point function is subtracted,

$$\tilde{O} = O - \langle O \rangle.$$

These operators are generalized free fields with $O(1)$ two point functions. Using properties of Type III₁ algebras, they were able to propose a natural time like evolution operator for an observer falling into the black hole.

Witten later proposed to include the Hamiltonian into the algebra [3]. In the strict $N \rightarrow \infty$ limit,

$$[U, \tilde{O}] = 0$$

where U is the properly normalized Hamiltonian with a definite large N limit, $U = \frac{H - \langle H \rangle}{N}$. Thus the algebra at this level is in fact a tensor product of the GFF algebra and the algebra of bounded operators of U , $\mathcal{A}_{\tilde{O}} \otimes \mathcal{A}_U$ and still a Type III₁ algebra.

Another property of Type III₁ von Neumann algebras is that one can extend it to include the outer automorphism of the algebra and changes it to a Type II_∞ von Neumann algebra. This algebra will be acting on a different Hilbert space and does not describe the original system¹. But, as one includes $1/N$ corrections to the operators exterior to the black hole, it was found that the

center mode U will become an outer automorphism to $\mathcal{A}_{\tilde{O}}$. Thus the new Type II_∞ algebra in fact describes the algebra of operators as one backs away from strict $N \rightarrow \infty$ limit and considers $1/N$ corrections. Entropy can be associated to this algebra of operators and it can be interpreted as the generalized entropy for the black hole. This construction is also done for maximally extended Schwarzschild black hole in flat space-time and deSitter space-time [4].

The purpose of this paper is to analyze this construction in the case of AdS-Rindler wedge. This corresponds to a spherical region in the boundary CFT and one needs to analyze single trace operators in this sub-region. They form a Type III₁ algebra and there is a center for these operators. But there is a difference with the previous discussion in that the two point function of the center mode will still be divergent after one naively normalizes it, in a similar way U is defined. In fact the two point function is given by the area of the horizon. We will renormalize this area and thus the two point function then we will be able to proceed with the construction that goes parallel to the previous discussions for deSitter and black holes in AdS and flat space-times.

II. THE TYPE III₁ NATURE OF THE ADS-RINDLER WEDGE

Consider two time-like separated points in the boundary CFT, p and q , where p is to the future of q . The intersection of the causal past of p and the causal future of q is defined to be the causal wedge D_q^p . To describe the physics in this diamond, it is enough to know the physical data on a space-like slice in the diamond called the Cauchy surface. This Cauchy slice can be chosen to be finite spherical region at some time slice in the boundary. The causal wedge of D_q^p , what is called the AdS-Rindler wedge, is the bulk region that is causally connected D_q^p . In other words, the fields in this bulk region can be explicitly expressed by evolving the boundary operators in D_q^p in to the bulk using the bulk equations of motion at the time slice[5–7]. Since one can think of the causal

¹ Still, by doing an additional crossed product by the dual group of the automorphism to this new algebra, one can get to the tensor product of the original Type III₁ algebra and an algebra of bounded operators on $\mathcal{L}^2(G)$. To study Type III₁ algebras like this was the original purpose of the crossed product construction.

wedge as a causal diamond in the bulk (with the same time-like separated points but now the causal past and future of p and q respectively, include points also in the bulk, i.e, the diamond includes causal paths from q to p that extend in the the bulk), any point in the wedge can be described only using the data on the time slice.

AdS-Rindler wedge is considered as the dual to the boundary causal diamond of a finite spherical region. It is the bulk region that can be reconstructed from this boundary sub-region.² Since the algebra of operators in a sub-region of a quantum field theory in flat space-time is expected to be Type III₁[8], one can conclude that the algebra of operators in the AdS-Rindler wedge is also Type III₁. Without even referring to the boundary, one can see that the algebra should be Type III₁ since in the bulk one has a local quantum field theory on a curved space-time in the strict large N limit. So a sub region like AdS-Rindler wedge will be Type III₁.

A. Explicit construction of the algebra

Once we have established the type of the algebra of operators in AdS-Rindler, $\tilde{\mathcal{A}}_0$, it follows that the Hilbert space that describes the system will have to be built out of a thermofield double state of the naive Hilbert space one would come up with. The reason for this can be understood if one thinks of the quantum field theory on AdS-Rindler space as a QFT at temperature $T = 1/2\pi$, which arises as a result of the Rindler horizon.

In the usual quantum field theory at zero temperature [9], the way to construct a Hilbert space is by first choosing a state that one is interested in, $|\xi\rangle$ ³, and start applying with the modes in the algebra of the system. There will be infinite number of modes, but in most interesting systems it will be enough to act with finite but arbitrary number of modes.⁴ This will create for us the pre-Hilbert space. To create the Hilbert space, one requires any Cauchy sequence to converge, i.e, one adds the limit points of any Cauchy sequence,

$$\{|\psi_n\rangle\}_{n \in \mathbb{N}},$$

in the pre Hilbert space, this is called its Hilbert space completion.

This Hilbert space, $\mathcal{H}_{|\xi\rangle}$, will not involve states that differ from the original state, $|\xi\rangle$, by the action of infinite number of modes. But with such states, one can create other Hilbert spaces in the same way (by acting with arbitrary but finite number operators and taking

the Hilbert space completion). These Hilbert spaces are totally independent⁵ and can be treated as different super selection sectors[10]. Naively one would think that the full Hilbert space of the system would be the space that includes all the Hilbert spaces whose construction was given just now.

$$\mathcal{H}' = \bigoplus_{|\xi\rangle} \mathcal{H}_{|\xi\rangle}$$

This Hilbert space, called non-separable Hilbert space, has uncountably infinite dimension and is hard to deal with. But, since the super selection rules apply to the 'small Hilbert spaces', it would be enough to consider only these super selection sectors whose dimension is countably infinite.

It is this construction that fails in the case of a QFT at non zero temperature. The reason is thermal fluctuations will take us outside the super selection sectors. Thus, it seems that one necessary have to deal with the non-separable Hilbert space. A way around this was found by von Neumann [11] by using the thermofield double state at temperature T to construct a separable Hilbert space but now the algebra of operators will not act irreducibly on the Hilbert space. Rather than choosing some state to build the Hilbert space on, now one chooses a thermal density matrix at a given temperature. Thus thinking the AdS-Rindler wedge in the strict large N limit, $N = \infty$, as a quantum field theory on a curved space-time at some temperature T , one uses the thermofield double state at temperature $T = 1/2\pi$ to construct the Hilbert space on which the Type III₁ algebra will act.

The algebra $\tilde{\mathcal{A}}_0$ contains bulk fluctuations acting in the AdS-Rindler wedge. Consider the following metric for the AdS-Rindler,

$$ds^2 = -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} + r^2 dH_{d-1}^2 \quad (1)$$

where $dH_{d-1}^2 = d\chi^2 + \chi^2 d\Omega_{d-2}^2$ is the standard metric on the hyperbolic ball H_{d-1} .

A free massless scalar field on this metric is given by,

$$\phi(t, r, \alpha) = \sum_{\lambda} \int_0^{\infty} \frac{d\omega d\lambda}{4\pi^2} (f_{\omega, \lambda}(t, r, \alpha) a_{\omega, \lambda} + f_{\omega, \lambda}^*(t, r, \alpha) a_{\omega, \lambda}^\dagger) \quad (2)$$

where the sum is over degeneracy for λ and $f(t, r, \alpha)$ solves the Klein-Gordon equation and is of the form,

$$f_{\omega, \lambda}(t, r, \alpha) = e^{i\omega t} Y_{\lambda}(\alpha) \psi_{\omega, \lambda}(r)$$

and $Y_{\lambda}(\alpha)$ is eigenfunction of the Laplacian on H_{d-1} and $\psi_{\omega, \lambda}(r)$ is given in terms of hypergeometric functions up to some prefactors that make sure that it has

² More precisely, what is dual to a boundary sub-region is the entanglement wedge but in the simple cases where the state is the vacuum, the entanglement wedge and the causal wedge coincide.

³ It is taken to be the vacuum in most cases.

⁴ Physically the intuition is that any experimenter will not be able to act with infinite number operators.

⁵ The fact that they are independent will necessarily depend on the algebra of operators.

the expected asymptotic behaviour. The elements of \mathcal{A}_0 , $F(a, a^\dagger)$, are linear combinations of a finite string of operators like

$$(a_{\omega_{i_1}, \lambda_{i_1}}^\dagger)^{i_1} \dots (a_{\omega_{i_r}, \lambda_{i_r}}^\dagger)^{i_r} (a_{\omega_{k_1}, \lambda_{k_1}})^{k_1} \dots (a_{\omega_{k_s}, \lambda_{k_s}})^{k_s}.$$

with complex coefficients.

Naively, one would start with the vacuum, a state annihilated by all the $a_{\omega, \lambda}$'s and then consider all states that are created by the action of the $a_{\omega, \lambda}^\dagger$'s.⁶ Considering the vacuum of the full bulk (the AdS vacuum), $|vac\rangle$, one can see that these modes are thermally populated,

$$\langle vac | a_{\omega, \lambda}^\dagger a_{\omega', \lambda'} | vac \rangle = \frac{1}{e^{2\pi\omega} - 1} \delta(\lambda - \lambda') \delta(\omega - \omega') \quad (3)$$

But because of the thermal fluctuations, states that can only be reached from the AdS-Rindler vacuum by the action of infinite number of $a_{\omega, \lambda}^\dagger$'s, should also be included. But, as remarked before, rather than considering the full non-separable Hilbert space, one will proceed to construct the Hilbert space on top of the thermofield double state.

The AdS-Rindler vacuum is $|\tilde{0}\rangle = \bigotimes_{\omega, \lambda} |\tilde{0}\rangle_{\omega, \lambda}$ and an arbitrary state created by the action of $j_{\omega, \lambda}$ number of $a_{\omega, \lambda}^\dagger$'s on the above state is,

$$|\tilde{j}\rangle = \bigotimes_{\omega, \lambda} |\tilde{j}_{\omega, \lambda}\rangle.$$

The Hilbert space \mathcal{H}_0 spanned by these states $|\tilde{j}\rangle$ is the non separable Hilbert space. We consider a state,

$$|\tilde{\Psi}_n\rangle = \frac{1}{\sqrt{\tilde{Z}_n}} \bigotimes_{\omega, \lambda} \sum_{j_{\omega, \lambda}} e^{-\pi\omega j_{\omega, \lambda}} |\tilde{j}_{\omega, \lambda}\rangle |\tilde{j}_{\omega, \lambda}\rangle' \quad (4)$$

where \tilde{Z}_n is such that $\langle \tilde{\Psi}_n | \tilde{\Psi}_n \rangle = 1$ and the state $|\tilde{j}_{\omega, \lambda}\rangle'$ is an element of \mathcal{H}'_0 , a copy of \mathcal{H}_0 . Thus we define the thermofield double state $|\tilde{\Psi}\rangle$ as a map from $\tilde{\mathcal{A}}_0$ to \mathbb{R} given by,

$$\langle \tilde{\Psi} | F(a, a^\dagger) | \tilde{\Psi} \rangle = \lim_{n \rightarrow \infty} \langle \tilde{\Psi}_n | F(a, a^\dagger) | \tilde{\Psi}_n \rangle. \quad (5)$$

⁷ Measurements made by an observer restricted to this bulk sub region is thus given by such expectation value.

⁶ From the full bulk perspective (in contrast to the just the AdS-Rindler subregion), this vacuum is not well defined, it has a firewall (infinite stress energy tensor) at the entangling surface with the complement region. The reason is that there will be no entanglement between the two complementary regions and the two point functions for this state between the two regions will not have the correct short distance behaviour.

⁷ There should also be an upper limit on the sum over $j_{\omega, \lambda}$ which we did not emphasize here. A more careful statement would be to consider $|\tilde{\Psi}_n\rangle = \frac{1}{\sqrt{\tilde{Z}_n}} \bigotimes_{\omega, \lambda} |\omega, \lambda\rangle$ and define $|\omega, \lambda\rangle_k = \sum_{j_{\omega, \lambda}} e^{-\pi\omega j_{\omega, \lambda}} |\tilde{j}_{\omega, \lambda}\rangle |\tilde{j}_{\omega, \lambda}\rangle'$ and $|\omega, \lambda\rangle$ is defined by $\langle \omega, \lambda | F(a_{\omega, \lambda}, a_{\omega, \lambda}^\dagger) | \omega, \lambda \rangle = \lim_{k \rightarrow \infty} \langle \omega, \lambda |_k F(a_{\omega, \lambda}, a_{\omega, \lambda}^\dagger) | \omega, \lambda \rangle_k$.

The Hilbert space $\mathcal{H}_\Psi \subset \mathcal{H}_0$ is generated by states given by the action of a finite but arbitrary string of a and a^\dagger 's on $|\tilde{\Psi}\rangle$ and their Hilbert space completion.

The boundary analog of the algebra of operators and Hilbert space construction can be discussed following the extrapolate dictionary,

$$\lim_{r \rightarrow \infty} r^\Delta \phi(r, x) = O(x),$$

Δ being the conformal dimension of the conformal field theory single trace operator O . Then it follows that,

$$O_{\omega, \lambda} = \int dt d\alpha e^{-i\omega t} Y_\lambda^*(\alpha) O(t, \alpha) = N_{\omega, \lambda} a_{\omega, \lambda} \quad (6)$$

Then, the non-separable Hilbert space \mathcal{H}_0 is spanned by $|j\rangle = \bigotimes_{\omega, \lambda} |j_{\omega, \lambda}\rangle$ constructed from the vacuum of the boundary subregion, $|0\rangle = \bigotimes_{\omega, \lambda} |0\rangle_{\omega, \lambda}$, by the action of $j_{\omega, \lambda}$ raising operators, $O_{\omega, \lambda}^\dagger$.

We also define a thermofield double state $|\Psi\rangle$ as a map from \mathcal{A}'_0 , the algebra of single trace operators, to \mathbb{R} exactly like 4 and 5 but now $a_{\omega, \lambda}$ and $a_{\omega, \lambda}^\dagger$ are replaced by $O_{\omega, \lambda}$ and $O_{\omega, \lambda}^\dagger$.

Single trace operators are gauge invariant operators that can be constructed by considering operators written in terms of trace of the fields and their derivatives with no explicit N dependence in matrix theories. In these theories, where the fields transform in the adjoint representation of, for example $SU(N)$ or $U(N)$, the fields can be normalized such that the action is given with an explicit N factor multiplied by a gauge invariant term with no explicit N dependence. Well known examples include, free Yang Mills, $\mathcal{N} = 4$ super Yang Mills and so on. These operators have $O(N)$ one point function and two point functions with a leading $O(N^2)$ disconnected term. Thus to make sense of the large N limit, one needs to consider the subtracted single trace operators, $\tilde{O} = O - \langle O \rangle$. These operators close and are generalized free fields in the large N limit. Thus, since we are studying the $N = \infty$ theory, one should use \tilde{O} to construct \mathcal{A}'_0 which are operators with a well defined large N limit.

The Hilbert space, \mathcal{H}_Ψ is now constructed from the action of operators in \mathcal{A}'_0 on $|\Psi\rangle$ and limit points of a Cauchy sequence of states

The algebra \mathcal{A}'_0 should also be completed in the weak sense, i.e, if a sequence of expectation values converges, $\lim_{n \rightarrow \infty} \langle \Psi | F_n | \Psi \rangle = \langle \Psi | F | \Psi \rangle$ for $F_n \in \mathcal{A}'_0$, then one adds the operator F to get the von Neumann algebra \mathcal{A}_0 . This is the algebra of single trace operators.

But this algebra of single trace operators is not all there is, there is a conserved charge associated with the conformal Killing vector, χ^μ , that preserve the causal diamond of the spherical boundary sub region and its complement. It is given by an integral that involves the stress energy tensor, χ^μ and the normal vector to the boundary Cauchy surface. Since it is a symmetry it annihilates the full vacuum, $|vac\rangle$. In the bulk this is the boost operator that

preserves Rindler horizon. The integral can be naturally divided into a part that acts in the subregion and a part that acts in the complement, $H_\chi = 2\pi(K - \bar{K})$. The operator K is the boost operator of the boundary subregion and the algebra of single trace operators will not capture it. In the strict large N limit, this corresponds in the bulk to the area of the horizon, loosely speaking⁸, which is also not captured by the bulk fluctuations.

In the normalization that single trace operator, O , have $O(N)$ one point function and leading disconnected $O(N^2)$ two point function, the stress energy tensor will have the same form as the action, a gauge invariant term multiplied by N . The generator of the conformal transformation that leaves the diamond of the boundary subregion B invariant is given by,

$$K = \int_B T_{\mu\nu} \chi^\mu dB^\nu \quad (7)$$

and will also have the explicit N dependence. This explicit N dependence will give K one point and connected two point functions of $O(N^2)$. A simple subtraction like what we did for the single trace operators will not result in a generalized free field, since the two point function will still be $O(N)$. Thus one defines, the operator,

$$\tilde{X} = \frac{K - \langle K \rangle}{N} \quad (8)$$

Up till now one can draw a direct parallel between the case of large black holes in AdS, with the Hamiltonian being the corresponding K . [3] The Hamiltonian is also given as an integral of the stress energy tensor and has $O(N^2)$ one point and connected two point function (above Hawking-Page temperature)⁹. The only difference would be in the case of the black hole there is a finite horizon area. But in the present case, the horizon is of infinite volume. Thus, even though for the Hamiltonian one would be satisfied with 8, for the present case there will still be an additional divergence coming from horizon which was not present for the black hole and strictly speaking, the mode \tilde{X} is not yet well defined.

From the bulk perspective, this can be seen from the computation of $\langle K \rangle$ and $\langle K^2 \rangle - \langle K \rangle^2$ in [13]. To leading order in G_N ,

$$\langle K \rangle = \langle (K - \langle K \rangle)^2 \rangle = \frac{A(\Sigma)}{4G_N} \quad (9)$$

where $A(\Sigma)$ is the volume of the Rindler horizon Σ which is a $d - 1$ dimensional hyperbolic hyper-surface. Following the normalization 8 and using $G_N \sim 1/N^2$ one gets

$\langle \tilde{X} \rangle = 0$ and $\langle \tilde{X}^2 \rangle \sim A(\Sigma)$. The two point function is infinite since $A(\Sigma)$ is infinite.

Even though \tilde{X} is well defined \tilde{X}^2 is not well defined, it has divergent expectation value. If there was any way to renormalize it and define a new operator X with a well defined large N limit for the one and two point function¹⁰, then one can include to our strict large N limit algebra of operators, bounded functions of this mode and get the full algebra $\mathcal{A} = \tilde{\mathcal{A}}_0 \otimes \mathcal{A}_X$. This algebra of operators will act on an extended Hilbert space, $\mathcal{H} = \mathcal{H}_\Psi \otimes \mathcal{L}^2(\mathbb{R})$ which in general include some entangled state between square integrable functions of X and states in \mathcal{H}_Ψ .

B. The renormalization of the infinite volume

We now claim that there is a natural renormalization of this additional divergence mentioned in the previous section. We start by noticing 9 that the expectation value and the fluctuations of K are given by the area of a $d - 1$ dimensional hyperbolic surface with boundary. A hyperbolic manifold with boundary will have, in general, infinite volume. But there is a renormalization of this volume which would follow from looking at the Einstein-Hilbert action of these manifolds. The action for a manifold M with Euclidean signature is given by,

$$S = -\frac{1}{16\pi G_N} \int_M d^{d-1}x \sqrt{g} (R - 2\Lambda) - \frac{1}{8\pi G_N} \int_{\partial M} d^{d-2}x \sqrt{\gamma} K \quad (10)$$

where Λ is the cosmological constant that is assumed to be negative, g and R are the metric and Ricci scalar on M while γ and K are the induced metric and the mean curvature on its boundary.

Einstein equations will then imply that the bulk part of the action is given by the volume of the manifold M (up to some prefactors). Since the M is non-compact, this volume diverges. But this divergence is such that if one takes a compact sub-manifold of M and take its boundary to approach ∂M , it goes like integral of local terms (more specifically, the first and second fundamental form) on the boundary of the compact sub manifold. Thus, this would imply to a renormalization where one subtracts the divergent terms and take the limit of the boundary of the sub manifold to ∂M . Thus giving a notion of renormalized volume for the hyperbolic manifold M .

The renormalization procedure is discussed in several papers[14, 15] but we will revise it here. It essentially involves four steps. The first is to take M to be in the metric,

$$ds^2 = \frac{1}{4\rho^2} d\rho^2 + \frac{h_{ij}}{\rho} dx_i dx_j \quad (11)$$

⁸ There are several subtleties concerning the diverging behaviour of the area term which we will discuss in the following sections.

⁹ One argument for the Type III₁ nature of the AdS-Rindler wedge and also black hole exterior in AdS is the continuous spectrum of the modular Hamiltonian (boost operator) (the Hamiltonian for the black hole case). This continuous spectrum of the Hamiltonian is in fact associated to the appearance of horizon in the bulk[12].

¹⁰ Here we assume large N factorization for X , so it is enough to renormalize the two point function.

as ρ goes to zero (the boundary), h goes to γ , the boundary metric. Then one solves Einstein's equation for the above metric.

$$\begin{aligned} \rho(2h'' - 2h'h^{-1}h' + \text{Tr}(h^{-1}h')h') \\ + R_\gamma - (d-2)h' - \text{Tr}(h^{-1}h')h &= 0 \\ (h^{-1})^{jk}(D_i h'_{jk} - D_k h'_{ij}) &= 0 \\ \text{Tr}(h^{-1}h'') - \frac{1}{2}\text{Tr}(h^{-1}h'h^{-1}h') &= 0 \end{aligned} \quad (12)$$

where R_γ is the boundary Ricci scalar and D_i is covariant derivative in the boundary metric while prime is derivative with respect to ρ . We have also used $\Lambda = \frac{-d(d-1)}{2}$.

The important next step is to note that the above equations can be solved by considering the following expansion for h ,

$$h = \gamma + \rho\gamma_2 + \rho^2\gamma_4 + \dots \quad (13)$$

where for even d , there is an additional term at the $\rho^{d/2}$ order which is $\rho^{d/2}\log\rho \kappa_2$. γ_k is given covariantly in terms of γ and contains k derivatives with respect to x_i .

Considering the simplest case of hyperbolic 3 manifolds, i.e, taking the full bulk to be AdS_5 , Einstein's equations become $R - 2\Lambda = -4$ and one can see that the action is just the volume of the manifold M ¹¹. Now we consider the action of a sub manifold $N_\epsilon \subset M$ that is regularized at $\rho = \epsilon$ for some cut off $\epsilon > 0$. The action of N_ϵ will have both bulk and boundary term, at $\rho = \epsilon$. The expansion 13 implies a similar expansion for \sqrt{g} and the action will be as follows,

$$S = \frac{V(N_\epsilon)}{4\pi G} = \frac{1}{4\pi G} \int_{\partial N_\epsilon} d^2x \frac{1}{\epsilon} a_0 + \log\epsilon a_2 + L_{finite}. \quad (14)$$

where $a_0 = \sqrt{\gamma}$ and $a_2 = -\frac{1}{4}\sqrt{\gamma}R_\gamma$. Thus adding counter terms to remove the divergent terms and taking the limit $\epsilon \rightarrow 0$, one gets a renormalized volume of the manifold M .

For our present case, this hyperbolic manifold is the horizon of the AdS-Rindler subregion. If we renormalize \tilde{X}^2 by subtracting the above divergent terms, then the renormalized mode X is well defined with finite one and two point functions and the algebra given above, $\mathcal{A} = \tilde{\mathcal{A}}_0 \otimes \mathcal{A}_X$ makes sense.

III. ADDING GRAVITY AND MAKING THE ALGEBRA TYPE II_∞

To move away from the strict N limit, one will have to include in $\tilde{\mathcal{A}}_0$ linear combinations of single trace operators with coefficients that have asymptotic power expansion in $1/N$ around $1/N = 0$. In addition, one have

¹¹ There are also additional boundary terms but we are going to use local boundary counter terms to renormalize the action, for 3 manifolds, the boundary term which are also given by the same terms will be canceled directly

to add $1/N$ corrections to X . The way to do this is to note that modular Hamiltonian at finite N generates modular time translation within the causal diamond of the boundary sub region.

$$\left[\frac{K - \langle K \rangle}{N}, O\right] = \frac{1}{N} \partial_\tau O. \quad (15)$$

Thus one adds to X an operator that have well defined large N limit and also have no divergence coming from infinite horizon area, i.e, H_X . Checking with 15, one has a new operator, $X + \frac{H_X}{2\pi N}$ that generates time translation for the single trace operators. Any other operator that one can add consistent with 15, like operators from the the complement region, can be removed by making use of the conjugate operator of X , that is $\Pi = \frac{d}{idX}$. By conjugating the algebra with $e^{i\Pi\tilde{O}/N}$, one can remove an additional operator from the complement, like \tilde{O}/N .^[3]¹² Thus, even at perturbative order in $1/N$, the new mode does not have divergences and is well defined.

Bounded operators for this mode will form the algebra $\mathcal{A}_{X + \frac{H_X}{2\pi N}}$. Since this enlargement of the algebra upgrades the center mode to an outer automorphism for $\tilde{\mathcal{A}}_0$, as is well known in the axiomatic quantum field theory literature and very well elaborated in a paper [3], the full algebra will be a Type II_∞ crossed product algebra $\tilde{\mathcal{A}}_0 \rtimes \mathcal{A}_{X + \frac{H_X}{2\pi N}}$. This algebra again will act on the Hilbert space, $\mathcal{H} = \mathcal{H}_\Psi \otimes \mathcal{L}^2(\mathbb{R})$ but now one can associate a density matrix for any state. A certain class of states one can consider are of the form $|\tilde{\Psi}\rangle = |\Psi\rangle \otimes g(X)^{1/2}$. Using such states, a trace can be defined for what are called trace class operators[3]. An element of the crossed product algebra, $\tilde{O} = \int ds O(s)e^{isY}$ where $Y = X + \frac{H_X}{2\pi N}$ is trace class if the trace of the operator defined as,

$$\text{Tr}(\tilde{O}) = \langle \tilde{\Psi} | \tilde{O} K^{-1} | \tilde{\Psi} \rangle,$$

is finite, where $K = \frac{g(Y)}{e^Y}$. This implies the density matrix for such states is in fact K . One can also define a density matrix, ρ_Φ , for a general state $|\tilde{\Phi}\rangle \in \mathcal{H}$. Thus, it is possible to define a von Neumann entropy, $S_\Phi = -\text{Tr}\rho_\Phi \log\rho_\Phi = -\langle \tilde{\Phi} | \log\rho_\Phi | \tilde{\Phi} \rangle$, can be associated to any state in the Hilbert space.[3]

IV. DISCUSSION

This analysis has been for AdS-Rindler wedge at perturbative order in $1/N$ around $N = \infty$. But for finite N , the algebra of the AdS-Rindler is expected to be again Type III₁. The reason is from the boundary perspective, one is looking at the algebra of a local region

¹² This is because algebras are defined up to conjugation.

of a quantum field theory which is in general Type III₁. On the other hand, the space-time description of the theory will not be precise for finite value say $N = 243$. Thus the renormalization procedure for the area of the Rindler horizon will make less and less sense as one backs away from the N going to infinity limit. Thus we expect that, to make sense of the theory non perturbatively, one would have to work with \tilde{X} and not the renormalized mode; this restores the Type III₁ nature of the boundary sub region for finite N . But still a more careful analysis has to be done to understand this better.

It would also be interesting to compare the von Neumann entropy given in the previous section to the

usual entropy proposed by Hubeny, Rangamani and Takayanagi[16].

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