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Zero modes and low-energy resolvent expansion for three dimensional Schrödinger operators with point interactions

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Abstract We investigate the low-energy behavior of the resolvent of Schrödinger operators with finitely many point interactions in three dimensions. We also discuss the occurrence and the multiplicity of zero energy obstructions.

1 Introduction and main results

A central topic in quantum mechanics is the study of quantum systems subject to very short-range interactions, supported around a submanifold of the ambient space. A relevant situation occurs when the *singular* interaction is supported on a set of points in the Euclidean space $\mathbb{R}^d$. This leads to consider, formally, operators of the form

$$\text{“}-\Delta + \sum_{y\in Y} \mu_y\, \delta_y(\cdot)\text{”}, \tag{1}$$

where Y is a discrete subset of $\mathbb{R}^d$, and μ_y, $y \in Y$, are real coupling constants.

Heuristically, (1) can be interpreted as the Hamiltonian for a non-relativistic quantum particle interacting with “point obstacles” of strengths μ_y, located at $y \in Y$.

From a mathematical point of view, Schrödinger operators with point (delta-like) interactions have been intensively studied, since the first rigorous realization by Berezin and Faddeev [5], and subsequent characterizations by many other authors [2, 32, 16, 17, 8, 23] (see the surveys [11, 3], the monograph of Albeverio, Gesztesy, Høegh-Krohn, and Holden [4], and references therein for a thorough discussion).

In this work we focus on the case of finitely many point interactions in three dimensions. Our aim is to provide a detailed spectral analysis at the bottom of the continuous spectrum, i.e. at zero energy. A similar analysis has been done in [6]

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for the two dimensional case, with application to the L^p-boundedness of the wave operators.

We start by recalling some well-known facts on the construction and the main properties of Schrödinger operators with point interactions.

We fix a natural number $N \geqslant 1$ and the set $Y = \{y_1, \dots, y_N\} \subseteq \mathbb{R}^3$ of distinct centers of the singular interactions. Consider

$$T_Y := \overline{(-\Delta) \restriction C_0^\infty(\mathbb{R}^3 \backslash \{Y\})} \tag{2}$$

as an operator closure with respect to the Hilbert space $L^2(\mathbb{R}^3)$. It is a closed, densely defined, non-negative, symmetric operator on $L^2(\mathbb{R}^3)$, with deficiency index N. Hence, it admits an N^2-real parameter family of self-adjoint extensions. Among these, there is an N-parameter family of *local* extension, denoted by

$$\{-\Delta_{\alpha,Y} \,\big|\, \alpha \equiv (\alpha_1, \dots, \alpha_N) \in (\mathbb{R} \cup \{\infty\})^N\}, \tag{3}$$

whose domain of self-adjointness is qualified by certain local boundary conditions at the singularity centers.

The self-adjoint operators $-\Delta_{\alpha,Y}$ provide rigorous realizations of the formal Hamiltonian (1), the coupling parameters α_j, $j = 1, \dots, N$, being now proportional to the inverse scattering length of the interaction at the center y_j. In particular, if for some $j \in \{1, \dots, N\}$ one has $\alpha_j = \infty$, then no actual interaction is present at the point y_j, and in practice things are as if one discards the point y_j. When all $\alpha_j = \infty$, one recovers the Friedrichs extension of T_Y, namely the self-adjoint realization of $-\Delta$ on $L^2(\mathbb{R}^3)$. Owing to the discussion above, we may henceforth assume, without loss of generality, that α runs over $\mathbb{R}^N$.

We review the basic properties of $-\Delta_{\alpha,Y}$, from [4, Section II.1.1] and [26] (see also [11, 9, 18, 12]). We introduce first some notations.

For $z \in \mathbb{C}$ and $x, y, y' \in \mathbb{R}^3$, set

$$\mathscr{G}_z^y(x) \;:=\; \frac{e^{\mathrm{i}z|x-y|}}{4\pi|x-y|}, \quad \mathscr{G}_z^{yy'} \;:=\; \begin{cases} \dfrac{e^{\mathrm{i}z|y-y'|}}{4\pi|y-y'|} & \text{if } y' \neq y \\ 0 & \text{if } y' = y, \end{cases} \tag{4}$$

and

$$\Gamma_{\alpha,Y}(z) \;:=\; \Big(\Big(\alpha_j - \frac{\mathrm{i}z}{4\pi}\Big)\delta_{j,k} - \mathscr{G}_z^{y_j y_k}\Big)_{j,k=1,\dots,N}. \tag{5}$$

The function $z \mapsto \Gamma_{\alpha,Y}(z)$ has values in the space of $N \times N$ symmetric, complex valued matrices and is clearly entire, whence $z \mapsto \Gamma_{\alpha,Y}(z)^{-1}$ is meromorphic in $\mathbb{C}$. It is known that $\Gamma_{\alpha,Y}(z)^{-1}$ has at most N poles in the open upper half-plane $\mathbb{C}^+$, which are all located along the positive imaginary semi-axis. We denote by $\mathscr{E}^+$ the set of such poles. Moreover, we denote by $\mathscr{E}^0$ the set of poles of $\Gamma_{\alpha,Y}(z)^{-1}$ on the real line. Observe that $\mathscr{E}^0$ is finite and symmetric with respect to $z = 0$. Actually, either $\mathscr{E}^0 = \emptyset$ or $\mathscr{E}^0 = \{0\}$. This follows by a generalization of the Rellich Uniqueness Theorem [29, Theorem 2.4], valid for a large class of compactly supported perturbations of the Laplacian, introduced by Sjöstrand and Zworski in [30]. For an introduction to the classical theory of the Rellich Uniqueness Theorem, we refer to the monograph of Lax and Phillips [22]. More recently, the absence of non-zero real poles for $\Gamma_{\alpha,Y}^{-1}$ has been proved through different techniques by Galtbayar-Yajima [14], and by the author in collaboration with Michelangeli [24].

The following facts are known.

Proposition 1

(i) The domain of $-\Delta_{\alpha,Y}$ *has the following representation, for any* $z \in \mathbb{C}^+ \backslash \mathscr{E}^+$*:*

$$\mathscr{D}(-\Delta_{\alpha,Y}) \;=\; \Big\{ g = F_z + \sum_{j,k=1}^{N} (\Gamma_{\alpha,Y}(z)^{-1})_{jk} F_z(y_k) \mathscr{G}_z^{y_j} \,,\, F_z \in H^2(\mathbb{R}^3) \Big\}\,. \tag{6}$$

Equivalently, for any $z \in \mathbb{C}^+ \backslash \mathscr{E}^+$*,*

$$\mathscr{D}(-\Delta_{\alpha,Y}) \;=\; \left\{ g = F_z + \sum_{j=1}^{N} c_j \mathscr{G}_z^{y_j} \;\middle|\; \begin{array}{c} F_z \in H^2(\mathbb{R}^3) \\ (c_1,\dots,c_N) \in \mathbb{C}^N \\ \begin{pmatrix} F_z(y_1) \\ \vdots \\ F_z(y_N) \end{pmatrix} = \Gamma_{\alpha,Y}(z) \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} \end{array} \right\}. \tag{7}$$

At fixed z, the decompositions above are unique.

(ii) With respect to the decompositions (6)-(7)*, one has*

$$(-\Delta_{\alpha,Y} - z^2)\, g \;=\; (-\Delta - z^2)\, F_z\,. \tag{8}$$

(iii) For $z \in \mathbb{C}^+ \backslash \mathscr{E}^+$*, we have the resolvent identity*

$$(-\Delta_{\alpha,Y} - z^2)^{-1} - (-\Delta - z^2)^{-1} \;=\; \sum_{j,k=1}^{N} (\Gamma_{\alpha,Y}(z)^{-1})_{jk}\, |\mathscr{G}_z^{y_j}\rangle\langle\overline{\mathscr{G}_z^{y_k}}|\,. \tag{9}$$

(iv) The spectrum $\sigma(-\Delta_{\alpha,Y})$ *of* $-\Delta_{\alpha,Y}$ *consists of at most N non-positive eigenvalues and the absolutely continuous part* $\sigma_{\mathrm{ac}}(-\Delta_{\alpha,Y}) = [0,\infty)$*, the singular continuous spectrum is absent.*

Parts (i) and (ii) of Proposition 1 above originate from [17] and are discussed in [4, Theorem II.1.1.3], in particular (7) is highlighted in [11]. Part (iii) was first proved in [16, 17] (see also [4, equation (II.1.1.33)]). Part (iv) is discussed in [4, Theorem II.1.1.4], where it is stated that $\sigma_p(-\Delta_{\alpha,Y}) \subset (-\infty,0)$. An errata at the end of the monograph (see also [13, 15]) specifies that a zero eigenvalue embedded in the continuous spectrum can actually occur: in fact for every $N \geq 2$ one can find a configuration Y of the N centers, and coupling parameters $\alpha_1, \dots \alpha_N$ such that $0 \in \sigma_p(-\Delta_{\alpha,Y})$ – see the discussion in Section 4.

Next, let us discuss in detail the spectral properties of $-\Delta_{\alpha,Y}$, whose resolvent is characterized by (9) as an explicit rank-N perturbation of the free resolvent. For negative eigenvalues, the situation is well-understood [4, Theorem II.1.1.4].

Proposition 2 *There is a one to one correspondence between the poles* $\mathrm{i}\lambda \in \mathscr{E}^+$ *of* $\Gamma_{\alpha,Y}(z)^{-1}$ *and the negative eigenvalues* $-\lambda^2$ *of* $-\Delta_{\alpha,Y}$*, counting the multiplicity. The eigenfunctions associated to the eigenvalue* $-\lambda^2 < 0$ *have the form*

$$\psi \;=\; \sum_{j=1}^{N} c_j \mathscr{G}_{\mathrm{i}\lambda}^{y_j},$$

where $(c_1,\ldots,c_N) \in Ker\,\Gamma_{\alpha,Y}(\mathrm{i}\lambda) \setminus \{0\}$*.*

Our main purpose is to analyze the spectral behavior of $-\Delta_{\alpha,Y}$ at $z = 0$, and more generally when z approaches the real line. The starting point is a classical version of the Limiting Absorption Principle for the free Laplacian. Given $\sigma > 0$, we consider the Banach space

$$\mathbf{B}_\sigma := \mathscr{B}(L^2(\mathbb{R}^3, \langle x\rangle^{2+\sigma} dx); L^2(\mathbb{R}^3, \langle x\rangle^{-2-\sigma} dx)), \tag{10}$$

where $\langle x\rangle := \sqrt{1+|x|^2}$, and $\mathscr{B}(X;Y)$ denotes the space of linear bounded operators from X to Y. We have the following result [1, 21, 19].

Proposition 3 (Limiting Absorption Principle for $-\Delta$**)** *Let* $\sigma > 0$*. For any* $z \in \mathbb{C}^+$*, we have* $(-\Delta - z^2)^{-1} \in \mathbf{B}_\sigma$*. Moreover, the map* $\mathbb{C}^+ \ni z \mapsto (-\Delta - z^2)^{-1} \in \mathbf{B}_\sigma$ *can be continuously extended to the real line.*

Owing to the resolvent formula (9), and observing that for any $z \in \mathbb{C}^+ \cup \mathbb{R}$ the projectors $|\mathscr{G}_z^{y_j}\rangle\langle\overline{\mathscr{G}_z^{y_k}}|$ belong to $\mathbf{B}_\sigma$, it is easy to deduce that also $-\Delta_{\alpha,Y}$ satisfies a Limiting Absorption Principle.

Proposition 4 (Limiting Absorption Principle for $-\Delta_{\alpha,Y}$**)** *Let* $\sigma > 0$*. For every* $z \in \mathbb{C}^+ \setminus \mathscr{E}^+$*, we have* $(-\Delta_{\alpha,Y} - z^2)^{-1} \in \mathbf{B}_\sigma$*. Moreover, the map*

$$\mathbb{C}^+ \setminus \mathscr{E}^+ \ni z \mapsto (-\Delta_{\alpha,Y} - z^2)^{-1} \in \mathbf{B}_\sigma$$

can be continuously extended to $\mathbb{R} \setminus \mathscr{E}^0$*.*

Our main result is a resolvent expansion in a neighborhood of $z = 0$, which in view of the previous discussion is the only possible singular point on the real line for the map $z \mapsto (-\Delta_{\alpha,Y} - z^2)^{-1} \in \mathbf{B}_\sigma$.

Theorem 1 *In a (real) neighborhood of* $z = 0$*, we have the expansion*

$$(-\Delta_{\alpha,Y} - z^2)^{-1} = \frac{R_{-2}}{z^2} + \frac{R_{-1}}{z} + R_0(z), \tag{11}$$

where $R_{-2}, R_{-1} \in \mathbf{B}_\sigma$ *and* $z \mapsto R_0(z)$ *is a continuous* $\mathbf{B}_\sigma$*-valued map. Moreover,* $R_{-2} \neq 0$ *if and only if zero is an eigenvalue for* $-\Delta_{\alpha,Y}$*.*

Remark 1 For Schrödinger operators of the form $-\Delta + V$, the Limiting Absorption Principle and the analogous of Theorem 1 can be proved under suitable short-range assumptions on the scalar potential V (see e.g. the classical papers [1, 19]). In this case, moreover, it is well-known that $R_{-1} \neq 0$ if and only if there exists a generalized eigenfunction at $z=0$ (a *zero-energy resonance* for $-\Delta+V$), namely a function $\psi \in L^2(\mathbb{R}^3, \langle x\rangle^{-1-\sigma}dx) \setminus L^2(\mathbb{R}^3)$, for any $\sigma > 0$, which satisfies $(-\Delta+V)\psi = 0$ as a distributional identity on $\mathbb{R}^3$. As it will be clear from the proof of Theorem 1, a similar characterization holds true also for $-\Delta_{\alpha,Y}$ (see Remark 2).

2 Asymptotic for $\Gamma_{\alpha,Y}(z)^{-1}$ as $z \to 0$

We fix $N \geq 1$, $\alpha \in \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^3$, and we set $\Gamma(z) := \Gamma_{\alpha,Y}(z)$.

We shall use the notation $O(z^k)$, $k \in \mathbb{Z}$, to denote a meromorphic $M^N(\mathbb{C})$-valued function whose Laurent expansion in a neighborhood of $z=0$ contains only terms of degree $\geq k$. In particular, $O(1)$ denotes an analytic map in a neighborhood of $z=0$. We also write $\Theta(z^k)$ to denote a function of the form Az^k, with $A \in M^N(\mathbb{C}) \setminus \{0\}$.

In a neighborhood of $z=0$, we can expand

$$\Gamma(z) = \Gamma_0 + z\Gamma_1 + z^2\Gamma_2 + O(z^3).$$

Explicitly, we have

$$(\Gamma_0)_{jk} = \alpha_j \delta_{jk} - \mathscr{G}_0^{y_j y_k}, \quad (\Gamma_1)_{jk} = (4\pi i)^{-1}, \quad (\Gamma_2)_{jk} = (8\pi)^{-1}|y_j - y_k|.$$

In particular, Γ_0, Γ_2 are real symmetric matrices, while Γ_1 is skew-Hermitian, i.e. $\Gamma_1^* = -\Gamma_1$. Our aim is to characterize the small z behavior of $\Gamma(z)^{-1}$. Preliminary, we recall the following useful result due to Jensen and Nenciu [20].

Lemma 1 (Jensen-Nenciu) *Let A be a closed operator in a Hilbert space $\mathscr{H}$ and P a projection, such that $A+P$ has a bounded inverse. Then A has a bounded inverse if and only if*

$$B = P - P(A+P)^{-1}P$$

has a bounded inverse in $P\mathscr{H}$, and in this case

$$A^{-1} = (A+P)^{-1} + (A+P)^{-1}P(B \restriction P\mathscr{H})^{-1}P(A+P)^{-1}.$$

We can state now the main result of this Section.

Proposition 5 *In a neighborhood of $z=0$ we have the Laurent expansion*

$$\Gamma(z)^{-1} = \frac{A_{-2}}{z^2} + \frac{A_{-1}}{z} + O(1), \tag{12}$$

where $A_{-2}, A_{-1} \in M^N(\mathbb{C})$. Moreover,

(i) $A_{-2} \neq 0$ if and only if $Ker\,\Gamma_0 \cap Ker\,\Gamma_1 \neq \{0\}$,
(ii) $A_{-1} \neq 0$ if and only if $Ker\,\Gamma_0 \not\subseteq Ker\,\Gamma_1$.

Proof If $\Gamma_0 = \Gamma(0)$ is non-singular, then $\Gamma(z)^{-1}$ is analytic in a sufficiently small neighborhood of $z = 0$. Assume now that Γ_0 is singular. Let us distinguish two cases:

Case 1: $Ker\Gamma_0 \cap Ker\Gamma_1 = \{0\}$. Let us set $\Gamma_{\leq 1}(z) := \Gamma_0 + z\Gamma_1$, and observe that for z small enough, $z \neq 0$, the matrix $\Gamma_{\leq 1}(z)$ is non-singular. Suppose indeed that $\Gamma_{\leq 1}(z)v = 0$ for some $v \in \mathbb{C}^N$. If $\Gamma_0 v \neq 0$, then for small z we also have $\Gamma_{\leq 1}(z)v \neq 0$, a contradiction. Hence $\Gamma_0 v = 0$, which for $z \neq 0$ implies $\Gamma_1 v = 0$, and using the hypothesis $Ker\Gamma_0 \cap Ker\Gamma_1 = \{0\}$ we deduce that $v = 0$. Observe also that for z small enough, $z \neq 0$, the matrix $\Gamma(z)$ in non-singular, with $\Gamma(z)^{-1} = \Gamma_{\leq 1}(z)^{-1} + O(1)$.

In order to invert $\Gamma_{\leq 1}(z)$, we use the Jensen-Nenciu Lemma. Let $P : \mathbb{C}^N \to \mathbb{C}^N$ be the orthogonal projection onto $Ker\Gamma_0$. Since $\Gamma_0^* = \Gamma_0$, we have that $\Gamma_0 + P$ is non-singular, whence the same is $\Gamma_{\leq 1}(z) + P$ for small z, with $(\Gamma_{\leq 1}(z) + P)^{-1} = O(1)$. More precisely,

$$\begin{aligned}(\Gamma_{\leq 1}(z) + P)^{-1} &= [I + z(\Gamma_0 + P)^{-1}\Gamma_1]^{-1}[\Gamma_0 + P]^{-1} \\ &= [I - z(\Gamma_0 + P)^{-1}\Gamma_1][\Gamma_0 + P]^{-1} + O(z^2).\end{aligned} \tag{13}$$

By Lemma 1 we get

$$\begin{aligned}&\Gamma_{\leq 1}(z)^{-1} = (\Gamma_{\leq 1}(z) + P)^{-1} \\ &\quad + (\Gamma_{\leq 1}(z) + P)^{-1}P\Big(\big(P - P(\Gamma_{\leq 1}(z) + P)^{-1}P\big) \restriction P\mathbb{C}^N\Big)^{-1}P(\Gamma_{\leq 1}(z) + P)^{-1}.\end{aligned} \tag{14}$$

Observe that $(\Gamma_0 + P)^{-1}P = P$, and since $\Gamma_0^* = \Gamma_0$ we also have $P(\Gamma_0 + P)^{-1} = P$. Using these relations and (13), we compute

$$P - P(\Gamma_{\leq 1}(z) + P)^{-1}P = zP\Gamma_1 P + O(z^2).$$

Substituting into (14) we obtain

$$\begin{aligned}\Gamma_{\leq 1}^{-1}(z) &= (\Gamma_{\leq 1}(z) + P)^{-1} \\ &\quad + (\Gamma_{\leq 1}(z) + P)^{-1}P\Big(\big(zP\Gamma_1 P \restriction P\mathbb{C}^N\big)^{-1} + O(1)\Big)P(\Gamma_{\leq 1}(z) + P)^{-1} \\ &= z^{-1}P(P\Gamma_1 P \restriction P\mathbb{C}^N)^{-1}P + O(1) = \Theta(z^{-1}) + O(1).\end{aligned} \tag{15}$$

Case 2: $Ker\Gamma_0 \cap Ker\Gamma_1 \neq \{0\}$. We start by proving that $Ker\Gamma_1 \cap Ker\Gamma_2 = \{0\}$. Since Γ_2 is real symmetric, and Γ_1 is purely imaginary and skew-symmetric, it is sufficient to show that the quadratic form associated to Γ_2 is strictly negative on

$$(Ker\Gamma_1 \cap \mathbb{R}^N) \setminus \{0\} = \big\{v \in \mathbb{R}^N \setminus \{0\} \,|\, v_1 + \ldots + v_N = 0\big\}.$$

To this aim, we prove preliminary that for any $v \in \mathbb{R}^N$, with $v_1 + \ldots + v_N = 0$,

$$\langle \Gamma_2 v, v\rangle := (8\pi)^{-1} \sum_{1 \leq j,k \leq N} |y_j - y_k| v_j v_k \leq 0. \tag{16}$$

The key point is to use the so called *averaging trick*. By rotational and scaling invariance, we can see that there exists a positive constant c such that, for any $y \in \mathbb{R}^3$,

$$\int_{S^2} |\langle w, y\rangle| dw = c|y|.$$

It follows that

$$(8\pi)^{-1} \sum_{1\le j,k\le N} |y_j - y_k| v_j v_k = (8\pi c)^{-1} \int_{S^2} \sum_{1\le j,k\le N} |\langle w, y_j - y_k\rangle| v_j v_k dw, \tag{17}$$

and then it is sufficient to prove that, for a fixed $w \in S^2$,

$$\sum_{1\le j,k\le N} |\tilde{y}_j - \tilde{y}_k| v_j v_k \le 0,$$

where we set $\tilde{y}_j := \langle w, y_j\rangle$ for $j = 1, \dots, N$. We have

$$\begin{aligned} \sum_{1\le j,k\le N} |\tilde{y}_j - \tilde{y}_k| v_j v_k &= 2 \sum_{1\le j,k\le N} \max\{\tilde{y}_j - \tilde{y}_k, 0\} v_j v_k \\ &= 2 \int_{t\in\mathbb{R}} \sum_{1\le j,k\le N} [\tilde{y}_k < t < \tilde{y}_j] v_j v_k, \end{aligned} \tag{18}$$

where we use the Iverson bracket notation $[P]$, which equals 1 if the statement P is true and 0 if it is false. So it is enough to prove that, for almost every $t \in \mathbb{R}$,

$$\sum_{\tilde{y}_k < t < \tilde{y}_j} v_j v_k \le 0.$$

For every $t \in \mathbb{R} \setminus \{\tilde{y}_1, \dots \tilde{y}_N\}$, define $J_t := \{ j \,|\, \tilde{y}_j > t\}$, $K_t := \{k \,|\, \tilde{y}_k < t\}$. We have

$$\sum_{\tilde{y}_k < t < \tilde{y}_j} v_j v_k = \sum_{j\in J_t, k\in K_t} v_j v_k = \Big(\sum_{j\in J_t} v_j\Big)\Big(\sum_{k\in K_t} v_k\Big) = -\Big(\sum_{j\in J_t} v_j\Big)^2 \le 0, \tag{19}$$

where we use, in the last equality, the hypothesis $v_1 + \ldots + v_N = 0$.

Assume now that we have the equality in (16), for a suitable vector $v \in \mathbb{R}^N$ with $v_1 + \ldots v_N = 0$. It follows from the identity (17) that for almost every $w \in S^2$

$$\sum_{1\le j,k\le N} |\langle w, y_j - y_k\rangle| v_j v_k = 0. \tag{20}$$

In particular, we can choose $w \in S^2$ satisfying (20), and such that the quantities $\tilde{y}_j = \langle w, y_j\rangle$ are pairwise distinct, say $\tilde{y}_1 > \tilde{y}_2 > \ldots > \tilde{y}_N$. Owing to (18)-(19), we deduce that

$$\sum_{\tilde{y}_k < t < \tilde{y}_j} v_j v_k = 0, \tag{21}$$

for every t in a full-measure set $\mathscr{T} \subset \mathbb{R}$. In particular, choosing $t_1, \dots t_{N-1} \in \mathscr{T}$, with $t_n \in (\tilde{y}_{n+1}, \tilde{y}_n)$ for $n = 1, \dots, N-1$, we obtain from (21) and (19) that

$$\sum_{j=1}^{n} v_j = 0 \quad \forall n \in \{1, \dots, N\}.$$

This implies $v = 0$, concluding the proof that $Ker\,\Gamma_1 \cap Ker\,\Gamma_2 = \{0\}$.

Now, let us set $\Gamma_{\leq 2}(z) := \Gamma_{\leq 1}(z) + z^2\Gamma_2$. Arguing as in Case 1, and using the property $Ker\,\Gamma_1 \cap Ker\,\Gamma_2 = \{0\}$, we deduce that for z small enough, $z \neq 0$, the matrix $\Gamma_{\leq 2}(z)$ is non-singular. In particular, for $z \neq 0$ small enough, also $\Gamma(z)$ is non-singular, with $\Gamma(z)^{-1} = \Gamma_{\leq 2}(z)^{-1} + O(1)$.

In order to invert $\Gamma_{\leq 2}(z)$, we use the Jensen-Nenciu Lemma. Let $P : \mathbb{C}^N \to \mathbb{C}^N$ be the orthogonal projection onto $Ker\,\Gamma_0 \cap Ker\,\Gamma_1$. Owing to the relations $\Gamma_0^* = \Gamma_0$, $\Gamma_1^* = -\Gamma_1$, we deduce that for z small enough $\Gamma_{\leq 1}(z) + P$ is non-singular, with

$$(\Gamma_{\leq 1}(z) + P)^{-1} = \begin{cases} \Theta(z^{-1}) + O(1) & Ker\,\Gamma_0 \nsubseteq Ker\,\Gamma_1 \\ O(1) & Ker\,\Gamma_0 \subseteq Ker\,\Gamma_1 \end{cases}. \tag{22}$$

For small z, also $\Gamma_{\leq 2}(z) + P$ is non-singular, with

$$(\Gamma_{\leq 2}(z) + P)^{-1} = (\Gamma_{\leq 1}(z) + P)^{-1} + O(1).$$

With similar computations as in Case 1, we get

$$\begin{aligned} \Gamma_{\leq 2}(z)^{-1} &= (\Gamma_{\leq 2}(z) + P)^{-1} + z^{-2}P(P\Gamma_2 P \restriction P\mathbb{C}^N)^{-1}P + O(1) \\ &= \begin{cases} \Theta(z^{-2}) + \Theta(z^{-1}) + O(1) & Ker\,\Gamma_0 \nsubseteq Ker\,\Gamma_1 \\ \Theta(z^{-2}) + O(1) & Ker\,\Gamma_0 \subseteq Ker\,\Gamma_1 \end{cases}. \end{aligned} \tag{23}$$

Expansion (12) is thus proved in any case. Moreover, statements (i) and (ii) easily follow from the discussion above. □

3 Proof of the main Theorem

This Section is devoted to the proof of Theorem 1. Let us fix $N \geq 1$, $\alpha \in \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^3$, and set $\Gamma(z) := \Gamma_{\alpha,Y}(z)$. Preliminary, observe that the low-energy expansion (11) follows by combining the resolvent formula (9) with the small z expansion (12) for $\Gamma(z)^{-1}$. We prove now that $R_{-2} \neq 0$ if and only if $0 \in \sigma(-\Delta_{\alpha,Y})$, which in view of Proposition 5, part (i), is equivalent to prove that $Ker\,\Gamma_0 \cap Ker\,\Gamma_1 \neq \{0\}$ if and only if $0 \in \sigma(-\Delta_{\alpha,Y})$.

Suppose first that there exists $c = (c_1, \dots, c_N) \neq 0 \in Ker\,\Gamma_0 \cap Ker\,\Gamma_1$. We are going to show that the non-zero function

$$\psi := \sum_{j=1}^{N} c_j \mathscr{G}_0^{y_j} \tag{24}$$

belongs to $Ker\,(-\Delta_{\alpha,Y})$. First of all, observe that the condition $\Gamma_1\, c = 0$ is equivalent to $c_1 + \ldots + c_N = 0$, which implies $\psi \in L^2(\mathbb{R}^3)$.

Let us fix $z \in \mathbb{C}^+ \setminus \mathscr{E}^+$, and write

$$\psi = F_z + \sum_{j=1}^{N} c_j \mathscr{G}_z^{y_j},$$

where

$$F_z := \sum_{j=1}^{N} c_j (\mathscr{G}_0^{y_j} - \mathscr{G}_z^{y_j}).$$

Observe that $F_z \in H^2(\mathbb{R}^3)$. Moreover, for every $k \in \{1, \ldots, N\}$,

$$F_z(y_k) = \sum_{j=1}^{N} c_j (\mathscr{G}_0^{y_j y_k} - \mathscr{G}_z^{y_j, y_k}) = \sum_{k=1}^{N} \Gamma(z)_{kj} c_j,$$

where in the second equality we use that $\Gamma_0\, c = \Gamma_1\, c = 0$. By virtue of representation (7), we conclude that $\psi \in \mathscr{D}(-\Delta_{\alpha,Y})$. Moreover, formula (8) yields

$$-\Delta_{\alpha,Y}\, \psi = (-\Delta - z^2) F_z + z^2 \sum_{j=1}^{N} c_j \mathscr{G}_z^{y_j} = \sum_{j=1}^{N} c_j \big[(-\Delta - z^2) \mathscr{G}_z^{y_j} - \Delta \mathscr{G}_0^{y_j} \big] = 0,$$

which shows that $\psi \in Ker(-\Delta_{\alpha,Y})$.

Let us discuss now the opposite implication. To this aim, consider a function $\psi \in Ker\,(-\Delta_{\alpha,Y}) \setminus \{0\}$. For a fixed $z = i\lambda \in \mathbb{C}^+ \setminus \mathscr{E}^+$, we can write

$$\psi = F_{i\lambda} + \sum_{j=1}^{N} c_j \mathscr{G}_{i\lambda}^{y_j}, \tag{25}$$

for some non-zero $F_z \in H^2(\mathbb{R}^3)$, where

$$c_j = \sum_{k=1}^{N} \Gamma(z)^{-1}_{jk} F_z(y_k).$$

Observe that the c_j's are necessarily independent of z, since $\mathscr{G}_{i\lambda}^{y_j} \notin H^2(\mathbb{R}^3)$ for any j. Moreover, the condition $\psi \in L^2(\mathbb{R}^3)$ implies $c_1 + \ldots + c_n = 0$, namely $\Gamma_1 c = 0$. Owing to (8) and the representation (25), the relation $-\Delta_{\alpha,Y}\, \psi = 0$ is equivalent to

$$-\Delta\, F_{i\lambda} = \lambda^2 \sum_{j=1}^{N} c_j \mathscr{G}_{i\lambda}^{y_j}. \tag{26}$$

We show now that $\|F_{i\lambda}\|_{H^2} \to 0$ as $\lambda \downarrow 0$, whence also $F_{i\lambda} \to 0$ as $\lambda \downarrow 0$, uniformly on compact subsets of $\mathbb{R}^3$. This implies

$$\Gamma_0 c = \lim_{\lambda \downarrow 0} \Gamma(i\lambda) c = 0,$$

and the identity

$$\psi = \sum_{j=1}^{N} c_j \mathscr{G}_0^{y_j},$$

which conclude the proof.

In order to show that $\|F_{i\lambda}\|_{H^2} \to 0$ as $\lambda \downarrow 0$, we start with the estimate

$$\|\Delta F_{i\lambda}\|_{L^2} = \|\lambda^2 \Delta(-\Delta + \lambda^2)^{-1} \psi\|_{L^2} \le \lambda^2 \|\psi\|_{L^2}. \tag{27}$$

Observe moreover that $\widehat{F_{i\lambda}}(p) = \lambda^2 (p^2 + \lambda^2)^{-1} \widehat{\psi}(p)$. By dominate convergence we get $\|F_{i\lambda}\|_{L^2} = o(1)$, which combined with (27) yields $\|F_{i\lambda}\|_{H^2} = o(1)$, as desired.

Remark 2 By Proposition 5(ii), there is a $\Theta(z^{-1})$ term in the expansion of $\Gamma(z)^{-1}$ at $z = 0$ if and only if there exists $c \in \mathbb{R}^n$ such that $\Gamma_0 c = 0$, $\Gamma_1 c \neq 0$. In this case, the function defined by (24) belongs to $L^2(\mathbb{R}^3, \langle x \rangle^{-1-\sigma} dx) \setminus L^2(\mathbb{R}^3)$, for any $\sigma > 0$, and formally satisfies $-\Delta_{\alpha,Y} \psi = 0$, whence ψ can be interpreted as a zero energy resonance for $-\Delta_{\alpha,Y}$. Hence, as anticipated in Remark 1, we have that $R_{-1} \neq 0$ in expansion (11) if and only if there exists a zero energy resonance, analogously to the case of classical Schrödinger operators.

4 Occurrence and multiplicity of zero energy obstructions

In this Section we discuss the occurrence and the multiplicity of obstructions at zero energy for the resolvent of $-\Delta_{\alpha,Y}$, depending on the choice of the set Y of centers of interactions and the coupling parameters $\alpha_1, \dots \alpha_N$.

In the single center case, it is easy to check that the only possible obstruction at $z = 0$ is a resonance, attained if and only if $\alpha = 0$. In general, for any N and for any given configuration of the centers, there exists a measure zero set of choices of the parameters $\alpha_1, \dots, \alpha_N$ which leads to a zero-energy resonance. By means of the discussion in Section 2 and Section 3, we can define the multiplicity of the zero-energy resonance as

$$r_{\alpha,Y} := dim\,(Ker\,\Gamma_0) - dim\,(Ker\,\Gamma_0 \cap Ker\,\Gamma_1).$$

We conjecture that, as N increases, one can find Y and α such that $r_{\alpha,Y}$ becomes arbitrarily large.

As anticipated in Section 1, when $N = 2$ we can find a simple zero eigenvalue by choosing $\alpha_1 = \alpha_2 = -(4\pi d)^{-1}$, where d is the distance between the two centers. For $N \ge 3$, a zero eigenvalue occurs for specific geometric configurations of the centers of interactions and for a measure zero set of choices of $\alpha_1, \dots, \alpha_N$. Owing to the discussion in Section 2 and Section 3, the multiplicity of the zero eigenvalue is given by

$$e_{\alpha,Y} := dim Ker(-\Delta_{\alpha,Y}) = dim(Ker\Gamma_0 \cap Ker\Gamma_1).$$

Let us discuss now the maximal possible value for $e_{\alpha,Y}$ as the number of centers of interactions increases.

- $N = 3$. We can take Y as the vertices of an equilateral triangle of side-length one, and $\alpha_1 = \alpha_2 = \alpha_3 = -(4\pi)^{-1}$. With this choice we get $e_{\alpha,Y} = 2$.
- $N = 4$. We can take Y as the vertices of a regular tetrahedron of side-length one, and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -(4\pi)^{-1}$. With this choice we get $e_{\alpha,Y} = 3$.
- $N = 5$. Observe that we can not find five points in $\mathbb{R}^3$ with constant pairwise distances. It easily follows that the maximal value for $e_{\alpha,Y}$ is still three.

One could conjecture that for $N \geq 4$ the maximal value of $e_{\alpha,Y}$ is three. Nevertheless, it is also conceivable that for large N there exist complicated geometrical configurations which lead to a higher multiplicity. Such kind of mechanism is well-known in similar contexts. Consider, for example, the problem in combinatorics to determine the chromatic number of the *unit distance graph* on $\mathbb{R}^3$, that is the graph with vertices set $V = \mathbb{R}^3$ and edges set $E = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, |x-y| = 1\}$. Owing to the compactness principle by De Bruijn and Erdős [10] this is equivalent, under the axiom of choice, to determine the highest chromatic number of a finite graph embedded in $\mathbb{R}^3$ in such a way all its edges have length one. For a graph with N vertices, we have the following situation:

- $N = 3$. We can consider an equilateral triangle of side-length one, which has chromatic number three.
- $N = 4$. We can consider a regular tetrahedron of side-length one, which has chromatic number four.
- $N = 5$. The highest possible chromatic number is still four.
- $N = 14$. There is a configuration of 14 points in $\mathbb{R}^3$, the Moser-Raiskii spindle, with chromatic number five [28, 31].
- For large N, the highest possible chromatic number is known to be between 6 and 12 [25, 27, 7].

It is evident that there are similarities between the two problems, and it would be interesting to understand if they are actually related. In particular, one may take Y as the vertices of the Moser-Raiskii spindle and wondering whether there exists $\alpha = (\alpha_1, \dots, \alpha_{14})$ such that $e_{\alpha,Y} = 4$.

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