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SEMISTABLE PRINCIPAL HIGGS BUNDLES

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Abstract. We give a Miyaoka-type semistability criterion for principal Higgs G-bundles $\mathfrak{E}$ on complex projective manifolds of any dimension, i.e., we prove that $\mathfrak{E}$ is semistable and the second Chern class of its adjoint bundle vanishes if and only if certain line bundles, obtained from the characters of the parabolic subgroups of G, are numerically effective. We also give alternative characterizations in terms of a notion of numerical effectiveness of Higgs vector bundles we have recently introduced.

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1. Introduction.

In 1987 Miyaoka gave a criterion for the semistability of a vector bundle V on a projective curve in terms of the numerical effectiveness of a suitable divisorial class (the relative anti-canonical divisor of the projectivization $\mathbb{P}V$ of V). Recently several generalizations of this criterion have been formulated [7, 2, 3], dealing with principal bundles, higher dimensional varieties, and considering also the case of bundles on compact Kähler manifolds. In this paper we prove a Miyaoka-type criterion for principal Higgs bundles on complex projective manifolds. Let us give a rough anticipation of this result. Given a principal Higgs G-bundle E on a complex projective manifold X, with Higgs field ϕ, and a parabolic subgroup P of G, we introduce a subscheme $R_P(E,\phi)$ of the bundle E/P whose sections parametrize reductions of the structure group G to P that are compatible with the Higgs field ϕ. Then in Theorem 4.6 we prove the equivalence of the following conditions: for every reduction of G to a parabolic subgroup P, and every dominant character of P, a certain associated line bundle on $R_P(E,\phi)$ is numerically effective; (E,ϕ) is semistable as a principal Higgs bundle, and the second Chern class of the adjoint bundle $\mathrm{Ad}(E)$ (with real coefficients) vanishes. We first prove this fact when X is a curve (so that the condition involving the second Chern class is void) and then extend it to complex projective manifolds of arbitrary dimension.

We also formulate an equivalent criterion which states that $\mathfrak{E} = (E,\phi)$ is semistable, and $c_2(\mathrm{Ad}(E)) = 0$, if and only if the adjoint Higgs bundle $\mathrm{Ad}(\mathfrak{E})$ is numerically effective (as a Higgs bundle) in a sense that we introduced in a previous paper [5].

Since a principal Higgs bundle with zero Higgs field is exactly a principal bundle, all results we prove in this paper hold true for principal bundles. In this way sometimes we just recover well-known results or some of the results in [2] with their known proofs, sometimes we provide simpler proofs, while at times the results are altogether new.

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2. Semistable principal bundles

In this short section we recall some basics about principal bundles, notably the definition of semistable principal bundle. Let X be a smooth complex projective variety, G a complex reductive algebraic group, and $\pi\colon E \to X$ a principal G-bundle on X. If $\rho\colon G \to \mathrm{Aut}(Y)$

is a representation of G as automorphisms of a variety Y, we may construct the associated bundle $E(\rho) = E \times_\rho Y$, the quotient of $E \times Y$ under the action of G given by $(u, y) \mapsto (ug, \rho(g^{-1})$ for $g \in G$. If $Y = \mathfrak{g}$ is the Lie algebra of G, and ρ is the adjoint action of G on $\mathfrak{g}$, one gets the the adjoint bundle of E, denoted by $\mathrm{Ad}(E)$. Another important example is obtained when ρ is given by a group homomorphism $\lambda\colon G \to G'$; in this case the associated bundle $E' = E \times_\lambda G'$ is a principal G'-bundle. We say that the structure group G of E has been extended to G'.

If E is a principal G-bundle on X, and F a principal G'-bundle on X, a morphism $E \to F$ is pair (f, f'), where $f'\colon G \to G'$ is a group homomorphism, and $f\colon E \to F$ is a morphism of bundles on X which is is f'-equivariant, i.e., $f(ug) = f(u)f'(g)$. Note that this induces a vector bundle morphism $\tilde{f}\colon \mathrm{Ad}(E) \to \mathrm{Ad}(F)$ given by $\tilde{f}(u, \alpha) = (f(u), f'_*(\alpha))$, where $f'_*\colon \mathfrak{g} \to \mathfrak{g}'$ is the morphism induced on the Lie algebras. As an example, consider a principal G-bundle E, a group homomorphism $\lambda\colon G \to G'$, and the extended bundle E'. There is a natural morphism $(f, \lambda)\colon E \ \to E'$, where $f = \mathrm{id} \times \lambda$ if we identify E with $E \times_{\mathrm{id}} G$.

If K is a closed subgroup of G, a *reduction* of the structure group G of E to K is a principal K-bundle F over X together with an injective K-equivariant bundle morphism $F \to E$. Let $E(G/K)$ denote the bundle over X with standard fibre G/K associated to E via the natural action of G on the homogeneous space G/K. There is an isomorphism $E(G/K) \simeq E/K$ of bundles over X. Moreover, the reductions of the structure group of E to K are in a one-to-one correspondence with sections $\sigma\colon X \to E(G/K) \simeq E/K$.

We first recall the definition of semistable principal bundle when the base variety X is a curve. Let $T_{E/K,X}$ be the vertical tangent bundle to the bundle $\pi_K\colon E/K \to X$.

Definition 2.1. *Let E be a principal G-bundle on a smooth connected projective curve X. We say that E is semistable if for every proper parabolic subgroup $P \subset G$, and every reduction $\sigma\colon X \to E/P$, the pullback $\sigma^*(T_{E/P,X})$ has nonnegative degree.*

When X is a higher dimensional variety, the definition must be somewhat refined; the introduction of an open dense subset whose complement has codimension at least two should be compared with the definition of semistable vector bundle, which involves non-locally free subsheaves (which are locally free exactly on open subsets of that kind).

Definition 2.2. *Let X be a polarized smooth projective variety. A principal G-bundle E on X is semistable if and only if for any proper parabolic subgroup $P \subset G$, any open dense* *subset $U \subset X$ such that* $\operatorname{codim}(X-U) \geq 2$*, and any reduction* $\sigma\colon U \to (E/P)_{|U}$ *of G to P on U, one has* $\deg \sigma^*(T_{E/P,X}) \geq 0$.

Here it is important that we may assume that a line bundle defined on an open dense subset of X extends uniquely to the whole of X, so that we may consistently consider its degree. This is discussed in detail in [15].

3. Principal Higgs bundles

We switch now to principal Higgs bundles. Let X be a smooth complex projective variety, and G a reductive complex algebraic group.

Definition 3.1. *A principal Higgs G-bundle $\mathfrak{E}$ is a pair (E,ϕ), where E is a principal G-bundle, and ϕ is a global section of* $\operatorname{Ad}(E)\otimes\Omega^1_X$ *such that* $[\phi,\phi]=0$.

When G is the general linear group, under the identification $\operatorname{Ad}(E)\simeq\operatorname{End}(V)$, where V is the vector bundle corresponding to E, this agrees with the usual definition of Higgs vector bundle.

Let K be a closed subgroup of G, and $\sigma\colon X\to E(G/K)\simeq E/K$ a reduction of the structure group of E to K. So one has a principal K-bundle F on X, and a principal bundle morphism $i_\sigma\colon F\to E$, and an injective morphism of bundles $\operatorname{Ad}(F)\to\operatorname{Ad}(E)$. Let $\Pi_\sigma\colon \operatorname{Ad}(E)\otimes\Omega^1_X\to(\operatorname{Ad}(E)/\operatorname{Ad}(F))\otimes\Omega^1_X$ be the induced projection.

Definition 3.2. *A section $\sigma\colon X\to E/K$ is a* Higgs reduction *of (E,ϕ) if $\phi\in\ker\Pi_\sigma$.*

Remark 3.3. Let us again consider the case when G is the general linear group $Gl(n,\mathbb{C})$, and let us assume that K is a (parabolic) subgroup such that G/K is the Grassmann variety $\operatorname{Gr}_k(\mathbb{C}^n)$ of k-dimensional quotients of $\mathbb{C}^n$. If V is the vector bundle corresponding to E, a reduction σ of G to K corresponds to a rank $n-k$ subbundle W of V, and the fact that σ is a Higgs reduction means that W is ϕ-invariant, i.e., $\phi(W)\subset W\otimes\Omega^1_X$. △

We want to show that the choice of ϕ singles out a subscheme of the variety E/K, which describes the Higgs reductions of the pair (E,ϕ). We start by noting that there is a natural morphism $\chi\colon \pi_K^*\operatorname{Ad}(E)\to T_{E/K,X}$. Then ϕ determines a section $\chi(\phi)$ of $T_{E/K,X}\otimes\Omega^1_{E/K}$.

Definition 3.4. *The* scheme of Higgs reductions of $\mathfrak{E}=(E,\phi)$ to K *is the closed subscheme $R_K(\mathfrak{E})$ of E/K given by the zero locus of $\chi(\phi)$.*

This construction is compatible with base change, i.e., if $f\colon Y \to X$ is a morphism of smooth complex projective varieties, and $f^*(\mathfrak{E})$ is the pullback of $\mathfrak{E}$ to Y, then $R_K(f^*(\mathfrak{E})) \simeq Y \times_X R_K(\mathfrak{E})$. By construction, $\sigma\colon X \to E(G/K) \simeq E/K$ is a Higgs reduction if and only if it takes values in the scheme $R_K(\mathfrak{E})$.

Also, one should note that the scheme of Higgs reductions is in general singular, so that in order to consider Higgs bundles on it one needs to use the theory of the de Rham complex for arbitrary schemes, as developed by Grothendieck [11].

For some time we restrict our attention to the case when X is a curve. We start by introducing a notion of semistability fr principal Higgs bundles.

Definition 3.5. *Let X be a smooth projective curve. A principal Higgs G-bundle $\mathfrak{E} = (E, \phi)$ is stable (semistable) if for every parabolic subgroup $P \subset G$ and every Higgs reduction $\sigma\colon X \to R_P(\mathfrak{E})$ one has $\deg \sigma^*(T_{E/P,X}) > 0$ ($\deg \sigma^*(T_{E/P,X}) \geq 0$).*

Remark 3.6. A notion of semistability for principal Higgs bundles was introduced by Simpson in [**?**]. According to that definition, a principal Higgs G-bundle $\mathfrak{E}$ is semistable if there exists a faithful linear representation $\rho\colon G \to \mathrm{Aut}(W)$ such that that associated Higgs vector bundle $\mathfrak{W} = \mathfrak{E} \times \rho W$ is semistable. In Corollary 4.5 we shall establish the equivalence between the two definitions. For the moment we only notice that the equivalence is clear when G is the general linear group $Gl(n, \mathbb{C})$ (cf. Remark 3.11). △

Lemma 3.7. *Let $f\colon X' \to X$ be a nonconstant morphism of smooth projective curves, and $\mathfrak{E}$ a principal Higgs G-bundle on X. The pullback Higgs bundle $f^*\mathfrak{E}$ is semistable if and only if $\mathfrak{E}$ is.*

Proof. As we shall prove in Lemma 4.4 in the case of X of arbitrary dimension, a principal Higgs bundle $\mathfrak{E}$ is semistable if and only if the adjoint Higgs bundle $\mathrm{Ad}(\mathfrak{E})$ is semistable (as a Higgs vector bundle). In view of this result, our claim reduces to the analogous statement for Higgs vector bundles, which was proved in [7]. □

If $\mathfrak{E} = (E, \phi)$ is a principal Higgs G-bundle on X, and K is a closed subgroup of G, we may associate with every character χ of K a line bundle $L_\chi = E \times_\chi \mathbb{C}$ on E/K, where we regard E as a principal K-bundle on E/K.

The following result extends to Higgs bundles Lemma 2.1 of [14], and its proof follows easily from that of that Lemma.

Proposition 3.8. *A principal Higgs G-bundle $\mathfrak{E} = (E, \phi)$ is semistable if and only if for every parabolic subgroup $P \subset G$, every nontrivial dominant character χ of P, and every Higgs reduction $\sigma \colon X \to R_P(\mathfrak{E})$, the line bundle $\sigma^*(L^*_\chi)$ has nonnegative degree.*

Proof. Let $\alpha_1, \dots, \alpha_r$ be simple roots of the Lie algebra $\mathfrak{g}$. We may assume that P is a maximal parabolic subgroup corresponding to a root α_i. It has been proven in [14, Lemma 2.1] that the determinant of the vertical tangent bundle $T_{E/P,X}$ is associated to the principal P-bundle $E \to E/P$ via a character that may be expressed as $\mu = -m\lambda_i$, where λ_i is the weight corresponding to α_i, and $m \geq 0$. Thus, if $\sigma \colon X \to R_P(\mathfrak{E})$ is a Higgs reduction, $\deg(\sigma^*(L^*_\mu)) \geq 0$ if and only if $\deg \sigma^*(T_{E/P,X}) \geq 0$. □

Remark 3.9. By "root" of $\mathfrak{g}$ we mean a root of the semisimple part of $\mathfrak{g}$ extended by zero on the centre. △

We may now state and prove a Miyaoka-type semistability criterion for principal Higgs bundles (over projective curves). This generalizes Proposition 2.1 of [2], and, of course, Miyaoka's original criterion in [12].

Theorem 3.10. *A principal Higgs G-bundle $\mathfrak{E} = (E, \phi)$ on a smooth projective curve X is semistable if and only if for every parabolic subgroup $P \subset G$, and every nontrivial dominant character χ of P, the line bundle L^*_χ restricted to $R_P(\mathfrak{E})$ is nef.*

Proof. Assume that $\mathfrak{E}$ is semistable and that $L^*_{\chi|R_P(\mathfrak{E})}$ is not nef. Then there is an irreducible curve $Y \subset R_P(\mathfrak{E})$ such that $[Y] \cdot c_1(L^*_\chi) < 0$. Since χ is dominant, the line bundle L^*_χ is nef when restricted to a fibre of the projection $E/P \to X$, so that the curve Y cannot be contained in such a fibre. Then Y surjects onto X. One can choose a morphism of smooth projective curves $h \colon Y' \to X$ such that $\tilde{Y} = Y' \times_X Y$ is a disjoint union of smooth curves in $h^*(R_P(\mathfrak{E}))$, each mapping isomorphically onto X. Using Lemma 3.7, in this way we may assume that Y is the image of a section $\sigma \colon X \to R_P(\mathfrak{E})$. But then the claim follows from Proposition 3.8.

The converse is obvious in view of Proposition 3.8. □

Remark 3.11. Let G be the linear group $\mathrm{Gl}(n, \mathbb{C})$. If $\mathfrak{E} = (E, \phi)$ is a principal Higgs G-bundle, and V is the rank n vector bundle corresponding to E, then the identification $\mathrm{Ad}(E) \simeq \mathrm{End}(E)$ makes ϕ into a Higgs morphism $\tilde{\phi}$ for V. A simple calculation shows that the semistability of $\mathfrak{E}$ is equivalent to the semistability of the Higgs vector bundle $(V, \tilde{\phi})$.

If P is such that the quotient G/P is the $(n-1)$-dimensional projective space, the bundle E/P is isomorphic to the projectivization $\mathbb{P}V \to X$ of V (regarded as the space whose sections classify rank 1 locally-free quotients of V). More generally, let P_k be a parabolic subgroup such that G/P_k is the Grassmannian of rank k quotient spaces of $\mathbb{C}^n$. In this case E/P_k is the Grassmann bundle $\mathrm{Gr}_k(V)$ of rank k locally free quotients of V. Then Theorem 3.10 corresponds to the result given in [7], according to which (V,ϕ) is semistable if and only if certain numerical classes θ_k in a closed subscheme of $\mathrm{Gr}_k(V)$ are nef (see [7, 5, 6] for details). △

4. The higher-dimensional case

In this section we consider the case of a base variety X which is a complex projective manifold of any dimension. Let X be equipped with a polarization H, and let G be a reductive complex algebraic group.

Definition 4.1. *A principal Higgs G-bundle $\mathfrak{E} = (E,\phi)$ is stable (semistable) if and only if for any proper parabolic subgroup $P \subset G$, any open dense subset $U \subset X$ such that* $\operatorname{codim}(X - U) \geq 2$*, and any Higgs reduction $\sigma\colon U \to R_P(\mathfrak{E})_{|U}$ of G to P on U, one has* $\deg \sigma^*(T_{E/P,X}) \geq 0$.

Remark 4.2. The arguments in the proof of Lemma 3.8 go through also in the higher dimensional case, allowing one to show that a principal Higgs G-bundle $\mathfrak{E}$ is semistable (according to Definition 4.1) if and only if for any proper parabolic subgroup $P \subset G$, any nontrivial dominant character χ of P, any open dense subset $U \subset X$ such that $\operatorname{codim}(X - U) \geq 2$, and any Higgs reduction $\sigma\colon U \to R_P(\mathfrak{E})_{|U}$ of G to P on U, the line bundle $\sigma^*(L_\chi^*)$ has positive (nonnegative) degree. △

If $\mathfrak{E}$ is a principal Higgs G-bundle, we denote by $\mathrm{Ad}(\mathfrak{E})$ the Higgs vector bundle given by the adjoint bundle $\mathrm{Ad}(E)$ equipped with the induced Higgs morphism.

We also introduce the notion of extension of the structure group for a principal Higgs G-bundle $\mathfrak{E} = (E,\phi)$. Given a group homomorphism $\lambda\colon G \to G'$, we consider the extended principal bundle E' and equip it with the Higgs field $\phi' = \tilde{f}(\phi)$, where $\tilde{f}\colon \mathrm{Ad}(E) \to \mathrm{Ad}(E')$ is the morphism introduced at the beginning of Section 2. We then set $\mathfrak{E}' = (E',\phi')$.

It is known that the extension of the structure group of a semistable principal bundle is still semistable [13]. The same is true in the Higgs case.

Proposition 4.3. *Let $\lambda\colon G \to G'$ be a homomorphism of connected reductive algebraic groups which maps the connected component of the centre of G into the connected component of the centre of G'. If $\mathfrak{E}$ is a semistable principal Higgs G-bundle, and $\mathfrak{E}'$ is obtained by extending the structure group G to G' by λ, then $\mathfrak{E}'$ is semistable.*

Proof. For the sake of simplicity we assume that X is a curve, in the higher dimensional case the proof is the same but applied to the kind of open subsets of X previously introduced. Let P' be a parabolic subgroup of G', and let $\sigma'\colon X \to R_{P'}(\mathfrak{E}')$ be a Higgs reduction of G' to P'. The inverse image $P = \lambda^{-1}(P')$ is a parabolic subgroup of G. Let $\mathfrak{F}'$ be the reduction of the structure group G' of $\mathfrak{E}'$ to P', and let $\mathfrak{F}$ be the inverse image in $\mathfrak{E}$ of $\mathfrak{F}'$ via the natural morphism $f\colon \mathfrak{E} \to \mathfrak{E}'$. Then $\mathfrak{F}$ is a reduction of the structure group G of $\mathfrak{E}$ to P, and corresponds to a section $\sigma\colon X \to R_P(\mathfrak{E})$ such that $\sigma' = \bar{f} \circ \sigma$, where $\bar{f}\colon E/P \to E'/P'$ is the morphism induced by f and λ.

Let χ' be a dominant character of P'; the composition $\chi = \chi' \circ \lambda$ is a dominant character of P, and one has $\sigma^*(L_\chi) \simeq (\sigma')^*(L_{\chi'})$. The claim then follows from Proposition 3.8. □

The following result generalizes Proposition 2.10 of [1] to the Higgs case.

Lemma 4.4. $\mathfrak{E}$ *is semistable if and only if* $\mathrm{Ad}(\mathfrak{E})$ *is semistable (as a Higgs vector bundle).*

Proof. Let us at first note that if G is the general linear group $Gl(n, \mathbb{C})$, the result holds true quite trivially: $\mathfrak{E}$ is semistable if and only if the corresponding Higgs vector bundle $\mathfrak{V}$ is semistable, and one knows that $\mathrm{Ad}(\mathfrak{E}) \simeq \mathrm{End}(\mathfrak{V})$ is semistable if and only if $\mathfrak{V}$ is.

The proof of the general statement in one direction follows quite closely Proposition 2.10 in [1]. Assume that $\mathfrak{E}$ is not semistable. Then, according to Remark 4.2, there exist a proper parabolic subgroup $P \subset G$, an open dense subset $U \subset X$ such that $\operatorname{codim}(X - U) \geq 2$, and a Higgs reduction $\sigma\colon U \to R_P(\mathfrak{E})_{|U}$ of G to P on U such that

$$\deg \sigma^*(T_{E/P,X}) < 0\,. \tag{1}$$

On the other hand, the restriction of the vertical tangent bundle $T_{E/P,X}$ to $R_P(\mathfrak{E})$ has a natural structure of Higgs bundle, and the pullback $\sigma^*(T_{E/P,X})$ is a Higgs quotient of $\mathrm{Ad}(\mathfrak{E})$. Since G is reductive, $\mathrm{Ad}(\mathfrak{E})$ has degree zero, and then it is not semistable due to condition (1).

To prove to converse let $G' = Gl(\mathfrak{g})$. The principal Higgs G'-bundle $\mathfrak{E}'$ obtained by extending the structure group of $\mathfrak{E}$ to G' is the bundle of linear frames of $\mathrm{Ad}(E)$ with its natural Higgs morphism. If $\mathfrak{E}$ is semistable, by Proposition 4.3, with λ given by the adjoint representation, the principal Higgs bundle $\mathfrak{E}'$ is semistable as well, so that $\mathrm{Ad}(\mathfrak{E})$ is semistable. □

As promised in Remark 3.6 we establish the equivalence of our definition of semistable principal Higgs bundle with the one which is current in the literature.

Corollary 4.5. *A principal Higgs G-bundle $\mathfrak{E} = (E, \phi)$ is semistable if and only if there is a faithful linear representation $\rho\colon G \to \mathrm{Aut}(W)$ such that the associated Higgs vector bundle $\mathfrak{W} = \mathfrak{E} \times_\rho W$ is semistable.*

Proof. Assume that $\mathfrak{E}$ is semistable. The structure group G may be expressed as a semidirect product $G = Z_0\, G'$, where Z_0 is the connected component of the centre of G which contains the identity, and G' is semisimple with trivial centre. Since parabolic subgroups of G and G' are in a one-to-one correspondence (cf. [14, §2]), we may assume that the structure group of $\mathfrak{E}$ is G'. Then, by considering a monomorphism $\rho\colon G' \to Gl(n, \mathbb{C})$ for a suitable n, we may apply Proposition 4.3, thus proving our claim.

Conversely, assume that ρ exists such that $\mathfrak{W}$ is semistable. Now, $\mathrm{Ad}(\mathfrak{E})$ is a Higgs subbundle of $\mathrm{End}(\mathfrak{W})$. Since the latter is semistable, and $c_1(\mathrm{Ad}(\mathfrak{E})) = c_1(\mathrm{End}(\mathfrak{W})) = 0$, then $\mathrm{Ad}(\mathfrak{E})$ is semistable, so that $\mathfrak{E}$ is semistable as well. □

We may now prove a version of Miyaoka's semistability criterion which works for principal Higgs bundles on projective varieties of any dimension.

Theorem 4.6. *Let $\mathfrak{E}$ be a principal Higgs G-bundle $\mathfrak{E} = (E, \phi)$ on X. The following conditions are equivalent.*

- (i) *For every parabolic subgroup $P \subset G$ and any nontrivial dominant character χ of P, the line bundle L^*_χ restricted to $R_P(\mathfrak{E})$ is numerically effective;*
- (ii) *for every morphism $f\colon C \to X$, where C is a smooth projective curve, the pullback $f^*(\mathfrak{E})$ is semistable;*
- (iii) *$\mathfrak{E}$ is semistable and $c_2(\mathrm{Ad}(E)) = 0$ in $H^4(X, \mathbb{R})$.*

Proof. Assume that condition (i) holds, and let $f\colon C \to X$ be as in the statement. The line bundle L'_χ on $f^*(E)/P$ given by the character χ is a pullback of L_χ. Then $L'_{\chi | R_P(f^*\mathfrak{E})}$ is nef, so that by Theorem 3.10, $f^*(\mathfrak{E})$ is semistable. Thus (i) implies (ii).

We prove that (ii) implies (iii). Since $f^*(\mathfrak{E})$ is semistable, by Lemma 4.4, the adjoint Higgs bundle $\mathrm{Ad}(f^*(\mathfrak{E}))$ is semistable. By results proved in [5] we have that $\mathrm{Ad}(\mathfrak{E})$ is semistable, and $c_2(\mathrm{Ad}(E)) = 0$. Again using Lemma 4.4, we have that $\mathfrak{E}$ is semistable.

Next we prove that (iii) implies (ii). This is proved by reversing the previous arguments: $\mathrm{Ad}(\mathfrak{E})$ is semistable by Lemma 4.4; thus, since $c_2(\mathrm{Ad}(E)) = 0$, by results in [5] the Higgs vector bundle $\mathrm{Ad}(f^*(\mathfrak{E}))$ is semistable, and then $f^*(\mathfrak{E})$ is semistable by Lemma 4.4.

Finally, we show that (ii) implies (i). Let C' be a curve in $R_P(\mathfrak{E})$. Possibly by replacing it with its normalization, we may assume it is smooth. We may also assume that C' is not contained in a fibre of the projection $\pi_P \colon R_P(\mathfrak{E}) \to X$. Then $\pi_P \colon C' \to C$ is a finite cover. We may choose a smooth projective curve C'' and a morphism $h \colon C'' \to C$ such that $\tilde{C} = C'' \times_C C'$ is an unramified cover. Then every sheet C_j of $\tilde{C}$ is the image of a section σ_j of $R_P(h^*\mathfrak{E})$. Since $h^*\mathfrak{E}$ is semistable by Lemma 3.7, we have $\deg \sigma_j^*(L_\chi^*) \geq 0$ by Lemma 3.8. This implies (i). □

Remark 4.7. We have used the fact that since G is reductive, we have $\mathrm{Ad}(E) \simeq \mathrm{Ad}(E)^*$, so that

$$\Delta(\mathrm{Ad}(E)) = c_2(\mathrm{Ad}(E)) - \frac{r-1}{2r}(c_1(\mathrm{Ad}(E)))^2 = c_2(\mathrm{Ad}(E))$$

where $r = \dim(G)$. △

5. Another semistability criterion

In this section we prove another semistability criterion, which states that a principal Higgs bundle $\mathfrak{E} = (E, \phi)$ is semistable and $c_2(\mathrm{Ad}(E)) = 0$ if and only if the adjoint Higgs bundle $\mathrm{Ad}(\mathfrak{E})$ is numerically flat, in a sense that we introduced in [5]. This calls for a brief reminder of the notion of numerical effectiveness for Higgs bundles (a notion that we shall call "H-nefness"). Let us remark beforehand that our definition of H-nefness for Higgs bundles requires to consider Higgs bundles on singular schemes. For such spaces there is well-behaved theory of the de Rham complex [11], which is all one needs to define Higgs bundles.

Let X be a scheme over the complex numbers, and E a rank r vector bundle on X. For every positive integer s less than r, let $\mathrm{Gr}_s(E)$ denote the Grassmann bundle of rank s quotients of E, with projection $p_s : \mathrm{Gr}_s(E) \to X$. There is a universal exact sequence of vector bundles on $\mathrm{Gr}_s(E)$

$$0 \to S_{r-s,E} \xrightarrow{\psi} p_s^*(E) \xrightarrow{\eta} Q_{s,E} \to 0 \tag{2}$$

where $S_{r-s,E}$ is the universal rank $r-s$ subbundle and $Q_{s,E}$ is the universal rank s quotient bundle [10].

Given a Higgs bundle $\mathfrak{E}$, we may construct closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subset \mathrm{Gr}_s(E)$ parametrizing rank s locally-free Higgs quotients (these are the counterparts in the vector bundle case of the schemes of Higgs reductions we have introduced previously). We define $\mathfrak{Gr}_s(\mathfrak{E})$ as the closed subscheme of $\mathrm{Gr}_s(E)$ where the composed morphism

$$(\eta \otimes 1) \circ p_s^*(\phi) \circ \psi \colon S_{r-s,E} \to Q_{s,E} \otimes p_s^*(\Omega_X)$$

vanishes. (This scheme has already been introduced by Simpson in a particular case, e.g., semistable Higgs bundles with vanishing Chern classes, cf. [17, Cor. 9.3]. We denote by ρ_s the projection $\mathfrak{Gr}_s(\mathfrak{E}) \to X$. The restriction of (2) to the scheme $\mathfrak{Gr}_s(\mathfrak{E})$ provides the exact sequence of vector bundles

$$0 \to S_{r-s,\mathfrak{E}} \to \rho_s^*(\mathfrak{E}) \to Q_{s,\mathfrak{E}} \to 0\,. \tag{3}$$

The Higgs morphism ϕ of $\mathfrak{E}$ induces by pullback a Higgs morphism $\Phi\colon \rho_s^*(\mathfrak{E}) \to \rho_s^*(\mathfrak{E}) \otimes \Omega_{\mathfrak{Gr}_s(\mathfrak{E})}$. Due to the condition $(\eta \otimes 1) \circ p_s^*(\phi) \circ \psi = 0$ which is satisfied on $\mathfrak{Gr}_s(\mathfrak{E})$, the morphism Φ sends $S_{r-s,\mathfrak{E}}$ to $S_{r-s,\mathfrak{E}} \otimes \Omega_{\mathfrak{Gr}_s(\mathfrak{E})}$. As a result, $S_{r-s,\mathfrak{E}}$ is a Higgs subbundle of $\rho_s^*(\mathfrak{E})$, and the quotient $Q_{s,\mathfrak{E}}$ has a structure of Higgs bundle. Thus (3) is an exact sequence of Higgs bundles.

We recall from [5] the notion of H-nef Higgs bundle.

Definition 5.1. *A Higgs bundle $\mathfrak{E}$ of rank one is said to be Higgs-numerically effective (for short, H-nef) if it is numerically effective in the usual sense. If $\mathrm{rk}\,\mathfrak{E} \geq 2$ we require that:*

(i) *all bundles $Q_{s,\mathfrak{E}}$ are Higgs-nef;*
(ii) *the line bundle $\det(E)$ is nef.*

If both $\mathfrak{E}$ and $\mathfrak{E}^*$ are Higgs-numerically effective, $\mathfrak{E}$ is said to be Higgs-numerically flat (H-nflat).

We are now in position to state and prove the additional semistability criterion we promised.

Theorem 5.2. *Let $\mathfrak{E}$ be a principal Higgs bundle $\mathfrak{E} = (E, \phi)$ on a polarized smooth complex projective variety X. The following conditions are equivalent.*

(i) *$\mathfrak{E}$ is semistable and $c_2(\mathrm{Ad}(E)) = 0$ in $H^4(X, \mathbb{R})$;*
(ii) *the adjoint Higgs bundle $\mathrm{Ad}(\mathfrak{E})$ is H-nflat.*

Proof. At first we prove this theorem when X is a curve. In this case the claim is the following: $\mathfrak{E}$ is semistable if and only if $\mathrm{Ad}(\mathfrak{E})$ is H-nflat. In view of Lemma 4.4, this amounts to proving that $\mathrm{Ad}(\mathfrak{E})$ is semistable if and only if it is H-nflat. Since $c_1(\mathrm{Ad}(E)) = 0$ this holds true ([5], Corollaries 3.4 and 3.6).

Let us assume now that $\dim(X) > 1$. If condition (i) holds, then $\mathfrak{E}_{|C}$ is semistable for any embedded curve C (as usual, if C is not smooth one replaces it with its normalization). Thus $\mathrm{Ad}(\mathfrak{E})_{|C}$ is semistable, hence H-nflat. But this implies that $\mathrm{Ad}(\mathfrak{E})$ is H-nflat as well.

Conversely, if $\mathrm{Ad}(\mathfrak{E})$ is H-nflat, then it is semistable (see [5]), so that $\mathfrak{E}$ is semistable. Moreover, all Chern classes of H-nflat Higgs bundles vanish, so that $c_2(\mathrm{Ad}(E)) = 0$. □

Remark 5.3. This characterization shows that the numerically flat principal G-bundles defined in [4] for semisimple structure groups G are no more than the class of principal bundles singled out by one of the equivalent conditions of Theorem 4.6 (cf. [4, Thm. 2.5]). △

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