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GROTHENDIECK DUALITY FOR PROJECTIVE DELIGNE-MUMFORD STACKS

FABIO NIRONI

Abstract. We develop Grothendieck duality for projective Deligne-Mumford stacks, in particular we prove the existence of a dualizing complex for a morphism from a projective stack to a scheme and for a proper representable morphism of algebraic stacks. In the first case we explicitly compute the dualizing complex and prove that Serre duality holds for smooth projective stacks in its usual form. We prove also that a projective stack has dualizing sheaf if and only if it is Cohen-Macaulay, it has a dualizing sheaf that is an invertible sheaf if and only if it is Gorenstein and for local complete intersections we explicitly compute the invertible sheaf. As an application of this general machinery we compute the dualizing sheaf of a nodal projective curve.

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Overview

This paper is devoted to the study of Grothendieck/Serre duality for projective Deligne-Mumford stacks; to be more specific, we prove Grothendieck duality for morphisms from a projective Deligne-Mumford stack to a scheme, and Grothendieck duality for proper representable morphisms.

The first section is foundational, it deals with the existence of the dualizing complex through the abstract machinery developed by Deligne in [Har66, Appendix] and refined by Neeman in [Nee96]. Having studied the property of flat base change of the dualizing complex 1.23, we are able to prove Serre Duality for smooth projective stacks and duality for finite morphisms. We obtain that the dualizing sheaf for a smooth projective stack is the canonical bundle shifted by the dimension of the stack. For a closed embedding $i\colon \mathcal{X} \to \mathcal{Y}$ in a smooth projective stack $\mathcal{Y}$ the dualizing complex of $\mathcal{X}$ is $\mathcal{E}xt^{\bullet}_{\mathcal{Y}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{Y}})$ where $\omega_{\mathcal{Y}}$ is the canonical bundle. This is a coherent sheaf if $\mathcal{X}$ is Cohen-Macaulay, an invertible sheaf if it is Gorenstein.

In the second section we use this abstract machinery to compute the dualizing sheaf of a projective nodal curve. We prove that the dualizing sheaf of a curve without smooth orbifold points is just the pullback of the dualizing sheaf of its moduli space. Smooth orbifold points give a non trivial contribution that can be computed using the root construction (Cadman [Cad07], Abramovich-Graber-Vistoli [AGV06]). We compute also the dualizing sheaf of a local complete intersection proving that it is the determinant of the cotangent complex shifted by the dimension of the stack as it is in the scheme theoretic setup.

This paper has been partially motivated by our study on semistable sheaves on projective stacks in [NF08]. In particular we have used Grothendieck duality to handle the definition of dual sheaf in the case of sheaves of non maximal dimension. Given a d-dimensional sheaf $\mathcal{F}$ on a projective Cohen-Macaulay stack $p\colon \mathcal{X} \to \operatorname{Spec} k$ over k an algebraically closed field, the dual $\mathcal{F}^D$ is defined to be $R\mathcal{H}om_{\mathcal{X}}(\mathcal{F}, p^! k)$ (as usual). If the sheaf $\mathcal{F}$ is torsion free on a smooth stack this is just $\mathcal{F}^\vee \otimes \omega_{\mathcal{X}}$ where $\omega_{\mathcal{X}}$ is the canonical bundle. Using Grothendieck duality we can prove that there is a natural morphism $\mathcal{F} \to \mathcal{F}^{DD}$ which is injective if and only if the sheaf is pure. We use this basic result in the GIT study of the moduli scheme of semistable pure sheaves [NF08, Lem 6.10].

Acknowledgments

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1. Foundation of duality for stacks

1.1. **History.** Serre duality for stacks can be easily proven with some ad hoc argument in specific examples, such as orbifold curves, gerbes, toric stacks and others; however a general enough proof requires some abstract machineries. Hartshorne's approach in Residues and Duality [Har66] is not suitable to be generalized to algebraic stacks (not in an easy way at least). In the appendix of the same book Deligne proves (in few pages) the following statement:

Lemma 1.1. *Let X be a quasi-compact scheme (non necessarily noetherian) and $QCoh_X$ the category of quasi coherent sheaves on X. Let $F\colon QCoh_X^\circ \to \mathfrak{Set}$ be a left exact contravariant functor sending filtered colimits to filtered limits, then the functor F is representable.*

Using this statement it's easy to prove the following:

Theorem 1.2. *Let $p\colon X \to Y$ be a morphism of separated noetherian schemes, F a sheaf on X and $C^\bullet(F)$ a functorial resolution of F acyclic with respect to p_*. Moreover let G be a quasi coherent sheaf on Y and $I^\bullet$ an injective resolution of G:*

(1) the functor $\operatorname{Hom}_Y(p_* C^q(F), I^p)$ is representable for every q, p and represented by a quasi coherent injective $p^!_q I^p$.

(2) the injective quasicoherent double-complex $p^!_q I^p$ defines a functor $p^!\colon D_{qc}(Y) \to D_{qc}(X)$ which is right adjoint of Rp_*:

$$R\operatorname{Hom}_Y(Rp_* F, G) \cong R\operatorname{Hom}_X(F, p^! G)$$

The proof of the first lemma relies on the fact that every finite presentation sheaf on X can be obtained as a colimit of sheaves of the kind $j_!\mathcal{O}_U$ where $j\colon U \to X$ is an open immersion, and every quasi coherent sheaf is a filtered colimit of finite presentation sheaves. In the case of noetherian algebraic stacks every quasi coherent sheaf is again a filtered colimit of coherent sheaves, however the first statement is trivially false.

We will prove Grothendieck duality for stacks using a further generalization of the previous technique. In [Nee96] Neeman proved Grothendieck duality using Brown representability theorem (adapted to triangulated category) and Bousfield localization.

Theorem 1.3. *Let $\mathcal{T}$ be a triangulated category which is compactly generated and $H\colon \mathcal{T}^\circ \to \mathfrak{Ab}$ be a homological functor. If the natural map:*

$$H\Big(\coprod_{\lambda\in\Lambda} x_\lambda\Big) \to \prod_{\lambda\in\Lambda} H(x_\lambda)$$

is an isomorphism for every small coproduct in $\mathcal{T}$, then H is representable.

If a scheme has an ample line bundle then it's easy to prove that $D_{qc}(X)$ is compactly generated. Every scheme admits locally an ample line bundle (take an affine cover then the structure sheaf is ample); to verify that local implies global Bousfield localization is used.

In the case of stacks we will prove that if $\mathcal{X}$ has a generating sheaf and X an ample invertible sheaf then $D_{qc}(\mathcal{X})$ is compactly generated, so that we can use Brown representability. It's true again that every Deligne-Mumford stack $\mathcal{X}$ has *étale* locally a generating sheaf [OS03, Prop 5.2], however the argument used by Neeman to prove that local implies global heavily relies on Zariski topology and cannot be generalized to stacks in an evident way.

Once existence and uniqueness are proved, we will be able to determine the shape of the dualizing functor in many examples computing it on injective sheaves. In many examples we will be able to compute locally a functor that behaves like a dualizing functor on injective sheaves; since sheaves can be glued we will assemble a global functor and we will be able to argue it is *the* dualizing complex by uniqueness (examples of this procedure are Prop 1.12, Cor 1.22, Lem 1.25, Thm 1.37).

1.2. Existence.

Assumption. In this section every stack and every scheme is separated and quasi-compact. Noetherianity will be explicitly stated if needed.

In this section we will prove Grothendieck duality for three classes of morphisms: morphisms from a scheme to an algebraic stack, morphisms from a projective stack to an algebraic stack and representable proper morphisms of algebraic stacks. We will derive also some properties tightly related to existence and not depending on the geometric properties of the objects involved. We will denote with $D(\mathcal{X})$ the derived category of quasicoherent sheaves on $\mathcal{X}$, we drop the notation $D_{qc}(\mathcal{X})$ of the old literature for the derived category of sheaves with quasicoherent cohomology since it is proven to be equivalent to $D(\mathcal{X})$ in [BN93, Cor 5.5].

Lemma 1.4. *Let $\pi\colon \mathcal{X} \to X$ be an algebraic stack with moduli space X. The functor $\pi_*\colon D(\mathcal{X}) \to D(X)$ respects small coproducts, that is the natural morphism:*

$$\coprod_{\lambda\in\Lambda} \pi_* x_\lambda \to \pi_* \coprod_{\lambda\in\Lambda} x_\lambda \tag{1.1}$$

is an isomorphism for every small Λ.

Proof. We recall that the category of sheaves of modules is an abelian category that satisfies the axiom $(AB4)$; in particular the coproduct is left exact, and as a matter of fact exact. We recall also that the coproduct in a derived category is just the coproduct of complexes. We choose a smooth presentation $X_0 \to \mathcal{X}$ and we associate to it the simplicial nerve $X^\bullet$. Let $f^i \colon X_i \to X$ be the obvious composition. For every quasicoherent sheaf $\mathcal{F}$ on $\mathcal{X}$ represented by $\mathcal{F}^\bullet$ on $X^\bullet$ we have a resolution (see [Ols07, Lem 2.5]):

$$0 \to \pi_* \mathcal{F} \to f^0_* \mathcal{F}_0 \to f^1_* \mathcal{F}_1 \to \dots \tag{1.2}$$

We just need to keep the first three terms in this sequence and the result follows from left exactness of the coproduct, the analogous result for schemes [Nee96, Lem 1.4] and the existence of the natural arrow (1.1). □

Corollary 1.5. *Let $\pi \colon \mathcal{X} \to X$ be as in the previous statement and $f \colon \mathcal{X} \to Y$ be a separated morphism to a scheme Y. The functor $Rf_* \colon D(\mathcal{X}) \to D(Y)$ respects small coproducts.*

Proof. This is an immediate consequence of the universal property of the moduli scheme X and the previous lemma. □

Corollary 1.6. *Let $\pi \colon \mathcal{X} \to X$ be as in the previous statement and $f \colon \mathcal{X} \to \mathcal{Y}$ is a separated morphism to an algebraic stack $\mathcal{Y}$. The functor $Rf_* \colon D(\mathcal{X}) \to D(\mathcal{Y})$ respects small coproducts.*

Proof. To prove this we compute the push-forward as in [LMB00, Lem 12.6.2] and use the previous corollary. □

Let $\pi \colon \mathcal{X} \to X$ be an algebraic stack with moduli space X. Let $\mathcal{E}$ be a generating sheaf of $\mathcal{X}$ and $\mathcal{O}_X(1)$ an ample invertible sheaf of X. In the following we will indicate this set of hypothesis with the sign $(*)$

Lemma 1.7. *Let the stack $\pi \colon \mathcal{X} \to X$, the sheaves $\mathcal{E}$ and $\mathcal{O}_X(1)$ satisfy $(*)$. The derived category $D(\mathcal{X})$ is compactly generated and the set $T = \{\mathcal{E} \otimes \pi^* \mathcal{O}_X(n)[m] \mid m, n \in \mathbb{Z}\}$ is a generating set.*

Proof. Same proof as in [Nee96, Ex 1.10], but using that every quasi coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ can be written as a quotient of $\mathcal{E} \otimes \pi^* \mathcal{O}_X(-t)$ for some integer number t. □

Remark 1.8. The most important class of algebraic stacks $\mathcal{X}$ satisfying conditions in the previous lemma is composed by projective stacks and more generally families of projective stacks; the second class we have in mind is given by stacks of the kind $[\operatorname{Spec} B/G] \to \operatorname{Spec} A$ where G is a linearly reductive group scheme on $\operatorname{Spec} A$, which is the structure of a tame stack étale locally on its moduli space.

Proposition 1.9. *Let $\pi \colon \mathcal{X} \to X$, $\mathcal{E}$ and $\mathcal{O}_X(1)$ satisfy $(*)$, let the morphism $f \colon \mathcal{X} \to \mathcal{Y}$ be separated and $\mathcal{Y}$ an algebraic stack. The functor $Rf_* \colon D(\mathcal{X}) \to D(\mathcal{Y})$ has a right adjoint $f^! \colon D(\mathcal{Y}) \to D(\mathcal{X})$.*

Proof. This is a formal consequence of Brown representability [Nee96, Thm 4.1], Corollary 1.6 and Lemma 1.7. □

Proposition 1.10. *Let $\mathcal{X}$ be an algebraic stack and $f \colon Z \to \mathcal{X}$ a separated morphism from a scheme Z. The functor $Rf_* \colon D(Z) \to D(\mathcal{X})$ has a right adjoint $f^! \colon D(Z) \to D(\mathcal{X})$.*

Proof. We only need to prove that the functor Rf_* respects coproducts and then use [Nee96, Thm 4.1] again. Let $X^\bullet$ be an étale presentation and $Z^\bullet$ the pullback presentation of Z and F a quasi coherent sheaf on Z. Denote with f_i the morphism $Z_i \to X_i$. The sheaf $(f_*F)|_{X_0}$ is just $f_0^*(F|_{Z_0})$. Coproducts commute with pullback because it has a right adjoint, so the result follows. □

Remark 1.11. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphisms of stacks such that $f^!$ exists. It is clear that the existence of a right adjoint is enough to guarantee uniqueness. Moreover assume we have a composition $g \circ f\colon \mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ such that both $g^!$ and $f^!$ exist. The two functors Rg_*Rf_* and $R(g\circ f)_*$ are canonically isomorphic. Duality gives us a canonical isomorphism:

$$f^!g^! \xrightarrow{\eta_{f,g}} (g\circ f)^! \tag{1.3}$$

If we want to explicitly compute Serre duality for a smooth proper Deligne-Mumford stack we need to prove existence of duality in one last case, which is duality for proper representable morphisms of noetherian algebraic stacks in general (no projectivity is assumed). Let $f\colon \mathcal{Y} \to \mathcal{X}$ be such a morphism and choose a smooth presentation $X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0 \xrightarrow{p_0} \mathcal{X}$ and produce the pullback presentation $Y_1 \underset{v}{\overset{u}{\rightrightarrows}} Y_0 \xrightarrow{p_0} \mathcal{Y}$. Let I be an injective sheaf on $\mathcal{X}$ and $\alpha\colon s^*I_0 \to t^*I_0$ the isomorphism defining the sheaf I. Using flat base change theorem for schemes [Ver69, Thm 2] we can produce the following chain of isomorphisms:

$$u^*f_0^!I_0 \xrightarrow{c_s} f_1^!s^*I_0 \xrightarrow{f_1^!\alpha} f_1^!t^*I_0 \xleftarrow{c_t} v^*f_0^!I_0$$

Call this isomorphism β. It satisfies the cocycle condition because α does and the isomorphisms c_t, c_s defined in [Ver69, pg 401] satisfy it according to the same reference (see next section for a recall of the construction of c_s). The data $\beta, f_0^!I_0$ define a complex of injective sheaves on $\mathcal{Y}$ and we will denote it with $f^\natural I$.

Proposition 1.12. *Let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism as above, the functor $Rf_*\colon D^+(\mathcal{Y}) \to D^+(\mathcal{X})$ admits a right adjoint $f^!\colon D^+(\mathcal{X}) \to D^+(\mathcal{Y})$. Let $\mathcal{F}^\bullet \in D^+(\mathcal{X})$ and $I^\bullet$ an injective complex quasi-isomorphic to it. The derived functor $f^!\mathcal{F}^\bullet$ is computed by $f^\natural I^\bullet$.*

Proof. Let J be an injective sheaf on $\mathcal{Y}$ and I an injective sheaf on $\mathcal{X}$. Keeping notations introduced above we can write the following exact sequence:

$$0 \longrightarrow \mathcal{H}om_{\mathcal{X}}(f_*J, I) \longrightarrow p_{0*}\mathcal{H}om_{X_0}(p_0^*f_*J, p_0^*I) \longrightarrow p_{0*}s_*\mathcal{H}om_{X_1}(s^*p_0^*f_*J, s^*p_0^*I)$$

Using duality for proper morphisms of schemes and flat base change for the twisted inverse immage we have the following commutative square:

$$\begin{array}{ccc}
p_{0*}\mathcal{H}om_{X_0}(p_0^*f_*J, p_0^*I) & \longrightarrow & p_{0*}s_*\mathcal{H}om_{X_1}(s^*p_0^*f_*J, s^*p_0^*I) \\
\downarrow \wr & & \downarrow \wr \\
p_{0*}\mathcal{H}om_{X_0}(f_{0*}q_0^*J, p_0^*I) & & p_{0*}s_*\mathcal{H}om_{X_1}(f_{1*}u^*q_0^*J, s^*p_0^*I) \\
\downarrow \wr & & \downarrow \wr \\
p_{0*}f_{0*}\mathcal{H}om_{X_0}(q_0^*J, f_0^!p_0^*I) & & p_{0*}s_*f_{1*}\mathcal{H}om_{X_1}(u^*q_0^*J, f_1^!s^*p_0^*I) \\
\downarrow \wr & & \downarrow \wr \\
f_*q_{0*}\mathcal{H}om_{X_0}(q_0^*J, f_0^!p_0^*I) & \longrightarrow & f_*q_{0*}u_*\mathcal{H}om_{X_1}(u^*q_0^*J, u^*f_0^!p_0^*I)
\end{array}$$

In the picture we have applied duality for f_0, f_1 but there are no higher derived push-forwards for the two morphisms because both $\mathcal{H}om_{X_0}(q_0^*J, f_0^! p_0^* I)$ and $\mathcal{H}om_{X_1}(u^* q_0^* J, f_1^! s^* p_0^* I)$ are injective. The morphism $f_* q_{0*} \mathcal{H}om_{X_0}(q_0^* J, f_0^! p_0^* I) \to f_* q_{0*} u_* \mathcal{H}om_{X_1}(u^* q_0^* J, u^* f_0^! p_0^* I)$ in the picture is clearly induced by β defined above so that its kernel is $f_* \mathcal{H}om_{\mathcal{Y}}(J, f^\natural I)$. This gives us a duality isomorphism:

$$\mathcal{H}om_{\mathcal{X}}(f_* J, I) \to f_* \mathcal{H}om_{\mathcal{Y}}(J, f^\natural I)$$

The result follows. □

Remark 1.13. Actually we have proven something stronger then bare duality, we have a sheaf version of the result. We will obtain an analogous sheaf version of the duality for the morphism from a stack to a scheme in the next section.

To conclude we study the behavior of the twisted inverse image with respect to the tensor product

Proposition 1.14. *Let $f\colon \mathcal{X} \to Y$ be a morphism from an algebraic stack to a scheme. Suppose Rf_* has a right adjoint $f^!$, then for every $F, G \in D^b(Y)$ there is a natural morphism:*

$$Lf^* F \overset{L}{\otimes} f^! G \to f^!(F \otimes G)$$

moreover if G is compact it is an isomorphism. In particular we have the following natural isomorphism:

$$Lf^* F \overset{L}{\otimes} f^! \mathcal{O}_{\mathcal{Y}} \to f^! F$$

If $f^! G$ is of finite Tor-dimension we can have $F \in D^+(Y)$.

Proof. The proof mostly relies on the existence of $f^!$ and a general enough projection formula. Putting together [OS03, Cor 5.3] and [Nee96, Prop 5.3] we actually have a general enough projection formula that is a natural isomorphism $F \overset{L}{\otimes} Rf_* \mathcal{G} \to Rf_*(Lf^* F \overset{L}{\otimes} \mathcal{G})$ for $\mathcal{G} \in D(\mathcal{X})$. Since we are working on a site we cannot use the fancy stuff of [Nee96, Thm 5.4] to define derived functors in non bounded derived categories. For this reason we have more restrictive conditions on F, G. Once everything is well defined the proof goes just like in [Nee96, Thm 5.4]. □

Definition 1.15. Let $f\colon \mathcal{X} \to S$ be an S-stack, f the structure morphism. Suppose $f^!$ exists, we will call $f^! \mathcal{O}_S$ the dualizing complex of $\mathcal{X}$.

This definition agrees with literature.

1.3. **Duality and flat base change.** Now that we have proven existence our aim is to explicitly write $f^!$ (*twisted inverse image*) in some interesting case. At the end of the next section we will obtain Serre duality for smooth projective stacks and Grothendieck duality for finite morphisms. To achieve this, we will use existence and a result of flat base-change in the same spirit as [Ver69, Thm 2]. In this section every scheme and stack is again noetherian and derived categories are bounded below.

We start proving base change for open immersions. We anticipate a technical lemma which is a variation of [Ver69, Lem 2]

Lemma 1.16. *Let $\mathcal{X}$ be an algebraic stack and $i\colon \mathcal{U} \to \mathcal{X}$ an open substack. Let $\mathcal{I}$ be an ideal sheaf defining the complementary of $\mathcal{U}$. For any $\mathcal{F} \in D^+_{qc}(\mathcal{X})$ the canonical* *morphisms:*

$$\lim_{n\to\infty} \operatorname{Ext}^p_{\mathcal{X}}(\mathcal{I}^n, \mathcal{F}) \to H^p(\mathcal{U}, i^*\mathcal{F}) \tag{1.4}$$

$$\lim_{n\to\infty} \mathcal{E}xt^p_{\mathcal{X}}(\mathcal{I}^n, \mathcal{F}) \to R^p i_* i^*\mathcal{F} \tag{1.5}$$

are isomorphisms for every p. Moreover if $\mathcal{G}$ is a bounded above complex over $\mathcal{X}$ with coherent cohomology we can generalize the first isomorphism to the following:

$$\lim_{n\to\infty} \operatorname{Ext}^p_{\mathcal{X}}(\mathcal{G} \overset{L}{\otimes} \mathcal{I}^n, \mathcal{F}) \to \operatorname{Ext}^p_{\mathcal{U}}(i^*\mathcal{G}, i^*\mathcal{F}) \tag{1.6}$$

Proof. The statement is the derived version of [Har66, App Prop 4]. This last proposition holds also for stacks. To see this we can use the usual trick of writing $\operatorname{Hom}_{\mathcal{X}}$ as the kernel of an opportune morphism between Hom_{X_0} and Hom_{X_1} for some given presentation $X^\bullet$. □

Consider the following cartesian square:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{i} & \mathcal{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & Y \end{array}$$

where $\pi\colon \mathcal{X} \to X$ is an algebraic stack satisfying the set of hypothesis $(*)$, the morphisms f, g are proper, i, j are flat morphisms and Y, U are quasi-compact separated schemes. We can define a canonical morphism $c_j\colon i^*f^! \to g^!j^*$. We can actually define it in two equivalent ways according to [Ver69, pg 401]. We recall here the two construction of this morphism for completeness, and to include a small modification that occurs in the case of stacks.

(1) Since j is flat we have that j^* is the left adjoint of Rj_* so that we have a unit and a counit:

$$\phi_j\colon \mathrm{id} \to Rj_*j^* \qquad \psi_j\colon j^*Rj_* \to \mathrm{id}$$

We can also apply theorem [NF08, Thm 1.9] and obtain an isomorphism: $\sigma\colon j^*Rf_* \to Rg_*i^*$. The right adjoint of this gives as $\widetilde{\sigma}\colon Ri_*g^! \to f^!Rj_*$. The canonical morphism we want is now the composition:

$$i^*f^! \xrightarrow{i^*f^!\circ\phi_j} i^*f^!Rj_*j^* \xrightarrow{i^*\circ\widetilde{\sigma}\circ j^*} i^*Ri_*g^!j^* \xrightarrow{\psi_i\circ g^!j^*} g^!i^*$$

(2) Since Rf_* has a right adjoint $f^!$ we have a unit and a counit:

$$\mathrm{cotr}_f\colon \mathrm{id} \to f^!Rf_* \qquad \mathrm{tr}_f\colon Rf_*f^! \to \mathrm{id}$$

We can now consider the following composition:

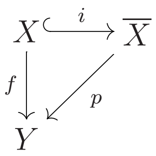
$$i^*f^! \xrightarrow{\mathrm{cotr}_g\circ i^*f^!} g^!Rg_*i^*f^! \xrightarrow{g^!\circ\sigma\circ f^!} g^!j^*Rf_*f^! \xrightarrow{g^!j^*\,\mathrm{tr}_f} g^!j^*$$

According to [Ver69, pg 401] both these two compositions define c_j; moreover it satisfies a cocycle condition when composing two base changes.

Proposition 1.17. *In the above setup, assume also that j is an open immersion, the canonical morphism $c_j\colon i^*f^! \to g^!j^*$ is an isomorphism.*

Proof. Same proof as in [Ver69, Thm 2, case 1] but using Lemma 1.16. □

Definition 1.18. We recall that a morphism of schemes (or stacks also) $f\colon X \to Y$ is *compactifiable* if it can be written as an open immersion i followed by a proper morphism p.



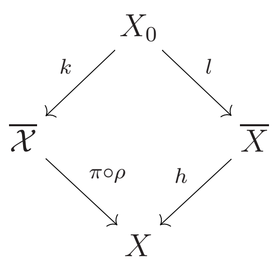
Deligne defined in [Har66, Appendix] a notion of duality for compactifiable morphisms (duality with compact support) of separated noetherian schemes. First of all we observe that given an open immersion i or more generally an object in a site, the functor i^* has a left adjoint $i_!\colon \mathrm{QCoh}_X \to \mathrm{QCoh}_{\overline{X}}$ which is an exact functor (see for instance [Mil80, II Rem 3.18] for a general enough construction). Given f compactifiable we can define the derived functor $Rf_! = R(p_* i_!) = (Rp_*)i_!$. It is clear that this last functor has a right adjoint in derived category that is $i^*p^!$ and we will denote it as $f^!$. Deligne proved that this definition of $f^!$ is independent from the chosen compactification and well behaved with respect to composition of morphisms.

The functor $i_!$ is actually everything left of all the local cohomology mess in Residues and Duality. To prove Serre duality for stacks we will use $i_!$ for both open immersions and étale maps, being confident that they are compatible in the following sense:

Proposition 1.19. *[Mil80, VI Thm 3.2.b] Let $f\colon X \to Y$ be an étale morphism of noetherian separated schemes. Split it in an open immersion i followed by a finite morphism g then we have $f_! = g_* i_!$.*

Let $\mathcal{X}$ be a Deligne-Mumford stack satisfying $(*)$ and $f_0\colon X_0 \to \mathcal{X}$ an étale atlas. The morphism $X_0 \to X$ is quasi finite and a fortiori compactifiable. We can choose a compactification using Zariski main theorem and split the morphism as $X_0 \xrightarrow{k} \overline{X} \xrightarrow{h} X$ where h is finite and k is an open immersion. We can also apply Zariski main theorem for stacks [LMB00, Thm 16.5] to the morphism f_0 and obtain a different compactification $X_0 \xrightarrow{l} \overline{\mathcal{X}} \xrightarrow{\rho} \mathcal{X} \xrightarrow{\pi} X$ where l is open, ρ is finite and $\overline{\mathcal{X}}$ is a Deligne-Mumford stack. Now we are ready to prove that these two different compactifications are equivalent from the point of view of duality with compact support.

Proposition 1.20. *Consider the commutative square:*



Let $F \in D^+(X)$, there is a canonical isomorphism $k^*(\pi \circ \rho)^! F \cong l^* h^! F$

Proof. Same proof as in [Ver69, Cor 1] but using base change result in 1.17. □

We can now generalize to stacks Proposition 1.19:

Corollary 1.21. *In the setup of the previous Proposition, for every $F \in D^+(X)$ there is a canonical isomorphism $f_0^* \pi^! F \cong l^* h^! F$*

Proof. We use the previous proposition, the representability of f_0 and the analogous result for schemes. $\square$

As a further consequence we can prove Grothendieck duality in its sheaf version:

Corollary 1.22. *Let $f\colon \mathcal{X} \to Y$ be a proper morphism from an algebraic stack satisfying $(*)$, Y be a scheme and $\mathcal{F} \in D^+(\mathcal{X})$, $G \in D^+(Y)$. The natural morphism:*

$$Rf_* R\mathcal{H}om_{\mathcal{X}}(\mathcal{F}, f^! G) \longrightarrow R\mathcal{H}om_Y(Rf_*\mathcal{F}, Rf_* f^! G) \xrightarrow{\mathrm{tr}_f} R\mathcal{H}om_Y(Rf_*\mathcal{F}, G)$$

is an isomorphism.

Proof. Take $I^\bullet, J^\bullet$ injective complexes quasi isomorphic to $\mathcal{F}, G$. Let $j\colon U \to Y$ be an étale morphism $X^\bullet$ an étale presentation of $\mathcal{X}$. We can construct the following:

$$\begin{array}{ccccccc}
U_1 & \overset{v}{\underset{u}{\rightrightarrows}} & U_0 & \xrightarrow{l} & \mathcal{U} & \xrightarrow{g} & U \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\
X_1 & \overset{t}{\underset{s}{\rightrightarrows}} & X_0 & \xrightarrow{h} & \mathcal{X} & \xrightarrow{f} & Y
\end{array}$$

As usual we have the exact sequence:

$$0 \to f_* \mathcal{H}om_{\mathcal{X}}(I^p, f^! J^q) \to f_* h_* \mathcal{H}om_{X_0}(h^* I^p, h^* f^! J^q) \to f_* h_* s_* \mathcal{H}om_{X_1}(s^* h^* I^p, s^* h^* f^! J^q)$$

for every p, q. We first use flat base change to obtain $i^* f_* h_* \mathcal{H}om_{X_0}(h^* I^p, h^* f^! J^q) \cong g_* l_* k^* \mathcal{H}om_{X_0}(h^* I^p, h^* f^! J^q) \cong g_* l_* \mathcal{H}om_{U_0}(k^* h^* I^p, k^* h^* f^! J^q)$. Now we observe that $i^* = i^!$ if we consider an étale morphism as a compactifiable morphism [Mil80, V Prop 1.13], and using 1.20 we obtain $k^* h^* f^! = l^* g^! i^*$. Eventually we have:

$$i^* f_* h_* \mathcal{H}om_{X_0}(h^* I^p, h^* f^! J^q) \cong g_* l_* \mathcal{H}om_{U_0}(l^* j^* I^p, l^* g^! i^* J^q)$$

With the same argument we have also:

$$i^* f_* h_* s_* \mathcal{H}om_{X_1}(s^* h^* I^p, s^* h^* f^! J^q) \cong g_* l_* u_* \mathcal{H}om_{U_1}(u^* l^* j^* I^p, u^* l^* g^! i^* J^q)$$

and eventually:

$$i^* f_* \mathcal{H}om_{\mathcal{X}}(I^p, f^! J^q) \cong g_* \mathcal{H}om_{\mathcal{U}}(j^* I^p, g^! i^* J^q)$$

Now we just take global sections of this and use the non sheaf version of Grothendieck duality to complete the proof. $\square$

We have now all the ingredients to prove the flat base change result:

Theorem 1.23. *Consider the following cartesian square:*

$$\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{i} & \mathcal{X} \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{j} & Y
\end{array}$$

where $\mathcal{X}$ is an algebraic stack satisfying $(*)$, the morphisms f, g are proper and i, j are flat. The canonical morphism $c_j\colon i^* f^! \to g^! j^*$ is an isomorphism.

Proof. Same proof as in [Ver69, Thm 2, case 2] but using the stacky Corollary 1.22. $\square$

1.4. **Duality for smooth morphisms.** In this section $\mathcal{X} \to \operatorname{Spec} k$ is a smooth projective Deligne-Mumford stack over a field if not differently specified. It clearly satisfies $(*)$.

We start with two local results that don't rely on smoothness.

Lemma 1.24. *Let $f\colon \mathcal{Y} \to \mathcal{Z}$ be a representable finite étale morphism of noetherian algebraic stacks (non necessarily smooth), then the functor $f^!$ is the same as f^*.*

Proof. If $\mathcal{Y}$ and $\mathcal{Z}$ are two schemes the result is true, then the result for stacks follows by Proposition 1.12. □

Let $\pi\colon \mathcal{X} \to X$ be a non necessarily smooth Deligne-Mumford stack with moduli scheme; using the previous lemma we can study the étale local structure of $\pi^!$. The morphism π étale locally is the same as $\rho\colon [\operatorname{Spec} B/G] \to \operatorname{Spec} A$ where G is a finite group and $\operatorname{Spec} A$ is the moduli scheme. Let $\operatorname{Spec} B \xrightarrow{p} [\operatorname{Spec} B/G]$ be the obvious étale (and finite) atlas and s, t source and target in the presentation (both étale and finite). Let I be an injective A-module. Consider the following chain of isomorphisms:

$$s^*(\rho\circ p)^! I \xrightarrow[\sim]{\eta_{s,\rho\circ p}} (\rho\circ p\circ s)^! I \xleftarrow[\sim]{\eta_{t,\rho\circ p}} t^*(\rho\circ p)^! I$$

where we have replaced $s^!, t^!$ with s^*, t^* using the lemma, and every isomorphism is given by equation (1.3). Call this isomorphism γ. The data of γ and $(\rho\circ p)^! I$ define a complex of injective sheaves on $[\operatorname{Spec} B/G]$ and we will denote it $\rho^\nabla I$. Let $F \in D^+(\operatorname{Spec} A)$ quasi isomorphic to a complex of injectives $I^\bullet$; the injective complex $\rho^\nabla I^\bullet$ on $[\operatorname{Spec} B/G]$ defines a functor $\rho^\nabla\colon D^+(\operatorname{Spec} A) \to D^+([\operatorname{Spec} B/G])$

Lemma 1.25. *The functor ρ^∇ above is actually $\rho^!$.*

Proof. We start observing that the twisted inverse image $(\rho\circ p)^! I$ is just the B-module $\operatorname{Hom}_A(B, I)$, the twisted inverse image $(\rho\circ p\circ s)^! I$ is $\operatorname{Hom}_A(B\otimes_A \mathcal{O}_G, I)$. The natural isomorphism for the composition of twisted inverse images $s^*(\rho\circ p)^! I \cong (\rho\circ p\circ s)^! I$ is just the canonical isomorphism $\operatorname{Hom}_A(B, I)\otimes_A \mathcal{O}_G \xrightarrow{\delta} \operatorname{Hom}_A(B\otimes_A \mathcal{O}_G, I)$. Let M be a B-module with α a coaction of $\mathcal{O}_G$. From the exact sequence in (1.2) and using duality we obtain the following exact diagram:

$$\begin{array}{ccc}
\operatorname{Hom}_A(M\otimes_A \mathcal{O}_G, I) & \xrightarrow{\operatorname{Hom}(\alpha-\iota,\mathrm{id})} & \operatorname{Hom}_A(M, I) \\
\downarrow \wr & & \\
\operatorname{Hom}_{B\otimes_A\mathcal{O}_G}(M\otimes_A \mathcal{O}_G, \operatorname{Hom}_A(B\otimes_A \mathcal{O}_G, I)) & & \\
{\scriptstyle \operatorname{Hom}(\mathrm{id},\delta^{-1})}\downarrow \wr & & \downarrow \wr \\
\operatorname{Hom}_{B\otimes_A\mathcal{O}_G}(M\otimes_A \mathcal{O}_G, \operatorname{Hom}_A(B, I)\otimes \mathcal{O}_G) & & \\
\| & & \\
\operatorname{Hom}_B(M, \operatorname{Hom}_A(B, I))\otimes_A \mathcal{O}_G & \longrightarrow & \operatorname{Hom}_B(M, \operatorname{Hom}_A(B, I))
\end{array}$$

The cokernel of the first horizontal arrow is $\operatorname{Hom}_A(M^G, I)$ while the cokernel of the last horizontal arrow is just $\operatorname{Hom}_B^G(M, \operatorname{Hom}_A(B, I))$ where the coaction of $\mathcal{O}_G$ on M is α and the coaction on $\operatorname{Hom}_A(B, I)$ is the one of $\rho^\nabla I$. The diagram induces an isomorphism[1]:

$$\operatorname{Hom}_A(M^G, I) \to \operatorname{Hom}_B^G(M, \operatorname{Hom}_A(B, I))$$

[1]Despite of the unhappy notation Hom_B^G is not a B-module but a $B^G = A$-module.

By uniqueness of the adjoint we conclude that ρ^∇ is exactly $\rho^!$. $\square$

Remark 1.26. Let F be a quasicoherent sheaf on $\operatorname{Spec} A$; if we know for some reason that $\rho^! F$ is a quasi coherent sheaf itself, then we can conclude that it glues as $\rho^!$ of an injective sheaf in the previous lemma.

We can now start using smoothness hypothesis on $\mathcal{X}$. We recall a result of Verdier in [Ver69, Thm 3]:

Theorem 1.27. *Let $f\colon X \to Y$ a proper morphism of Noetherian schemes and $j\colon U \to X$ an open immersion such that $f \circ j$ is smooth of relative dimension n. There exists a canonical isomorphism:*

$$j^* f^! \mathcal{O}_Y \xrightarrow{can} \omega_{U/Y}[n] \tag{1.7}$$

where $\omega_{U/Y}$ is the canonical sheaf.

In order to use Lemma 1.25 we need to explicitly know the isomorphism $f^! \circ g^! \cong (g \circ f)^!$ when f, g are compactifiable smooth morphisms of schemes. For this purpose we state a compactified version of a statement of Hartshorne [Har66, III Prop 2.2]:

Lemma 1.28. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be smooth compactifiable morphisms of noetherian schemes of relative dimensions n, m respectively. There is a natural isomorphism $\zeta\colon \omega_{X/Z} \to \omega_{x/Y} \otimes f^* \omega_{Y/Z}$. Called $\eta_{f,g}\colon f^! \circ g^! \cong (g \circ f)^!$ the natural isomorphism obtained for adjunction from $R(g \circ f)_! \cong Rf_! \circ Rg_!$ (same as equation (1.3)) we have the following commutative diagram:*

$$\begin{array}{ccc}
\omega_{X/Z} & \xrightarrow{\quad\zeta\quad} & \omega_{X/Y} \otimes f^* \omega_{Y/Z} \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle can} \\
(g \circ f)^! \mathcal{O}_Z & \xleftarrow{\quad\eta_{f,g}\quad} & f^!(g^! \mathcal{O}_Z) = f^! \mathcal{O}_Y \otimes f^* g^! \mathcal{O}_Z
\end{array}$$

With this machinery we can solve the local situation:

Proposition 1.29. *Let $\rho\colon \mathcal{X} = [\operatorname{Spec} B/G] \to \operatorname{Spec} A$ be an n-dimensional smooth Deligne-Mumford stack over $\operatorname{Spec} k$ with structure map $\sigma\colon \operatorname{Spec} A \to \operatorname{Spec} k$ a compactifiable morphism; $p\colon X_0 \to \mathcal{X}$ an étale atlas. The dualizing complex $\rho^! \sigma^! k$ is canonically isomorphic to $\omega_{\mathcal{X}}[n]$*

Proof. According to Theorem 1.27 we have that $(\sigma \circ \rho \circ p)^! k = \omega_B[n]$ (it's important to remember that duality along σ is duality with compact support). By Lemma 1.28 and using that s, t are finite we have:

$$s^*(\sigma \circ \rho \circ p)^! k = s^* \omega_B[n] \xrightarrow{\text{can}} \omega_{B \times G}[n] = (\sigma \circ \rho \circ p \circ s)^! k$$

and the same for t^*. Since the canonical isomorphism is the one described in 1.28 and using Lemma 1.25 we obtain that $(\sigma \circ \rho)^! k$ is canonically isomorphic to $\omega_{\mathcal{X}}[n]$. $\square$

Remark 1.30. Suppose now to change the atlas in the previous proposition to some scheme W_0 étale over $\mathcal{X}$ but not necessarily finite. We have a new étale presentation $W_1 \underset{u}{\overset{v}{\rightrightarrows}} W_0 \xrightarrow{\tau} \mathcal{X}$. The two morphisms $\sigma \circ \rho \circ \tau\colon W_0 \to \operatorname{Spec} k$ and $\sigma \circ \rho \circ \tau \circ u\colon W_1 \to \operatorname{Spec} k$ are both compactifiable. Using duality with compact support we have canonical isomorphisms $u^*(\sigma \circ \rho \circ \tau)^! k \cong (\sigma \circ \rho \circ \tau \circ u)^! k$ and the same with v^*. Recall now that for an étale morphism the twisted inverse image (duality with compact support) is the same as the pullback; according to 1.28 and using the previous proposition the two isomorphisms are the two canonical isomorphisms $u^* \omega_{W_0} \cong \omega_{W_1}$ and $v^* \omega_{W_0} \cong \omega_{W_1}$.

To deal with the global case we have to study more the two natural isomorphisms we have: $\eta_{f,g}$ for the composition of twisted inverse images $f^!, g^!$ and c_j for the flat base change by a map j of a twisted inverse image. They are compatible according to a pentagram relation.

Lemma 1.31. *Consider the following diagram of noetherian schemes where horizontal arrows are compactifiable morphisms, vertical arrows are flat and the two squares cartesian:*

$$\begin{array}{ccccc} X' & \xrightarrow{g} & Y' & \xrightarrow{\rho} & Z' \\ {\scriptstyle i}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \sigma}\downarrow \\ X & \xrightarrow[f]{} & Y & \xrightarrow[\pi]{} & Z \end{array}$$

Let $F \in D^+(Z)$, the following pentagram relation holds:

(1.8)
$$\begin{array}{ccccc} & & (\rho \circ g)^! \sigma^* F & & \\ & \nearrow^{c_\sigma} & & \nwarrow^{\eta_{g,\rho}} & \\ i^*(\pi \circ f)^! F & & & & g^! \rho^! \sigma^* F \\ \uparrow^{i^*\eta_{f,\pi}} & & & & \uparrow^{g^! c_\sigma} \\ i^* f^! \pi^! F & & \xrightarrow[c_h]{} & & g^! h^* \pi^! F \end{array}$$

We are ready for the global case:

Theorem 1.32 (Smooth Serre duality)**.** *Let $\sigma\colon \mathcal{X} \to \operatorname{Spec} k$ be a smooth projective Deligne-Mumford stack of dimension n. The complex $\sigma^! k$ is canonically isomorphic to the complex $\omega_{\mathcal{X}}[n]$.*

Proof. We start with a picture that reproduces the local setup and summarizes all the morphisms we are going to use:

(1.9)
$$\begin{array}{ccccccccc} Y_1 & \overset{v}{\underset{u}{\rightrightarrows}} & Y_0 & \xrightarrow{g} & \coprod_i [\operatorname{Spec} B_i/G_i] & \xrightarrow{\rho} & \coprod \operatorname{Spec} A_i & & \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \sigma} & \searrow & \\ X_1 & \overset{t}{\underset{s}{\rightrightarrows}} & X_0 & \xrightarrow{f} & \mathcal{X} & \xrightarrow{\pi} & X & \longrightarrow & \operatorname{Spec} k \end{array}$$

We denote with k_X the dualizing complex of the scheme X. First we observe that $\pi^! k_X$ is a sheaf. Indeed we have an isomorphism $c_\sigma\colon \rho^! \sigma^* k_X \to h^* \pi^! k_X$ for Theorem 1.23; according to Proposition 1.29 the complex $\rho^! \sigma^* k_X$ is a sheaf and since h is faithfully flat $\pi^! k_X$ must be a sheaf itself. Denote with ξ the isomorphism on the double intersection X_1 defining the sheaf $\pi^! k_X$. Using again Proposition 1.29 and the following remark we have a commutative diagram expressing the isomorphism ξ in relation with the base change isomorphism c_σ:

$$
\begin{array}{ccccc}
u^*h_0^*(\pi\circ f)^!k_X & \xrightarrow{u^*c_\sigma} & u^*(\rho\circ g)^!\sigma^*k_X & & \\
\downarrow{\scriptstyle u^*h_0^*\eta_{f,\pi}} & & \big\downarrow & \searrow{\scriptstyle \eta_{u,\rho\circ g}} & \\
u^*h_0^*f^*\pi^!k_X & & & & (\rho\circ g\circ u)^!\sigma^*k_X \\
\downarrow{\scriptstyle h_1^*\xi} & & & \nearrow{\scriptstyle \eta_{v,\rho\circ g}} & \\
v^*h_0^*f^*\pi^!k_X & & & & \\
\uparrow{\scriptstyle v^*h_0^*\eta_{f,\pi}} & & & & \\
v^*h_0^*(\pi\circ f)^!k_X & \xrightarrow{v^*c_\sigma} & v^*(\rho\circ g)^!\sigma^*k_X & &
\end{array}
$$

Now we can use two times the pentagram relation for compactifiable morphisms in (1.8) (remember that $u^!, v^!, t^!, s^! = u^*, v^*, t^*, s^*$) and obtain the following commutative diagram:

$$
\tag{1.10}
\begin{array}{ccccc}
u^*h_0^*(\pi\circ f)^!k_X & & \xrightarrow{u^*c_\sigma} & & u^*(\rho\circ g)^!\sigma^*k_X \\
\| & & & & \searrow{\scriptstyle \eta_{u,\rho\circ g}} \\
h_1^*s^*(\pi\circ f)^!k_X & \searrow{\scriptstyle h_1^*\eta_{s,\pi\circ f}} & & & \\
 & & h_1^*(\pi\circ f\circ s)^!k_X & \xrightarrow{c_\sigma} & (\rho\circ g\circ u)^!\sigma^*k_X \\
h_1^*t^*(\pi\circ f)^!k_X & \nearrow{\scriptstyle h_1^*\eta_{t,\pi\circ f}} & & & \\
\| & & & & \nearrow{\scriptstyle \eta_{v,\rho\circ g}} \\
v^*h_0^*(\pi\circ f)^!k_X & & \xrightarrow{v^*c_\sigma} & & v^*(\rho\circ g)^!\sigma^*k_X
\end{array}
$$

Comparing the two commutative diagrams we have the following commutative square:

$$
\tag{1.11}
\begin{array}{ccc}
h_1^*s^*f^*\pi^!k_X & \xrightarrow{h_1^*s^*\eta_{f,\pi}} & h_1^*s^*(\pi\circ f)^!k_X \\
\downarrow{\scriptstyle h_1^*\xi} & & \downarrow{\scriptstyle h_1^*(\eta_{t,\pi\circ f}^{-1}\circ\eta_{s,\pi\circ f})} \\
h_1^*t^*f^*\pi^!k_X & \xrightarrow{h_1^*t^*\eta_{f,\pi}} & h_1^*t^*(\pi\circ f)^!k_X
\end{array}
$$

First of all we observe that the sheaf $(\pi\circ f)^!k_X$ glued by $\eta_{t,\pi\circ f}^{-1}\circ\eta_{s,\pi\circ f}$ is exactly $\omega_{\mathcal{X}}$ according to Lemma 1.28. The commutative square tells us that the isomorphism $\eta_{f,\pi}$ is an isomorphism between the dualizing sheaf and $\omega_{\mathcal{X}}$ once it is restricted to Y_0; unfortunately Y_0 is finer then the atlas we are using (X_0) and we actually don't know if this isomorphism descends. To make it descend we produce a finer presentation of $\mathcal{X}$, that is we use Y_0 as an atlas and we complete the presentation to the groupoid $\overline{X}_2 \rightrightarrows \overline{X}_1 \underset{u'}{\overset{v'}{\rightrightarrows}} Y_0$. This gives us an arrow λ from $\overline{X}_1$ to Y_1 and an arrow from $\overline{X}_2$ to Y_2. We can take the square in 1.11 and pull it back with λ^* to $\overline{X}_1$. Now the isomorphism $h_0^*\eta_{f,\pi}$ descends to an isomorphism of sheaves on $\mathcal{X}$. The main problem now is that we don't know if $\lambda^* h_1^* \xi$ and $\lambda^* h_1^*(\eta^{-1}_{t,\pi\circ f} \circ \eta_{s,\pi\circ f})$ are still the gluing isomorphisms of respectively the dualizing sheaf and $\omega_{\mathcal{X}}$. For what concerns the second we have the following commutative square:

$$\begin{array}{ccc}
\lambda^* h_1^* s^*(\pi\circ f)^! k_X & \xrightarrow{u'^*\eta_{h_0,\pi\circ f}} & u'^*(\pi\circ f\circ h_0)^! k_X \\
{\scriptstyle \lambda^* h_1^*\eta_{s,\pi\circ f}}\downarrow & & \downarrow{\scriptstyle \eta_{u',\pi\circ f\circ h_0}} \\
\lambda^* h_1^*(\pi\circ f\circ s)^! k_X & \xrightarrow{v'^*\eta_{h_0,\pi\circ f}} & v'^*(\pi\circ f\circ h_0)^! k_X
\end{array}$$

and an analogous one for t, v'. This square implies that the sheaf $h_0^*(\pi\circ f)^! k_X$ with the gluing isomorphism $\lambda^* h_1^* \eta^{-1}_{t,\pi\circ f} \circ \eta_{s,\pi\circ f}$ is canonically isomorphic via $h_0^*\eta_{h_0,\pi\circ f}$ to the sheaf given by $(\pi\circ f\circ h_0)^! k_X$ and the gluing isomorphism $\eta^{-1}_{v',\pi\circ f\circ h_0} \circ \eta_{u',\pi\circ f\circ h_0}$ which is actually $\omega_{\mathcal{X}}$. For what concerns the dualizing sheaf we start considering the following picture:

(1.12)
$$\begin{array}{ccccc}
\overline{X}_1 & \xrightarrow{\lambda} & Y_1 & \xrightarrow{v} & Y_0 \\
{\scriptstyle \lambda}\downarrow & & {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h_0} \\
Y_1 & \xrightarrow{h_1} & X_1 & \xrightarrow{t} & X_0 \\
{\scriptstyle u}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle f} \\
Y_0 & \xrightarrow{h_0} & X_0 & \xrightarrow{f} & \mathcal{X}
\end{array}$$

where every single square is 2-cartesian. Only the square at the bottom right has a non trivial canonical two-arrow, let's call it γ. If we think of $\pi^! k_X$ as a sheaf on the étale site of $\mathcal{X}$, the gluing isomorphism ξ on the presentation $X_1 \rightrightarrows X_0$ is induced by the two-arrow γ. If we change the presentation to $\overline{X}_1 \rightrightarrows Y_0$ the gluing isomorphism is induced by $\gamma * \mathrm{id}_{h_1\circ\lambda}$ according to the picture 1.12; this implies that the induced isomorphism is exactly $\lambda^* h_1^* \xi$. We can conclude that the sheaf $h_0^* f^* \pi^! k_X$ with gluing data $\lambda^* h_1^* \xi$ is again the dualizing sheaf. □

Keeping all the notations of the previous theorem we have the following non-smooth result:

Theorem 1.33. *Let $\sigma\colon \mathcal{X} \to \operatorname{Spec} k$ be a projective Deligne-Mumford stack. Let F be a quasicoherent sheaf on the moduli scheme X, assume that $\pi^! F$ is a quasicoherent sheaf on $\mathcal{X}$ then its equivariant structure is given by the isomorphism $\eta^{-1}_{t,\pi\circ f} \circ \eta_{s,\pi\circ f}$.*

Proof. First of all we observe that $\pi^! F$ being a sheaf can be checked étale locally as in the previous theorem, to be more specific it is enough to know that $\rho^! \sigma^* F$ is a sheaf. We achieve the result of the theorem repeating the same proof as in the smooth case and keeping in mind Remark 1.26. □

1.5. **Duality for finite morphisms.** We are going to prove that given $f\colon \mathcal{X} \to \mathcal{Y}$ a representable finite morphism of Deligne-Mumford stacks the functor $f^!$ is perfectly analogous to the already familiar one in the case of schemes.

To start with the proof we first need to state a couple of results, well known in the scheme-theoretic set up ([EGAII, Prop 1.3.1] and [EGAII, Prop 1.4.1]), in the stack-theoretic set-up. The first one is taken from [LMB00, Prop 14.2.4].

Lemma 1.34. *Let $\mathcal{X}$ be an algebraic stack over a scheme S. There is an equivalence of categories between the category of algebraic stacks $\mathcal{Y}$ together with a finite schematically representable S-morphism $f\colon \mathcal{X} \to \mathcal{Y}$ and quasicoherent $\mathcal{O}_{\mathcal{X}}$-algebras. This equivalence* *associates to the stack* $\mathcal{Y}$ *and the morphism* f *the sheaf of algebras* $f_*\mathcal{O}_\mathcal{Y}$*; to a sheaf of algebras* $\mathcal{A}$ *the affine morphism* $f_\mathcal{A}\colon \operatorname{Spec}\mathcal{A} \to \mathcal{X}$*.*

From this lemma we can deduce the following result on quasicoherent sheaves:

Lemma 1.35. *Let* $\mathcal{X}$ *be as above and* $\mathcal{A}$ *an* $\mathcal{O}_\mathcal{X}$*-algebra. There is an equivalence of categories between the category of quasicoherent* $\mathcal{A}$*-modules and the category of quasicoherent sheaves on* $\operatorname{Spec}\mathcal{A}$*. Denoted with* f *the affine morphism* $\operatorname{Spec} A \to \mathcal{X}$*, and given* $\mathcal{F}$ *a quasicoherent sheaf on* $\operatorname{Spec} A$*, the equivalence associates to* $\mathcal{F}$ *the sheaf* $\overline{f}_*\mathcal{F}$ *that is the sheaf* $f_*\mathcal{F}$ *with its natural structure of* $\mathcal{A}$*-algebra. The inverted equivalence is the left-adjoint of* $\overline{f}_*$*, we will denote it with* $\overline{f}^*$ *and it maps the category* $QCoh_{f_*\mathcal{O}_\mathcal{Y}}$ *to* $QCoh_{\mathcal{O}_\mathcal{Y}}$*.*

We need also a couple of properties of the functor $\overline{f}^*$:

Lemma 1.36. *(1) The functor* $\overline{f}^*$ *is exact.*

(2) Let $f\colon \mathcal{X} \to \mathcal{Y}$ *be an affine morphism of algebraic stacks and consider a base change p:*

$$\begin{array}{ccc} \mathcal{X}_0 & \xrightarrow{q} & \mathcal{X} \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}_0 & \xrightarrow{p} & \mathcal{Y} \end{array}$$

The following base change rule holds:

$$\overline{f}^* p^* \cong q^* \overline{f}_0^* \tag{1.13}$$

and the isomorphism is canonical.

Proof. See [Har66, III.6] for some more detail. ☐

Given a finite representable morphism of algebraic stacks $f\colon \mathcal{Y} \to \mathcal{X}$ we are now able to define the following functor:

$$f^\flat\mathcal{F} = \overline{f}^* R\mathcal{H}om_\mathcal{X}(f_*\mathcal{O}_\mathcal{Y}, \mathcal{F}) = R(\overline{f}^* \mathcal{H}om_\mathcal{X}(f_*\mathcal{O}_\mathcal{Y}, \mathcal{F})); \quad \mathcal{F} \in D^+(\mathcal{X}) \tag{1.14}$$

where the complex $R\mathcal{H}om_\mathcal{X}(f_*\mathcal{O}_\mathcal{Y}, \mathcal{F})$ must be considered as a complex of $f_*\mathcal{O}_\mathcal{Y}$-modules.

Theorem 1.37 (finite duality)**.** *Let* $f\colon \mathcal{Y} \to \mathcal{X}$ *be a finite representable morphism of algebraic stacks. The twisted inverse image* $f^!$ *is the functor* $f^\flat$*.*

Proof. Let I be an injective quasicoherent sheaf on $\mathcal{X}$ defined by the couple (I_0, α) on a presentation of $\mathcal{X}$ (We keep notations in Proposition 1.12). We start observing that the sheaf $\mathcal{H}om_\mathcal{X}(f_*\mathcal{O}_\mathcal{Y}, I)$ is determined, on the same presentation by the following isomorphism:

$$s^*\mathcal{H}om_{X_0}(f_{0*}\mathcal{O}_{Y_0}, I_0) \xrightarrow[\sim]{b_s} \mathcal{H}om_{X_1}(f_{1*}\mathcal{O}_{Y_1}, s^*I_0) \xrightarrow[\sim]{\widetilde{\alpha}} \mathcal{H}om_{X_1}(f_{1*}\mathcal{O}_{Y_1}, t^*I_0) \dots$$

$$\dots \xleftarrow[\sim]{b_t} t^*\mathcal{H}om_{X_0}(f_{0*}\mathcal{O}_{Y_0}, I_0)$$

where b_s, b_t are the two natural isomorphisms and $\widetilde{\alpha}$ is induced by α. Applying $\overline{f}_1^*$ to this isomorphism we obtain the one defining $f^\flat I$. According to equation (1.13) we have $\overline{f}_1^* s^* \cong u^*\overline{f}_0^*$ and $\overline{f}_1^* t^* \cong v^*\overline{f}_0^*$. The composed isomorphism:

$$u^*\overline{f}_0^*\mathcal{H}om_{X_0}(f_{0*}\mathcal{O}_{Y_0}, I_0) \xrightarrow{\text{can}} \overline{f}_1^* s^*\mathcal{H}om_{X_0}(f_{0*}\mathcal{O}_{Y_0}, I_0) \xrightarrow{\overline{f}_1^* b_s} \overline{f}_1^*\mathcal{H}om_{X_1}(f_{1*}\mathcal{O}_{Y_1}, s^*I_0)$$

is the same as c_s and we can repeat the argument for t. Comparing with the isomorphism called β in Proposition 1.12 we prove the claim. ☐

To be more explicit we need some vanishing result for $R\mathcal{H}om_{\mathcal{X}}$ like in Hartshorne [Har77, Lem 7.3]. First a technical lemma:

Lemma 1.38. *Let $\mathcal{X}$ be a projective Deligne-Mumford stack, $\mathcal{O}_X(1)$ and $\mathcal{E}$ as in $(*)$. Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on $\mathcal{X}$. For every integer i There is an integer $q_0 > 0$ such that for every $q \geq q_0$:*

$$\mathrm{Ext}^i_{\mathcal{X}}(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(q)) \cong \Gamma(\mathcal{X}, \mathcal{E}xt^i_{\mathcal{X}}(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(q)))$$

Proof. Same proof as in [Har77, Prop 6.9] with obvious modifications. □

Lemma 1.39. *Let $\mathcal{Y}$ be a codimension r closed substack in an n-dimensional smooth projective stack $\mathcal{X}$. Then $\mathcal{E}xt^i_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}}) = 0$ for all $i < r$.*

Proof. The proof goes more or less like in [Har77, Lem 7.3]. Denote with F^i the coherent sheaf $\mathcal{E}xt^i_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}})$. For q large enough the coherent sheaf $F_{\mathcal{E}}(F^i)(q)$ is generated by the global sections; if we can prove that $\Gamma(X, F_{\mathcal{E}}(F^i)(q)) = 0$ for $q >> 0$ we have also that $F_{\mathcal{E}}(F^i) = 0$. In Lemma [NF08, Lem 3.4] we have proven that $\pi \operatorname{supp} F^i = \operatorname{supp} F_{\mathcal{E}}(F^i)$, using this result we conclude that if $F_{\mathcal{E}}(F^i)(q)$ has no global sections for q big enough the sheaf F^i is the zero sheaf. We can now study the vanishing of $\Gamma(X, F_{\mathcal{E}}(F^i)(q))$:

$$\Gamma(X, F_{\mathcal{E}}(F^i)(q)) = \Gamma(\mathcal{X}, \mathcal{E}xt^i_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}} \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(q))) = \mathrm{Ext}^i_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}} \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(q))$$

The last equality holds for a possibly bigger q according to Lemma 1.38. Applying smooth Serre duality we have the following isomorphism:

$$\mathrm{Ext}^i_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}} \otimes \mathcal{E}^\vee \otimes \pi^*\mathcal{O}_X(q)) \cong H^{n-i}(\mathcal{X}, \mathcal{O}_{\mathcal{Y}} \otimes \mathcal{E} \otimes \pi^*\mathcal{O}_X(-q))$$

This last cohomology group is the same as $H^{n-i}(X, F_{\mathcal{E}}(\mathcal{O}_{\mathcal{Y}})(-q))$; using again [NF08, Lem 3.4] we have that the dimension of $F_{\mathcal{E}}(\mathcal{O}_{\mathcal{Y}})(-q)$ is $n - r$ so that the cohomology group vanishes for $i < r$. □

Proposition 1.40. *Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a codimension r closed immersion of an equidimensional Cohen-Macaulay algebraic stack $\mathcal{Y}$ in a smooth projective stack $\mathcal{X} \to \operatorname{Spec} k$ of dimension n. The quasicoherent $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{E}xt^r_{\mathcal{X}}(\mathcal{O}_{\mathcal{Y}}, \omega_{\mathcal{X}})[n-r]$ is the dualizing complex of $\mathcal{Y}$.*

Proof. This is an immediate consequence of smooth Serre duality, Theorem 1.37 and Lemma 1.39. □

Corollary 1.41 (Serre Duality)**.** *Let $\mathcal{X}$ be a projective stack of pure dimension n and $i \colon \mathcal{X} \to \mathcal{P}$ a codimension r closed embedding in a smooth proper Deligne-Mumford stack $f \colon \mathcal{P} \to \operatorname{Spec} k$. The complex $\mathcal{E}xt^\bullet_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})[n]$ is the dualizing complex of $\mathcal{X}$:*

$$i^! f^! k = \mathcal{E}xt^\bullet_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})[n]$$

if $\mathcal{X}$ is also Cohen-Macaulay the dualizing complex is just the dualizing sheaf $\mathcal{E}xt^r_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})$.

Proof. We have to prove that $\mathcal{E}xt^j_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}}) = 0$ for $j > r$. We prove that for every point of $\mathcal{X}$ the stalk of $\mathcal{E}xt^q_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})$ vanishes for $q > r$. We use that for x a point in $\mathcal{X}$ we have $\mathcal{E}xt^j_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})_x = \mathrm{Ext}^j_{\mathcal{O}_{\mathcal{P},i(x)}}(\mathcal{O}_{\mathcal{X},x}, \omega_{\mathcal{P},i(x)})$. The stack $\mathcal{P}$ étale locally is $[\operatorname{Spec} C/G]$ where C is regular and G a finite group, since a closed embedding is given by a sheaf of ideals we can assume that $i \colon \mathcal{X} \to \mathcal{P}$ is given locally by:

$$[\operatorname{Spec} B/G] \xrightarrow{i} [\operatorname{Spec} C/G]$$

where B is local and Cohen-Macaulay. We denote with ω_C the canonical sheaf of $[\operatorname{Spec} C/G]$. As usual we have the long exact sequence:

$$0 \longrightarrow \operatorname{Hom}_C^G(B,\omega_C) \longrightarrow \operatorname{Hom}_C(B,\omega_C) \longrightarrow \operatorname{Hom}_C(B,\omega_C)\otimes G \longrightarrow$$
$$\longrightarrow \operatorname{Hom}_C(B,\omega_C)\otimes G\otimes G \longrightarrow \cdots$$

where the arrows are induced by the coactions of ω_C and of the structure sheaf of $[\operatorname{Spec} B/G]$. If we replace ω_C with an injective (equivariant) resolution $I^\bullet$ we obtain a double complex spectral sequence $E_1^{p,q} = \operatorname{Ext}^p_C(B,\omega_C)\otimes G^{\otimes q}$ abouting to the equivariant $R\operatorname{Hom}_C^G(B,\omega_C)$ (the sheaves $\operatorname{Hom}_C^G(B,I^\bullet)$ are considered as B^G-modules) that is the stalk of the curly Ext. Since B is Cohen-Macaulay of the same dimension as $\mathcal{X}$ we have $\operatorname{Ext}^p_C(B,\omega_C)=0$ for $p>r$ and the desired result follows. □

It is important to stress that the previous Corollary 1.41 holds for every projective stack in characteristic zero; indeed according to [Kre06] such a stack can be embedded in a smooth proper Deligne-Mumford stack. To conclude we prove that π_* maps the dualizing sheaf of a projective Deligne-Mumford stack to the dualizing sheaf of its moduli scheme.

Proposition 1.42. *Let $\mathcal{X}$ be a projective Cohen-Macaulay Deligne-Mumford stack with moduli scheme $\pi\colon \mathcal{X}\to X$. Denote with $\omega_{\mathcal{X}}$ the dualizing sheaf of $\mathcal{X}$. The quasi coherent sheaf $\pi_*\omega_{\mathcal{X}} =: \omega_X$ is the dualizing sheaf of X.*

Proof. First of all we observe that the moduli scheme X is projective and Cohen-Macaulay so that we already know that its dualizing complex is actually a sheaf ω_X. We also know that $\pi^!\omega_X=\omega_{\mathcal{X}}$. We have just to prove that $\pi_*\pi^!\omega_X=\omega_X$. Let F be a quasi coherent sheaf on X, by duality we know that $R\operatorname{Hom}_X(F,\pi_*\pi^!\omega_X) = R\operatorname{Hom}_{\mathcal{X}}(L\pi^*F,\pi^!\omega_X) = R\operatorname{Hom}_X(\pi_*L\pi^*F,\omega_X)$. However we already know that $\pi_*\pi^*=\operatorname{id}$ and taking a locally free resolution (X is projective) we have also $\pi_*L\pi^*=\operatorname{id}$. Using duality on X and uniqueness of the dualizing sheaf we obtain $\pi_*\pi^!\omega_X=\omega_X$. □

Corollary 1.43. *Let $X\to \operatorname{Spec} k$ be a variety with finite quotient singularities. Denote with $\pi\colon \mathcal{X}^{can}\to X$ the canonical stack associated to X as in [FMN07, Rem 4.9] and with ω_{can} its canonical bundle. The coherent sheaf $\pi_*\omega_{can}$ is the dualizing sheaf of X.*

Proof. We just observe that $\mathcal{X}^{\mathrm{can}}$ is smooth so that its dualizing sheaf is canonical bundle and apply the previous corollary. □

2. Applications and computations

2.1. **Duality for nodal curves.** Despite being probably already known, it is a good exercise to compute the dualizing sheaf for a nodal curve using the machinery developed so far. First of all we specify that by nodal curve we mean a non necessarily balanced nodal curve. We can assume from the beginning that the curve has generically trivial stabilizer. If it is not the case, we can always rigidify the curve and treat the gerbe separately. We assume also that if the node is reducible none of the two components has a non trivial generic stabilizer. With this assumption a stacky node étale locally looks like $\left[\operatorname{Spec}\frac{k[x,y]}{(xy)}/\mu_{a,k}\right]$. The action of $\mu_{a,k}$ is given by:

$$\frac{k[k,y]}{(xy)} \longrightarrow \frac{k[x,y]}{(xy)}\otimes \mu_{a,k}$$
$$x,y \longmapsto \lambda^i x, \lambda^j y$$

where $(i,j)=1$; $i,j\neq 0 \mod a$ and a is coprime with the characteristic of k so that the stack is tame. The result of this section is the following theorem:

Theorem 2.1. *Let $\mathcal{C}$ be a proper tame nodal curve as specified above. Let $\pi\colon \mathcal{C}\to C$ be its moduli space. Let D be the effective Cartier divisor of C marking the orbifold points, and denote with $\mathcal{D}=\pi^{-1}(D)_{red}$. Denote also with ω_C the dualizing sheaf of C and with $\omega_{\mathcal{C}}=\pi^!\omega_C$ the dualizing sheaf of $\mathcal{C}$. The following relation holds:*

$$\omega_{\mathcal{C}}(\mathcal{D})=\pi^*\omega_C(D)$$

We start proving this theorem with the following local computation:

Lemma 2.2. *Consider the orbifold node $\mathcal{Y}:=\left[\operatorname{Spec}\frac{k[x,y]}{(xy)}/\mu_{a,k}\right]$ described above. Let $\rho\colon\mathcal{Y}\to Y:=\operatorname{Spec}\frac{k[u,v]}{(uv)}$ be the moduli scheme, then we have $\omega_{\mathcal{Y}}=\rho^!\mathcal{O}_Y=\mathcal{O}_{\mathcal{Y}}$.*

Proof. We will denote the ring $\frac{k[x,y]}{(xy)}$ with B and $\frac{k[u,v]}{(uv)}$ with A. We also choose $\operatorname{Spec}B$ as atlas for the stack. Let α,β be the smallest positive integers such that $i\alpha=0 \mod a$, $j\beta=0 \mod a$, then the morphism from the atlas to the moduli scheme is the following:

$$\begin{array}{ccc} A & \xrightarrow{p_0} & B \\ u,v & \longmapsto & x^\alpha, y^\beta \end{array}$$

The dualizing sheaf for $\operatorname{Spec}A$ is isomorphic to the structure sheaf, so it's enough to compute duality for the structure sheaf denoted as the free A-module $\langle e\rangle$. According to 1.25 we first need to compute the B-module $R\mathcal{H}om_A(B,\langle e\rangle)$. We take the infinite projective resolution of B as an A-module:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{\oplus(\alpha+\beta-2)} & \longrightarrow & A^{\oplus(\alpha+\beta-2)} & \longrightarrow & A^{\oplus(\alpha+\beta-1)} & \longrightarrow & B \longrightarrow 0 \\ & & & & & & e_0 & \longmapsto & 1 \\ \cdots & & e_l & \longmapsto & ue_l,\ e_l & \longmapsto & ve_l,\ e_l & \longmapsto & x^l \\ \cdots & & f_m & \longmapsto & vf_m,\ f_m & \longmapsto & uf_m,\ f_m & \longmapsto & y^m \end{array}$$

where $1\le l\le\alpha-1$, $1\le m\le\beta-1$. We apply the functor $\operatorname{Hom}_A(\cdot,\langle e\rangle)$ and compute cohomology. The complex is obviously acyclic as expected, and h^0 is the A-module $\bigoplus_l(u)e_l^\vee\oplus\bigoplus_m(v)f_m^\vee\oplus e_0^\vee$. The A-module h^0 is naturally a sub-module of $\operatorname{Hom}_A(B,\langle e\rangle)$ and its B-module structure is induced by the natural B-module structure of this last one. Let $g_l^\vee\in\operatorname{Hom}_A(B,\langle e\rangle)$ the morphism such that $g_l^\vee(x^l)=e$ and zero otherwise, $h_m^\vee$ the morphism such that $h_m^\vee(y^m)=e$ and zero otherwise, and $g_0^\vee$ such that $g_0^\vee(1)=e$. The B-module $\operatorname{Hom}_A(B,\langle e\rangle)$ can be written as $\frac{\langle g_{\alpha-1}^\vee,h_{\beta-1}^\vee\rangle}{x^{\alpha-1}g_{\alpha-1}^\vee-y^{\beta-1}h_{\beta-1}^\vee}$. The B module structure of h^0 is then given by:

$$\begin{array}{ccc} \langle e_0^\vee\rangle & \rightarrowtail & \frac{\langle g_{\alpha-1}^\vee,h_{\beta-1}^\vee\rangle}{x^{\alpha-1}g_{\alpha-1}^\vee-y^{\beta-1}h_{\beta-1}^\vee} \\ e_0^\vee & \longmapsto & x^{\alpha-1}g_{\alpha-1}^\vee \end{array}$$

Eventually we have $\overline{p}_0^*R\operatorname{Hom}_A(B,\langle e\rangle)=\langle e_0^\vee\rangle$. To compute the equivariant structure of $\langle e_0^\vee\rangle$ we follow the recipe in Lemma 1.25. We find out that the coaction on $e_0^\vee,e_l^\vee,f_m^\vee$ is as follows:

$$\begin{array}{ccc} e_0^\vee & \longrightarrow & e_0^\vee \\ e_l^\vee & \longrightarrow & \lambda^{-il}e_l^\vee \\ f_m^\vee & \longrightarrow & \lambda^{-jm}e_m^\vee \end{array}$$

so that the equivariant structure of $\langle e_0^\vee\rangle$ is the trivial one and $\rho^!\omega_Y$ is then canonically isomorphic to the structure sheaf. $\square$

Remark 2.3. We can notice that the assumptions on the two integers i, j have never been used in the previous proof, however they are going to be necessary in what follows.

With the following proposition we take care of smooth orbifold points.

Proposition 2.4. *Let $\mathcal{X} \to \operatorname{Spec} k$ be a projective Deligne-Mumford stack that is generically a scheme, let $D = \sum_{i=1}^d D_i$ be a simple normal crossing divisor whose support does not contain any orbifold structure. Let $\mathbf{a} = (a_1, \dots, a_d)$ positive integers. Denote with $\mathcal{X}_{\mathbf{a},D} = \sqrt[\mathbf{a}]{D/\mathcal{X}} \xrightarrow{\tau} \mathcal{X}$ and with $\mathcal{D}_i = (\tau^{-1}D_i)_{red}$. For every $\mathcal{F}$ quasicoherent sheaf on $\mathcal{X}$ the object $\tau^!\mathcal{F} \in D(\mathcal{X}_{\mathbf{a},D})$ is the quasicoherent sheaf $\tau^*\mathcal{F}(\sum_{i=1}^d (a_i - 1)\mathcal{D}_i)$.*

Proof. Since τ is a flat morphism we already know that $\tau^!$ maps quasicoherent sheaves to quasicoherent sheaves. The precise statement can be retrieved using some of the computations in [AGV06, Thm 7.2.1] and the machinery in Theorem 1.33 (details left to the reader). $\square$

proof of Theorem 2.1. For the moment we can assume that the curve has no other orbifold points then the nodes and without loss of generality we can assume that there is only one node. First we prove that $\pi^!\omega_C = \pi^*\omega_C$, then we can add smooth orbifold points in a second time applying the root construction; the formula claimed in the theorem follows then from Proposition 2.4. First of all we take an étale cover $\mathcal{Y}$ of $\mathcal{C}$ in this way: we choose an étale chart of the node that is an orbifold node like the one in Lemma 2.2 and we complete the cover with a chart that is the curve $\mathcal{C}$ minus the node, denoted with C_0. The setup is summarized by the following cartesian square:

$$\begin{array}{ccc} C_0 \coprod [\operatorname{Spec} B/\mu_a] & \xrightarrow{i \coprod \overline{\sigma}} & \mathcal{C} \\ \downarrow{\scriptstyle \mathrm{id} \coprod p_0} & & \downarrow{\scriptstyle \pi} \\ C_0 \coprod \operatorname{Spec} A & \xrightarrow{i \coprod \sigma} & C \end{array}$$

where $B = \frac{k[x,y]}{(xy)}$, $A = \frac{k[u,v]}{(uv)}$, the map p_0 sends u, v to x^α, y^β, the map i is the inclusion of C_0 in C, and σ is étale. We use as an atlas $C_0 \coprod \operatorname{Spec} B$ with the obvious map to $\mathcal{C}$. Completing the presentation we obtain the following groupoid:

$$C_0 \coprod \operatorname{Spec} B \times \mu_a \coprod \operatorname{Spec} B \times_{\sigma\circ p_0, C} C_0 \overset{s}{\underset{t}{\rightrightarrows}} C_0 \coprod \operatorname{Spec} B$$

We can divide it in three pieces: one is the trivial groupoid over C_0, the second is $\operatorname{Spec} B \times \mu_a \rightrightarrows \operatorname{Spec} B$ where the arrows are action and projection; the last one is $\operatorname{Spec} B \times_{\sigma\circ p_0, C} C_0 \overset{p_0\circ q_1}{\underset{q_2 p_0}{\rightrightarrows}} C_0 \coprod \operatorname{Spec} B$, where q_1 is projection to $\operatorname{Spec} B$, q_2 is projection to C_0 and $\overline{p}_0$ fits inside the following cartesian square:

$$\begin{array}{ccccc} \operatorname{Spec} B \times_{\sigma\circ p_0, C} C_0 & \xrightarrow{\overline{p}_0} & \operatorname{Spec} A \times_{\sigma, C} C_0 & \xrightarrow{q_2} & C_0 \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle i} \\ \operatorname{Spec} B & \xrightarrow{p_0} & \operatorname{Spec} A & \xrightarrow{\sigma} & C \end{array} \tag{2.1}$$

Now we check if the dualizing sheaf glues like $\pi^*\omega_C$ on this presentation and we achieve this using Theorem 1.33. The result is trivially true for the first piece of the presentation. For the second piece of the presentation it is implied immediately by Lemma 2.2. For what concerns the last piece of presentation we have that $\pi^!\omega_C$ glues with the canonical isomorphism $\overline{p}_0^! q_2^* i^*\omega_C \cong q_1^* p_0^! \sigma^*\omega_C$ where the canonical isomorphism comes from the cartesian square in picture (2.1). However we have $q_1^* p_0^! \sigma^*\omega_C = q_1^* p_0^! \mathcal{O}_A \otimes q_1^* p_0^* \sigma^*\omega_C$ and $\overline{p}_0^! \mathcal{O}_{\operatorname{Spec} A\times_{\sigma,C} C_0} \otimes \overline{p}_0^* q_2^* i^*\omega_C p_0^! \sigma^*\omega_C$ where the canonical isomorphism comes from the cartesian square in picture (2.1). However we have $q_1^* p_0^! \sigma^*\omega_C = q_1^* p_0^! \mathcal{O}_A \otimes q_1^* p_0^* \sigma^*\omega_C$ and $\overline{p}_0^! q_2^* i^*\omega_C = \overline{p}_0^! \mathcal{O}_{\operatorname{Spec} A\times_{\sigma,C} C_0} \otimes \overline{p}_0^* q_2^* i^*\omega_C$. According to Lemma 2.2 the sheaves $p_0^!\mathcal{O}_A$ and $\overline{p}_0^! \mathcal{O}_{\operatorname{Spec} A\times_{\sigma,C} C_0}$ are respectively equal to $\mathcal{O}_B$ and $\mathcal{O}_{\operatorname{Spec} B\times_{\sigma\circ p_0,C} C_0}$. Eventually the gluing isomorphism is just the identity and we can conclude that the dualizing sheaf is $\pi^*\omega_C$. □

2.2. **Other examples of singular curves.** In the previous section we have seen that nodal curves, balanced or not, have a dualizing sheaf that is an invertible sheaf and carry a trivial representation on the fiber on the node. It is not difficult to find examples of singularities where the representation on the fiber of the singularity is non trivial. What follows is a collection of computations of duality with compact support, performed with the same technique used in Lemma 2.2. These examples are mere applications of Lemma 1.25 and we will be able to retrieve these results with a better technique in the next section.

Example 2.5. **(Cusp over a line)** Let B be the cusp $k[x,y]/(y^2-x^3)$ with an action of $\mu_{2,k}$ given by $y\mapsto \lambda y$ and $x\mapsto x$. The moduli scheme of the quotient stack is the affine line $k[x] =: A$ with the morphism:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ x & \longmapsto & x \end{array}$$

The morphism f is flat and the dualizing sheaf restricted to the atlas is the B-module $\operatorname{Hom}_A(B,A)$. As a B-module this is just $\langle e_1^\vee\rangle$ where $e_1^\vee(y)=1$ and zero otherwise. The coaction is given by $e_1^\vee \mapsto \lambda^{-1}e_1^\vee$.

Example 2.6. **(Tac-node over a node)** Let B be the tac-node $k[x,y]/(y^2-x^4)$ with an action of $\mu_{2,k}$ given by $y\mapsto y$ and $x\mapsto \lambda x$. The moduli scheme of the quotient stack is the node $k[u,y]/(y^2-u^2) =: A$ with the morphism:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u,y & \longmapsto & x^2,y \end{array}$$

The stack is reducible. The morphism f is flat again and the dualizing sheaf is the B-module $\langle e_1^\vee\rangle$ ($e_1^\vee(x)=1$ and otherwise zero) with the coaction $e_1^\vee\mapsto\lambda^{-1}e_1^\vee$.

These two examples look pretty similar but they are actually of a quite different nature. With a simple computation we obtain that the tac-node is actually a root construction over the node $\sqrt[2]{\mathcal{O}_A, 0/\operatorname{Spec} A}$. For a root construction we expected that kind of dualizing sheaf from, the already studied, smooth case (Lemma 2.4). The cusp is not a root construction, however it is flat on the moduli scheme anyway, and the dualizing sheaf is the same we have for the root construction.

With the following example we see that the dualizing sheaf can be the structure sheaf for nodes other than $xy=0$.

Example 2.7. **(Tac-node over a cusp (an irreducible node))** Let B be the tac-node $k[x,y]/(y^2-x^4)$ with an action of $\mu_{2,k}$ given by $y\mapsto\lambda y$ and $x\mapsto\lambda x$. The moduli scheme of the quotient stack is the cusp $k[u,t]/(t^2-u^3) =: A$ with the morphism:

$$\begin{aligned} A &\xrightarrow{f} B \\ u,t &\longmapsto x^2, xy \end{aligned}$$

This one stack is irreducible, none of $y-x^2$ and $y+x^2$ can be a closed substack. With a computation very similar to the one in Lemma 2.2 we obtain that the dualizing sheaf is $\langle e_0^\vee \rangle$ ($e_0^\vee(1)=1$ and otherwise zero) with the trivial coaction.

2.3. **Local complete intersections.** According to Corollary 1.41 every Cohen-Macaulay projective Deligne-Mumford stack (in characteristic zero) admits a dualizing coherent sheaf; using Corollary 1.21 it's immediate to prove that if the stack also have a Gorenstein atlas then the dualizing sheaf is an invertible sheaf. The aim of this section is to reconstruct the classic duality result for local complete intersections as in [Har77, III.7.11].

Theorem 2.8. *Let $\mathcal{X}$ be a projective Deligne-Mumford stack that has a regular codimension r closed embedding in a smooth projective Deligne-Mumford stack $\mathcal{P}$. Denote with $\mathcal{I}$ the ideal sheaf defining the closed-embedding and with $\omega_{\mathcal{P}}$ the canonical sheaf of $\mathcal{P}$. The dualizing sheaf of $\mathcal{X}$ is $\omega_{\mathcal{P}} \otimes \wedge^r (\mathcal{I}/\mathcal{I}^2)^\vee$.*

Proof. Our task is to compute $\mathcal{E}xt^r_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})$. We can produce an étale cover of $\mathcal{P}$ so that it is locally $[\operatorname{Spec} C/G]$ where C is a regular ring and G a finite group. We can assume that the regular closed embedding is locally:

$$[\operatorname{Spec} B/G] \xrightarrow{i} [\operatorname{Spec} C/G]$$

where B is defined by $(f_1,\dots,f_r)$ a regular sequence in C. Once we have fixed a basis for the ideal sheaf $\mathcal{I}$, the coaction of G is also determined on that basis. We denote with β_f the coaction on the basis $(f_1,\dots,f_r)$, and it is an r-dimensional representation. We denote with ω_C the canonical sheaf on $\operatorname{Spec} C$, and it also comes with a coaction that is a one-dimensional representation ρ_C (we are assuming that ω_C is free). We now take the Koszul resolution $K^\bullet$ of B. The coactions of G on ω_C and $\mathcal{I}$ induce a coaction of G on $K^\bullet$ and a coaction on the complex $\operatorname{Hom}_C(K^\bullet,\omega_C)$. In particular the coaction on $\operatorname{Hom}_C(\wedge^r C^{\oplus r},\omega_C)$ is the representation $\rho_C \otimes \det \beta_f^\vee$. Both the induced coaction (denoted with $\gamma^\bullet$) and the trivial coaction are morphisms of complexes:

$$0 \longrightarrow \operatorname{Hom}_C^G(K^\bullet,\omega_C) \longrightarrow \operatorname{Hom}_C(K^\bullet,\omega_C) \overset{\gamma^\bullet}{\underset{}{\rightrightarrows}} \operatorname{Hom}_C(K^\bullet,\omega_C)\otimes \mathcal{O}_G$$

and the equalizer is the C^G-module of equivariant morphisms. The cohomology of this first complex computes the global sections of $\mathcal{E}xt^\bullet_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}},\omega_C)$ restricted to $[\operatorname{Spec} B/G]$. We can also compute cohomology of the second and third complex and we have arrows between cohomologies induced by both the coaction $\gamma^\bullet$ and the trivial coaction:

$$h^\bullet(\operatorname{Hom}_C(K^\bullet,\omega_C)) \overset{h^\bullet(\gamma^\bullet)}{\rightrightarrows} h^\bullet(\operatorname{Hom}_C(K^\bullet,\omega_C))\otimes \mathcal{O}_G$$

This gives the sheaves $h^\bullet(\operatorname{Hom}_C(K^\bullet,\omega_C))$ an equivariant structure; eventually these sheaves with the equivariant structure are $\mathcal{E}xt^\bullet_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}},\omega_C)$ restricted to $[\operatorname{Spec} B/G]$. However, we already know that $h^r(\operatorname{Hom}_C(K^\bullet,\omega_C)) = \frac{\omega_C}{(f_1,\dots,f_r)\omega_C}$ and the others are zero. It is easy to check that the equivariant structure of the non vanishing one is the representation $\rho_C \otimes \det \beta_f^\vee$. To summarize, the dualizing sheaf restricted to $[\operatorname{Spec} B/G]$ is isomorphic to $\omega_C \otimes_C B$ where B has the non necessarily trivial coaction $\det \beta_f^\vee$. As in the case of schemes this isomorphism is not canonical. If we change the basis $(f_1,\dots,f_r)$ to a new one $(g_1, \dots, g_r)$ where $f_i = \delta_{i,j} g_j$ we produce an automorphism on the Koszul complex $K^\bullet$; in particular the last term of the complex $\wedge^r C^{\oplus r}$ carries an automorphism given by $\det \delta$. In the equivariant setup, also the representation β_f is affected by a change of basis. In particular the new basis carries a representation β_g such that $(a^*\delta) \circ \beta_g = \beta_f \circ \delta$, where a denotes the action of G on $\operatorname{Spec} C$.

As in the case of schemes the sheaf $\wedge^r (\mathcal{I}/\mathcal{I}^2)^\vee$ is trivial on $[\operatorname{Spec} B/G]$ and the change of basis $(f_1, \dots, f_r) \mapsto (g_1, \dots, g_r)$ induces an automorphism on the sheaf that is multiplication by $\det \delta^{-1}$. Moreover it is straightforward to check that the equivariant structure of the sheaf is given by the representation $\det \beta_f^\vee$. It is also obvious that the representation changes, after a change of basis, according to the formula $(a^*\delta) \circ \beta' = \beta \circ \delta$. Eventually we can conclude that there is an isomorphism between $\mathcal{E}xt^r_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_C)|_{[\operatorname{Spec} B/G]}$ and $\omega_{\mathcal{P}} \otimes \wedge^r (\mathcal{I}/\mathcal{I}^2)^\vee|_{[\operatorname{Spec} B/G]}$ that doesn't depend on the choice of the basis of $\mathcal{I}$, so to speak a canonical isomorphism. This implies that we can glue these local isomorphisms to obtain a global one. □

In the proof of this theorem we have seen how to compute the sheaf $\mathcal{E}xt^r_{\mathcal{P}}(\mathcal{O}_{\mathcal{X}}, \omega_{\mathcal{P}})$ locally on a stack that is $[\operatorname{Spec} B/G]$ using the Koszul resolution. Even if the stack is not locally complete intersection but Cohen-Macaulay we can use the same technique to compute $\mathcal{E}xt^r$, replacing the Koszul complex with some other equivariant resolution. This approach is obviously a much faster and reliable technique than the one used in section 2.2.

Example 2.9. **(A non Gorenstein example)** Let B be the triple point $k[u, v, t]/(uv - t^2, ut - v^2, vt - u^2)$ with an action of $\mu_{a,k}$ given by $u, v, t \mapsto \lambda u, \lambda v, \lambda t$ and we study duality of the quotient stack. This is non Gorenstein since the reducible ideal (uvt) is a system of parameters and B is Cohen-Macaulay (we apply the Ubiquity Theorem in [Bas63]). We can close it in $k[u, v, t]$ with the same action of $\mu_{a,k}$. The canonical sheaf of this quotient stack is the free module $\langle du \wedge dv \wedge dt \rangle$ with the coaction $du \wedge dv \wedge dt \mapsto \lambda^{-3} du \wedge dv \wedge dt$. It is completely straightforward to check that the dualizing sheaf of B is the B-module:

$$\omega_B := \frac{\langle e_1, e_2 \rangle}{(te_1 + ue_2, ve_1 + te_2, ue_1 + ve_2)}$$

It is also easy to check that the equivariant resolution of $(uv - t^2, ut - v^2, vt - u^2)$ induces a coaction $e_i \mapsto \lambda^3 e_i$ for $i = 1, 2$ that compensates the coaction carried by $\langle du \wedge dv \wedge dt \rangle$. Eventually the dualizing sheaf of $[\operatorname{Spec} B/G]$ is ω_B with the trivial coaction.

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Fabio Nironi, SISSA, Via Beirut 2-4, 34014 Trieste, Italy
E-mail address: fabio.nironi@gmail.com