# ON THE 1D WAVE EQUATION IN TIME-DEPENDENT DOMAINS AND THE PROBLEM OF DEBOND INITIATION

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ABSTRACT. Motivated by a debonding model for a thin film peeled from a substrate, we analyse the one-dimensional wave equation, in a time-dependent domain which is degenerate at the initial time. In the first part of the paper we prove existence for the wave equation when the evolution of the domain is given; in the second part of the paper, the evolution of the domain is unknown and is governed by an energy criterion coupled with the wave equation. Our existence result for such coupled problem is a contribution to the study of crack initiation in dynamic fracture.

**Keywords:** Wave equation in time-dependent domains; Singularities; Dynamic debonding; Dynamic energy release rate; Griffith's criterion; Dynamic fracture; Crack initiation; Thin films.
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## INTRODUCTION

When partial differential equations are set in domains with singularities, their resolution may be challenging even in simple contexts. An example is provided in this work, where we deal with the one-dimensional, linear, undamped wave equation, in a time-dependent domain which is degenerate at the initial time. Precisely, we study the following equation:

$$u_{tt}(t,x) - u_{xx}(t,x) = 0, \quad t > 0,\ 0 < x < \ell(t), \tag{0.1a}$$

where $t \mapsto \ell(t)$ is a given Lipschitz function with $\ell(0) = 0$ and $0 \le \dot{\ell} < 1$. The equation is complemented with a time-dependent Dirichlet condition at the fixed endpoint $x = 0$ and with a homogeneous Dirichlet condition at the moving boundary $x = \ell(t)$:

$$u(t,0) = w(t), \quad t > 0, \tag{0.1b}$$

$$u(t,\ell(t)) = 0, \quad t > 0, \tag{0.1c}$$

where $w$ is given in such a way that $w(0) = 0$. The latter assumption ensures that the two boundary conditions are compatible. This problem is motivated by a debonding model for a thin film peeled from a substrate, as we describe below.

Hyperbolic equations in time-dependent domains may be solved by means of changes of variables when the time-dependence of the domains is sufficiently regular and the domains are not degenerate, see e.g. a general approach in [9] and the more recent papers [19, 8, 2] in arbitrary dimension. As we shall see, in the debonding problem the function $\ell$ is not expected to have such a high regularity. Therefore, problem (0.1) was studied in [7], where existence and uniqueness were proved assuming that $\ell$ is Lipschitz and $\ell(t) \ge \ell(0) > 0$, with initial conditions on $(0, \ell(0))$. In this work we focus on the difficulties of the case $\ell(0) = 0$.

The strategy of [7] relies on the specific one-dimensional geometry of the problem. Indeed, D'Alembert's formula guarantees that any solution of (0.1a) can be written in the form $u(t,x) = f(t-x) + g(t+x)$. Setting $x = 0$, by (0.1b) we get $g(t) = w(t) - f(t)$; setting $x = \ell(t)$, by (0.1c) we obtain the "bounce formula"

$$f(t+\ell(t)) = w(t+\ell(t)) + f(t-\ell(t)).$$

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If one constructs a function $f$ fulfilling such requirement, the corresponding function $u$ solves the original problem. Let us define

$$\varphi(t) := t - \ell(t), \quad \psi(t) := t + \ell(t), \quad \omega(s) := \varphi \circ \psi^{-1}(s).$$

Notice that $\omega(s) < s$. Then the "bounce formula" gives a recursive rule

$$f(s) = w(s) + f(\omega(s)) = w(s) + w(\omega(s)) + f(\omega^2(s)) = \dots \tag{0.2}$$

In the case $\ell(0) > 0$, the recursion is finite, i.e., for every $s$ there is $k$ such that $\omega^k(s)$ lies in the interval $[-\ell(0), \ell(0)]$, where $f$ can be determined by using the initial conditions: see Figure 1. This allows one to find explicitly the function $f$ and a corresponding solution to (0.1) via D'Alembert's formula [7].

This argument cannot be reproduced in the case $\ell(0) = 0$. Indeed, in Remark 1.3 we observe that the recursion is infinite and $\omega^k(s) \searrow 0$ as $k \to \infty$ for every $s$. We could solve the problem by setting

$$f(s) := \sum_{k=0}^{\infty} w(\omega^k(s)) \text{ for } s > 0, \quad f(0) = 0, \tag{0.3}$$

provided the series is summable. However, we are able to employ such formula only in the simplest case where the graph of $\ell$ is a straight line (Section 1.1). In that case, using (0.3) we prove that there is a unique solution $u$ to (0.1) of class $H^1$ (in the space/time product); but we also find infinitely many solutions of class $H^1_{\mathrm{loc}}$ with a singularity at $(t, x) = (0, 0)$, see Remark 1.4. This motivates the choice of $H^1$ as space where solutions are seeked.

In general, the series in (0.3) may be not summable. We show that it diverges e.g. when $\ell$ is quadratic in a neighbourhood of zero and $w$ is linear (Section 1.2). In this case, we are unable to say whether a solution exists. To rule out this example, we assume that $\dot{\ell}$ is bounded from below by a positive constant in a neighbourhood of zero. Under this hypothesis, in Theorem 1.6 we prove that there exists a solution to (0.1). To this end, we do not argue by recursion as in (0.2): our strategy is to obtain the "bounce formula" by applying a fixed point theorem; our new proof is independent of the one given in [7]. It turns out that the solution is unique provided a further integrability condition is required.

In the second part of the paper we deal with a free boundary problem where $\ell$ is unknown. Indeed, (0.1) is coupled with a flow rule that governs the evolution of $\ell$. The coupled problem has a mechanical interpretation as a debonding model: the function $u$ describes the profile of a thin film, initially attached onto a substrate parametrised on the $x$-axis; the film is peeled from one endpoint due to the loading $w$, so it is debonded in the interval $(0, \ell(t))$, see Figure 3 on page 10. The debonding front $\ell$ has to be determined under the constraint that it is a nondecreasing function of time, i.e., the debonded part cannot be glued again. The energetic principle that governs its evolution is called Griffith's criterion: in this context, it takes the form of an ordinary differential equation coupled with (0.1), as described in detail in Section 2.

In this work we assume the initial condition $\ell(0) = 0$, i.e., the film is completely attached onto the substrate at time zero. For this reason, as in the first part of the paper, we cannot apply a recursive technique as done in [7], where the coupled problem was solved for $\ell(0) > 0$. We propose an approximation procedure, starting from solutions $(u^\delta, \ell^\delta)$ such that $\ell^\delta$ is linear in a small interval $(0, \delta)$; then we pass to the limit as $\delta \to 0$. By assuming suitable controls on the oscillation of the data in a neighbourhood of zero, we show that the functions $(u^\delta, \ell^\delta)$ and their derivatives are bounded in an interval independent of $\delta$ (Proposition 2.6). In particular, the limit evolution $(u, \ell)$ satisfies $0 < c_0 < \dot{\ell} < L < 1$ for small times, that is the condition assumed in the first part of the paper. Thus we prove the existence of a solution to the coupled problem (Theorem 2.7 and Corollary 2.8). We are unable to prove uniqueness because we pass to subsequences in the approximation scheme.

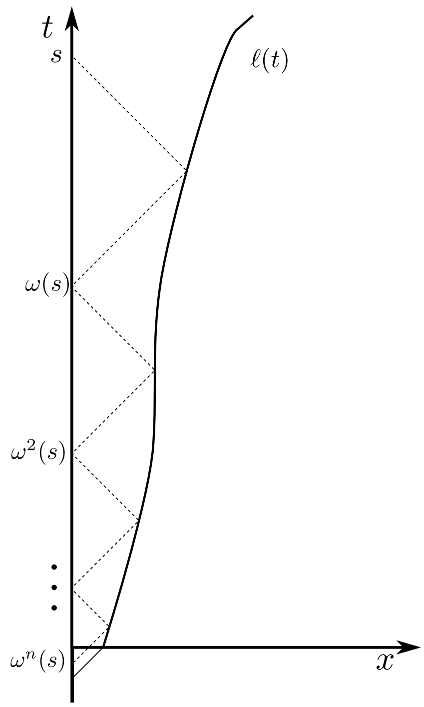


FIGURE 1. Finite iteration of the "bounce formula" in the case $\ell(0) > 0$.

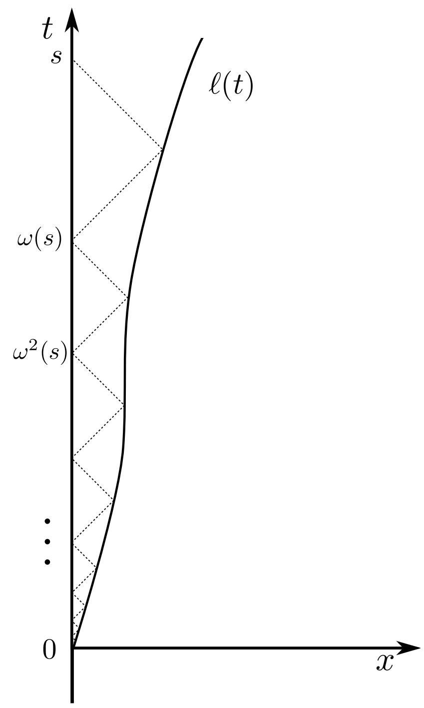


FIGURE 2. Infinite iteration of the "bounce formula" in the case $\ell(0) = 0$..

We finally underline that the debonding model analysed in the present paper can be regarded as a basic version of dynamic fracture, as already remarked in [11]: in fact, both these models feature a strong coupling between the wave equation (corresponding to momentum balance), satisfied in a time-dependent domain, and a propagation rule for the evolution of such domain. In higher dimension, existence for dynamic fracture or delamination was proved under the assumption that the crack path is known in advance [4, 5, 20, 21, 22] or that the crack is smooth [6]. The only results without strong geometrical assumptions on the crack were given for phase-field approximations [14, 18]. In contrast, in the simplified version of dynamic fracture discussed here, it is possible to predict the evolution of the debonding front without any a priori assumption, see also [7, 17].

In this paper we give a preliminary theoretical contribution to the study of initiation in dynamics, that is to determine whether fracture may be nucleated in a sound material. In dynamics, phase-field models may *not* give an accurate description of crack initiation [13]. On the other hand, in the quasistatic theory of fracture, where inertia is neglected [1], it was proved that cracks appear with velocity zero at the points where the loadings have strong singularities, otherwise the propagation is brutal, i.e., the crack velocity is infinite [3]. In contrast, our new result shows that, in a sharp-interface model, Griffith's criterion is able to predict dynamic debond initiation with well bounded, positive speed.

## 1. The wave equation in a time-dependent domain

In the present section we study the one-dimensional wave equation in a time-dependent domain $(0,\ell(t))$, whose evolution is a priori known. We consider the following system:

$$u_{tt}(t,x) - u_{xx}(t,x) = 0, \quad t>0,\ 0<x<\ell(t), \tag{1.1a}$$

$$u(t,0) = w(t), \quad t>0, \tag{1.1b}$$

$$u(t,\ell(t)) = 0, \quad t>0, \tag{1.1c}$$

In (1.1c), a Dirichlet boundary condition is prescribed on the moving boundary $x=\ell(t)$. In this work we consider the case where the domain is degenerate at $t=0$ (so that there are no initial conditions). Specifically, $\ell\colon [0,+\infty)\to[0,+\infty)$ is a given Lipschitz function and satisfies

$$0\le \dot\ell(t) < 1, \quad \text{for a.e. } t>0, \tag{1.2a}$$

$$\ell(0) = 0, \tag{1.2b}$$

$$\ell(t) > 0, \quad \text{for every } t>0. \tag{1.2c}$$

Notice that $\dot\ell$ is bounded by the wave speed. In (1.1b), at the fixed endpoint $x=0$ a time-dependent boundary condition is imposed, given by

$$w\in \widetilde H^1(0,+\infty), \tag{1.3a}$$

where

$$\widetilde H^1(0,+\infty) := \{f\in H^1_{\mathrm{loc}}(0,+\infty) : f\in H^1(0,T) \text{ for every } T>0\}.$$

We assume that

$$w(t)\ge 0 \quad \text{for every } t, \tag{1.3b}$$

as well as the compatibility condition

$$w(0)=0. \tag{1.3c}$$

These hypotheses are motivated by a model for the debonding of a film from a substrate, described in the next section. Our analysis extends previous results contained in [7], which assumed $\ell(0)>0$.

It is useful to introduce functions $\varphi, \psi \colon [0,\infty) \to [0,\infty)$ defined by

$$\varphi(t) := t - \ell(t) \quad \text{and} \quad \psi(t) := t + \ell(t). \tag{1.4}$$

Since $\varphi$ and $\psi$ are increasing by (1.2a), we can define

$$\omega \colon [0,+\infty) \to [0,+\infty), \quad \omega(t) := \varphi \circ \psi^{-1}(t). \tag{1.5}$$

Observe that $\omega(t) = 0$ if and only if $t = 0$ and

$$0 < \omega(t_2) - \omega(t_1) \leq t_2 - t_1, \quad \text{for every } 0 \leq t_1 < t_2. \tag{1.6}$$

In (1.6), the second inequality holds as equality if and only if $\ell$ is constant in $(\psi^{-1}(t_1), \psi^{-1}(t_2))$. In particular, we have

$$0 < \omega(t) < t, \quad \text{for every } t > 0. \tag{1.7}$$

We define

$$\Omega := \{(t,x) : t > 0\,,\, 0 < x < \ell(t)\},$$

and, for every $T > 0$,

$$\Omega_T := \{(t,x) : 0 < t < T\,,\, 0 < x < \ell(t)\}.$$

Let

$$\widetilde{H}^1(\Omega) := \{u \in H^1_{\mathrm{loc}}(\Omega) : u \in H^1(\Omega_T) \text{ for every } T > 0\}.$$

**Definition 1.1.** *We say that $u \in \widetilde{H}^1(\Omega)$ (resp. $u \in H^1(\Omega_T)$) is a solution to problem* (1.1) *if $u_{tt} - u_{xx} = 0$ holds in the sense of distributions in $\Omega$ (resp. in $\Omega_T$) and the boundary conditions* (1.1b)*&*(1.1c) *are satisfied in the sense of traces.*

A well-known property of (1.1a) is that its solutions can be written in the form $u(t,x) = f(t-x) + g(t+x)$ (D'Alembert's formula). By (1.1b) we get $g(t) = w(t) - f(t)$, so

$$u(t,x) = w(t{+}x) - f(t{+}x) + f(t{-}x), \quad \text{for a.e. } (t,x) \in \Omega. \tag{1.8}$$

By (1.1c) we obtain the "bounce formula"

$$f(t{+}\ell(t)) = w(t{+}\ell(t)) + f(t{-}\ell(t)), \quad \text{for every } t > 0. \tag{1.9}$$

More precisely, in [7, Proposition 1.4] it is shown the following fact.

**Proposition 1.2.** *Given $T > 0$, $u \in H^1(\Omega_T)$ is a solution to* (1.1) *in the sense of Definition 1.1 if and only if there exists a function of one variable $f \in H^1_{\mathrm{loc}}(0, T + \ell(T))$ such that*

$$\int_0^{T-\ell(T)} \dot f(s)^2 (\varphi^{-1}(s) - s)\,\mathrm{d}s + \int_{T-\ell(T)}^{T} \dot f(s)^2 (T - s)\,\mathrm{d}s < +\infty,$$
$$\int_0^{T+\ell(T)} (\dot w(s) {-} \dot f(s))^2 ((s \wedge T) - \psi^{-1}(s))\,\mathrm{d}s < +\infty,$$

*whose continuous representative satisfies* (1.9)*; in this case, the solution is given by* (1.8)*.*

Notice that any function $f$ satisfying (1.8) and (1.9) may be chosen up to an additive constant.

**Remark 1.3** (Representation formula for $f$)**.** We seek an explicit formula for a function $f$ satisfying (1.9): this will show some difficulties coming from the fact that $\ell(0) = 0$. Using the change of variables $s = \psi(t)$ we re-write (1.9) as

$$f(s) = w(s) + f(\omega(s)), \quad \text{for every } s > 0. \tag{1.10}$$

Recalling that $\omega(s) > 0$ for every $s > 0$, we may iterate the previous formula and obtain for every $N \geq 1$

$$f(s) = \sum_{k=0}^{N-1} w(\omega^k(s)) + f(\omega^N(s)). \tag{1.11}$$

See Figure 2. Observe that for every $s > 0$ we have

$$\omega^k(s) \to 0, \text{ as } k \to \infty.$$

Indeed, let $s_k := \omega^k(s)$. By (1.7), the sequence $\{s_k\}$ is decreasing and thus $s_k \to \bar{s}$ for some $\bar{s} \geq 0$. By continuity of $\omega$,

$$\omega(\bar{s}) = \omega(\lim_k s_k) = \lim_k \omega(s_k) = \lim_k s_{k+1} = \bar{s}.$$

This implies that $\bar{s} = 0$.

If

$$\sum_{k=0}^{\infty} w(\omega^k(s)) < \infty, \tag{1.12}$$

the function $f$ defined by

$$f(s) := \sum_{k=0}^{\infty} w(\omega^k(s)) \text{ for } s > 0, \quad f(0) = 0, \tag{1.13}$$

satisfies (1.9) and thus provides a solution to problem (1.1) via (1.8).

We first show that the series in (1.12) is finite in an example where $\ell$ describes a straight line; next we see that the series diverges in an example where $\ell$ is quadratic close to the origin. This motivates us to consider functions $\ell$ such that $\dot{\ell}$ is bounded from below by a positive constant in a neighbourhood of zero: in this case we prove that there is a unique function $f \in \widetilde{H}^1(0, +\infty)$ such that $f(0) = 0$ and (1.9) holds, by means of a contraction argument (Theorem 1.6).

1.1. **The straight line case.** Before proceeding with a general analysis, we first discuss the sample case of $\ell(t) = pt$, where $0 < p < 1$, and $w(t) = \beta t$, where $\beta > 0$. This is a simple case where the representation formula (1.13) can be derived.

A solution to the wave equation in the domain $\Omega$ is $u(t, x) = \beta\big(t - \frac{x}{p}\big)$. In the next Remark we observe that this is indeed the unique solution of class $\widetilde{H}^1$. By (1.8), we have

$$u(t, x) = \beta \left(t - \frac{x}{p}\right) = \beta(t + x) - f(t{+}x) + f(t{-}x) \quad \text{for every } (t, x) \in \Omega.$$

Recalling that $\omega(t) = \frac{1-p}{1+p}t$, we set

$$f(s) = \sum_{k=0}^{\infty} w(\omega^k(s)) = \sum_{k=0}^{\infty} \beta\omega^k(s) = \sum_{k=0}^{\infty} \beta \left(\frac{1-p}{1+p}\right)^k s = \beta\, \frac{1+p}{2p}\, s. \tag{1.14}$$

Therefore, in the case of a given evolution of the form $\ell(t) = pt$ and of a boundary condition linear in time, formula (1.13) is explicit.

**Remark 1.4** (Non-uniqueness of $H^1_{\text{loc}}$ solutions)**.** We show that, even in the case of the straight line, we find more than one solution (in fact, infinitely many) to (1.1), when $u$ is only required to be in $H^1_{\text{loc}}(\Omega)$. On the other hand, in this case there is only one solution in $\widetilde{H}^1(\Omega)$.

We assume again $\ell(t) = pt$, where $0 < p < 1$, and $w(t) = \beta t$, where $\beta > 0$. Problem (1.1) admits a unique solution if and only if the following problem has $u(t, x) \equiv 0$ as its unique solution:

$$u_{tt}(t, x) - u_{xx}(t, x) = 0, \tag{1.15a}$$

$$u(t, 0) = 0, \tag{1.15b}$$

$$u(t, pt) = 0. \tag{1.15c}$$

By D'Alembert's formula and using condition (1.15b), we find

$$u(t,x) = f(t-x) - f(t+x). \tag{1.16}$$

Moreover, the boundary condition on $(t, pt)$ implies that $f((1-p)t) = f((1+p)t)$. We define $\mu := \frac{1-p}{1+p}$ and obtain

$$f(\mu t) = f(t). \tag{1.17}$$

We now look for a solution $f$ of the form $f(t) = F(\log t)$, so that condition (1.17) can be written as

$$F(\log t) = F(\log t + \log \mu).$$

This implies that any smooth $(\log \mu)$-periodic function $F$ gives a solution to problem (1.15).

On the other hand, we can prove that there is a unique $u \in \widetilde{H}^1(\Omega)$ solution to (1.15). Indeed, by Proposition 1.2, if $u \in \widetilde{H}^1(\Omega)$ we have

$$+\infty > \int_0^1 t \dot{f}(t)^2 \, \mathrm{d}t = \int_0^1 \frac{\dot{F}(\log t)^2}{t} \, \mathrm{d}t = \int_{-\infty}^0 \dot{F}(s)^2 \, \mathrm{d}s. \tag{1.18}$$

Since $F$ is periodic, also $\dot{F}$ is periodic. Therefore (1.18) gives $\dot{F} = 0$ almost everywhere. This implies that $F$ is constant and thus the same holds for $f$. Finally, by (1.16) we obtain $u = 0$ and therefore there is a unique solution $u \in \widetilde{H}^1(\Omega)$ to (1.15). On the other hand, if we seek solutions $u \in H^1_{\mathrm{loc}}(\Omega)$, then $F$ need not be constant and infinitely many solutions to (1.15) are thus found.

1.2. **An example where the representation formula does not hold.** There are choices of the data $\ell$ and $w$ such that the series in (1.12) diverges. In this case, it is *not* possible to find a solution to (1.1) by using (1.8), (1.9), and (1.13). In this case, we do not know if problem (1.1) has a solution in the sense of Definition 1.1. If a solution exists, any function $f$ satisfying (1.9) would be in $H^1_{\mathrm{loc}} \setminus \widetilde{H}^1$, with $f(s) \to -\infty$ as $s \to 0$ by (1.11).

We study the case where $\ell(t) = t^2$ in a neighbourhood of zero and $w(t) = t$. In this case (1.4) reads

$$\varphi(t) = t - t^2 \quad \text{and} \quad \psi(t) = t + t^2.$$

In order to compute $\psi^{-1}$, we consider the equation $s = \psi(t) = t + t^2$, which has

$$\psi^{-1}(s) = \frac{-1 + \sqrt{1+4s}}{2}$$

as its only positive solution. Therefore,

$$\omega(s) = -1 - s + \sqrt{1+4s}.$$

In order to compute $\varphi^{-1}$, we consider the equation $s = \varphi(t) = t - t^2$. Notice that the curves $t \mapsto t^2$ and $t \mapsto t - s$ intersect in exactly two points (for $s$ sufficiently small); the point with the smallest coordinate is given by

$$\varphi^{-1}(s) = \frac{1 - \sqrt{1-4s}}{2}.$$

Therefore,

$$\omega^{-1}(s) = 1 - s - \sqrt{1-4s}.$$

**Proposition 1.5.** *For every $s > 0$ sufficiently small,*

$$\sum_{k=0}^{\infty} w(s_k) = \sum_{k=0}^{\infty} s_k = +\infty, \tag{1.19}$$

*where $s_k := \omega^k(s)$.*

*Proof.* By (1.2a), we observe that $s_k$ is decreasing in $k$ and converges to zero. It is easy to see that $\omega^{-1}(s) \leq s + 4s^2$ for $s$ sufficiently small; hence,

$$0 \leq k\left(\frac{s_k}{s_{k+1}} - 1\right) \leq k\left(\frac{s_{k+1} + 4s_{k+1}^2}{s_{k+1}} - 1\right) = 4k\, s_{k+1} =: r_k\,.$$

Let $M \geq 4$ be such that $s_1 = \omega(s) > \frac{1}{M}$. Fix $k$. If $r_k \geq \frac{4}{M}$, then

$$s_{k+1} \geq \frac{1}{M}\frac{1}{k} > \frac{1}{M}\frac{1}{k+1}\,.$$

In contrast, if $r_k < \frac{4}{M}$, then

$$s_{k+1} > s_k\frac{Mk}{Mk+4} \geq s_k\frac{Mk}{Mk+M} = s_k\frac{k}{k+1}\,.$$

In particular,

$$\text{if } s_k > \frac{1}{M}\frac{1}{k}\,, \quad \text{then } s_{k+1} > \frac{1}{M}\frac{1}{k+1}\,.$$

By induction on $k$, we obtain

$$s_k > \frac{1}{M}\frac{1}{k}, \quad \text{for every } k,$$

thus the series in (1.19) diverges. □

In an example where $\ell(t) = t^2$, we have shown that the series in (1.12) diverges, therefore it is not possible to solve the problem via formula (1.13). It is an open question if a solution to the wave equation exists in this case. We leave open also the question whether a similar behaviour is obtained for $\ell(t) = t^p$ for $p \in (1, 2)$.

1.3. **Existence of solutions in domains growing sufficiently fast.** To rule out the previous example, in addition to (1.2) we assume

$$0 < c_0 \leq \dot{\ell}(t) < 1 \quad \text{for a.e. } t > 0. \tag{1.20}$$

This condition implies that $\omega$ is a contraction with

$$0 < \dot{\omega}(t) \leq \frac{1 - c_0}{1 + c_0}\,. \tag{1.21}$$

In this case, we obtain existence of solutions $u \in \widetilde{H}^1(\Omega)$ according to Definition 1.1; we have uniqueness of solutions provided we require that the function $f$ of (1.8)–(1.9) is of class $H^1$ around zero. We recall that in [7, Theorem 1.8], existence and uniqueness are proved for $\ell(0) > 0$, assuming $0 \leq \dot{\ell} < 1$; combining the two results, we solve the case where (1.20) only holds in a neighbourhood of zero.

**Theorem 1.6.** *Assume* (1.2) , (1.3)*, and* (1.20)*. Then, there exists a unique solution* $u \in \widetilde{H}^1(\Omega)$ *to problem* (1.1) *such that* (1.8) *holds with* $f \in \widetilde{H}^1(0, +\infty)$*.*

*Proof.* Fixed $T > 0$, we introduce the space

$$H_0 := \{f \in H^1(0, T) : f(0) = 0\}.$$

We show that there exists a unique $f \in H_0$ such that (1.9) is satisfied. Hence, (1.8) provides a solution to problem (1.1), extended to $\Omega$ by arbitrarity of $T$.

We consider the map $S\colon H_0 \to H_0$ defined by

$$S(g) := g \circ \omega + w,$$

for every $g \in H_0$, see (1.10). We notice that $S$ maps $H_0$ into itself because $\omega \in C^{0,1}([0,T])$ and $w \in H^1(0,T)$, with $\omega(0) = 0$ and $w(0) = 0$. By (1.21), we observe that

$$\begin{aligned}
\|\tfrac{\mathrm{d}}{\mathrm{d}t}(Sf_1 - Sf_2)\|^2_{L^2(0,T)} &= \left\|\left(\dot{f}_1 \circ \omega - \dot{f}_2 \circ \omega\right)\dot{\omega}\right\|^2_{L^2(0,T)} \\
&= \int_0^T \left|\dot{f}_1(\omega(t)) - \dot{f}_2(\omega(t))\right|^2 \dot{\omega}(t)\,(\dot{\omega}(t)\,\mathrm{d}t) \\
&\le \frac{1-c_0}{1+c_0}\int_0^{\omega(T)} |f_1(s) - f_2(s)|^2\,\mathrm{d}s \\
&\le \frac{1-c_0}{1+c_0}\|f_1 - f_2\|^2_{H^1(0,T)}\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|Sf_1 - Sf_2\|^2_{L^2} &\le \int_0^T |f_1(\omega(t)) - f_2(\omega(t))|^2\,\mathrm{d}t \\
&= \int_0^T |(f_1(\omega(t)) - f_2(\omega(t))) - (f_1(0) - f_2(0))|^2\,\mathrm{d}t \\
&= \int_0^T \left|\int_0^{\omega(t)} \frac{\mathrm{d}}{\mathrm{d}s}(f_1(s) - f_2(s))\,\mathrm{d}s\right|^2 \mathrm{d}t \\
&\le \int_0^T \|f_1 - f_2\|^2_{H^1(0,T)}\,(\omega(t) - 0)\,\mathrm{d}t \\
&\le \|f_1 - f_2\|^2_{H^1(0,T)}\,\frac{1-c_0}{1+c_0}\int_0^T t\,\mathrm{d}t \\
&= \frac{T^2}{2}\,\frac{1-c_0}{1+c_0}\,\|f_1 - f_2\|^2_{H^1(0,T)}\,.
\end{aligned}$$

Notice that we used Hölder's inequality and (1.21). If $T$ is sufficiently small, then $S$ is a contraction in $H_0$. By the Contraction Lemma, there exists a unique function $f \in H_0$ such that $Sf(t) = f(\omega(t)) + w(t) = f(t)$, that is (1.9). By formula (1.8), we obtain a solution $u \in H^1(\Omega_{T-\ell(T)})$ to problem (1.1). We finally notice that, for $T$ large, $S$ is not a contraction in $H_0$. However, we can argue as above in a smaller interval and then extend $f$ to $[0,T]$ using an iterative argument based on (1.9), see also [7, Theorem 1.8]. □

## 2. The wave equation coupled with Griffith's criterion

In Theorem 1.6 we established the existence for the wave equation in a time-dependent domain whose evolution is given. We now deal with the case where the domain's evolution is unknown and governed by an energetic criterion, motivated by a mechanical problem for the debonding of a thin film initially attached onto a rigid substrate and peeled from one endpoint.

The rigid substrate is represented by the horizontal half plane $\{(x,y,z) : x \ge 0,\ z = 0\}$, where the film is parametrised in its reference configuration. We assume that all deformations are of the type $(x,y) \mapsto (x + h(t,x), y, u(t,x))$, where $h, u$ are functions to be determined. Notice that the second component is fixed and the displacement is $(h(t,x), 0, u(t,x))$. Due to this simplification, we obtain a one-dimensional model, see Figure 3. This model was studied in [10, 15] in particular cases (where the displacement's derivatives are piecewise constant). General results were given in [7, 16, 17] in the case where a portion of the film is already debonded at the initial time.

Here we consider the problem of initiation, i.e., the film is completely attached onto the substrate at time zero. Precisely, we assume that the debonded part of the film corresponds to

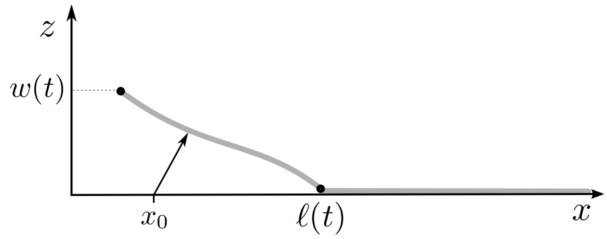


FIGURE 3. The curve $x \mapsto (x + h(t,x), u(t,x))$ gives the section of the film in its deformed configuration. The vector applied to the point $x_0$ is the displacement $(h(t,x_0), u(t,x_0))$.

an interval $(0,\ell(t))$, where $t \mapsto \ell(t)$ has to be determined under the constraint that $t \mapsto \ell(t)$ is non-decreasing and $\ell(0) = 0$. We have $h(t,x) = u(t,x) = 0$ in $(\ell(t),+\infty)$, i.e., the film is glued onto the substrate for $x \geq \ell(t)$. We assume that the film is perfectly flexible and inextensible, so the following formula hold for the horizontal displacement:

$$h(t,x) = \frac{1}{2}\int_x^{+\infty} u_x(t,\xi)^2 \,\mathrm{d}\xi.$$

Hence, the main unknowns are $u$ and $\ell$. The vertical displacement $u$ solves (1.1), where $w$ is a time-dependent external loading applied to the endpoint $x = 0$, from where the film is peeled. We assume that the speed of sound is constant in the film (and normalised to one). The flow rule for the evolution of $\ell$ is Griffith's criterion, which we are going to present.

2.1. **Griffith's criterion.** We first introduce the internal energy

$$\mathcal{E}(t;\ell,w) := \frac{1}{2}\int_0^{\ell(t)} u_x(t,x)^2 \,\mathrm{d}x + \frac{1}{2}\int_0^{\ell(t)} u_t(t,x)^2 \,\mathrm{d}x, \tag{2.1}$$

defined as a functional depending on $\ell$ and $w$. Definition (2.1) is well posed in the class of functions $(\ell,w)$ satisfying (1.2), (1.3), and (1.20), so Theorem (1.6) provides a unique displacement $u$ solving the momentum equation. In fact, properties (1.2) and (1.20) will be a by-product of our construction for the debonding evolution $\ell$.

Let us now consider the situation where we already know the evolution of the debonding front $\ell$ up to a time $t_0 > 0$ and we have to extend it for $t > t_0$. Fixed $\alpha \in (0,1)$, we extend $\ell$ to a virtual path $\lambda \in C^{0,1}([0,+\infty))$ with $\lambda(t) = \ell(t)$ for every $0 \leq t \leq t_0$, $\dot\lambda < 1$ for a.e. $t > 0$, and

$$\frac{1}{h}\int_{t_0}^{t_0+h} \left|\dot\lambda(t) - \alpha\right| \mathrm{d}t \to 0, \quad \text{as } h \to 0^+.$$

We want to study the behaviour of (2.1) with respect to variations of $\lambda$, without accounting for oscillations of external loading. Therefore, we extend $w$ after time $t_0$ by setting

$$z(t) := \begin{cases} w(t), & t \leq t_0, \\ w(t_0), & t > t_0. \end{cases}$$

The dynamic energy release rate $G_\alpha(t_0)$ at time $t_0$, for initial debonding speed $\alpha \in (0,1)$, is

$$G_\alpha(t_0) := \lim_{t \to t_0^+} \frac{\mathcal{E}(t_0;\lambda,z) - \mathcal{E}(t;\lambda,z)}{(t-t_0)\alpha}$$

and corresponds to a (sort of) partial derivative of (2.1) with respect to $\ell$. In [7, Section 2] it is proved that such limit exists and that the following formula holds:

$$G_\alpha(t) = 2\,\frac{1-\alpha}{1+\alpha}\dot f(t-\ell(t))^2, \tag{2.2}$$

where $f$ is the function appearing in (1.8). In particular, $G_\alpha$ explicitly depends on $\alpha$, that is the slope of $\lambda$ at $t_0^+$. For $\alpha = 0$, we define

$$G_0(t) := 2\dot f(t-\ell(t))^2.$$

The debonding process is activated if the energy release rate is sufficiently large: more precisely, it has to reach a value, denoted by $\kappa$, that is a material parameter for the toughness of the glue between the film and the substrate. We assume

$$\kappa \in \widetilde{C}^{0,1}(0,+\infty), \quad 0 < c_1 \le \kappa(x) \le c_2 < +\infty \text{ for every } x, \tag{2.3}$$

where

$$\widetilde{C}^{0,1}(0,+\infty) := \{f \in C^0([0,+\infty)) : f \in C^{0,1}([0,T]) \text{ for every } T > 0\}.$$

Griffith's criterion for the evolution of the debonding front $t \mapsto \ell(t)$ then reads as follows:

$$\dot\ell(t) \ge 0, \tag{2.4a}$$

$$G_{\dot\ell(t)}(t) \le \kappa(\ell(t)), \tag{2.4b}$$

$$\dot\ell(t)[G_{\dot\ell(t)}(t) - \kappa(\ell(t))] = 0. \tag{2.4c}$$

It asserts that the debonding may only increase, the energy release rate is bounded by the toughness (otherwise it would be convenient to debond a longer piece of the film), and the debonding may start only if the dynamic energy release rate is critical (i.e., (2.4b) holds as equality). Combining (2.2) with (2.4), we re-write Griffith's criterion as a Cauchy problem:

$$\dot\ell(t) = \frac{2\dot f(t-\ell(t))^2 - \kappa(\ell(t))}{2\dot f(t-\ell(t))^2 + \kappa(\ell(t))} \vee 0, \tag{2.5a}$$

$$\ell(0) = 0. \tag{2.5b}$$

**Remark 2.1.** By the "bounce formula" (1.9), see also (1.10), we obtain we obtain for a.e. $s > 0$

$$\dot f(s) = \dot w(s) + \dot f(\omega(s))\,\dot\omega(s) = \dot w(s) + \dot f(\omega(s))\,\frac{1-\dot\ell(\psi^{-1}(s))}{1+\dot\ell(\psi^{-1}(s))}\,. \tag{2.6}$$

By (2.5), we write (2.6) as

$$\dot f(s) = \begin{cases} \dot w(s) + \dot f(\omega(s)), & \text{if } \dot\ell(\psi^{-1}(s)) = 0, \\[2mm] \dot w(s) + \dfrac{\kappa(\psi^{-1}(s))}{2\dot f(\omega(s))}\,, & \text{if } \dot\ell(\psi^{-1}(s)) \ne 0, \end{cases} \tag{2.7}$$

for a.e. $s > 0$.

In the case

$$\ell(0) = \ell_0 > 0, \tag{2.8}$$

the coupled problem (1.1)&(2.5a) was analysed in [7, Theorem 3.5], with initial conditions

$$u(0,x) = u_0(x), \quad x \in (0,\ell_0), \tag{2.9a}$$

$$u_t(0,x) = u_1(x), \quad x \in (0,\ell_0), \tag{2.9b}$$

and with the following compatibility condition,

$$u_0(0) = w(0) \quad \text{and} \quad u_0(\ell_0) = 0. \tag{2.10}$$

We summarise here the existence and uniqueness result.

**Theorem 2.2.** *Let $\kappa$ be as in* (2.3)*. Let $u_0 \in C^{0,1}([0,\ell_0])$, $u_1 \in L^\infty(0,\ell_0)$, and $w \in \widetilde{C}^{0,1}(0,+\infty)$ such that* (2.10) *holds. Then, there exists a unique pair $(u,\ell) \in \widetilde{H}^1(\Omega)\times\widetilde{C}^{0,1}(0,+\infty)$ solving the coupled problem* (1.1)&(2.5a) *with initial conditions* (2.9) *and* (2.8)*. Moreover, for every $T>0$, it holds $u \in C^{0,1}(\overline{\Omega}_T)$ and there exists a constant $L_T$ such that $0 \le \dot{\ell}(t) \le L_T < 1$ for a.e. $t \in [0,T]$.*

2.2. **Existence for the coupled problem.** Our aim is now to construct a pair $(u,\ell)$ solving the coupled problem (1.1)&(2.5). To this end, we first consider again a case where the evolution of the debonding front turns out to be a straight line, cf. Section 1.1. More precisely, assuming that the loading is linear in time and that the toughness is constant, we find a solution where the debonding evolves linearly; there is only one solution of this type, however we cannot exclude that there are solution with nonlinear debonding growth.

**Lemma 2.3.** *Assume that the local toughness $\kappa$ is constant and that the loading is given by $w(t)=\beta t$ with $\beta>0$. Let*

$$p := \frac{\beta}{\sqrt{2\kappa+\beta^2}}\,, \quad u(t,x) := \beta\left(t-\frac{x}{p}\right), \quad \ell(t) := pt.$$

*Then $(u,\ell)$ is a solution to the coupled problem* (1.1)&(2.5)*. Moreover, it is the only solution such that $\ell$ is linear.*

*Proof.* We seek solutions such that $\ell(t)=qt$, with $q>0$. In Section 1.1 we showed that the only $\widetilde{H}^1$ solution to the wave equation in $(0,qt)$ is $u(t,x)=\beta\big(t-\frac{x}{q}\big)$. Thus, by (1.14) and (2.2), we obtain that the dynamic energy release rate is

$$G_{\dot{\ell}(t)}(t) = \frac{\beta^2}{2}\,\frac{1-q^2}{q^2}\,.$$

By (2.4c) the last expression must be equal to $\kappa$. This implies that $q=p$, where $p$ is defined in the statement. $\square$

Given a local toughness $\kappa$ as in (2.3) and an external loading $w \in \widetilde{C}^{0,1}(0,+\infty)$ satisfying (1.3), we construct an approximate solution to the coupled problem. Let $\delta>0$, set

$$\kappa_\delta := \kappa(\delta), \quad \beta_\delta := \frac{w(\delta)}{\delta}, \quad p_\delta := \frac{\beta_\delta}{\sqrt{2\kappa_\delta+\beta_\delta^2}}\,, \tag{2.11}$$

and define

$$\kappa^\delta(x) := \begin{cases} \kappa_\delta, & \text{if } 0 \le x < \delta,\\ \kappa(x), & \text{if } x \ge \delta, \end{cases} \qquad w^\delta(t) := \begin{cases} \beta_\delta\, t, & \text{if } 0 \le t < \delta,\\ w(t), & \text{if } t \ge \delta. \end{cases} \tag{2.12}$$

By Lemma 2.3, the pair $(u^\delta,\ell^\delta)$ defined by

$$u^\delta(t,x) := \beta_\delta\left(t-\frac{x}{p_\delta}\right), \quad \ell^\delta(t) := p_\delta\, t \tag{2.13}$$

solves the coupled problem with data $\kappa^\delta$ and $w^\delta$ for $x \in (0,\ell^\delta(t))$ and $t \in (0,\delta)$. Notice that, by (1.14), the function $f_\delta$, related to $u_\delta$ by formula (1.8), reads as

$$f^\delta(t) = \frac{\sqrt{2\kappa_\delta+\beta_\delta^2}+\beta_\delta}{2}\,t, \quad \text{for } 0 \le t \le \delta + p_\delta\,\delta. \tag{2.14}$$

At time $t = \delta$ we can explicitly compute

$$u_0^\delta(x) := u(\delta, x) = \beta_\delta\, \delta - \sqrt{2\kappa_\delta + \beta_\delta^2}\, x, \quad u_1^\delta(x) := u_t(\delta, x) = \beta_\delta, \quad \ell_0^\delta := \ell^\delta(\delta) = p_\delta\, \delta.$$

Since $u_0^\delta$ satisfies (2.10), by Theorem 2.2 with data given by $\kappa^\delta$, $\ell_0^\delta$, $u_0^\delta$, $u_1^\delta$, and $w^\delta$, we extend $(u^\delta, \ell^\delta)$ to a solution of (1.1)&(2.5) for every $t > 0$, with Lipschitz regularity. Notice that we are unable to prove uniqueness here.

Our strategy is then to study the limit as $\delta \to 0$ in order to obtain a limit pair $(u, \ell)$ solution to the coupled problem (1.1)&(2.5) corresponding to the original data $\kappa$ and $w$. To this end, we shall make some regularity assumptions on $\kappa$ and $w$ in a small neighbourhood of zero. Such assumptions involve some constants whose properties are stated in the following technical Lemma, which will be later applied with $\kappa_0 = \kappa(0)$ and $\beta_0 = \dot{w}(0)$, see (2.21)–(2.22). Notice that, in view of formula (2.7), we are interested in studying the function

$$\rho(y) := \beta_0 + \frac{\kappa_0}{2y}\,, \tag{2.15}$$

which has a fixed point at $\nu := \frac{1}{2}(\sqrt{2\kappa_0 + \beta_0^2} + \beta_0)$.

**Lemma 2.4.** *Given $\kappa_0, \beta_0 > 0$, there exist $\lambda, \mu$ such that*

$$\sqrt{\frac{\kappa_0}{2}} < \lambda < \nu = \rho(\nu) < \mu, \tag{2.16a}$$

$$\lambda < \rho(\mu), \tag{2.16b}$$

$$\rho(\lambda) < \mu. \tag{2.16c}$$

*Proof.* We can re-write (2.16b)&(2.16c) in the equivalent form:

$$\frac{\kappa_0}{2}\frac{1}{\mu - \beta_0} < \lambda < \beta_0 + \frac{\kappa_0}{2\mu}\,. \tag{2.17}$$

We choose $\mu$ such that $\mu > \nu$; since $\nu > \beta_0$, then the left-hand side of the previous inequality is positive. In particular, we have

$$\frac{\kappa_0}{2}\frac{1}{\mu - \beta_0} < \beta_0 + \frac{\kappa_0}{2\mu}\,.$$

Indeed, this is equivalent to

$$2\mu^2 - 2\beta_0\mu - \kappa_0 > 0,$$

which is always satisfied because of the choice $\mu > \nu$. Therefore, there are values of $\lambda$ satisfying (2.17). In particular, (2.16b) implies $\lambda < \nu$ since $\rho$ is decreasing.

In order to enforce the first inequality in (2.16a), we additionaly impose that

$$\sqrt{\frac{\kappa_0}{2}} < \left(\frac{\kappa_0}{2}\frac{1}{\mu - \beta_0}\right),$$

which is equivalent to

$$\mu < \beta_0 + \sqrt{\frac{\kappa_0}{2}}\,. \tag{2.18}$$

Notice that (2.18) is compatible with the requirement $\mu > \nu$, since $\nu < \beta_0 + \sqrt{\frac{\kappa_0}{2}}$. □

Given $\kappa_0, \beta_0 > 0$, let us define the following constants:

(1) Fix $\lambda, \mu$ such that (2.16) holds. More precisely, set $\mu > \nu$ satisfying (2.18) and $\lambda$ satisfying (2.17).

(2) Fix $\eta, \sigma > 0$ such that

$$\eta\beta_0 + \sigma\,\frac{\kappa_0}{2\mu} < \Big(\beta_0 + \frac{\kappa_0}{2\mu} - \lambda\Big) \wedge \Big(\mu - \beta_0 - \frac{\kappa_0}{2\lambda}\Big), \tag{2.19a}$$

$$\frac{\kappa_0}{2}(1+\sigma) < \lambda^2. \tag{2.19b}$$

Notice that the right hand side of (2.19a) is positive by choice of $\lambda$ and $\mu$; moreover, (2.19b) is compatible with $\sigma > 0$ because of (2.16a).

(3) Fix $\gamma > 0$ such that

$$\gamma > \frac{\log 3}{|\log\theta|}\,, \quad \text{where } \theta := \frac{\kappa_0}{2\lambda^2}(1+\sigma). \tag{2.20}$$

Notice that $\theta \in (0,1)$ by (2.19b).

We are in a position to introduce the assumptions on $\kappa$ and $w$: there is $\delta_0 > 0$ such that

$$\kappa \in \widetilde{C}^{0,1}(0,+\infty),\ c_1 \le \kappa \le c_2, \tag{2.21a}$$

$$\kappa(0) = \kappa_0 > 0, \tag{2.21b}$$

$$(1-\sigma)\kappa_0 \le \kappa(x) \le (1+\sigma)\kappa_0, \quad \text{for every } x \in [0,\delta_0], \tag{2.21c}$$

$$|\dot\kappa(x)| \le C_1\, x^\gamma, \quad \text{for a.e. } x \in (0,\delta_0), \tag{2.21d}$$

and

$$w \in \widetilde{C}^{0,1}(0,+\infty) \cap C^{1,1}([0,\delta_0]), \tag{2.22a}$$

$$w(0) = 0,\ \dot w(0) = \beta_0 > 0, \tag{2.22b}$$

$$(1-\eta)\dot w(0) < \dot w(t) < (1+\eta)\dot w(0), \quad \text{for every } t \in [0,\delta_0], \tag{2.22c}$$

$$|\ddot w(s)| \le C_2\, t^\gamma, \quad \text{for a.e. } t \in (0,\delta_0), \tag{2.22d}$$

where $C_1, C_2 > 0$.

We now analyse the properties of the function $f$ of (1.8). The following result provides conditions to deduce boundedness of $\dot f$ if it is controlled in a first time interval.

**Proposition 2.5.** *Assume* (2.16), (2.19), (2.21), *and* (2.22). *Let* $(u,\ell)$ *be solution to* (1.1)&(2.5) *and let* $f$ *be as in* (1.8). *Then, the following implications hold true for a.e.* $t \in (0,\delta_0)$*:*

$$\lambda \le \dot f(\varphi(t)) \le \mu \quad \Rightarrow \quad \dot\ell(t) > 0, \tag{2.23a}$$

$$\lambda \le \dot f(\omega(t)) \le \mu \quad \Rightarrow \quad \lambda \le \dot f(t) \le \mu. \tag{2.23b}$$

*Proof.* Fix $s > 0$. By (2.2) we have

$$G_{\dot\ell(\psi^{-1}(s))}(\psi^{-1}(s)) = 2\,\frac{1-\dot\ell(\psi^{-1}(s))}{1+\dot\ell(\psi^{-1}(s))}\,\dot f(\omega(s))^2 \ge 2\,\frac{1-\dot\ell(\psi^{-1}(s))}{1+\dot\ell(\psi^{-1}(s))}\,\lambda^2,$$

provided $\dot f(\omega(s)) \ge \lambda$. Moreover, by (2.4b) and (2.21c), if $\psi^{-1}(s) < \delta_0$ we get

$$G_{\dot\ell(\psi^{-1}(s))}(\psi^{-1}(s)) \le \kappa(\psi^{-1}(s)) \le (1+\sigma)\kappa_0.$$

By (2.19b) it then follows that

$$\frac{1-\dot\ell(\psi^{-1}(s))}{1+\dot\ell(\psi^{-1}(s))} < \frac{(1+\sigma)\kappa_0}{2\lambda^2} < 1.$$

Therefore, $\dot\ell(\psi^{-1}(s)) > 0$. Setting $t = \psi^{-1}(s)$ we obtain the conclusion of (2.23a).

In the second part of the proof we set $t = s$. Recalling that $\dot\ell(\psi^{-1}(t)) > 0$, we appy formula (2.7) and use (2.22c), which holds for $t < \delta_0$. Hence, the conclusion of (2.23b) is true provided

$$(1-\eta)\dot w(0) + \frac{\kappa_0(1-\sigma)}{2\mu} \geq \lambda \quad \text{and} \quad (1+\eta)\dot w(0) + \frac{\kappa_0(1+\sigma)}{2\lambda} \leq \mu.$$

These two conditions are satisfied by (2.19a). □

We now come back to the approximate solutions $(u^\delta, \ell^\delta)$ defined above and to the corresponding functions $f^\delta$. In the next proposition we give a uniform estimate on the derivatives of $f^\delta$ and $\ell^\delta$.

**Proposition 2.6.** *Assume* (2.16)*,* (2.19)*,* (2.20)*,* (2.21)*, and* (2.22)*. For every $\delta \in (0, \delta_0)$, let $(u^\delta, \ell^\delta)$ be the solution to* (1.1)*&*(2.5)*, corresponding to data $\kappa^\delta$ and $w^\delta$ given in* (2.12)*, such that* (2.13) *hold; let $f^\delta$ be related to $u^\delta$ by* (1.8)*. Then, there exist $\delta_1 \in (0, \delta_0)$ and $M > 0$, independent of $\delta$, such that*

$$\|f^\delta\|_{C^{1,1}([0,\delta_0])} \leq M, \quad \|\ell^\delta\|_{C^{1,1}([0,\delta_0])} \leq M,$$

*for $\delta \in (0, \delta_1)$.*

*Proof.* Recall that $f^\delta(0) = \ell^\delta(0) = 0$ for every $\delta$. We have $f^\delta \in C^{1,1}([0, \delta_0])$ by (1.9), (2.14), and (2.22a). Therefore, $\ell^\delta \in C^{1,1}([0, \delta_0])$ by (2.5a) and (2.21a). By (2.14),

$$\dot f^\delta(s) = \frac{\sqrt{2\kappa_\delta + \beta_\delta^2} + \beta_\delta}{2}, \quad \text{for every } s \in (0, \delta). \tag{2.24}$$

*Bound on the first derivatives.* By (2.11), (2.21), and (2.22), $\dot f^\delta(s) \to \nu$ for every $s \in (0, \delta)$, where $\nu$ is the fixed point of the function $\rho$ defined in (2.15). Then, there exists $\delta_1 > 0$ such that

$$\lambda < \dot f^\delta(s) < \mu, \quad \text{for every } s \in (0, \delta), \text{ for } \delta \in (0, \delta_1).$$

with bounds uniform in $\delta$. We may choose $\delta_1$ such that $\delta_1 < \delta_0$. We apply Proposition 2.5 and obtain that

$$\dot\ell^\delta(s) > 0 \quad \text{and} \quad \lambda < \dot f^\delta(s) < \mu, \quad \text{for every } s \in (0, \delta_0), \text{ for } \delta \in (0, \delta_1).$$

Using (2.5a), we deduce bounds on the debonding speed $\dot\ell^\delta$: since $y \mapsto \frac{2y^2-\kappa}{2y^2+\kappa}$ is increasing and $y \mapsto \frac{2\lambda^2-y}{2\lambda^2+y}$ is decreasing,

$$0 < c_0 := \frac{2\lambda^2 - \kappa_0(1+\sigma)}{2\lambda^2 + \kappa_0(1+\sigma)} \leq \dot\ell^\delta(s) \leq \frac{2\mu^2 - \kappa_0(1-\sigma)}{2\mu^2 + \kappa_0(1-\sigma)}, \quad \text{for every } s \in (0, \delta_0), \text{ for } \delta \in (0, \delta_1). \tag{2.25}$$

In analogy with (1.4)–(1.5), we consider $\omega_\delta(s) := \varphi_\delta(\psi_\delta^{-1}(s))$, where $\varphi_\delta(s) := s - \ell^\delta(s)$ and $\psi_\delta(s) := s + \ell^\delta(s)$. Since $\dot\omega_\delta = \frac{1-\dot\ell^\delta}{1+\dot\ell^\delta}$ and $y \mapsto \frac{1-y}{1+y}$ is decreasing, by (2.25) we obtain

$$\frac{\kappa_0}{2\mu^2}(1-\sigma) \leq \dot\omega_\delta(s) \leq \frac{1-c_0}{1+c_0} = \frac{\kappa_0}{2\lambda^2}(1+\sigma) = \theta, \quad \text{for every } s \in (0, \delta_0), \text{ for } \delta \in (0, \delta_1),$$

cf. (2.20) for the definition of $\theta$. In particular, since $\omega_\delta(0) = 0$ for every $\delta > 0$,

$$\omega_\delta(s) \leq \theta s, \quad \text{for every } s \in [0, \delta_0], \text{ for } \delta \in (0, \delta_1). \tag{2.26}$$

*Bound on the second derivatives.* We now seek a uniform bound for the second derivative of $f^\delta$. Starting from (2.6) and using (2.22a), we obtain for a.e. $s \in (0, \delta_0)$

$$\ddot{f}^\delta(s) = \ddot{w}^\delta(s) + \dot{\omega}_\delta^2(s)\ddot{f}^\delta(\omega_\delta(s)) - \frac{2\ddot{\ell}^\delta(\psi_\delta^{-1}(s))}{(1+\ell^\delta(\psi_\delta^{-1}(s)))^3}\,\dot{f}^\delta(\omega_\delta(s)).$$

We now compute the second derivative of $\ell^\delta(t)$, recalling (2.21a). Deriving (2.5) we get for a.e. $t \in (0, \delta_0)$

$$\ddot{\ell}^\delta(t) = \frac{8\kappa(\ell^\delta(t))(1-\dot{\ell}^\delta(t))\dot{f}^\delta(t-\ell^\delta(t))}{[2\dot{f}^\delta(t-\ell^\delta(t))^2+\kappa(\ell^\delta(t))]^2}\,\ddot{f}^\delta(t-\ell^\delta(t)) - \frac{4\dot{\kappa}(\ell^\delta(t))\dot{\ell}^\delta(t)}{[2\dot{f}^\delta(t-\ell^\delta(t))^2+\kappa(\ell^\delta(t))]^2}\,\dot{f}^\delta(t-\ell^\delta(t))^2. \tag{2.27}$$

In order to infer the latter equality we have employed the fact that $\dot{\ell}^\delta \geq c_0 > 0$ by (2.25), thus $\kappa(\ell^\delta(t))$ is well defined for a.e. $t$. For $t = \psi_\delta^{-1}(s)$, the two equalities give

$$\begin{aligned}\ddot{f}^\delta(s) &= \ddot{w}^\delta(s) + \left(\dot{\omega}_\delta^2(s) - \frac{16\kappa(\ell(\psi_\delta^{-1}(s)))\dot{f}^\delta(\omega_\delta(s))^2}{[2\dot{f}^\delta(\omega_\delta(s))^2+\kappa(\psi_\delta^{-1}(s))]^2}\,\frac{1-\dot{\ell}^\delta(\psi_\delta^{-1}(s))}{(1+\dot{\ell}^\delta(\psi_\delta^{-1}(s)))^3}\right)\ddot{f}^\delta(\omega_\delta(s))\\
&\quad + \frac{8\dot{\kappa}(\ell^\delta(\psi_\delta^{-1}(s)))}{[2\dot{f}^\delta(\omega_\delta(s))^2+\kappa(\psi_\delta^{-1}(s))]^2}\,\frac{\dot{\ell}^\delta(\psi_\delta^{-1}(s))}{(1+\dot{\ell}^\delta(\psi_\delta^{-1}(s)))^3}\,\dot{f}^\delta(\omega_\delta(s))^3\\
&=:\ \ddot{w}^\delta(s) + A\ddot{f}^\delta(\omega_\delta(s)) + B.\end{aligned}$$

Using $\dot{\omega}_\delta \leq 1$ and the inequality $\frac{ab}{(a+b)^2} \leq \frac{1}{4}$ with $a = 2\dot{f}^\delta(\omega_\delta(s))^2$ and $b = \kappa(\psi_\delta^{-1}(s))$, we obtain

$$|A| \leq \left|\dot{\omega}_\delta^2(s)\right| + \left|\frac{16\kappa(\ell(\psi_\delta^{-1}(s)))\dot{f}^\delta(\omega_\delta(s))^2}{[2\dot{f}^\delta(\omega_\delta(s))^2+\kappa(\psi_\delta^{-1}(s))]^2}\,\frac{1-\dot{\ell}^\delta(\psi_\delta^{-1}(s))}{(1+\dot{\ell}^\delta(\psi_\delta^{-1}(s)))^3}\right| \leq 3.$$

Moreover, since $\dot{f}^\delta$ and $\dot{\ell}^\delta$ are bounded in $(0, \delta_0)$, by (2.21a) and (2.21d) we find $K > 0$ such that

$$|B| = \left|\frac{8\dot{\kappa}(\ell^\delta(\psi_\delta^{-1}(s)))}{[2\dot{f}^\delta(\omega_\delta(s))^2+\kappa(\psi_\delta^{-1}(s))]^2}\,\frac{\dot{\ell}^\delta(\psi_\delta^{-1}(s))}{(1+\dot{\ell}^\delta(\psi_\delta^{-1}(s)))^3}\,\dot{f}^\delta(\omega_\delta(s))^3\right| \leq K\ell^\delta(\psi_\delta^{-1}(s))^\gamma.$$

Since $\ell^\delta(\psi_\delta^{-1}(s)) \leq s$, the last two inequalities provide

$$|\ddot{f}^\delta(s)| \leq |\ddot{w}^\delta(s)| + Ks^\gamma + 3|\ddot{f}^\delta(\omega_\delta(s))|.$$

Hence, we can iterate the previous formula and obtain

$$|\ddot{f}^\delta(s)| \leq |\ddot{w}^\delta(s)| + 3|\ddot{w}^\delta(\omega_\delta(s))| + Ks^\gamma + 3K\omega_\delta(s)^\gamma + 9|\ddot{f}^\delta(\omega_\delta^2(s))|.$$

Notice that there exists $n_\delta \in \mathbb{N}$ such that $\omega_\delta^k(s) \in (0, \delta)$ for every $s \in (0, \delta_0)$ and every $k \geq n_\delta$. For every such $k$, $\dot{f}^\delta(\omega_\delta^k(s))$ is constant, its value being given by (2.24). Therefore, for a.e. $s \in (0, \delta_0)$ we get $\ddot{f}^\delta(\omega_\delta^k(s)) = 0$ for every $k \geq n_\delta$. Finally, by (2.22d) and (2.26) we obtain for a.e. $s \in (0, \delta_0)$

$$\begin{aligned}|\ddot{f}^\delta(s)| &\leq \sum_{k=0}^{n_\delta} 3^k|\ddot{w}^\delta(\omega_\delta^k(s))| + K\sum_{k=0}^{n_\delta} 3^k|\omega_\delta^k(s)|^\gamma\\
&\leq (C_2+K)\sum_{k=0}^{n_\delta} 3^k|\theta^k s|^\gamma \leq (C_2+K)\,\delta_0^\gamma\sum_{k=0}^{\infty}(3\theta^\gamma)^k \leq M.\end{aligned}$$

In order to deduce that the last sum is bounded by $M > 0$, we have recalled that $3\theta^\gamma < 1$ by (2.20). Using the bound on $\ddot{f}^\delta(s)$ and (2.27), we also see that $\ddot{\ell}^\delta(s)$ is bounded. □

Let us comment on the latter result: starting from evolutions $\ell^\delta$ that are linear in small intervals $(0,\delta)$, we are able to infer estimates, uniform in $\delta$, satisfied in a larger interval $(0,\delta_0)$ independent of $\delta$. To this end, a key assumption is that $\kappa$ and $\dot w$ are positive and bounded in $(0,\delta_0)$. In particular, in (2.25) we have proved that $\dot\ell$ is bounded from below by a positive constant in $(0,\delta_0)$ and thus satisfies the assumption (1.20) made in the first part of the paper.

We are in a position to prove our main result.

**Theorem 2.7.** *In the assumptions of Proposition 2.6, there exist a sequence $\delta_n \to 0$ and functions $f$, $\ell$, such that $f^{\delta_n}$, $\ell^{\delta_n}$, converge to $f$, $\ell$, respectively, in $C^1([0,\delta_0])$. Setting $u$ as in* (1.8)*, $(u,\ell)$ is a solution to the coupled problem* (1.1)*&*(2.5) *in $\Omega_{\delta_0}$.*

*Proof.* In view of Proposition 2.6, by the Ascoli-Arzelà Theorem there exists a sequence $\delta_n$ such that $\ell^{\delta_n}$ converges to a limit evolution $\ell$ in $C^1([0,\delta_0])$. Since $\ell^\delta$ is monotone non-decreasing for every $\delta$, then the limit $\ell$ is monotone non-decreasing as well. Up to extracting a further subsequence (not relabelled), there exists $f$ such that $f^{\delta_n}$ converges to $f$ in $C^1([0,\delta_0])$. By continuity and uniform convergence of $f^{\delta_n}$, $\ell^{\delta_n}$, and $w^{\delta_n}$, we can pass to the limit in the "bounce formula",

$$f^{\delta_n}(t+\ell^{\delta_n}(t)) = w^{\delta_n}(t+\ell^{\delta_n}(t)) + f^{\delta_n}(t-\ell^{\delta_n}(t)),$$

so as $n\to\infty$ we obtain that $f$ satisfies (1.9). Thus the function $u$ defined as in (1.8) is a solution to (1.1) in $\Omega_{\delta_0}$ by Proposition 1.2. Finally, using again continuity and uniform convergence of $\dot f^{\delta_n}$, $\ell^{\delta_n}$, $\dot\ell^{\delta_n}$, and $w^{\delta_n}$, we can pass to the limit in

$$\begin{aligned} \dot\ell^{\delta_n}(t) &= \frac{2\dot f^{\delta_n}(t-\ell^{\delta_n}(t))^2 - \kappa^{\delta_n}(\ell^{\delta_n}(t))}{2\dot f^{\delta_n}(t-\ell^{\delta_n}(t))^2 + \kappa^{\delta_n}(\ell^{\delta_n}(t))} \vee 0, \\ \ell(0) &= 0, \end{aligned}$$

finding that $\ell$ solves (2.5) in $[0,\delta_0]$. □

Combining the last result with Theorem (2.2), we obtain existence globally in time.

**Corollary 2.8.** *Assume* (2.16)*,* (2.19)*,* (2.20)*,* (2.21)*, and* (2.22)*. Then, there exists a pair $(u,\ell) \in \widetilde H^1(\Omega)\times \widetilde C^{0,1}(0,+\infty)$ solving the coupled problem* (1.1)*&*(2.5)*. Moreover, for every $T>0$, it holds $u\in C^{0,1}(\overline\Omega_T)$ and there exists a constant $L_T$ such that $0\le \dot\ell(t)\le L_T<1$ for a.e. $t\in[0,T]$.*

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