Twisted Covariance vs Weyl Quantisation

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Abstract

In this letter we wish to clarify in which sense the tensor nature of the commutation relations

$$[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu}$$

underlying Minkowski spacetime quantisation cannot be suppressed even in the twisted approach to Lorentz covariance. We then address the *vexata quaestio:* "why θ?"

1 Introduction

We consider spacetime quantisation induced in a specific reference frame by commutation relations of the form

$$[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu}$$

among the coordinates, for some fixed arbitrary choice of the real antisymmetric matrix θ. Ordinary functions of the classical Minkowski spacetime are quantised according to a formally covariant version of the Weyl prescription, as suggested in a more general (and fully covariant) context by [1]. According to a remark of [2], the associated twisted product is form–invariant under a correspondingly twisted action of Poincaré covariance. This is usually interpreted as a fundamental breakdown of Lorentz covariance, embodied by the asserted non tensor character of the invariant matrix θ.

In the next section we will instead unveil the hidden tensor character of θ, which will necessarily emerge from the interplay between twisted covariance and Weyl quantisation. The simple argument will rely on the assumption that all the observers adopt some *a priori* unspecified Weyl quantisation as their quantisation prescription.

As a consequence of this remark, the twisted covariant approach can be recognised (see [3] for more details) as essentially equivalent to superposing a non invariant constrain on the fully covariant DFR model [1], determined by an arbitrary choice of the tensor θ in an arbitrary frame of reference. In the last section we will comment on the implications of this choice.

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2 Covariance

Fix an observer, say Jim, in his own frame of reference, once and for all. Given (in that frame) the commutation relations

$$[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu},$$

the Weyl quantisation

$$W(f) = \int dk \, \check{f}(k) e^{ik_\mu \mathbf{x}^\mu}$$

(where $\check{f}(k) = (2\pi)^{-4} \int dx \, f(x) e^{-ik_\mu x^\mu}$) induces a symbolic calculus

$$W(f)W(g) = W(f \star_\theta g)$$

in terms of a twisted product which may be written in the form

$$f \star_\theta g = m_\theta(f \otimes g) = m(F_\theta f \otimes g),$$

where m is the ordinary pointwise multiplication (= restriction to the diagonal set $x = y$), and

$$(F_\theta f \otimes g)(x, y) =$$
$$= \frac{2}{(2\pi)^4 |\det \theta|} \iint da\, db\, f(x + a) g(y + b) e^{2ia_\mu (\theta^{-1})^{\mu\nu} b_\nu}$$

fulfils $F_\theta^{-1} = F_{-\theta}$; a degenerate θ would require some *proviso*; see [3]. $\mathcal{A}_\theta$ will denote the Weyl algebra of symbols.

Let $(\alpha(L)f)(x) = f(L^{-1}x)$ be the action of the Poincaré group on $\mathcal{A}_\theta$, where $L = (\Lambda, a)$. According to a remark of [2], we may deform the ordinary coproduct in the sense of Drinfeld twists, thus obtaining a twisted action of the Poincarè transformations

$$\alpha_\theta^{(2)}(L) = F_\theta^{-1}(\alpha(L) \otimes \alpha(L))F_\theta$$

on $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$, which is compatible with the twisted product in the sense that

$$m_\theta(\alpha_\theta^{(2)}(L)f \otimes g) = \alpha(L)m_\theta(f \otimes g); \tag{1}$$

the above compatibility condition is called twisted covariance.

This fact is commonly interpreted as a fundamental breakdown of ordinary Lorentz covariance, responsible of which should be the asserted non tensor character of the matrix θ.

Already at a formal level, this view is at least questionable: with

$$\theta'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\tau \theta^{\rho\tau}; \tag{2}$$

then the straightforward commutation rule

$$(\alpha(L) \otimes \alpha(L))F_\theta = F_{\theta'}(\alpha(L) \otimes \alpha(L))$$

entails that

$$m_\theta(\alpha_\theta^{(2)}(L)f \otimes g) \equiv (\alpha(L)f) \star_{\theta'} (\alpha(L)g),$$

where of course $\star_{\theta'}$ is the product twisted with the transformed matrix. The above identity is then identical to the more appealing

$$(f \star_\theta g)' \equiv f' \star_{\theta'} g',$$

where primed functions are obtained by means of ordinary Poincaré action.

In other words, the formalism of twisted covariance is completely equivalent to the formalism where the Poincaré action is untwisted, and the matrix θ is treated as a tensor.

Since however the relativity principle is *a priori* broken by the choice of a particular θ in a particular frame, formal covariance is void of meaning and cannot be taken alone as a guidance. In order to decide which of the two formalisms is more tailored to the conceptual framework, we need to rely on the physical interpretation of $i\theta$ as the commutator of the coordinates, and ask ourselves which commutation rules are observed in a different frame.

We then adopt the point of view of twisted covariance and assume that θ is invariant (not a tensor). Consider another observer — Jane — in the reference frame connected to Jim's by L. Jane also is doing physics and she writes down her own Weyl quantisation $W'(f) = \int dk \, \check{f}(k) e^{ik\mathbf{x}'}$ in terms of the coordinates $\mathbf{x}'$ in her frame. We make no *a priori* assumptions on the commutation relations for $\mathbf{x}'$.

Now we use twisted covariance (1): we must have

$$W'(m_\theta(\alpha_\theta^{(2)}(L)f \otimes g) = W'(\alpha(L)f)W'(\alpha(L)g).$$

We compute

$$W'(m_\theta^{(2)}(\alpha_\theta^{(2)}(L)f \otimes g)) =$$
$$= \iint dh dk \; e^{i(k+h)_\mu(\mathbf{x}'-a)^\mu} e^{-\frac{i}{2}(\varLambda^{-1}h)_\mu \theta^{\mu\nu}(\varLambda^{-1}k')_\nu}$$
$$\check{f}(\varLambda^{-1}h)\check{g}(\varLambda^{-1}(k')),$$
$$W'(\alpha^{(1)}(L)f)W'(\alpha^{(1)}(L)g) =$$
$$= \iint dh dk \; e^{-i(h+k)_\mu a^\mu} e^{ih_\mu \mathbf{x}'^\mu} e^{ik_\mu \mathbf{x}'^\mu} \check{f}(\varLambda^{-1}h)\check{g}(\varLambda^{-1}k),$$

from which (and the arbitrarity of f, g) the Weyl relations for the $\mathbf{x}'^\mu$'s are immediately recovered:

$$e^{ih_\mu \mathbf{x}'^\mu} e^{ik_\mu \mathbf{x}'^\mu} = e^{-\frac{i}{2}h_\mu \theta'^{\mu\nu}k_\nu} e^{i(h+k)_\mu \mathbf{x}'^\nu}, \tag{3}$$

where θ' is given by (2).

Equation (3) is the Weyl form of the relations

$$[\mathbf{x}'^\mu, \mathbf{x}'^\nu] = i\theta'^{\mu\nu}.$$

The intrinsic tensor nature of $i\theta$ as the commutator of the coordinates is then established in the framework of twisted covariance. This speaks in favour of the formalism of covariant twisted products and untwisted Poincaré actions, which is simpler to deal with than twisted covariance. Indeed, there is no evident reason why unprivileged observers should prefer to use different matrices for the commutation relations of the coordinates and for the associated twisted product.

3 Why θ?

Let us first clarify the issues at hand by means of an elementary example (the "Newtonian example", in what follows). Consider the Newton laws, together with Galilei covariance and the relativity principle. A non covariant modification of the theory could be obtained by complementing the three Newton laws with a criterion for *a priori* selecting a non invariant set of solutions. This of course should be done by assigning a rejection rule in some specific reference frame (Jim's frame, to fix ideas). In this case the equations would be formally covariant: a different observer (Jane, say) agreeing with Jim's choice would translate the constrain on the allowed solutions in her own coordinates. For example, if Jim discards the solutions $(x(t), y(t), z(t))$ such that $z(0) < 0$, and if Jane's frame is rotated by $180°$ w.r.t. Jim's, then Jane would discard solutions with $z'(0) > 0$ in her coordinates x', y', z'. While such a choice would be perfectly acceptable if motivated by contingent reasons external to the general theory (e.g. interest in some specific problem), the promotion of such a selection criterion to a new fundamental law of mechanics would be highly questionable, since it would severely break the relativity principle; non invariant constrains are very bad candidates to be general laws. For example, it would allow to give an absolute criterion for classifying the observers. The first question one should ask would be: "What's wrong with the discarded solutions?"

If one agrees on treating properly the Weyl quantisation in all reference frames, the situation now resembles our Newtonian example: the formalism is essentially covariant, but we may classify the observers according to which θ' they see. Within a fully covariant theory, instead, all transformed $\Lambda\theta\Lambda^t$ should be available at once together with θ to each observer, and in particular to Jim (precisely like all initial positions of motions should be available to all observers in the Newtonian example). The latter is precisely the point of view adopted by [1].

Indeed, by carefully rephrasing our simple remark, it is possible to show [3] that, if Weyl quantisation is treated properly and θ fulfils the DFR stability condition, the formalism of twisted covariance is equivalent to the fully covariant DFR formalism [1] up to discarding a huge, non invariant set of admissible localisation states: only localisation states ω which are pure on the centre of the DFR algebra and such that $\omega(-i[\mathbf{x}^\mu, \mathbf{x}^\nu]) = \theta^{\mu\nu}$ in Jim's frame are allowed for.

We are thus facing a precise analogue of our non covariant modification of the Newtonian example. Hence we ask the natural question: "What's wrong with the discarded localisation states?"

References

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