# PIMSNER ALGEBRAS AND GYSIN SEQUENCES FROM PRINCIPAL CIRCLE ACTIONS

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Abstract. A self Morita equivalence over an algebra $B$, given by a $B$-bimodule $E$, is thought of as a line bundle over $B$. The corresponding Pimsner algebra $\mathcal{O}_E$ is then the total space algebra of a noncommutative principal circle bundle over $B$. A natural Gysin-like sequence relates the $KK$-theories of $\mathcal{O}_E$ and of $B$. Interesting examples come from $\mathcal{O}_E$ a quantum lens space over $B$ a quantum weighted projective line (with arbitrary weights). The $KK$-theory of these spaces is explicitly computed and natural generators are exhibited.

Contents



*Date*: v1 September 2014; v2 February 2015.

2010 *Mathematics Subject Classification.* 19K35, 55R25, 46L08, 58B32.

*Key words and phrases.* KK-theory, Pimsner algebras, Gysin sequences, circle actions, quantum principal bundles, quantum lens spaces, quantum weighted projective spaces.

*Thanks.* GL was partially supported by the Italian Project "Prin 2010-11 – Operator Algebras, Noncommutative Geometry and Applications".

## 1. Introduction

In the present paper we put in close relation two notions that seem to have touched each other only occasionally in the recent literature. These are the notion of a Pimsner (or Cuntz-Krieger-Pimsner) algebra on one hand and that of a noncommutative (in general) principal circle bundle on the other.

At the $C^*$-algebraic level one needs a self Morita equivalence of a $C^*$-algebra $B$, thus we look at a full Hilbert $C^*$-module $E$ over $B$ together with an isomorphism of $B$ with the compacts on $E$. Through a natural universal construction this data gives rise to a $C^*$-algebra, the *Pimsner algebra* $\mathcal{O}_E$ generated by $E$. In the case where both $E$ and its Hilbert $C^*$-module dual $E^*$ are finitely generated projective over $B$ one obtains that the $*$-subalgebra generated by the elements of $E$ and $B$ becomes the total space of a noncommutative principal circle bundle with base space $B$.

At the purely algebraic level we start from a $\mathbb{Z}$-graded $*$-algebra $\mathscr{A}$ which forms the total space of a *quantum principal circle bundle* with base space the $*$-subalgebra of invariant elements $\mathscr{A}_{(0)}$ and with a coaction of the Hopf algebra $\mathcal{O}(U(1))$ coming from the $\mathbb{Z}$-grading. Provided that $\mathscr{A}$ comes equipped with a $C^*$-norm, which is compatible with the circle action likewise defined by the $\mathbb{Z}$-grading, we show that the closure of $\mathscr{A}$ has the structure of a Pimsner algebra. Indeed, the first spectral subspace $\mathscr{A}_{(1)}$ is then finitely generated and projective over the algebra $\mathscr{A}_{(0)}$. The closure $E$ of $\mathscr{A}_{(1)}$ will become a Hilbert $C^*$-module over $B$, the closure of $\mathscr{A}_{(0)}$, and the couple $(E, B)$ will lend itself to a Pimsner algebra construction.

The commutative version of this part of our program was spelled out in [11, Prop. 5.8]. This amounts to showing that the continuous functions on the total space of a (compact) principal circle bundle can be described as a Pimsner algebra generated by a classical line bundle over the compact base space.

With a Pimsner algebra there come two natural six term exact sequences in $KK$-theory, which relate the $KK$-theories of the Pimsner algebra $\mathcal{O}_E$ with that of the $C^*$-algebra of (the base space) scalars $B$. The corresponding sequences in $K$-theory are noncommutative analogues of the Gysin sequence which in the commutative case relates the $K$-theories of the total space and of the base space. The classical cup product with the Euler-class is in the noncommutative setting replaced by a Kasparov product with the identity minus the generating Hilbert $C^*$-module $E$. Predecessors of these six term exact sequences are the Pimsner-Voiculescu six term exact sequences of [19] for crossed products by the integers.

Interesting examples are quantum lens spaces over quantum weighted projective lines. The latter spaces $W_q(k,l)$ are defined as fixed points of weighted circle actions on the quantum 3-sphere $S^3_q$. On the other hand, quantum lens spaces $L_q(dlk; k, l)$ are fixed points for the action of a finite cyclic group on $S^3_q$. For general $(k,l)$ coprime positive integers and any positive integer $d$, the coordinate algebra of the lens space is a quantum principal circle bundle over the corresponding coordinate algebra for the quantum weighted projective space, thus generalizing the cases studied in [5].

At the $C^*$-algebra level the lens spaces are given as Pimsner algebras over the $C^*$-algebra of the continuous functions over the weighted projective spaces (see §6).

Using the associated exact sequences coming from the construction of [18], we explicitly compute in §7 the $KK$-theory of these spaces for general weights. A central character in this computation is played by an integer matrix whose entries are index pairings. These are in turn computed by pairing the corresponding Chern-Connes characters in cyclic theory. The computation of the $KK$-theory of our class of $q$-deformed lens spaces is, to the best of our knowledge, a novel one. Also, it is worth emphasizing that the quantum lens spaces and weighted projective spaces are in general not $KK$-equivalent to their commutative counterparts.

Pimsner algebras were introduced in [18]. This notion gives a unifying framework for a range of important $C^*$-algebras including crossed products by the integers, Cuntz-Krieger algebras [9, 8], and $C^*$-algebras associated to partial automorphisms [10]. Generalized crossed products, a notion which is somewhat easier to handle, were independently invented in [3]. More recently, Katsura has constructed Pimsner algebras for general $C^*$-correspondences [15]. In the present paper we work in a simplified setting (see Assumption 2.1 below) which is close to the one of [3].

*Acknowledgments.* We are very grateful to Georges Skandalis for many suggestions and to Ralf Meyer for useful discussions. We thank Tomasz Brzeziński for making us aware of the reference [17]. This paper was finished at the Hausdorff Research Institute for Mathematics in Bonn during the 2014 Trimester Program "Non-commutative Geometry and its Applications". We thank the organizers of the Program for the kind invitation and all people at HIM for the nice hospitality.

## 2. Pimsner algebras

We start by reviewing the construction of Pimsner algebras associated to Hilbert $C^*$-modules as given in [18]. Rather than the full fledged generality we aim at a somewhat simplified version adapted to the context of the present paper, and motivated by our geometric intuition coming from principal circle bundles.

Our reference for the theory of Hilbert $C^*$-modules is [16]. Throughout this section $E$ will be a countably generated (right) Hilbert $C^*$-module over a separable $C^*$-algebra $B$, with $B$-valued (and right $B$-linear) inner product denoted $\langle \cdot, \cdot \rangle_B$; or simply $\langle \cdot, \cdot \rangle$ to lighten notations. Also, $E$ is taken to be full, that is the ideal $\langle E, E \rangle := \mathrm{span}_{\mathbb{C}} \big\{ \langle \xi, \eta \rangle \,|\, \xi, \eta \in E \big\}$ is dense in $B$.

Given two Hilbert $C^*$-modules $E$ and $F$ over the same algebra $B$, we denote by $\mathscr{L}(E, F)$ the space of bounded *adjointable* homomorphisms $T : E \to F$. For each of these there exists a homomorphism $T^* : F \to E$ (the adjoint) with the property that $\langle T^* \xi, \eta \rangle = \langle \xi, T \eta \rangle$ for any $\xi \in F$ and $\eta \in E$. Given any pair $\xi \in F, \eta \in E$, an adjointable operator $\theta_{\xi,\eta} : E \to F$ is defined by

$$\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle \,, \quad \forall \, \zeta \in E \,.$$

The closed linear subspace of $\mathscr{L}(E, F)$ spanned by elements of the form $\theta_{\xi,\eta}$ as above is denoted $\mathscr{K}(E, F)$, the space of compact homomorphisms. When $E = F$, it results that $\mathscr{L}(E) := \mathscr{L}(E, E)$ is a $C^*$-algebra with $\mathscr{K}(E) := \mathscr{K}(E, E) \subseteq \mathscr{L}(E)$ the (sub) $C^*$-algebra of compact endomorphisms of $E$.

2.1. **The algebras and their universal properties.** On top of the above basic conditions, the following will remain in effect as well:

**Assumption 2.1.** *There is a $*$-homomorphism $\phi : B \to \mathscr{L}(E)$ which induces an isomorphism $\phi : B \to \mathscr{K}(E)$.*

Next, let $E^*$ be the dual of $E$ (when viewed as a Hilbert $C^*$-module):

$$E^* := \big\{ \phi \in \mathrm{Hom}_B(E, B) \,|\, \exists\, \xi \in E \text{ with } \phi(\eta) = \langle \xi, \eta \rangle \ \forall \eta \in E \big\} .$$

Thus, with $\xi \in E$, if $\lambda_\xi : E \to B$ is the operator defined by $\lambda_\xi(\eta) = \langle \xi, \eta \rangle$, for all $\eta \in E$, every element of $E^*$ is of the form $\lambda_\xi$ for some $\xi \in E$. By its definition, $E^* := \mathscr{K}(E, B)$. The dual $E^*$ can be given the structure of a (right) Hilbert $C^*$-module over $B$. Firstly, the right action of $B$ on $E^*$ is given by

$$\lambda_\xi \, b := \lambda_\xi \circ \phi(b) .$$

Then, with operator $\theta_{\xi,\eta} \in \mathscr{K}(E)$ for $\xi, \eta \in E$, the inner product on $E^*$ is given by

$$\langle \lambda_\xi, \lambda_\eta \rangle := \phi^{-1}(\theta_{\xi,\eta}) ,$$

and $E^*$ is full as well. With the $*$-homomorphism $\phi^* : B \to \mathscr{L}(E^*)$ defined by $\phi^*(b)(\lambda_\xi) := \lambda_{\xi \cdot b^*}$, the pair $(\phi^*, E^*)$ satisfies the conditions in Assumption 2.1.

We need the interior tensor product $E \widehat{\otimes}_\phi E$ of $E$ with itself over $B$. As a first step, one constructs the quotient of the vector space tensor product $E \otimes_{\mathrm{alg}} E$ by the ideal generated by elements of the form

$$\xi b \otimes \eta - \xi \otimes \phi(b)\eta , \qquad \text{for} \quad \xi, \eta \in E , \quad b \in B . \tag{2.1}$$

There is a natural structure of right module over $B$ with the action given by

$$(\xi \otimes \eta) b = \xi \otimes (\eta b) , \qquad \text{for} \quad \xi, \eta \in E , \quad b \in B ,$$

and a $B$-valued inner product given, on simple tensors, by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \phi(\langle \xi_1, \xi_2 \rangle) \eta_2 \rangle \tag{2.2}$$

and extended by linearity. The inner product is well defined and has all required properties; in particular, the null space $N = \{\zeta \in E \otimes_{\mathrm{alg}} E \,;\, \langle \zeta, \zeta \rangle = 0\}$ is shown to coincide with the subspace generated by elements of the form in (2.1). One takes $E \otimes_\phi E := E \otimes_{\mathrm{alg}} E/N$ and defines $E \widehat{\otimes}_\phi E$ to be the Hilbert module obtained by completing with respect to the norm induced by (2.2). The construction can be iterated and, for $n > 0$, we denote by $E^{\widehat{\otimes}_\phi n}$, the $n$-fold interior tensor power of $E$ over $B$. Like-wise, $(E^*)^{\widehat{\otimes}_{\phi^*} n}$ denotes the $n$-fold interior tensor power of $E^*$ over $B$.

To lighten notation, in the following we define, for each $n \in \mathbb{Z}$, the modules

$$E^{(n)} := \begin{cases} E^{\widehat{\otimes}_\phi n} & n > 0 \\ B & n = 0 \\ (E^*)^{\widehat{\otimes}_{\phi^*}(-n)} & n < 0 \end{cases} .$$

Clearly, $E^{(1)} = E$ and $E^{(-1)} = E^*$. We define the Hilbert $C^*$-module over $B$:

$$E_\infty := \bigoplus_{n \in \mathbb{Z}} E^{(n)} .$$

For each $\xi \in E$ we have a bounded adjointable operator $S_\xi : E_\infty \to E_\infty$ defined component-wise by

$$\begin{aligned} S_\xi(b) &:= \xi \cdot b\,, && b \in B\,, \\ S_\xi(\xi_1 \otimes \cdots \otimes \xi_n) &:= \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n\,, && n > 0\,, \\ S_\xi(\lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_{-n}}) &:= \lambda_{\xi_2 \cdot \phi^{-1}(\theta_{\xi_1,\xi})} \otimes \lambda_{\xi_3} \otimes \cdots \otimes \lambda_{\xi_{-n}}\,, && n < 0\,. \end{aligned}$$

In particular, $S_\xi(\lambda_{\xi_1}) = \phi^{-1}(\theta_{\xi,\xi_1}) \in B$.

The adjoint of $S_\xi$ is easily found to be given by $S_{\lambda_\xi} := S_\xi^* : E_\infty \to E_\infty$:

$$\begin{aligned} S_{\lambda_\xi}(b) &:= \lambda_\xi \cdot b\,, && b \in B\,, \\ S_{\lambda_\xi}(\xi_1 \otimes \ldots \otimes \xi_n) &:= \phi(\langle \xi, \xi_1 \rangle)(\xi_2) \otimes \xi_3 \otimes \ldots \otimes \xi_n\,, && n > 0\,, \\ S_{\lambda_\xi}(\lambda_{\xi_1} \otimes \ldots \otimes \lambda_{\xi_{-n}}) &:= \lambda_\xi \otimes \lambda_{\xi_1} \otimes \ldots \otimes \lambda_{\xi_{-n}}\,, && n < 0\,; \end{aligned}$$

and in particular $S_{\lambda_\xi}(\xi_1) = \langle \xi, \xi_1 \rangle \in B$.

From its definition, each $E^{(n)}$ has a natural structure of Hilbert $C^*$-module over $B$ and, with $\mathscr{K}$ again denoting the Hilbert $C^*$-module compacts, we have isomorphisms

$$\mathscr{K}(E^{(n)}, E^{(m)}) \simeq E^{(m-n)}\,.$$

**Definition 2.2.** The *Pimsner algebra* of the pair $(\phi, E)$ is the smallest $C^*$-subalgebra of $\mathscr{L}(E_\infty)$ which contains the operators $S_\xi : E_\infty \to E_\infty$ for all $\xi \in E$. The Pimsner algebra is denoted by $\mathcal{O}_E$ with inclusion $\widetilde{\phi} : \mathcal{O}_E \to \mathscr{L}(E_\infty)$.

There is an injective $*$-homomorphism $i : B \to \mathcal{O}_E$. This is induced by the injective $*$-homomorphism $\phi : B \to \mathscr{L}(E_\infty)$ defined component-wise by

$$\begin{aligned} \phi(b)(b') &:= b \cdot b'\,, \\ \phi(b)(\xi_1 \otimes \ldots \otimes \xi_n) &:= \phi(b)(\xi_1) \otimes \xi_2 \otimes \ldots \otimes \xi_n\,, \\ \phi(b)(\lambda_{\xi_1} \otimes \ldots \otimes \lambda_{\xi_n}) &:= \phi^*(b)(\lambda_{\xi_1}) \otimes \lambda_{\xi_2} \otimes \ldots \otimes \lambda_{\xi_n} = \lambda_{\xi_1 \cdot b^*} \otimes \lambda_{\xi_2} \otimes \ldots \otimes \lambda_{\xi_n}\,, \end{aligned}$$

and which factorizes through the Pimsner algebra $\mathcal{O}_E \subseteq \mathscr{L}(E_\infty)$. Indeed, for all $\xi, \eta \in E$ it holds that $S_\xi S_\eta^* = i(\phi^{-1}(\theta_{\xi,\eta}))$, that is the operator $S_\xi S_\eta^*$ on $E_\infty$ is right-multiplication by the element $\phi^{-1}(\theta_{\xi,\eta}) \in B$.

A Pimsner algebra is universal in the following sense [18, Thm. 3.12]:

**Theorem 2.3.** *Let $C$ be a $C^*$-algebra and let $\sigma : B \to C$ be a $*$-homomorphism. Suppose that there exist elements $s_\xi \in C$ for all $\xi \in E$ such that*

(1) $\alpha s_\xi + \beta s_\eta = s_{\alpha\xi + \beta\eta}$ *for all* $\alpha, \beta \in \mathbb{C}$ *and* $\xi, \eta \in E$ ,

(2) $s_\xi \sigma(b) = s_{\xi b}$ *and* $\sigma(b) s_\xi = s_{\phi(b)(\xi)}$ *for all* $\xi \in E$ *and* $b \in B$ ,

(3) $s_\xi^* s_\eta = \sigma(\langle \xi, \eta \rangle)$ *for all* $\xi, \eta \in E$ ,

(4) $s_\xi s_\eta^* = \sigma\big(\phi^{-1}(\theta_{\xi,\eta})\big)$ *for all* $\xi, \eta \in E$ .

*Then there is a unique $*$-homomorphism $\widetilde{\sigma} : \mathcal{O}_E \to C$ with $\widetilde{\sigma}(S_\xi) = s_\xi$ for all $\xi \in E$.*

Also, in the context of this theorem the identity $\widetilde{\sigma} \circ i = \sigma$ follows automatically.

**Remark 2.4.** In the paper [18], the pair $(\phi, E)$ was referred to as a *Hilbert bimodule*, since the map $\phi$ (taken to be injective there) naturally endows the right Hilbert module $E$ with a left module structure. As mentioned, our Assumption 2.1 simplifies the construction to a great extent (see also [3]). For the pair $(\phi, E)$ with a general $*$-homomorphism $\phi : B \to \mathscr{L}(E)$, (in particular, a non necessarily injective one), the name $C^*$-*correspondence* over $B$ has recently emerged as a more common one, reserving the terminology Hilbert bimodule to the more restrictive case where one has both a left and a right inner product satisfying an extra compatibility relation.

2.2. **Six term exact sequences.** With a Pimsner algebra there come two six term exact sequences in $KK$-theory. Firstly, since $\phi : B \to \mathscr{L}(E)$ factorizes through the compacts $\mathscr{K}(E) \subseteq \mathscr{L}(E)$, the following class is well defined.

**Definition 2.5.** The class in $KK_0(B, B)$ defined by the even Kasparov module $(E, \phi, 0)$ (with trivial grading) will be denoted by $[E]$.

Next, let $P : E_\infty \to E_\infty$ denote the orthogonal projection with
$$\mathrm{Im}(P) = \big( \oplus_{n=1}^\infty E^{(n)} \big) \oplus B \subseteq E_\infty \, .$$
Notice that $[P, S_\xi] \in \mathscr{K}(E_\infty)$ for all $\xi \in E$ and thus $[P, S] \in \mathscr{K}(E_\infty)$ for all $S \in \mathcal{O}_E$.

Then, let $F := 2P - 1 \in \mathscr{L}(E_\infty)$ and recall that $\widetilde{\phi} : \mathcal{O}_E \to \mathscr{L}(E_\infty)$ is the inclusion.

**Definition 2.6.** The class in $KK_1(\mathcal{O}_E, B)$ defined by the odd Kasparov module $(E_\infty, \widetilde{\phi}, F)$ will be denoted by $[\partial]$.

For any separable $C^*$-algebra $C$ we then have the group homomorphisms
$$[E] : KK_*(B, C) \to KK_*(B, C) \, , \qquad [E] : KK_*(C, B) \to KK_*(C, B)$$
and
$$[\partial] : KK_*(C, \mathcal{O}_E) \to KK_{*+1}(C, B) \, , \qquad [\partial] : KK_*(B, C) \to KK_{*+1}(\mathcal{O}_E, C) \, ,$$
which are induced by the Kasparov product.

The six term exact sequences in $KK$-theory given in the following theorem were constructed by Pimsner, see [18, Thm. 4.8].

**Theorem 2.7.** *Let $\mathcal{O}_E$ be the Pimsner algebra of the pair $(\phi, E)$ over the $C^*$-algebra $B$. If $C$ is any separable $C^*$-algebra, there are two exact sequences:*
$$\begin{array}{ccccc}
KK_0(C,B) & \xrightarrow{1-[E]} & KK_0(C,B) & \xrightarrow{i_*} & KK_0(C,\mathcal{O}_E) \\
{\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\
KK_1(C,\mathcal{O}_E) & \xleftarrow[i_*]{} & KK_1(C,B) & \xleftarrow[1-[E]]{} & KK_1(C,B)
\end{array}$$
*and*
$$\begin{array}{ccccc}
KK_0(B,C) & \xleftarrow[1-[E]]{} & KK_0(B,C) & \xleftarrow[i^*]{} & KK_0(\mathcal{O}_E,C) \\
\downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\uparrow \\
KK_1(\mathcal{O}_E,C) & \xrightarrow{i^*} & KK_1(B,C) & \xrightarrow{1-[E]} & KK_1(B,C)
\end{array}$$

*with $i^*$, $i_*$ the homomorphisms in $KK$-theory induced by the inclusion $i : B \to \mathcal{O}_E$.*

**Remark 2.8.** For $C = \mathbb{C}$, the first sequence above reduces to

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{1-[E]} & K_0(B) & \xrightarrow{i_*} & K_0(\mathcal{O}_E) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ K_1(\mathcal{O}_E) & \xleftarrow[i_*]{} & K_1(B) & \xleftarrow[1-[E]]{} & K_1(B) \end{array} .$$

This could be considered as a generalization of the classical *Gysin sequence* in $K$-theory (see [14, IV.1.13]) for the 'line bundle' $E$ over the 'noncommutative space' $B$ and with the map $1 - [E]$ having the role of the *Euler class* $\chi(E) := 1 - [E]$ of the line bundle $E$. The second sequence would then be an analogue in $K$-homology:

$$\begin{array}{ccccc} K^0(B) & \xleftarrow[1-[E]]{} & K^0(B) & \xleftarrow[i^*]{} & K^0(\mathcal{O}_E) \\ \downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\uparrow \\ K^1(\mathcal{O}_E) & \xrightarrow{i^*} & K^1(B) & \xrightarrow{1-[E]} & K^1(B) \end{array} .$$

Examples of Gysin sequences in $K$-theory were given in [2] for line bundles over quantum projective spaces and leading to a class of quantum lens spaces. These examples will be generalized later on in the paper to a class of quantum lens spaces as circle bundles over quantum weighted projective spaces with arbitrary weights.

## 3. Pimsner algebras and circle actions

An interesting source of Pimsner algebras consists of $C^*$-algebras which are equipped with a circle action and subject to an extra completeness condition on the associated spectral subspaces. We now investigate this relationship.

Throughout this section $A$ will be a $C^*$-algebra and $\{\sigma_z\}_{z\in S^1}$ will be a strongly continuous action of the circle $S^1$ on $A$.

### 3.1. **Algebras from actions.**

For each $n \in \mathbb{Z}$, define the spectral subspace

$$A_{(n)} := \left\{ \xi \in A \mid \sigma_z(\xi) = z^{-n}\, \xi \quad \text{for all } z \in S^1 \right\} .$$

Then the invariant subspace $A_{(0)} \subseteq A$ is a $C^*$-subalgebra and each $A_{(n)}$ is a (right) Hilbert $C^*$-module over $A_{(0)}$ with right action induced by the algebra structure on $A$ and $A_{(0)}$-valued inner product just $\langle \xi, \eta \rangle := \xi^*\, \eta$, for all $\xi, \eta \in A_{(n)}$.

**Assumption 3.1.** *The data $(A, \sigma_z)$ as above is taken to satisfy the conditions:*

1. *The $C^*$-algebra $A_{(0)}$ is separable.*
2. *The Hilbert $C^*$-modules $A_{(1)}$ and $A_{(-1)}$ are full and countably generated over the $C^*$-algebra $A_{(0)}$.*

**Lemma 3.2.** *With the $*$-homomorphism $\phi : A_{(0)} \to \mathscr{L}(A_{(1)})$ simply defined by $\phi(a)(\xi) := a\,\xi$, the pair $(\phi, A_{(1)})$ satisfies the conditions of Assumption 2.1.*

*Proof.* To prove that $\phi : A_{(0)} \to \mathscr{L}(A_{(1)})$ is injective, let $a \in A_{(0)}$ and suppose that $a\,\xi = 0$ for all $\xi \in A_{(1)}$. It then follows that $a\,\xi\,\eta^* = 0$ for all $\xi, \eta \in A_{(1)}$. But this implies that $a\,\langle v, w\rangle = 0$ for all $v, w \in A_{(-1)}$. Since $A_{(-1)}$ is full this shows that $a = 0$. We may thus conclude that $\phi : A_{(0)} \to \mathscr{L}(A_{(1)})$ is injective, and the image of $\phi$ is therefore closed.

To conclude that $\mathscr{K}(A_{(1)}) \subseteq \phi(A_{(0)})$ it is now enough to show that the operator $\theta_{\xi,\eta} \in \phi(A_{(0)})$ for all $\xi, \eta \in A_{(1)}$. But this is clear since $\theta_{\xi,\eta} = \phi(\xi\,\eta^*)$.

To prove that $\phi(A_{(0)}) \subseteq \mathscr{K}(A_{(1)})$ it suffices to check that $\phi(\langle v, w\rangle) \in \mathscr{K}(A_{(1)})$ for all $v, w \in A_{(-1)}$ (again since $A_{(-1)}$ is full). But this is true being $\phi(\langle v, w\rangle) = \theta_{v^*,w^*}$. □

The condition that both $A_{(1)}$ and $A_{(-1)}$ are full over $A_{(0)}$ has the important consequence that the action $\{\sigma_z\}_{z\in S^1}$ is semi-saturated in the sense of the following:

**Definition 3.3.** A circle action $\{\sigma_z\}_{z\in S^1}$ on a $C^*$-algebra $A$ is called *semi-saturated* if $A$ is generated, as a $C^*$-algebra, by the fixed point algebra $A_{(0)}$ together with the first spectral subspace $A_{(1)}$.

**Proposition 3.4.** *Suppose that $A_{(1)}$ and $A_{(-1)}$ are full over $A_{(0)}$. Then the circle action $\{\sigma_z\}_{z\in S^1}$ is semi-saturated.*

*Proof.* With $\mathrm{cl}(\cdot)$ refering to the norm-closure, we show that the Banach algebra

$$\mathrm{cl}\Big(\sum_{n=0}^{\infty} A_{(n)}\Big) \subseteq A$$

is generated by $A_{(1)}$ and $A_{(0)}$. A similar proof in turn shows that

$$\mathrm{cl}\Big(\sum_{n=0}^{\infty} A_{(-n)}\Big) \subseteq A$$

is generated by $A_{(-1)}$ and $A_{(0)}$. Since the span $\sum_{n\in\mathbb{Z}} A_{(n)}$ is norm-dense in $A$ (see [10, Prop. 2.5]), this proves the proposition. We show by induction on $n \in \mathbb{N}$ that

$$(A_{(1)})^n := \mathrm{span}\big\{x_1 \cdot \ldots \cdot x_n \mid x_1, \ldots, x_n \in A_{(1)}\big\}$$

is dense in $A_{(n)}$. For $n = 1$ the statement is void.

Suppose thus that the statement holds for some $n \in \mathbb{N}$. Then, let $x \in A_{(n+1)}$ and choose a countable approximate identity $\{u_m\}_{m\in\mathbb{N}}$ for the separable $C^*$-algebra $A_{(0)}$. Let $\varepsilon > 0$ be given. We need to construct an element $y \in (A_{(1)})^{n+1}$ such that

$$\|x - y\| < \varepsilon\ .$$

To this end we first remark that the sequence $\{x \cdot u_m\}_{m\in\mathbb{N}}$ converges to $x \in A_{(n+1)}$. Indeed, this follows due to $x^*x \in A_{(0)}$ and since, for all $m \in \mathbb{N}$,

$$\|x \cdot u_m - x\|^2 = \|u_m x^* x u_m + x^* x - x^* x u_m - u_m x^* x\|\ .$$

We may thus choose an $m \in \mathbb{N}$ such that

$$\|x \cdot u_m - x\| < \varepsilon/3\ .$$

Since $A_{(1)}$ is full over $A_{(0)}$, there are elements $\xi_1, \ldots, \xi_k$ and $\eta_1, \ldots, \eta_k \in A_{(1)}$ so that

$$\|x \cdot u_m - \sum_{j=1}^{k} x \cdot \xi_j^* \cdot \eta_j\| < \varepsilon/3 \ .$$

Furthermore, since $x \cdot \xi_j^* \in A_{(n)}$ we may apply the induction hypothesis to find elements $z_1, \ldots, z_k \in (A_{(1)})^n$ such that

$$\| \sum_{j=1}^{k} x \cdot \xi_j^* \cdot \eta_j - \sum_{j=1}^{k} z_j \cdot \eta_j \| < \varepsilon/3 \ .$$

Finally, it is straightforward to verify that for the element

$$y := \sum_{j=1}^{k} z_j \cdot \eta_j \in (A_{(1)})^{n+1}$$

it holds that: $\|x - y\| < \varepsilon$. This proves the present proposition. □

Having a semi-saturated action one is lead to the following theorem [3, Thm. 3.1].

**Theorem 3.5.** *The Pimsner algebra $\mathcal{O}_{A_{(1)}}$ is isomorphic to $A$. The isomorphism is given by $S_\xi \mapsto \xi$ for all $\xi \in A_{(1)}$.*

3.2. **$\mathbb{Z}$-graded algebras.** In much of what follows, the $C^*$-algebras of interest with a circle action, will come from closures of dense $\mathbb{Z}$-graded $*$-algebras, with the $\mathbb{Z}$-grading defining the circle action in a natural fashion.

Let $\mathscr{A} = \oplus_{n\in\mathbb{Z}} \mathscr{A}_{(n)}$ be a $\mathbb{Z}$-graded unital $*$-algebra. The grading is compatible with the involution $^*$, this meaning that $x^* \in \mathscr{A}_{(-n)}$ whenever $x \in \mathscr{A}_{(n)}$ for some $n \in \mathbb{Z}$. For $w \in S^1$, define the $*$-automorphism $\sigma_w : \mathscr{A} \to \mathscr{A}$ by

$$\sigma_w : x \mapsto w^{-n} x \qquad \text{for} \quad x \in \mathscr{A}_{(n)} \quad n \in \mathbb{Z} \, .$$

We will suppose that we have a $C^*$-norm $\| \cdot \| : \mathscr{A} \to [0, \infty)$ on $\mathscr{A}$ satisfying

$$\|\sigma_w(x)\| \leq \|x\| \qquad \text{for all} \quad w \in S^1 \quad x \in \mathscr{A} \, ,$$

thus the action has to be isometric. The completion of $\mathscr{A}$ is denoted by $A$.

The following standard result is here for the sake of completeness and its use below. The proof relies on the existence of a conditional expectation naturally associated to the action.

**Lemma 3.6.** *The collection $\{\sigma_w\}_{w\in S^1}$ extends by continuity to a strongly continuous action of $S^1$ on $A$. Each spectral subspace $A_{(n)}$ agrees with the closure of $\mathscr{A}_{(n)} \subseteq A$.*

*Proof.* Once $\mathscr{A}_{(n)}$ is shown to be dense in $A_{(n)}$ the rest follows from standard arguments. Thus, for $n \in \mathbb{Z}$, define the bounded operator $E_{(n)} : A \to A_{(n)}$ by

$$E_{(n)} : x \mapsto \int_{S^1} w^n \, \sigma_w(x) \, \mathrm{d}w \, ,$$

where the integration is carried out with respect to the Haar-measure on $S^1$. We have that $E_{(n)}(x) = x$ for all $x \in A_{(n)}$ and then that $\|E_{(n)}\| \leq 1$. This implies that $\mathscr{A}_{(n)} \subseteq A_{(n)}$ is dense. □

Let now $d \in \mathbb{N}$ and consider the unital $*$-subalgebra $\mathscr{A}^{1/d} := \oplus_{n\in\mathbb{Z}} \mathscr{A}_{(nd)} \subseteq \mathscr{A}$. Then $\mathscr{A}^{1/d}$ is a $\mathbb{Z}$-graded unital $*$-algebra as well and we denote the associated circle action by $\sigma_w^{1/d} : \mathscr{A}^{1/d} \to \mathscr{A}^{1/d}$. Let $w \in S^1$ and choose a $z \in S^1$ such that $z^d = w$. Then

$$\sigma_w^{1/d}(x_{nd}) = w^n \cdot x_{nd} = z^{nd} \cdot x_{nd} = \sigma_z(x_{nd}) \, , \qquad \text{for all } \; x_{nd} \in \mathscr{A}_{(nd)} \, ,$$

and it follows that $\sigma_w^{1/d}(x) = \sigma_z(x)$ for all $x \in \mathscr{A}^{1/d}$. With the $C^*$-norm obtained by restriction $\| \cdot \| : \mathscr{A}^{1/d} \to [0, \infty)$, it follows in particular that

$$\|\sigma_w^{1/d}(x)\| \leq \|x\|$$

by our standing assumption on the compatibility of $\{\sigma_w\}_{w\in S^1}$ with the norm on $\mathscr{A}$. The $C^*$-completion of $\mathscr{A}^{1/d}$ is denoted by $A^{1/d}$.

**Proposition 3.7.** *Suppose that $\{\sigma_w\}_{w\in S^1}$ is semi-saturated on $A$ and let $d \in \mathbb{N}$. Then we have unitary isomorphisms of Hilbert $C^*$-modules*

$$(A_{(1)})^{\widehat{\otimes}_\phi d} \simeq (A^{1/d})_{(1)} \qquad \text{and} \qquad (A_{(-1)})^{\widehat{\otimes}_\phi d} \simeq (A^{1/d})_{(-1)}$$

*induced by the product $\psi : x_1 \otimes \ldots \otimes x_d \mapsto x_1 \cdot \ldots \cdot x_d$ .*

*Proof.* We only consider the case of $A_{(1)}$ since the the proof for $A_{(-1)}$ is the same.

Observe firstly that $(\mathscr{A}^{1/d})_{(1)} = \mathscr{A}_{(d)}$. Thus Lemma 3.6 yields $A_{(d)} = (A^{1/d})_{(1)}$. This implies that the product $\psi : (\mathscr{A}_{(1)})^{\otimes_{\mathscr{A}_{(0)}} d} \to (\mathscr{A}^{1/d})_{(1)}$ is a well-defined homomorphism of right modules over $\mathscr{A}_{(0)}$ (here "$\otimes_{\mathscr{A}_{(0)}}$" refers to the algebraic tensor product of bimodules over $\mathscr{A}_{(0)}$). Furthermore, since

$$\langle x_1 \otimes \ldots \otimes x_d, y_1 \otimes \ldots \otimes y_d \rangle = x_d^* \cdot \ldots \cdot x_1^* \cdot y_1 \cdot \ldots \cdot y_d \, ,$$

we get that $\psi$ extends to a homomorphism $\psi : (A_{(1)})^{\widehat{\otimes}_\phi d} \to A_{(1)}^{1/d}$ of Hilbert $C^*$-modules over $A_{(0)}$ with $\langle \psi(\xi), \psi(\eta) \rangle = \langle \xi, \eta \rangle$ for all $\xi, \eta \in (A_{(1)})^{\widehat{\otimes}_\phi d}$.

It is therefore enough to show that $\mathrm{Im}(\psi) \subseteq (A^{1/d})_{(1)}$ is dense. But this is a consequence of [10, Prop. 4.8]. □

**Lemma 3.8.** *Suppose that $\{\sigma_w\}_{w\in S^1}$ satisfies the conditions of Assumption 3.1. Then $\{\sigma_w^{1/d}\}_{w\in S^1}$ satisfies the conditions of Assumption 3.1 for all $d \in \mathbb{N}$.*

*Proof.* We only need to show that the Hilbert $C^*$-modules $A_{(d)}$ and $A_{(-d)}$ are full and countably generated over $A_{(0)}$.

By Proposition 3.4 we have that $\{\sigma_w\}_{w\in S^1}$ is semi-saturated. It thus follows from Proposition 3.7 that

$$A_{(d)} \simeq (A_{(1)})^{\widehat{\otimes}_\phi d} \qquad \text{and} \qquad A_{(-d)} \simeq (A_{(-1)})^{\widehat{\otimes}_\phi d} \, . \tag{3.1}$$

Since both $A_{(1)}$ and $A_{(-1)}$ are full and countably generated by assumption these unitary isomorphisms prove the lemma. □

The following result is a stronger version of Theorem 3.5 since it incorporates all the spectral subspaces and not just the first one.

**Theorem 3.9.** *Suppose that the circle action $\{\sigma_w\}_{w\in S^1}$ on $A$ satisfies the conditions in Assumption 3.1. Then the Pimsner algebra $\mathcal{O}_{A_{(d)}} \simeq \mathcal{O}_{(A_{(1)})^{\widehat{\otimes} d}}$ is isomorphic to the $C^*$-algebra $A^{1/d}$ for all $d \in \mathbb{N}$. The isomorphism is given by $S_\xi \mapsto \xi$ for all $\xi \in A_{(d)}$.*

*Proof.* This follows by combining Lemma 3.8, Proposition 3.7 and Theorem 3.5. □

We finally investigate what happens when the $C^*$-norm on $\mathscr{A} = \oplus_{n\in\mathbb{Z}}\mathscr{A}_{(n)}$ is changed. Thus, let $\|\cdot\|' : \mathscr{A} \to [0,\infty)$ be an alternative $C^*$-norm on $\mathscr{A}$ satisfying

$$\|\sigma_w(x)\|' \leq \|x\|' \qquad \text{for all } w \in S^1 \text{ and } x \in \mathscr{A}\,.$$

The corresponding completion $A'$ will carry an induced circle action $\{\sigma'_w\}_{w\in S^1}$. The next theorem can be seen as a manifestation of the gauge-invariant uniqueness theorem, [15, Thm. 6.2 and Thm. 6.4]. This property was indirectly used already in [18, Thm. 3.12] for the proof of the universal properties of Pimsner algebras.

**Theorem 3.10.** *Suppose that $\|x\| = \|x\|'$ for all $x \in \mathscr{A}_{(0)}$. Then $\{\sigma_w\}_{w\in S^1}$ satisfies the conditions of Assumption 3.1 if and only if $\{\sigma'_w\}$ satisfies the conditions of Assumption 3.1. And in this case, the identity map $\mathscr{A} \to \mathscr{A}$ induces an isomorphism $A \to A'$ of $C^*$-algebras. In particular, we have that $\|x\| = \|x\|'$ for all $x \in \mathscr{A}$.*

*Proof.* Remark first that the identity map $\mathscr{A}_{(n)} \to \mathscr{A}_{(n)}$ induces an isometric isomorphism of Hilbert $C^*$-modules $A_{(n)} \to A'_{(n)}$ for all $n \in \mathbb{Z}$. This is a consequence of the identity $\|x\| = \|x\|'$ for all $x \in \mathscr{A}_{(0)}$. But then we also have isomorphisms

$$(A_{(1)})^{\widehat{\otimes}_\phi n} \simeq (A'_{(1)})^{\widehat{\otimes}_\phi n} \qquad \text{and} \qquad (A_{(-1)})^{\widehat{\otimes}_\phi n} \simeq (A'_{(-1)})^{\widehat{\otimes}_\phi n}$$

for all $n \in \mathbb{N}$. These observations imply that $\{\sigma_w\}_{w\in S^1}$ satisfies the conditions of Assumption 3.1 if and only if $\{\sigma'_w\}$ satisfies the conditions of Assumption 3.1. But it then follows from Theorem 3.5 that

$$A \simeq \mathcal{O}_{A_{(1)}} \simeq \mathcal{O}_{A'_{(1)}} \simeq A'\,,$$

with corresponding isomorphism $A \simeq A'$ induced by the identity map $\mathscr{A} \to \mathscr{A}$. □

## 4. Quantum principal bundles and $\mathbb{Z}$-graded algebras

We start by recalling the definition of a quantum principal $U(1)$-bundle.

Later on in the paper we shall exhibit a novel class of quantum lens spaces as principal $U(1)$-bundles over quantum weighted projective lines with arbitrary weights.

### 4.1. Quantum principal bundles.

Define the unital complex algebra

$$\mathcal{O}(U(1)) := \mathbb{C}[z, z^{-1}]/\langle 1 - zz^{-1}\rangle$$

where $\langle 1 - zz^{-1}\rangle$ denotes the ideal generated by $1 - zz^{-1}$ in the polynomial algebra $\mathbb{C}[z, z^{-1}]$ in two variables. The algebra $\mathcal{O}(U(1))$ is a Hopf algebra by defining, for all $n \in \mathbb{Z}$, coproduct $\Delta : z^n \mapsto z^n \otimes z^n$, antipode $S : z^n \mapsto z^{-n}$ and counit $\varepsilon : z^n \mapsto 1$. We simply write $\mathcal{O}(U(1)) = \big(\mathcal{O}(U(1)), \Delta, S, \varepsilon\big)$ for short.

Let $\mathscr{A}$ be a complex unital algebra and suppose in addition that it is a right comodule algebra over $\mathcal{O}(U(1))$, that is we have a homomorphism of unital algebras

$$\Delta_R : \mathscr{A} \to \mathscr{A} \otimes \mathcal{O}(U(1)) \,,$$

which also provides a coaction of the Hopf algebra $\mathcal{O}(U(1))$ on $\mathscr{A}$.

Let $\mathscr{B} := \{x \in \mathscr{A} \mid \Delta_R(x) = x \otimes 1\}$ denote the unital subalgebra of $\mathscr{A}$ consisting of coinvariant elements for the coaction.

**Definition 4.1.** One says that the datum $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{B}\big)$ is a *quantum principal $U(1)$-bundle* when the *canonical map*

$$\mathrm{can} : \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A} \to \mathscr{A} \otimes \mathcal{O}(U(1)) \,, \qquad x \otimes y \mapsto x \cdot \Delta_R(y) \,,$$

is an isomorphism.

**Remark 4.2.** One ought to qualify Definition 4.1 by saying that the quantum principal bundle is 'for the universal differential calculus' [6]. In fact, the definition above means that the right comodule algebra $\mathscr{A}$ is a *$\mathscr{B}$-Galois extension*, and this is equivalent (in the present context) by [12, Prop. 1.6] to the bundle being a quantum principal bundle for the universal differential calculus.

4.2. **Relation with $\mathbb{Z}$-graded algebras.** We now provide a detailed analysis of the case where the quantum principal bundle structure comes from a $\mathbb{Z}$-grading of the 'total space' algebra. This will lead to an alternative characterization of quantum $U(1)$-principal bundles in this setting. While this description is not new (see for instance [21, Lemma 5.1]), it is certainly more manageable. In particular, we will apply it in §6 below for the case of quantum lens spaces as $U(1)$-principal bundles over quantum weighted projective lines.

Let $\mathscr{A} = \oplus_{n\in\mathbb{Z}} \mathscr{A}_{(n)}$ be a $\mathbb{Z}$-graded unital algebra and let $\mathcal{O}(U(1))$ be the Hopf algebra defined in the previous section. Define the unital algebra homomorphism

$$\Delta_R : \mathscr{A} \to \mathscr{A} \otimes \mathcal{O}(U(1)) \qquad x \mapsto x \otimes z^{-n} \,, \ \text{ for } \ x \in \mathscr{A}_{(n)} \,.$$

It is then clear that $\Delta_R$ turns $\mathscr{A}$ into a right comodule algebra over $\mathcal{O}(U(1))$. The unital subalgebra of coinvariant elements coincides with $\mathscr{A}_{(0)}$.

**Theorem 4.3.** *The triple $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal $U(1)$-bundle if and only if there exist finite sequences*

$$\{\xi_j\}_{j=1}^N \,, \ \{\beta_i\}_{i=1}^M \ \textit{in } \mathscr{A}_{(1)} \qquad \textit{and} \qquad \{\eta_j\}_{j=1}^N \,, \ \{\alpha_i\}_{i=1}^M \ \textit{in } \mathscr{A}_{(-1)}$$

*such that there hold identities:*

$$\sum_{j=1}^N \xi_j \eta_j = 1_{\mathscr{A}} = \sum_{i=1}^M \alpha_i \beta_i \,.$$

*Proof.* Suppose first that $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal $U(1)$-bundle. Thus, that the canonical map

$$\mathrm{can} : \mathscr{A} \otimes_{\mathscr{A}_{(0)}} \mathscr{A} \to \mathscr{A} \otimes \mathcal{O}(U(1))$$

is an isomorphism. For each $n \in \mathbb{Z}$, define the idempotents

$$P_{(n)} : \mathcal{O}(U(1)) \to \mathcal{O}(U(1)) , \qquad P_{(n)} : z^m \mapsto \delta_{nm} z^m \qquad \text{and}$$

$$E_{(n)} : \mathscr{A} \to \mathscr{A} , \qquad E_{(n)} : x_m \mapsto \delta_{nm} x_m$$

where $x_m \in \mathscr{A}_{(m)}$ and where $\delta_{nm} \in \{0, 1\}$ denotes the Kronecker delta. Clearly,

$$\mathrm{can} \circ (1 \otimes E_{(-n)}) = (1 \otimes P_{(n)}) \circ \mathrm{can} : \mathscr{A} \otimes_{\mathscr{A}_{(0)}} \mathscr{A} \to \mathscr{A} \otimes \mathcal{O}(U(1)) . \tag{4.1}$$

for all $n \in \mathbb{Z}$. Let us now define the element

$$\gamma := \mathrm{can}^{-1}(1_{\mathscr{A}} \otimes z) = \sum_{j=1}^{N} \gamma_j^0 \otimes \gamma_j^1 .$$

It then follows from (4.1) that

$$\gamma = (1 \otimes E_{(-1)})(\gamma) = \sum_{j=1}^{N} \gamma_j^0 \otimes E_{(-1)}(\gamma_j^1)$$

To continue, we remark that

$$m(\gamma) = m \circ \mathrm{can}^{-1}(1_{\mathscr{A}} \otimes z) = (\mathrm{id} \otimes \varepsilon)(1_{\mathscr{A}} \otimes z) = 1_{\mathscr{A}}$$

where $m : \mathscr{A} \otimes_{\mathscr{A}_{(0)}} \mathscr{A} \to \mathscr{A}$ is the algebra multiplication. And this implies that

$$1_{\mathscr{A}} = \sum_{j=1} \gamma_j^0 \cdot E_{(-1)}(\gamma_j^1) = \sum_{j=1}^{N} E_{(1)}(\gamma_j^0) \cdot E_{(-1)}(\gamma_j^1) .$$

We therefore put,

$$\xi_j := E_{(1)}(\gamma_0^j) \qquad \text{and} \qquad \eta_j := E_{(-1)}(\gamma_1^j) , \qquad \text{for all } j = 1, \dots, N .$$

Next, we define the element

$$\delta := \mathrm{can}^{-1}(1_{\mathscr{A}} \otimes z^{-1}) = \sum_{i=1}^{M} \delta_i^0 \otimes \delta_i^1 .$$

An argument similar to the one before then shows that $\sum_{i=1}^{M} \alpha_i \cdot \beta_i = 1_{\mathscr{A}}$, with

$$\alpha_i := E_{(-1)}(\delta_i^0) \qquad \text{and} \qquad \beta_i := E_{(1)}(\delta_i^1) , \qquad \text{for all } i = 1, \dots, M .$$

This proves the first half of the theorem.

To prove the second half we suppose that there exist sequences $\{\xi_j\}_{j=1}^N$, $\{\beta_i\}_{i=1}^M$ in $\mathscr{A}_{(1)}$ and $\{\eta_j\}_{j=1}^N$, $\{\alpha_i\}_{i=1}^M$in $\mathscr{A}_{(-1)}$ such that $\sum_{j=1}^N \xi_j \eta_j = 1_{\mathscr{A}} = \sum_{i=1}^M \alpha_i \beta_i$.

We then define the map $\mathrm{can}^{-1} : \mathscr{A} \otimes \mathcal{O}(U(1)) \to \mathscr{A} \otimes_{\mathscr{A}_{(0)}} \mathscr{A}$ by the formula

$$\mathrm{can}^{-1} : x \otimes z^n \mapsto \begin{cases} \sum_{J \in \{1,\dots,N\}^n} x\, \xi_{j_1} \cdot \ldots \cdot \xi_{j_n} \otimes \eta_{j_n} \cdot \ldots \cdot \eta_{j_1} , & \text{for } n \geq 0 \\ \\ \sum_{I \in \{1,\dots,M\}^{-n}} x\, \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_{-n}} \otimes \beta_{i_{-n}} \cdot \ldots \cdot \beta_{i_1} , & \text{for } n \leq 0 \end{cases} .$$

It is then straightforward to check that

$$\mathrm{can}^{-1} \circ \mathrm{can} = \mathrm{id} \qquad \text{and} \qquad \mathrm{can} \circ \mathrm{can}^{-1} = \mathrm{id} .$$

This ends the proof of the theorem. □

**Remark 4.4.** The above theorem shows that $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal $U(1)$-bundle if and only if $\mathscr{A}$ is *strongly* $\mathbb{Z}$-graded, see [17, Lem. I.3.2]. Our next corollary is thus a consequence of [17, Cor. I.3.3]. We present a proof here since we need the explicit form of the idempotents later on.

**Corollary 4.5.** *With the same conditions as in Theorem 4.3. The right-modules $\mathscr{A}_{(1)}$ and $\mathscr{A}_{(-1)}$ are finitely generated and projective over $\mathscr{A}_{(0)}$.*

*Proof.* With the $\xi$'s and the $\eta$'s as above, define the module homomorphisms

$$\Phi_{(1)} : \mathscr{A}_{(1)} \to (\mathscr{A}_{(0)})^N , \qquad \Phi_{(1)}(\zeta) = \begin{pmatrix} \eta_1\,\zeta \\ \eta_2\,\zeta \\ \vdots \\ \eta_N\,\zeta \end{pmatrix} \qquad \text{and}$$

$$\Psi_{(1)} : (\mathscr{A}_{(0)})^N \to \mathscr{A}_{(1)} , \qquad \Psi_{(1)} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \xi_1\,x_1 + \xi_2\,x_2 + \cdots + \xi_N\,x_N .$$

It then follows that $\Psi_{(1)}\Phi_{(1)} = \mathrm{id}_{\mathscr{A}_{(1)}}$. Thus $E_{(1)} := \Phi_{(1)}\Psi_{(1)}$ is an idempotent in $M_N(\mathscr{A}_{(0)})$ and this proves the first half of the corollary.

Similarly, with the $\alpha$'s and the $\beta$'s as above, define the module homomorphisms

$$\Phi_{(-1)} : \mathscr{A}_{(-1)} \to \mathcal{O}(W_q(k,l))^2 , \qquad \Phi_{(-1)}(\zeta) = \begin{pmatrix} \beta_1\,\zeta \\ \beta_2\,\zeta \\ \vdots \\ \beta_M\,\zeta \end{pmatrix} \qquad \text{and}$$

$$\Psi_{(-1)} : \mathcal{O}(W_q(k,l))^2 \to \mathscr{A}_{(-1)} , \qquad \Psi_{(-1)} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{pmatrix} = \alpha_1\,x_1 + \alpha_2\,x_2 + \cdots + \alpha_M\,x_M .$$

Now one gets $\Psi_{(-1)}\Phi_{(-1)} = \mathrm{id}_{\mathscr{A}_{(-1)}}$. Thus $E_{(-1)} := \Phi_{(-1)}\Psi_{(-1)}$ is an idempotent in $M_M(\mathscr{A}_{(0)})$ as well. This finishes the proof of the corollary. □

Let $d \in \mathbb{N}$ and consider the $\mathbb{Z}$-graded unital $\mathbb{C}$-algebra $\mathscr{A}^{1/d} := \oplus_{n\in\mathbb{Z}}\mathscr{A}_{(dn)}$.

As a consequence of Theorem 4.3 we obtain the following:

**Proposition 4.6.** *Suppose $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal $U(1)$-bundle. Then $\big(\mathscr{A}^{1/d}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal $U(1)$-bundle for all $d \in \mathbb{N}$.*

*Proof.* Let the finite sequences $\{\xi_j\}_{j=1}^N$, $\{\beta_i\}_{i=1}^M$ in $\mathscr{A}_{(1)}$ and $\{\eta_j\}_{j=1}^N$, $\{\alpha_i\}_{i=1}^M$ in $\mathscr{A}_{(-1)}$ be as in Theorem 4.3. For each multi-index $J \in \{1,\ldots,N\}^d$ and each multi-index

$I \in \{1, \ldots, M\}^d$ define the elements

$$\xi_J := \xi_{j_1} \cdot \ldots \cdot \xi_{j_d}\,, \qquad \beta_I := \beta_{i_d} \cdot \ldots \cdot \beta_{i_1} \in \mathscr{A}_{(d)} \qquad \text{and}$$
$$\eta_J := \eta_{j_d} \cdot \ldots \cdot \eta_{j_1}\,, \qquad \alpha_I := \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_d} \in \mathscr{A}_{(-d)}\,.$$

It is then clear that

$$\sum_{J \in \{1,\ldots,N\}^d} \xi_J\, \eta_J = 1_{\mathscr{A}^{1/d}} = \sum_{I \in \{1,\ldots,M\}^d} \alpha_I\, \beta_I\ .$$

This proves the proposition by an application of Theorem 4.3. □

Remark that it follows from Proposition 4.6 and Corollary 4.5 that when $\big(\mathscr{A}, \mathcal{O}(U(1)), \mathscr{A}_{(0)}\big)$ is a quantum principal bundle then the right modules $\mathscr{A}_{(d)}$ and $\mathscr{A}_{(-d)}$ are finitely generated projective over $\mathscr{A}_{(0)}$ for all $d \in \mathbb{N}$.

## 5. Quantum weighted projective lines

We recall the definition of the quantum weighted projective lines as fixed point algebras of circle actions on the quantum 3-sphere. These algebras play the role of the coordinate functions on the base space which parametrizes the lines generating the quantum lens spaces (as total spaces). Corresponding $C^*$-algebras will be the analogues of continuous functions on the base and total space respectively. The latter $C^*$-algebra will be given as a Pimsner algebra coming from the line bundles.

### 5.1. Coordinate algebras.

Let $n \in \mathbb{N}_0$ and let $q \in (0, 1)$.

**Definition 5.1.** The coordinate algebra $\mathcal{O}(S_q^{2n+1})$ of the *quantum sphere* $S_q^{2n+1}$ is the universal unital $*$-algebra with generators $z_0, \ldots, z_n$ and relations

$$z_i z_j = q z_j z_i \ \text{ for } \ i < j\ , \qquad z_i z_j^* = q z_j^* z_i \ \text{ for } \ i \neq j\,,$$
$$z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{m=i+1}^{n} z_m z_m^*\ , \qquad \sum_{m=0}^{n} z_m z_m^* = 1\,.$$

This algebra was introduced in [22]. Next, let $L = (l_0, \ldots, l_n) \in \mathbb{N}^{n+1}$ be fixed. We then have a circle action $\{\sigma_w^L\}_{w \in S^1}$ on $\mathcal{O}(S_q^{2n+1})$ defined on generators by

$$\sigma_w^L : z_i \mapsto w^{l_i} z_i \qquad \text{for all } \ i \in \{0, \ldots, n\}\,.$$

**Definition 5.2.** The coordinate algebra $\mathcal{O}(W_q(L))$ of the *quantum weighted projective space* $W_q(L)$ is the fixed point algebra of the circle action $\{\sigma_w^L\}_{w \in S^1}$. Thus

$$\mathcal{O}(W_q(L)) := \big\{x \in \mathcal{O}(S_q^{2n+1}) \mid \sigma_w^L(x) = x \text{ for all } w \in S^1\big\}\,.$$

From now on, we will suppose that $n = 1$ and that $k := l_0$ and $l := l_1$ are coprime. By [5, Thm. 2.1], the algebraic quantum projective line $\mathcal{O}(W_q(k, l))$ agrees with the unital $*$-subalgebra of $\mathcal{O}(S_q^3)$ generated by the elements $z_0^l (z_1^*)^k$ and $z_1 z_1^*$.

Alternatively, one may identify $\mathcal{O}(W_q(k,l))$ with the universal unital $*$-algebra with generators $a, b$, subject to the relations

$$b^* = b \ , \qquad ba = q^{-2l} \ ab \ ,$$
$$aa^* = q^{2kl} \ b^k \prod_{m=0}^{l-1} (1 - q^{2m} b) \ , \qquad a^* a = b^k \prod_{m=1}^{l} (1 - q^{-2m} b) \, .$$

The identification is just $a \mapsto z_0^l (z_1^*)^k$ and $b \mapsto z_1 z_1^*$ (we have exchanged the names of generators with respect to [5]). In particular $\mathcal{O}(W_q(1,1)) = \mathcal{O}(\mathbb{C}P^1_q)$, while $\mathcal{O}(W_q(1,l))$ was named *quantum teardrop* in [5].

5.2. **$C^*$-completions.** We fix $k, l \in \mathbb{N}$ to be coprime positive integers.

**Definition 5.3.** The algebra of continuous functions on the *quantum weighted projective line* $W_q(k,l)$ is the universal enveloping $C^*$-algebra, denoted $C(W_q(k,l))$, of the coordinate algebra $\mathcal{O}(W_q(k,l))$.

Let $\mathscr{K}$ denote the $C^*$-algebra of compact operators on the separable Hilbert space $l^2(\mathbb{N}_0)$ of all square summable sequences indexed by $\mathbb{N}_0$, with orthonormal basis $\{e_p\}_{p \in \mathbb{N}_0}$. It was shown in [5, Prop. 5.1] that $C(W_q(k,l))$ is isomorphic to the unital $C^*$-algebra

$$\widetilde{\oplus_{s=1}^l \mathscr{K}} \subseteq \mathscr{L}\big( \oplus_{s=1}^l l^2(\mathbb{N}_0) \big) \, ,$$

where $\widetilde{\cdot}$ denotes the unitalization functor. The isomorphism is induced by the direct sum of representations $\oplus_{s=1}^l \pi_s : \mathcal{O}(W_q(k,l)) \to \mathscr{L}\big( \oplus_{s=1}^l l^2(\mathbb{N}_0) \big)$ where each $\pi_s$ is defined on generators by

$$\begin{aligned} &\pi_s(z_1 z_1^*)(e_p) := q^{2s} \ q^{2lp} \ e_p \ , \qquad \pi_s(z_0^l (z_1^*)^k)(e_0) := 0 \, , \\ &\pi_s(z_0^l (z_1^*)^k)(e_p) := q^{k(lp+s)} \ \prod_{m=1}^{l} (1 - q^{2(lp+s-m)})^{1/2} \ e_{p-1} \ , \ p \geq 1 \, . \end{aligned} \tag{5.1}$$

Notice that the $C^*$-algebra $C(W_q(k,l))$ does not depend on $k$. As a consequence one has the following corollary due to Brzeziński and Fairfax, see [5, Cor. 5.3].

**Corollary 5.4.** *The $K$-groups of $C(W_q(k,l))$ are:*

$$K_0(C(W_q(k,l))) = \mathbb{Z}^{l+1} \, , \qquad K_1(C(W_q(k,l))) = 0 \, .$$

Notice that the $K$-theory groups of the quantum weighted projective lines do not agree with the $K$-theory groups of their commutative counterparts: In the commutative case, the $K_0$-group is given by $K_0(C(W(k,l))) = \mathbb{Z}^2$ independently of both weights $k$ and $l$, see [1, Prop. 2.5].

**Definition 5.5.** The algebra of continuous functions on the *quantum 3-sphere* $S_q^3$ is the universal enveloping $C^*$-algebra, $C(S_q^3)$, of the coordinate algebra $\mathcal{O}(S_q^3)$.

The (weighted) circle action $\{\sigma_w^{(k,l)}\}_{w \in S^1}$ on $\mathcal{O}(S_q^3)$ will be denoted simply by $\{\sigma_w\}_{w \in S^1}$. It induces a strongly continuous circle action on $C(S_q^3)$. We let $C(S_q^3)_{(0)}$ denote the fixed point algebra of this action.

**Lemma 5.6.** *The inclusion $\mathcal{O}(W_q(k,l)) \subseteq \mathcal{O}(S^3_q)$ induces an isomorphism of unital $C^*$-algebras,*

$$i : C(W_q(k,l)) \to C(S^3_q)_{(0)} \, .$$

*Proof.* Clearly, one has $\mathrm{Im}(i) \subseteq C(S^3_q)_{(0)}$ and $\mathrm{Im}(i)$ is dense by the argument used in the proof of Lemma 3.6.

It therefore suffices to show that $i : C(W_q(k,l)) \to C(S^3_q)$ is injective. To this end, consider the $*$-homomorphism $\pi := \oplus_{s=1}^{l} \pi_s : \mathcal{O}(W_q(k,l)) \to \mathscr{L}\big( \oplus_{s=1}^{l} l^2(\mathbb{N}_0)\big)$. Then, by [5, Prop. 2.4] there exist a $*$-homomorphism $\rho : \mathcal{O}(S^3_q) \to \mathscr{L}(l^2(\mathbb{N}_0))$ and an isomorphism $\phi : \mathscr{L}\big( \oplus_{s=1}^{l} l^2(\mathbb{N}_0)\big) \to \mathscr{L}(l^2(\mathbb{N}_0))$ such that

$$\phi \circ \pi = \rho \circ i : \mathcal{O}(W_q(k,l)) \to \mathscr{L}(l^2(\mathbb{N}_0)) \, .$$

Let now $x \in \mathcal{O}(W_q(k,l))$. It follows from the above, that

$$\|x\| = \|\pi(x)\| = \|(\phi \circ \pi)(x)\| = \|(\rho \circ i)(x)\| \leq \|i(x)\| \, .$$

This proves that $i : C(W_q(k,l)) \to C(S^3_q)_{(0)}$ is an isometry and it is therefore injective. □

Let $\mathscr{L}^1$ denotes the trace class operators on the Hilbert space $l^2(\mathbb{N}_0)$.

**Lemma 5.7.** *The $*$-homomorphism $\pi := \oplus_{s=1}^{l} \pi_s : \mathcal{O}(W_q(k,l)) \to \widetilde{\oplus_{s=1}^{l} \mathscr{K}}$ factorizes through the unital $*$-subalgebra $\widetilde{\oplus_{s=1}^{l} \mathscr{L}^1}$.*

*Proof.* Let $s \in \{1, \ldots, l\}$. We only need to show that $\pi_s(z_0^l (z_1^*)^k), \pi_s(z_1 z_1^*) \in \mathscr{L}^1$.

With notation $a := z_0^l (z_1^*)^k$ and $b := z_1 z_1^*$, the operator $\pi_s(b) : l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$ is positive and diagonal with eigenvalues $\{q^{2s} \, q^{2lp}\}_{p=0}^{\infty}$ each of multiplicity 1.

It is immediate to show that $\pi_s(b)^{1/2} \in \mathscr{L}^1$. Indeed, from (5.1),

$$\mathrm{Tr}(\pi_s(b)^{1/2}) = \sum_{p=0}^{\infty} \, q^s \, q^{lp} = q^s \, (1-q^l)^{-1} < \infty \, ,$$

having restricted the deformation parameter to $q \in (0,1)$. From $\pi_s(b)^{1/2} \in \mathscr{L}^1$ the inclusion $\pi_s(b) \in \mathscr{L}^1$ follows as well.

To obtain that $\pi_s(a) \in \mathscr{L}^1$ we need to verify that $|\pi_s(a)| \in \mathscr{L}^1$. Now, recall that

$$a^* a = b^k \cdot \prod_{m=1}^{l} (1 - q^{-2m} b) \, .$$

Using this relation, we may compute the absolute value:

$$|\pi_s(a)| = \pi_s(b)^{k/2} \cdot \Big( \prod_{m=1}^{l} (1 - q^{-2m} \pi_s(b)) \Big)^{1/2} \, .$$

Since $\mathscr{L}^1$ is an ideal in $\mathscr{L}(l^2(\mathbb{N}_0))$ we may thus conclude that $|\pi_s(a)| \in \mathscr{L}^1$. □

## 6. Quantum lens spaces

We define 3-dimensional quantum lens spaces $\mathcal{O}\big(L_q(dlk;k,l)\big)$ as fixed point algebras for the action of a finite cyclic group on the coordinate algebra of the quantum 3-sphere. We show that these spaces are quantum principal bundles over quantum weighted projective spaces. Our examples are more general than those of [5]. As said the enveloping $C^*$-algebras of the lens spaces will be given as Pimsner algebras.

### 6.1. **Coordinate algebras.**

Let $k,l \in \mathbb{N}$ be coprime positive integers. For each $d \in \mathbb{N}$ define the action of the cyclic group $\mathbb{Z}/(dlk)\mathbb{Z}$ on the quantum sphere $S^3_q$,

$$\alpha^{1/d} : \mathbb{Z}/(dlk)\mathbb{Z} \times \mathcal{O}(S^3_q) \to \mathcal{O}(S^3_q)\,,$$

by letting on generators:

$$\alpha^{1/d}(1,z_0) := \exp\Big(\frac{2\pi \mathrm{i}}{dl}\Big)\, z_0 \qquad \text{and} \qquad \alpha^{1/d}(1,z_1) := \exp\Big(\frac{2\pi \mathrm{i}}{dk}\Big)\, z_1\,. \tag{6.1}$$

**Definition 6.1.** The coordinate algebra for the *quantum lens space* $L_q(dlk;k,l)$ is the fixed point algebra of the action $\alpha^{1/d}$. This unital $*$-algebra is denoted by $\mathcal{O}\big(L_q(dlk;k,l)\big)$. Thus

$$\mathcal{O}\big(L_q(dlk;k,l)\big) := \big\{x \in \mathcal{O}(S^3_q) \mid \alpha^{1/d}(1,x) = x\big\}\,.$$

The elements $z_0^l(z_1^*)^k$ and $z_1z_1^*$, generating the weighted projective space algebra $\mathcal{O}(W_q(k,l))$, are clearly invariant leading, for any $d \in \mathbb{N}$, to an algebra inclusion

$$\mathcal{O}(W_q(k,l)) \hookrightarrow \mathcal{O}\big(L_q(dlk;k,l)\big)\,.$$

Next, for each $n \in \mathbb{N}_0$, consider the subspaces of $\mathcal{O}(S^3_q)$ given by

$$\begin{aligned}
\mathscr{A}_{(n)}(k,l) &:= \sum_{j=0}^{n} (z_0^*)^{lj}(z_1^*)^{k(n-j)} \cdot \mathcal{O}(W_q(k,l))\,,\\
\mathscr{A}_{(-n)}(k,l) &:= \sum_{j=0}^{n} (z_0)^{lj}(z_1)^{k(n-j)} \cdot \mathcal{O}(W_q(k,l))\,.
\end{aligned} \tag{6.2}$$

By construction these subspaces are in fact right-modules over $\mathcal{O}(W_q(k,l))$.

Recall that the algebra $\mathcal{O}(S^3_q)$ admits [23] a vector space basis given by the vectors $\{e_{p,r,s} \mid p \in \mathbb{Z},\ r,s \in \mathbb{N}_0\}$, where

$$e_{p,r,s} = \begin{cases} z_0^p z_1^r (z_1^*)^s & \text{for} \quad p \geq 0 \\ (z_0^*)^{-p} z_1^r (z_1^*)^s & \text{for} \quad p \leq 0 \end{cases}.$$

**Lemma 6.2.** *Let $n \in \mathbb{Z}$. It holds that*

$$\begin{aligned}
e_{p,r,s} \in \mathscr{A}_{(n)}(k,l) &\Leftrightarrow pk + (r-s)l = -nkl \\
&\Leftrightarrow \sigma^{k,l}_w(e_{p,r,s}) = w^{-nkl} e_{p,r,s}\,,\ \forall\, w \in S^1\,.
\end{aligned}$$

*As a consequence, it holds that*

$$x \in \mathscr{A}_{(n)}(k,l) \Leftrightarrow \sigma^{k,l}_w(x) = w^{-nkl} x\,,\ \forall\, w \in S^1\,.$$

*Proof.* Clearly one has that

$$e_{p,r,s} \in \mathscr{A}_{(n)}(k,l) \Rightarrow pk + (r-s)l = -nkl$$
$$\Leftrightarrow \sigma_w^{k,l}(e_{p,r,s}) = w^{-nkl} e_{p,r,s}\,, \ \forall\, w \in S^1\,.$$

Thus, it only remains to prove the implication

$$pk + (r-s)l = -nkl \Rightarrow e_{p,r,s} \in \mathscr{A}_{(n)}(k,l)\,.$$

Then, suppose $pk+(r-s)l = -nkl$. Since $k,l \in \mathbb{N}$ are coprime there exists integers $d_0, d_1 \in \mathbb{Z}$ such that $p = d_0 l$ and $(r-s) = d_1 k$. Furthermore, $d_0 + d_1 = -n$. Suppose first that $(r-s)\,,\ p \geq 0$. Then,

$$e_{p,r,s} = z_0^p z_1^{(r-s)} (z_1 z_1^*)^s = z_0^{ld_0} z_1^{kd_1} (z_1 z_1^*)^s \in \mathscr{A}_{(-d_0-d_1)}(k,l) = \mathscr{A}_{(n)}(k,l)\,.$$

Suppose next that $p \geq 0$ and $(r-s) \leq 0$. Then,

$$e_{p,r,s} = z_0^p (z_1^*)^{s-r} (z_1 z_1^*)^r = z_0^{ld_0} (z_1^*)^{-d_1 k} (z_1 z_1^*)^r\,.$$

We now have two sub-cases: Either $d_0 \geq -d_1$ or $-d_1 \geq d_0$. When $d_0 \geq -d_1$, it follows from the above that

$$e_{p,r,s} = z_0^{l(d_0+d_1)} z_0^{-d_1 l} (z_1^*)^{-d_1 k} (z_1 z_1^*)^r \in \mathscr{A}_{(n)}(k,l)\,.$$

On the other hand, if $-d_1 \geq d_0$, we have that

$$e_{p,r,s} = z_0^{ld_0} (z_1^*)^{kd_0} (z_1^*)^{(-d_1-d_0)k} (z_1 z_1^*)^r \in \mathscr{A}_{(n)}(k,l)\,.$$

The remaining two cases (when $p \leq 0$ and $(r-s) \geq 0$ and when $p\,,\ (r-s) \leq 0$) follow by similar arguments. This proves the lemma. □

**Proposition 6.3.** *The subspaces $\{\mathscr{A}_{(dn)}(k,l)\}_{n\in\mathbb{Z}}$ gives $\mathcal{O}(L_q(dlk;k,l))$ the structure of a $\mathbb{Z}$-graded unital $*$-algebra.*

*Proof.* We need to prove that the vector space sum provides a bijection

$$\oplus_{n\in\mathbb{Z}} \mathscr{A}_{(dn)}(k,l) \to \mathcal{O}(L_q(dlk;k,l))\,.$$

Suppose thus that $\sum_{n\in\mathbb{Z}} x_n = 0$ where $x_n \in \mathscr{A}_{(dn)}(k,l)$ for all $n \in \mathbb{Z}$ and $x_n = 0$ for all but finitely many $n \in \mathbb{Z}$. It then follows from Lemma 6.2 that the terms $x_n$ lie in different homogeneous spaces for the circle action $\{\sigma_w^{k,l}\}_{w\in S^1}$ on $\mathcal{O}(S_q^3)$. We may then conclude that $x_n = 0$ for all $n \in \mathbb{Z}$. This proves the claimed injectivity.

Next, let $x \in \mathcal{O}(L_q(dlk;k,l))$. Without loss of generality we may take $x = e_{p,r,s}$ for some $p \in \mathbb{Z}$ and $r,s \in \mathbb{N}_0$. The fact that $x \in \mathcal{O}(L_q(dlk;k,l))$ then means that

$$p/(dl) + (r-s)/(dk) \in \mathbb{Z} \ \Leftrightarrow\ pk + (r-s)l \in (dkl)\,\mathbb{Z}$$

It then follows from Lemma 6.2 that $e_{p,r,s} \in \sum_{n\in\mathbb{Z}} \mathscr{A}_{(dn)}(k,l)$. This proves surjectivity.

Finally, let $x \in \mathscr{A}_{(dn)}(k,l)$ and $y \in \mathscr{A}_{(dm)}(k,l)$. It only remains to prove that $x^* \in \mathscr{A}_{(-dn)}(k,l)$ and $x\,y \in \mathscr{A}_{(d(n+m))}(k,l)$. But these properties also follow immediately from Lemma 6.2 since $\sigma_w^{k,l}$ is a $*$-automorphism of $\mathcal{O}(S_q^3)$ for each $w \in S^1$. □

6.2. **Lens spaces as quantum principal bundles.** The right-modules $\mathscr{A}_{(1)}(k,l)$ and $\mathscr{A}_{(-1)}(k,l)$ play a central role. Recall from (6.2) that they are given by

$$\mathscr{A}_{(1)}(k,l) := (z_1^*)^k \cdot \mathcal{O}(W_q(k,l)) + (z_0^*)^l \cdot \mathcal{O}(W_q(k,l)) \qquad \text{and}$$
$$\mathscr{A}_{(-1)}(k,l) := z_1^k \cdot \mathcal{O}(W_q(k,l)) + z_0^l \cdot \mathcal{O}(W_q(k,l)) \,.$$

**Proposition 6.4.** *There exist elements*

$$\xi_1, \xi_2, \beta_1, \beta_2 \in \mathscr{A}_{(1)}(k,l) \qquad \textit{and} \qquad \eta_1, \eta_2, \alpha_1, \alpha_2 \in \mathscr{A}_{(-1)}(k,l)$$

*such that*

$$\xi_1\eta_1 + \xi_2\eta_2 = 1 = \alpha_1\beta_1 + \alpha_2\beta_2$$

*Proof.* Firstly, a repeated use of the defining relations of the algebra $\mathcal{O}(S_q^3)$ leads to

$$(z_0^*)^l z_0^l = \prod_{m=1}^{l} (1 - q^{-2m} z_1 z_1^*) \,.$$

Then, define the polynomial $F \in \mathbb{C}[X]$ by the formula

$$F(X) := \Big(1 - \prod_{m=1}^{l} (1 - q^{-2m} X)\Big)/X \,.$$

Since $z_1 z_1^* = z_1^* z_1$ one has that

$$(z_0^*)^l z_0^l + z_1^* \ F(z_1 z_1^*) \ z_1 = 1 \,.$$

In particular, this implies that

$$\begin{aligned} 1 &= \big((z_0^*)^l z_0^l + z_1^* \ F(z_1 z_1^*) \ z_1\big)^k = \sum_{j=0}^{k} \big((z_0^*)^l z_0^l\big)^j \ \big(z_1^* \ F(z_1 z_1^*) \ z_1\big)^{k-j} \binom{k}{j} \\ &= (z_1^*)^k \big(F(z_1 z_1^*)\big)^k z_1^k + \sum_{j=1}^{k} \big((z_0^*)^l z_0^l\big)^j \ \big(1 - (z_0^*)^l z_0^l\big)^{k-j} \binom{k}{j} \\ &= (z_1^*)^k \big(F(z_1 z_1^*)\big)^k z_1^k + (z_0^*)^l \left\{ \sum_{j=1}^{k} \big(z_0^l (z_0^*)^l\big)^{j-1} \big(1 - z_0^l (z_0^*)^l\big)^{k-j} \binom{k}{j} \right\} z_0^l \,. \end{aligned}$$

Define now the polynomial $G \in \mathbb{C}[X]$ by the formula

$$G(X) := (1 - (1-X)^k)/X = \sum_{j=1}^{k} X^{j-1} (1-X)^{k-j} \binom{k}{j} \,, \tag{6.3}$$

so that

$$\sum_{j=1}^{k} \big(z_0^l (z_0^*)^l\big)^{j-1} \big(1 - z_0^l (z_0^*)^l\big)^{k-j} \binom{k}{j} = G\big(z_0^l (z_0^*)^l\big) \,.$$

And this enables us to write the above identities as

$$1 = (z_1^*)^k \big(F(z_1 z_1^*)\big)^k z_1^k + (z_0^*)^l G\big(z_0^l (z_0^*)^l\big) z_0^l \,. \tag{6.4}$$

Notice that both $F(z_1 z_1^*)$ and $G\big(z_0^l (z_0^*)^l\big)$ belong to $\mathcal{O}(W_q(k,l))$. We thus define

$$\xi_1 := (z_1^*)^k \big(F(z_1 z_1^*)\big)^k \,, \qquad \eta_1 := z_1^k \,,$$
$$\xi_2 := (z_0^*)^l \, G\big(z_0^l (z_0^*)^l\big) \,, \qquad \eta_2 := z_0^l$$

and this proves the first half of the proposition.

To prove the second half, we consider instead the identity

$$z_0^l (z_0^*)^l = \prod_{m=0}^{l-1} (1 - q^{2m} z_1^* z_1) \,,$$

which again follows by a repeated use of the defining identities for $\mathcal{O}(S_q^3)$.

The polynomial $\widetilde{F} \in \mathbb{C}[X]$ is now given by the formula

$$\widetilde{F}(X) := \Big(1 - \prod_{m=0}^{l-1} (1 - q^{2m} X)\Big)/X \,.$$

and we obtain that

$$z_0^l (z_0^*)^l + z_1 \tilde{F}(z_1 z_1^*) z_1^* = 1 \,.$$

By taking $k^{\text{th}}$ powers and computing as above, this yields that

$$1 = z_1^k \big(\tilde{F}(z_1 z_1^*)\big)^k (z_1^*)^k + z_0^l \left\{ \sum_{j=1}^{k} \binom{k}{j} \big((z_0^*)^l z_0^l\big)^{j-1} \big(1 - (z_0^*)^l z_0^l\big)^{k-j} \right\} (z_0^*)^l \,.$$

This identity may be rewritten as

$$1 = z_1^k \big(\widetilde{F}(z_1 z_1^*)\big)^k (z_1^*)^k + z_0^l G\big((z_0^*)^l z_0^l\big) (z_0^*)^l \,,$$

where $G \in \mathbb{C}[X]$ is again the one defined by (6.3).
Since both $\widetilde{F}(z_1 z_1^*)$ and $G\big((z_0^*)^l z_0^l\big)$ belong to $\mathcal{O}(W_q(k,l))$ we define

$$\alpha_1 := z_1^k \big(\widetilde{F}(z_1 z_1^*)\big)^k \,, \qquad \beta_1 := (z_1^*)^k \,,$$
$$\alpha_2 := z_0^l \, G\big((z_0^*)^l z_0^l\big) \,, \qquad \beta_2 := (z_0^*)^l \,.$$

This ends the proof of the present proposition. □

The next proposition is now an immediate consequence of Proposition 6.3, Proposition 6.4, Theorem 4.3, and Proposition 4.6.

**Proposition 6.5.** *The triple* $\big(\mathcal{O}(L_q(dlk;k,l), \mathcal{O}(U(1)), \mathcal{O}(W_q(k,l))\big)$ *is a quantum principal* $U(1)$*-bundle for each* $d \in \mathbb{N}$.

6.3. $C^*$**-completions.** We fix $k, l \in \mathbb{N}$ to be coprime positive integers. Let $d \in \mathbb{N}$. With $C(S_q^3)$ the $C^*$-algebra of continuous functions on the quantum sphere $S_q^3$, the action of the cyclic group $\mathbb{Z}/(dlk)\mathbb{Z}$ given on generators in (6.1) results into an action

$$\alpha^{1/d} : \mathbb{Z}/(dkl)\mathbb{Z} \times C(S_q^3) \to C(S_q^3) \,.$$

**Definition 6.6.** The $C^*$-algebra of continuous functions on the *quantum lens space* $L_q(dlk;k,l)$ is the fixed point algebra of this action. It is denoted by $C(S_q^3)^{1/d}$. Thus

$$C(S_q^3)^{1/d} := \big\{ x \in C(S_q^3) \mid \alpha^{1/d}(1,x) = x \big\} \,.$$

**Lemma 6.7.** *The $C^*$-quantum lens space $C(S^3_q)^{1/d}$ is the closure of the algebraic quantum lens space $\mathcal{O}(L_q(dkl;k,l))$ with respect to the universal $C^*$-norm on $\mathcal{O}(S^3_q)$.*

*Proof.* This follows by applying the bounded operator $E_{1/d} : C(S^3_q) \to C(S^3_q)^{1/d}$,

$$E_{1/d} : x \mapsto \frac{1}{dkl} \sum_{m=1}^{dkl} \alpha^{1/d}([m], x)\,,$$

with $[m]$ denoting the residual class in $\mathbb{Z}/(dkl)\mathbb{Z}$ of the integer $m$. □

Alternatively, and in parallel with Definition 5.3, we could define the $C^*$-quantum lens space as the universal enveloping $C^*$-algebra of the algebraic quantum lens space $\mathcal{O}(L_q(dkl;k,l))$. We will denote this $C^*$-algebra by $C(L_q(dkl;k,l))$.

**Lemma 6.8.** *For all $d \in \mathbb{N}$, the identity map $\mathcal{O}(L_q(dkl;k,l)) \to \mathcal{O}(L_q(dkl;k,l))$ induces an isomorphisms of $C^*$-algebras,*

$$C(S^3_q)^{1/d} \simeq C(L_q(dkl;k,l))\,.$$

*Proof.* We use Theorem 3.10. Indeed, let $d \in \mathbb{N}$ and let $\|\cdot\| : \mathcal{O}(S^3_q) \to [0,\infty)$ and $\|\cdot\|' : \mathcal{O}(L_q(dkl;k,l)) \to [0,\infty)$ denote the universal $C^*$-norms of the two different unital $*$-algebras in question. We then have $\|x\| \leq \|x\|'$ for all $x \in \mathcal{O}(L_q(dkl;k,l))$ since the inclusion $\mathcal{O}(L_q(dkl;k,l)) \to \mathcal{O}(S^3_q)$ induce a $*$-homomorphism $C(L_q(dkl;k,l)) \to C(S^3_q)^{1/d}$. But we also have $\|x\|' \leq \|x\|$ since the restriction $\|\cdot\| : \mathcal{O}(W_q(k,l)) \to [0,\infty)$ is the maximal $C^*$-norm on $\mathcal{O}(W_q(k,l))$ by Lemma 5.6. □

From now on, to lighten the notation, denote by $B := C(W_q(k,l))$ the $C^*$-quantum weighted projective line. Furthermore, let $E$ denote the Hilbert $C^*$-module over $B$ obtained as the closure of the module $\mathscr{A}_{(1)}(k,l)$ in the universal $C^*$-norm on the quantum sphere $\mathcal{O}(S^3_q)$. As usual, we let $\phi : B \to \mathscr{L}(E)$ denote the $*$-homomorphism induced by the left multiplication $B \times C(S^3_q) \to C(S^3_q)$.

We are ready to realize the $C^*$-quantum lens spaces as Pimsner algebras.

**Theorem 6.9.** *For all $d \in \mathbb{N}$, there is an isomorphism of $C^*$-algebras,*

$$\mathcal{O}_{E^{\widehat{\otimes}_\phi d}} \simeq C(S^3_q)^{1/d}\,,$$

*given by*

$$S_{\xi_1 \otimes \ldots \otimes \xi_d} \mapsto \xi_1 \cdot \ldots \cdot \xi_d \quad \text{for all} \quad \xi_1, \ldots, \xi_d \in E\,.$$

*Proof.* Recall from Proposition 6.3 that, for all $d \in \mathbb{N}$, it holds that

$$\mathcal{O}(L_q(dlk;k,l)) \simeq \oplus_{n\in\mathbb{Z}} \mathscr{A}_{(dn)}(k,l)\,.$$

Let us denote by $\{\rho_w\}_{w\in S^1}$ the associated circle action on $\mathcal{O}(L_q(dlk;k,l))$. Then, we have $\|\rho_w(x)\| \leq \|x\|$ for all $x \in \mathcal{O}(L_q(dlk;k,l))$ and all $w \in S^1$, where $\|\cdot\|$ is the norm on $C(S^3_q)^{1/d}$ (the restriction of the maximal $C^*$-norm on $C(S^3_q)$). To see this, choose a $z \in S^1$ such that $z^{dkl} = w$. Then $\sigma^{(k,l)}_z(x) = \rho_w(x)$, where the weighted circle action $\sigma^{(k,l)} : S^1 \times C(S^3_q) \to C(S^3_q)$ is the one defined at the beginning of §5.1.

An application of Theorem 3.9 now shows that $\mathcal{O}_{E^{\widehat{\otimes}_\phi d}} \simeq C(S^3_q)^{1/d}$ for all $d \in \mathbb{N}$, provided that $\{\rho_w\}_{w\in S^1}$ satisfies the conditions of Assumption 3.1. To this end, taking into account the analysis of the coordinate algebra $\mathcal{O}(L_q(lk;k,l))$ provided in §6.1, the only non-trivial thing to check is that the collections

$$\langle E, E\rangle := \operatorname{span}\big\{\xi^*\eta \mid \xi,\eta \in E\big\} \qquad \text{and} \qquad \langle E^*, E^*\rangle := \operatorname{span}\big\{\xi\eta^* \mid \xi,\eta \in E\big\}$$

are dense in $C(W_q(k,l))$. But this follows at once from Proposition 6.4. □

## 7. $KK$-theory of quantum lens spaces

We now combine the results obtained until this point and, using methods coming from the Pimsner algebra constructions, we are able to compute the $KK$-theory of the quantum lens spaces $L_q(dkl;k,l)$ for any coprime $k,l \in \mathbb{N}$ and any $d \in \mathbb{N}$.

As before we let $E$ denote the Hilbert $C^*$-module over the quantum weighted projective line $C(W_q(k,l))$ which is obtained as the closure of $\mathscr{A}_{(1)}(k,l)$ in $C(S^3_q)$.

The two polynomials in $\mathcal{O}(W_q(k,l))$ in the proof of Proposition 6.4, written as

$$\begin{aligned} (F(z_1z_1^*))^k &= \Big(\big(1-(z_0^*)^l z_0^l\big)/(z_1z_1^*)\Big)^k \qquad \text{and} \\ G\big(z_0^l(z_0^*)^l\big) &= \big(1-(1-z_0^l(z_0^*)^l)^k\big)/\big(z_0^l(z_0^*)^l\big)\,, \end{aligned}$$

are manifestly positive, since $\|z_1z_1^*\| \leq 1$ and thus also $\|z_0^l(z_0^*)^l\|, \|(z_0^*)^l z_0^l\| \leq 1$ in $C(W_q(k,l))$. Thus it makes sense to take their square roots:

$$\begin{aligned} \xi_1 &:= F(z_1z_1^*)^{k/2} = \Big(\big(1-(z_0^*)^l z_0^l\big)/(z_1z_1^*)\Big)^{k/2} \in C(W_q(k,l)) \qquad \text{and} \\ \xi_0 &:= G\big(z_0^l(z_0^*)^l\big)^{1/2} = \Big(\big(1-(1-z_0^l(z_0^*)^l)^k\big)/\big(z_0^l(z_0^*)^l\big)\Big)^{1/2} \in C(W_q(k,l))\,. \end{aligned}$$

Next, define the morphism of Hilbert $C^*$-modules $\Psi : E \to C(W_q(k,l))^2$ by

$$\Psi : \eta \mapsto \begin{pmatrix} \xi_1 z_1^k\ \eta \\ \xi_0 z_0^l\ \eta \end{pmatrix},$$

whose adjoint $\Psi^* : C(W_q(k,l))^2 \to E$ is then given by

$$\Psi^* : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto (z_1^*)^k \xi_1\ x + (z_0^*)^l \xi_0\ y\,.$$

It then follows from (6.4) that $\Psi^*\Psi = \mathrm{id}_E$. The associated orthogonal projection is

$$P := \Psi\Psi^* = \begin{pmatrix} \xi_1\ (z_1z_1^*)^k\ \xi_1 & \xi_1\ z_1^k(z_0^*)^l\ \xi_0 \\ \xi_0\ z_0^l(z_1^*)^k\ \xi_1 & \xi_0\ z_0^l(z_0^*)^l\ \xi_0 \end{pmatrix} \in M_2(C(W_q(k,l)))\,. \tag{7.1}$$

7.1. **Fredholm modules over quantum weighted projective lines.** We recall [7, Chap. IV] that an *even Fredholm module* over a $*$-algebra $\mathscr{A}$ is a datum $(H,\rho,F,\gamma)$ where $H$ is a Hilbert space of a representation $\rho$ of $\mathscr{A}$, the operator $F$ on $H$ is such that $F^2 = F$ and $F^2 = 1$, with a $\mathbb{Z}/2\mathbb{Z}$-grading $\gamma$, $\gamma^2 = 1$, which commutes with the representation and such that $\gamma F + F\gamma = 0$. Finally, for all $a \in \mathscr{A}$ the commutator $[F,\rho(a)]$ is required to be compact. The Fredholm module is said to be 1-*summable* if the commutator $[F,\rho(a)]$ is trace class for all $a \in \mathscr{A}$.

Now, the quantum sphere $S^3_q$ is the 'underlying manifold' of the quantum group $\mathrm{SU}_q(2)$. The latter's counit when restricted to the subalgebra $\mathcal{O}(W_q(k,l))$ yields a one-dimensional representation $\varepsilon : \mathcal{O}(W_q(k,l)) \to \mathbb{C}$, simply given on generators by,

$$\varepsilon(z_1 z_1^*) = \varepsilon(z_0^l (z_1^*)^k) := 0\,, \quad \varepsilon(1) = 1\,.$$

Next, let $H := l^2(\mathbb{N}_0) \otimes \mathbb{C}^2$. We use the subscripts "+" and "−" to indicate that the corresponding spaces are thought of as being even or odd respectively, for a $\mathbb{Z}/2\mathbb{Z}$-grading $\gamma$:$H_\pm$ will be two copies of $H$. For each $s \in \{1, \ldots, l\}$, with the $*$-representation $\pi_s$ given in (5.1), define the even $*$-homomorphism

$$\rho_s : \mathcal{O}(W_q(k,l)) \to \mathscr{L}\big(H_+ \oplus H_-\big)\,, \quad \rho_s : x \mapsto \begin{pmatrix} \pi_s(\Psi x \Psi^*) & 0 \\ 0 & \varepsilon(\Psi x \Psi^*) \end{pmatrix}.$$

We are slightly abusing notation here: the element $\Psi x \Psi^*$ is a $2 \times 2$ matrix, hence $\pi_s$ and $\varepsilon$ have to be applied component-wise. Next, define

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2}$$

**Lemma 7.1.** *The datum $\mathscr{F}_s := \big(H_+ \oplus H_-, \rho_s, F, \gamma\big)$, defines an even 1-summable Fredholm module over the coordinate algebra $\mathcal{O}(W_q(k,l))$.*

*Proof.* It is enough to check that $\pi_s(\Psi z_1 z_1^* \Psi^*), \pi_s(\Psi z_0^l (z_1^*)^k \Psi^*) \in \mathscr{L}^1(H)$ and furthermore that $\pi_s(P) - \varepsilon(P) \in \mathscr{L}^1(H)$, for $P$ the projection in (7.1).

That the two operators involving the generators $z_1 z_1^*$ and $z_0^l (z_1^*)^k$ lie in $\mathscr{L}^1(H)$ follows easily from Lemma 5.7. To see that $\pi_s(P) - \varepsilon(P) \in \mathscr{L}^1(H)$ note that

$$\varepsilon(P) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The desired inclusion then follows since Lemma 5.7 yields that the operators $\pi_s(z_1 z_1^*)^k$, $\pi_s(z_0^l (z_1^*)^k)$, and $\pi_s(1 - z_0^l (z_0^*)^l)$ are of trace class. □

For $s = 0$, we take

$$\rho_0 := \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix} : C(W_q(k,l)) \to \mathscr{L}(\mathbb{C} \oplus \mathbb{C})$$

and define the even 1-summable Fredholm module

$$\mathscr{F}_0 := \big(\mathbb{C}_+ \oplus \mathbb{C}_-, \rho_0, F, \gamma\big)\,.$$

**Remark 7.2.** The 1-summable $l + 1$ Fredholm modules over $\mathcal{O}(W_q(k,l))$ we have defined are different from the 1-summable Fredholm modules defined in [5, §4]. The present Fredholm modules are obtained by "twisting" the Fredholm modules in [5] with the Hilbert $C^*$-module $E$.

7.2. **Index pairings.** Recall the representations $\pi_s$ of $C(W_q(k,l))$ given in (5.1).

For each $r \in \{1, \ldots, l\}$, let $p_r \in C(W_q(k,l))$ denote the orthogonal projection defined by the requirement

$$\pi_s(p_r) = \begin{cases} e_{00} & \text{for} \quad s = r \\ 0 & \text{for} \quad s \neq r \end{cases}, \tag{7.3}$$

where $e_{00} : l^2(\mathbb{N}_0) \to l^2(\mathbb{N}_0)$ denotes the orthogonal projection onto the closed subspace $\mathbb{C}e_0 \subseteq l^2(\mathbb{N}_0)$. For $r = 0$, let $p_0 = 1 \in C(W_q(k,l))$. The classes of these $l+1$ projections $\{p_r,\ r = 0,1,\ldots,l\}$ form a basis for the group $K_0(C(W_q(k,l)))$ given in Corollary 5.4.

On the other hand we have the classes in the $K$-homology group $K^0(C(W_q(k,l)))$ represented by the even 1-summable Fredholm modules $\mathscr{F}_s$, $s = 0,\ldots,l$, which we described in the previous paragraph.

We are interested in computing the index pairings

$$\langle [\mathscr{F}_s],[p_r]\rangle := \tfrac{1}{2}\operatorname{Tr}\big(\gamma F[F,\rho_s(p_r)]\big) \in \mathbb{Z}\,, \qquad \text{for } r,s \in \{0,\ldots,l\}\,.$$

**Proposition 7.3.** *It holds that:*

$$\langle [\mathscr{F}_s],[p_r]\rangle = \begin{cases} 1 & \text{for } \quad s = r \\ 1 & \text{for } \quad r = 0 \\ 0 & \qquad \text{else} \end{cases} .$$

*Proof.* Suppose first that $r,s \in \{1,\ldots,l\}$. We then have:

$$\langle [\mathscr{F}_s],[p_r]\rangle = \operatorname{Tr}\big(\pi_s(\Psi p_r \Psi^*)\big)\,,$$

and the above operator trace is well-defined since $\pi_s(\Psi p_r \Psi^*)$ is an orthogonal projection in $M_2(\mathscr{K})$ and it is therefore of trace class. We may then compute as follows:

$$\begin{aligned}\operatorname{Tr}\big(\pi_s(\Psi p_r \Psi^*)\big) &= \operatorname{Tr}\big(\pi_s(\xi_1 z_1^k p_r (z_1^*)^k \xi_1)\big) + \operatorname{Tr}\big(\pi_s(\xi_0 z_0^l p_r (z_0^*)^l \xi_0)\big) \\ &= \operatorname{Tr}\big(\pi_s(p_r (z_1^*)^k \xi_1^2 z_1^k)\big) + \operatorname{Tr}\big(\pi_s(p_r (z_0^*)^l \xi_0^2 z_0^l)\big) \\ &= \operatorname{Tr}\big(\pi_s(p_r)\big) = \delta_{sr}\,,\end{aligned}$$

where the second identity follows from [20, Cor. 3.8] and $\delta_{sr} \in \{0,1\}$ denotes the Kronecker delta.

If $r \in \{1,\ldots,l\}$ and $s = 0$, then $\rho_0(p_r) = 0$ and thus $\langle [\mathscr{F}_0],[p_r]\rangle = 0$.

Next, suppose that $r = s = 0$. Then

$$\langle [\mathscr{F}_0],[p_0]\rangle = \operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1\,.$$

Finally, suppose that $r = 0$ and $s \in \{1,\ldots,l\}$. We then compute

$$\begin{aligned}\langle [\mathscr{F}_s],[p_0]\rangle &= \operatorname{Tr}\big(\pi_s(P) - \varepsilon(P)\big) = \operatorname{Tr}\big(\pi_s(\xi_1^2 (z_1 z_1^*)^k)\big) + \operatorname{Tr}\big(\pi_s(\xi_0 z_0^l (z_0^*)^l \xi_0) - 1\big) \\ &= \operatorname{Tr}\big(\pi_s(1 - (z_0^*)^l z_0^l)^k\big) - \operatorname{Tr}\big(\pi_s(1 - z_0^l (z_0^*)^l)^k\big)\,.\end{aligned}$$

We will prove in the next lemma that this quantity is equal to 1. This will complete the proof of the present proposition. □

**Lemma 7.4.** *It holds that:*

$$\operatorname{Tr}\big(\pi_s(1 - (z_0^*)^l z_0^l)^k\big) - \operatorname{Tr}\big(\pi_s(1 - z_0^l (z_0^*)^l)^k\big) = \operatorname{Tr}\big(\pi_s([z_0^l,(z_0^*)^l])\big) = 1\,.$$

*Proof.* Notice firstly that $\pi_s\big(1-(z_0^*)^l z_0^l\big), \pi_s\big(1-z_0^l (z_0^*)^l\big) \in \mathscr{L}^1(l^2(\mathbb{N}_0))$ by Lemma 5.7. It then follows by induction that

$$\operatorname{Tr}\big(\pi_s(1 - (z_0^*)^l z_0^l)^k\big) - \operatorname{Tr}\big(\pi_s(1 - z_0^l (z_0^*)^l)^k\big) = \operatorname{Tr}\big(\pi_s([z_0^l,(z_0^*)^l])\big)\,.$$

Indeed, with $x := z_0^l$, for all $j \in \{2, 3, \ldots\}$, one has that,

$$\begin{aligned}
&\mathrm{Tr}\big(\pi_s(1-x^*x)^j\big) - \mathrm{Tr}\big(\pi_s(1-xx^*)^j\big) \\
&\qquad = \mathrm{Tr}\big(\pi_s(1-x^*x)^{j-1}\big) - \mathrm{Tr}\big(\pi_s(xx^*(1-xx^*)^{j-1})\big) - \mathrm{Tr}\big(\pi_s(1-xx^*)^j\big) \\
&\qquad = \mathrm{Tr}\big(\pi_s(1-x^*x)^{j-1}\big) - \mathrm{Tr}\big(\pi_s(1-xx^*)^{j-1}\big) .
\end{aligned}$$

It therefore suffices to show that $\mathrm{Tr}\big(\pi_s([z_0^l, (z_0^*)^l])\big) = 1$. Now, one has:

$$[z_0^l, (z_0^*)^l] = \sum_{m=0}^{l} (-1)^m q^{m(m-1)} \binom{l}{m}_{q^2} (1 - q^{-2ml})\ (z_1 z_1^*)^m$$

where the notation $\binom{l}{m}_{q^2}$ refers to the $q^2$-binomial coefficient, defined by the identity

$$\prod_{m=1}^{l} (1 + q^{2(m-1)} Y) = \sum_{m=0}^{l} q^{m(m-1)} \binom{l}{m}_{q^2} Y^m$$

in the polynomial algebra $\mathbb{C}[Y]$. Then, as in [5, Prop. 4.3] one computes:

$$\begin{aligned}
\mathrm{Tr}\big(\pi_s([z_0^l, (z_0^*)^l])\big) &= \sum_{m=1}^{l} (-1)^m q^{m(m-1)} \binom{l}{m}_{q^2} (1 - q^{-2ml}) \frac{q^{2ms}}{1 - q^{2ml}} \\
&= 1 - \sum_{m=0}^{l} (-1)^m q^{m(m-1)} \binom{l}{m}_{q^2} q^{2m(s-l)} \\
&= 1 - \prod_{m=1}^{l} (1 - q^{2(s-m)}) = 1 ,
\end{aligned}$$

since, due to $s \in \{1, \ldots, l\}$ one of the factors in the product must vanish. □

**Remark 7.5.** The non-vanishing of the pairings in Proposition 7.3 for $r = 0$ means that the class of the projection $P$ in (7.1) is non-trivial in $K_0(C(W_q(k,l)))$. (In this case the pairings are computing the couplings of the Fredholm modules of [5, §4] with the projection $P$.) Geometrically this means that the line bundle $\mathscr{A}_{(1)}(k,l)$ over $\mathcal{O}(W_q(k,l))$ and then the quantum principal $U(1)$-bundle $\mathcal{O}(W_q(k,l))) \hookrightarrow \mathcal{O}(L_q(dlk); k, l)$ are non-trivial.

7.3. **Gysin sequences.** To ease the notation, we now let $C(W_q) := C(W_q(k,l))$ and $C(L_q(d)) := C(L_q(dkl; k, l))$. Also as before we let $E$ denote the Hilbert $C^*$-module over $C(W_q)$ obtained as the closure of $\mathscr{A}_{(1)}(k,l)$ in $C(S_q^3)$. The $*$-homomorphism $\phi : C(W_q) \to \mathscr{L}(E)$ is induced by the product on $C(S_q^3)$.

For each $d \in \mathbb{N}$, let $[E^{\widehat{\otimes} d}] \in KK(C(W_q), C(W_q))$ denote the class of the Hilbert $C^*$-module $E^{\widehat{\otimes}_\phi d}$ as in Definition 2.5. And recall from Theorem 6.9 that the Pimnser algebra $\mathcal{O}_{E^{\widehat{\otimes}_\phi d}}$ can be identified with $C(L_q(d))$:

$$\mathcal{O}_{E^{\widehat{\otimes}_\phi d}} \simeq C(L_q(d)) .$$

Then, given any separable $C^*$-algebra $B$, by Theorem 2.7 we obtain two six term exact sequences:

$$\begin{array}{ccccc} KK_0(B,C(W_q)) & \xrightarrow{1-[E^{\widehat{\otimes} d}]} & KK_0(B,C(W_q)) & \xrightarrow{i_*} & KK_0\big(B,C(L_q(d))\big) \\ {\scriptstyle [\partial]}\big\uparrow & & & & \big\downarrow{\scriptstyle [\partial]} \\ KK_1(B,C(L_q(d))) & \xleftarrow[i_*]{} & KK_1(B,C(W_q)) & \xleftarrow[1-[E^{\widehat{\otimes} d}]]{} & KK_1(B,C(W_q)) \end{array} \tag{7.4}$$

and

$$\begin{array}{ccccc} KK_0(C(W_q),B) & \xleftarrow[1-[E^{\widehat{\otimes} d}]]{} & KK_0(C(W_q),B) & \xleftarrow[i^*]{} & KK_0\big(C(L_q(d)),B\big) \\ \big\downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\big\uparrow \\ KK_1\big(C(L_q(d)),B\big) & \xrightarrow{i^*} & KK_1(C(W_q),B) & \xrightarrow{1-[E^{\widehat{\otimes} d}]} & KK_1(C(W_q),B) \end{array} \;. \tag{7.5}$$

We will refer to these two sequences as the *Gysin sequences* (in $KK$-theory) for the quantum lens space $L_q(dkl;k,l)$.

**Remark 7.6.** For $B=\mathbb{C}$, the first sequence above was first constructed in [2] for quantum lens spaces in any dimension $n$ (and not just for $n=1$) but with weights all equal to one; so that the 'base space' was a quantum projective space.

7.4. **Computing the $KK$-theory of quantum lens spaces.** We recall from [5, Prop. 5.1] that $C(W_q)$ is isomorphic to $\widetilde{\mathscr{K}^l}$ (see also §5.2). In particular, this means that $C(W_q)$ is $KK$-equivalent to $\mathbb{C}^{l+1}$.

To show this equivalence explicitly, for each $s \in \{0,\dots,l\}$ we define a $KK$-class $[\Pi_s] \in KK(C(W_q),\mathbb{C})$ via the Kasparov module $\Pi_s \in \mathbb{E}(C(W_q),\mathbb{C})$ given by:

$$\Pi_s := \big(l^2(\mathbb{N}_0)_+ \oplus l^2(\mathbb{N}_0)_-, \widetilde{\pi}_s, F, \gamma\big) \quad \text{for } s \neq 0 \quad \text{and} \quad \Pi_0 := (\mathbb{C},\varepsilon,0) \quad \text{for } s=0\,,$$

with $F$ and $\gamma$ the canonical operators in (7.2). The representation is

$$\widetilde{\pi}_s = \begin{pmatrix} \pi_s & 0 \\ 0 & \varepsilon \end{pmatrix},$$

with the representation $\pi_s$ given by (5.1) and $\varepsilon$ is (induced by) the counit.

Furthermore, for each $r \in \{0,\dots,l\}$ we define the $KK$-class $[I_r] \in KK(\mathbb{C},C(W_q))$ by the Kasparov module

$$I_r := \big(C(W_q), i_r, 0\big) \in \mathbb{E}(\mathbb{C},C(W_q))\,,$$

where $i_r : \mathbb{C} \to C(W_q)$ is the $*$-homomorphism defined by $i_r : 1 \mapsto p_r$ with the orthogonal projections $p_r \in C(W_q)$ given in (7.3).

Upon collecting these classes as

$$[\Pi] := \oplus_{s=0}^{l}[\Pi_s] \in KK(C(W_q),\mathbb{C}^{l+1}) \quad \text{and} \quad [I] := \oplus_{r=0}^{l}[I_r] \in KK(\mathbb{C}^{l+1},C(W_q))\,,$$

it follows that $[I]\widehat{\otimes}_{C(W_q)}[\Pi] = [1_{\mathbb{C}^{l+1}}]$ and that $[\Pi]\widehat{\otimes}_{\mathbb{C}^{l+1}}[I] = [1_{C(W_q)}]$, from stability of $KK$-theory (see [4, Cor. 17.8.8]).

We need a final tensoring with the Hilbert $C^*$-module $E$. This yields a class

$$[I_r]\widehat{\otimes}_{C(W_q)}[E]\widehat{\otimes}_{C(W_q)}[\Pi_s] \in KK(\mathbb{C},\mathbb{C})\,,$$

for each $s, r \in \{0, \dots, l\}$. Then, we let $M_{sr} \in \mathbb{Z}$ denote the corresponding integer in $KK(\mathbb{C}, \mathbb{C}) \simeq \mathbb{Z}$, with $M := \{M_{sr}\}_{s,r=0}^l \in M_{l+1}(\mathbb{Z})$ the corresponding matrix.

As a consequence the six term exact sequence in (7.4) becomes

$$\begin{array}{ccccc} \oplus_{r=0}^l K^0(B) & \xrightarrow{1-M^d} & \oplus_{s=0}^l K^0(B) & \longrightarrow & KK_0\big(B, C(L_q(d))\big) \\ \uparrow & & & & \downarrow \\ KK_1(B, C(L_q(d))) & \longleftarrow & \oplus_{s=0}^l K^1(B) & \xleftarrow[1-M^d]{} & \oplus_{r=0}^l K^1(B) \end{array} \tag{7.6}$$

while, with $M^t \in M_{l+1}(\mathbb{Z})$ denoting the matrix transpose of $M \in M_{l+1}(\mathbb{Z})$, the six term exact sequence in (7.5) becomes

$$\begin{array}{ccccc} \oplus_{s=0}^l K_0(B) & \xleftarrow[1-(M^t)^d]{} & \oplus_{r=0}^l K_0(B) & \longleftarrow & KK_0\big(C(L_q(d)), B\big) \\ \downarrow & & & & \uparrow \\ KK_1\big(C(L_q(d)), B\big) & \longrightarrow & \oplus_{r=0}^l K_1(B) & \xrightarrow{1-(M^t)^d} & \oplus_{s=0}^l K_1(B) \end{array} . \tag{7.7}$$

In order to proceed we therefore need to compute the matrix $M \in M_{l+1}(\mathbb{Z})$.

**Lemma 7.7.** *The Kasparov product $[E]\widehat{\otimes}_{C(W_q)}[\Pi_s] \in KK(C(W_q), \mathbb{C})$ is represented by the Fredholm module $\mathscr{F}_s$ in Lemma 7.1 for each $s \in \{0, \dots, l\}$.*

*Proof.* Recall firstly that the class $[E] \in KK(C(W_q), C(W_q))$ is represented by the Kasparov module

$$\big(E, \phi, 0\big) \in \mathbb{E}(C(W_q), C(W_q)),$$

where $\phi : C(W_q) \to \mathscr{L}(E)$ is induced by the product on the algebra $C(S_q^3)$. It then follows from the observations in the beginning of §7 that $(E, \phi, 0)$ is equivalent to the Kasparov module

$$\big(C(W_q)^2, \Psi\phi\Psi^*, 0\big) \in \mathbb{E}(C(W_q), C(W_q)).$$

Suppose next that $s = 0$. The Kasparov product $[E]\widehat{\otimes}_{C(W_q)}[\Pi_0]$ is then represented by the Kasparov module

$$\big(C(W_q)^2\widehat{\otimes}_\varepsilon\mathbb{C}, \Psi\phi\Psi^* \otimes 1, 0\big) \in \mathbb{E}(C(W_q), \mathbb{C}),$$

which is equivalent to the Kasparov module

$$\Big(\mathbb{C}_+ \oplus \mathbb{C}_-, \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big).$$

This proves the claim of the lemma in this case.

Suppose thus that $s \in \{1, \dots, l\}$. The Kasparov product $[E]\widehat{\otimes}_{C(W_q)}[\Pi_s]$ is then represented by the Kasparov module given by the $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space

$$\big(C(W_q)^2\widehat{\otimes}_{\pi_s} l^2(\mathbb{N}_0)\big)_+ \oplus \big(C(W_q)^2\widehat{\otimes}_\varepsilon l^2(\mathbb{N}_0)\big)_- \simeq H_+ \oplus H_-$$

with associated $*$-homomorphism

$$\rho_s = \begin{pmatrix} \pi_s(\Psi\phi\Psi^*) & 0 \\ 0 & \varepsilon(\Psi\phi\Psi^*) \end{pmatrix} : C(W_q) \to \mathscr{L}\big(H_+ \oplus H_-\big),$$

and with Fredholm operator $F$ and grading $\gamma$ the canonical ones in (7.2). This proves the claim of the lemma in these cases as well. □

The results of Lemma 7.7 and Proposition 7.3 now yield the following:

**Proposition 7.8.** *The matrix* $M = \{M_{sr}\} \in M_{l+1}(\mathbb{Z})$ *has entries*

$$M_{sr} = \langle [\mathscr{F}_s], [I_r] \rangle = \begin{cases} 1 & \text{for} \quad s = r \\ 1 & \text{for} \quad r = 0 \\ 0 & \quad \text{else} \end{cases} .$$

A combination of Proposition 7.8 and the six term exact sequences in (7.6) and (7.7) then allows us to compute the $K$-theory and the $K$-homology of the quantum lens space $L_q(dlk; k, l)$ for all $d \in \mathbb{N}$.

When $B = \mathbb{C}$, the sequence in (7.6) reduces to

$$0 \longrightarrow K_1(C(L_q(d)) \longrightarrow \mathbb{Z}^{l+1} \xrightarrow{1-M^d} \mathbb{Z}^{l+1} \longrightarrow K_0(C(L_q(d)) \longrightarrow 0$$

while the one in (7.7) becomes

$$0 \longleftarrow K^1(C(L_q(d)) \longleftarrow \mathbb{Z}^{l+1} \xleftarrow{1-(M^t)^d} \mathbb{Z}^{l+1} \longleftarrow K^0(C(L_q(d)) \longleftarrow 0\,.$$

Let us use the notation $\iota : \mathbb{Z} \to \mathbb{Z}^l$, $1 \mapsto (1, \dots, 1)$ for the diagonal inclusion and let $\iota^t : \mathbb{Z}^l \to \mathbb{Z}$ denote the transpose, $\iota^t : (m_1, \dots, m_l) \mapsto m_1 + \ldots + m_l$.

**Theorem 7.9.** *Let* $k, l \in \mathbb{N}$ *be coprime and let* $d \in \mathbb{N}$*. Then*

$$K_0\big(C(L_q(dlk; k, l))\big) \simeq \mathrm{Coker}(1 - M^d) \simeq \mathbb{Z} \oplus \big(\mathbb{Z}^l / \mathrm{Im}(d \cdot \iota)\big)$$
$$K_1\big(C(L_q(dlk; k, l))\big) \simeq \mathrm{Ker}(1 - M^d) \simeq \mathbb{Z}^l$$

*and*

$$K^0\big(C(L_q(dlk; k, l))\big) \simeq \mathrm{Ker}(1 - (M^t)^d) \simeq \mathbb{Z} \oplus \big(\mathrm{Ker}(\iota^t)\big)$$
$$K^1\big(C(L_q(dlk; k, l))\big) \simeq \mathrm{Coker}(1 - (M^t)^d) \simeq \mathbb{Z}/(d\mathbb{Z}) \oplus \mathbb{Z}^l\,.$$

We finish by stressing that the results on the $K$-theory and $K$-homology of the lens spaces $L_q(dlk; k, l)$ are different from the ones obtained for instance in [13]. In fact our lens spaces are not included in the class of lens spaces considered there. Thus, for the moment, there seems to be no alternative method which results in a computation of the $KK$-groups of these spaces.

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