VISCO-ENERGETIC SOLUTIONS FOR A MODEL OF CRACK GROWTH IN BRITTLE MATERIALS

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Abstract. Visco-energetic solutions have been recently advanced as a new solution concept for rate-independent systems, alternative to energetic solutions/quasistatic evolutions and balanced viscosity solutions. In the spirit of this novel concept, we revisit the analysis of the variational model proposed by Francfort and Marigo for the quasi-static crack growth in brittle materials, in the case of antiplane shear. In this context, visco-energetic solutions can be constructed by perturbing the time incremental scheme for quasistatic evolutions by means of a viscous correction inspired by the term introduced by Almgren, Taylor, and Wang in the study of mean curvature flows. With our main result we prove the existence of a visco-energetic solution with a given initial crack. We also show that, if the cracks have a finite number of tips evolving smoothly on a given time interval, visco-energetic solutions comply with Griffith's criterion.

Keywords: variational models, energy minimization, visco-energetic solutions, crack propagation, Griffith's criterion.

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1. Introduction

The variational approach to brittle fracture, based on the classical theory by Griffith [Gri20], was initiated more than twenty years ago by Francfort and Marigo [FM98] (cf. also [BFM08]). In these models crack growth results from a trade-off between the competing mechanisms of

- energy conservation, with the *driving energy* given by the stored elastic energy;
- energy dissipation, which takes into account the *dissipated energy* spent to open the crack.

In the case of *antiplane shear* the reference configuration is represented by a bounded, connected, Lipschitz domain $\Omega \subset \mathbb{R}^2$, the displacement $u\colon \Omega \to \mathbb{R}$ is scalar and the cracks are represented by compact subsets K of $\overline{\Omega}$. The evolution is triggered by a prescribed time dependent boundary condition $u = g(t)$ on a subset $\partial_D\Omega$ of $\partial\Omega$. According to the model by Francfort and Marigo, for a linearly elastic homogeneous isotropic material the competing energy terms are

$$\textit{driving energy:}\quad \mathcal{E}(t,K) := \min\Big\{ \int_{\Omega\setminus K} \tfrac{1}{2}|\nabla u|^2\,\mathrm{d}x \,:\, u = g(t) \text{ on } \partial_D\Omega \setminus K \Big\}, \tag{1.1}$$

dissipated energy: $\mathcal{H}^1(K(t)\setminus K(s))$, where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

For simplicity the elastic constant and the toughness of the material are normalized to 1.

If $\mathcal{K}(\overline{\Omega})$ denotes the collection of all compact subsets of $\overline{\Omega}$, a quasistatic evolution for the brittle fracture model proposed in [FM98] is a function $K : [0,T] \to \mathcal{K}(\overline{\Omega})$ fulfilling the following conditions:

(I) *irreversibility*: $K(s) \subset K(t)$ for all $0 \le s \le t \le T$;

(S) *stability*: at every $t \in [0,T]$ we have

$$\mathcal{E}(t,K(t)) \le \mathcal{E}(t,K') + \mathcal{H}^1(K'\setminus K(t)) \qquad \text{for all } K' \in \mathcal{K}(\overline{\Omega}) \text{ with } K' \supset K(t), \tag{1.2}$$

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namely the release of potential energy when passing from the current state $K(t)$ to any other state $K' \in \mathcal{K}(\overline{\Omega})$ is smaller than the energy dissipated, that is why (1.2) can also be understood as a stability condition;

(E) *energy-dissipation balance*:

$$\mathcal{E}(t, K(t)) + \mathcal{H}^1(K(t)\backslash K(s)) = \mathcal{E}(s, K(s)) + \int_s^t \partial_t \mathcal{E}(r, K(r))\,\mathrm{d}r \qquad \text{for all } 0 \le s \le t \le T, \tag{1.3}$$

involving the stored energy at the process times s and t, the energy dissipated in the time interval $[s,t]$, and the work of the external forces represented by the integral term.

In [DMT02b] this crack model in the antiplane case was analyzed imposing a bound on the number of connected components of the crack, thus replacing the space $\mathcal{K}(\overline{\Omega})$ by the space $\mathcal{K}_m(\overline{\Omega})$ of all compact subsets of $\overline{\Omega}$ with at most m connected components and finite 1-dimensional Hausdorff measure. The existence of quasistatic evolutions satisfying (I), (S), and (E) was proved by constructing discrete-time approximate solutions: given a partition $0 = t_\tau^0 < t_\tau^1 < \ldots < t_\tau^{N_\tau} = T$ of $[0,T]$, the time incremental minimization scheme

$$K_\tau^i \in \mathrm{Argmin}\big\{\mathcal{E}(t_\tau^i, K) + \mathcal{H}^1(K\backslash K_\tau^{i-1}) \,:\, K \in \mathcal{K}_m(\overline{\Omega}),\ K \supset K_\tau^{i-1}\big\} \qquad \text{for } i = 1, \ldots, N_\tau \tag{1.4}$$

provides an approximate solution which converges to a continuous-time solution as the time step tends to 0. Ever since, the analysis of quasistatic evolutions for crack propagation models has been extended in several directions, cf., e.g., [Cha03, FL03, DMFT05, DML10, FS18]. Furthermore, thanks to their flexibility and robustness, the notion of quasistatic evolution and the parallel concept of *energetic solution* [MT99, MT04] have been extensively applied to a broad class of rate-independent systems (cf. [MR15] for a survey).

Nonetheless, it has been known for some time that quasistatic evolutions/energetic solutions have a drawback. Namely, when the energy functional driving the system is nonconvex, such evolutions, as functions of time, may have 'too early' and 'too long' jumps between energy wells, cf., e.g., [KMZ08, Ex. 6.3], and the full characterization of energetic solutions to 1-dimensional rate-independent systems proved in [RS13]. Essentially, this is due to the *global* character of the stability condition (S), which involves the overall energy landscape. These considerations have motivated the quest of alternative weak solvability notions based on local, rather than global, minimality.

To our knowledge, the first attempt in this direction dates back to [DMT02a], where, as an alternative to (1.4), the following time incremental minimization scheme was proposed for brittle fracture growth (still in the two-dimensional antiplane case):

$$(u_\tau^i, K_\tau^i) \in \mathrm{Argmin}\big\{\mathcal{E}(t_\tau^i, K) + \mathcal{H}^1(K\backslash K_\tau^{i-1}) + \lambda\|u - u_\tau^{i-1}\|^2_{L^2(\Omega)} \,:\, K \in \mathcal{K}_m(\overline{\Omega}),\ K \supset K_\tau^{i-1}, u \in H^1(\Omega\backslash K)\big\}$$

for $i = 1, \ldots, N_\tau$, with $\lambda > 0$ a *fixed* constant. The additional L^2-contribution penalizes the L^2-distance of the updated discrete displacement u_τ^i from the previous u_τ^{i-1}, and thus enforces *locality* on the time discrete level. We also record the notion of fracture evolution by local minimality advanced in [Lar10].

A more general approach to a reformulation of rate-independent evolution devoid of unnatural jumps was pioneered in [EM06]. It stemmed from the idea that rate-independent evolution originates in the limit of systems governed by two time scales: the 'fast' inner scale of the system and the 'slow', but dominant, time scale of the external forces. In this perspective, viscous dissipation is negligible during a time interval in which the system evolves continuously, but it is expected to enter into the system behavior at jumps. Thus, one selects those solutions to the original rate-independent system that arise as limits of solutions to the viscously regularized system. This procedure leads to an alternative solution concept featuring a *local*, in place of a global, stability/minimality condition, and an energy-dissipation balance that provides a description of the system behavior at jumps, with the possible onset of 'viscous behavior'. On the one hand, the vanishing-viscosity technique has been formalized in an abstract setting in [MRS12, MRS16] (cf. also [Neg14]), that codified the properties of these 'vanishing-viscosity solutions' in the notion of *balanced viscosity* solution. On the other hand, it has been developed and refined in various concrete applications, cf., e.g., [DDS11, BFM12, KRZ13, CL16]).

As far as brittle fracture models are concerned, however, the vanishing-viscosity approach has been carried out either assuming that the crack path is a priori known, or in specific geometric settings, cf., e.g., [TZ09, Cag08, KMZ08, KZM10, LT11, LT13, Alm17, CL17, ALL20]). These restrictions are related to the fact that the construction of balanced viscosity solutions ultimately relies on the validity of a suitable chain rule for the energy functional driving the system, which seems to be hard to obtain for more general fracture models.

That is why, finding an appropriate mathematical formulation for the evolution of brittle fracture as an alternative to the notion of quasistatic evolution, without specific assumptions on the cracks, is still an up-to-date and challenging issue. In this paper we aim to contribute to it by showing how the concept of *visco-energetic* solution to a rate-independent system, recently introduced in [MS18], can be successfully applied to the two-dimensional antiplane model first addressed in [DMT02b].

As we will see, visco-energetic solutions have a structure in between that of energetic and balanced viscosity solutions. This intermediate character is also apparent in their characterization, obtained for one-dimensional systems in [Min17], in the results of [RS17], and in their applicability to rate-independent systems in damage, plasticity, and delamination, cf. [Ros19]. Here we are going to demonstrate that the model by Francfort and Marigo, at least in the versions considered in [DMT02b] and [Cha03], provides yet another example of rate-independent process for which visco-energetic solutions are an adequate tool, while the balanced viscosity concept fails to apply.

Visco-energetic evolution of brittle fracture. Visco-energetic (hereafter often abbreviated as VE) solutions were introduced in [MS18] in the context of an abstract rate-independent system whose state space is a Hausdorff topological space (X,σ), endowed with

(1) a driving energy functional $\mathcal{E}\colon [0,T]\times X \to (-\infty+\infty]$;
(2) a (possibly asymmetric, quasi-)distance $\mathsf{d}\colon X\times X\to[0,+\infty]$ that encodes the energy dissipation of the system.

In the spirit of [DMT02a], the key idea at the core of VE concept is to enforce locality by suitably perturbing the time incremental minimization scheme. In the general context addressed in [MS18], this perturbation is obtained by means of

(3) a *viscous correction*, namely a lower semicontinuous functional $\delta\colon X\times X\to[0,+\infty]$, compatible with d in a suitable sense.

The above elements constitute a *viscously corrected* rate-independent system $(X,\mathcal{E},\sigma,\mathsf{d},\delta)$.

Revisiting the brittle fracture model analyzed in [DMT02b] within the approach of [MS18], we work in the ambient space

$$X=\mathcal{K}(\overline{\Omega}),\ \text{endowed with the topology of the Hausdorff distance } \mathsf{h}. \tag{1.5}$$

The evolution is driven by the energy functional $\mathcal{E}:[0,T]\times\mathcal{K}(\overline{\Omega})\to[0,+\infty)$ defined in (1.1).

Instead of imposing a bound on the number of connected components of the crack, we penalize the nucleation of new connected components by means of the quasi-distance $\alpha(K,K')$ defined as the number of connected components of K' disjoint from K. Indeed, we fix a constant $\lambda>0$ and we consider the dissipation distance $\mathsf{d}\colon\mathcal{K}(\overline{\Omega})\times\mathcal{K}(\overline{\Omega})\to[0,+\infty]$ defined by

$$\mathsf{d}(K,K'):=\mathcal{H}^1(K'\backslash K)+\lambda\,\alpha(K,K') \tag{1.6}$$

if $K\subset K'$, and set equal to $+\infty$ otherwise. The additional term $\lambda\,\alpha(K,K')$ controls the number of connected components and is important in order to obtain the lower semicontinuity of d with respect to the Hausdorff distance, cf. Proposition 3.6 ahead. The constant λ plays the role of the energetic cost of the nucleation of a new connected component of the crack.

We have chosen as viscous correction the functional $\delta\colon\mathcal{K}(\overline{\Omega})\times\mathcal{K}(\overline{\Omega})\to[0,+\infty]$ given by

$$\delta(K,K'):=\int_{K'\backslash K}\operatorname{dist}(x,K)\,\mathrm{d}\mathcal{H}^1(x)+\mu\,\alpha(K,K')\qquad\text{if } K\subset K', \tag{1.7}$$

for some $\mu > 0$. In the regular case considered in Section 6, when K' is close to K the integral contribution to δ is approximately the sum of the squares of the length increments of the branches of the crack. A similar term has already been used in [LT11, LT13] to study a viscosity-driven model of crack growth. The integral in (1.7) is well defined for arbitrary compact sets K and K', with no structural assumptions. This term was introduced by ALMGREN, TAYLOR and WANG in [ATW93], cf. also [DG93, LS95], where it plays the role of a sort of L^2-distance between K and K' in the *Minimizing Movement* scheme for the mean curvature flow. The term $\mu\,\alpha(K,K')$ has a technical role related to the lower semicontinuity properties of δ. The constant μ can be interpreted as an additional energetic cost due to the nucleation of a new connected component.

Hereafter, we shall refer to the quintuple

$$(\mathcal{K}(\overline{\Omega}), \mathcal{E}, \mathsf{h}, \mathsf{d}, \delta) \quad \text{as a } \textit{viscously corrected} \text{ rate-independent system for brittle fracture.}$$

Along the footsteps of [MS18] we construct discrete solutions by solving time incremental minimization scheme

$$K_\tau^i \in \mathrm{Argmin}_{K\in\mathcal{K}(\overline{\Omega})} \left(\mathcal{E}(t_\tau^i, K) + \mathsf{d}(K_\tau^{i-1}, K) + \delta(K_\tau^{i-1}, K)\right) \qquad \text{for } i = 1, \dots, N_\tau, \tag{1.8}$$

with $K_\tau^0 := K_0$ the initial crack. It turns out that for every $i = 1, \dots, N_\tau$ (1.8) admits a solution thanks to the aforementioned lower semicontinuity properties of d and δ, and of the energy $\mathcal{E}$. Our main result, **Theorem 4.5** ahead, states that there exists a vanishing sequence $(\tau_j)_j$ of time steps along which the discrete solutions $(K_{\tau_j})_j$, defined by piecewise constant interpolation of the minimizers $(K_\tau^i)_{i=1}^{N_\tau}$, converge to a visco-energetic solution of the viscously corrected system $(\mathcal{K}(\overline{\Omega}), \mathcal{E}, \mathsf{h}, \mathsf{d}, \delta)$. The latter is a curve $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$, with jump set J_K, complying with the following conditions:

(I) *irreversibility*: $K(s) \subset K(t)$ for all $0 \le s \le t \le T$;

(S_{VE}) $(\mathsf{d}{+}\delta)$-*stability*: at every $t \in [0,T] \setminus \mathrm{J}_K$ there holds

$$\mathcal{E}(t, K(t)) \le \mathcal{E}(t, K') + \mathsf{d}(K(t), K') + \delta(K(t), K') \qquad \text{for all } K' \in \mathcal{K}(\overline{\Omega}) \text{ with } K' \supset K(t);$$

(E_{VE}) the *energy-dissipation balance*

$$\mathcal{E}(t, K(t)) + \mathrm{Var}_{\mathsf{d}}(K, [s,t]) + \mathrm{Jmp}_{\mathsf{c}}(K; [s,t]) = \mathcal{E}(s, K(s)) + \int_s^t \partial_t \mathcal{E}(r, K(r))\,\mathrm{d}r \qquad \text{for all } 0 \le s \le t \le T.$$

In (E_{VE}), the contribution $\mathrm{Jmp}_{\mathsf{c}}$ keeps track of the energy dissipated at jumps and is defined in terms of the 'visco-energetic' cost c introduced in (1.9) below.

As we have mentioned before, the structure of this solution concept is in between those of quasistatic evolutions (cf. (1.2) & (1.3)) and of balanced viscosity solutions. On the one hand, the stability condition (S_{VE}), though featuring the viscous correction δ and holding only outside the jump set J_K, still retains a *global* character. On the other hand, in the energy balance (E_{VE}) the dissipation of energy is not only recorded by the d-induced total variation functional $\mathrm{Var}_{\mathsf{d}}$, but, like in the case of balanced viscosity solutions, also by an additional term that measures the energy dissipated at the jump points of K in the interval $[s,t]$, i.e. $\mathrm{Jmp}_{\mathsf{c}}(K; [s,t])$. The jump cost $\mathrm{Jmp}_{\mathsf{c}}$ is, in turn, defined in terms of a functional c obtained by minimizing a suitable transition cost along curves ϑ connecting the two end-points $K(t-)$ and $K(t+)$ of the curve K at $t \in \mathrm{J}_K$, namely

$$\mathsf{c}(t, K(t-), K(t+)) := \inf \left\{ \mathrm{Trc}_{\mathrm{VE}}(t; \vartheta, E) \,:\, E \Subset \mathbb{R},\ \vartheta \in \mathrm{C}(E; \mathcal{K}(\overline{\Omega})),\ \vartheta(E^\pm) = K(t\pm) \right\}, \tag{1.9}$$

where $E^- := \inf E$ and $E^+ := \sup E$. The transition cost

$$\mathrm{Trc}_{\mathrm{VE}}(t; \vartheta, E) := \lambda\Big(\mathrm{Var}_\alpha(\vartheta, E) - \alpha(\vartheta(E^-), \vartheta(E^+))\Big) + \mathrm{GapVar}_\delta(\vartheta, E) + \sum_{s\in E\setminus\{\sup E\}} \mathscr{R}(t, \vartheta(s))$$

features

(1) the α-total variation of the curve ϑ;

(2) a quantity related to the 'gaps', or 'holes', of the set E (which is just an arbitrary compact subset of $\mathbb{R}$ and may have a more complicated structure than an interval);

(3) a functional $\mathscr{R}\colon [0,T]\times\mathcal{K}(\overline{\Omega})\to[0,\infty)$ that keeps track of the violation of the VE-stability condition along the curve ϑ, as it fulfills $\mathscr{R}(t,\vartheta(s))>0$ if and only if $\vartheta(s)$ does not comply with $(\mathrm{S}_{\mathrm{VE}})$ at the process time t.

It is in terms of the cost c that the VE concept offers an alternative description of the system behavior at jumps, in comparison with quasistatic evolutions. Indeed, VE solutions satisfy the jump conditions

$$\mathcal{E}(t,K(t-))-\mathcal{E}(t,K(t+))=\mathsf{d}(K(t-),K(t+))+\mathsf{c}(t,K(t-),K(t+))\qquad\text{for all } t\in\mathrm{J}_K,$$

cf. Proposition 4.6 ahead. Thus, the release of elastic energy at a jump point is not only balanced by the length of the crack opening (like it would be for quasistatic evolutions), but also by the 'visco-energetic' cost between the two end-points $K(t-)$ and $K(t+)$.

Nonetheless, if, along the footsteps of [DMT02b], we assume that, on some interval $(\tau_0,\tau_1)\subset[0,T]$ the VE solution constructed in Theorem 4.5 has the additional property that the cracks $K(t)$ have a fixed number of tips, which evolve smoothly on the interval (τ_0,τ_1) along simple and disjoint paths, then we can prove that Griffith's criterion for crack growth is satisfied, cf. Theorem 6.5 ahead. This result is completely analogous to [DMT02b, Thm. 8.4] for quasistatic evolutions. It reflects the fact that VE solutions essentially differ from quasistatic evolutions in the description of the energetic behavior of the system at jumps, cf. the characterization provided by Proposition 4.6 ahead.

Plan of the paper. In Section 2 we recall some preliminary results on Hausdorff convergence, and on the properties of the elastic energy, proved in [DMT02b]. Then, in Section 3 we introduce the dissipation distance d and the viscous correction δ, and settle their basic properties. Section 4 is devoted to the precise definition of visco-energetic solutions and to the statement of our main existence result, Theorem 4.5. In Section 5, the proof of Theorem 4.5 is carried out by showing that the viscously corrected system for brittle fracture $(\mathcal{K}(\overline{\Omega}),\mathcal{E},\mathsf{h},\mathsf{d},\delta)$ satisfies the conditions of the general existence result [MS18, Thm. 3.9], which thus applies yielding the existence of VE solutions. The main result of Section 6, Theorem 6.5, provides a characterization of the behavior at the crack tips of a VE solution $K:[0,T]\to\mathcal{K}(\overline{\Omega})$ in an interval (τ_0,τ_1) during which K evolves *continuously* as a function of time and the crack set $K(t)$ fulfills suitable geometric conditions. Finally, in Section 7 we show how, relying on the results from [Cha03], our existence result for VE solutions can be extended to the planar case of linearized elasticity.

2. Notation and preliminaries

Throughout the paper, Ω is a fixed bounded connected open subset of $\mathbb{R}^2$ with Lipschitz boundary. As in [DMT02b], we shall additionally suppose that the boundary of Ω decomposes into

- a Neumann part $\partial_N\Omega$, which is a (possibly empty) relatively open subset of $\partial\Omega$ with a finite number of connected components;
- the Dirichlet part $\partial_D\Omega:=\partial\Omega\setminus\overline{\partial_N\Omega}$; it turns out that $\partial_D\Omega$ is also a relatively open subset of $\partial\Omega$ with a finite number of connected components.

The one-dimensional Hausdorff measure is denoted by $\mathcal{H}^1$. The set of all compact subsets of $\overline{\Omega}$ is denoted by $\mathcal{K}(\overline{\Omega})$, whereas $\mathcal{K}_m(\overline{\Omega})$ is the set of all compact subsets K of $\overline{\Omega}$ with at most m connected components and $\mathcal{H}^1(K)<+\infty$.

The space $\mathcal{K}(\overline{\Omega})$ is endowed with the *Hausdorff distance* h, defined by

$$\mathsf{h}(H,K):=\max\big\{\sup_{x\in H}\mathrm{dist}(x,K),\sup_{y\in K}\mathrm{dist}(y,H)\big\}\qquad\text{for all } H,\,K\in\mathcal{K}(\overline{\Omega}),\tag{2.1}$$

where, as usual, $\mathrm{dist}(x,K):=\min_{y\in K}|x-y|$, with the convention that $\mathrm{dist}(x,\emptyset)=\mathrm{diam}(\Omega)$ and $\sup\emptyset=0$, so that $\mathsf{h}(\emptyset,K)=0$ if $K=\emptyset$ and $\mathsf{h}(\emptyset,K)=\mathrm{diam}(\Omega)$ if $K\neq\emptyset$. Given $(K_n)_n,K\subset\mathcal{K}(\overline{\Omega})$, we will often write $K_n\xrightarrow{\mathsf{h}}K$ whenever $\mathsf{h}(K_n,K)\to0$. With a slight abuse of notation, the topology induced by the Hausdorff distance is still denoted by h, and the corresponding product topology on $[0,T]\times\mathcal{K}(\overline{\Omega})$ by $\mathsf{h}_{\mathbb{R}}$. The following compactness theorem is well known (see, e.g., [Rog70, Blaschke's Selection Theorem]).

Theorem 2.1. *The metric space* $(\mathcal{K}(\overline{\Omega}), \mathsf{h})$ *is compact.*

We will also make use of the following result (cf. [DMT02b, Cor. 3.3, 3.4]), derived from the Gołąb theorem (cf., e.g., [MS95, Thm. 10.19]). It shows that the class $\mathcal{K}_m(\overline{\Omega})$ is closed w.r.t. Hausdorff convergence, and that, with respect to this notion of convergence the Hausdorff measure $\mathcal{H}^1$ is lower semicontinuous on $\mathcal{K}_m(\overline{\Omega})$ (while it is not lower semicontinuous on $\mathcal{K}(\overline{\Omega})$).

Theorem 2.2. *Let* $m \geq 1$ *and* $(K_n)_n \subset \mathcal{K}_m(\overline{\Omega})$.

(i) If $\mathsf{h}(K_n, K) \to 0$ *as* $n \to \infty$ *for some* $K \in \mathcal{K}(\overline{\Omega})$, *then* $K \in \mathcal{K}_m(\overline{\Omega})$ *and*

$$\mathcal{H}^1(K \cap U) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \cap U)$$

for every open set $U \subset \mathbb{R}^2$.

(ii) In addition, suppose that $(H_n)_n, H \in \mathcal{K}(\overline{\Omega})$, *with* $\mathsf{h}(H_n, H) \to 0$ *as* $n \to \infty$. *Then,*

$$\mathcal{H}^1(K \backslash H) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \backslash H_n).$$

Deny-Lions spaces and elastic energy. Along the footsteps of [DMT02b], we will work with the *Deny-Lions* space [DL54]

$$L^{1,2}(A) := \{u \in L^2_{\mathrm{loc}}(A) \,:\, \nabla u \in L^2(A; \mathbb{R}^2)\}, \tag{2.2}$$

for a given $A \subset \mathbb{R}^2$. If A is bounded with Lipschitz boundary, then $L^{1,2}(A) = H^1(A)$, see [DMT02b, Prop. 2.2]. For a given $g \in H^1(\Omega)$ and a given $K \in \mathcal{K}_m(\overline{\Omega})$, let us now introduce the space of *admissible displacements*

$$\mathcal{V}(g, K) := \{v \in L^{1,2}(\Omega \backslash K) \,:\, v = g \ \text{ on } \partial_D\Omega \setminus K\}. \tag{2.3}$$

In (2.3) the equality $v = g$ is to be interpreted in the sense of traces. Note that the trace of v on $\partial_D\Omega \setminus K$ is well defined, since $\partial\Omega$ is Lipschitz (see e.g. [DMT02b, Prop. 2.2]).

As discussed in [DMT02b, Sec. 4], the minimum problem

$$\min_{v \in \mathcal{V}(g,K)} \int_{\Omega\backslash K} \tfrac{1}{2}|\nabla v|^2 \,\mathrm{d}x \qquad \text{has a solution.} \tag{2.4}$$

We mention in advance that this minimization problem will be involved in the definition of the energy functional $\mathcal{E}$ driving our system. It may happen that the minimizer is not unique, but, by strict convexity, any two minimizers have the same gradient on $\Omega \backslash K$. The following result, proved in [DMT02b, Thm. 5.1], shows the continuous dependence of these gradients on the set K and on the boundary datum g, and will ensure the continuity properties of the energy functional $\mathcal{E}$

Proposition 2.3. *Let* $m \geq 1$ *and let* $(K_n)_n$, $K \in \mathcal{K}(\overline{\Omega})$ *fulfill* $\sup_{n\in\mathbb{N}} \mathcal{H}^1(K_n) < +\infty$ *and* $\mathsf{h}(K_n, K) \to 0$ *as* $n \to \infty$. *Let* $(g_n)_n$, $g \in H^1(\Omega)$ *with* $g_n \to g$ *strongly in* $H^1(\Omega)$. *Let* $(u_n)_n$, u *fulfill*

$$u_n \in \mathrm{Argmin}_{v \in \mathcal{V}(g_n, K_n)} \int_{\Omega\backslash K_n} \tfrac{1}{2}|\nabla v|^2 \,\mathrm{d}x \qquad \textit{for all } n \in \mathbb{N}, \qquad u \in \mathrm{Argmin}_{v \in \mathcal{V}(g,K)} \int_{\Omega\backslash K} \tfrac{1}{2}|\nabla v|^2 \,\mathrm{d}x.$$

Then,

$$\nabla u_n \to \nabla u \qquad \textit{as } n \to \infty \textit{ in } L^2(\Omega; \mathbb{R}^2), \tag{2.5}$$

where ∇u_n *and* ∇u *are regarded as functions defined a.e. in* Ω.

3. Setup for visco-energetic solutions for brittle fracture

In this section we precisely define the

(1) driving *energy functional* $\mathcal{E}$ (cf. (3.1)),
(2) *dissipation quasi-distance* d (cf. (3.7)),
(3) *viscous correction* δ (cf. (3.31)).

intervening in our notion of visco-energetic evolution of brittle fracture. Upon introducing $\mathcal{E}$, d, and δ, we will also settle some of their basic properties; in particular, those underlying the definition of VE solution. Further properties will be investigated in Section 5 ahead, when carrying out the proof of our existence result Theorem 4.5.

The energy functional. Throughout the paper $g \in \mathrm{C}^1([0,T]; H^1(\Omega))$ is a fixed function, whose trace on $\partial_D\Omega$ plays the role of a time-dependent Dirichlet loading acting on $\partial_D\Omega$. The energy functional $\mathcal{E}\colon [0,T]\times\mathcal{K}(\overline{\Omega}) \to [0,+\infty)$ is defined by

$$\mathcal{E}(t,K) := \min_{u\in\mathcal{V}(g(t),K)} \int_{\Omega\setminus K} \tfrac{1}{2}|\nabla u|^2 \,\mathrm{d}x, \tag{3.1}$$

where the space of admissible displacements is given by (2.3). As we will see in Proposition 5.1 ahead, $\mathcal{E}$ is lower semicontinuous on $[0,T]\times\mathcal{K}(\overline{\Omega})$, w.r.t. to the product topology $\mathsf{h}_{\mathbb{R}}$ on $[0,T]\times\mathcal{K}(\overline{\Omega})$, along sequences with bounded d-distance from some reference set $K_o \in \mathcal{K}_m(\overline{\Omega})$. A straightforward calculation shows that the power functional $\partial_t\mathcal{E}(t,K)$ exists for every $t\in(0,T)$ and all $K\in\mathcal{K}(\overline{\Omega})$ and that

$$\partial_t\mathcal{E}(t,K) := \int_{\Omega\setminus K} \nabla\dot{g}(t)\cdot\nabla u(t)\,\mathrm{d}x, \tag{3.2}$$

where $\dot{g}(t) \in H^1(\Omega)$ is the time derivative of the function g and $u(t) \in \mathcal{V}(g(t),K)$ is a solution of the minimum problem (3.1); the formula for $\partial_t\mathcal{E}(t,K)$ is well given since $\nabla u(t)$ does not depend on the choice of the minimizer, cf. Section 2. The upcoming Proposition 5.1 will collect all properties of $\mathcal{E}$ and $\partial_t\mathcal{E}$ that are relevant for our analysis.

The dissipation quasi-distance. Preliminarily, let us introduce a quasi-distance between two sets H and K that keeps track of the (number of) connected components of K disjoint from H. More precisely, $\alpha\colon \mathcal{K}(\overline{\Omega})\times\mathcal{K}(\overline{\Omega}) \to [0,+\infty]$ is defined in this way:

$$\alpha(H,K) \text{ is the number of the connected components of } K \text{ that do not intersect } H \tag{3.3}$$

if $H\subset K$, while $\alpha(H,K) := +\infty$ otherwise.

Our first result shows that α satisfies the triangle inequality.

Lemma 3.1. *The function* $\alpha\colon \mathcal{K}(\overline{\Omega})\times\mathcal{K}(\overline{\Omega}) \to [0,+\infty]$ *fulfills*

$$\alpha(H,L) \le \alpha(H,K)+\alpha(K,L) \qquad \textit{for all } H,\ K,\ L\in\mathcal{K}(\overline{\Omega}). \tag{3.4}$$

Proof. It is enough to show (3.4) in the case in which $\alpha(H,K)<+\infty$ and $\alpha(K,L)<+\infty$ so that, in particular, $H\subset K\subset L$. Hence, a subfamily of the connected components of L which are disjoint from H consists of connected components of L which are disjoint from K. Let $n=\alpha(K,L)$ and suppose that L has at least j connected components, $L_1,\dots,L_j$, disjoint from H and that the connected components of L disjoint from K coincide with the sets $L_{j-n+1},\dots,L_j$. We now have to prove that $\alpha(H,K)\ge j-n$. For this, it suffices to consider the connected components $\{L_1,\dots,L_{j-n}\}$ intersecting K. For every $\ell\in\{1,\dots,j-n\}$ we have that L_ℓ intersects at least a connected component K_ℓ of K; since $K\subset L$, we ultimately have that $K_\ell\subset L_\ell$. Since $L_\ell\cap H=\emptyset$, also $K_\ell\cap H=\emptyset$. Therefore, each connected component K_ℓ, $\ell\in\{1,\dots,j-n\}$, contributes to the number of connected components of K disjoint from H, which yields that $\alpha(H,K)\ge j-n$. Since this holds for every $j\le\alpha(H,L)$, we obtain (3.4). □

Secondly, we prove that α is lower semicontinuous w.r.t. Hausdorff convergence.

Lemma 3.2. *For all sequences* $(K_n)_n$, $(H_n)_n\subset\mathcal{K}(\overline{\Omega})$ *we have that*

$$\Big(K_n \xrightarrow{\mathsf{h}} K,\ H_n \xrightarrow{\mathsf{h}} H\Big) \;\Rightarrow\; \alpha(H,K) \le \liminf_{n\to\infty}\alpha(H_n,K_n)\,. \tag{3.5}$$

Proof. Preliminarily, we prove the following

Claim: *for every connected component K^ℓ of K and every $x \in K^\ell$ there exists a sequence $(K_n^\ell)_n$ such that K_n^ℓ is a connected component of K_n for every $n \in \mathbb{N}$ and $K_n^\ell \xrightarrow{\mathsf{h}} \widehat{K}^\ell$ as $n \to \infty$ for some connected set $\widehat{K}^\ell \in \mathcal{K}(\overline{\Omega})$ such that $x \in \widehat{K}^\ell \subset K^\ell$.*

Indeed, since $K_n \xrightarrow{\mathsf{h}} K$, there exists a sequence $(x_n)_n$ such that $x_n \to x$ and $x_n \in K_n$ for every $n \in \mathbb{N}$. Let $\widehat{K}_n^\ell$ be the connected component of K_n containing x_n. By the Blaschke Selection and the Gołąb Theorems, up to a (not relabeled) subsequence, the sets $(\widehat{K}_n^\ell)_n$ converge to a connected set $\widehat{K}^\ell$, which clearly contains x. Thus, $\widehat{K}^\ell \subset K^\ell$.

We are now in a position to prove (3.5). Indeed, suppose that there are h connected components $K^1, \ldots, K^h$ of K disjoint from H. For each $\ell \in \{1, \ldots, h\}$, select a point $x_\ell \in K^\ell$ and consider the connected sets $(K_n^\ell)_n$ and $\widehat{K}^\ell$ whose existence is ensured by the previously proved claim. Then,

$$\forall \ell \in \{1, \ldots, h\} \quad \exists \bar{n}_\ell \in \mathbb{N} \ \forall n \geq \bar{n}_\ell : \quad K_n^\ell \cap H_n = \emptyset$$

(otherwise, we would have $\widehat{K}^\ell \cap H \neq \emptyset$, hence $K^\ell \cap H \neq \emptyset$). Thus, setting $\bar{n} := \max_{\ell \in \{1,\ldots,h\}} \bar{n}_\ell$, we have

$$\alpha(H_n, K_n) \geq h \qquad \text{for all } n \geq \bar{n},$$

and (3.5) follows. □

Our next key result shows that, if H has a finite number of connected components and $K \in \mathcal{K}(\overline{\Omega})$ contains H and fulfills $\alpha(H, K) < +\infty$, then K also has a finite number of connected components.

Lemma 3.3. *Let $H \in \mathcal{K}_h(\overline{\Omega})$ for some $h \geq 1$, and let $K \in \mathcal{K}(\overline{\Omega})$ contain H and fulfill $\alpha(H, K) = i < +\infty$. Then,*

$$K \in \mathcal{K}_m(\overline{\Omega}) \qquad \textit{with } m = h + i\,. \tag{3.6}$$

Proof. It suffices to observe that there are i connected components of K disjoint from H; each of the remaining connected components of K intersects H and in fact contains at least one connected component of H, since $K \supset H$. Hence, K has at most h connected components intersecting H. □

For a given $\lambda > 0$, we are now in a position to define the (asymmetric) *dissipation quasi-distance* d by

$$\mathsf{d} : \mathcal{K}(\overline{\Omega}) \times \mathcal{K}(\overline{\Omega}) \to [0, +\infty], \qquad \mathsf{d}(H, K) := \mathcal{H}^1(K \backslash H) + \lambda\, \alpha(H, K)\,. \tag{3.7}$$

Remark 3.4. The contribution $\lambda\alpha$ to d will have the role of controlling the growth of the number of connected components of the visco-energetic fracture evolution $[0, T] \ni t \mapsto K(t)$. It is exploiting this term that we may prove the lower semicontinuity of d w.r.t. to Hausdorff convergence, in fact extending the Gołąb Theorem, cf. Proposition 3.6 ahead. The constant λ can be interpreted as the nucleation cost of each new connected component of the crack set.

Obviously,

$$\mathsf{d}(K, K) = 0 \qquad \text{for every } K \in \mathcal{K}(\overline{\Omega}). \tag{3.8a}$$

On the other hand, d separates the points of $\mathcal{K}(\overline{\Omega})$, namely for every $H,\ K \in \mathcal{K}(\overline{\Omega})$

$$\mathsf{d}(H, K) = 0 \text{ implies } H = K\,. \tag{3.8b}$$

Indeed, $\mathsf{d}(H, K) = 0$ implies that all the connected components of K have non-empty intersection with H. Then, from $H \subset K$ and $\mathcal{H}^1(K \backslash H) = 0$ we conclude that $H = K$. As an immediate consequence of Lemma 3.1 we have that d satisfies the triangle inequality.

Proposition 3.5. *The function $\mathsf{d} \colon \mathcal{K}(\overline{\Omega}) \times \mathcal{K}(\overline{\Omega}) \to [0, +\infty]$ fulfills*

$$\mathsf{d}(H, L) \leq \mathsf{d}(H, K) + \mathsf{d}(K, L) \qquad \textit{for all } H,\ K,\ L \in \mathcal{K}(\overline{\Omega}). \tag{3.9}$$

The lower semicontinuity of d w.r.t. Hausdorff convergence will be a consequence of the following result, which in fact generalizes the Gołąb Theorem.

Proposition 3.6. *Let* $(H_n, K_n)_n \subset \mathcal{K}(\overline{\Omega}) \times \mathcal{K}(\overline{\Omega})$ *be a sequence such that* $H_n \xrightarrow{\mathsf{h}} H$ *and* $K_n \xrightarrow{\mathsf{h}} K$. *Suppose that the number of connected components of* K_n *disjoint from* H_n *is uniformly bounded w.r.t.* $n \in \mathbb{N}$. *Then,*

$$\mathcal{H}^1((K\backslash H) \cap U) \leq \liminf_{n\to\infty} \mathcal{H}^1((K_n\backslash H_n) \cap U). \tag{3.10}$$

for every open set $U \subset \Omega$.

Proof. Clearly, passing to a subsequence it is not restrictive to assume that there exists $\tilde{k} \in \mathbb{N}$ such that

$$K_n \text{ has } \tilde{k} \text{ connected components disjoint from } H_n \text{ for all } n \in \mathbb{N}. \tag{3.11}$$

For clarity, first of all we will show that (3.10) holds with $U = \Omega$, and then we shall point out how the proof can be adapted to yield the localized inequality (3.10).

Claim 1: we have

$$\mathcal{H}^1(K\backslash H) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n\backslash H_n). \tag{3.12}$$

Clearly, we may suppose that the right-hand side is finite and, up to a further extraction, that

$$\lim_{n\to\infty} \mathcal{H}^1(K_n\backslash H_n) < +\infty\,. \tag{3.13}$$

For every $n \in \mathbb{N}$, let $\widetilde{\mathcal{C}}_n$ denote the collection of the connected components of K_n that do not intersect H_n. Due to (3.11), $\widetilde{\mathcal{C}}_n$ has $\tilde{k}$ elements, denoted as $\widetilde{C}_n^1, \ldots, \widetilde{C}_n^{\tilde{k}}$. Up to a subsequence we may suppose that

$$\widetilde{C}_n^i \to \widetilde{C}^i \qquad \text{for } i = 1, \ldots, \tilde{k}, \tag{3.14}$$

for some $\widetilde{C}^i \in \mathcal{K}(\overline{\Omega})$. Let us now fix two open sets V and V' such that $H \subset V' \Subset V$ and let

$$\eta := \inf_{x\in V'} \text{dist}(x, \Omega\backslash V) > 0.$$

Since $H_n \xrightarrow{\mathsf{h}} H$ and $H \subset V'$, for n sufficiently large we have $H_n \subset V'$; for simplicity and without loss of generality, hereafter we shall suppose that $H_n \subset V'$ for every $n \in \mathbb{N}$.

Let us now consider the family $\widehat{\mathcal{C}}_n$ of the connected components C of K_n such that

$$C \setminus V \neq \emptyset, \qquad C \cap H_n \neq \emptyset. \tag{3.15}$$

We will now show that

$$\mathcal{H}^1(C\backslash V') \geq \eta \qquad \text{for all } C \in \widehat{\mathcal{C}}_n\,. \tag{3.16}$$

Indeed, let us consider the 1-Lipschitz function $f\colon \Omega \to [0, +\infty)$ defined by $f(x) := \text{dist}(x, V')$. Since C is a connected set, $f(C)$ is an interval. It follows from the second of (3.15) and the fact that $H_n \subset V'$ that $0 \in f(C)$. Furthermore, by the first of (3.15) and since $\eta = \text{dist}(V', \Omega\backslash V)$, we also have that $\eta \in f(C)$, so that $[0, \eta] \subset f(C)$. In particular, for every $t \in (0, \eta]$ there exists $x \in C$ such that $f(x) = d(x, V') = t$, so that $x \in C\backslash V'$. Therefore, $(0, \eta] \subset f(C\backslash V')$. Since f is 1-Lipschitz, we then have $\eta \leq \mathcal{H}^1(f(C\backslash V')) \leq \mathcal{H}^1(C\backslash V')$, i.e., (3.16).

Since $H_n \subset V'$, for every $C \in \widehat{\mathcal{C}}_n$ there holds $C \setminus V' \subset K_n \setminus H_n$ and therefore

$$\sum_{C\in\widehat{\mathcal{C}}_n} \mathcal{H}^1(C\backslash V') \leq \mathcal{H}^1(K_n\backslash H_n) \leq M, \tag{3.17}$$

with $M = \sup_n \mathcal{H}^1(K_n\backslash H_n) < +\infty$ by (3.13). Combining (3.16) and (3.17) we then infer that $\widehat{\mathcal{C}}_n$ has at most $\frac{M}{\eta}$ elements. We may then suppose, up to a subsequence, that $\widehat{\mathcal{C}}_n$ consists of exactly $\hat{k} \in \mathbb{N}$ elements $\widehat{C}_n^1, \ldots, \widehat{C}_n^{\hat{k}}$ for every n. There exist compact and connected subsets $\widehat{C}^j \in \mathcal{K}(\overline{\Omega})$, $j = 1, \ldots, \hat{k}$, such that $\widehat{C}_n^j \xrightarrow{\mathsf{h}} \widehat{C}^j$ as $n \to \infty$; moreover, it follows from (3.15) that

$$\widehat{C}^j \setminus V \neq \emptyset, \qquad \widehat{C}^j \cap H \neq \emptyset. \tag{3.18}$$

We will now prove that

$$K \setminus \overline{V} \subset \bigcup_{i=1}^{\tilde{k}} (\widetilde{C}^i \setminus \overline{V}) \cup \bigcup_{j=1}^{\hat{k}} (\widehat{C}^j \setminus \overline{V}) \,. \tag{3.19}$$

Indeed, for every $x \in K \setminus \overline{V}$ there exists a sequence $(x_n)_n$ such that $x_n \to x$ as $n \to \infty$ and $x_n \in K_n \setminus \overline{V}$ for sufficiently big n. Let C_n be the connected component of K_n containing x_n. There exists $C^* \in \mathcal{K}_1(\overline{\Omega})$ such that, up to a subsequence, $C_n \xrightarrow{\mathsf{h}} C^*$, so that $x \in C^*$. Now, for every $n \in \mathbb{N}$ we either have $C_n \cap H_n = \emptyset$ or $C_n \cap H_n \neq \emptyset$. In the former case, $C_n \in \widetilde{\mathcal{C}}_n$. In the latter case, since $x_n \in C_n \setminus V \neq \emptyset$, we have $C_n \in \widehat{\mathcal{C}}_n$. If $C_n \in \widetilde{\mathcal{C}}_n = \{\widetilde{C}_n^1, \ldots, \widetilde{C}_n^{\tilde{k}}\}$ for infinitely many indexes n, there exists $i_0 \in \{1, \ldots, \tilde{k}\}$ such that $C_n = \widetilde{C}_n^{i_0}$ for infinitely many n so that $C^* = \widetilde{C}^{i_0}$ and, ultimately, $x \in \widetilde{C}^{i_0} \subset \cup_{i=1}^{\tilde{k}} \widetilde{C}^i$. If $C_n \in \widehat{\mathcal{C}}_n = \{\widehat{C}_n^1, \ldots, \widehat{C}_n^{\hat{k}}\}$ for infinitely many indexes, then there exists $j_0 \in \{1, \ldots, \hat{k}\}$ such that $C_n = \widehat{C}_n^{j_0}$ for infinitely many n, so that $C^* = \widehat{C}^{j_0}$ and thus $x \in \widehat{C}^{j_0} \subset \cup_{j=1}^{\hat{k}} \widehat{C}^j$. We have thus proved (3.19).

By the local version of the Gołąb Theorem (cf. Theorem 2.2i), we have

$$\begin{aligned} &\mathcal{H}^1(\widetilde{C}^i \setminus \overline{V}) \leq \liminf_{n\to\infty} \mathcal{H}^1(\widetilde{C}_n^i \setminus \overline{V}) \quad \text{for all } i = 1, \ldots, \tilde{k} \quad \text{and} \\ &\mathcal{H}^1(\widehat{C}^j \setminus \overline{V}) \leq \liminf_{n\to\infty} \mathcal{H}^1(\widehat{C}_n^j \setminus \overline{V}) \quad \text{for all } j = 1, \ldots, \hat{k}. \end{aligned} \tag{3.20}$$

Hence, from (3.19) and (3.20) we deduce that

$$\begin{aligned} \mathcal{H}^1(K \setminus \overline{V}) &\leq \sum_{i=1}^{\tilde{k}} \mathcal{H}^1(\widetilde{C}^i \setminus \overline{V}) + \sum_{j=1}^{\hat{k}} \mathcal{H}^1(\widehat{C}^j \setminus \overline{V}) \leq \sum_{i=1}^{\tilde{k}} \liminf_{n\to\infty} \mathcal{H}^1(\widetilde{C}_n^i \setminus \overline{V}) + \sum_{j=1}^{\hat{k}} \liminf_{n\to\infty} \mathcal{H}^1(\widehat{C}_n^j \setminus \overline{V}) \\ &\leq \liminf_{n\to\infty} \left(\sum_{i=1}^{\tilde{k}} \mathcal{H}^1(\widetilde{C}_n^i \setminus \overline{V}) + \sum_{j=1}^{\hat{k}} \mathcal{H}^1(\widehat{C}_n^j \setminus \overline{V}) \right) . \end{aligned} \tag{3.21}$$

Now, for every n the connected components $\widetilde{C}_n^1, \ldots, \widetilde{C}_n^{\tilde{k}}$ are pairwise disjoint, and so are the sets $\widehat{C}_n^1, \ldots, \widehat{C}_n^{\hat{k}}$. Furthermore, by the very definition of $\widetilde{\mathcal{C}}_n$ and $\widehat{\mathcal{C}}_n$ we also have that $\widetilde{C}_n^i \cap \widehat{C}_n^j = \emptyset$ for every $i = 1, \ldots, \tilde{k}$ and $j = 1, \ldots, \hat{k}$. Therefore, (3.21) leads to

$$\mathcal{H}^1(K \setminus \overline{V}) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \setminus \overline{V}) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \setminus H_n), \tag{3.22}$$

where the latter inequality holds due to the fact that $H_n \subset V$ (at least for sufficiently large n).

Finally, let $(V_m)_m$ be a sequence of open sets containing H such that $H = \cap_{m=1}^{\infty} \overline{V}_m$. It follows from (3.22) that $\mathcal{H}^1(K \setminus \overline{V}_m) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \setminus H_n)$ for every $m \in \mathbb{N}$ so that, taking the limit as $m \to \infty$ we ultimately have $\mathcal{H}^1(K \setminus H) \leq \liminf_{n\to\infty} \mathcal{H}^1(K_n \setminus H_n)$, i.e., (3.12).

Claim 2: the localized inequality (3.10) holds. It is sufficient to repeat the arguments up to (3.19), which can be localized, yielding for every open set $U \subset \Omega$

$$(K \setminus \overline{V}) \cap U \subset \bigcup_{i=1}^{\tilde{k}} ((\widetilde{C}^i \setminus \overline{V}) \cap U) \cup \bigcup_{j=1}^{\tilde{k}} ((\widehat{C}^j \setminus \overline{V}) \cap U) \,.$$

Then, by the Gołąb Theorem 2.2 the analogues of (3.20) hold for $\mathcal{H}^1((\widetilde{C}^i \setminus \overline{V}) \cap U)$ and $\mathcal{H}^1((\widehat{C}^j \setminus \overline{V}) \cap U)$, yielding the corresponding estimate for $\mathcal{H}^1((K \setminus \overline{V}) \cap U)$ (cf. (3.21)). Hence, the analogue of (3.22) holds, i.e.

$$\mathcal{H}^1((K \setminus \overline{V}) \cap U) \leq \liminf_{n\to\infty} \mathcal{H}^1((K_n \setminus H_n) \cap U).$$

From the above inequality it is then possible to infer (3.10) by the very same arguments as in Claim 1. □

Recalling that the quasi-distance α is lower semicontinuous w.r.t. Hausdorff convergence by Lemma 3.2, we immediately deduce the following result from Proposition 3.6.

Corollary 3.7. *The function* $\mathsf{d}\colon \mathcal{K}(\overline{\Omega}) \times \mathcal{K}(\overline{\Omega}) \to [0, +\infty]$ *is lower semicontinuous w.r.t. the Hausdorff distance.*

It follows from (3.8), Proposition 3.5, and Corollary 3.7, that the quasi-distance d on $\mathcal{K}(\overline{\Omega})\times\mathcal{K}(\overline{\Omega})$ satisfies the basic conditions required in [MS18, Sec. 2.1].

Curves with bounded d-variation. As we shall see in the next section, VE solutions to the viscously corrected system for brittle fracture are curves

$$K\colon [0,T]\to\mathcal{K}(\overline{\Omega}) \qquad \text{such that} \qquad \mathrm{Var}_{\mathsf{d}}(K,[0,T])<+\infty \tag{3.23}$$

where, for a subset $E\subset[0,T]$, we define

$$\mathrm{Var}_{\mathsf{d}}(K,E):=\sup\Big\{\sum_{j=1}^{N}\mathsf{d}(K(t_{j-1}),K(t_j))\ :\ t_0\le t_1\le\ldots\le t_N,\ t_j\in E \text{ for } j=0,\ldots,N\Big\}, \tag{3.24}$$

with the convention that $\mathrm{Var}_{\mathsf{d}}(K,\emptyset)=0$. Let us now gain further insight into the properties of curves satisfying (3.23).

First of all, from (3.23) it clearly follows that $\mathsf{d}(K(s),K(t))<+\infty$ for every $0\le s\le t\le T$, so that

$$K(s)\subset K(t) \qquad \text{for all } 0\le s\le t\le T, \tag{3.25}$$

i.e., the function K is *increasing* w.r.t. set inclusion.

The next result shows that crack evolutions with finite d-variation are (h,d)*-regulated* in the sense of [MS18, Definition 2.3], namely at every $t\in(0,T)$ the left and the right limits of K w.r.t. h exist and satisfy a 'compatibility' condition w.r.t. the dissipation distance d.

Lemma 3.8. *Let $K\colon[0,T]\to\mathcal{K}(\overline{\Omega})$ satisfy (3.23). For $t\in[0,T]$, set*

$$K(t-):=\mathrm{cl}\left(\cup_{s<t}K(s)\right), \qquad K(t+):=\cap_{s>t}K(s), \tag{3.26}$$

with the conventions $K(0-):=K(0)$ and $K(T+):=K(T)$. Then,

$$\begin{aligned} &K(s)\overset{\mathsf{h}}{\to}K(t-) \quad \text{as } s\to t_- \quad \text{for all } t\in(0,T],\\ &K(s)\overset{\mathsf{h}}{\to}K(t+) \quad \text{as } s\to t_+ \quad \text{for all } t\in[0,T) \end{aligned} \tag{3.27}$$

and, in addition,

$$\begin{aligned} &\lim_{s\to t_-}\mathsf{d}(K(s),K(t-))=\lim_{s\to t_-}\mathcal{H}^1(K(t-)\backslash K(s))=\lim_{s\to t_-}\alpha(K(s),K(t-))=0 \quad \text{for all } t\in(0,T],\\ &\lim_{s\to t_+}\mathsf{d}(K(s),K(t+))=\lim_{s\to t_+}\mathcal{H}^1(K(s)\backslash K(t+))=\lim_{s\to t_+}\alpha(K(t+)),K(s))=0 \quad \text{for all } t\in[0,T). \end{aligned} \tag{3.28}$$

Furthermore, there holds $K(t-)\subset K(t)\subset K(t+)$ for all $t\in[0,T]$. Let $\Theta:=\{t\in(0,T)\ :\ K(t-)=K(t)=K(t+)\}$. Then,

$$\text{the set } \mathrm{J}_K:=[0,T]\setminus\Theta \text{ is at most countable, and } K(t_n)\overset{\mathsf{h}}{\to}K(t) \text{ for every } t\in\Theta \text{ and every } t_n\to t. \tag{3.29}$$

Proof. Properties (3.27) are an immediate consequence of definitions (3.26) and of the definition of Hausdorff distance, while (3.29) has been proved in [DMT02b, Prop. 6.1], only relying on the monotonicity property (3.25).

Let us now exploit (3.23) in order to check (3.28) for $K(t-)$ (the proof of the assertion for $K(t+)$ follows the same lines). For every $s\in[0,T]$, let $V(s):=\mathrm{Var}_{\mathsf{d}}(K,[0,s])$. Since V is monotone increasing and (3.23) holds, we have that $V(t-):=\lim_{s\to t_-}V(s)<+\infty$. For every $0<s<s_1<t$ we have that $\mathsf{d}(K(s),K(s_1))\le V(s_1)-V(s)$. Passing to the limit as $s_1\to t_-$ and using the semicontinuity of d (cf. Corollary 3.7), we conclude that

$$\mathsf{d}(K(s),K(t-))\le V(t-)-V(s)\,.$$

Hence, taking the limit as $s\to t_-$ we conclude that $\lim_{s\to t-}\mathsf{d}(K(s),K(t-))=0$. This concludes the proof. □

Our final result shows that curves with bounded d-total variation starting from a crack with finitely many connected components evolve with a uniform-in-time bound on the number of their connected components.

Lemma 3.9. *Let* $K\colon[0,T]\to\mathcal{K}(\overline{\Omega})$ *fulfill* (3.23), *and suppose that* $K(0)\in\mathcal{K}_h(\overline{\Omega})$ *for some* $h\geq 1$. *Then,*

$$K(t)\in\mathcal{K}_m(\overline{\Omega})\qquad \text{for all } t\in[0,T],\ \text{with } m\leq h+\tfrac{1}{\lambda}\mathrm{Var}_{\mathsf{d}}(K,[0,T]). \tag{3.30}$$

Proof. Since $\lambda\,\alpha(K(0),K(t))\leq\mathrm{Var}_{\mathsf{d}}(K,[0,t])$, we can apply Lemma 3.3. □

The viscous corrrection. We consider the viscous correction $\delta\colon\mathcal{K}(\overline{\Omega})\times\mathcal{K}(\overline{\Omega})\to[0,+\infty]$ defined by

$$\delta(H,K):=\int_{K\setminus H}\mathrm{dist}(x,H)\,\mathrm{d}\mathcal{H}^1(x)+\mu\,\alpha(H,K)\,, \tag{3.31}$$

where $\mu>0$ is a prescribed constant, which plays the role of a nucleation cost for each new connected component of the crack set. As before, we adopt the convention that $\mathrm{dist}(x,\emptyset)=\mathrm{diam}(\Omega)$. The first property to be satisfied for δ to be an *admissible* viscous correction is lower semicontinuity w.r.t. Hausdorff convergence. As we will see in the proof of Proposition 3.10, the contribution of the quasi-distance α, modulated by whatever positive coefficient μ, has again a key role in ensuring lower semicontinuity, as it controls the growth of the number of connected components of K disjoint from H.

Proposition 3.10. *(1) Let* $H\in\mathcal{K}(\overline{\Omega})$ *be fixed. Then for all* $(K_n)_n\subset\mathcal{K}(\overline{\Omega})$ *such that the number of connected components of* K_n *disjoint from* H *is uniformly bounded w.r.t.* n, *we have that*

$$K_n\xrightarrow{\mathsf{h}}K\ \Rightarrow\ \int_{K\setminus H}\mathrm{dist}(x,H)\,\mathrm{d}\mathcal{H}^1(x)\leq\liminf_{n\to\infty}\int_{K_n\setminus H}\mathrm{dist}(x,H)\,\mathrm{d}\mathcal{H}^1(x)\,. \tag{3.32}$$

Moreover,

$$\delta(H,K)\leq\liminf_{n\to\infty}\delta(H,K_n). \tag{3.33}$$

(2) For all $(H_n)_n$, $H\in\mathcal{K}(\overline{\Omega})$, *and* $(K_n)_n$, $K\in\mathcal{K}(\overline{\Omega})$ *we have*

$$\Big(H_n\xrightarrow{\mathsf{h}}H,\ K_n\xrightarrow{\mathsf{h}}K\Big)\ \Rightarrow\ \delta(H,K)\leq\liminf_{n\to\infty}\delta(H_n,K_n). \tag{3.34}$$

Proof. ▷ (1): Let us observe that for every lower semicontinuous nonnegative function $f\colon\Omega\to[0,+\infty)$ we have

$$\int_{K\setminus H}f\,\mathrm{d}\mathcal{H}^1(x)\leq\liminf_{n\to\infty}\int_{K_n\setminus H}f\,\mathrm{d}\mathcal{H}^1(x).$$

Indeed, by the lower semicontinuity of f the set $U_t=\{x\in\Omega: f(x)>t\}$ is open and by Proposition 3.6, since the number of connected components of K_n disjoint from H is uniformly bounded, we have $\mathcal{H}^1((K\setminus H)\cap U_t)\leq\liminf_{n\to\infty}\mathcal{H}^1((K_n\setminus H)\cap U_t)$. Therefore, by the Fatou Lemma we have

$$\begin{aligned}\int_{K\setminus H}f(x)\,\mathrm{d}\mathcal{H}^1(x)&=\int_0^{+\infty}\mathcal{H}^1(\{x: f(x)>t\}\cap(K\setminus H))\,\mathrm{d}t\\&\leq\liminf_{n\to\infty}\int_0^{+\infty}\mathcal{H}^1(\{x: f(x)>t\}\cap(K_n\setminus H))\,\mathrm{d}t=\liminf_{n\to\infty}\int_{K_n\setminus H}f(x)\,\mathrm{d}\mathcal{H}^1(x)\,.\end{aligned}$$

Choosing $f(x)=\mathrm{dist}(x,H)$ we obtain (3.32).

Clearly, (3.33) immediately follows: as we may suppose that $\liminf_{n\to\infty}\delta(H,K_n)<\infty$, up to the extraction of a further subsequence, we have that $\sup_n\mu\,\alpha(H,K_n)\leq\sup_n\delta(H,K_n)<\infty$; then, it suffices to recall that, by Lemma 3.2, $\liminf_{n\to\infty}\alpha(H,K_n)\geq\alpha(H,K)$.

▷ (2): We may of course suppose $\sup_n\delta(H_n,K_n)<\infty$. By Lemma 3.2, $\liminf_{n\to\infty}\alpha(H_n,K_n)\geq\alpha(H,K)$. In order to show the lower semicontinuity of the first contribution to δ, let us introduce the set $H^\varepsilon=\{x\in\overline{\Omega}\,:\,\mathrm{dist}(x,H)\leq\varepsilon\}$ for every fixed $\varepsilon>0$. We have that $H_n\subset H^\varepsilon$ for n large enough; in what follows, for simplicity we will suppose that $H_n\subset H^\varepsilon$ for all n. Thus $\mathrm{dist}(x,H^\varepsilon)\leq\mathrm{dist}(x,H_n)$ for all $x\in\overline{\Omega}$. Then,

$$\int_{K_n\setminus H_n}\mathrm{dist}(x,H^\varepsilon)\,\mathrm{d}\mathcal{H}^1(x)\leq\int_{K_n\setminus H_n}\mathrm{dist}(x,H_n)\,\mathrm{d}\mathcal{H}^1(x)\,.$$

Therefore,

$$\begin{aligned}\liminf_{n\to\infty}\int_{K_n\setminus H_n}\mathrm{dist}(x,H_n)\,\mathrm{d}\mathcal{H}^1(x) &\geq \liminf_{n\to\infty}\int_{K_n\setminus H_n}\mathrm{dist}(x,H^\varepsilon)\,\mathrm{d}\mathcal{H}^1(x)\\ &\geq \liminf_{n\to\infty}\int_{K_n\setminus H^\varepsilon}\mathrm{dist}(x,H^\varepsilon)\,\mathrm{d}\mathcal{H}^1(x)\\ &\geq \int_{K\setminus H^\varepsilon}\mathrm{dist}(x,H^\varepsilon)\,\mathrm{d}\mathcal{H}^1(x)\geq\int_{K\setminus H}\mathrm{dist}(x,H^\varepsilon)\,\mathrm{d}\mathcal{H}^1(x)\,,\end{aligned}$$

where for the last-but-one inequality we have applied (3.32). This is possible since the boundedness of $\alpha(H_n,K_n)$ and the inclusions $H_n\subset H^\varepsilon$ for all $n\in\mathbb{N}$ imply that the number of connected components of K_n disjoint from H^ε is uniformly bounded w.r.t. $n\in\mathbb{N}$. Since $\varepsilon>0$ is arbitrary, we may pass to the limit as $\varepsilon\downarrow 0$ via the Fatou Lemma to obtain (3.34). □

In Section 5 we will gain further insight into the properties of δ, cf. Proposition 5.2 ahead.

4. VE solutions: definition and main results

In this Section we give the Definition of visco-energetic solution to the system $(\mathcal{K}(\overline{\Omega}),\mathcal{E},\mathsf{h},\mathsf{d},\delta)$ for brittle fracture and state our main existence result.

4.1. **Definition of VE solution.** The definition of the VE concept (cf. Def. 4.3 ahead) hinges on a notion of stability, introduced in Def. 4.1 below, that involves both the dissipation quasi-distance d and its viscous correction δ, and on an energy-dissipation distance featuring a cost that suitably measures the energy dissipated at jumps.

Stable sets in the visco-energetic sense. With the viscous correction δ defined in (3.31) at hand, we introduce the 'corrected' dissipation $\mathsf{D}\colon\mathcal{K}(\overline{\Omega})\times\mathcal{K}(\overline{\Omega})\to[0,+\infty]$

$$\mathsf{D}(H,K):=\mathsf{d}(H,K)+\delta(H,K)=\begin{cases}\mathcal{H}^1(K\setminus H)+\int_{K\setminus H}\mathrm{dist}(x,H)\,\mathrm{d}\mathcal{H}^1(x)+(\lambda+\mu)\alpha(H,K) & \text{if } H\subset K,\\ +\infty & \text{otherwise.}\end{cases}$$

We are now in a position to introduce the notion of stability in the visco-energetic sense.

Definition 4.1. *Let $Q\geq 0$. We say that $(t,K)\in[0,T]\times\mathcal{K}(\overline{\Omega})$ is* (D,Q)-stable *if it satisfies*

$$\mathcal{E}(t,K)\leq\mathcal{E}(t,K')+\mathsf{D}(K,K')+Q\qquad\text{for all } K'\in\mathcal{K}(\overline{\Omega}).\tag{4.1}$$

If $Q=0$, we will simply say that (t,K) is D-stable. We denote by $\mathscr{S}_{\mathsf{D}}$ the collection of all the D-stable points, and by $\mathscr{S}_{\mathsf{D}}(t):=\{K\in\mathcal{K}(\overline{\Omega})\,:\,(t,K)\in\mathscr{S}_{\mathsf{D}}\}$ its section at the process time $t\in[0,T]$. Analogously, with the symbols $\mathscr{S}_{\mathsf{D}}^Q$ and $\mathscr{S}_{\mathsf{D}}^Q(t)$ we will denote the (D,Q)-stable sets and their sections.

We introduce the *residual stability function* $\mathscr{R}\colon[0,T]\times\mathcal{K}(\overline{\Omega})\to[0,+\infty]$ via

$$\begin{aligned}\mathscr{R}(t,K):=\sup_{K'\in\mathcal{K}(\overline{\Omega})}\{\mathcal{E}(t,K)-\mathcal{E}(t,K')-\mathsf{D}(K,K')\}=\mathcal{E}(t,K)-\mathscr{Y}(t,K)\quad\text{with}\\ \mathscr{Y}(t,K)=\inf_{K'\in\mathcal{K}(\overline{\Omega})}\left(\mathcal{E}(t,K')+\mathsf{D}(K,K')\right)\,.\end{aligned}\tag{4.2}$$

By the properties of $\mathcal{E}$ (cf. Section 5.1) and the lower semicontinuity of d and δ,

$$M(t,K):=\mathrm{Argmin}_{K'\in\mathcal{K}(\overline{\Omega})}\left(\mathcal{E}(t,K')+\mathsf{D}(K,K')\right)\neq\emptyset.\tag{4.3}$$

Observe that $\mathscr{R}$ in fact records the failure of the stability condition at a given point $(t,K)\in[0,T]\times\mathcal{K}(\overline{\Omega})$, since

$$\mathscr{R}(t,K)\geq 0\text{ for all }(t,K)\in[0,T]\times\mathcal{K}(\overline{\Omega}),\quad\text{with }\mathscr{R}(t,K)=0\text{ if and only if }(t,K)\in\mathscr{S}_{\mathsf{D}}\,.\tag{4.4}$$

Furthermore, $\mathscr{R}$ is lower semicontinuous w.r.t. the product topology $\mathsf{h}_{\mathbb{R}}$ on $[0,T]\times\mathcal{K}(\overline{\Omega})$ if and only if for every $Q\geq 0$ the (D,Q)-stable sets are $\mathsf{h}_{\mathbb{R}}$-closed.

The visco-energetic cost c. It is defined by minimizing a suitable *transition cost* functional over a class of curves, connecting the left- and right-limits $K(t-)$ and $K(t+)$ at a jump point $t \in \mathrm{J}_K$. Such curves are in general defined on a compact subset $E \subset \mathbb{R}$ with a possibly more complicated structure than that of an interval. They are continuous w.r.t. the Hausdorff topology h, increasing in the sense of (3.25), and satisfying the following additional continuity condition w.r.t. the dissipation distance d

$$\forall\, \varepsilon > 0\ \exists\, \eta > 0\ :\ \mathsf{d}(\vartheta(s_0), \vartheta(s_1)) \leq \varepsilon \quad \text{for all } s_0, s_1 \in E \text{ with } s_0 \leq s_1 \leq s_0 + \eta\,. \tag{4.5}$$

Such conditions define the space

$$\mathrm{C}_{\mathsf{h},\mathsf{d}}(E; \mathcal{K}(\overline{\Omega})) := \{\vartheta \in \mathrm{C}(E; (\mathcal{K}(\overline{\Omega}), \mathsf{h}))\ :\ \vartheta \text{ fulfills (4.5) and } \vartheta(s) \subset \vartheta(t) \text{ for all } s, t \in E \text{ with } s \leq t\}\,. \tag{4.6}$$

Definition 4.2. *Let E be a compact subset of $\mathbb{R}$, let $E^- := \inf E$, $E^+ := \sup E$, and let $\vartheta \in \mathrm{C}_{\mathsf{h},\mathsf{d}}(E; \mathcal{K}(\overline{\Omega}))$. For every $t \in [0, T]$ we define the* transition cost function

$$\mathrm{Trc}_{\mathrm{VE}}(t, \vartheta, E) := \lambda\Big(\mathrm{Var}_\alpha(\vartheta, E) - \alpha(\vartheta(E^-), \vartheta(E^+))\Big) + \mathrm{GapVar}_\delta(\vartheta, E) + \sum_{s \in E \setminus \{E^+\}} \mathscr{R}(t, \vartheta(s)) \quad \textit{with} \tag{4.7}$$

(1) $\mathrm{Var}_\alpha(\vartheta, E)$ the α-total variation of the curve ϑ, which is defined, in analogy with (3.24)*, as*

$$\mathrm{Var}_\alpha(\vartheta, E) := \sup\left\{ \sum_{j=1}^N \alpha(\vartheta(t_{j-1}), \vartheta(t_j))\ :\ t_0 \leq t_1 \leq \ldots \leq t_N,\ t_j \in E \textit{ for } j = 0, \ldots, N \right\} \tag{4.8}$$

with the convention that $\mathrm{Var}_\alpha(K, \emptyset) = 0$;

(2) $\mathrm{GapVar}_\delta(\vartheta, E) := \sum_{I \in \mathscr{H}(E)} \delta(\vartheta(I^-), \vartheta(I^+))$, where $I^- := \inf I$, $I^+ := \sup I$, and $\mathscr{H}(E)$ is the collection of the connected components of $[E^-, E^+] \setminus E$;

(3) the (possibly infinite) sum

$$\sum_{s \in E \setminus \{E^+\}} \mathscr{R}(t, \vartheta(s)) := \begin{cases} \sup\{\sum_{s \in P} \mathscr{R}(t, \vartheta(s))\ :\ P \textit{ finite, } P \subset E \setminus \{E^+\}\} & \textit{if } E \neq \emptyset, \\ 0 & \textit{otherwise.} \end{cases}$$

We can now introduce the *visco-energetic jump dissipation cost* $\mathsf{c}\colon [0, T] \times \mathcal{K}(\overline{\Omega}) \times \mathcal{K}(\overline{\Omega}) \to [0, +\infty]$ between the two end-points of a jump of an *increasing* curve $K : [0, T] \to \mathcal{K}(\overline{\Omega})$. Namely, for all K_-, $K_+ \in \mathcal{K}(\overline{\Omega})$, we set

$$\mathsf{c}(t, K_-, K_+) := \inf\{\mathrm{Trc}_{\mathrm{VE}}(t, \vartheta, E)\ :\ E \Subset \mathbb{R},\ \vartheta \in \mathrm{C}_{\mathsf{h},\mathsf{d}}(E; \mathcal{K}(\overline{\Omega})),\ \ \vartheta(E^-) = K_-,\ \vartheta(E^+) = K_+\}, \tag{4.9}$$

with the convention $\inf \emptyset = +\infty$. Along the footsteps of [MS18], we define the *jump variation* functional, defined along a curve $K\colon [0, T] \to \mathcal{K}(\overline{\Omega})$ via

$$\begin{aligned} \mathrm{Jmp}_{\mathsf{c}}(K; [t_0, t_1]) := \mathsf{c}(t_0, K(t_0), K(t_0+)) + &\sum_{t \in \mathrm{J}_K \cap (t_0, t_1)} (\mathsf{c}(t, K(t-), K(t)) + \mathsf{c}(t, K(t), K(t+))) \\ &+ \mathsf{c}(t_1, K(t_1-), K(t_1)) \quad \text{for all } [t_0, t_1] \subset [0, T]. \end{aligned} \tag{4.10}$$

We are now in a position to define the concept of visco-energetic solution of the system $(\mathcal{K}(\overline{\Omega}), \mathcal{E}, \mathsf{h}, \mathsf{d}, \delta)$, featuring the D-stability condition (4.11) below, required outside the jump set J_K of the curve K, and the energy balance (4.12), where the energy dissipated at jumps is recorded by the jump dissipation cost introduced in (4.10).

Definition 4.3 (Visco-energetic solution)**.** *A curve $K\colon [0, T] \to \mathcal{K}(\overline{\Omega})$ is a visco-energetic (*VE*) solution of the system $(\mathcal{K}(\overline{\Omega}), \mathcal{E}, \mathsf{h}, \mathsf{d}, \delta)$ for brittle fracture if it satisfies*

- *the monotonicity condition* (3.25)*;*
- *the D-stability condition*

$$\mathcal{E}(t, K(t)) \leq \mathcal{E}(t, K') + \mathsf{D}(K(t), K') \quad \textit{for all } K' \in \mathcal{K}(\overline{\Omega}) \textit{ and all } t \in [0, T] \setminus \mathrm{J}_K, \tag{4.11}$$

- *the* (d,c)*-energy-dissipation balance*

$$\mathcal{E}(t,K(t))+\mathrm{Var}_{\mathsf{d}}(K,[0,t])+\mathrm{Jmp}_{\mathsf{c}}(K;[0,t])=\mathcal{E}(0,K(0))+\int_0^t\partial_t\mathcal{E}(s,K(s))\,\mathrm{d}s\quad \textit{for all } t\in[0,T]\,. \tag{4.12}$$

Remark 4.4. Taking into account the definitions of the corrected dissipation distance D and of the dissipation-quasidistance d in (3.7), the D-stability condition (4.11) rephrases as

$$\mathcal{E}(t,K(t))\le\mathcal{E}(t,K')+\mathsf{d}(K(t),K')+\delta(K(t),K')\quad\text{for all } K'\in\mathcal{K}(\overline{\Omega}) \text{ and all } t\in[0,T]\setminus \mathrm{J}_K,$$

while, taking also into account Definition 4.2 and the monotonicity of $t\mapsto K(t)$, see (3.25), the energy-dissipation balance (4.12) can be written as

$$\mathcal{E}(t,K(t))+\mathcal{H}^1(K(t)\backslash K(0))+\lambda\mathrm{Var}_{\alpha}(K,[0,t])+\mathrm{Jmp}_{\mathsf{c}}(K;[0,t])=\mathcal{E}(0,K(0))+\int_0^t\partial_t\mathcal{E}(s,K(s))\,\mathrm{d}s \tag{4.13}$$

for all $t\in[0,T]$.

4.2. **Existence and properties of** VE **solutions.** This section collects all of our results on VE solutions for the system $(\mathcal{K}(\overline{\Omega}),\mathcal{E},\mathsf{h},\mathsf{d},\delta)$, with $\mathcal{E}$, h, d and δ defined in (3.1), (2.1), (3.7), (3.31), respectively: first and foremost, the existence Theorem 4.5.

As mentioned in the Introduction, VE solutions are constructed as follows: for a given partition $\mathscr{P}_\tau=\{0=t_\tau^0<t_\tau^1<\ldots<t_\tau^{N_\tau}=T\}$ of the interval $[0,T]$ with time step $\tau:=\max_{i=1,\ldots N_\tau}(t_\tau^i-t_\tau^{i-1})$, and an assigned datum $K_0\in\mathcal{K}(\overline{\Omega})$, we consider the minimum problem

$$K_\tau^i\in\mathrm{Argmin}_{K\in\mathcal{K}(\overline{\Omega})}\big\{\mathcal{E}(t_\tau^i,K)+\mathsf{d}(K_\tau^{i-1},K)+\delta(K_\tau^{i-1},K)\big\}\qquad\text{for } i=1,\ldots,N_\tau, \tag{4.14}$$

which admits a solution thanks to the previously proved lower semicontinuity properties of d and δ, and the lower semicontinuity/coercivity properties of $\mathcal{E}$ that will be precisely stated in Section 5.1. We introduce the (left-continuous) piecewise constant interpolant of the elements $(K_\tau^i)_{i=1}^{N_\tau}$

$$K_\tau\colon[0,T]\to\mathcal{K}(\overline{\Omega})\qquad K_\tau(0):=K_0,\qquad K_\tau(t):=K_\tau^i\quad\text{if } t\in(t_\tau^{i-1},t_\tau^i]\,. \tag{4.15}$$

We are now in a position to give our existence result, stating the convergence of the above interpolants to a VE solution. Let us mention in advance that, starting from an initial datum $K_0\in\mathcal{K}_h(\overline{\Omega})$ for some $h\ge1$, we construct a fracture evolution with values in some $\mathcal{K}_m(\overline{\Omega})$, also providing an explicit bound on the index m, cf. (4.17) below.

Theorem 4.5 (Existence of VE solutions)**.** *Assume that the time-dependent Dirichlet loading fulfills*

$$g\in\mathrm{C}^1([0,T];H^1(\Omega)). \tag{4.16}$$

Let $K_0\in\mathcal{K}_h(\overline{\Omega})$ *for some* $h\ge1$*. Then, there exists a visco-energetic solution* K *of the system* $(\mathcal{K}(\overline{\Omega}),\mathcal{E},\mathsf{h},\mathsf{d},\delta)$ *for brittle fracture with such that* $K(0)=K_0$*. Moreover, every* VE *solution* K *with* $K(0)=K_0$ *satisfies for every* $t\in[0,T]$

$$K(t)\in\mathcal{K}_m(\overline{\Omega}),\ \textit{with } m\le h+\tfrac{1}{\lambda}\exp(C_PT)(\mathcal{E}(0,K_0)+1), \tag{4.17}$$

where C_P *is the constant defined in* (5.6) *ahead. Finally, for every sequence* $(\tau_k)_k$ *of time steps with* $\tau_k\downarrow0$ *as* $k\to\infty$ *there exist a (not relabeled) subsequence of* K_{τ_k} *and a* VE *solution* K *such that*

$$K_{\tau_k}(t)\xrightarrow{\mathsf{h}}K(t)\qquad\textit{for all } t\in[0,T]. \tag{4.18}$$

The *proof* of Theorem 4.5 will be carried out in Section 5 based on some preliminary results in which we are going to show that the dissipation distance d defined (3.7), the viscous correction δ in (3.31), and the driving energy functional $\mathcal{E}$ in (3.1) satisfy a series of conditions that are at the heart of the general existence result [MS18, Thm. 3.9], see conditions $<\mathbf{A}>$, $<\mathbf{B}>$, and $<\mathbf{C}>$ stated at the beginning of Section 5. Relying on these properties, we will deduce the proof of Theorem 4.5 from [MS18, Thm. 3.9].

In [MS18] several results on the characterization of the VE concept, and on *optimal jump transitions*, were proved. As we will see in Section 5.3, such results also hold for our specific rate-independent system for brittle fracture, cf. Propositions 4.6 and 4.7 below.

Proposition 4.6 provides a twofold characterization of visco-energetic solutions. First of all, in analogy to the properties of energetic and balanced viscosity solutions, for a curve $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ that is stable in the visco-energetic sense, the validity of the energy balance (4.12) is equivalent to the corresponding energy inequality $\leq$ (cf. (4.19)). It is also equivalent to the validity of an energy-dissipation inequality that solely involves the dissipation distance d, cf. (4.20) below, joint with jump conditions that also feature the VE cost c. As we have recalled in the Introduction, the notion of quasistatic evolution in brittle fracture features (4.20), joint with a d-stability condition. Therefore, the characterization provided by Proposition 4.6(2) highlights that VE solutions essentially differ from quasistatic evolutions in the description of the energetic behavior of the system at jumps.

Proposition 4.6. [MS18, Prop. 3.8] *Let the assumptions of Thm. 4.5 hold. A curve $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ satisfying the D-stability condition* (4.11) *is a* VE *solution of the system $(\mathcal{K}(\overline{\Omega}), \mathcal{E}, \mathsf{h}, \mathsf{d}, \delta)$ for brittle fracture if and only if K satisfies, in addition,*

(1) either the (d,c)-energy-dissipation upper estimate

$$\mathcal{E}(T,K(T)) + \mathcal{H}^1(K(T)\backslash K(0)) + \lambda \mathrm{Var}_\alpha(K,[0,T]) + \mathrm{Jmp}_{\mathsf{c}}(K;[0,T]) \leq \mathcal{E}(0,K(0)) + \int_0^T \partial_t \mathcal{E}(s,K(s))\,\mathrm{d}s; \quad (4.19)$$

(2) or the d-energy-dissipation upper estimate for every $[s,t] \subset [0,T]$

$$\mathcal{E}(t,K(t)) + \mathcal{H}^1(K(t)\backslash K(s)) + \lambda \mathrm{Var}_\alpha(K,[s,t]) \leq \mathcal{E}(s,K(s)) + \int_s^t \partial_t \mathcal{E}(r,K(r))\,\mathrm{d}r, \quad (4.20)$$

joint with the following jump conditions at every jump point $t \in \mathrm{J}_K$:

$$\begin{aligned}
\mathcal{E}(t,K(t-)) - \mathcal{E}(t,K(t)) &= \mathcal{H}^1(K(t)\backslash K(t-)) + \lambda\alpha(K(t-),K(t)) + \mathsf{c}(t,K(t-),K(t)),\\
\mathcal{E}(t,K(t)) - \mathcal{E}(t,K(t+)) &= \mathcal{H}^1(K(t+)\backslash K(t)) + \lambda\alpha(K(t),K(t+)) + \mathsf{c}(t,K(t),K(t+)),\\
\mathcal{E}(t,K(t-)) - \mathcal{E}(t,K(t+)) &= \mathcal{H}^1(K(t+)\backslash K(t-)) + \lambda\alpha(K(t-),K(t+)) + \mathsf{c}(t,K(t-),K(t+)).
\end{aligned} \quad (4.21)$$

Finally, let us gain further insight into the description of the system behavior at jumps provided by the VE concept, via the properties of *optimal jump transitions.* Given $t \in [0,T]$ and K_-, $K_+ \in \mathcal{K}(\overline{\Omega})$, an admissible transition curve $\vartheta \in \mathrm{C}_{\mathsf{h},\mathsf{d}}(E;\mathcal{K}(\overline{\Omega}))$, with $E \Subset \mathbb{R}$, is an optimal transition between K_- and K_+ at time $t \in [0,T]$ if it is a minimizer for $\mathsf{c}(t,K_-,K_+)$, namely

$$\vartheta(E^-) = K_-, \quad \vartheta(E^+) = K_+, \quad \mathrm{Trc}_{\mathrm{VE}}(t,\vartheta,E) = \mathsf{c}(t,K_-,K_+). \quad (4.22)$$

Furthermore, we say that ϑ is a

(1) *sliding transition*, if $\mathscr{R}(t,\vartheta(s)) = 0$ for all $s \in E$;

(2) *viscous transition*, if $\mathscr{R}(t,\vartheta(s)) > 0$ for all $s \in E \setminus \{E^-,E^+\}$.

We have the following result, cf. [MS18, Thm. 3.14, Rmk. 3.15, Cor. 3.17, Prop. 3.18].

Proposition 4.7. *Let $K : [0,T] \to \mathcal{K}(\overline{\Omega})$ be a* VE *solution of the system $(\mathcal{K}(\overline{\Omega}), \mathcal{E}, \mathsf{h}, \mathsf{d}, \delta)$ for brittle fracture. Then,*

(1) at every jump point $t \in \mathrm{J}_K$ there exists an optimal jump transition ϑ between $K(t-)$ and $K(t+)$ such that $\vartheta(s) = K(t)$ for some $s \in E$;

(2) for a viscous transition ϑ between $K(t-)$ and $K(t+)$ the set $E \setminus \{E^-,E^+\}$ is discrete, i.e., all of its points are isolated: namely, ϑ is a pure jump *transition. In fact, ϑ may be represented as a finite, or countable, sequence $(\vartheta_n)_{n\in O}$, with O a compact interval of $\mathbb{Z} \cup \{\pm\infty\}$, satisfying*

$$\vartheta_n \in M(t,\vartheta_{n-1}) = \mathrm{Argmin}_{K' \in \mathcal{K}(\overline{\Omega})} \left(\mathcal{E}(t,K') + \mathsf{D}(\vartheta_{n-1},K')\right) \quad \text{for all } n \in O \setminus \{O^-\}; \quad (4.23)$$

(3) any optimal jump transition can be canonically decomposed into an (at most) countable collection of sliding and viscous transitions.

5. Proofs of the main results

As previously mentioned, prior to carrying out the proof of Thm. 4.5, in Sections 5.1 and 5.2 ahead we shall check that the system $(\mathcal{K}(\overline{\Omega}), \mathcal{E}, \mathsf{h}, \mathsf{d}, \delta)$ given by (3.7), (3.31), and (3.1) complies with a series of conditions that were proposed in [MS18, Sec. 2.2, Sec. 3.1, Sec. 3.3] as a basis for the existence of VE solutions. Such conditions will also involve the perturbed functional $\mathcal{F}\colon [0,T]\times\mathcal{K}(\overline{\Omega}) \to [0,+\infty]$

$$\mathcal{F}(t,K) := \mathcal{E}(t,K) + \mathsf{d}(K_o,K) \tag{5.1}$$

with $K_o \in \mathcal{K}_h(\overline{\Omega})$ for some $h \geq 1$, an *arbitrary* reference point. By a sublevel of $\mathcal{F}$ we mean a set of the form

$$\{(t,K) \in [0,T]\times\mathcal{K}(\overline{\Omega}) \ : \ \mathcal{F}(t,K) \leq r\}$$

for some $r > 0$. The abstract conditions from [MS18] read as follows:

$<\mathbf{A}>$: the energy functional $\mathcal{E}\colon [0,T]\times\mathcal{K}(\overline{\Omega}) \to [0,+\infty)$ is lower semicontinuous w.r.t. the product topology $\mathsf{h}_{\mathbb{R}}$ on the sublevels of $\mathcal{F}$, which are $\mathsf{h}_{\mathbb{R}}$-compact; at *all* $(t,K)\in[0,T]\times\mathcal{K}(\overline{\Omega})$ there exists $\partial_t\mathcal{E}(t,K)$; $\partial_t\mathcal{E}\colon[0,T]\times\mathcal{K}(\overline{\Omega})\to\mathbb{R}$ is upper semicontinuous w.r.t. $\mathsf{h}_{\mathbb{R}}$ on the sublevels of $\mathcal{F}$, and

$$\exists\, C_P > 0 \ \ \forall\,(t,K)\in[0,T]\times\mathcal{K}(\overline{\Omega}) \ : \qquad |\partial_t\mathcal{E}(t,K)| \leq C_P(\mathcal{E}(t,K){+}1)\,. \tag{5.2}$$

$<\mathbf{B}>$: the viscous correction δ is left-d-continuous, namely for all sequences $(K_n)_n$, $K\in\mathcal{K}(\overline{\Omega})$

$$\Big(K_n \xrightarrow{\mathsf{h}} K \text{ and } \mathsf{d}(K_n,K)\to 0 \text{ as } n\to\infty\ \Big) \ \Rightarrow \ \lim_{n\to\infty}\delta(K_n,K) = 0 \tag{5.3}$$

and for every $(t,K)\in\mathscr{S}_{\mathsf{D}}$ there holds

$$\limsup_{(s,H)\overset{\simeq}{\to}(t,K)} \frac{\mathcal{E}(s,H)-\mathcal{E}(s,K)}{\mathsf{d}(H,K)} \leq 1\,, \tag{5.4}$$

where we have used the place-holder

$$(s,H)\overset{\simeq}{\to}(t,K) \text{ for } (s\to t,\ H\xrightarrow{\mathsf{h}}K,\ \mathsf{d}(H,K)\to 0,\ (s,H)\in\mathscr{S}_{\mathsf{D}},\ s\leq t).$$

$<\mathbf{C}>$ For every $Q\geq 0$ the (D,Q)-quasistable sets $\mathscr{S}_{\mathsf{D}}^{Q}$ have $\mathsf{h}_{\mathbb{R}}$-closed intersections with the sublevels of the functional $\mathcal{F}$.

As observed in [MS18], (5.4) in particular guarantees that D-stability yields local d-stability.

Relying on the validity of properties $<\mathbf{A}>$, $<\mathbf{B}>$, and $<\mathbf{C}>$, in Section 5.3 ahead we shall conclude the proofs of Theorem 4.5 and Propositions 4.6 and 4.7.

5.1. Verification of properties $<\mathbf{A}>$, $<\mathbf{B}>$, and $<\mathbf{C}>$.

Propositions 5.1, 5.2, and 5.3 ahead state the validity of properties $<\mathbf{A}>$, $<\mathbf{B}>$, and $<\mathbf{C}>$, respectively, for our system $(\mathcal{K}(\overline{\Omega}),\mathcal{E},\mathsf{h},\mathsf{d},\delta)$ for brittle fracture. Throughout the proof of Propositions 5.1 and 5.3, we will repeatedly use that, for sequences $(t_n,K_n)_n$ in the sublevels of the functional $\mathcal{F}$ defined in (5.1), there holds

$$\sup_n \mathcal{H}^1(K_n) < +\infty \qquad \text{and} \qquad \exists\, m\geq 1 \ : \ (K_n)_n\subset\mathcal{K}_m(\overline{\Omega}) \tag{5.5}$$

as a consequence of Lemma 3.3.

Proposition 5.1. *Under the assumptions of Thm. 4.5, the functional* $\mathcal{E}\colon[0,T]\times\mathcal{K}(\overline{\Omega})\to[0,+\infty)$ *defined in* (3.1) *and* $\partial_t\mathcal{E}$ *are continuous w.r.t. the* $\mathsf{h}_{\mathbb{R}}$*-topology on the sublevels of* $\mathcal{F}$ *and* $\partial_t\mathcal{E}$ *fulfills* (5.2).

Proof. Let $(t_n,K_n)_n\subset[0,T]\times\mathcal{K}(\overline{\Omega})$ with $\sup_n\mathcal{F}(t_n,K_n)<+\infty$ converge to some (t,K) w.r.t. the $\mathsf{h}_{\mathbb{R}}$-topology. It follows from (5.5) and Theorem 2.2 that $K\in\mathcal{K}_m(\overline{\Omega})$, too. Since $g\in\mathrm{C}^1([0,T];H^1(\Omega))$, we have that $g(t_n)\to g(t)$ in $H^1(\Omega)$. Therefore, thanks to (5.5) we may apply Proposition 2.3 and conclude that any sequence $(u_n)_n$ with $u_n\in\mathrm{Argmin}_{v\in\mathcal{V}(g(t_n),K_n)}\int_{\Omega\setminus K_n}\frac12|\nabla v|^2\,\mathrm{d}x$ fulfills $\nabla u_n\to\nabla u$ as $n\to\infty$ in $L^2(\Omega;\mathbb{R}^2)$, with $u\in\mathrm{Argmin}_{v\in\mathcal{V}(g(t),K)}\int_{\Omega\setminus K}\frac12|\nabla v|^2\,\mathrm{d}x$. Then,

$$\mathcal{E}(t_n,K_n) = \int_{\Omega\setminus K_n}\tfrac12|\nabla u_n|^2\,\mathrm{d}x \to \int_{\Omega\setminus K}\tfrac12|\nabla u|^2\,\mathrm{d}x = \mathcal{E}(t,K) \qquad \text{as } n\to\infty.$$

Since $g \in \mathrm{C}^1([0,T];H^1(\Omega))$, formula (3.2) gives $\partial_t \mathcal{E}(t,K)$ at all $(t,K) \in [0,T] \times \mathcal{K}(\overline{\Omega})$. We have

$$|\partial_t \mathcal{E}(t,K)| \leq \int_{\Omega\setminus K} |\nabla \dot{g}(t)||\nabla u| \,\mathrm{d}x \leq \|\nabla \dot{g}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^2))} \left(\int_{\Omega\setminus K} \tfrac{1}{2}|\nabla u|^2 \,\mathrm{d}x + \tfrac{1}{2}\mathcal{L}^2(\Omega\setminus K) \right).$$

Then, estimate (5.2) follows with

$$C_P = \|\nabla \dot{g}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^2))} \left(\tfrac{1}{2}\mathcal{L}^2(\Omega) \vee 1 \right) \tag{5.6}$$

The very same arguments used for the continuity of $\mathcal{E}$, combined with the fact that $\dot{g} \in \mathrm{C}^0([0,T];H^1(\Omega))$, in fact yield the $\mathsf{h}_\mathbb{R}$-continuity of $\partial_t\mathcal{E}$. This concludes the proof. □

With the following result we check the validity of condition $<\mathbf{B}>$; we shall in fact prove the stronger condition (5.7) below.

Proposition 5.2. *The dissipation distance* d *defined in* (3.7) *and the viscous correction* δ *in* (3.31) *fulfill*

$$\lim_{n\to\infty} \frac{\delta(K_n,K)}{\mathsf{d}(K_n,K)} = 0 \qquad \text{for all } (K_n)_n,\ K \in \mathcal{K}_m(\overline{\Omega}) \text{ such that } K_n \stackrel{\mathsf{h}}{\to} K \text{ and } \lim_{n\to\infty} \mathsf{d}(K_n,K) = 0. \tag{5.7}$$

In particular, conditions (5.3) *and* (5.4) *are satisfied.*

Proof. Since $\mathsf{d}(K_n,K) \to 0$ as $n \to \infty$, we have that, for n sufficiently large, $K_n \subset K$ and the integers $\alpha(K_n,K)$ are 0. Therefore, it is sufficient to observe that

$$\frac{\delta(K_n,K)}{\mathsf{d}(K_n,K)} \leq \frac{1}{\mathcal{H}^1(K\setminus K_n)} \int_{K\setminus K_n} \mathrm{dist}(x,K_n)\,\mathrm{d}\mathcal{H}^1(x) \leq \mathsf{h}(K_n,K) \longrightarrow 0 \quad \text{as } n \to \infty, \tag{5.8}$$

where the last inequality follows by the definition of Hausdorff distance. □

We conclude this section with a discussion on the closedness of the intersection of the Q-stable sets with the sublevels of the functional $\mathcal{F}$ introduced in (5.1). It is immediate to see that this property is guaranteed by the following condition: given a sequence $(t_n,K_n)_n \subset \mathscr{S}_{\mathsf{D}}^Q$, for some $Q \geq 0$, such that $(t_n,K_n) \stackrel{\mathsf{h}_\mathbb{R}}{\to} (t,K)$ as $n \to \infty$ and $\sup_n \mathcal{F}(t_n,K_n) < +\infty$, for every $K' \in \mathcal{K}(\overline{\Omega})$, with $K' \supset K$ and $\mathsf{d}(K,K') < +\infty$, we can exhibit a *recovery sequence* $(K'_n)_n$ such that $K'_n \supset K_n$ and

$$\begin{aligned} &\limsup_{n\to\infty} \left(\mathcal{E}(t_n,K'_n) - \mathcal{E}(t_n,K_n) + \mathsf{d}(K_n,K'_n) + \delta(K_n,K'_n) + Q\right) \\ &\leq \mathcal{E}(t,K') - \mathcal{E}(t,K) + \mathsf{d}(K,K') + \delta(K,K') + Q\,. \end{aligned} \tag{5.9}$$

In this way, we obtain $\mathcal{E}(t,K') - \mathcal{E}(t,K) + \mathsf{d}(K,K') + \delta(K,K') + Q \geq 0$ for all $K' \in \mathcal{K}(\overline{\Omega})$, whence $(t,K) \in \mathscr{S}_{\mathsf{D}}^Q$. Indeed, in Proposition 5.3 below we shall obtain (5.9) in a stronger form.

Proposition 5.3. *Let* $(t_n,K_n)_n \subset \mathscr{S}_{\mathsf{D}}^Q$ *be a sequence of* Q*-stable points fulfilling* $\sup_n \mathcal{F}(t_n,K_n) < +\infty$. *Suppose that* $(t_n,K_n) \stackrel{\mathsf{h}_\mathbb{R}}{\to} (t,K)$. *Then, for every* $K' \in \mathcal{K}(\overline{\Omega})$ *with* $K' \supset K$ *and* $\mathsf{d}(K,K') < +\infty$ *there exists a sequence* $(K'_n)_n$ *such that* $K'_n \supset K_n$ *and the following convergences hold as* $n \to \infty$*:*

$$K'_n \stackrel{\mathsf{h}}{\to} K', \tag{5.10a}$$

$$\mathcal{E}(t_n,K'_n) \to \mathcal{E}(t,K'), \tag{5.10b}$$

$$\mathsf{d}(K_n,K'_n) \to \mathsf{d}(K,K'), \tag{5.10c}$$

$$\delta(K_n,K'_n) \to \delta(K,K'). \tag{5.10d}$$

In particular, condition $<\mathbf{C}>$ *is valid.*

The *proof* shall be carried out in the upcoming Section 5.2.

5.2. **Proof of Proposition 5.3.** Since $\sup_n \mathcal{F}(t_n, K_n) < +\infty$, we have that $(K_n) \subset \mathcal{K}_m(\overline{\Omega})$ for some $m \geq 1$. Along the footsteps of [DMT02b], first of all we shall exhibit a recovery sequence for a fixed competitor set $K' = J$ that is, additionally, connected, i.e. $J \in \mathcal{K}_1(\overline{\Omega})$, cf. the upcoming Lemma 5.4. Then, in Lemma 5.5 we will address the general case in which the competitor set is in $\mathcal{K}_p(\overline{\Omega})$ for some $p \geq 1$. The proof of Proposition 5.3 will be then carried out at the end of this section. The proofs of Lemmas 5.4 and 5.5 strongly rely on the arguments for [DMT02b, Lemmas 3.8 & 3.5].

Lemma 5.4. *Let $m \in \mathbb{N}$, let $(K_n)_n$, $K \in \mathcal{K}_m(\overline{\Omega})$ fulfill $\mathsf{h}(K_n, K) \to 0$ as $n \to \infty$, and let $J \in \mathcal{K}_1(\overline{\Omega})$ with $J \supset K$. Then, there exists a sequence $(J_n)_n \subset \mathcal{K}_1(\overline{\Omega})$ such that $J_n \supset K_n$ and*

$$\mathsf{h}(J_n, J) \to 0 \qquad \textit{as } n \to \infty, \tag{5.11a}$$

$$\mathcal{H}^1(J_n \backslash K_n) \to \mathcal{H}^1(J \backslash K), \qquad \textit{as } n \to \infty, \tag{5.11b}$$

$$\int_{J_n \backslash K_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \to \int_{J \backslash K} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x) \qquad \textit{as } n \to \infty\,. \tag{5.11c}$$

Proof. If $K = \emptyset$, it is sufficient to define $J_n := J$. Indeed, from $\mathsf{h}(K_n, \emptyset) \to 0$ we deduce that $K_n = \emptyset$ for n sufficiently large, and then the convergences properties (5.11) are trivially satisfied.

Let us now assume $K \neq \emptyset$, and let $K^1, \ldots, K^i$, $1 \leq i \leq m$, be its connected components. First of all, in Step 1 we will provide the construction of a recovery sequence with the desired properties (5.11) for a carefully chosen set $\widehat{J}$, such that $\widehat{J}$ coincides with K, if K is connected, and $\widehat{J}$ is a suitable subset of J containing K (cf. (5.12b)) in the general case.
Step 1**.** If $i = 1$, let us set

$$\widehat{J} := K = K^1. \tag{5.12a}$$

If $i \geq 2$, we apply [DMT02b, Lemma 3.7] to conclude that there exists a finite family of indices $(\sigma_j)_{j=0}^{\ell}$, with $\{\sigma_0, \ldots, \sigma_\ell\} = \{1, \ldots, i\}$, and a family $(\Gamma_j)_{j=1}^{\ell}$ of connected components of $J \setminus K$, such that $K^{\sigma_{j-1}} \cap \overline{\Gamma}_j \neq \emptyset \neq K^{\sigma_j} \cap \overline{\Gamma}_j$ for $j = 1, \ldots, \ell$, namely $\overline{\Gamma}_j$ connects $K^{\sigma_{j-1}}$ to K^{σ_j}. In this case, we set

$$\widehat{J} := K \cup \bigcup_{j=1}^{\ell} \overline{\Gamma}_j \tag{5.12b}$$

and prove the following
Claim: *there exists a sequence $(\widehat{J}_n)_n \subset \mathcal{K}_1(\overline{\Omega})$ such that $\widehat{J}_n \supset K_n$ and*

$$\mathsf{h}(\widehat{J}_n, \widehat{J}) \to 0 \qquad \text{as } n \to \infty, \tag{5.13a}$$

$$\mathcal{H}^1(\widehat{J}_n \backslash K_n) \to \mathcal{H}^1(\widehat{J} \backslash K) \qquad \text{as } n \to \infty, \tag{5.13b}$$

$$\int_{\widehat{J}_n \backslash K_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \to \int_{\widehat{J} \backslash K} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x) \qquad \text{as } n \to \infty\,. \tag{5.13c}$$

To carry out the construction of the sets $\widehat{J}_n$, we proceed in the following way. Given the connected components $(K^l)_{l=1}^i$ of K, we choose $\varepsilon > 0$ such that the sets $\{x \in \overline{\Omega} \,:\, \mathrm{dist}(x, K^l) \leq \varepsilon\}$ are pairwise disjoint, and we set

$$\widetilde{K}_n^l := \{x \in K_n \,:\, \mathrm{dist}(x, K^l) \leq \varepsilon\}.$$

Following [DMT02b], we observe that, for sufficiently large n, we have that $K_n = \widetilde{K}_n^1 \cup \ldots \cup \widetilde{K}_n^i$, $\widetilde{K}_n^l \in \mathcal{K}_m(\overline{\Omega})$, and $\mathsf{h}(\widetilde{K}_n^l, K^l) \to 0$ as $n \to \infty$ for all $l \in \{1, \ldots, i\}$. We now apply [DMT02b, Lemma 3.6] and for all $l \in \{1, \ldots, i\}$ we find a sequence $(\widehat{K}_n^l)_n \subset \mathcal{K}_1(\overline{\Omega})$ such that $\widehat{K}_n^l \supset \widetilde{K}_n^l$,

$$\mathsf{h}(\widehat{K}_n^l, K^l) \to 0, \quad \text{and} \quad \mathcal{H}^1(\widehat{K}_n^l \backslash \widetilde{K}_n^l) \to 0 \qquad \text{as } n \to \infty. \tag{5.14a}$$

Therefore, $\mathsf{h}(\widehat{K}_n^l, \widetilde{K}_n^l) \leq \mathsf{h}(\widehat{K}_n^l, K^l) + \mathsf{h}(\widetilde{K}_n^l, K^l) \to 0$, as $n \to \infty$. This implies that

$$\int_{\widehat{K}_n^l \backslash \widetilde{K}_n^l} \mathrm{dist}(x, \widetilde{K}_n^l)\, \mathrm{d}\mathcal{H}^1(x) \to 0 \qquad \text{as } n \to \infty. \tag{5.14b}$$

In the case $i = 1$ (namely, $K = K^1 \in \mathcal{K}_1(\overline{\Omega})$), we define $\widehat{J}_n := \widehat{K}^1_n \in \mathcal{K}_1(\overline{\Omega})$. Then, properties (5.13) are satisfied: indeed, in this case $\widehat{J} = K$, so that the first of (5.14a) yields (5.13a). Furthermore, $\widetilde{K}^1_n := \{x \in K_n \,:\, \mathrm{dist}(x, K^1) = \mathrm{dist}(x, K) \le \varepsilon\}$ coincides with K_n for n large enough. Therefore, $\widehat{J}_n \supset K_n$ and $\mathcal{H}^1(\widehat{J}_n \backslash K_n) = \mathcal{H}^1(\widehat{K}^1_n \backslash \widetilde{K}^1_n) \to 0$ by the second of (5.14a). Property (5.13c) then follows from (5.14b), as $\widehat{J} \setminus K = \emptyset$.

Suppose now that K is not connected, namely $i \ge 2$. Then, the set $\widehat{J}$ is given by (5.12b). For every $j = 1, \ldots, \ell$, we fix $x^j \in K^{\sigma_{j-1}} \cap \overline{\Gamma}_j$ and $y^j \in K^{\sigma_j} \cap \overline{\Gamma}_j$. Since $\mathsf{h}(\widehat{K}^l_n, K^l) \to 0$ as $n \to \infty$ for all $l \in \{1, \ldots, i\}$, we have that there exist sequences $(x^j_n)_n$, $(y^j_n)_n$ with $x^j_n \in \widehat{K}^{\sigma_{j-1}}_n$ and $y^j_n \in \widehat{K}^{\sigma_j}_n$ for all $n \in \mathbb{N}$, such that $x^j_n \to x^j$ and $y^j_n \to y^j$ as $n \to \infty$. Since Ω has a Lipschitz boundary, there exist arcs X^j_n and Y^j_n in $\overline{\Omega}$, connecting x^j_n to x^j and y^j_n to y^j, respectively, such that $\mathcal{H}^1(X^j_n) \to 0$ and $\mathcal{H}^1(Y^j_n) \to 0$ as $n \to \infty$. We set for every $n \in \mathbb{N}$

$$\widehat{J}_n := \bigcup_{l=1}^{i} \widehat{K}^l_n \cup \bigcup_{j=1}^{\ell} X^j_n \cup \bigcup_{j=1}^{\ell} \overline{\Gamma}_j \cup \bigcup_{j=1}^{\ell} Y^j_n . \tag{5.15}$$

It has been shown in the proof of [DMT02b, Lemma 3.8] that $\widehat{J}_n \in \mathcal{K}_1(\overline{\Omega})$ for sufficiently large n, and that (5.13a) and (5.13b) hold. It remains to check (5.13c). With this aim, we observe that by (5.15) we have

$$\begin{aligned}
\int_{\widehat{J}_n \backslash K_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) &= \int_{\widehat{J}_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \\
&= \sum_{l=1}^{i} \int_{\widehat{K}^l_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) + \sum_{j=1}^{\ell} \int_{X^j_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \\
&\quad + \sum_{j=1}^{\ell} \int_{\Gamma_j} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) + \sum_{j=1}^{\ell} \int_{Y^j_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \\
&\doteq S^1_n + S^2_n + S^3_n + S^4_n
\end{aligned}$$

(where the integrals are taken over Γ_j since $\mathcal{H}^1(\overline{\Gamma}_j) = \mathcal{H}^1(\Gamma_j)$ by [DMT02b, Prop. 2.5]). As for the first summand, observe that

$$\int_{\widehat{K}^l_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \le \int_{\widehat{K}^l_n} \mathrm{dist}(x, \widetilde{K}^l_n)\, \mathrm{d}\mathcal{H}^1(x) \longrightarrow 0 \text{ as } n \to \infty \quad \text{for every } l = 1, \ldots, i,$$

where the inequality is due to the fact that $K_n = \widetilde{K}^1_n \cup \ldots \cup \widetilde{K}^i_n \supset \widetilde{K}^l_n$, while the convergence to 0 is proved in (5.14b). Therefore, $S^1_n \to 0$ as $n \to \infty$. We trivially estimate

$$\int_{X^j_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \le \mathrm{diam}(\Omega) \cdot \mathcal{H}^1(X^j_n) \to 0 \text{ as } n \to \infty \quad \text{for all } j = 1, \ldots, \ell,$$

and we handle the terms $\int_{Y^j_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x)$ in the same way. We thus conclude that $S^2_n \to 0$ and $S^4_n \to 0$ as $n \to \infty$. Finally, we observe that

$$\limsup_{n\to\infty} \int_{\Gamma_j} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \le \int_{\Gamma_j} \limsup_{n\to\infty} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \le \int_{\Gamma_j} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x) \quad \text{for all } j = 1, \ldots, \ell,$$

where the first inequality follows from the Fatou Lemma, and the second one is a straightforward consequence of the fact that $K_n \to K$ w.r.t. the Hausdorff distance. All in all, we conclude that

$$\begin{aligned}
\limsup_{n\to\infty} \int_{\widehat{J}_n \backslash K_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \le \limsup_{n\to\infty} \sum_{j=1}^{\ell} \int_{\Gamma_j} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) &= \sum_{j=1}^{\ell} \int_{\Gamma_j} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x) \\
&= \int_{\widehat{J} \backslash K} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x),
\end{aligned}$$

where the last equality is due to (5.12b). Then, (5.13c) ensues, since $\liminf_{n\to\infty} \int_{\widehat{J}_n \backslash K_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x)$ is estimated from below by $\int_{\widehat{J} \backslash K} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x)$ thanks to Proposition 3.10.

Step 2. Let us now carry out the construction of the recovery sequence $(J_n)_n$ for a given $J \in \mathcal{K}_1(\overline{\Omega})$ with $J \supset K$. Let $\widehat{J}$ be the set introduced in (5.12). Observe that J is locally connected (see [CD97, Lemma 1]), hence the connected components of $J \setminus \widehat{J}$ are open in the relative topology of J. Therefore, since J is separable, $J \setminus \widehat{J}$ has at most countably many connected components $(C_\ell)_{\ell \in L}$, with L a finite or an infinite subset of $\mathbb{N}$. It follows from the proof of [DMT02b, Lemma 3.7] that each component C_ℓ is open in J and satisfies $\overline{C}_\ell \cap K \neq \emptyset$. Let us fix a point $z^\ell \in \overline{C}_\ell \cap K$ for every $\ell \in L$. From $\mathsf{h}(K_n, K) \to 0$ as $n \to \infty$ we deduce that there exists a sequence $(z_n^\ell)_n$ such that $z_n^\ell \in K_n$ for all $n \in \mathbb{N}$ and $z_n^\ell \to z^\ell$ as $n \to \infty$. Since Ω is Lipschitz, for every $\ell \in L$ there exists an arc $Z_n^\ell \subset \overline{\Omega}$ connecting z_n^ℓ to z_ℓ, and such that $\mathcal{H}^1(Z_n^\ell) \to 0$ as $n \to \infty$. Finally, along the footsteps of [DMT02b] we observe that there exists a sequence $(\Lambda_n)_n \subset \mathbb{N}$ such that

$$\lim_{n\to\infty} \sum_{\ell=1}^{\Lambda_n} \mathcal{H}^1(Z_n^\ell) = 0$$

(in fact, if the set L consists of $1 \le \Lambda < +\infty$ elements, then we take $\Lambda_n = \Lambda$).

We claim that the recovery sequence

$$J_n := \widehat{J}_n \cup \bigcup_{\ell=1}^{\Lambda_n} Z_n^\ell \cup \bigcup_{\ell=1}^{\Lambda_n} \overline{C}_\ell \tag{5.16}$$

complies with (5.11). In fact, it is sufficient to check (5.11c), as (5.11a) and (5.11b) have been proved in [DMT02b, Lemma 3.8]. With this aim, we observe that

$$\begin{aligned}
&\int_{J_n \setminus K_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \\
&= \int_{\widehat{J}_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) + \sum_{\ell=1}^{\Lambda_n} \int_{Z_n^\ell} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) + \sum_{\ell=1}^{\Lambda_n} \int_{C_\ell} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) =: S_5^n + S_6^n + S_7^n\,.
\end{aligned}$$

It follows from (5.13c) that

$$\limsup_{n\to\infty} S_5^n \le \int_{\widehat{J} \setminus K} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x)\,.$$

We estimate

$$S_6^n \le \mathrm{diam}(\Omega) \sum_{\ell=1}^{\Lambda_n} \mathcal{H}^1(Z_n^\ell) \to 0 \qquad \text{as } n \to \infty.$$

Finally, we observe that

$$\begin{aligned}
\limsup_{n\to\infty} S_7^n = \limsup_{n\to\infty} \sum_{\ell=1}^{\Lambda_n} \int_{C_\ell} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) &\le \limsup_{n\to\infty} \int_{\bigcup_{\ell\in L} C_\ell} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \\
&\le \int_{\bigcup_{\ell\in L} C_\ell} \limsup_{n\to\infty} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) \\
&\le \int_{\bigcup_{\ell\in L} C_\ell} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x),
\end{aligned}$$

again by the Fatou Lemma and the fact that $\mathsf{h}(K_n, K) \to 0$ as $n \to \infty$. All in all, we conclude that

$$\begin{aligned}
\limsup_{n\to\infty} \int_{J_n \setminus K_n} \mathrm{dist}(x, K_n)\, \mathrm{d}\mathcal{H}^1(x) &\le \int_{\widehat{J} \setminus K} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x) + \int_{\bigcup_{\ell\in L} C_\ell} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x) \\
&= \int_{J \setminus K} \mathrm{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x)\,,
\end{aligned}$$

namely, an inequality in (5.11c). The converse inequality follows from Proposition 3.10. This concludes the proof. □

As in [DMT02b, Lemma 3.5], we now extend the construction of the recovery sequence to the case the 'competitor set' J has at most p connected components, with $p \geq 1$.

Lemma 5.5. *Let $m, p \in \mathbb{N}$, $m, p \geq 1$, let $(K_n)_n$, $K \in \mathcal{K}_m(\overline{\Omega})$ fulfill $\mathsf{h}(K_n, K) \to 0$ as $n \to \infty$, and let $K' \in \mathcal{K}_p(\overline{\Omega})$ with $K' \supset K$. Then, there exists a sequence $(K'_n)_n \subset \mathcal{K}_p(\overline{\Omega})$ such that $K'_n \supset K_n$ and properties* (5.10) *hold.*

Proof. As in the proof of [DMT02b, Lemma 3.5], we consider the connected components $J^1, \ldots, J^i$, $1 \leq i \leq p$, of the set K', we fix $\varepsilon > 0$ such that the sets $\{x \in \overline{\Omega} : \operatorname{dist}(x, J^l) \leq \varepsilon\}$, $l \in \{1, \ldots, i\}$, are pairwise disjoint, and we define

$$\widehat{K}^l_n := \{x \in K_n : \operatorname{dist}(x, J^l) \leq \varepsilon\}, \qquad l \in \{1, \ldots, i\}.$$

Following [DMT02b], we observe that, for n large enough, the sets $\widehat{K}^l_n$ are in $\mathcal{K}_m(\overline{\Omega})$, $K_n = \bigcup_{l=1}^i \widehat{K}^l_n$, and $\mathsf{h}(\widehat{K}^l_n, K^l) \to 0$ as $n \to \infty$, with $K^l := K \cap J^l$. If $K^l = \emptyset$, we set $J^l_n \equiv J^j$ for all $n \in \mathbb{N}$. If $K^l \neq \emptyset$, we apply Lemma 5.4 to the connected sets J^l and to the sequences $(\widehat{K}^l_n)_n$, $l \in \{1, \ldots, i\}$, and for each $l \in \{1, \ldots, i\}$ we find a sequence $(J^l_n)_n \subset \mathcal{K}_1(\overline{\Omega})$ such that

$$\begin{gathered} J^l_n \supset \widehat{K}^l_n, \quad \mathsf{h}(J^l_n, J^l) \to 0, \quad \mathcal{H}^1(J^l_n \backslash \widehat{K}^l_n) \to \mathcal{H}^1(J^l \backslash K^l), \\ \limsup_{n\to\infty} \int_{J^l_n \backslash \widehat{K}^l_n} \operatorname{dist}(x, \widehat{K}^l_n)\, \mathrm{d}\mathcal{H}^1(x) \leq \int_{J^l \backslash K^l} \operatorname{dist}(x, K^l)\, \mathrm{d}\mathcal{H}^1(x)\,. \end{gathered} \tag{5.17}$$

Note that, for n large enough, the sets $(J^l_n)_{l=1}^i$ are pairwise disjoint.

Then, we define the recovery sequence $(K'_n)_n$ for the set K' in this way:

$$K'_n := J^1_n \cup \ldots \cup J^i_n\,.$$

By construction $K'_n \supset \bigcup_{l=1}^i \widehat{K}^l_n = K_n$ and $\mathsf{h}(K'_n, K') \to 0$ as $n \to \infty$, namely (5.10a) holds. Then, (5.10b) follows from (5.10a) and Proposition 5.1. Furthermore,

$$\begin{aligned} \mathcal{H}^1(K'_n \backslash K_n) &= \mathcal{H}^1\Big(\bigcup_{l=1}^i J^l_n \backslash \bigcup_{j=1}^i \widehat{K}^j_n\Big) = \mathcal{H}^1\Big(\bigcup_{l=1}^i (J^l_n \backslash \widehat{K}^l_n)\Big) \\ &\leq \sum_{l=1}^i \mathcal{H}^1(J^l_n \backslash \widehat{K}^l_n) \longrightarrow \sum_{l=1}^i \mathcal{H}^1(J^l \backslash K^l) = \mathcal{H}^1(K' \setminus K)\,. \end{aligned} \tag{5.18}$$

where the second equality follows from the fact that $J^l_n \setminus \widehat{K}^j_n = J^l_n$ for $l \neq j$, analogously, we have $J^l \backslash K^j = J^l$ for $l \neq j$, which gives the very last equality.

Now, we calculate $\alpha(K_n, K'_n) = \alpha(\bigcup_{l=1}^i \widehat{K}^l_n, \bigcup_{l=1}^i J^l_n)$, namely the number of connected components Λ of $\bigcup_{l=1}^i J^l_n$ such that $\Lambda \cap \bigcup_{l=1}^i \widehat{K}^l_n = \emptyset$. Since for n large enough the sets J^l_n, $l = 1, \ldots, i$, are connected and pairwise disjoint, each Λ must coincide with a set $J^{\bar{l}}_n$, for some $\bar{l} \in \{1, \ldots, i\}$, that fulfills $J^{\bar{l}}_n \cap \widehat{K}^l_n = \emptyset$ for every $l \in \{1, \ldots, i\}$. Recall that, for n sufficiently large, $J^{\bar{l}}_n \cap \widehat{K}^l_n = J^{\bar{l}} \cap K^l = \emptyset$ for $l \neq \bar{l}$, and that, $J^{\bar{l}}_n \cap \widehat{K}^{\bar{l}}_n = \emptyset$ if and only if $J^{\bar{l}} \cap K^{\bar{l}} = \emptyset$. Then, we easily conclude that

$$\alpha(K_n, K'_n) = \alpha\Big(\bigcup_{l=1}^i K^l, \bigcup_{l=1}^i J^l\Big) = \alpha(K, K') \qquad \text{for } n \text{ large enough.}$$

Thus, we infer the validity of property (5.10c).

With the same arguments we find that

$$\begin{aligned} &\limsup_{n\to\infty} \int_{K'_n \backslash \widehat{K}_n} \operatorname{dist}(x, \widehat{K}^l_n)\, \mathrm{d}\mathcal{H}^1(x) = \limsup_{n\to\infty} \sum_{l=1}^i \int_{J^l_n \backslash \widehat{K}^l_n} \operatorname{dist}(x, \widehat{K}^l_n)\, \mathrm{d}\mathcal{H}^1(x) \\ &\leq \sum_{l=1}^i \int_{J^l \backslash K^l} \operatorname{dist}(x, K^l)\, \mathrm{d}\mathcal{H}^1(x) = \int_{K' \backslash K} \operatorname{dist}(x, K)\, \mathrm{d}\mathcal{H}^1(x), \end{aligned}$$

where the last inequality follows from (5.17). This gives an inequality in (5.10d); the converse one is again due to Proposition 3.10. This concludes the proof. □

Proof of Proposition 5.3. By (5.5) and by Theorem 2.2(i), the set K belongs to $\mathcal{K}_m(\overline{\Omega})$ for some m. Since $\lambda\alpha(K,K') \le \mathsf{d}(K,K') < +\infty$, we have that $K' \in \mathcal{K}_p(\overline{\Omega})$ with $p = m+\alpha(K,K')$. Then, the conclusion follows from Lemma 5.5. □

5.3. **Proofs.** In this section we prove Theorem 4.5.

Proof of Theorem 4.5. The first and the last statement follow from the general existence result [MS18, Thm. 3.9], which applies to the system $(\mathcal{K}(\overline{\Omega}), \mathcal{E}, \mathsf{h}, \mathsf{d}, \delta)$ for brittle fracture thanks to the validity of conditions $<\mathbf{A}>$, $<\mathbf{B}>$, and $<\mathbf{C}>$, proved in Propositions 5.1, 5.2, and 5.3.

We now prove the explicit bound (4.17) on the number m of connected components. Indeed, from the energy-dissipation balance (4.12) we infer that

$$\mathrm{Var}_{\mathsf{d}}(K,[0,t]) \le \mathcal{E}(t,K(t)) + \mathrm{Var}_{\mathsf{d}}(K,[0,t]) + \mathrm{Jmp}_{\mathsf{c}}(K;[0,t]) \le \mathcal{E}(0,K_0) + \int_0^t C_P(\mathcal{E}(s,K(s))+1)\,\mathrm{d}s \qquad (5.19)$$

where the last inequality follows from (3.2). Then, by the Gronwall Lemma we infer that

$$\mathcal{E}(t,K(t)) \le (\mathcal{E}(0,K_0)+1)\exp(C_P t) - 1 \qquad \text{for all } t \in [0,T],$$

which, inserted in (5.19), yields

$$\mathrm{Var}_{\mathsf{d}}(K,[0,T]) \le (\mathcal{E}(0,K_0)+1)\exp(C_P T) - 1.$$

Therefore, taking into account Lemma 3.9, we infer that

$$K(t) \in \mathcal{K}_m(\overline{\Omega}) \quad \text{for all } t \in [0,T] \quad \text{with } m \le h + \tfrac{1}{\lambda}\exp(C_P T)(\mathcal{E}(0,K_0)+1), \qquad (5.20)$$

(with C_P the positive constant in (5.6)), i.e., (3.6). This concludes the proof of Theorem 4.5. □

6. Behavior near the crack tips

In the same spirit of [DMT02b, Sec. 8], in this section we describe the *singularity* at the crack tips of the displacement $u(t)$ associated with a VE solution $K(t)$ to the system $(\mathcal{K}(\overline{\Omega}), \mathcal{E}, \mathsf{h}, \mathsf{d}, \delta)$. This will be examined in an interval (τ_0,τ_1) during which K evolves *continuously* as a function of time. Furthermore, along the footsteps of [DMT02b], we confine the discussion to the case in which the (moving part of the) crack set consists of a *finite* family of simple arcs, whose endpoints are the moving tips of the crack, as specified in Hypothesis 6.1 below. In Theorem 6.5 below we will show that the VE solution K complies with Griffith's criterion for crack growth.

Let us specify the structural condition on the crack $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$.

Hypothesis 6.1. *We suppose that $K\colon [0,T] \to \mathcal{K}(\overline{\Omega})$ fulfills the following condition on some $(\tau_0,\tau_1) \subset [0,T]$: there exists a finite family $(\Gamma_i)_{i=1}^p$ of arcs contained in Ω and parameterized by arc length by C^2 bijective functions $\gamma_i\colon [\sigma_i^0,\sigma_i^1] \to \Gamma_i$ such that*

$$K(t) = K(\tau_0) \cup \bigcup_{i=1}^p \Gamma_i(\sigma_i(t)) \qquad \text{for all } t \in (\tau_0,\tau_1), \qquad (6.1)$$

where, for $i = 1,\dots,p$, $\sigma_i\colon [\tau_0,\tau_1] \to [\sigma_i^0,\sigma_i^1]$ are non-decreasing continuous functions such that $\sigma_i(\tau_0) = \sigma_i^0$ and $\sigma_i^0 < \sigma_i(t) < \sigma_i^1$, while $\Gamma_i(\sigma) = \{\gamma_i(s) : \sigma_i^0 \le s \le \sigma\}$. We also assume that the arcs $(\Gamma_i)_{i=1}^p$ are pairwise disjoint, and that $\Gamma_i \cap K(t_0) = \{\gamma_i(\sigma_i^0)\}$ for every $i = 1,\dots,p$.

Hence, for $t \in (\tau_0,\tau_1)$ the fracture grows along the branches Γ_i, $i = 1,\dots,p$, and the points $\gamma_i(\sigma_i(t))$ are the moving crack tips. The compliance with Griffith's criterion stated in Theorem 6.5 ahead will be expressed in terms of conditions involving the *stress intensity factors* of the displacements $u(t)$ at the crack tips. We briefly recall some preliminary facts about this notion.

Basics on the stress intensity factor. The notion of stress intensity factor is based on the following result.

Proposition 6.2. *Let $B \subset \mathbb{R}^2$ be an open ball, and let $\gamma\colon [\sigma_0, \sigma_1] \to \mathbb{R}^2$ be a simple path of class C^2 parameterized by arc length, such that $\gamma(\sigma_0) \in \partial B$, $\gamma(\sigma_1) \in \partial B$, and $\gamma(\sigma) \in B$ for all $\sigma \in (\sigma_0, \sigma_1)$. In addition, assume that γ is not tangent to ∂B at σ_0 and σ_1.*

Given $\sigma \in (\sigma_0, \sigma_1)$, let $\Gamma(\sigma) := \{\gamma(\mathsf{s}) : \sigma_0 \le s \le \sigma\}$ and let $u \in L^{1,2}(B\backslash\Gamma(\sigma))$ satisfy

$$\int_{B\backslash\Gamma(\sigma)} \nabla u \cdot \nabla z \, \mathrm{d}x = 0 \qquad \text{for all } z \in L^{1,2}(B\backslash\Gamma(\sigma)) \text{ with } z = 0 \text{ on } \partial B\backslash\Gamma(\sigma).$$

Then, there exists a unique constant $\kappa = \kappa(u, \sigma) \in \mathbb{R}$ such that

$$u - 2\kappa\sqrt{\rho/\pi}\sin(\theta/2) \in H^2(B\backslash\Gamma(\sigma)) \cap H^{1,\infty}(B\backslash\Gamma(\sigma)), \tag{6.2}$$

where $\rho(x) = |x - \gamma(\sigma)|$ and $\theta(x)$ is the continuous function on $B\backslash\Gamma(\sigma)$ that coincides with the oriented angle between $\dot\gamma(\sigma)$ and $x - \gamma(\sigma)$, and vanishes on the points of the form $x = \gamma(\sigma) + \varepsilon\dot\gamma(\sigma)$ for $\varepsilon > 0$ small enough.

Proof. Since the connected components of $B\backslash\Gamma(\sigma)$ have Lipschitz boundary, the space $L^{1,2}(B\backslash\Gamma(\sigma))$ coincides with $H^1(B\backslash\Gamma(\sigma))$. Then the proof of (6.2) can be found in [Gri85, Theorem 4.4.3.7 and Section 5.2] and [MS89, Appendix 1]. □

The constant κ is proportional to the stress intensity factor considered in the engineering literature. It is related to the derivative of the energy with respect to the crack length, as we shall see in Proposition 6.3 below.

Given an open subset $A \subset \Omega$ with Lipschitz boundary, a compact set $K \subset \overline{\Omega}$, and a function $g\colon \partial A\backslash K \to \mathbb{R}$, we define

$$\widetilde{\mathcal{E}}(A; g, K) := \min_{v \in \widetilde{\mathcal{V}}(A;g,K)} \int_{A\backslash K} \tfrac{1}{2}|\nabla v|^2 \, \mathrm{d}x, \tag{6.3}$$

where

$$\widetilde{\mathcal{V}}(A; g, K) := \{v \in L^{1,2}(A\backslash K) : v = g \text{ on } \partial A \setminus K\}. \tag{6.4}$$

The following result can be obtained by adapting the proof of [Gri92, Thm. 6.4.1].

Proposition 6.3. *Let B and γ be as in Proposition 6.2 and let $g : \partial B\backslash\{\gamma(\sigma_0)\} \to \mathbb{R}$ be a function. For every $\sigma \in (\sigma_0, \sigma_1)$ suppose that $\widetilde{\mathcal{V}}(B; g, \Gamma(\sigma)) \neq \emptyset$ and let $u(\sigma) \in \mathrm{Argmin}_{v \in \widetilde{\mathcal{V}}(B;g,\Gamma(\sigma))} \int_{B\backslash\Gamma(\sigma)} \frac{1}{2}|\nabla v|^2 \, \mathrm{d}x$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\widetilde{\mathcal{E}}(B; g, \Gamma(\sigma)) = -\kappa(u(\sigma), \sigma)^2 \qquad \text{for every } \sigma \in (\sigma_0, \sigma_1), \tag{6.5}$$

with κ defined by (6.2).

Localization of the stability condition. We now prove that the VE-stability inequality can be localized, in the spirit of [DMT02b, Lemma 8.5].

Lemma 6.4. *Assume that $(t, K) \in [0, T] \times \mathcal{K}(\overline{\Omega})$ is D-stable and let $u \in \mathrm{Argmin}_{v \in \mathcal{V}(g(t),K)} \int_{\Omega\backslash K} \frac{1}{2}|\nabla v|^2 \, \mathrm{d}x$. Then, for every open subset $A \subset \Omega$ with Lipschitz boundary we have*

$$\widetilde{\mathcal{E}}(A; u, K) \le \widetilde{\mathcal{E}}(A; u, K') + \mathcal{H}^1(K'\backslash K) + \int_{K'\backslash K} \mathrm{dist}(x, K \cap A) \, \mathrm{d}x + (\lambda + \mu)\alpha(K \cap \overline{A}, (K' \cup K) \cap \overline{A}) \tag{6.6}$$

for all $K' \in \mathcal{K}(\overline{A})$ with $K' \supset K \cap \overline{A}$.

Proof. Let $K' \in \mathcal{K}(\overline{A})$. It follows from (4.1), with K' replaced by $K' \cup K$, that

$$\mathcal{E}(t, K) \le \mathcal{E}(t, K' \cup K) + \mathcal{H}^1(K'\backslash K) + \int_{K'\backslash K} \mathrm{dist}(x, K) \, \mathrm{d}x + (\lambda + \mu)\alpha(K, K' \cup K). \tag{6.7}$$

We repeat the very same calculations as in the proof of [DMT02b, Lemma 8.5], obtaining that

$$\mathcal{E}(t, K' \cup K) - \mathcal{E}(t, K) \le \widetilde{\mathcal{E}}(A; u, K') - \widetilde{\mathcal{E}}(A; u, K). \tag{6.8}$$

As for the third term on the right-hand side of (6.7), we observe that

$$\int_{K'\setminus K} \operatorname{dist}(x,K)\,\mathrm{d}x \le \int_{K'\setminus K} \operatorname{dist}(x,K\cap A)\,\mathrm{d}x. \tag{6.9}$$

Finally, let us prove that

$$\alpha(K,K'\cup K) \le \alpha(K\cap\overline{A},(K'\cup K)\cap\overline{A})\,. \tag{6.10}$$

It is enough to show that every connected component of $K'\cup K$ disjoint from K is a connected component of $(K'\cup K)\cap\overline{A}$. If C is a connected component of $K'\cup K$ and does not intersect K, then $C\subset K'\subset\overline{A}$, hence $C\subset(K'\cup K)\cap\overline{A}$. If C' is a connected set such that $C\subset C'\subset(K'\cup K)\cap\overline{A}$, then $C'\subset K'\cup K$ and hence $C'=C$. This shows that C is a connected component of $(K'\cup K)\cap\overline{A}$ and concludes the proof of (6.10), which, together with (6.7)–(6.8), yields (6.6). □

Griffith's condition at the crack tips. Our result for a VE solution $K\colon[0,T]\to\mathcal{K}(\overline{\Omega})$ satisfying, in addition, the structural condition stated in Hypothesis 6.1, involves the constants the $\kappa_i=\kappa_i(u(t),\sigma_i(t))$ satisfying (6.2) at the tips $\gamma_i(\sigma_i(t))$ of the branches of the crack, where $u(t)$ is the corresponding minimal displacement (cf. (6.11) below).

Theorem 6.5. *Let $K\colon[0,T]\to\mathcal{K}(\overline{\Omega})$ be a* VE *solution of the system for brittle fracture $(\mathcal{K}(\overline{\Omega}),\mathcal{E},\mathsf{h},\mathsf{d},\delta)$, with time-dependent boundary datum $g\in\mathrm{C}^1([0,T];H^1(\Omega))$, and for every $t\in[0,T]$ let*

$$u(t)\in\operatorname{Argmin}_{v\in\mathcal{V}(g(t),K(t))}\int_{\Omega\setminus K(t)}\tfrac{1}{2}|\nabla v|^2\,\mathrm{d}x\,. \tag{6.11}$$

Assume that K satisfies Hypothesis 6.1 on some $(\tau_0,\tau_1)\subset[0,T]$, with arcs Γ_i and functions σ_i, $i=1,\dots,p$. Then,

$$\dot\sigma_i(t)\ge 0 \qquad \textit{for a.a. } t\in(\tau_0,\tau_1), \tag{6.12a}$$

$$1-\kappa_i(u(t),\sigma_i(t))^2\ge 0 \qquad \textit{for all } t\in(\tau_0,\tau_1), \tag{6.12b}$$

$$\left(1-\kappa_i(u(t),\sigma_i(t))^2\right)\dot\sigma_i(t)=0 \qquad \textit{for a.a. } t\in(\tau_0,\tau_1) \tag{6.12c}$$

for every $i=1,\dots,p$.

Remark 6.6. Following [DMT02b], we observe that (6.12a) states that the length of every branch of the crack is non-decreasing, in accordance with the irreversibility of the crack growth process; (6.12b) imposes that the absolute value of the stress intensity factor, at each tip, be less or equal than 1; by (6.12c), the stress intensity factor reaches the threshold values ±1 as soon as the tip moves with positive velocity. In fact, conditions (6.12) rephrase Griffith's criterion in our context.

Therefore, Theorem 6.5 ensures that a VE solution complying with Hyp. 6.1 satisfies Griffith's criterion in the interval (τ_0,τ_1) during which it evolves *continuously* as a function of time, like it happens for the quasistatic evolutions considered in [DMT02b, Thm. 8.4]. This is consistent with the fact that the most relevant difference between VE solutions and quasistatic evolutions resides in the jump behavior, as highlighted by Proposition 4.6.

Proof of Theorem 6.5. As in the proof of [DMT02b, Thm. 8.4], we fix an arbitrary $t\in(\tau_0,\tau_1)$ and consider a family of open balls $B_1,\dots,B_p$ centered at the points $\gamma_i(\sigma_i(t))$. Up to choosing their radii sufficiently small, we have that $\overline{B}_i\subset\Omega$ and $\overline{B}_i\cap K(\tau_0)=\overline{B}_i\cap\overline{B}_j=\overline{B}_i\cap\Gamma_j=\emptyset$ for $j\ne i$. Furthermore, we may assume that, for every $i=1,\dots,p$,

$$B_i\cap\Gamma_i=\{\gamma_i(\sigma)\,:\,\rho_i^0<\sigma<\rho_i^1\}$$

for suitable ρ_i^0 and ρ_i^1 such that $\sigma_i^0<\rho_i^0<\sigma_i(t)<\rho_i^1<\sigma_i^1$, and that the arcs Γ_i intersect ∂B_i only at the points $\gamma_i(\rho_i^0)$ and $\gamma_i(\rho_i^1)$ with a transversal intersection. Then, taking into account Hypothesis 6.1, we conclude that

$$\overline{B}_i\cap K(s)=\overline{B}_i\cap\Gamma_i(\sigma_i(s))=\{\gamma_i(\sigma)\,:\,\rho_i^0\le\sigma\le\sigma_i(s)\} \tag{6.13}$$

whenever $\sigma_i(s) \in (\rho_i^0, \rho_i^1)$. In particular, (6.13) holds at $s = t$ and for s sufficiently close to t, since σ_i is continuous at t.

It follows from Lemma 6.4 that, for every $i = 1, \dots, p$,

$$\widetilde{\mathcal{E}}(B_i; u(t), K(t)) \le \widetilde{\mathcal{E}}(B_i; u(t), K') + \mathcal{H}^1(K' \setminus K(t)) + \int_{K' \setminus K(t)} \mathrm{dist}(x, K(t) \cap B_i)\,\mathrm{d}x + (\lambda+\mu)\alpha(K(t) \cap \overline{B}_i, (K' \cup K(t)) \cap \overline{B}_i)$$

for all $K' \in \mathcal{K}(\overline{B}_i)$ with $K' \supset K(t) \cap \overline{B}_i$, where $\widetilde{\mathcal{E}}$ is the localized energy functional defined in (6.3). Choosing $K' = \Gamma_i(\sigma) \cap \overline{B}_i = \{\gamma_i(\rho) : \rho_i^0 \le \rho \le \sigma\}$ with $\sigma \in [\sigma_i(t), \rho_i^1]$, and recalling that $\mathcal{H}^1(\Gamma_i(\sigma) \setminus \Gamma_i(\sigma_i(t))) = \sigma - \sigma_i(t)$ and $\alpha(\Gamma_i(\sigma_i(t)) \cap \overline{B}_i, \Gamma_i(\sigma) \cap \overline{B}_i) = 0$, we deduce that

$$\widetilde{\mathcal{E}}(B_i; u(t), \Gamma_i(\sigma_i(t)) \le \widetilde{\mathcal{E}}(B_i; u(t), \Gamma_i(\sigma)) + \sigma - \sigma_i(t) + \int_{\Gamma_i(\sigma) \setminus \Gamma_i(\sigma_i(t))} \mathrm{dist}(x, \Gamma_i(\sigma_i(t)) \cap B_i)\,\mathrm{d}x. \tag{6.14}$$

for all $\sigma \in [\sigma_i(t), \rho_i^1]$. Taking into account that

$$\lim_{\sigma \to \sigma_i(t)} \frac{1}{\sigma - \sigma_i(t)} \int_{\Gamma_i(\sigma) \setminus \Gamma_i(\sigma_i(t))} \mathrm{dist}(x, \Gamma_i(\sigma_i(t)) \cap B_i)\,\mathrm{d}x = 0,$$

from (6.14) we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} \widetilde{\mathcal{E}}(B_i; u(t), \Gamma_i(\sigma))\big|_{\sigma = \sigma_i(t)} + 1 \ge 0 \qquad \text{for all } i = 1, \dots, p.$$

Then, (6.12b) follows from Proposition 6.3 applied with $g = u(t)$.

For every $[s,t] \subset (\tau_0, \tau_1)$ we have that $\alpha(K(s), K(t)) = 0$ by Hypothesis 6.1. Hence $\mathrm{Var}_\alpha(K, [s,t]) = 0$, so that $\mathrm{Var}_{\mathsf{d}}(K, [s,t]) = \mathcal{H}^1(K(t) \setminus K(s))$. Since K evolves continuously in time on the interval (τ_0, τ_1), the energy-dissipation balance (4.12) reduces to

$$\mathcal{E}(t, K(t)) + \mathcal{H}^1(K(t) \setminus K(s)) = \mathcal{E}(s, K(s)) + \int_s^t \partial_t \mathcal{E}(r, K(r))\,\mathrm{d}r \quad \text{for all } [s,t] \subset (\tau_0, \tau_1)\,. \tag{6.15}$$

From (6.15), with the very same arguments as in the proof of [DMT02b, Thm. 8.4] we deduce (6.12c). □

7. Extension to 2D linearized elasticity

In [Cha03] the existence of *quasistatic evolutions* for fracture, proved in the scalar setting in [DMT02b], was extended to the vectorial, still two-dimensional, setting of linearized elasticity. The argument relied on a density result of $H^1(A; \mathbb{R}^2)$-fields in the space of fields whose symmetrized gradient is in $L^2(A; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, proved by the author in the case $A \subset \mathbb{R}^2$ is a bounded open set whose complement has a finite number of connected components.

We will now briefly explain how the arguments in [Cha03] also allow us to prove the existence of visco-energetic solutions for the *vectorial* (2D, linearized elasticity) version of the system for brittle fracture, that we address in a domain $\Omega \subset \mathbb{R}^2$ still complying with the conditions expounded at the beginning of Section 2. The viscously corrected system for brittle fracture is now given by the quadruple $(\mathcal{K}(\overline{\Omega}), \mathcal{E}_{\mathrm{LE}}, \mathsf{h}, \mathsf{d}, \delta)$ in which

- the dissipation quasi-distance d, and the viscous correction δ are still given by (3.7), and (3.31), respectively;
- the driving energy functional $\mathcal{E}_{\mathrm{LE}}[0,T] \times \mathcal{K}(\overline{\Omega}) \to [0, +\infty)$ is defined by

$$\mathcal{E}_{\mathrm{LE}}(t, K) := \min_{u \in \mathcal{V}_{\mathrm{LE}}(g(t), K)} \int_{\Omega \setminus K} \tfrac{1}{2} \mathbb{C} e(u) : e(u)\,\mathrm{d}x, \tag{7.1}$$

  where $\mathbb{C}$ is the elasticity tensor and $e(u)$ denotes the symmetric part of ∇u,

$$g \in \mathrm{C}^1([0,T]; H^1(\Omega; \mathbb{R}^2)),$$

  and the space for admissible displacements is now given by

$$\mathcal{V}_{\mathrm{LE}}(g(t), K) := \{v \in \mathrm{LD}(\Omega \setminus K) \; : \; v = g(t) \; \text{ on } \partial_D \Omega \setminus K\}\,.$$

Here, following [Cha03], for a given $A \subset \mathbb{R}^2$ we denote by $\mathrm{LD}(A)$ the space

$$\mathrm{LD}(A) := \{v \in L^2_{\mathrm{loc}}(A;\mathbb{R}^2) \,:\, e(v) \in L^2(A;\mathbb{R}^{2\times 2}_{\mathrm{sym}})\}\,.$$

We will denote by $\mathcal{F}_{\mathrm{LE}}$ the functional associated with $\mathcal{E}_{\mathrm{LE}}$ and a reference point K_o as in (5.1).

As we have seen in Section 5.3, in order to prove the existence of VE solutions it is sufficient to show that the system for brittle fracture $(\mathcal{K}(\overline{\Omega}), \mathcal{E}_{\mathrm{LE}}, \mathsf{h}, \mathsf{d}, \delta)$ complies with conditions $< \mathbf{A} >$, $< \mathbf{B} >$, and $< \mathbf{C} >$ listed at the beginning of Section 5. Now, the viscous correction δ obviously still enjoys property $< \mathbf{B} >$. As for $< \mathbf{A} >$, it follows from the following analogue of Proposition 5.1.

Proposition 7.1. *The functional* $\mathcal{E}_{\mathrm{LE}} \colon [0,T] \times \mathcal{K}(\overline{\Omega}) \to [0,+\infty)$ *defined in* (7.1) *is continuous w.r.t. the* $\mathsf{h}_{\mathbb{R}}$*-topology on sublevels of the functional* $\mathcal{F}_{\mathrm{LE}}$ *Moreover,* $\partial_t \mathcal{E}_{\mathrm{LE}} \colon [0,T] \times \mathcal{K}(\overline{\Omega}) \to \mathbb{R}$ *is given by*

$$\partial_t \mathcal{E}_{\mathrm{LE}}(t,K) = \int_{\Omega \setminus K} \mathbb{C}e(u) : e(\dot{g}(t))\,\mathrm{d}x \qquad \textit{for all } u \in \mathcal{V}_{\mathrm{LE}}(g(t),K),$$

is also continuous w.r.t. the $\mathsf{h}_{\mathbb{R}}$*-topology on sublevels of* $\mathcal{F}_{\mathrm{LE}}$*, and fulfills estimate* (5.2).

The *proof* of Proposition 7.1 follows from the arguments in [Cha03, Thm. 3]. Finally, Proposition 5.3, guaranteeing the validity of property $< \mathbf{C} >$, carries over to the present setting: in particular, the construction of the recovery sequence $(K_n)_n$ fulfilling (5.9) developed throughout Section 5.2 is still appropriate for this vectorial setting thanks to the aforementioned continuity properties of $\mathcal{E}_{\mathrm{LE}}$.

That is why the analogue of our existence Theorem 4.5 holds for the system $(\mathcal{K}(\overline{\Omega}), \mathcal{E}_{\mathrm{LE}}, \mathsf{h}, \mathsf{d}, \delta)$.

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