EXISTENCE FOR CONSTRAINED DYNAMIC GRIFFITH FRACTURE WITH A WEAK MAXIMAL DISSIPATION CONDITION

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Abstract. There are very few existence results for fracture evolution, outside of globally minimizing quasi-static evolutions. Dynamic evolutions are particularly problematic, due to the difficulty of showing energy balance, as well as of showing that solutions obey a maximal dissipation condition, or some similar condition that prevents stationary cracks from always being solutions. Here we introduce a new weak maximal dissipation condition and show that it is compatible with cracks constrained to grow smoothly on a smooth curve. In particular, we show existence of dynamic fracture evolutions satisfying this maximal dissipation condition, subject to the above smoothness constraints, and exhibit explicit examples to show that this maximal dissipation principle can indeed rule out stationary cracks as solutions.

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1. Introduction

In this paper, we formulate and study a model for dynamic Griffith fracture with prescribed crack path, but unknown crack speed, and with antiplane displacement. In particular, we consider a domain $\Omega \subset \mathbb{R}^2$ and a smooth curve $\Gamma \subset \Omega$ that contains the crack at all times. As described in detail below, the crack evolutions we are considering are characterized by their length $s(t)$ at time t. Given such a (sufficiently regular) function s, and appropriate initial and boundary data, one can find the corresponding displacement u by solving the scalar wave equation off the corresponding crack set, with zero Neumann condition on the crack (see [4] and [7]). We note that, due to the uniqueness proved in [7], this displacement is determined by s.

There are other conditions on s that need to be satisfied in order to have a reasonable model for crack evolution. First, energy balance: the energy of the displacement-crack pair at time t (including the energy dissipated by the crack, which in the Griffith theory is proportional to its length) is equal to the initial energy, plus the total work done between time zero and time t (see, e.g., [11]). Second, since a stationary crack will always satisfy the above criteria, one needs to add a principle requiring that, in certain situations, the crack must grow. Such a principle, termed *maximal dissipation*, was formulated in [13], in a weak sense that does not require regularity of the crack.

Existence of crack-displacement pairs satisfying the above criteria is completely open, and the goal of this paper is to make progress in this direction. Below, after a short overview of recent mathematical progress in fracture evolution, we will formulate an even weaker, but we believe natural, maximal dissipation condition. We then consider a class of crack evolutions with uniform bounds on certain derivatives. Restricted to this class, we will prove existence of an evolution satisfying the wave equation, energy balance, and our weak maximal dissipation condition.

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Recent progress on the mathematical analysis of fracture evolution was initiated by [10], who formulated a precise mathematical model for quasi-static Griffith fracture evolution, with no regularity restriction on possible cracks. This led to a seqence of papers analyzing variants of that model, among them [8], [9], [3], [12], [5].

Despite these successes, dynamic fracture evolution (*i.e.*, the inclusion of inertial effects) remains an outstanding challenge. Indeed, the assumption in quasi-static models of negligible inertia is invalid, even with slowly varying loads or boundary conditions, if the initial state is not a global minimizer, or if the crack speed is not negligible. However, formulating a mathematically precise model, which makes no assumption on crack regularity, is difficult. As already described, one needs a principle that rules out a stationary crack always being a solution. In the context of quasi-static evolution, such a principle is minimality, but minimality is not a useful principle when the displacement is following elastodynamics, instead of minimization. An exception to this is given by phase-field models, which were formulated and studied in [2, 14]. These papers led to the maximal dissipation approach proposed in [13].

A first step in proving existence of dynamic fracture evolutions is to show that there is no regularity restriction required on the crack set in order to solve for the corresponding elastodynamics. That is, given any arbitrary and possibly growing family of crack sets $(\Gamma(t) : t \in [0,T])$, with $\mathcal{H}^{N-1}(\Gamma(T))$ finite, can one solve, in an appropriate weak sense, a wave equation "off" of the crack set, with zero Neumann condition on it? This was answered in the affirmative in [4], though no uniqueness was shown without an additional dissipation term. However, in [6], it was shown that with sufficient regularity of $t \mapsto \Gamma(t)$, there is uniqueness and, critically, continuous dependence on the data. We will use these latter results extensively below.

The spirit of this maximal dissipation condition is simply that the crack must run as fast as possible, consistent with energy balance. Below we give a precise and equivalent version of that condition, which suggests the weaker form we study here. First, we define a class of admissible evolutions, namely, those evolutions such that, together with the corresponding displacement, energy balance is satisfied. An admissible evolution s is said to satisfy maximal dissipation on $[0,T]$ if, for all $\eta > 0$ and all $\tau \in [0, T-\eta]$, there is no admissible σ such that $\sigma = s$ on $[0,\tau]$, $\sigma \geq s$ on $[\tau, \tau+\eta]$, and $\tau = \inf\{\sigma > s\}$.

Up to additional regularity assumptions on s and its competitors σ, our weaker version depends on the above η: An admissible evolution s is said to satisfy our η-maximal dissipation condition on $[0,T]$ if the above is satisfied with respect to this specified value of η.

Our main result, in Section 3, is the existence of an η-maximal dissipation solution, for any prescribed boundary and initial conditions. Furthermore, in Section 4, we give some explicit examples to show that this maximal dissipation condition is meaningful, in that if the elastic singularity of the initial data is large enough, it is possible to grow the crack while balancing energy. In particular, this shows that a stationary crack does not always satisfy η-maximal dissipation, and therefore it does not always satisfy maximal dissipation.

2. Notation and preliminary results

Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary. We assume that the crack evolves along a prescribed simple curve Γ of class $C^{2,1}$ contained in Ω except for its endpoints, which belong to $\partial\Omega$. We also assume that Γ divides Ω into two subsets Ω^+ and Ω^-, both having a Lipschitz boundary. Let us fix a (possibly empty) Borel subset $\partial_D\Omega$ of $\partial\Omega$, where we will prescribe a time dependent boundary condition. On the complement $\partial_N\Omega := \partial\Omega\backslash\partial_D\Omega$ we will prescribe the natural Neumann homogeneous boundary condition.

Let $\gamma\colon [a,b] \to \overline{\Omega}$ be an arc-length parametrization of Γ with $a < 0 < b$ and $\gamma(a), \gamma(b) \in \partial\Omega$. For every $s \in [a,b]$ we set $\Gamma_s = \gamma([a,s])$, $\Omega_s := \Omega \setminus \Gamma_s$, and $H^1_D(\Omega_s) = \{u \in H^1(\Omega_s) :$

$u = 0$ $\mathcal{H}^1$-a.e. on $\partial_D\Omega\}$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure and the values of u on $\partial_D\Omega$ are defined using the trace operator from $H^1(\Omega_s)$ to $L^2(\partial\Omega)$.

Given a function $u \in H^1(\Omega_s)$ for some $s \in [a,b]$, it is convenient to regard its gradient ∇u as an element of $L^2(\Omega;\mathbb{R}^2)$, by extending it to 0 on Γ_s. To underline the fact that this extension is not the distributional gradient of any extension of u, we shall denote it by $\widehat{\nabla} u$.

Let us fix $T > 0$. The external loading f satisfies

$$f \in L^2((0,T);L^2(\Omega))\,. \tag{2.1}$$

The Dirichlet boundary condition is prescribed using a function w, which, in order to use the results of [6], needs the regularity assumptions

$$w \in L^2((0,T);H^2(\Omega_0)) \cap H^1((0,T);H^1(\Omega_0)) \cap H^2((0,T);L^2(\Omega_0))\,. \tag{2.2}$$

The initial conditions u^0 and u^1 for the displacement and for its velocity satisfy

$$u^0 - w(0) \in H^1_D(\Omega_0) \text{ and } u^1 \in L^2(\Omega)\,. \tag{2.3}$$

In order to use the results of [6], throughout the paper we fix two parameters δ, M with $0 < \delta < 1$ and $M > 0$. The uniform regularity assumptions on the length of the cracks are prescribed in the following definition.

Definition 2.1. For every interval $[t_1,t_2] \subset [0,T]$, the class $\mathcal{C}_{\delta,M}([t_1,t_2])$ is composed of all functions satisfying the following conditions:

$$s \in C^{1,1}([t_1,t_2];[0,b]) \tag{2.4}$$

$$0 \le \dot{s}(t) \le 1-\delta, \text{ for every } t \in [t_1,t_2] \tag{2.5}$$

$$|\ddot{s}(t)| \le M \text{ for a.e. } t \in [t_1,t_2] \tag{2.6}$$

$$\sup_{\phi \in C^1_c((t_1,t_2)),\|\phi\|_\infty<1} \int_{t_1}^{t_2} \ddot{s}(t)\dot{\phi}(t)dt \le M\,. \tag{2.7}$$

Inequality (2.7) is equivalent to saying that the third derivative of s in the sense of distributions is a bounded Radon measure on (t_1,t_2) and that its total variation on (t_1,t_2) is bounded by M.

Given a function $s \in \mathcal{C}_{\delta,M}([0,T])$, we consider the corresponding displacement u, that satisfies the wave equation

$$\ddot{u}(t,x) - \Delta u(t,x) = f(t,x) \qquad \text{for } t \in (0,T) \text{ and } x \in \Omega_{s(t)}\,, \tag{2.8}$$

with Dirichlet boundary condition on $\partial_D\Omega$

$$u(t,x) = w(t,x) \qquad \text{for } t \in (0,T) \text{ and } x \in \partial_D\Omega\,, \tag{2.9}$$

Neumann boundary condition

$$\partial_\nu u(t,x) = 0 \qquad \text{for } t \in (0,T) \text{ and } x \in \partial_N\Omega \cup \Gamma_{s(t)}\,, \tag{2.10}$$

and initial conditions

$$u(0,x) = u^0(x) \quad \text{and} \quad \dot{u}(0,x) = u^1(x) \qquad \text{for } x \in \Omega_{s(0)}\,. \tag{2.11}$$

In the following definition we introduce the notion of weak solution of this problem.

Definition 2.2. Assume that f, w, u^0, u^1 satisfy (2.1)-(2.3) and let $s \in \mathcal{C}_{\delta,M}([0,T])$. We say that u is a weak solution of the wave equation (2.8) with boundary and initial conditions

(2.9)-(2.11) on the time-dependent cracking domains $t \mapsto \Omega_{s(t)}$, if

$$u \in C^1([0,T]; L^2(\Omega)) \tag{2.12}$$

$$u(t) - w(t) \in H^1_D(\Omega_{s(t)}) \quad \text{for every } t \in [0,T] \tag{2.13}$$

$$\widehat{\nabla} u \in C^0([0,T]; L^2(\Omega;\mathbb{R}^2)) \tag{2.14}$$

$$\dot{u} \in AC([t,T]; H^{-1}_D(\Omega_{s(t)})) \quad \text{for every } t \in [0,T) \tag{2.15}$$

$$\tfrac{1}{h}(\dot{u}(t+h) - \dot{u}(t)) \rightharpoonup \ddot{u}(t) \text{ weakly in } H^{-1}_D(\Omega_{s(t)}), \text{ as } h \to 0, \text{ for a.e. } t \in [0,T] \tag{2.16}$$

$$t \mapsto \|\ddot{u}(t)\|_{H^{-1}_D(\Omega_{s(t)})} \text{ is integrable on } (0,T), \tag{2.17}$$

$$u(0) = u^0 \text{ and } \dot{u}(0) = u^1 \text{ in } L^2(\Omega), \tag{2.18}$$

and for a.e. $t \in [0,T]$

$$\langle \ddot{u}(t), \varphi\rangle + \langle \widehat{\nabla} u(t), \widehat{\nabla}\varphi\rangle = \langle f(t), \varphi\rangle \quad \text{for every } \varphi \in H^1_D(\Omega_{s(t)}). \tag{2.19}$$

Here and in the rest of the paper $\langle\cdot,\cdot\rangle$ denotes a duality product between spaces that are clear from the context. For instance, its first occurrence in (2.19) refers to the duality between $H^{-1}_D(\Omega_{s(t)})$ and $H^1_D(\Omega_{s(t)})$, its second occurrence refers to the duality between $L^2(\Omega;\mathbb{R}^2)$ and $L^2(\Omega;\mathbb{R}^2)$, and its third occurrence refers to the duality between $L^2(\Omega)$ and $L^2(\Omega)$.

Definition 2.2 is a more regular version of the definition introduced in [4]. The following existence and uniqueness result is proved in [6, Example 1.14 and Corollary 2.3].

Theorem 2.3. *Under the assumptions of Definition 2.2 there exists a unique weak solution of problem* (2.8)-(2.11) *on the time-dependent cracking domains* $t \mapsto \Omega_{s(t)}$.

The following continuous dependence result is proved in [6, Theorem 3.1 and Example 3.3].

Theorem 2.4. *Assume* $s_k \in \mathcal{C}_{\delta,M}([0,T])$ *converges uniformly to some* $s \in \mathcal{C}_{\delta,M}([0,T])$. *For every* k, *let* u_k *be the weak solution of problem* (2.8)-(2.11) *on the cracking domain* $t \mapsto \Omega_{s_k(t)}$ *and let* u *be the weak solution of problem* (2.8)-(2.11) *on the cracking domain* $t \mapsto \Omega_{s(t)}$. *Then*

$$u_k(t,\cdot) \to u(t,\cdot) \text{ strongly in } L^2(\Omega), \tag{2.20}$$

$$\widehat{\nabla} u_k(t,\cdot) \to \widehat{\nabla} u(t,\cdot) \text{ strongly in } L^2(\Omega;\mathbb{R}^2), \tag{2.21}$$

$$\dot{u}_k(t,\cdot) \to \dot{u}(t,\cdot) \text{ strongly in } L^2(\Omega), \tag{2.22}$$

for every $t \in [0,T]$.

Besides the class $\mathcal{C}_{\delta,M}([0,T])$, we can consider the class $\mathcal{C}^{\rm piec}_{\delta,M}([0,T])$ defined in the following way: $s \in \mathcal{C}^{\rm piec}_{\delta,M}([0,T])$ if and only if $s \in C^0([0,T])$ and there exists a finite subdivision $0 = T_0 < T_1 < \cdots < T_k = T$ such that

$$s|_{[T_{j-1},T_j]} \in \mathcal{C}_{\delta,M}([T_{j-1},T_j]) \text{ for every } j = 1,\dots,k.$$

If f, w, u^0, u^1 satisfy (2.1)-(2.3) and $s \in \mathcal{C}^{\rm piec}_{\delta,M}([0,T])$, then Definition 2.2 still provides a notion of weak solution of the wave equation (2.8) with boundary and initial conditions (2.9)-(2.11) on the time-dependent cracking domains $t \mapsto \Omega_{s(t)}$. The existence and uniqueness of such a solution is a direct consequence of Theorem 2.3, applied to each interval $[T_{j-1}, T_j]$ of the subdivision.

3. Energy balance and maximal dissipation

We start by discussing the issue of the energy balance for weak solutions of (2.8) with f, w, u^0, u^1 satisfying (2.1)-(2.3) and with a general $s \in \mathcal{C}^{\rm piec}_{\delta,M}([0,T])$. The sum of the elastic and kinetic energies of a solution u at time t is given by $\mathcal{E}(\widehat{\nabla} u(t), \dot{u}(t))$, where

$$\mathcal{E}(\Psi, v) := \frac{1}{2}\|\Psi\|^2 + \frac{1}{2}\|v\|^2 \tag{3.1}$$

for every $\Psi \in L^2(\Omega;\mathbb{R}^2)$ and every $v \in L^2(\Omega)$. Here and in the rest of the paper $\|\cdot\|$ refers to the L^2 norm on sets that are clear from the context.

The work of the external forces on the solution u over a time interval $[t_1,t_2] \subset [0,T]$ is given by

$$\mathcal{W}_{load}(u;t_1,t_2) := \int_{t_1}^{t_2} \langle f(t), \dot{u}(t)\rangle dt\,, \tag{3.2}$$

which is well defined by (2.1) and (2.12). One would expect that the work on the solution u due to the varying boundary conditions w over a time interval $[t_1,t_2] \subset [0,T]$ is given by

$$\mathcal{W}_{bdry}(u;t_1,t_2) = \int_{t_1}^{t_2} \langle \partial_\nu u(t), \dot{w}(t)\rangle_{\partial_D\Omega} dt\,, \tag{3.3}$$

where $\langle\cdot,\cdot\rangle_{\partial_D\Omega}$ is the duality pairing between $L^2(\partial_D\Omega)$ and $L^2(\partial_D\Omega)$. Since $\dot{w}(t) \in H^1(\Omega_0)$ by (2.2), its trace on $\partial_D\Omega$ belongs to $L^2(\partial_D\Omega)$.

Unfortunately, under the assumptions (2.12)-(2.19) the trace of the normal derivative $\partial_\nu u(t)$ cannot be defined, not even in a weaker sense, because $\Delta u(t)$, in general, does not belong to $L^2(\Omega_{s(t)})$. This difficulty is solved by the following result.

Proposition 3.1. *Assume that there exists an open neighborhood U of the closure of $\partial_D\Omega$ such that $u \in L^2((t_1,t_2);H^2(U\cap\Omega\setminus\Gamma)) \cap H^2((t_1,t_2);L^2(U\cap\Omega))$. Then $\partial_\nu u \in L^2((t_1,t_2);L^2(\partial_D\Omega))$ and*

$$\begin{aligned} \mathcal{W}_{bdry}(u;t_1,t_2) &= \langle \dot{u}(t_2), \dot{w}(t_2)\rangle - \langle \dot{u}(t_1), \dot{w}(t_1)\rangle \\ &- \int_{t_1}^{t_2} \langle \ddot{w}(t), \dot{u}(t)\rangle dt - \int_{t_1}^{t_2} \langle f(t), \dot{w}(t)\rangle dt + \int_{t_1}^{t_2} \langle \widehat{\nabla} u(t), \nabla \dot{w}(t)\rangle dt\,. \end{aligned} \tag{3.4}$$

Since the right-hand side of (3.4) is well defined under the weaker assumptions (2.12)-(2.18), when u is an arbitrary weak solution of (2.8) we consider (3.4) as the weak definition of the work $\mathcal{W}_{bdry}(u;t_1,t_2)$ due to the varying boundary conditions w.

Proof of Proposition 3.1. Since $u \in L^2((t_1,t_2);H^2(U\cap\Omega^\pm))$, and U is a neighborhood of $\partial_D\Omega$, by the continuity of the trace operator we have $\partial_\nu u \in L^2((t_1,t_2);L^2(\partial_D\Omega))$. Moreover our assumptions, together with (2.19), imply that

$$\ddot{u}(t) - \Delta u(t) = f(t) \quad \text{in } L^2(U\cap\Omega\setminus\Gamma) \tag{3.5}$$

for a.e. $t \in (t_1,t_2)$. Let us fix $\varphi \in C^1_c(\mathbb{R}^2)$ with $\operatorname{supp}\varphi \subset U$ and $\varphi = 1$ on $\partial_D\Omega$. Integrating by parts with respect to space and using (3.5), we obtain

$$\begin{aligned} \langle \partial_\nu u(t), \dot{w}(t)\rangle_{\partial_D\Omega} &= \langle \partial_\nu u(t), \varphi\dot{w}(t)\rangle_{\partial_D\Omega} = \langle \Delta u(t), \varphi\dot{w}(t)\rangle + \langle \widehat{\nabla} u(t), \nabla(\varphi\dot{w}(t))\rangle \\ &= \langle \ddot{u}(t), \varphi\dot{w}(t)\rangle - \langle f(t), \varphi\dot{w}(t)\rangle + \langle \widehat{\nabla} u(t), \nabla(\varphi\dot{w}(t))\rangle\,. \end{aligned} \tag{3.6}$$

Since $(1-\varphi)\dot{w}(t) \in H^1_D(\Omega_{s(t)})$ for a.e. $t \in (0,T)$, from (2.19) we get

$$\langle \ddot{u}(t), (1-\varphi)\dot{w}(t)\rangle + \langle \widehat{\nabla} u(t), \nabla((1-\varphi)\dot{w}(t))\rangle - \langle f(t), (1-\varphi)\dot{w}(t)\rangle = 0 \tag{3.7}$$

for a.e. $t \in (0,T)$. Adding the left-hand side of (3.7) to the right-hand side of (3.6), from (3.3) we deduce that

$$\mathcal{W}_{bdry}(u;t_1,t_2) = \int_{t_1}^{t_2} \langle \ddot{u}(t), \dot{w}(t)\rangle dt - \int_{t_1}^{t_2} \langle f(t), \dot{w}(t)\rangle dt + \int_{t_1}^{t_2} \langle \widehat{\nabla} u(t), \nabla \dot{w}(t)\rangle dt$$

Integrating by parts with respect to time we finally obtain (3.4). □

The total work on the solution u over a time interval $[t_1,t_2] \subset [0,T]$ is defined by

$$\mathcal{W}(u;t_1,t_2) := \mathcal{W}_{load}(u;t_1,t_2) + \mathcal{W}_{bdry}(u;t_1,t_2)\,,$$

where $\mathcal{W}_{load}$ is defined by (3.2) and $\mathcal{W}_{bdry}$ is defined by (3.4).

Remark 3.2. If $s(\cdot)$ is constant in the interval $[t_1,t_2]\subset[0,T]$, then every solution u of the wave equation (2.8) according to Definition 2.2, satisfies the energy equality

$$\mathcal{E}(\widehat{\nabla}u(t_2),\dot u(t_2))-\mathcal{E}(\widehat{\nabla}u(t_1),\dot u(t_1))=\mathcal{W}(u;t_1,t_2)\,.$$

This result is well known for the wave equation in a sufficiently smooth domain. For an arbitrary bounded open set one can follow the lines of the proof of [7, Lemma 1]. An alternative way is to perturb the wave equation with a small damping term and use the fact that the energy balance always holds for the damped wave equation (see [4, Theorem 3.2]).

We now introduce the class of admissible functions s which satisfy the energy-dissipation balance.

Definition 3.3. Assume that f,w,u^0,u^1 satisfy (2.1)-(2.3) and let $s^0\in[0,b]$. The class $\mathcal{S}$ is composed of all functions $s\in\mathcal{C}^{piec}_{\delta,M}([0,T])$, with $s(0)=s^0$, such that the unique weak solution u of (2.8)-(2.11) on the time-dependent cracking domains $t\mapsto\Omega_{s(t)}$ satisfies the energy-dissipation balance

$$\mathcal{E}(\widehat{\nabla}u(t_2),\dot u(t_2))-\mathcal{E}(\widehat{\nabla}u(t_1)),\dot u(t_1))+s(t_2)-s(t_1)=\mathcal{W}(u;t_1,t_2)\tag{3.8}$$

for every interval $[t_1,t_2]\subset[0,T]$.

The meaning of (3.8) is conservation of energy: The work done on the system is balanced by the change in mechanical energy $\mathcal{E}(\widehat{\nabla}u(t_2),\dot u(t_2))-\mathcal{E}(\widehat{\nabla}u(t_1)),\dot u(t_1))$ and by the energy dissipated in the process of crack production in the same time interval $[t_1,t_2]$, which is assumed to be given by $s(t_2)-s(t_1)$, consistent with the Griffith theory.

Remark 3.4. Note that the class $\mathcal{S}$ is not empty: by Remark 3.2 it contains at least the constant function $s(t)=s^0$ for all $t\in[0,T]$. We shall see in Section 4 that there are examples where $\mathcal{S}$ contains nonconstant functions.

To define the notion of η-maximal dissipation solution of the dynamic crack evolution, for every $s\in\mathcal{S}$, $\eta\in[0,T]$, and $\tau\in[0,T-\eta]$ we introduce the class $\mathcal{A}(s,\eta,\tau)$ of admissible comparison functions, defined as the set of $\sigma\in\mathcal{S}$ with $\sigma|_{[0,\tau]}=s|_{[0,\tau]}$, $\sigma|_{[\tau,\tau+\eta]}\in\mathcal{C}_{\delta,M}([\tau,\tau+\eta])$, and $\dot\sigma(\tau+)=\dot s(\tau+)$, where $\tau+$ denotes the limit from the right.

Definition 3.5. Assume that f,w,u^0,u^1 satisfy (2.1)-(2.3) and let $s^0\in[0,b]$ and $\eta\in[0,T]$. We say that s is an *η-maximal dissipation* solution of the dynamic crack evolution problem corresponding to these data if $s\in\mathcal{S}$ and for every $\tau\in[0,T-\eta]$ there is no $\sigma\in\mathcal{A}(s,\eta,\tau)$ such that

$$\sigma(t)\ge s(t)\text{ for every }t\in[\tau,\tau+\eta]\quad\text{and}\quad\tau=\inf\{t\in[0,T]:\sigma(t)>s(t)\}\,.\tag{3.9}$$

Theorem 3.6. *Given f, w, u^0, u^1, s^0, and η as in Definition 3.5, there exists an η-maximal dissipation solution of the dynamic crack evolution problem corresponding to these data.*

Proof. Let us fix a finite subdivision $0=T_0<T_1<\cdots<T_k=T$ such that $T_j-T_{j-1}\le\eta$ for every $j=1,\dots,k$. The solution will be constructed recursively in the intervals $[T_{j-1},T_j]$. Let $\mathcal{S}_1$ be the set of all functions $s\in\mathcal{C}_{\delta,M}([T_0,T_1])$, with $s(T_0)=s^0$, such that the unique weak solution u of (2.8)-(2.11) on the time-dependent cracking domains $t\mapsto\Omega_{s(t)}$, $t\in[T_0,T_1]$, satisfies the energy-dissipation balance (3.8) for every interval $[t_1,t_2]\subset[T_0,T_1]$. In Lemma 3.7 below we shall prove that there exists $s_1\in\mathcal{S}_1$ such that

$$\int_{T_0}^{T_1}s_1(t)\,dt=\max_{s\in\mathcal{S}_1}\int_{T_0}^{T_1}s(t)\,dt\,.$$

Finally, we define u_1 to be the unique weak solution u of (2.8)-(2.11) on the time-dependent cracking domains $t\mapsto\Omega_{s_1(t)}$, $t\in[T_0,T_1]$.

Suppose now that $s_j \in \mathcal{C}_{\delta,M}([T_{j-1},T_j])$ and $u_j \in C^1([T_{j-1},T_j];L^2(\Omega))$ are defined for some $1 \le j < k$. Then we define $\mathcal{S}_{j+1}$ as the set of all functions $s \in \mathcal{C}_{\delta,M}([T_j,T_{j+1}])$, with $s(T_j) = s_j(T_j)$, such that the unique weak solution u of (2.8)-(2.11), with u^0 and u^1 replaced by $u_j(T_j)$ and $\dot u_j(T_j)$, on the time-dependent cracking domains $t \mapsto \Omega_{s(t)}$, $t \in [T_j,T_{j+1}]$, satisfies the energy-dissipation balance (3.8) for every interval $[t_1,t_2] \subset [T_j,T_{j+1}]$. As above, there exists $s_{j+1} \in \mathcal{S}_{j+1}$ such that

$$\int_{T_j}^{T_{j+1}} s_{j+1}(t)\,dt = \max_{s\in\mathcal{S}_{j+1}} \int_{T_j}^{T_{j+1}} s(t)\,dt\,. \tag{3.10}$$

Finally, we define u_{j+1} to be the unique weak solution u of (2.8)-(2.11), with u^0 and u^1 replaced by $u_j(T_j)$ and $\dot u_j(T_j)$, on the time-dependent cracking domains $t \mapsto \Omega_{s_{j+1}(t)}$, $t \in [T_j,T_{j+1}]$.

After k steps we have constructed s_j and u_j on each interval $[T_{j-1},T_j]$, $j = 1,\dots,k$, in such a way that $s_j(T_j) = s_{j+1}(T_j)$, $u_j(T_j) = u_{j+1}(T_j)$, and $\dot u_j(T_j) = \dot u_{j+1}(T_j)$ for $j = 1,\dots,k-1$. Moreover $s_1(0) = s^0$, $u_1(0) = u^0$, and $\dot u_1(0) = u^1$. We now set $s(t) := s_j(t)$ and $u(t) := u_j(t)$ for $t \in [T_{j-1},T_j]$, $j = 1,\dots,k$. The previous remark shows that $s \in \mathcal{S}$ and that u is the unique weak solution of (2.8)-(2.11) on the time-dependent cracking domains $t \mapsto \Omega_{s(t)}$, $t \in [0,T]$.

To show that s satisfies the maximality property in Definition 3.5, we assume by contradiction that there exists $\tau \in [0,T-\eta]$ and $\sigma \in \mathcal{A}(s,\eta,\tau)$ which satisfy (3.9). Let $j \in \{0,\dots,k-1\}$ be such that $\tau \in [T_j,T_{j+1})$. Since $T_{j+1} - T_j \le \eta$, we have $T_{j+1} \le \tau + \eta$. Using the definition of $\mathcal{A}(s,\eta,\tau)$ given before Definition 3.5 it is easy to see that $\sigma \in \mathcal{C}_{\delta,M}([T_j,\tau+\eta])$. This implies, in particular, that $\sigma \in \mathcal{C}_{\delta,M}([T_j,T_{j+1}])$. Since $\sigma|_{[0,\tau]} = s|_{[0,\tau]}$, by the uniqueness of the solution to the wave equation we deduce that $u_\sigma(T_j) = u(T_j) = u_j(T_j)$ and $\dot u_\sigma(T_j) = \dot u(T_j) = \dot u_j(T_j)$, where u_σ and u denote the unique solutions to the wave equation on $[0,T]$ in the cracking domains corresponding to σ and s, respectively. This implies that $\sigma|_{[T_j,T_{j+1}]} \in \mathcal{S}_{j+1}$, hence (3.10) gives

$$\int_{T_j}^{T_{j+1}} s(t)\,dt = \int_{T_j}^{T_{j+1}} s_{j+1}(t)\,dt \ge \int_{T_j}^{T_{j+1}} \sigma(t)\,dt\,.$$

Since $\sigma(t) \ge s(t)$ for every $t \in [T_j,T_{j+1}]$, and both functions are continuous in $[T_j,T_{j+1}]$, we conclude that $\sigma(t) = s(t)$ for every $t \in [T_j,T_{j+1}]$, which contradicts the second property in (3.9). □

To conclude the proof of Theorem 3.6 we need the following lemma.

Lemma 3.7. *For every $j = 1,\dots,k$ there exists $s_j \in \mathcal{S}_j$ such that*

$$\int_{T_{j-1}}^{T_j} s_j(t)\,dt = \max_{s\in\mathcal{S}_j} \int_{T_{j-1}}^{T_j} s(t)\,dt\,. \tag{3.11}$$

Proof. Fix $j = 1,\dots,k$ and set $I_{max} := \sup_{s\in\mathcal{S}_j} \int_{T_{j-1}}^{T_j} s(t)dt$ and, for every $n \in \mathbb{N}$, let $s_n \in \mathcal{S}_j$ be such that

$$\int_0^T s_n(t)dt \ge I_{max} - \frac{1}{n}\,. \tag{3.12}$$

By the compactness of $\mathcal{C}_{\delta,M}([T_{j-1},T_j])$ there exists a subsequence of s_n, not relabeled, and a function $s \in \mathcal{C}_{\delta,M}([T_{j-1},T_j])$ such that $s_n \to s$ in $C^1([T_{j-1},T_j])$.

Since $s_n \in \mathcal{S}_j$, the weak solutions u_n to the wave equation corresponding to s_n and to the data f, w, $u_{j-1}(T_{j-1})$, and $\dot u_{j-1}(T_{j-1})$ (u^0 and u^1 if $j = 1$), satisfy the energy equality

$$\mathcal{E}(\widehat\nabla u_n(t_2),\dot u_n(t_2)) - \mathcal{E}(\widehat\nabla u_n(t_1),\dot u_n(t_1)) + s_n(t_2) - s_n(t_1) = \mathcal{W}(u_n;t_1,t_2)$$

for every interval $[t_1,t_2]\subset[T_{j-1},T_j]$. Let u be the solution to the wave equation corresponding to s and to the same data f,w,u^0,u^1. Passing to the limit thanks to Theorem 2.4, by (2.20)-(2.22), we obtain (3.8), so that $s\in\mathcal{S}_j$. On the other hand, from (3.12) we get $\displaystyle\int_0^T s(t)dt=I_{max}$, which gives (3.11). □

4. An Example

In this section we present an example in which the crack length increases in time and the energy-dissipation balance is satisfied. In this example $0\in\Omega$, $\partial_D\Omega=\partial\Omega$, Γ is the intersection of $\overline{\Omega}$ with the x_1-axis, and the parametrization is given by $\gamma(s)=(s,0)\in\Omega$ for every $s\in(a,b)$, so that $\Omega_s=\Omega\setminus\{(x_1,0):x_1\le s\}$. We also assume that the crack tip moves with constant speed $0<c<1$, with $cT<b$, hence $s(t)=ct$ for every $t\in[0,T]$. Finally, we assume that there is no external loading, so $f=0$.

Let us consider the function $\psi\colon\mathbb{R}^2\setminus\{(x_1,0):x_1\le0\}\to\mathbb{R}$ defined by

$$\psi(x):=\mathcal{I}m(\sqrt{x_1+ix_2})=\sqrt{\rho}\sin(\theta/2)\,,\tag{4.1}$$

where $\mathcal{I}m$ denotes the imaginary part and ρ, θ are the polar coordinates of $x=(x_1,x_2)$, with $-\pi<\theta<\pi$. Since $\Delta\psi=0$ in $\mathbb{R}^2\setminus\{(x_1,0):x_1\le0\}$, and $\partial_\nu\psi=0$ on $\{(x_1,0):x_1<0\}$, it is easy to check that for every $\kappa\in\mathbb{R}$ the function

$$u_\kappa(t,x):=\kappa\,\psi\left(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\right),\tag{4.2}$$

defined for $t\in[0,T]$ and $x\in\Omega_{ct}$, satisfies (2.8) and (2.10). Since $\psi\in H^1(B\setminus\{(x_1,0):x_1\le0\})$ for every bounded open set $B\subset\mathbb{R}^2$, we conclude that u_κ satisfies (2.12), (2.14), (2.15), (2.16), (2.17), and (2.19).

Let us fix an open neighborhood U of $\partial\Omega$, such that $U\setminus\Gamma\subset\Omega_s$ for every $s\in[0,cT]$, and a function $\varphi\in C_c^\infty(\mathbb{R}^2)$, with $\operatorname{supp}\varphi\in U$ and $\varphi=1$ on $\partial\Omega$. Then the function

$$w_\kappa(t,x):=\varphi(x)u_\kappa(t,x)$$

belongs to $C^\infty([0,T]\times\mathbb{R}^2)$ and u_κ satisfies (2.9) and (2.13).

We want to determine κ so that the energy-dissipation balance (3.8) be satisfied for every interval $[t_1,t_2]\subset[0,T]$. Taking into account Proposition 3.1, the energy-dissipation balance becomes

$$\frac12\|\widehat{\nabla}u_\kappa(t)\|^2+\frac12\|\dot u_\kappa(t)\|^2+ct=\frac12\|\widehat{\nabla}u_\kappa(0)\|^2+\frac12\|\dot u_\kappa(0)\|^2+\int_0^t\langle\partial_\nu u_\kappa(\tau),\dot u_\kappa(\tau)\rangle_{\partial\Omega}\,d\tau\tag{4.3}$$

for every $t\in[0,T]$.

Let

$$\Omega_{ct}:=\Omega\setminus\{(x_1,0):x_1\le ct\}\quad\text{and}\quad\Omega_{ct}^\varepsilon=\Omega_{ct}\setminus E_{ct}^\varepsilon\,,$$

where

$$E_{ct}^\varepsilon=\{x=(x_1,x_2)\in\mathbb{R}^2:\frac{(x_1-ct)^2}{1-c^2}+x_2^2\le\varepsilon^2\}\,.$$

On Ω_{ct}^ε we multiply both terms of equation (2.8) by $\dot u_\kappa(t)$, which is smooth in this set. Integrating by parts we obtain

$$\frac12\int_{\Omega_{ct}^\varepsilon}\frac{d}{dt}|\dot u_\kappa(t)|^2dx+\frac12\int_{\Omega_{ct}^\varepsilon}\frac{d}{dt}|\widehat{\nabla}u_\kappa(t)|^2dx=\int_{\partial\Omega}\partial_\nu u_\kappa(t)\,\dot u_\kappa(t)\,d\mathcal{H}^1-\int_{\partial E_{ct}^\varepsilon}\partial_\nu u_\kappa(t)\,\dot u_\kappa(t)\,d\mathcal{H}^1,$$

where ν is the outer unit normal to Ω and to E_{ct}^ε. On the other hand, using the identity

$$\frac{d}{dt}\int_{\Omega_{ct}^\varepsilon}v\,dx=-c\int_{\partial E_{ct}^\varepsilon}v\nu_1\,d\mathcal{H}^1\,,$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_{\Omega^\varepsilon_{ct}}|\dot u_\kappa(t)|^2dx+\frac{1}{2}\int_{\Omega^\varepsilon_{ct}}|\widehat\nabla u_\kappa(t)|^2dx\right)\\
&=\frac{1}{2}\int_{\Omega^\varepsilon_{ct}}\frac{d}{dt}|\dot u_\kappa(t)|^2dx+\frac{1}{2}\int_{\Omega^\varepsilon_{ct}}\frac{d}{dt}|\widehat\nabla u_\kappa(t)|^2dx-\frac{c}{2}\int_{\partial E^\varepsilon_{ct}}|\dot u_\kappa(t)|^2\nu_1d\mathcal{H}^1-\frac{c}{2}\int_{\partial E^\varepsilon_{ct}}|\widehat\nabla u_\kappa(t)|^2\nu_1d\mathcal{H}^1\\
&=\int_{\partial\Omega}\partial_\nu u_\kappa(t)\,\dot u_\kappa(t)\,d\mathcal{H}^1-\int_{\partial E^\varepsilon_{ct}}\Big(\partial_\nu u_\kappa(t)\,\dot u_\kappa(t)+\frac{c}{2}|\dot u_\kappa(t)|^2\nu_1+\frac{c}{2}|\widehat\nabla u_\kappa(t)|^2\nu_1\Big)\,d\mathcal{H}^1\,.
\end{aligned}$$

Integrating on $[t_1,t_2]\subset[0,T]$ we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus E^\varepsilon_{ct_2}}|\widehat\nabla u_\kappa(t_2)|^2dx+\frac{1}{2}\int_{\Omega\setminus E^\varepsilon_{ct_2}}|\dot u_\kappa(t_2)|^2dx-\frac{1}{2}\int_{\Omega\setminus E^\varepsilon_{ct_1}}|\widehat\nabla u_\kappa(t_1)|^2dx-\frac{1}{2}\int_{\Omega\setminus E^\varepsilon_{ct_1}}|\dot u_\kappa(t_1)|^2dx\\
&=\int_{t_1}^{t_2}\!\!\int_{\partial\Omega}\partial_\nu u_\kappa(t)\,\dot u_\kappa(t)\,d\mathcal{H}^1\,dt-\int_{t_1}^{t_2}\!\!\int_{\partial E^\varepsilon_{ct}}\Big(\partial_\nu u_\kappa(t)\,\dot u_\kappa(t)+\frac{c}{2}|\dot u_\kappa(t)|^2\nu_1+\frac{c}{2}|\widehat\nabla u_\kappa(t)|^2\nu_1\Big)d\mathcal{H}^1dt\,.
\end{aligned}$$

Note that by the definition of u_κ

$$\int_{E^\varepsilon_{ct_1}}(|\dot u_\kappa(t_1)|^2+|\widehat\nabla u_\kappa(t_1)|^2)dx=\int_{E^\varepsilon_{ct_2}}(|\dot u_\kappa(t_2)|^2+|\widehat\nabla u_\kappa(t_2)|^2)dx\,.$$

Adding these terms to the previous equality we obtain

$$\begin{aligned}
&\frac{1}{2}\|\widehat\nabla u_\kappa(t_2)\|^2+\frac{1}{2}\|\dot u_\kappa(t_2)\|^2-\frac{1}{2}\|\widehat\nabla u_\kappa(t_1)\|^2-\frac{1}{2}\|\dot u_\kappa(t_1)\|^2\\
&=\int_{t_1}^{t_2}\!\!\int_{\partial\Omega}\partial_\nu u_\kappa(t)\,\dot u_\kappa(t)\,d\mathcal{H}^1\,dt-\int_{t_1}^{t_2}\!\!\int_{\partial E^\varepsilon_{ct}}\Big(\partial_\nu u_\kappa(t)\,\dot u_\kappa(t)+\frac{c}{2}|\dot u_\kappa(t)|^2\nu_1+\frac{c}{2}|\widehat\nabla u_\kappa(t)|^2\nu_1\Big)d\mathcal{H}^1dt\,.
\end{aligned}$$

Therefore, the energy-dissipation balance (4.3) is satisfied provided that

$$c(t_2-t_1)=\int_{t_1}^{t_2}\int_{\partial E^\varepsilon_{ct}}\Big(\partial_\nu u_\kappa(t)\,\dot u_\kappa(t)+\frac{c}{2}|\dot u_\kappa(t)|^2\nu_1+\frac{c}{2}|\widehat\nabla u_\kappa(t)|^2\nu_1\Big)\,d\mathcal{H}^1\,dt\,. \tag{4.4}$$

Using (4.2), we express the integrands in terms of the function ψ defined in (4.1), and we obtain

$$\begin{aligned}
I&:=\int_{\partial E^\varepsilon_{ct}}\Big(\partial_\nu u_\kappa(t)\,\dot u_\kappa(t)+\frac{c}{2}|\dot u_\kappa(t)|^2\nu_1+\frac{c}{2}|\widehat\nabla u_\kappa(t)|^2\nu_1\Big)\,d\mathcal{H}^1\\
&=\kappa^2\int_{\partial E^\varepsilon_{ct}}\Big[\Big(\frac{1}{\sqrt{1-c^2}}\partial_1\psi\Big(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\Big)\nu_1+\partial_2\psi\Big(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\Big)\nu_2\Big)\frac{-c}{\sqrt{1-c^2}}\partial_1\psi\Big(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\Big)\\
&\qquad+\frac{c}{2}\left(\frac{1}{1-c^2}\Big(\partial_1\psi\Big(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\Big)\Big)^2+\Big(\partial_2\psi\Big(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\Big)\Big)^2\right)\nu_1\\
&\qquad+\frac{c}{2}\frac{c^2}{1-c^2}\Big(\partial_1\psi\Big(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\Big)\Big)^2\nu_1\Big]\,d\mathcal{H}^1\\
&=c\kappa^2\int_{\partial E^\varepsilon_{ct}}\Big[-\frac{1}{2}\Big(\partial_1\psi\Big(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\Big)\Big)^2\nu_1+\frac{1}{2}\Big(\partial_2\psi\Big(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\Big)\Big)^2\nu_1\\
&\qquad+\frac{-1}{\sqrt{1-c^2}}\partial_1\psi\Big(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\Big)\partial_2\psi\Big(\frac{x_1-ct}{\sqrt{1-c^2}},x_2\Big)\nu_2\Big]\,d\mathcal{H}^1
\end{aligned}$$

We now parametrize $\partial E^\varepsilon_{ct}$ in order to compute the integrals:

$$x_1-ct=\varepsilon\sqrt{1-c^2}\cos\theta\,,\quad x_2=\varepsilon\sin\theta\,,\quad \theta\in(-\pi,\pi)\,.$$

Then the outer unit normal and the length element are given by

$$\nu(x)=\frac{(\cos\theta,\sqrt{1-c^2}\sin\theta)}{\sqrt{(1-c^2)\sin^2\theta+\cos^2\theta}}\quad\text{and}\quad d\mathcal{H}^1=\sqrt{(1-c^2)\sin^2\theta+\cos^2\theta}\,d\theta\,.$$

Hence

$$\begin{aligned} I &= c\kappa^2 \int_{-\pi}^{\pi} \varepsilon \Big[-\frac{1}{2}\Big(\partial_1\psi(\varepsilon\cos\theta,\varepsilon\sin\theta)\Big)^2\cos\theta + \frac{1}{2}\Big(\partial_2\psi(\varepsilon\cos\theta,\varepsilon\sin\theta)\Big)^2\cos\theta \\ &\qquad + \frac{-1}{\sqrt{1-c^2}}\partial_1\psi(\varepsilon\cos\theta,\varepsilon\sin\theta)\partial_2\psi(\varepsilon\cos\theta,\varepsilon\sin\theta)\sqrt{1-c^2}\sin\theta\Big]\,d\theta \\ &= \frac{c}{2}\kappa^2 \int_{-\pi}^{\pi} \Big[-\Big(\partial_1\psi(\cos\theta,\sin\theta)\Big)^2\cos\theta + \Big(\partial_2\psi(\cos\theta,\sin\theta)\Big)^2\cos\theta \\ &\qquad -2\partial_1\psi(\cos\theta,\sin\theta)\partial_2\psi(\cos\theta,\sin\theta)\sin\theta\Big]d\theta\,, \end{aligned}$$

where we have used the fact that $\partial_1\psi$ and $\partial_2\psi$ are positively homogeneous of degree $-1/2$ by (4.1). Substituting in (4.4) we deduce that the energy-dissipation balance (4.3) is satisfied provided that

$$\begin{aligned} 1 = \frac{\kappa^2}{2} \int_{-\pi}^{\pi} \Big[&-\Big(\partial_1\psi(\cos\theta,\sin\theta)\Big)^2\cos\theta + \Big(\partial_2\psi(\cos\theta,\sin\theta)\Big)^2\cos\theta \\ &-2\partial_1\psi(\cos\theta,\sin\theta)\partial_2\psi(\cos\theta,\sin\theta)\sin\theta\Big]d\theta\,. \end{aligned}$$

This shows that κ does not depend on the crack speed c (nor on t_1, t_2, and ε, which can be deduced also by simpler arguments).

Using the explicit expression (4.1) of the function ψ we obtain that

$$\partial_1\psi(\cos\theta,\sin\theta) = -\frac{\operatorname{sign}\theta}{2\sqrt{2}}\sqrt{1-\cos\theta} \qquad \text{and} \qquad \partial_2\psi(\cos\theta,\sin\theta) = \frac{1}{2\sqrt{2}}\sqrt{1+\cos\theta}\,.$$

Hence the energy-dissipation balance (4.3) is satisfied if

$$1 = \frac{\kappa^2}{2} \int_{-\pi}^{\pi} \Big[-\frac{1}{8}(1-\cos\theta)\cos\theta + \frac{1}{8}(1+\cos\theta)\cos\theta + \frac{2}{8}\sin^2\theta\Big]d\theta = \frac{\kappa^2}{2}\frac{\pi}{2}\,,$$

and we obtain that in this case

$$\kappa^2 = \frac{4}{\pi}\,. \tag{4.5}$$

We remark that this result does not agree with formula (81) in [15]. The discrepancy can be explained by the fact that we multiply equation (2.8) by $\dot u_\kappa(t)$ and integrate by parts only in Ω^ε_{ct}, *i.e.*, far from the singularity of $u_\kappa(t)$, while in [15, Proof of Lemma 5.5] this is done on Ω_{ct}, where the product $\nabla\dot u_\kappa(t)$ is not in L^2 and $\nabla u_\kappa(t)\nabla\dot u_\kappa(t)$ is not integrable.

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