HOMOGENIZATION OF VARIATIONAL PROBLEMS UNDER MANIFOLD CONSTRAINTS

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Abstract. Homogenization of integral functionals is studied under the constraint that admissible maps have to take their values into a given smooth manifold. The notion of tangential homogenization is defined by analogy with the tangential quasiconvexity introduced by Dacorogna, Fonseca, Malý & Trivisa [18]. For energies with superlinear or linear growth, a Γ-convergence result is established in Sobolev spaces. In the case of energies with linear growth, the homogenization problem is also studied in the space of functions of bounded variation.

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1. Introduction

The homogenization theory aims to find an effective description of materials whose heterogeneities scale is much smaller than the size of the body. The simplest example is periodic homogenization for which the microstructure is assumed to be periodically distributed within the material. In the framework of the Calculus of Variations, periodic homogenization problems rest on the study of equilibrium states, or minimizers, of integral functionals of the form

$$\int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u\right) dx\,, \quad u : \Omega \to \mathbb{R}^d\,, \tag{1.1}$$

under suitable boundary conditions, where $\Omega \subset \mathbb{R}^N$ is a bounded open set and $f : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0, +\infty)$ is some oscillating integrand with respect to the first variable. To understand the asymptotic behavior of (almost) minimizers of such energies, it is convenient to perform a Γ-convergence analysis (see [19] for a detailed description of this subject) which is an adequate theory to study such variational problems. It is usual to assume that the integrand f satisfies uniform p-growth and p-coercivity conditions (with $1 \le p < +\infty$) so that one should ask the admissible fields to belong to the Sobolev space $W^{1,p}$. For energies with superlinear growth, *i.e.*, $p > 1$, this problem has a quite long history, and we refer to [34] in the convex case. Then it has received the most general answer in the independent works of [13] and [36], showing that such materials asymptotically behave like homogeneous ones. These results have been subsequently generalized into a lot of different manners. Let us mention [15] where the authors add a surface energy term allowing for fractured media. In that case, Sobolev spaces are not adapted to take into account eventual discontinuities of the deformation field across the cracks. For energies growing linearly, the pathological nature of $W^{1,1}$ leads to a relaxation in the space of functions with Bounded Variation. These kind of homogenization problems have been successively studied in [11], [20], and in [12] with an extra surface energy term.

In many applications admissible fields have to satisfy additional constraints. This is for example the case in the study of equilibria for liquid crystals, in ferromagnetism or for magnetostrictive materials where the order parameters take their values into a given manifold. It then becomes necessary to understand the behaviour of integral functionals of the type (1.1) under this additional constraint. For fixed $\varepsilon > 0$, the possible lack of lower semicontinuity of the energy may prevent the existence of minimizers (with eventual boundary conditions). It leads to compute its relaxation under the manifold constraint. In the framework of Sobolev spaces, it has been studied in [18,3], and the relaxed energy is obtained by replacing the integrand by its tangential quasiconvexification which is the analogue of the quasiconvex envelope in the non constrained case. The analysis for the linear growth case has been performed in [2] assuming that the manifold is the unit sphere of $\mathbb{R}^d$. The case of general manifolds has been recently treated in [35] where the author makes a further isotropy assumption on the integrand (see also the Appendix). We finally mention a slightly different problem originally introduced in [16,9], where the energy is assumed to be finite only for smooth maps. The most recent generalizations can be found in [28,30,31] where the study is performed within the framework of Cartesian Currents (see [29]). It shows the emergence in the relaxation process of non local effects of topological nature related to the non density of smooth maps (see [8,10]).

The aim of this paper is to treat the problem of manifold constrained homogenization, *i.e.*, the asymptotic as $\varepsilon \to 0$ of energies of the form (1.1) defined on manifold valued Sobolev spaces. Let us make the idea more precise. We consider a connected submanifold $\mathcal{M}$ of $\mathbb{R}^d$ of class $\mathcal{C}^1$ without boundary. The class of admissible maps we are interested in is defined as

$$W^{1,p}(\Omega;\mathcal{M}) := \left\{u \in W^{1,p}(\Omega;\mathbb{R}^d) : \ u(x) \in \mathcal{M} \text{ for } \mathcal{L}^N\text{-a.e. } x \in \Omega\right\}.$$

The function $f : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ is assumed to be a Carathéodory integrand satisfying

(H_1) for every $\xi \in \mathbb{R}^{d\times N}$ the function $f(\cdot,\xi)$ is 1-periodic, *i.e.* if $\{e_1,\dots,e_N\}$ denotes the canonical basis of $\mathbb{R}^N$, one has $f(y+e_i,\xi) = f(y,\xi)$ for every $i = 1,\dots,N$ and $y \in \mathbb{R}^N$;

(H_2) there exist $0 < \alpha \le \beta < +\infty$ and $1 \le p < +\infty$ such that

$$\alpha|\xi|^p \le f(y,\xi) \le \beta(1+|\xi|^p) \quad \text{for a.e. } y \in \mathbb{R}^N \text{ and all } \xi \in \mathbb{R}^{d\times N}.$$

For $\varepsilon > 0$, we define the functionals $\mathcal{F}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$ by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\varepsilon},\nabla u\right) dx & \text{if } u \in W^{1,p}(\Omega;\mathcal{M}), \\ +\infty & \text{otherwise}. \end{cases}$$

For energies with superlinear growth, we have the following result.

Theorem 1.1. *Let $\mathcal{M}$ be a connected $\mathcal{C}^1$-submanifold of $\mathbb{R}^d$ without boundary, and $f : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ be a Carathéodory function satisfying (H_1) and (H_2) with $1 < p < +\infty$. Then the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ Γ-converges for the strong L^p-topology to the functional $\mathcal{F}_{\rm hom} : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined by*

$$\mathcal{F}_{\rm hom}(u) := \begin{cases} \displaystyle\int_\Omega Tf_{\rm hom}(u,\nabla u)\, dx & \text{if } u \in W^{1,p}(\Omega;\mathcal{M}), \\ +\infty & \text{otherwise}, \end{cases}$$

where for every $s \in \mathcal{M}$ and $\xi \in [T_s(\mathcal{M})]^N$,

$$Tf_{\rm hom}(s,\xi) = \lim_{t\to+\infty} \inf_{\varphi} \left\{ ⨍_{(0,t)^N} f(y,\xi+\nabla\varphi(y))\, dy : \varphi \in W_0^{1,\infty}((0,t)^N;T_s(\mathcal{M})) \right\} \tag{1.2}$$

is the tangentially homogenized energy density.

If the integrand f has a linear growth in the ξ-variable, *i.e.*, if f satisfies (H_2) with $p = 1$, we assume in addition that $\mathcal{M}$ is compact, and that

(H_3) there exists $L > 0$ such that

$$|f(y,\xi) - f(y,\xi')| \le L|\xi - \xi'|, \quad \text{for a.e. } y \in \mathbb{R}^N \text{ and all } \xi,\ \xi' \in \mathbb{R}^{d\times N}.$$

Then the following representation result on $W^{1,1}(\Omega;\mathcal{M})$ holds:

Theorem 1.2. *Let $\mathcal{M}$ be a connected and compact $\mathcal{C}^1$-submanifold of $\mathbb{R}^d$ without boundary, and $f:\mathbb{R}^N\times\mathbb{R}^{d\times N}\to[0,+\infty)$ be a Carathéodory function satisfying (H_1) to (H_3) with $p=1$. Then the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ Γ-converges for the strong L^1-topology at every $u\in W^{1,1}(\Omega;\mathcal{M})$ to $\mathcal{F}_{\rm hom}:W^{1,1}(\Omega;\mathcal{M})\to[0,+\infty)$, where*

$$\mathcal{F}_{\rm hom}(u):=\int_\Omega Tf_{\rm hom}(u,\nabla u)\,dx\,,$$

and $Tf_{\rm hom}$ is given by (1.2).

We would like to emphasize that the use of hypothesis (H_3) is not too restrictive. Indeed, the Γ-limit remains unchanged upon first relaxing the functional $\mathcal{F}_\varepsilon$ (at fixed $\varepsilon>0$) in $W^{1,1}(\Omega;\mathbb{R}^d)$. It would lead to replace the integrand f by its tangential quasiconvexification which, by virtue of the growth condition (H_1), does satisfy such a Lipschitz continuity assumption (see [18]).

In the case of an integrand with linear growth, the domain of the Γ-limit is obviously larger than the Sobolev space $W^{1,1}(\Omega;\mathcal{M})$. In view of the study performed in [35], the domain is exactly given by $BV(\Omega;\mathcal{M})$ defined by

$$BV(\Omega;\mathcal{M}):=\big\{u\in BV(\Omega;\mathbb{R}^d):\ u(x)\in\mathcal{M}\text{ for }\mathcal{L}^N\text{-a.e. }x\in\Omega\big\}\,.$$

We have extended Theorem 1.2 to BV-maps. More precisely, under the additional (standard) assumption,

(H_4) there exist $C>0$ and $0<q<1$ such that

$$|f(y,\xi)-f^\infty(y,\xi)|\le C(1+|\xi|^{1-q})\,,\qquad\text{for a.e. }y\in\mathbb{R}^N\text{ and all }\xi\in\mathbb{R}^{d\times N}\,,$$

where $f^\infty:\mathbb{R}^N\times\mathbb{R}^{d\times N}\to[0,+\infty)$ is the recession function of f defined by

$$f^\infty(y,\xi):=\limsup_{t\to+\infty}\frac{f(y,t\xi)}{t}\,,$$

we have obtained the following result.

Theorem 1.3. *Let $\mathcal{M}$ be a smooth compact and connected submanifold of $\mathbb{R}^d$ without boundary, and let $f:\mathbb{R}^N\times\mathbb{R}^{d\times N}\to[0,+\infty)$ be a Carathéodory function satisfying (H_1) to (H_4) with $p=1$. Then the family $\{\mathcal{F}_\varepsilon\}$ Γ-converges for the strong L^1-topology to the functional $\mathcal{F}_{\rm hom}:L^1(\Omega;\mathbb{R}^d)\to[0,+\infty]$ defined by*

$$\mathcal{F}_{\rm hom}(u):=\begin{cases}\displaystyle\int_\Omega Tf_{\rm hom}(u,\nabla u)\,dx+\int_{\Omega\cap S_u}\vartheta_{\rm hom}(u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}\\ \displaystyle\qquad\qquad+\int_\Omega Tf^\infty_{\rm hom}\left(\tilde u,\frac{dD^cu}{d|D^cu|}\right)d|D^cu| & \text{if }u\in BV(\Omega;\mathcal{M})\,,\\ +\infty & \text{otherwise}\,,\end{cases}$$

where $Tf_{\rm hom}$ is given in (1.2), $Tf^\infty_{\rm hom}$ is the recession function of $Tf_{\rm hom}$ defined for every $s\in\mathcal{M}$ and every $\xi\in[T_s(\mathcal{M})]^N$ by

$$Tf^\infty_{\rm hom}(s,\xi):=\limsup_{t\to+\infty}\frac{Tf_{\rm hom}(s,t\xi)}{t}\,,$$

and for all $(a,b,\nu)\in\mathcal{M}\times\mathcal{M}\times\mathbb{S}^{N-1}$,

$$\vartheta_{\rm hom}(a,b,\nu):=\lim_{t\to+\infty}\inf_\varphi\bigg\{\frac{1}{t^{N-1}}\int_{t\,Q_\nu}f^\infty(y,\nabla\varphi(y))\,dy:\varphi\in W^{1,1}(tQ_\nu;\mathcal{M})\,,\\ \varphi=a\text{ on }\partial(tQ_\nu)\cap\{x\cdot\nu>0\}\text{ and }\varphi=b\text{ on }\partial(tQ_\nu)\cap\{x\cdot\nu\le0\}\bigg\}\,,\quad(1.3)$$

Q_ν standing for any open unit cube in $\mathbb{R}^N$ centered at the origin with two of its faces orthogonal to ν.

The paper is organized as follows. We first review in Section 2 standard facts about of manifold valued Sobolev mappings and functions of bounded variation that will be used all the way through. The study of the energy densities $Tf_{\rm hom}$ and $\vartheta_{\rm hom}$ and their main properties is the object of Section 3. The proofs of

Theorems 1.1 and 1.2 are presented in Section 4. The homogenization in the space of functions of bounded variation is performed in Section 5. Finally we state in the Appendix a relaxation result for general manifolds and integrands which extends [2] and [35].

2. Preliminaries

2.1. *Notations*

We start by introducing some notations. Let Ω be a generic bounded open subset of $\mathbb{R}^N$. We write $\mathcal{A}(\Omega)$ for the family of all open subsets of Ω, and $\mathcal{B}(\Omega)$ for the σ-algebra of all Borel subsets of Ω. We also consider a countable subfamily $\mathcal{R}(\Omega)$ of $\mathcal{A}(\Omega)$ made of all finite unions of cubes with rational edge length centered at rational points of $\mathbb{R}^N$. We write $B^k(s,r)$ for the closed ball in $\mathbb{R}^k$ of center $s \in \mathbb{R}^k$ and radius $r > 0$. The unit sphere in $\mathbb{R}^k$ is denoted by $\mathbb{S}^{k-1} := \{s \in \mathbb{R}^k : |s| = 1\}$. Given $\nu \in \mathbb{S}^{N-1}$, Q_ν stands for an open unit cube in $\mathbb{R}^N$ centered at the origin with two of its faces orthogonal to ν and $Q_\nu(x_0,\rho) := x_0 + \rho\, Q_\nu$. Similarly $Q := (-1/2, 1/2)^N$ is the unit cube in $\mathbb{R}^N$ and $Q(x_0,\rho) := x_0 + \rho\, Q$.

The space of vector valued Radon measures in Ω with finite total variation is denoted by $\mathcal{M}(\Omega;\mathbb{R}^m)$. If $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ and $E \in \mathcal{B}(\Omega)$, $\mu \llcorner E$ stands for the restriction of μ to E, *i.e.*, $\mu \llcorner E(B) = \mu(E \cap B)$ for any $B \in \mathcal{B}(\Omega)$. We denote by $\mathcal{L}^N$ the Lebesgue measure in $\mathbb{R}^N$, and by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure. If $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ and $\lambda \in \mathcal{M}(\Omega)$ is a nonnegative Radon measure, we denote by $\frac{d\mu}{d\lambda}$ the Radon-Nikodým derivative of μ with respect to λ. By a generalization of Besicovitch Differentiation Theorem (see [5, Proposition 2.2]), there exists $E \in \mathcal{B}(\Omega)$ such that $\lambda(E) = 0$ and

$$\frac{d\mu}{d\lambda}(x) = \lim_{\rho \to 0^+} \frac{\mu(Q_\nu(x,\rho))}{\lambda(Q_\nu(x,\rho))}$$

for all $x \in \operatorname{Supp}\mu \setminus E$ and all $\nu \in \mathbb{S}^{N-1}$.

2.2. *Functions of bounded variation*

We say that $u \in L^1_{\rm loc}(\Omega;\mathbb{R}^d)$ has an approximate limit at $x \in \Omega$ if there exists $z \in \mathbb{R}^d$ such that

$$\lim_{\rho \to 0^+} \fint_{Q(x,\rho)} |u(y) - z|\, dy = 0\,. \tag{2.1}$$

The subset S_u of Ω is defined as the set of points where this property fails. It is well known that $S_u \in \mathcal{B}(\Omega)$, and from Lebesgue Differentiation Theorem that $\mathcal{L}^N(S_u) = 0$. The approximate limit z of u at $x \in \Omega \setminus S_u$ is denoted by $\tilde{u}(x)$, and the Borel map $x \mapsto \tilde{u}(x)\chi_{\Omega \setminus S_u}(x)$ is called the precise representative of u. The jump set J_u of u is defined as the set of points $x \in S_u$ for which the following property holds: there exist $a, b \in \mathbb{R}^d$ with $a \neq b$, and $\nu \in \mathbb{S}^{N-1}$ such that

$$\lim_{\rho \to 0^+} \fint_{Q^+_\nu(x,\rho)} |u(y) - a|\, dy = 0\,, \quad \lim_{\rho \to 0^+} \fint_{Q^-_\nu(x,\rho)} |u(y) - b|\, dy = 0\,, \tag{2.2}$$

where $Q^\pm_\nu(x,\rho) := \{y \in Q_\nu(x,\rho) : \pm(y - x)\cdot\nu > 0\}$. The triplet (a, b, ν) is uniquely determined by (2.2) up to a permutation of (a, b) and a change of sign of ν, and it is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

A function u is said to have bounded variation, and we write $u \in BV(\Omega;\mathbb{R}^d)$, if $u \in L^1(\Omega;\mathbb{R}^d)$ and if its distributional derivative $Du \in \mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ is a (matrix valued) Radon measure with finite total variation. For general properties of BV functions, we refer to [6]. We just recall here basic facts that will be useful in the sequel. The set S_u is countably $\mathcal{H}^{N-1}$-rectifiable and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$. By the Lebesgue Decomposition Theorem, the measure Du can be split into two mutually singular measures

$$Du = D^a u + D^s u\,,$$

where $D^a u$ and $D^s u$ are respectively the absolutely continuous part and the singular part of Du with respect to the Lebesgue measure $\mathcal{L}^N$. The Radon-Nikodým derivative of $D^a u$ with respect to $\mathcal{L}^N$ is denoted by ∇u, and it satisfies

$$\lim_{\rho \to 0^+} \fint_{Q(x,\rho)} \frac{|u(y) - \tilde{u}(x) - \nabla u(x)(y - x)|}{\rho}\, dy = 0 \tag{2.3}$$

for $\mathcal{L}^N$-a.e. $x \in \Omega \setminus S_u$. A point which enjoys property (2.3) is said to be a point of approximate differentiability. The measure $D^s u$ can in turn be decomposed into the sum of two mutually singular measures $D^s u = D^j u + D^c u$ where $D^j u$ is the jump part and $D^c u$ is the Cantor part. The jump part $D^j u$ is given by

$$D^j u := Du \llcorner S_u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \llcorner S_u \,,$$

and the Cantor part is defined as $D^c u := D^s u \llcorner (\Omega \setminus S_u)$. We recall Alberti Rank One Theorem (see [1]) which states that for $|D^c u|$-a.e. $x \in \Omega$,

$$A(x) := \frac{dD^c u}{d|D^c u|}(x)$$

is a rank one matrix.

2.3. *Manifold valued spaces*

In this paper, we are interested in Sobolev and BV maps taking their values into a given manifold. We consider a connected $\mathcal{C}^1$-submanifold $\mathcal{M}$ of $\mathbb{R}^d$ without boundary. The tangent space of $\mathcal{M}$ at $s \in \mathcal{M}$ is denoted by $T_s(\mathcal{M})$, $\mathrm{co}(\mathcal{M})$ stands for the convex hull of $\mathcal{M}$, and $\Pi_1(\mathcal{M})$ is the fundamental group of $\mathcal{M}$. For any $p \in [1, +\infty)$, we define

$$W^{1,p}(\Omega; \mathcal{M}) := \big\{ u \in W^{1,p}(\Omega; \mathbb{R}^d) : u(x) \in \mathcal{M} \text{ for } \mathcal{L}^N\text{-a.e. } x \in \Omega \big\} \,.$$

As for Sobolev spaces, we define the space of functions of bounded variation with values into the manifold $\mathcal{M}$ by

$$BV(\Omega; \mathcal{M}) := \big\{ u \in BV(\Omega; \mathbb{R}^d) : u(x) \in \mathcal{M} \text{ for } \mathcal{L}^N\text{-a.e. } x \in \Omega \big\} \,.$$

It can be shown that if $u \in W^{1,p}(\Omega; \mathcal{M})$, then $\nabla u(x) \in [T_{u(x)}(\mathcal{M})]^N$ for $\mathcal{L}^N$-a.e. $x \in \Omega$. The analogue statement for BV-maps is given in Lemma 2.1 below. For simplicity, we present a proof in the case $\mathcal{M}$ compact.

Lemma 2.1. *Assume that $\mathcal{M}$ is compact. For every $u \in BV(\Omega; \mathcal{M})$,*

$$\tilde{u}(x) \in \mathcal{M} \text{ for every } x \in \Omega \setminus S_u \,; \tag{2.4}$$

$$u^{\pm}(x) \in \mathcal{M} \text{ for every } x \in J_u \,; \tag{2.5}$$

$$\nabla u(x) \in [T_{u(x)}(\mathcal{M})]^N \text{ for } \mathcal{L}^N\text{-a.e. } x \in \Omega \,; \tag{2.6}$$

$$A(x) := \frac{dD^c u}{d|D^c u|}(x) \in [T_{\tilde{u}(x)}(\mathcal{M})]^N \text{ for } |D^c u|\text{-a.e. } x \in \Omega \,. \tag{2.7}$$

Proof. We first show (2.4). By definition of the space $BV(\Omega; \mathcal{M})$, $u(y) \in \mathcal{M}$ for a.e. $y \in \Omega$. Therefore for any $x \in \Omega \setminus S_u$, we have $|u(y) - \tilde{u}(x)| \geq \mathrm{dist}(\tilde{u}(x), \mathcal{M})$ for a.e. $y \in \Omega$. In view of (2.1), this yields $\mathrm{dist}(\tilde{u}(x), \mathcal{M}) = 0$, *i.e.*, $\tilde{u}(x) \in \mathcal{M}$. Arguing as for the approximate limit points, one obtains (2.5).

Now it remains to prove (2.6) and (2.7). We introduce the function $\Phi : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\Phi(s) = \chi(\delta^{-1} \mathrm{dist}(s, \mathcal{M})^2) \, \mathrm{dist}(s, \mathcal{M})^2 \,,$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}; [0,1])$ with $\chi(t) = 1$ for $|t| \leq 1$, $\chi(t) = 0$ for $|t| \geq 2$, and $\delta > 0$ is small enough so that $\Phi \in \mathcal{C}^1(\mathbb{R}^d)$. Note that for every $s \in \mathcal{M}$, $\Phi(s) = 0$ and

$$\mathrm{Ker}\, \nabla \Phi(s) = T_s(\mathcal{M}) \,. \tag{2.8}$$

By the Chain Rule formula in BV (see, *e.g.*, [6, Theorem 3.96]), $\Phi \circ u \in BV(\Omega)$ and

$$\begin{aligned} D(\Phi \circ u) &= \nabla \Phi(u) \nabla u \, \mathcal{L}^N \llcorner \Omega + \nabla \Phi(\tilde{u}) D^c u + \big(\Phi(u^+) - \Phi(u^-)\big) \otimes \nu_u \, \mathcal{H}^{N-1} \llcorner J_u \\ &= \nabla \Phi(u) \nabla u \, \mathcal{L}^N \llcorner \Omega + \nabla \Phi(\tilde{u}) A |D^c u| \,, \end{aligned}$$

thanks to (2.5). On the other hand, $\Phi \circ u = 0$ a.e. in Ω since $u(x) \in \mathcal{M}$ for a.e. $x \in \Omega$. Therefore $D(\Phi \circ u) \equiv 0$. Since $\mathcal{L}^N \llcorner \Omega$ and $|D^c u|$ are mutually singular measures, we infer that $\nabla \Phi(u(x)) \nabla u(x) = 0$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and $\nabla \Phi(\tilde{u}(x)) A(x) = 0$ for $|D^c u|$-a.e. $x \in \Omega$. Hence (2.6) and (2.7) follow from (2.8) together with (2.4). $\square$

In [8,10], density results of smooth functions between manifolds into Sobolev spaces have been established. In the following theorem, we summarize these results only for $p = 1$ (which will be the only case needed). Let $\mathcal{S}$ be the family of all finite unions of subsets contained in a $(N-2)$-dimensional submanifold of $\mathbb{R}^N$.

Theorem 2.1. *Let $\mathcal{D}(\Omega;\mathcal{M}) \subset W^{1,1}(\Omega;\mathcal{M})$ be defined by*

$$\mathcal{D}(\Omega;\mathcal{M}) := \begin{cases} W^{1,1}(\Omega;\mathcal{M}) \cap \mathcal{C}^\infty(\Omega;\mathcal{M}) & \text{if } \Pi_1(\mathcal{M}) = 0\,, \\ \big\{u \in W^{1,1}(\Omega;\mathcal{M}) \cap \mathcal{C}^\infty(\Omega\setminus\Sigma;\mathcal{M}) \text{ for some } \Sigma \in \mathcal{S}\big\} & \text{otherwise}. \end{cases}$$

Then $\mathcal{D}(\Omega;\mathcal{M})$ is dense in $W^{1,1}(\Omega;\mathcal{M})$ for the strong $W^{1,1}(\Omega;\mathbb{R}^d)$-topology.

We now present a useful projection technique (taken from [21] for $\mathcal{M} = \mathbb{S}^{d-1}$). It was first introduced in [32,33], and makes use of an averaging device going back to [24]. We sketch the proof for the convenience of the reader.

Proposition 2.1. *Let $\mathcal{M}$ be a compact connected m-dimensional smooth submanifold of $\mathbb{R}^d$ without boundary, and let $v \in W^{1,1}(\Omega;\mathbb{R}^d) \cap \mathcal{C}^\infty(\Omega\setminus\Sigma;\mathbb{R}^d)$ for some $\Sigma \in \mathcal{S}$ such that $v(x) \in \mathrm{co}(\mathcal{M})$ for a.e. $x \in \Omega$. Then there exists $w \in W^{1,1}(\Omega;\mathcal{M})$ satisfying $w = v$ a.e. in $\big\{x \in \Omega\setminus\Sigma : v(x) \in \mathcal{M}\big\}$ and*

$$\int_\Omega |\nabla w|\,dx \le C_\star \int_\Omega |\nabla v|\,dx\,, \tag{2.9}$$

for some constant $C_\star > 0$ which only depends on d and $\mathcal{M}$.

Proof. According to [33, Lemma 6.1] (which holds for $p = 1$), there exist a compact Lipschitz polyhedral set $X \subset \mathbb{R}^d$ of codimension greater or equal to 2, and a locally Lipschitz map $\pi : \mathbb{R}^d \setminus X \to \mathcal{M}$ such that

$$\int_{B^d(0,R)} |\nabla\pi(s)|\,ds < +\infty \qquad \text{for every } R < +\infty\,. \tag{2.10}$$

Moreover, in a neighborhood of $\mathcal{M}$ the mapping π is smooth of constant rank equal to m.

We argue as in the proof of [33, Theorem 6.2]. Let B be an open ball in $\mathbb{R}^d$ containing $\mathcal{M}\cup X$, and let $\delta > 0$ small enough so that the nearest point projection on $\mathcal{M}$ is a well defined smooth mapping in the δ-neighborhood of $\mathcal{M}$. Fix $\sigma < \inf\{\delta, \mathrm{dist}(\mathrm{co}(\mathcal{M}), \partial B)\}$ small enough, and for $a \in B^d(0,\sigma)$ we define the translates

$$B_a := a + B \quad \text{and} \quad X_a := a + X\,,$$

and the projection $\pi_a : B_a \setminus X_a \to \mathcal{M}$ by $\pi_a(s) := \pi(s-a)$. Since π has full rank and is smooth in a neighborhood of $\mathcal{M}$, by the Inverse Function Theorem the number

$$\Lambda := \sup_{a\in B^d(0,\sigma)} \mathrm{Lip}\big(\pi_{a|\mathcal{M}}\big)^{-1} \tag{2.11}$$

is finite and only depends on $\mathcal{M}$. Using Sard's lemma, one can show that $\pi_a \circ v \in W^{1,1}(\Omega;\mathcal{M})$ for $\mathcal{L}^d$-a.e. $a \in B^d(0,\sigma)$. Then Fubini's theorem together with the Chain Rule formula yields

$$\begin{aligned} \int_{B^d(0,\sigma)} \int_\Omega |\nabla(\pi_a\circ v)(x)|\,d\mathcal{L}^N(x)\,d\mathcal{L}^d(a) &\le \int_\Omega |\nabla v(x)| \left(\int_{B^d(0,\sigma)} |\nabla\pi_a(v(x))|\,d\mathcal{L}^d(a)\right) d\mathcal{L}^N(x) \\ &= \int_\Omega |\nabla v(x)| \left(\int_{B^d(0,\sigma)} |\nabla\pi(v(x)-a)|\,d\mathcal{L}^d(a)\right) d\mathcal{L}^N(x) \\ &\le \left(\int_B |\nabla\pi(s)|\,d\mathcal{L}^d(s)\right)\left(\int_\Omega |\nabla v(x)|\,d\mathcal{L}^N(x)\right). \end{aligned}$$

Therefore we can find $a \in B^d(0,\sigma)$ such that

$$\int_\Omega |\nabla(\pi_a\circ v)|\,dx \le C\mathcal{L}^d\left(B^d(0,\sigma)\right)^{-1} \int_\Omega |\nabla v|\,dx\,, \tag{2.12}$$

where we used (2.10). To conclude, it suffices to set $w := \big(\pi_{a|\mathcal{M}}\big)^{-1} \circ \pi_a \circ v$, and (2.9) arises as a consequence of (2.11) and (2.12). □

3. Properties of homogenized energy densities

In this section we present the main properties of the energy densities $Tf_{\rm hom}$ and $\vartheta_{\rm hom}$ defined in (1.2) and (1.3). In particular we will prove that each formula is well defined in the sense that both limits exist.

3.1. *The tangentially homogenized bulk energy*

We start by considering the bulk energy density

$$Tf_{\rm hom}(s,\xi) := \liminf_{t\to+\infty} \inf_{\varphi} \left\{ \fint_{(0,t)^N} f(y, \xi + \nabla\varphi(y))\,dy : \varphi \in W_0^{1,\infty}((0,t)^N; T_s(\mathcal{M})) \right\}$$

defined for $s \in \mathcal{M}$ and $\xi \in [T_s(\mathcal{M})]^N$. Our first concern is to show that the lim inf above is actually a limit. To this purpose we shall introduce a new energy density $\bar f$ for which we can apply classical homogenization theories.

For $s \in \mathcal{M}$ we denote by $P_s : \mathbb{R}^d \to T_s(\mathcal{M})$ the orthogonal projection from $\mathbb{R}^d$ into $T_s(\mathcal{M})$, and we set

$$\mathbf{P}_s(\xi) := (P_s(\xi_1), \dots, P_s(\xi_N)) \quad \text{for } \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^{d\times N}\,.$$

Given the Carathéodory integrand $f : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ satisfying assumptions (H_1) and (H_2) with $1 \le p < +\infty$, we define $\bar f : \mathbb{R}^N \times \mathcal{M} \times \mathbb{R}^{d\times N} \to [0,+\infty)$ by

$$\bar f(y,s,\xi) := f(y, \mathbf{P}_s(\xi)) + |\xi - \mathbf{P}_s(\xi)|^p\,. \tag{3.1}$$

The new integrand $\bar f$ is a Carathéodory function, and $\bar f(\cdot, s, \xi)$ is 1-periodic for every $(s,\xi) \in \mathcal{M} \times \mathbb{R}^{d\times N}$. By assumption (H_2), $\bar f$ also satisfies uniform p-growth and p-coercivity conditions, *i.e.*,

$$\alpha'|\xi|^p \le \bar f(y,s,\xi) \le \beta'(1+|\xi|^p) \quad \text{for every } (s,\xi) \in \mathcal{M}\times\mathbb{R}^{d\times N} \text{ and a.e. } y \in \mathbb{R}^N\,, \tag{3.2}$$

for some constants $0 < \alpha' \le \beta' < +\infty$.

Proposition 3.1. *Let $f : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ be a Carathéodory integrand satisfying (H_1) and (H_2) with $1 \le p < +\infty$. Then the following properties hold:*

(i) for every $s \in \mathcal{M}$ and $\xi \in [T_s(\mathcal{M})]^N$,

$$Tf_{\rm hom}(s,\xi) = \lim_{t\to+\infty} \inf_{\varphi} \left\{ \fint_{(0,t)^N} f(y, \xi + \nabla\varphi(y))\,dy : \varphi \in W_0^{1,\infty}((0,t)^N; T_s(\mathcal{M})) \right\}\,,$$

and

$$Tf_{\rm hom}(s,\xi) = \bar f_{\rm hom}(s,\xi)\,, \tag{3.3}$$

where

$$\bar f_{\rm hom}(s,\xi) := \lim_{t\to+\infty} \inf_{\varphi} \left\{ \fint_{(0,t)^N} \bar f(y,s,\xi + \nabla\varphi(y))\,dy : \varphi \in W_0^{1,\infty}((0,t)^N; \mathbb{R}^d) \right\}$$

is the usual homogenized energy density of $\bar f$ (see, e.g., [14, Chapter 14]);

(ii) the function $Tf_{\rm hom}$ is tangentially quasiconvex, i.e., for all $s \in \mathcal{M}$ and all $\xi \in [T_s(\mathcal{M})]^N$,

$$Tf_{\rm hom}(s,\xi) \le \int_Q Tf_{\rm hom}(s, \xi + \nabla\varphi(y))\,dy$$

for every $\varphi \in W_0^{1,\infty}(Q; T_s(\mathcal{M}))$. In particular $Tf_{\rm hom}(s,\cdot)$ is rank one convex;

(iii) there exists $C > 0$ such that

$$\alpha|\xi|^p \le Tf_{\rm hom}(s,\xi) \le \beta(1+|\xi|^p)\,, \tag{3.4}$$

and

$$|Tf_{\rm hom}(s,\xi) - Tf_{\rm hom}(s,\xi')| \le C(1+|\xi|^{p-1}+|\xi'|^{p-1})|\xi-\xi'| \tag{3.5}$$

for every $s \in \mathcal{M}$ and $\xi,\, \xi' \in [T_s(\mathcal{M})]^N$.

Proof. Fix $s \in \mathcal{M}$ and $\xi \in [T_s(\mathcal{M})]^N$. For any $t > 0$, we introduce

$$Tf_t(s,\xi) := \inf_{\varphi} \left\{ -\!\!\!\!\!\!\int_{(0,t)^N} f(y, \xi + \nabla\varphi)\, dy : \varphi \in W_0^{1,\infty}((0,t)^N; T_s(\mathcal{M})) \right\},$$

and

$$\bar{f}_t(s,\xi) := \inf_{\varphi} \left\{ -\!\!\!\!\!\!\int_{(0,t)^N} \bar{f}(y, s, \xi + \nabla\varphi)\, dy : \varphi \in W_0^{1,\infty}((0,t)^N; \mathbb{R}^d) \right\}.$$

By classical results (see, *e.g.*, [14, Proposition 14.4]), there exists $\lim_{t\to+\infty} \bar{f}_t(s,\xi)$ for every $s \in \mathcal{M}$ and $\xi \in [T_s(\mathcal{M})]^N$. Hence to prove *(i)*, it suffices to show that $Tf_t(s,\xi) = \bar{f}_t(s,\xi)$ for every $t > 0$. For any $\varphi \in W_0^{1,\infty}((0,t)^N; T_s(\mathcal{M}))$, we have

$$\bar{f}_t(s,\xi) \leq -\!\!\!\!\!\!\int_{(0,t)^N} \bar{f}(y, s, \xi + \nabla\varphi)\, dy = -\!\!\!\!\!\!\int_{(0,t)^N} f(y, \xi + \nabla\varphi)\, dy\,,$$

since $\xi + \nabla\varphi(y) \in [T_s(\mathcal{M})]^N$ for a.e. $y \in (0,t)^N$. Taking the infimum over all such φ's in the right hand side of the previous inequality yields $\bar{f}_t(s,\xi) \leq Tf_t(s,\xi)$. To prove the converse inequality we pick up $\psi \in W_0^{1,\infty}((0,t)^N; \mathbb{R}^d)$ and we set $\tilde{\psi} = P_s(\psi)$. One easily checks that $\tilde{\psi} \in W_0^{1,\infty}((0,t)^N; T_s(\mathcal{M}))$ and $\nabla\tilde{\psi} = \mathbf{P}_s(\nabla\psi)$ a.e. in $(0,t)^N$. Therefore

$$Tf_t(s,\xi) \leq -\!\!\!\!\!\!\int_{(0,t)^N} f(y, \xi + \nabla\tilde{\psi})\, dy = -\!\!\!\!\!\!\int_{(0,t)^N} f\big(y, \mathbf{P}_s(\xi + \nabla\psi)\big)\, dy \leq -\!\!\!\!\!\!\int_{(0,t)^N} \bar{f}(y, s, \xi + \nabla\psi)\, dy\,.$$

Then the converse inequality arises taking the infimum over all admissible ψ's.

By standard results $\bar{f}_{\rm hom}(s,\cdot)$ is a quasiconvex function for every $s \in \mathcal{M}$ (see, *e.g.*, [14, Theorem 14.5]). As a consequence, for any $s \in \mathcal{M}$, $\xi \in [T_s(\mathcal{M})]^N$ and $\varphi \in W_0^{1,\infty}(Q; T_s(\mathcal{M}))$, we have

$$Tf_{\rm hom}(s,\xi) = \bar{f}_{\rm hom}(s,\xi) \leq \int_Q \bar{f}_{\rm hom}(s, \xi + \nabla\varphi)\, dy = \int_Q Tf_{\rm hom}(s, \xi + \nabla\varphi)\, dy\,,$$

which proves that $Tf_{\rm hom}$ is tangentially quasiconvex. As a consequence of (3.3) and the fact that $\bar{f}_{\rm hom}(s,\cdot)$ is rank one convex, it follows that $Tf_{\rm hom}(s,\cdot)$ is rank one convex as well.

The proof of (3.4) is immediate in view of (H_1) and the definition of $Tf_{\rm hom}$. Moreover rank one convex functions satisfying uniform p-growth and p-coercivity conditions are p-Lipschitz (see, *e.g.*, [17, Lemma 2.2, Chap. 4]), and thus (3.5) holds. □

Remark 3.1. It readily follows from the previous proof that Proposition 3.1 still holds for any Caratéodory integrand $\hat{f} : \mathbb{R}^N \times \mathcal{M} \times \mathbb{R}^{d\times N} \to [0,+\infty)$ instead of $\bar{f}$, provided that: $\hat{f}(x,s,\xi) = f(y,\xi)$ for every $s \in \mathcal{M}$, every $\xi \in [T_s(\mathcal{M})]^N$ and a.e. $y \in \mathbb{R}^N$; $\hat{f}(\cdot, s, \cdot)$ satisfies (H_1) and (H_2) for every $s \in \mathcal{M}$ with uniform estimates with respect to s.

Remark 3.2. If $\dim(\mathcal{M}) = 1$ then $T_s(\mathcal{M})$ is a one dimensional linear subspace of $\mathbb{R}^d$ for every $s \in \mathcal{M}$. Hence, given $s \in \mathcal{M}$, we can identify $T_s(\mathcal{M})$ with $\mathbb{R}$ through some linear mapping $i_s : \mathbb{R} \to T_s(\mathcal{M})$. Using the application i_s, we can also identify $[T_s(\mathcal{M})]^N$ with $\mathbb{R}^N$ setting for $z = (z_1, \ldots, z_N) \in \mathbb{R}^N$, $i_s(z) := (i_s(z_1), \ldots, i_s(z_N))$. Define $\hat{f}(y,s,z) := f(y, i_s(z))$ for $(y,s,z) \in \Omega\times\mathcal{M}\times\mathbb{R}^N$. By (3.3) and [14, Remark 14.6], we can replace in formula (1.2) homogeneous boundary conditions by periodic boundary conditions, and the limit as $t \to +\infty$ by the infimum over all $t \in \mathbb{N}$. Moreover, in the scalar case the homogenization formula can be reduced to a single cell formula (see, *e.g.*, [14, Chapter 14]). Therefore

$$\begin{aligned}
Tf_{\rm hom}(s,\xi) &= \inf_{t\in\mathbb{N}} \inf \left\{ -\!\!\!\!\!\!\int_{(0,t)^N} f(y, \xi + \nabla\varphi)\, dy : \varphi \in W_\#^{1,\infty}((0,t)^N; T_s(\mathcal{M})) \right\} \\
&= \inf_{t\in\mathbb{N}} \inf \left\{ -\!\!\!\!\!\!\int_{(0,t)^N} \hat{f}(y, s, i_s^{-1}(\xi) + \nabla\phi)\, dy : \phi \in W_\#^{1,\infty}((0,t)^N) \right\} \\
&= \inf \left\{ \int_Q \hat{f}(y, s, i_s^{-1}(\xi) + \nabla\phi)\, dy : \phi \in W_\#^{1,\infty}(Q) \right\} \\
&= \inf \left\{ \int_Q f(y, \xi + \nabla\varphi)\, dy : \varphi \in W_\#^{1,\infty}(Q; T_s(\mathcal{M})) \right\}.
\end{aligned}$$

This remark states that whenever the manifold $\mathcal{M}$ is one dimensional, test functions in the minimization problem (1.2) are in fact scalar valued, and thus, one can compute the tangentially homogenized energy density over one single cell instead of an infinite set of cells. Note that this is not true in general even in the non constrained case (see, *e.g.*, the counter-example in [36, Theorem 4.3]).

We now present an elementary example with an explicit dependence on the s-variable showing that *tangential homogenization* does not reduce in general to standard homogenization. It based on a rank one laminate for which direct computations can be performed.

Example 3.1. Assume that $\mathcal{M} = \mathbb{S}^1$ and for $x \in \mathbb{R}^N$, $\xi = (\xi_{ij}) \in \mathbb{R}^{2\times N}$,

$$f(x,\xi) = \sum_{j=1}^{N} \big(a(x_1)|\xi_{1j}|^2 + b(x_1)|\xi_{2j}|^2\big)\,,$$

where $a, b \in L^\infty(\mathbb{R};\mathbb{R})$ are 1-periodic and bounded from below by a positive constant. Arguing as in Remark 3.2 and [19, Example 25.6], one may compute for $s = (s_1, s_2) \in \mathbb{S}^1$ and $\xi \in [T_s(\mathbb{S}^1)]^N$,

$$Tf_{\rm hom}(s,\xi) = \sum_{j=1}^{N} \alpha_j(s)\big(|\xi_{1j}|^2 + |\xi_{2j}|^2\big)\,,$$

with

$$\alpha_j(s) = \begin{cases} \displaystyle\left(\int_{-1/2}^{1/2} \frac{dt}{a(t)s_2^2 + b(t)s_1^2}\right)^{-1} & \text{if } j = 1\,, \\ \displaystyle\int_{-1/2}^{1/2} \big(a(t)s_2^2 + b(t)s_1^2\big)\,dt & \text{otherwise}\,. \end{cases}$$

Compare this result with [19, Example 25.6].

To treat the homogenization problem with $p = 1$, we will need to extend the function $\bar f$ to the whole space $\mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$. First we recall that the recession function h^∞ of a generic scalar function h defined on $\mathbb{R}^{d\times N}$ is given by

$$h^\infty(\xi) := \limsup_{t\to+\infty} \frac{h(t\xi)}{t}\,.$$

We may now state our extension procedure.

Lemma 3.1. *Assume that $\mathcal{M}$ is compact. Let $f : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ be a Carathéodory function satisfying (H_1) to (H_3) with $p = 1$. Then there exists a Carathéodory function $g : \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0,+\infty)$ such that*

$$g(y,s,\xi) = f(y,\xi) \quad \textit{and} \quad g^\infty(y,s,\xi) = f^\infty(y,\xi) \quad \textit{for } s \in \mathcal{M} \textit{ and } \xi \in [T_s(\mathcal{M})]^N\,, \tag{3.6}$$

and satisfying :

(i) g is 1-periodic in the first variable;

(ii) there exist $0 < \alpha' \le \beta'$ such that

$$\alpha'|\xi| \le g(y,s,\xi) \le \beta'(1+|\xi|) \quad \textit{for every } (s,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N} \textit{ and a.e. } y \in \mathbb{R}^N\,; \tag{3.7}$$

(iii) there exist $C > 0$ and $C' > 0$ such that

$$|g(y,s,\xi) - g(y,s',\xi)| \le C|s - s'|\,|\xi|\,, \tag{3.8}$$

and

$$|g(y,s,\xi) - g(y,s,\xi')| \le C'|\xi - \xi'| \tag{3.9}$$

for every s, $s' \in \mathbb{R}^d$, every $\xi \in \mathbb{R}^{d\times N}$ and a.e. $y \in \mathbb{R}^N$;

(iv) if in addition (H_4) holds, there exists $0 < q < 1$ and $C'' > 0$ such that

$$|g(y,s,\xi) - g^\infty(y,s,\xi)| \le C''(1 + |\xi|^{1-q}) \quad \textit{for every } (s,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N} \textit{ and a.e. } y \in \mathbb{R}^N\,. \tag{3.10}$$

Proof. For $\delta_0 > 0$ fixed, let $\mathcal{U} := \{s \in \mathbb{R}^d \,:\, \mathrm{dist}(s,\mathcal{M}) < \delta_0\}$ be the δ_0-neighborhood of $\mathcal{M}$. Choosing $\delta_0 > 0$ small enough, we may assume that the nearest point projection $\Pi : \mathcal{U} \to \mathcal{M}$ is a well defined Lipschitz mapping. Then the map $s \in \mathcal{U} \mapsto P_{\Pi(s)}$ is Lipschitz. Now we introduce a cut-off function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d;[0,1])$ such that $\chi(t) = 1$ if $\mathrm{dist}(s,\mathcal{M}) \le \delta_0/2$, and $\chi(s) = 0$ if $\mathrm{dist}(s,\mathcal{M}) \ge 3\delta_0/4$. We define

$$\mathbb{P}_s(\xi) := \chi(s)\mathbf{P}_{\Pi(s)}(\xi) \quad \text{for } (s,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}\,.$$

We consider the integrand $g : \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0,+\infty)$ given by

$$g(y,s,\xi) = f(y,\mathbb{P}_s(\xi)) + |\xi - \mathbb{P}_s(\xi)|\,.$$

One may check that g is a Carathéodory function, that $g(\cdot,s,\xi)$ is 1-periodic for every $(s,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}$, and that (H_2) yields (3.7). Then (3.8) and (3.9) follow from (H_3) and the Lipschitz continuity of $s \mapsto \mathbb{P}_s$. Next observe that

$$g^\infty(y,s,\xi) = f^\infty(y,s,\mathbb{P}_s(\xi)) + |\xi - \mathbb{P}_s(\xi)|\,.$$

Hence (3.6) is immediate while (3.10) is a consequence of (H_4). □

Remark 3.3. In view of (3.6), one may argue exactly as in the proof of (3.3) to show that

$$Tf_{\mathrm{hom}}(s,\xi) = g_{\mathrm{hom}}(s,\xi) \quad \text{for every } s \in \mathcal{M} \text{ and } \xi \in [T_s(\mathcal{M})]^N\,, \tag{3.11}$$

where

$$g_{\mathrm{hom}}(s,\xi) := \lim_{t\to+\infty} \inf_{\varphi} \left\{ \fint_{(0,t)^N} g(y,s,\xi+\nabla\varphi(y))\,dy \,:\, \varphi \in W_0^{1,\infty}((0,t)^N;\mathbb{R}^d) \right\}.$$

Hence upon extending Tf_{hom} by g_{hom} outside the set $\{(s,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N} \,:\, s \in \mathcal{M},\ \xi \in [T_s(\mathcal{M})]^N\}$, we can tacitly assume Tf_{hom} to be defined over the whole $\mathbb{R}^d \times \mathbb{R}^{d\times N}$.

Remark 3.4. Observe that, if f satisfies assumption (H_3), then f^∞ satisfies (H_3) as well. In particular the function f^∞ is Carathéodory, 1-periodic in the first variable, and positively 1-homogeneous with respect to the second variable. In view of the growth and coercivity condition (H_2) with $p = 1$, one gets that

$$\alpha|\xi| \le f^\infty(y,\xi) \le \beta|\xi| \quad \text{for all } \xi \in \mathbb{R}^{d\times N} \text{ and a.e. } y \in \mathbb{R}^N\,. \tag{3.12}$$

Then, as for f^∞, the function g^∞ is Carathéodory, 1-periodic in the first variable, and positively 1-homogeneous with respect to the second variable. Moreover,

$$\alpha'|\xi| \le g^\infty(y,s,\xi) \le \beta'|\xi| \quad \text{for every } (s,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N} \text{ and a.e. } y \in \mathbb{R}^N\,,$$

and g^∞ satisfies estimates analogue to (3.8) and (3.9). Hence we may apply Proposition 3.1 to f^∞ and classical homogenization results to g^∞. In view of (3.6), we also have

$$T(f^\infty)_{\mathrm{hom}}(s,\xi) = (g^\infty)_{\mathrm{hom}}(s,\xi) \quad \text{for every } s \in \mathcal{M} \text{ and } \xi \in [T_s(\mathcal{M})]^N\,,$$

again as in the proof of (3.3).

With the convention that g_{hom} extends Tf_{hom} to $\mathbb{R}^d \times \mathbb{R}^{d\times N}$, we have the following result.

Proposition 3.2. *Assume that $\mathcal{M}$ is compact. Let $f : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ be a Carathéodory function satisfying (H_1) to (H_3) with $p = 1$. Then the following properties hold:*

(i) there exists $C_1 > 0$ such that

$$|Tf_{\mathrm{hom}}(s,\xi) - Tf_{\mathrm{hom}}(s',\xi)| \le C_1|s-s'|(1+|\xi|)\,, \tag{3.13}$$

for every s, $s' \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{d\times N}$. In particular Tf_{hom} is continuous;

(ii) if in addition (H_4) holds, there exist $C_2 > 0$ and $0 < q < 1$ such that

$$|Tf^\infty_{\mathrm{hom}}(s,\xi) - Tf_{\mathrm{hom}}(s,\xi)| \le C_2(1+|\xi|^{1-q})\,, \tag{3.14}$$

for every $(s,\xi) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}$.

Proof. Fix $s, s' \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{d\times N}$. For any $\eta > 0$, we may find $k \in \mathbb{N}$ and $\varphi \in W_0^{1,\infty}((0,k)^N;\mathbb{R}^d)$ such that

$$\fint_{(0,k)^N} g(y,s,\xi+\nabla\varphi)\,dy \leq g_{\rm hom}(s,\xi) + \eta\,.$$

We infer from (3.7) that $\alpha'|\xi| \leq g_{\rm hom}(s,\xi) \leq \beta'(1+|\xi|)$ and consequently

$$\fint_{(0,k)^N} |\nabla\varphi|\,dy \leq C(1+|\xi|)\,,$$

for some constant $C > 0$ depending only on α' and β'. Then from (3.11) and (3.8) it follows that

$$\begin{aligned} Tf_{\rm hom}(s',\xi) - Tf_{\rm hom}(s,\xi) = g_{\rm hom}(s',\xi) - g_{\rm hom}(s,\xi) \leq \fint_{(0,k)^N} \big(g(y,s',\xi+\nabla\varphi) - g(y,s,\xi+\nabla\varphi)\big)\,dy + \eta \leq \\ \leq C|s-s'| \fint_{(0,k)^N} |\xi+\nabla\varphi|\,dy + \eta \leq C|s-s'|(1+|\xi|) + \eta\,. \end{aligned}$$

We deduce relation (3.13) inverting the roles of s and s', and sending η to zero. In particular, we obtain that $Tf_{\rm hom}$ is continuous as a consequence of (3.13) and (3.5).

To show (3.14), let us consider sequences $t_n \nearrow +\infty$, $k_n \in \mathbb{N}$ and $\varphi_n \in W_0^{1,\infty}((0,k_n)^N;T_s(\mathcal{M}))$ such that

$$Tf^\infty_{\rm hom}(s,\xi) = \lim_{n\to+\infty} \frac{Tf_{\rm hom}(s,t_n\xi)}{t_n}\,,$$

and

$$\fint_{(0,k_n)^N} f(y,t_n\xi + t_n\nabla\varphi_n)\,dy \leq Tf_{\rm hom}(s,t_n\xi) + \frac{1}{n}\,.$$

Then assumption (H_2) and (3.4) yield

$$\fint_{(0,k_n)^N} |\nabla\varphi_n|\,dy \leq C(1+|\xi|)\,, \tag{3.15}$$

for some constant $C > 0$ depending only on α and β. Using (H_4), we derive that

$$\begin{aligned} Tf_{\rm hom}(s,\xi) - Tf^\infty_{\rm hom}(s,\xi) &= \lim_{n\to+\infty}\left(Tf_{\rm hom}(s,\xi) - \frac{Tf_{\rm hom}(s,t_n\xi)}{t_n}\right) \\ &\leq \liminf_{n\to+\infty} \fint_{(0,k_n)^N} \left| f(y,\xi+\nabla\varphi_n) - \frac{f(y,t_n\xi+t_n\nabla\varphi_n)}{t_n}\right| dy \\ &\leq \liminf_{n\to+\infty} \left\{ \fint_{(0,k_n)^N} \left| f(y,\xi+\nabla\varphi_n) - f^\infty(y,\xi+\nabla\varphi_n)\right| dy + \right. \\ &\qquad \left. + \fint_{(0,k_n)^N} \left| f^\infty(y,\xi+\nabla\varphi_n) - \frac{f(y,t_n\xi+t_n\nabla\varphi_n)}{t_n}\right| dy \right\} \\ &\leq \liminf_{n\to+\infty} \left\{ C \fint_{(0,k_n)^N} (1+|\xi+\nabla\varphi_n|^{1-q})\,dy \right. \\ &\qquad \left. + \frac{C}{t_n} \fint_{(0,k_n)^N} (1+t_n^{1-q}|\xi+\nabla\varphi_n|^{1-q})\,dy \right\}, \end{aligned}$$

where we have used the fact that $f^\infty(y,\cdot)$ is positively homogeneous of degree one in the last inequality. Then (3.15) and Hölder's inequality lead to

$$Tf_{\rm hom}(s,\xi) - Tf^\infty_{\rm hom}(s,\xi) \leq C(1+|\xi|^{1-q})\,. \tag{3.16}$$

Conversely, given $k \in \mathbb{N}$ and $\varphi \in W_0^{1,\infty}((0,k)^N;T_s(\mathcal{M}))$, we deduce from (H_2) that

$$\frac{f(\cdot,t(\xi+\nabla\varphi(\cdot)))}{t} \leq \beta(1+|\xi+\nabla\varphi|) \in L^1((0,k)^N)$$

whenever $t > 1$. Then Fatou's lemma implies

$$Tf^\infty_{\rm hom}(s,\xi) := \limsup_{t\to+\infty} \frac{Tf_{\rm hom}(s,t\xi)}{t} \le \limsup_{t\to+\infty} \fint_{(0,k)^N} \frac{f(y,t\xi+t\nabla\varphi)}{t}\,dy \le \fint_{(0,k)^N} f^\infty(y,\xi+\nabla\varphi)\,dy\,.$$

Taking the infimum over all admissible φ's and letting $k \to +\infty$, we infer

$$Tf^\infty_{\rm hom}(s,\xi) \le T(f^\infty)_{\rm hom}(s,\xi)\,. \tag{3.17}$$

For $\eta > 0$ arbitrary small, consider $k \in \mathbb{N}$ and $\varphi \in W^{1,\infty}_0((0,k)^N;T_s(\mathcal{M}))$ such that

$$\fint_{(0,k)^N} f(y,\xi+\nabla\varphi)\,dy \le Tf_{\rm hom}(s,\xi)+\eta\,.$$

In view of (H_2) and (3.4), it turns out that

$$\fint_{(0,k)^N} |\nabla\varphi|\,dy \le C(1+|\xi|)\,, \tag{3.18}$$

where $C > 0$ only depends on α and β. Then it follows from (3.17) that

$$\begin{aligned} Tf^\infty_{\rm hom}(s,\xi) - Tf_{\rm hom}(s,\xi) &\le T(f^\infty)_{\rm hom}(s,\xi) - Tf_{\rm hom}(s,\xi) \le \\ &\le \fint_{(0,k)^N} |f^\infty(y,\xi+\nabla\varphi)-f(y,\xi+\nabla\varphi)|\,dy + \eta \le C\fint_{(0,k)^N} (1+|\xi+\nabla\varphi|^{1-q})\,dy+\eta\,, \end{aligned}$$

where we have used (H_4) in the last inequality. Using Hölder's inequality, relation (3.18) together with the arbitrariness of η yields

$$Tf^\infty_{\rm hom}(s,\xi) - Tf_{\rm hom}(s,\xi) \le C(1+|\xi|^{1-q})\,. \tag{3.19}$$

Gathering (3.16) and (3.19) we conclude the proof of (3.14). □

3.2. *The homogenized surface energy*

We now present the homogenized surface energy required to treat the homogenization problem with linear growth, *i.e.*, assumption (H_2) with $p = 1$, in the space of functions of bounded variations (see Section 5). Recall that in this case, the manifold $\mathcal{M}$ is assumed to be connected and compact.

Given $\nu = (\nu_1,\dots,\nu_N)$ an orthonormal basis of $\mathbb{R}^N$ and $(a,b) \in \mathcal{M}\times\mathcal{M}$, we use the notations

$$Q_\nu := \Big\{\alpha_1\nu_1+\ldots+\alpha_N\nu_N \;:\; \alpha_1,\dots,\alpha_N \in (-1/2,1/2)\Big\}\,,$$

and for $x \in \mathbb{R}^N$, we set $\|x\|_{\nu,\infty} := \sup_{i\in\{1,\dots,N\}} |x\cdot\nu_i|$, $x_\nu := x\cdot\nu_1$ and $x' := (x\cdot\nu_2)\nu_2+\ldots+(x\cdot\nu_N)\nu_N$ so that x can be identified to the pair (x',x_ν). Let $u_{a,b,\nu} : Q_\nu \to \mathcal{M}$ be the function defined by

$$u_{a,b,\nu}(x) := \begin{cases} a & \text{if } x_\nu > 0\,,\\ b & \text{if } x_\nu \le 0\,. \end{cases}$$

We introduce the class of functions

$$\mathcal{A}_t(a,b,\nu) := \Big\{\varphi \in W^{1,1}(tQ_\nu;\mathcal{M}) : \varphi = u_{a,b,\nu} \text{ on } \partial(tQ_\nu)\Big\}\,.$$

We have the following result.

Proposition 3.3. *For every $(a,b,\nu_1) \in \mathcal{M}\times\mathcal{M}\times\mathbb{S}^{N-1}$, there exists*

$$\vartheta_{\rm hom}(a,b,\nu_1) := \lim_{t\to+\infty} \inf_\varphi \left\{\frac{1}{t^{N-1}}\int_{tQ_\nu} f^\infty(y,\nabla\varphi(y))\,dy : \varphi \in \mathcal{A}_t(a,b,\nu)\right\}\,,$$

where $\nu = (\nu_1,\dots,\nu_N)$ is any orthonormal basis of $\mathbb{R}^N$ with first element equal to ν_1 (the limit being independent of such a choice).

The proof of Proposition 3.3 is quite indirect and is based on an analogous result for a similar surface energy density $\tilde\vartheta_{\rm hom}$ (see (3.20) below). We will prove in Proposition 3.4 that the two densities coincide.

Given a and $b \in \mathcal{M}$, we introduce the family of geodesic curves between a and b by

$$\mathcal{G}(a,b) := \left\{ \gamma \in \mathcal{C}^\infty(\mathbb{R};\mathcal{M}) :\ \gamma(t) = a \text{ if } t \geq 1/2,\ \gamma(t) = b \text{ if } t \leq -1/2 \text{ and } \int_{\mathbb{R}} |\dot\gamma|\,dt = \mathbf{d}_{\mathcal{M}}(a,b) \right\},$$

where $\mathbf{d}_{\mathcal{M}}$ denotes the geodesic distance on $\mathcal{M}$. We define for $\varepsilon > 0$ and $\nu = (\nu_1, \dots, \nu_N)$ an orthonormal basis of $\mathbb{R}^N$,

$$\mathcal{B}_\varepsilon(a,b,\nu) := \left\{ u \in W^{1,1}(Q_\nu;\mathcal{M})\ :\ u(x) = \gamma(x_\nu/\varepsilon) \text{ on } \partial Q_\nu \text{ for some } \gamma \in \mathcal{G}(a,b) \right\}.$$

Proposition 3.4. *For every $(a,b) \in \mathcal{M}\times\mathcal{M}$ and every orthonormal basis $\nu = (\nu_1,\dots,\nu_N)$ of $\mathbb{R}^N$, there exists the limit*

$$\tilde\vartheta_{\rm hom}(a,b,\nu) := \lim_{\varepsilon\to 0} \inf_u \left\{ \int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon}, \nabla u\right) dx : u \in \mathcal{B}_\varepsilon(a,b,\nu) \right\}. \tag{3.20}$$

Moreover $\tilde\vartheta_{\rm hom}(a,b,\nu)$ only depends on a, b and ν_1.

Proof. The proof follows the scheme of the one in [15, Proposition 2.2]. We fix a and $b \in \mathcal{M}$. For every $\varepsilon > 0$ and every orthonormal basis $\nu = (\nu_1,\dots,\nu_N)$ of $\mathbb{R}^N$, we set

$$I_\varepsilon(\nu) = I_\varepsilon(a,b,\nu) := \inf \left\{ \int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon}, \nabla u\right) dx : u \in \mathcal{B}_\varepsilon(a,b,\nu) \right\}.$$

We divide the proof into several steps.

Step 1. Let ν and ν' be two orthonormal basis of $\mathbb{R}^N$ with equal first vector, *i.e.*, $\nu_1 = \nu'_1$. Suppose that ν is a rational basis, *i.e.*, for all $i \in \{1,\dots,N\}$ there exists $\gamma_i \in \mathbb{R}\setminus\{0\}$ such that $v_i := \gamma_i\nu_i \in \mathbb{Z}^N$. We claim that

$$\limsup_{\varepsilon\to 0} I_\varepsilon(\nu') \leq \liminf_{\varepsilon\to 0} I_\varepsilon(\nu). \tag{3.21}$$

Define

$$P := \left\{ \alpha_2 v_2 + \ldots + \alpha_N v_N\ :\ \alpha_2,\dots,\alpha_N \in [-1/2, 1/2) \right\},$$

and observe that f^∞ is P-periodic in the first variable, *i.e.*, $f^\infty(y + l_2 v_2 + \ldots + l_N v_N, \xi) = f^\infty(y,\xi)$ for every $(y,\xi) \in \mathbb{R}^N \times \mathbb{R}^{d\times N}$ and every $l_2,\dots,l_N \in \mathbb{Z}$.

Let $0 < \eta < \varepsilon$ and let $u_\varepsilon \in \mathcal{B}_\varepsilon(a,b,\nu)$ be such that $u_\varepsilon(x) = \gamma_\varepsilon(x_\nu/\varepsilon)$ on ∂Q_ν for some $\gamma_\varepsilon \in \mathcal{G}(a,b)$ and

$$\int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) dx \leq I_\varepsilon(\nu) + \varepsilon.$$

For every $\lambda = (\lambda_2,\dots,\lambda_N) \in \mathbb{Z}^{N-1}$ set

$$x^{(\lambda)} := \eta(\lambda_2 v_2 + \ldots + \lambda_N v_N), \quad Q_\nu^{(\lambda)} := x^{(\lambda)} + \frac{\eta}{\varepsilon} Q_\nu.$$

We now choose the centers $x^{(\lambda)}$ properly. Let

$$\Lambda = \Lambda(\varepsilon,\eta) := \Bigg\{ \lambda \in \mathbb{Z}^{N-1}\ :\ Q_\nu^{(\lambda)} \subset Q_{\nu'} \text{ and } x^{(\lambda)} \in \sum_{i=2}^N l_i\left(\frac{\eta}{\varepsilon} + \eta\gamma_i\right)\nu_i + \eta P \text{ for some } (l_2,\dots,l_N) \in \mathbb{Z}^{N-1} \Bigg\}. \tag{3.22}$$

We can check that the elements of $\{Q_\nu^{(\lambda)}\}_{\lambda\in\Lambda}$ are pairwise disjoint. Moreover, setting S to be the hyperplane $\{x_\nu = 0\}$, we have

$$\lim_{\eta\to 0} \mathcal{H}^{N-1}\Big(S \cap (Q_{\nu'} \setminus \bigcup_{\lambda\in\Lambda} Q_\nu^{(\lambda)}) \Big) = 0 \tag{3.23}$$

or equivalently, $\lim_{\eta\to 0}(\eta/\varepsilon)^{N-1}\#\Lambda = 1$. We define $u_\eta : Q_{\nu'} \to \mathcal{M}$ by

$$u_\eta(x) := \begin{cases} u_\varepsilon\left(\dfrac{\varepsilon(x-x^{(\lambda)})}{\eta}\right) & \text{if } x \in Q_\nu^{(\lambda)} \text{ for some } \lambda\in\Lambda\,,\\[2mm] \gamma_\varepsilon\left(\dfrac{x_\nu}{\eta}\right) & \text{otherwise}\,. \end{cases}$$

Note that $u_\eta \in W^{1,1}(Q_{\nu'};\mathcal{M})$ since $x^{(\lambda)}\cdot\nu_1 = 0$ for every $\lambda\in\mathbb{Z}^{N-1}$. In addition, $u_\eta(x) = \gamma_\varepsilon(x_{\nu'}/\eta)$ on $\partial Q_{\nu'}$ since $\nu_1 = \nu'_1$. Hence $u_\eta \in \mathcal{B}_\eta(a,b,\nu')$, and thus

$$I_\eta(\nu') \le \int_{Q_{\nu'}} f^\infty\left(\frac{x}{\eta},\nabla u_\eta\right)dx = \sum_{\lambda\in\Lambda}\int_{Q_\nu^{(\lambda)}} f^\infty\left(\frac{x}{\eta},\nabla u_\eta\right)dx + \int_{Q_{\nu'}\setminus\bigcup_{\lambda\in\Lambda}Q_\nu^{(\lambda)}} f^\infty\left(\frac{x}{\eta},\nabla u_\eta\right)dx =: I_1 + I_2\,.$$

We estimate both integrals. Using the change of variables $x = x^{(\lambda)} + (\eta/\varepsilon)y$, the homogeneity and the P-periodicity of f^∞, we derive

$$\begin{aligned} I_1 &= \left(\frac{\eta}{\varepsilon}\right)^{N-1}\sum_{\lambda\in\Lambda}\int_{Q_\nu} f^\infty\left(\frac{y}{\varepsilon}+\lambda_2 v_2+\ldots+\lambda_N v_N,\nabla u_\varepsilon(y)\right)dy = \\ &= \left(\frac{\eta}{\varepsilon}\right)^{N-1}\#\Lambda\int_{Q_\nu} f^\infty\left(\frac{y}{\varepsilon},\nabla u_\varepsilon(y)\right)dy \le \left(\frac{\eta}{\varepsilon}\right)^{N-1}\#\Lambda\left(I_\varepsilon(\nu)+\varepsilon\right). \end{aligned} \tag{3.24}$$

From the growth condition (3.12), we infer that

$$\begin{aligned} I_2 &= \int_{Q_{\nu'}\setminus\bigcup_{\lambda\in\Lambda}Q_\nu^{(\lambda)}} f^\infty\left(\frac{x}{\eta},\frac{1}{\eta}\dot\gamma_\varepsilon\left(\frac{x_\nu}{\eta}\right)\otimes\nu_1\right)dx \le \frac{\beta}{\eta}\int_{Q_{\nu'}\setminus\bigcup_{\lambda\in\Lambda}Q_\nu^{(\lambda)}}\left|\dot\gamma_\varepsilon\left(\frac{x_\nu}{\eta}\right)\right|dx \\ &= \beta\mathcal{H}^{N-1}\Big(S\cap\big(Q_{\nu'}\setminus\bigcup_{\lambda\in\Lambda}Q_\nu^{(\lambda)}\big)\Big)\frac{1}{\eta}\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\left|\dot\gamma_\varepsilon\left(\frac{t}{\eta}\right)\right|dt \\ &= \beta\mathcal{H}^{N-1}\Big(S\cap\big(Q_{\nu'}\setminus\bigcup_{\lambda\in\Lambda}Q_\nu^{(\lambda)}\big)\Big)\int_{-\frac12}^{\frac12}|\dot\gamma_\varepsilon(s)|\,ds\,. \end{aligned} \tag{3.25}$$

Estimates (3.24) and (3.25) together with (3.23) yield

$$\limsup_{\eta\to 0} I_\eta(\nu') \le I_\varepsilon(\nu)+\varepsilon\,.$$

Then (3.21) follows taking the lim inf as $\varepsilon\to 0$.

Step 2. Let ν and ν' be two orthonormal rational basis of $\mathbb{R}^N$ with equal first vector. Then the limits $\lim_{\varepsilon\to 0} I_\varepsilon(\nu)$ and $\lim_{\varepsilon\to 0} I_\varepsilon(\nu')$ exist and are equal. Indeed, applying Step 1 with $\nu=\nu'$ yields the existence of the limits. Then inverting the roles of ν and ν' we deduce that they are equal.

Step 3. We claim that for every $\sigma>0$ there exists $\delta>0$ (independent of a and b) such that if ν and ν' are two orthonormal basis of $\mathbb{R}^N$ with $|\nu_i-\nu'_i|<\delta$ for every $i=1,\ldots,N$, then

$$\liminf_{\varepsilon\to 0} I_\varepsilon(\nu) - K\sigma \le \liminf_{\varepsilon\to 0} I_\varepsilon(\nu') \le \limsup_{\varepsilon\to 0} I_\varepsilon(\nu') \le \limsup_{\varepsilon\to 0} I_\varepsilon(\nu) + K\sigma$$

where K is a positive constant which only depends on $\mathcal{M}$, β and N.

We use the notation $Q_{\nu,\eta} := (1-\eta)Q_\nu$ where $0<\eta<1$. Let $\sigma>0$ be fixed and let $0<\eta<1$ be such that

$$\eta < \frac{1}{34} \quad\text{and}\quad \max\left\{1-(1-\eta)^{N-1}\,,\ \frac{(1-\eta)^{N-1}(1-2\eta)^{N-1}}{(1-3\eta)^{N-1}} - (1-2\eta)^{N-1}\right\} < \sigma. \tag{3.26}$$

Consider $\delta_0>0$ (that may be chosen so that $\delta_0\le\eta/(2\sqrt{N})$) such that for every $0<\delta\le\delta_0$ and every pair ν and ν' of orthonormal basis of $\mathbb{R}^N$ satisfying $|\nu_i-\nu'_i|\le\delta$ for $i=1,\ldots,N$, one has

$$Q_{\nu,3\eta}\subset Q_{\nu',2\eta}\subset Q_{\nu,\eta}\,, \tag{3.27}$$

and

$$\{x\cdot\nu'_1=0\}\cap\partial Q_{\nu,\eta}\subset\{|x\cdot\nu_1|\le 1/8\}\,.$$

Given $\varepsilon > 0$ small, we consider $u_\varepsilon \in \mathcal{B}_\varepsilon(a,b,\nu')$ such that

$$\int_{Q_{\nu'}} f^\infty\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) dx \leq I_\varepsilon(\nu') + \sigma\,,$$

where $u_\varepsilon(x) = \gamma_\varepsilon(x_{\nu'}/\varepsilon)$ for $x \in \partial Q_{\nu'}$. Now we construct $v_\varepsilon \in \mathcal{B}_{(1-2\eta)\varepsilon}(a,b,\nu)$ satisfying the boundary condition $v_\varepsilon(x) = \gamma_\varepsilon\big(x_\nu/(1-2\eta)\varepsilon\big)$ for $x \in \partial Q_\nu$. Consider $F_\eta : \mathbb{R}^N \to \mathbb{R}$,

$$F_\eta(x) := \left(\frac{1-2\|x'\|_{\nu,\infty}}{\eta}\right)\frac{x_{\nu'}}{1-2\eta} + \left(\frac{\eta - 1 + 2\|x'\|_{\nu,\infty}}{\eta}\right)\frac{x_\nu}{1-2\eta}\,,$$

and define

$$v_\varepsilon(x) := \begin{cases} u_\varepsilon\left(\dfrac{x}{1-2\eta}\right) & \text{if } x \in Q_{\nu',2\eta},\\[2mm] \gamma_\varepsilon\left(\dfrac{x_{\nu'}}{(1-2\eta)\varepsilon}\right) & \text{if } x \in Q_{\nu,\eta}\setminus Q_{\nu',2\eta}\,,\\[2mm] a & \text{if } x \in Q_\nu \setminus Q_{\nu,\eta} \text{ and } x_\nu \geq \dfrac{1}{4}\,,\\[2mm] \gamma_\varepsilon\left(\dfrac{F_\eta(x)}{\varepsilon}\right) & \text{if } x \in A_\eta := \big\{x : |x_\nu| \leq 1/4\big\} \cap (Q_\nu \setminus Q_{\nu,\eta})\,,\\[2mm] b & \text{if } x \in Q_\nu \setminus Q_{\nu,\eta} \text{ and } x_\nu \leq -\dfrac{1}{4}\,. \end{cases}$$

We can check that v_ε is well defined for ε small enough and that $v_\varepsilon \in \mathcal{B}_{(1-2\eta)\varepsilon}(a,b,\nu)$. Therefore

$$\begin{aligned} I_{(1-2\eta)\varepsilon}(\nu) &\leq \int_{Q_\nu} f^\infty\left(\frac{x}{(1-2\eta)\varepsilon}, \nabla v_\varepsilon\right) dx \\ &= \int_{Q_{\nu',2\eta}} f^\infty\left(\frac{x}{(1-2\eta)\varepsilon}, \nabla v_\varepsilon\right) dx + \int_{Q_{\nu,\eta}\setminus Q_{\nu',2\eta}} f^\infty\left(\frac{x}{(1-2\eta)\varepsilon}, \nabla v_\varepsilon\right) dx \\ &\quad + \int_{A_\eta} f^\infty\left(\frac{x}{(1-2\eta)\varepsilon}, \nabla v_\varepsilon\right) dx \\ &=: I_1 + I_2 + I_3\,. \end{aligned} \tag{3.28}$$

We now estimate these three integrals. First, we easily get that

$$I_1 = (1-2\eta)^{N-1}\int_{Q_{\nu'}} f^\infty\left(\frac{y}{\varepsilon}, \nabla u_\varepsilon\right) dy \leq I_\varepsilon(\nu') + \sigma\,. \tag{3.29}$$

In view of (3.27) we have $Q_{\nu,\eta} \subset (1-\eta)(1-2\eta)(1-3\eta)^{-1}Q_{\nu'} =: D_\eta$. Then we infer from the growth condition (3.12) together with Fubini's theorem that

$$\begin{aligned} I_2 &\leq \beta\int_{D_\eta\setminus Q_{\nu',2\eta}} |\nabla v_\varepsilon|\,dx = \frac{\beta}{(1-2\eta)\varepsilon}\int_{(D_\eta\setminus Q_{\nu',2\eta})\cap\{|x_{\nu'}|\leq(1-2\eta)\varepsilon/2\}} \left|\dot\gamma_\varepsilon\left(\frac{x_{\nu'}}{(1-2\eta)\varepsilon}\right)\right| dx \\ &= \beta\mathcal{H}^{N-1}\big((D_\eta\setminus Q_{\nu',2\eta})\cap\{x_{\nu'}=0\}\big)\frac{1}{(1-2\eta)\varepsilon}\int_{-(1-2\eta)\varepsilon/2}^{(1-2\eta)\varepsilon/2}\left|\dot\gamma_\varepsilon\left(\frac{t}{(1-2\eta)\varepsilon}\right)\right| dt \\ &= \beta\mathbf{d}_{\mathcal{M}}(a,b)\left(\frac{(1-\eta)^{N-1}(1-2\eta)^{N-1}}{(1-3\eta)^{N-1}} - (1-2\eta)^{N-1}\right). \end{aligned} \tag{3.30}$$

Now it remains to estimate I_3. To this purpose we first observe that (3.27) yields

$$\|\nabla F_\eta\|_{L^\infty(A_\eta;\mathbb{R}^N)} \leq C\,, \tag{3.31}$$

for some absolute constant $C > 0$, and

$$|\nabla F_\eta(x)\cdot\nu_1| \geq 1 \quad \text{for a.e. } x \in A_\eta\,. \tag{3.32}$$

Hence, thanks the growth condition (3.12), (3.31) and (3.32), we get that

$$\begin{aligned} I_3 \leq \beta\int_{A_\eta}|\nabla v_\varepsilon|\,dx &\leq \frac{C\beta}{\varepsilon}\int_{A_\eta}\left|\dot\gamma_\varepsilon\left(\frac{F_\eta(x)}{\varepsilon}\right)\right| dx \leq \frac{C\beta}{\varepsilon}\int_{A_\eta}\left|\dot\gamma_\varepsilon\left(\frac{F_\eta(x)}{\varepsilon}\right)\right||\nabla F_\eta(x)\cdot\nu_1|\,dx = \\ &= C\beta\int_{A'_\eta}\left(\frac{1}{\varepsilon}\int_{-1/4}^{1/4}\left|\dot\gamma_\varepsilon\left(\frac{F_\eta(t\nu_1+x')}{\varepsilon}\right)\right||\nabla F_\eta(t\nu_1+x')\cdot\nu_1|\,dt\right) d\mathcal{H}^{N-1}(x')\,, \end{aligned}$$

where we have set $A'_\eta := A_\eta \cap \{x_\nu = 0\}$, and used Fubini's theorem in the last equality. Changing variables $s = (1/\varepsilon)F_\eta(t\nu_1 + x')$, we obtain that for $\mathcal{H}^{N-1}$-a.e. $x' \in A'_\eta$,

$$\frac{1}{\varepsilon}\int_{-1/4}^{1/4}\left|\dot\gamma_\varepsilon\left(\frac{F_\eta(t\nu_1+x')}{\varepsilon}\right)\right| |\nabla F_\eta(t\nu_1+x')\cdot\nu_1|\,dt \leq \int_{\mathbb{R}} |\dot\gamma_\varepsilon(s)|\,ds = \mathbf{d}_{\mathcal{M}}(a,b)\,.$$

Consequently,

$$I_3 \leq C\beta\,\mathcal{H}^{N-1}(A'_\eta)\,\mathbf{d}_{\mathcal{M}}(a,b) = C\beta\big(1-(1-\eta)^{N-1}\big)\,\mathbf{d}_{\mathcal{M}}(a,b)\,. \tag{3.33}$$

In view of (3.28), (3.26) and estimates (3.29), (3.30) and (3.33), we conclude that

$$I_{(1-2\eta)\varepsilon}(\nu) \leq I_\varepsilon(\nu') + K\sigma\,,$$

where $K = 1 + \beta\Delta(1+C)$, Δ is the diameter of $\mathcal{M}$ and C is the constant given by (3.31). Finally, letting $\varepsilon \to 0$ we derive

$$\liminf_{\varepsilon\to 0} I_\varepsilon(\nu) \leq \liminf_{\varepsilon\to 0} I_\varepsilon(\nu') + K\sigma\,,$$

and

$$\limsup_{\varepsilon\to 0} I_\varepsilon(\nu) \leq \limsup_{\varepsilon\to 0} I_\varepsilon(\nu') + K\sigma\,.$$

The symmetry of the roles of ν and ν' allows us to invert them, thus concluding the proof of Step 3.

Step 4. Let ν and ν' be two orthonormal basis of $\mathbb{R}^N$ with equal first vector. We claim that the limits $\lim_{\varepsilon\to0} I_\varepsilon(\nu)$ and $\lim_{\varepsilon\to0} I_\varepsilon(\nu')$ exist and are equal. Indeed, let $\sigma > 0$ be fixed and let $\delta > 0$ as in Step 3. Let μ and μ' be two rational orthonormal basis of $\mathbb{R}^N$ such that $\mu_1 = \mu'_1$ and

$$|\mu_1 - \nu_1| < \delta\,, \quad |\mu_i - \nu_i| < \delta\,, \quad |\mu'_i - \nu'_i| < \delta \quad \text{for } i = 2,\dots,N.$$

By Step 2, $\lim_{\varepsilon\to0} I_\varepsilon(\mu) = \lim_{\varepsilon\to0} I_\varepsilon(\mu') =: \ell$. Then by Step 3 we infer that

$$\ell - K\sigma \leq \liminf_{\varepsilon\to0} I_\varepsilon(\nu) \leq \limsup_{\varepsilon\to0} I_\varepsilon(\nu) \leq \ell + K\sigma\,.$$

Hence $\limsup_{\varepsilon\to0} I_\varepsilon(\nu) - \liminf_{\varepsilon\to0} I_\varepsilon(\nu) \leq 2K\sigma$ and since σ is arbitrary we conclude that $\lim_{\varepsilon\to0} I_\varepsilon(\nu)$ exists. Arguing the same way for $I_\varepsilon(\nu')$, we obtain the existence of $\lim_{\varepsilon\to0} I_\varepsilon(\nu')$. In addition we derive from the estimate above that $|\ell - \lim_{\varepsilon\to0} I_\varepsilon(\nu)| \leq K\sigma$ and $|\ell - \lim_{\varepsilon\to0} I_\varepsilon(\nu')| \leq K\sigma$. Consequently, $|\lim_{\varepsilon\to0} I_\varepsilon(\nu) - \lim_{\varepsilon\to0} I_\varepsilon(\nu')| \leq 2K\sigma$ which proves that the two limits are equal since σ is arbitrary. □

Proof of Proposition 3.3. We use the notation of the previous proof. Given $\varepsilon > 0$ and an orthonormal basis $\nu = (\nu_1,\dots,\nu_N)$ of $\mathbb{R}^N$, we set

$$\begin{aligned} J_\varepsilon(\nu) = J_\varepsilon(a,b,\nu) &:= \inf\left\{\int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon},\nabla u\right)dx : u \in \mathcal{A}_1(a,b,\nu)\right\} \\ &= \inf\left\{\varepsilon^{N-1}\int_{\frac{1}{\varepsilon}Q_\nu} f^\infty(y,\nabla\varphi)\,dy \ : \ \varphi \in \mathcal{A}_{1/\varepsilon}(a,b,\nu)\right\}. \end{aligned}$$

We claim that

$$\lim_{\varepsilon\to0} J_\varepsilon(\nu) = \lim_{\varepsilon\to0} I_\varepsilon(\nu)\,. \tag{3.34}$$

For $0 < \varepsilon < 1$ we set $\tilde\varepsilon = \varepsilon/(1-\varepsilon)$, and we consider $u_{\tilde\varepsilon} \in \mathcal{B}_{\tilde\varepsilon}(a,b,\nu)$ satisfying

$$\int_{Q_\nu} f^\infty\left(\frac{x}{\tilde\varepsilon},\nabla u_{\tilde\varepsilon}\right)dx \leq I_{\tilde\varepsilon}(\nu) + \varepsilon\,,$$

where $u_{\tilde\varepsilon}(x) = \gamma_{\tilde\varepsilon}(x_\nu/\tilde\varepsilon)$ if $x \in \partial Q_\nu$, for some $\gamma_{\tilde\varepsilon} \in \mathcal{G}(a,b)$. We define for every $x \in Q_\nu$,

$$v_\varepsilon(x) := \begin{cases} u_{\tilde\varepsilon}\left(\dfrac{x}{1-\varepsilon}\right) & \text{if } x \in Q_{\nu,\varepsilon}\,,\\[2ex] \gamma_{\tilde\varepsilon}\left(\dfrac{x_\nu}{1-2\|x'\|_{\nu,\infty}}\right) & \text{otherwise}\,. \end{cases}$$

One may check that $v_\varepsilon \in \mathcal{A}_1(a,b,\nu)$, and hence

$$J_\varepsilon(\nu) \leq \int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx = \int_{Q_{\nu,\varepsilon}} f^\infty\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx + \int_{Q_\nu \setminus Q_{\nu,\varepsilon}} f^\infty\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx = I_1 + I_2\,.$$

We now estimate these two integrals. First, we have

$$I_1 = (1-\varepsilon)^{N-1}\int_{Q_\nu} f^\infty\left(\frac{y}{\tilde\varepsilon}, \nabla u_{\tilde\varepsilon}\right) dy \leq (1-\varepsilon)^{N-1}\big(I_{\tilde\varepsilon}(\nu) + \varepsilon\big)\,. \tag{3.35}$$

In view of the growth condition (3.12),

$$\begin{aligned} I_2 &\leq \beta \int_{Q_\nu \setminus Q_{\nu,\varepsilon}} \left|\dot\gamma_{\tilde\varepsilon}\left(\frac{x_\nu}{1-2\|x'\|_{\nu,\infty}}\right)\right| \left(\frac{1}{1-2\|x'\|_{\nu,\infty}} + \frac{|x_\nu||\nabla(\|x'\|_{\nu,\infty})|}{(1-2\|x'\|_{\nu,\infty})^2}\right) dx \\ &\leq 2\beta \int_{(Q_\nu\setminus Q_{\nu,\varepsilon})\cap\{|x_\nu|\leq(1-2\|x'\|_{\nu,\infty})/2\}} \left|\dot\gamma_{\tilde\varepsilon}\left(\frac{x_\nu}{1-2\|x'\|_{\nu,\infty}}\right)\right|\left(\frac{1}{1-2\|x'\|_{\nu,\infty}}\right) dx\,, \end{aligned}$$

where we have used the facts that $\dot\gamma_{\tilde\varepsilon}(x_\nu/(1-2\|x'\|_{\nu,\infty})) = 0$ in the set $\{|x_\nu| > (1-2\|x'\|_\infty)/2\}$ and $\|\nabla(\|x'\|_{\nu,\infty})\|_{L^\infty(Q_\nu;\mathbb{R}^N)} \leq 1$. Setting $Q'_\nu = Q_\nu \cap \{x_\nu = 0\}$ and $Q'_{\nu,\varepsilon} = Q_{\nu,\varepsilon}\cap\{x_\nu = 0\}$, we infer from Fubini's theorem that

$$\begin{aligned} I_2 \leq 2\beta \int_{Q'_\nu\setminus Q'_{\nu,\varepsilon}} \left(\int_{-(1-2\|x'\|_{\nu,\infty})/2}^{(1-2\|x'\|_{\nu,\infty})/2} \left|\dot\gamma_{\tilde\varepsilon}\left(\frac{t}{1-2\|x'\|_{\nu,\infty}}\right)\right|\left(\frac{1}{1-2\|x'\|_{\nu,\infty}}\right) dt\right) d\mathcal{H}^{N-1}(x') \leq \\ \leq 2\beta\,\mathcal{H}^{N-1}(Q'_\nu \setminus Q'_{\nu,\varepsilon})\,\mathbf{d}_\mathcal{M}(a,b) \leq 2\beta\mathbf{d}_\mathcal{M}(a,b)\big(1-(1-\varepsilon)^{N-1}\big)\,. \end{aligned} \tag{3.36}$$

In view of the estimates (3.35) and (3.36) obtained for I_1 and I_2, we derive that

$$\limsup_{\varepsilon\to 0} J_\varepsilon(\nu) \leq \lim_{\varepsilon\to 0} I_\varepsilon(\nu)\,. \tag{3.37}$$

Conversely, given $0 < \varepsilon < 1$, we consider $\tilde u_\varepsilon \in \mathcal{A}_1(a,b,\nu)$ such that

$$\int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon}, \nabla\tilde u_\varepsilon\right) dx \leq J_\varepsilon(\nu) + \varepsilon\,,$$

and $\gamma \in \mathcal{G}(a,b)$ fixed. We define for $x \in Q_\nu$,

$$w_\varepsilon(x) := \begin{cases} \tilde u_\varepsilon\left(\dfrac{x}{1-\varepsilon}\right) & \text{if } x \in Q_{\nu,\varepsilon}, \\ \gamma\left(\dfrac{x_\nu}{(1-\varepsilon)(2\|x'\|_{\nu,\infty} - 1 + \varepsilon)}\right) & \text{otherwise.} \end{cases}$$

We can check that $w_\varepsilon \in \mathcal{B}_{(1-\varepsilon)\varepsilon}(a,b,\nu)$ so that

$$\begin{aligned} I_{(1-\varepsilon)\varepsilon}(\nu) &\leq \int_{Q_\nu} f^\infty\left(\frac{x}{(1-\varepsilon)\varepsilon}\,, \nabla w_\varepsilon\right) dx \\ &= \int_{Q_{\nu,\varepsilon}} f^\infty\left(\frac{x}{(1-\varepsilon)\varepsilon}\,, \nabla w_\varepsilon\right) dx + \int_{Q_\nu\setminus Q_{\nu,\varepsilon}} f^\infty\left(\frac{x}{(1-\varepsilon)\varepsilon}\,, \nabla w_\varepsilon\right) dx\,. \end{aligned}$$

Arguing as previously, we infer that

$$\int_{Q_{\nu,\varepsilon}} f^\infty\left(\frac{x}{(1-\varepsilon)\varepsilon}\,, \nabla w_\varepsilon\right) dx = (1-\varepsilon)^{N-1}\int_{Q_\nu} f^\infty\left(\frac{y}{\varepsilon}, \nabla\tilde u_\varepsilon\right) dy \leq (1-\varepsilon)^{N-1}\big(J_\varepsilon(\nu)+\varepsilon\big)\,,$$

and

$$\int_{Q_\nu\setminus Q_{\nu,\varepsilon}} f^\infty\left(\frac{x}{(1-\varepsilon)\varepsilon}\,, \nabla w_\varepsilon\right) dx \leq 2\beta\mathbf{d}_\mathcal{M}(a,b)\big(1-(1-\varepsilon)^{N-1}\big)\,.$$

Consequently,

$$\lim_{\varepsilon\to 0} I_\varepsilon(\nu) \leq \liminf_{\varepsilon\to 0} J_\varepsilon(\nu)\,,$$

which, together with (3.37), completes the proof of Proposition 3.3. □

We now state the following properties of the surface energy density.

Proposition 3.5. *The function ϑ_{hom} is continuous on $\mathcal{M}\times\mathcal{M}\times\mathbb{S}^{N-1}$ and there exist constants $C_1>0$ and $C_2>0$ such that*

$$|\vartheta_{\rm hom}(a_1,b_1,\nu_1)-\vartheta_{hom}(a_2,b_2,\nu_1)|\leq C_1(|a_1-a_2|+|b_1-b_2|)\,, \tag{3.38}$$

and

$$\vartheta_{\rm hom}(a_1,b_1,\nu_1)\leq C_2|a_1-b_1| \tag{3.39}$$

for every $a_1,b_1,a_2,b_2\in\mathcal{M}$ and $\nu_1\in\mathbb{S}^{N-1}$.

Proof. We use the notation of the previous proof. By Proposition 3.3 together with steps 3 and 4 of the proof of Proposition 3.4, we get that $\vartheta_{\rm hom}(a,b,\cdot)$ is continuous on $\mathbb{S}^{N-1}$ uniformly with respect to a and b. Hence it is enough to show that (3.38) holds to get the continuity of $\vartheta_{\rm hom}$.

Step 1. We start with the proof of (3.38). Fix $\nu_1\in\mathbb{S}^{N-1}$ and let $\nu=(\nu_1,\nu_2,\dots,\nu_N)$ be any orthonormal basis of $\mathbb{R}^N$. For every $\varepsilon>0$, let $\tilde\varepsilon:=\varepsilon/(1-\varepsilon)$ and consider $\gamma_{\tilde\varepsilon}\in\mathcal{G}(a_1,b_1)$ and $u_{\tilde\varepsilon}\in\mathcal{B}_{\tilde\varepsilon}(a_1,b_1,\nu)$ such that $u_{\tilde\varepsilon}(x)=\gamma_{\tilde\varepsilon}(x_\nu/\tilde\varepsilon)$ for $x\in\partial Q_\nu$ and

$$\int_{Q_\nu} f^\infty\left(\frac{x}{\tilde\varepsilon},\nabla u_{\tilde\varepsilon}\right)dx\leq I_{\tilde\varepsilon}(a_1,b_1,\nu)+\varepsilon\,.$$

We shall now carefully modify $u_{\tilde\varepsilon}$ in order to get another function $v_\varepsilon\in\mathcal{A}_1(a_2,b_2,\nu)$. We will proceed as in the proofs of Propositions 3.3 and 3.4. Let $\gamma_a\in\mathcal{G}(a_2,a_1)$ and $\gamma_b\in\mathcal{G}(b_2,b_1)$, and define

$$v_\varepsilon(x):=\begin{cases} u_{\tilde\varepsilon}\left(\dfrac{x}{1-\varepsilon}\right) & \text{if } x\in Q_{\nu,\varepsilon}\,,\\ \gamma_{\tilde\varepsilon}\left(\dfrac{x_\nu}{1-2\|x'\|_{\nu,\infty}}\right) & \text{if } x\in A_1:=\left\{\dfrac{1-\varepsilon}{2}\leq\|x'\|_{\nu,\infty}<\dfrac12 \text{ and } |x_\nu|\leq-\|x'\|_{\nu,\infty}+\dfrac12\right\},\\ \gamma_a\left(\dfrac{2\|x\|_{\nu,\infty}-1}{\varepsilon}+\dfrac12\right) & \text{if } x\in A_2:=(Q_\nu\setminus Q_{\nu,\varepsilon})\cap\{x_\nu\geq\varepsilon/2\}\,,\\ \gamma_b\left(\dfrac{2\|x\|_{\nu,\infty}-1}{\varepsilon}+\dfrac12\right) & \text{if } x\in A_3:=(Q_\nu\setminus Q_{\nu,\varepsilon})\cap\{x_\nu\leq-\varepsilon/2\}\,,\\ \gamma_a\left(\dfrac{2\|x'\|_{\nu,\infty}-1}{2x_\nu}+\dfrac12\right) & \text{if } x\in A_4:=\left\{0<x_\nu\leq\dfrac{\varepsilon}{2} \text{ and } \dfrac12-x_\nu\leq\|x'\|_{\nu,\infty}<\dfrac12\right\},\\ \gamma_b\left(\dfrac{1-2\|x'\|_{\nu,\infty}}{2x_\nu}+\dfrac12\right) & \text{if } x\in A_5:=\left\{-\dfrac{\varepsilon}{2}<x_\nu\leq 0 \text{ and } \dfrac12+x_\nu\leq\|x'\|_{\nu,\infty}<\dfrac12\right\}. \end{cases}$$

One may check that the function v_ε has been constructed in such a way that $v_\varepsilon\in\mathcal{A}_1(a_2,b_2,\nu)$, and thus

$$J_\varepsilon(a_2,b_2,\nu)\leq\int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon},\nabla v_\varepsilon\right)dx\,. \tag{3.40}$$

Arguing exactly as in the proof of Proposition 3.4, one can show that

$$\int_{Q_{\nu,\varepsilon}} f^\infty\left(\frac{x}{\varepsilon},\nabla v_\varepsilon\right)dx\leq I_{\tilde\varepsilon}(a_1,b_1,\nu)+\varepsilon\,, \tag{3.41}$$

and

$$\int_{A_1} f^\infty\left(\frac{x}{\varepsilon},\nabla v_\varepsilon\right)dx\leq C\mathbf{d}_{\mathcal{M}}(a_1,b_1)(1-(1-\varepsilon)^{N-1}))\,. \tag{3.42}$$

Now we only estimate the integrals over A_2 and A_4, the ones over A_3 and A_5 being very similar. Define the Lipschitz function $F_\varepsilon:\mathbb{R}^N\to\mathbb{R}$ by

$$F_\varepsilon(x):=\frac{2\|x\|_{\nu,\infty}-1}{\varepsilon}+\frac12\,.$$

Using the growth condition (3.12) together with Fubini's theorem, and the fact that $A_2\subset F_\varepsilon^{-1}\big([-1/2,1/2)\big)$, we derive

$$\int_{A_2} f^\infty\left(\frac{x}{\varepsilon},\nabla v_\varepsilon\right)dx\leq\beta\int_{A_2}|\dot\gamma_a(F_\varepsilon(x))|\,|\nabla F_\varepsilon(x)|\,dx\leq$$
$$\leq\beta\int_{F_\varepsilon^{-1}([-1/2,1/2))}|\dot\gamma_a(F_\varepsilon(x))|\,|\nabla F_\varepsilon(x)|\,dx\leq\beta\int_{-1/2}^{1/2}|\,\dot\gamma_a(t)|\,\mathcal{H}^{N-1}(F_\varepsilon^{-1}\{t\})\,dt\,,$$

where we used the Coarea formula in the last inequality. We observe that for every $t \in (-1/2, 1/2)$, $F_\varepsilon^{-1}\{t\} = \partial Q_{\nu, \frac{\varepsilon(1-2t)}{2}}$ so that $\mathcal{H}^{N-1}(F_\varepsilon^{-1}\{t\}) \leq \mathcal{H}^{N-1}(\partial Q)$. Therefore

$$\int_{A_2} f^\infty\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx \leq \beta \mathcal{H}^{N-1}(\partial Q) \mathbf{d}_{\mathcal{M}}(a_1, a_2) . \tag{3.43}$$

Define now $G : \mathbb{R}^N \to \mathbb{R}$ by

$$G(x) := \frac{2\|x'\|_{\nu,\infty} - 1}{2x_\nu} + \frac{1}{2} .$$

The growth condition (3.12) and Fubini's theorem yield

$$\int_{A_4} f^\infty\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx \leq \beta \int_0^{\varepsilon/2} \left(\int_{G(\cdot, x_\nu)^{-1}([-1/2,1/2))} |\dot{\gamma}_a(G(x', x_\nu))| \, |\nabla G(x', x_\nu)| \, d\mathcal{H}^{N-1}(x') \right) dx_\nu .$$

As $|\nabla_{x'} G(x)| = 1/x_\nu$ and $|\nabla_{x_\nu} G(x)| \leq 1/x_\nu$ for a.e. $x \in A_4$, it follows that $|\nabla G(x)| \leq 2|\nabla_{x'} G(x)|$ for a.e. $x \in A_4$. Hence

$$\int_{A_4} f^\infty\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx \leq 2 \int_0^{\varepsilon/2} \left(\int_{G(\cdot, x_\nu)^{-1}([-1/2,1/2))} |\dot{\gamma}_a(G(x', x_\nu))| \, |\nabla_{x'} G(x', x_\nu)| \, d\mathcal{H}^{N-1}(x') \right) dx_\nu .$$

For every $x_\nu \in (0, \varepsilon/2)$ the function $G(\cdot, x_\nu) : \mathbb{R}^{N-1} \to \mathbb{R}$ is Lipschitz, and thus the Coarea formula implies

$$\begin{aligned} \int_{A_4} f^\infty\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx &\leq 2 \int_0^{\varepsilon/2} \left(\int_{-1/2}^{1/2} |\dot{\gamma}_a(t)| \, \mathcal{H}^{N-2}(\{x' : G(x', x_\nu) = t\}) \, dt \right) dx_\nu \\ &\leq C\varepsilon \, \mathbf{d}_{\mathcal{M}}(a_1, a_2) , \end{aligned} \tag{3.44}$$

where we used as previously the estimate $\mathcal{H}^{N-2}(\{x' : G(x', x_\nu) = t\}) \leq \mathcal{H}^{N-2}\big(\partial(\frac{-1}{2}, \frac{1}{2})^{N-1}\big)$. Gathering (3.40) to (3.44) and considering the analogous estimates for the integrals over A_3 and A_5 (with b_1 and b_2 instead of a_1 and a_2), we infer that

$$J_\varepsilon(a_2, b_2, \nu) \leq \int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) dx \leq I_{\tilde{\varepsilon}}(a_1, b_1, \nu) + C\big(\varepsilon + \mathbf{d}_{\mathcal{M}}(a_1, a_2) + \mathbf{d}_{\mathcal{M}}(b_1, b_2)\big) .$$

Taking the limit as $\varepsilon \to 0$, we get in light of Propositions 3.3 and 3.4 that

$$\vartheta_{\rm hom}(a_2, b_2, \nu) \leq \vartheta_{\rm hom}(a_1, b_1, \nu) + C\big(\mathbf{d}_{\mathcal{M}}(b_1, b_2) + \mathbf{d}_{\mathcal{M}}(a_1, a_2)\big) .$$

Since the geodesic distance on $\mathcal{M}$ is equivalent to the Euclidian distance, we conclude, possibly exchanging the roles of (a_1, b_1) and (a_2, b_2), that (3.38) holds.

Step 2. We now prove (3.39). Given an arbitrary orthonormal basis $\nu = (\nu_1, \ldots, \nu_N)$ of $\mathbb{R}^N$, let $\gamma \in \mathcal{G}(a_1, b_1)$ and define $u_\varepsilon(x) := \gamma(x_\nu/\varepsilon)$. Obviously $u_\varepsilon \in \mathcal{B}_\varepsilon(a_1, b_1, \nu)$. Using (3.34) together with the growth condition (3.12) satisfied by f^∞, we derive that

$$\vartheta_{\rm hom}(a_1, b_1, \nu_1) \leq \liminf_{\varepsilon \to 0} \int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) dx \leq \liminf_{\varepsilon \to 0} \frac{\beta}{\varepsilon} \int_{Q_\nu} \left|\dot{\gamma}\left(\frac{x \cdot \nu_1}{\varepsilon}\right)\right| dx = \beta \mathbf{d}_{\mathcal{M}}(a_1, b_1) .$$

Then (3.39) follows from the equivalence between $\mathbf{d}_{\mathcal{M}}$ and the Euclidian distance. □

4. Homogenization in Sobolev spaces

This section is devoted to the proofs of Theorems 1.1 and 1.2. We first show that a suitable functional larger than the Γ-limit is a measure. It will allow us to obtain the upper bound (see Lemma 4.2) through the blow-up method introduced in [25,26]. The lower bound will require different proofs in the cases $p > 1$ (Lemma 4.3) and $p = 1$ (Lemma 4.4).

Let us consider an arbitrary sequence $\{\varepsilon_n\} \searrow 0^+$. Along this sequence we define the $\Gamma(L^p)$-lower limit $\mathcal{F} : L^p(\Omega; \mathbb{R}^d) \to [0, +\infty]$ by

$$\mathcal{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n \to +\infty} \mathcal{F}_{\varepsilon_n}(u_n) \; : \; u_n \in W^{1,p}(\Omega; \mathcal{M}) , \, u_n \to u \text{ in } L^p(\Omega; \mathbb{R}^d) \right\} .$$

4.1. Localization

The idea is to localize the functionals $\{\mathcal{F}_{\varepsilon_n}\}_{n\in\mathbb{N}}$ on the family $\mathcal{A}(\Omega)$ of all open subsets of Ω. For every $u \in L^p(\Omega;\mathbb{R}^d)$ and every $A \in \mathcal{A}(\Omega)$, define

$$\mathcal{F}_{\varepsilon_n}(u,A) := \begin{cases} \displaystyle\int_A f\left(\frac{x}{\varepsilon_n},\nabla u\right)dx & \text{if } u \in W^{1,p}(\Omega;\mathcal{M}),\\ +\infty & \text{otherwise}. \end{cases}$$

Given a compact set $\mathcal{K} \subset \mathcal{M}$ and a subsequence $\{\varepsilon_k\} := \{\varepsilon_{n_k}\} \searrow 0^+$, we introduce for $u \in W^{1,p}(\Omega;\mathcal{M})$ and $A \in \mathcal{A}(\Omega)$,

$$\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A) := \inf_{\{u_k\}}\Bigg\{ \limsup_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k,A) :\ u_k \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^d),$$
$$u_k \to u \text{ uniformly and } u_k(x) = u(x) \text{ whenever } \operatorname{dist}(u(x),\mathcal{K}) > 1 \text{ for a.e. } x \in \Omega \Bigg\}.$$

A key point in the upcoming analysis is the following locality result.

Lemma 4.1. *For every $u \in W^{1,p}(\Omega;\mathcal{M})$, there exists a subsequence $\{\varepsilon_k\}$ such that the set function $\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$.*

Proof. From the p-growth condition (H_2) we infer that for any subsequence $\{\varepsilon_k\}$,

$$\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A) \le \beta\int_A (1+|\nabla u|^p)\,dx\,, \tag{4.1}$$

so it remains to prove the existence of a suitable subsequence $\{\varepsilon_k\}$ for which $\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\cdot)$ is (the trace of) a Radon measure.

Step 1. We start by proving that for any subsequence $\{\varepsilon_k\}$ the following subadditivity property holds:

$$\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A) \le \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,B) + \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A\setminus\overline{C}) \tag{4.2}$$

for every A, B and $C \in \mathcal{A}(\Omega)$ such that $\overline{C} \subset B \subset A$. Given $\eta > 0$ arbitrary, there exist sequences $\{u_k\}$, $\{v_k\} \subset W^{1,p}(\Omega;\mathcal{M})$ such that u_k and v_k converge weakly to u in $W^{1,p}(\Omega;\mathbb{R}^d)$, $u_k(x) = v_k(x) = u(x)$ if $\operatorname{dist}(u(x),\mathcal{K}) > 1$ for a.e. $x \in \Omega$, u_k and v_k are uniformly converging to u, and

$$\begin{cases} \displaystyle\limsup_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k,B) \le \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,B) + \eta\,,\\ \displaystyle\limsup_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(v_k,A\setminus\overline{C}) \le \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A\setminus\overline{C}) + \eta\,. \end{cases} \tag{4.3}$$

Let $\mathcal{K}' := \{s \in \mathcal{M} : \operatorname{dist}(s,\mathcal{K}) \le 1\}$, then $\mathcal{K}'$ is a compact subset of $\mathcal{M}$ and $u_k(x) = v_k(x) = u(x)$ if $u(x) \notin \mathcal{K}'$ for a.e. $x \in \Omega$.

Consider $L := \operatorname{dist}(C,\partial B)$, $M \in \mathbb{N}$, and for every $i \in \{0,\dots,M\}$ define

$$B_i := \left\{x \in B :\ \operatorname{dist}(x,\partial B) > \frac{iL}{M}\right\}.$$

Given $i \in \{0,\dots,M-1\}$ let $S_i := B_i \setminus \overline{B_{i+1}}$, and $\zeta_i \in \mathcal{C}^\infty_c(\Omega;[0,1])$ be a cut-off function satisfying

$$\zeta_i(x) = \begin{cases} 1 & \text{in } B_{i+1}\,,\\ 0 & \text{in } \Omega\setminus B_i\,, \end{cases} \quad \text{and} \quad |\nabla\zeta_i| \le \frac{2M}{L}\,.$$

By Lemma 3.2 and Remark 3.3 in [18], there exist $\delta > 0$, $c > 0$, and a uniformly continuously differentiable mapping $\Phi : D_\delta \times [0,1] \to \mathcal{M}$, where

$$D_\delta := \big\{(s_0,s_1) \in \mathcal{M}\times\mathcal{M} : \operatorname{dist}(s_0,\mathcal{K}') < \delta,\ \operatorname{dist}(s_1,\mathcal{K}') < \delta,\ |s_0 - s_1| < \delta\big\}\,,$$

such that

$$\Phi(s_0,s_1,0)=s_0\,,\quad \Phi(s_0,s_1,1)=s_1\,,\quad \frac{\partial\Phi}{\partial t}(s_0,s_1,t)\leq c|s_0-s_1|\,, \tag{4.4}$$

and

$$|\Phi(s_0,s_1,t)-s_0|\leq c|s_0-s_1|\,. \tag{4.5}$$

Since $\{u_k\}$ and $\{v_k\}$ are uniformly converging to u, one can choose k large enough to ensure that

$$\|u_k-u\|_{L^\infty(\Omega;\mathbb{R}^d)}<\delta\,,\quad \|v_k-u\|_{L^\infty(\Omega;\mathbb{R}^d)}<\delta \quad \text{and} \quad \|u_k-v_k\|_{L^\infty(\Omega;\mathbb{R}^d)}<\delta\,.$$

Therefore for a.e. $x\in\Omega$, $\operatorname{dist}(u_k(x),\mathcal{K}')<\delta$ and $\operatorname{dist}(v_k(x),\mathcal{K}')<\delta$ whenever $u(x)\in\mathcal{K}'$. Now we are allowed to define

$$w_{k,i}(x):=\begin{cases}\Phi(v_k(x),u_k(x),\zeta_i(x)) & \text{if } u(x)\in\mathcal{K}'\,,\\ u(x) & \text{if } u(x)\notin\mathcal{K}'\,,\end{cases}$$

and $w_{k,i}\in W^{1,p}(\Omega;\mathcal{M})$. Using the p-growth condition (H_2) together with (4.4), we derive

$$\int_A f\left(\frac{x}{\varepsilon_k},\nabla w_{k,i}\right)dx\leq\int_B f\left(\frac{x}{\varepsilon_k},\nabla u_k\right)dx+\int_{A\setminus\overline{C}} f\left(\frac{x}{\varepsilon_k},\nabla v_k\right)dx+ \\ +C_0\int_{S_i}(1+|\nabla u_k|^p+|\nabla v_k|^p+M^p|u_k-v_k|^p)\,dx\,,$$

for some constant $C_0>0$ independent of k, i and M. Summing up over $i\in\{0,\dots,M-1\}$ and dividing by M yields

$$\frac{1}{M}\sum_{i=0}^{M-1}\int_A f\left(\frac{x}{\varepsilon_k},\nabla w_{k,i}\right)dx\leq\int_B f\left(\frac{x}{\varepsilon_k},\nabla u_k\right)dx+\int_{A\setminus\overline{C}} f\left(\frac{x}{\varepsilon_k},\nabla v_k\right)dx+ \\ +\frac{C_0}{M}\int_{B\setminus\overline{C}}(1+|\nabla u_k|^p+|\nabla v_k|^p+M^p|u_k-v_k|^p)\,dx\,.$$

Hence one may find some $i_k\in\{0,\dots,M-1\}$ such that $\bar{w}_k:=w_{k,i_k}$ satisifies

$$\int_A f\left(\frac{x}{\varepsilon_k},\nabla \bar{w}_k\right)dx\leq\int_B f\left(\frac{x}{\varepsilon_k},\nabla u_k\right)dx+\int_{A\setminus\overline{C}} f\left(\frac{x}{\varepsilon_k},\nabla v_k\right)dx+ \\ +\frac{C_0}{M}\int_{B\setminus\overline{C}}(1+|\nabla u_k|^p+|\nabla v_k|^p+M^p|u_k-v_k|^p)\,dx\,. \tag{4.6}$$

From (4.4) and (4.5) we deduce that $\bar{w}_k\to u$ uniformly, $\bar{w}_k\rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^d)$, and $\bar{w}_k(x)=u(x)$ if $\operatorname{dist}(u(x),\mathcal{K})>1$ for a.e. $x\in\Omega$. Taking $\{\bar{w}_k\}$ as competitor for $\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A)$, and using (4.6) together with (4.3) leads to

$$\begin{aligned}\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A)&\leq\limsup_{k\to+\infty}\mathcal{F}_{\varepsilon_k}(\bar{w}_k,A)\\ &\leq\limsup_{k\to+\infty}\Big\{\mathcal{F}_{\varepsilon_k}(u_k,B)+\mathcal{F}_{\varepsilon_k}(v_k,A\setminus\overline{C})+\frac{C_0}{M}\int_{B\setminus\overline{C}}(1+|\nabla u_k|^p+|\nabla v_k|^p+M^p|u_k-v_k|^p)\,dx\Big\}\\ &\leq\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,B)+\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A\setminus\overline{C})+2\eta+\frac{C_0}{M}\sup_{k\in\mathbb{N}}\int_{B\setminus\overline{C}}(1+|\nabla u_k|^p+|\nabla v_k|^p)\,dx\,.\end{aligned}$$

Then property (4.2) arises sending first $M\to+\infty$, and then $\eta\to 0$.

Step 2. Now we complete the proof of Lemma 4.1. Using a standard diagonal argument, we construct a subsequence $\{\varepsilon_k\}\searrow 0^+$ and a sequence $\{u_k\}\subset W^{1,p}(\Omega,\mathcal{M})$ satisfying

$$\lim_{k\to+\infty}\mathcal{F}_{\varepsilon_k}(u_k,\Omega)=\inf_{\{v_k\}}\Big\{\liminf_{k\to+\infty}\mathcal{F}_{\varepsilon_k}(v_k,\Omega):\,v_k\rightharpoonup u\text{ weakly in }W^{1,p}(\Omega;\mathbb{R}^d)\,,\\ v_k\to u\text{ uniformly and }v_k(x)=u(x)\text{ whenever }\operatorname{dist}(u(x),\mathcal{K})>1\text{ for a.e. }x\in\Omega\Big\}\,.$$

By construction of $\{\varepsilon_k\}$ and $\{u_k\}$, we have $\lim_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k,\Omega) = \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\Omega)$. Up to the extraction of a further subsequence, we may assume that

$$f\left(\frac{\cdot}{\varepsilon_k}, \nabla u_k\right) \mathcal{L}^N \llcorner\, \Omega \overset{\star}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\Omega)\,,$$

for some nonnegative Radon measure $\mu \in \mathcal{M}(\Omega)$. By lower semicontinuity, we have

$$\mu(\Omega) \leq \lim_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k,\Omega) = \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\Omega)\,.$$

We claim that

$$\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A) = \mu(A) \qquad \text{for any } A \in \mathcal{A}(\Omega)\,.$$

We fix $A \in \mathcal{A}(\Omega)$ and we start by proving the inequality "$\leq$". Given $\eta > 0$ arbitrary we can select, in view of (4.1), $C \in \mathcal{A}(\Omega)$, $C \subset\subset A$, such that $\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u, A\setminus \overline{C}) \leq \eta$. Then inequality (4.2) implies that for any $B \in \mathcal{A}(\Omega)$, $C \subset\subset B \subset\subset A$,

$$\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A) \leq \eta + \limsup_{k\to+\infty} \mathcal{F}_{\varepsilon_k}(u_k,B) \leq \eta + \mu(\overline{B}) \leq \eta + \mu(A)\,,$$

and the conclusion follows from the arbitrariness of η.

Conversely, for any $B \in \mathcal{A}(\Omega)$, $B \subset\subset A$, we have

$$\mu(\Omega) \leq \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\Omega) \leq \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A) + \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\Omega\setminus\overline{B}) \leq \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A) + \mu(\Omega\setminus\overline{B}) \leq \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A) + \mu(\Omega) - \mu(B)\,.$$

Therefore $\mu(B) \leq \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,A)$ and the conclusion follows by inner regularity of μ. □

4.2. *The upper bound*

We now make use of the previous locality result to show the upper bound. This will be done thanks to a blow-up analysis in the spirit of [18, Theorem 3.1].

Lemma 4.2. *For every $p \in [1,+\infty)$ and $u \in W^{1,p}(\Omega;\mathcal{M})$, we have $\mathcal{F}(u) \leq \mathcal{F}_{\rm hom}(u)$.*

Proof. Step 1. Let $u \in W^{1,p}(\Omega;\mathcal{M})$. Given $R > 0$ arbitrary large, we set $\mathcal{K} := \mathcal{M} \cap B^d(0,R)$, and we consider the subsequence $\{\varepsilon_k\}$ given by Lemma 4.1. Obviously $\mathcal{F}(u) \leq \mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\Omega)$. We claim that

$$\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\Omega) \leq \int_\Omega \Big\{ \chi_R(|u|)\, Tf_{\rm hom}(u,\nabla u) + \beta\big(1-\chi_R(|u|)\big)\big(1+|\nabla u|^p\big)\Big\}\,dx\,, \tag{4.7}$$

where $\chi_R(t) = 1$ for $t \leq R$ and $\chi_R(t) = 0$ otherwise. We postpone the proof of (4.7) to the next step, and we complete now the proof of Lemma 4.2.

Consider a sequence $R_j \to +\infty$ as $j \to +\infty$. Since $\chi_{R_j} \to 1$ pointwise, we deduce from Fatou's lemma together with (3.4) that

$$\mathcal{F}(u) \leq \limsup_{j\to+\infty} \int_\Omega \Big\{ \chi_{R_j}(|u|)\, Tf_{\rm hom}(u,\nabla u) + \beta\big(1-\chi_{R_j}(|u|)\big)\big(1+|\nabla u|^p\big)\Big\}\,dx \leq \int_\Omega Tf_{\rm hom}(u,\nabla u)\,dx\,,$$

which is the announced estimate.

Step 2. Thanks to Lemma 4.1, to obtain (4.7) it suffices to prove that

$$\frac{d\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\cdot)}{d\mathcal{L}^N}(x_0) \leq \chi_R(|u(x_0)|)\, Tf_{\rm hom}(u(x_0),\nabla u(x_0)) + \beta\big(1-\chi_R(|u(x_0)|)\big)\big(1+|\nabla u(x_0)|^p\big)$$

for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$.

Let $x_0 \in \Omega$ be a Lebesgue point of u and ∇u such that $u(x_0) \in \mathcal{M}$, $\nabla u(x_0) \in [T_{u(x_0)}(\mathcal{M})]^N$, and the Radon-Nikodým derivative of $\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\cdot)$ with respect to the Lebesgue measure $\mathcal{L}^N$ exists. Note that almost every points in Ω satisfy these properties. Now set $s_0 := u(x_0)$ and $\xi_0 := \nabla u(x_0)$.

Case 1. Assume that $s_0 \notin \mathcal{K}$. Then, using (H_2), we derive that

$$\frac{d\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\rho\to 0^+} \frac{\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,Q(x_0,\rho))}{\rho^N} \leq \limsup_{\rho\to 0^+} \limsup_{k\to+\infty} \rho^{-N}\mathcal{F}_{\varepsilon_k}(u,Q(x_0,\rho)) \leq$$
$$\leq \lim_{\rho\to 0^+} \frac{\beta}{\rho^N}\int_{Q(x_0,\rho)} (1+|\nabla u|^p)\,dx = \beta\big(1+|\xi_0|^p\big)\,,$$

which is the desired estimate.

Case 2. Now we assume that $s_0 \in \mathcal{K}$. Fix $0<\eta<1$ arbitrary. By Proposition 3.1, claim *(i)*, there exist $j\in\mathbb{N}$ and $\varphi \in W^{1,\infty}_0((0,j)^N;T_{s_0}(\mathcal{M}))$ such that

$$⨍_{(0,j)^N} f(y,\xi_0+\nabla\varphi(y))\,dy \leq Tf_{\rm hom}(s_0,\xi_0)+\eta\,. \tag{4.8}$$

Extend φ to $\mathbb{R}^N$ by j-periodicity, and define $\varphi_k(x) := \xi_0\,x + \varepsilon_k\varphi(x/\varepsilon_k)$.

Let $\mathcal{U}$ be an open neighborhood of $\mathcal{M}$ such that the nearest point projection $\Pi:\mathcal{U}\to\mathcal{M}$ defines a $\mathcal{C}^1$-mapping. Fix $\sigma,\delta_0\in(0,1)$ such that $B^d(s_0,2\delta_0)\subset\mathcal{U}$, and consider $\delta=\delta(\sigma)\in(0,\delta_0)$ for which

$$|\nabla\Pi(s)-\nabla\Pi(s')|<\sigma \quad \text{for all } s,s'\in B^d(s_0,\delta_0) \text{ satisfying } |s-s'|<\delta\,. \tag{4.9}$$

Next we introduce a cut-off function $\zeta\in\mathcal{C}^\infty_c(\mathbb{R}^d;[0,1])$ satisfying

$$\zeta(x) = \begin{cases} 1 & \text{for } x\in B^d(0,\delta/4)\,,\\ 0 & \text{for } x\notin B^d(0,\delta/2)\,, \end{cases} \qquad \text{with} \quad |\nabla\zeta| \leq \frac{C}{\delta}\,,$$

and we define

$$w_k(x) := u(x) + \varepsilon_k\zeta(u(x)-s_0)\varphi(x/\varepsilon_k)\,.$$

Let $k_0\in\mathbb{N}$ be such that

$$\max\left\{\varepsilon_k\|\varphi\|_{L^\infty((0,j)^N;\mathbb{R}^d)}\|\nabla\zeta\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)},\,\frac{2\varepsilon_k\|\varphi\|_{L^\infty((0,j)^N;\mathbb{R}^d)}}{\delta}\right\}<1 \quad \text{for any } k\geq k_0\,. \tag{4.10}$$

Define for every $k\geq k_0$,

$$u_k(x) := \Pi(w_k(x))\,.$$

Remark that by (4.10), for a.e. $x\in\Omega$ and all $k\geq k_0$, one has $w_k(x)\in B^d(s_0,\delta)$ whenever $|u(x)-s_0|<\delta/2$ while $w_k(x)=u(x)$ when $|u(x)-s_0|\geq\delta/2$. Hence u_k is well defined, $\{u_k\}\subset W^{1,p}(\Omega;\mathcal{M})$, and for a.e. $x\in\Omega$, $u_k(x)=u(x)$ whenever $\operatorname{dist}(u(x),\mathcal{K})>1$. Moreover,

$$\|u_k-u\|_{L^\infty(\Omega;\mathbb{R}^d)} = \|\Pi(w_k)-\Pi(u)\|_{L^\infty(\{|u-s_0|<\delta/2\};\mathbb{R}^d)} \leq \varepsilon_k\,\|\nabla\Pi\|_{L^\infty(B^d(s_0,\delta_0);\mathbb{R}^d)}\|\varphi\|_{L^\infty((0,j)^N;\mathbb{R}^d)} \to 0$$

as $k\to+\infty$. Now the Chain Rule formula yields

$$\nabla u_k(x) = \nabla\Pi(w_k(x))\Big(\nabla u(x)+\varepsilon_k\big(\varphi(x/\varepsilon_k)\otimes\nabla\zeta(u(x)-s_0)\big)\nabla u(x)+\zeta(u(x)-s_0)\nabla\varphi(x/\varepsilon_k)\Big)\,,$$

and consequently

$$|\nabla u_k(x)| \leq \|\nabla\Pi\|_{L^\infty(B^d(s_0,\delta_0);\mathbb{R}^d)}\Big(\big(1+\varepsilon_k\|\varphi\|_{L^\infty((0,j)^N;\mathbb{R}^d)}\|\nabla\zeta\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)}\big)|\nabla u(x)|+\|\nabla\varphi\|_{L^\infty((0,j)^N;\mathbb{R}^{d\times N})}\Big)\,.$$

By (4.10) it follows that for any $k\geq k_0$,

$$|\nabla u_k(x)| \leq C_0(|\nabla u(x)-\xi_0|+1) \tag{4.11}$$

for some constant $C_0=C_0(s_0,\xi_0,\delta_0,\eta)>0$ independent of x and k. Hence the sequence $\{u_k\}$ is uniformly bounded in $W^{1,p}(\Omega;\mathbb{R}^d)$ so that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^d)$.

Then we observe that $|\nabla u_k|\leq 2C_0$ a.e. in $\{|\nabla u-\xi_0|<\sigma\}$ while

$$\|\nabla\varphi_k\|_{L^\infty(\Omega;\mathbb{R}^{d\times N})} \leq |\xi_0|+\|\nabla\varphi\|_{L^\infty((0,j)^N;\mathbb{R}^{d\times N})}\,.$$

Set

$$M := \max\left\{2C_0\,, |\xi_0| + \|\nabla\varphi\|_{L^\infty((0,j)^N;\mathbb{R}^{d\times N})}\right\}, \tag{4.12}$$

(which only depends on s_0, ξ_0, δ_0 and η) so that

$$|\nabla u_k| \le M \quad \text{and} \quad |\nabla \varphi_k| \le M \quad \text{ a.e. in } \{|\nabla u - \xi_0| < \sigma\}\,. \tag{4.13}$$

Next for a.e. $x \in \{|u - s_0| < \delta/4\} \cap \{|\nabla u - \xi_0| < \sigma\}$, we have $\zeta(u(x) - s_0) = 1$ and

$$\begin{aligned}|\nabla u_k(x) - \nabla\varphi_k(x)| &\le |\nabla\Pi(w_k)\nabla u(x) - \xi_0| + |\nabla\Pi(w_k)\nabla\varphi(x/\varepsilon_k) - \nabla\varphi(x/\varepsilon_k)|\\ &\le |\nabla\Pi(w_k) - \nabla\Pi(s_0)|\,|\nabla u(x)| + |\nabla\Pi(s_0)|\,|\nabla u(x) - \xi_0| +\\ &\qquad + |\nabla\Pi(w_k) - \nabla\Pi(s_0)|\,\|\nabla\varphi\|_{L^\infty((0,j)^N;\mathbb{R}^{d\times N})}\,,\end{aligned}$$

where, in the last inequality, we have used the fact that $\nabla\Pi(s_0)\nabla\varphi(y) = \nabla\varphi(y)$ since $\nabla\varphi(y) \in [T_{s_0}(\mathcal{M})]^N$ for a.e. $y \in \mathbb{R}^N$. Using (4.9) and the fact that $|w_k - s_0| < \delta$ a.e. in $\{|u - s_0| < \delta/4\} \cap \{|\nabla u - \xi_0| < \sigma\}$, we deduce

$$|\nabla u_k(x) - \nabla\varphi_k(x)| \le \big(|\nabla u(x)| + |\nabla\Pi(s_0)| + \|\nabla\varphi\|_{L^\infty((0,j)^N;\mathbb{R}^{d\times N})}\big)\sigma \le C_1\sigma \tag{4.14}$$

for a.e. $x \in \{|u - s_0| < \delta/4\} \cap \{|\nabla u - \xi_0| < \sigma\}$, where $C_1 = C_1(s_0, \xi_0, \delta_0, \eta) > 0$ is a constant independent of σ, k and x.

Now we estimate

$$\begin{aligned}\frac{d\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\cdot)}{d\mathcal{L}^N}(x_0) &= \lim_{\rho\to 0^+}\frac{\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u, Q(x_0,\rho))}{\rho^N}\\ &\le \limsup_{\rho\to 0^+}\limsup_{k\to+\infty}\frac{1}{\rho^N}\int_{Q(x_0,\rho)} f\left(\frac{x}{\varepsilon_k}, \nabla u_k\right)dx\\ &\le \limsup_{\rho\to 0^+}\limsup_{k\to+\infty}\frac{1}{\rho^N}\int_{Q(x_0,\rho)\cap\{|u-s_0|\ge\delta/4\}} f\left(\frac{x}{\varepsilon_k}, \nabla u_k\right)dx\\ &\quad + \limsup_{\rho\to 0^+}\limsup_{k\to+\infty}\frac{1}{\rho^N}\int_{Q(x_0,\rho)\cap\{|u-s_0|<\delta/4\}\cap\{|\nabla u-\xi_0|<\sigma\}} f\left(\frac{x}{\varepsilon_k}, \nabla u_k\right)dx\\ &\quad + \limsup_{\rho\to 0^+}\limsup_{k\to+\infty}\frac{1}{\rho^N}\int_{Q(x_0,\rho)\cap\{|u-s_0|<\delta/4\}\cap\{|\nabla u-\xi_0|\ge\sigma\}} f\left(\frac{x}{\varepsilon_k}, \nabla u_k\right)dx\\ &=: I_1 + I_2 + I_3\,.\end{aligned} \tag{4.15}$$

Thanks to (4.11), the p-growth condition (H_2) and our choice of x_0, we have

$$\begin{aligned}I_1 &\le C\limsup_{\rho\to 0^+}\frac{1}{\rho^N}\int_{Q(x_0,\rho)\cap\{|u-s_0|\ge\delta/4\}}(1 + |\nabla u(x) - \xi_0|^p)\,dx\\ &\le C\limsup_{\rho\to 0^+}\fint_{Q(x_0,\rho)}|\nabla u(x) - \xi_0|^p\,dx + \frac{4C}{\delta}\limsup_{\rho\to 0^+}\fint_{Q(x_0,\rho)}|u(x) - s_0|\,dx = 0\,,\end{aligned} \tag{4.16}$$

while

$$\begin{aligned}I_3 &\le C\limsup_{\rho\to 0^+}\frac{1}{\rho^N}\int_{Q(x_0,\rho)\cap\{|u-s_0|<\delta/4\}\cap\{|\nabla u-\xi_0|\ge\sigma\}}(1 + |\nabla u(x) - \xi_0|^p)\,dx\\ &\le C\limsup_{\rho\to 0^+}\fint_{Q(x_0,\rho)}|\nabla u(x) - \xi_0|^p\,dx + \frac{C}{\sigma}\limsup_{\rho\to 0^+}\fint_{Q(x_0,\rho)}|\nabla u(x) - \xi_0|\,dx = 0\,.\end{aligned} \tag{4.17}$$

Let us now treat the integral I_2. Since, for a.e. $y \in \mathbb{R}^N$, the function $f(y,\cdot)$ is continuous, it is uniformly continuous on $B^{d\times N}(0,M)$ where $M > 0$ is given in (4.12). Define the modulus of continuity of $f(y,\cdot)$ over $B^{d\times N}(0,M)$ by

$$\omega(y,t) := \sup\{|f(y,\xi) - f(y,\xi')| :\ \xi,\, \xi' \in B^{d\times N}(0,M) \text{ and } |\xi - \xi'| \le t\}\,.$$

It turns out that $\omega(y,\cdot)$ is increasing, continuous and $\omega(y,0) = 0$, while $\omega(\cdot,t)$ is measurable (since the supremum can be restricted to all admissible ξ and ξ' having rational entries) and 1-periodic. Thanks to (4.13) and (4.14) we get that

$$\left|f\left(\frac{x}{\varepsilon_k}, \nabla u_k(x)\right) - f\left(\frac{x}{\varepsilon_k}, \nabla\varphi_k(x)\right)\right| \le \omega\left(\frac{x}{\varepsilon_k}, C_1\sigma\right)$$

for a.e. $x \in Q(x_0,\rho)\cap\{|u-s_0|<\delta/4\}\cap\{|\nabla u-\xi_0|<\sigma\}$.

Integrating over the set $Q(x_0,\rho)\cap\{|u-s_0|<\delta/4\}\cap\{|\nabla u-\xi_0|<\sigma\}$, and taking the limit as $k\to+\infty$, we obtain in view of the Riemann-Lebesgue Lemma that

$$\limsup_{k\to+\infty}\rho^{-N}\int_{Q(x_0,\rho)\cap\{|u-s_0|<\delta/4\}\cap\{|\nabla u-\xi_0|<\sigma\}}\left|f\left(\frac{x}{\varepsilon_k},\nabla u_k(x)\right)-f\left(\frac{x}{\varepsilon_k},\nabla\varphi_k(x)\right)\right|dx\leq$$
$$\leq\limsup_{k\to+\infty}\rho^{-N}\int_{Q(x_0,\rho)}\omega\left(\frac{x}{\varepsilon_k},C_1\sigma\right)dx=\int_Q\omega(y,C_1\sigma)\,dy\,,$$

where we have used the fact that $y\mapsto\omega(y,C_1\sigma)$ is a measurable 1-periodic function. Observe that the Dominated Convergence Theorem together with $\omega(y,0)=0$ implies

$$\lim_{\sigma\to0^+}\int_Q\omega(y,C_1\sigma)\,dy=0\,. \tag{4.18}$$

We have obtained

$$I_2\leq\limsup_{\rho\to0^+}\limsup_{k\to+\infty}\frac{1}{\rho^N}\int_{Q(x_0,\rho)}f\left(\frac{x}{\varepsilon_k},\nabla\varphi_k\right)dx+\int_Q\omega(y,C_1\sigma)\,dy\,. \tag{4.19}$$

Using the definition of φ_k and the Riemann-Lebesgue Lemma, we infer from (4.8) that

$$\limsup_{\rho\to0^+}\limsup_{k\to+\infty}\frac{1}{\rho^N}\int_{Q(x_0,\rho)}f\left(\frac{x}{\varepsilon_k},\xi_0+\nabla\varphi\left(\frac{x}{\varepsilon_k}\right)\right)dx=$$
$$=\fint_{(0,j)^N}f(y,\xi_0+\nabla\varphi(y))\,dy\leq Tf_{\rm hom}(s_0,\xi_0)+\eta\,. \tag{4.20}$$

Hence gathering (4.15), (4.16), (4.17), (4.19) and (4.20) we deduce that

$$\frac{d\mathcal{F}^{\{\varepsilon_k\}}_{\mathcal{K}}(u,\cdot)}{d\mathcal{L}^N}(x_0)\leq Tf_{\rm hom}(s_0,\xi_0)+\int_Q\omega(y,C_1\sigma)\,dy+\eta\,.$$

Thanks to (4.18), the thesis follows sending first $\sigma\to0$, and then $\eta\to0$. □

4.3. *The lower bound*

We now investigate the Γ-lim inf inequality still through the blow-up method. In contrast with Lemma 4.2 we will distinguish energies with superlinear growth and energies with linear growth.

4.3.1. *The case of superlinear growth*

The case $p>1$ is based on an equi-integrability result known as Decomposition Lemma [27, Lemma 1.2], which allows to consider sequences with p-equi-integrable gradients. It enables to use properties valid up to sets where the energy remains small.

Lemma 4.3. *Assume $p\in(1,+\infty)$. Then $\mathcal{F}(u)\geq\mathcal{F}_{\rm hom}(u)$ for every $u\in W^{1,p}(\Omega;\mathcal{M})$.*

Proof. Let $u\in W^{1,p}(\Omega;\mathcal{M})$. By a standard diagonal argument, we first obtain a subsequence $\{\varepsilon_n\}$ (not relabeled) and $\{u_n\}\subset W^{1,p}(\Omega;\mathcal{M})$ such that $u_n\to u$ in $L^p(\Omega;\mathbb{R}^d)$ and

$$\mathcal{F}(u)=\lim_{n\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_n},\nabla u_n\right)dx<+\infty\,.$$

Define the sequence of nonnegative Radon measures

$$\mu_n:=f\left(\frac{\cdot}{\varepsilon_n},\nabla u_n\right)\mathcal{L}^N\llcorner\,\Omega\,.$$

Extracting a further subsequence if necessary, we may assume that there exists a nonnegative Radon measure $\mu\in\mathcal{M}(\Omega)$ such that $\mu_n\overset{*}{\rightharpoonup}\mu$ in $\mathcal{M}(\Omega)$. Using Lebesgue Differentiation Theorem one can split μ into the sum of two mutually disjoint nonnegative measures $\mu = \mu^a + \mu^s$ where $\mu^a \ll \mathcal{L}^N$ and μ^s is singular with respect to $\mathcal{L}^N$. Since $\mu^a(\Omega) \leq \mu(\Omega) \leq \mathcal{F}(u)$, it is enough to check that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq T f_{\rm hom}(u(x_0), \nabla u(x_0)) \quad \text{for } \mathcal{L}^N\text{-a.e. } x_0 \in \Omega\,.$$

Step 1. Select a point $x_0 \in \Omega$ which is a Lebesgue point of u and ∇u, a point of approximate differentiability of u (so that $u(x_0) \in \mathcal{M}$, $\nabla u(x_0) \in [T_{u(x_0)}(\mathcal{M})]^N$), and such that the Radon-Nikodým derivative of μ with respect to the Lebesgue measure $\mathcal{L}^N$ exists and is finite. Note that almost every points x_0 in Ω satisfy these properties. As in the proof of Lemma 4.2, set $s_0 := u(x_0)$ and $\xi_0 := \nabla u(x_0)$.

Let $\{\rho_k\} \searrow 0^+$ be such that $\mu(\partial Q(x_0, \rho_k)) = 0$ for every $k \in \mathbb{N}$. Using the integrand $\bar f$ defined in (3.1) one obtains

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{k\to+\infty} \frac{\mu(Q(x_0,\rho_k))}{\rho_k^N}\\ &= \lim_{k\to+\infty}\lim_{n\to+\infty} \frac{\mu_n(Q(x_0,\rho_k))}{\rho_k^N}\\ &= \lim_{k\to+\infty}\lim_{n\to+\infty} \int_Q f\left(\frac{x_0+\rho_k y}{\varepsilon_n}, \nabla u_n(x_0+\rho_k y)\right) dy\\ &= \lim_{k\to+\infty}\lim_{n\to+\infty} \int_Q \bar f\left(\frac{x_0+\rho_k y}{\varepsilon_n}, u_n(x_0+\rho_k y), \nabla u_n(x_0+\rho_k y)\right) dy\\ &= \lim_{k\to+\infty}\lim_{n\to+\infty} \int_Q \bar f\left(\frac{x_0+\rho_k y}{\varepsilon_n}, s_0+\rho_k v_{n,k}(y), \nabla v_{n,k}(y)\right) dy\,, \end{aligned} \tag{4.21}$$

where we have set $v_{n,k}(y) := \big[u_n(x_0+\rho_k y) - s_0\big]/\rho_k$. Note that since x_0 is a point of approximate differentiability of u and $u_n \to u$ in $L^p(\Omega;\mathbb{R}^d)$, we have

$$\lim_{k\to+\infty}\lim_{n\to+\infty} \int_Q |v_{n,k}(y) - \xi_0\, y|^p\, dy = \lim_{k\to+\infty} \int_{Q(x_0,\rho_k)} \frac{|u(y) - s_0 - \xi_0\,(y-x_0)|^p}{\rho_k^{N+p}}\, dy = 0\,. \tag{4.22}$$

Hence one can find a diagonal sequence $\varepsilon_k := \varepsilon_{n_k} < \rho_k^2$ such that, setting $v_k(y) := v_{n_k,k}(y)$ and $v_0(y) := \xi_0\, y$, $v_k \to v_0$ in $L^p(Q;\mathbb{R}^d)$ and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k\to+\infty} \int_Q \bar f\left(\frac{x_0+\rho_k y}{\varepsilon_k}, s_0+\rho_k v_k(y), \nabla v_k(y)\right) dy\,. \tag{4.23}$$

Next observe that $\{\nabla v_k\}$ is bounded in $L^p(Q;\mathbb{R}^{d\times N})$ thanks to the coercivity condition (3.2). By the Decomposition Lemma [27, Lemma 1.2] we now find a sequence $\{\bar v_k\} \subset W^{1,\infty}(Q;\mathbb{R}^d)$ such that $\bar v_k = v_0$ on a neighborhood of ∂Q, $\bar v_k \to v_0$ in $L^p(Q;\mathbb{R}^d)$, the sequence of gradients $\{|\nabla \bar v_k|^p\}$ is equi-integrable, and

$$\begin{aligned}\lim_{k\to+\infty} \int_Q \bar f\left(\frac{x_0+\rho_k y}{\varepsilon_k}, s_0+\rho_k v_k(y), \nabla v_k(y)\right) dy&\\ \geq \limsup_{k\to+\infty} \int_Q \bar f\left(\frac{x_0+\rho_k y}{\varepsilon_k}, s_0+\rho_k v_k(y), \nabla \bar v_k(y)\right) dy&\,. \end{aligned} \tag{4.24}$$

Step 2. Write

$$\frac{x_0}{\varepsilon_k} = m_k + s_k \quad \text{with } m_k \in \mathbb{Z}^N \text{ and } s_k \in [0,1)^N\,,$$

and define

$$x_k := \frac{\varepsilon_k}{\rho_k} s_k \to 0 \quad \text{and} \quad \delta_k := \varepsilon_k/\rho_k \to 0\,. \tag{4.25}$$

By the 1-periodicity of $\bar f$ with respect to its first variable, (4.23) and (4.24), we infer

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \limsup_{k\to+\infty} \int_Q \bar f\left(\frac{x_k+y}{\delta_k}, s_0+\rho_k v_k(y), \nabla \bar v_k(y)\right) dy\\ &\geq \limsup_{k\to+\infty} \int_{x_k+Q} \bar f\left(\frac{y}{\delta_k}, s_0+\rho_k v_k(y-x_k), \nabla \bar v_k(y-x_k)\right) dy\,. \end{aligned} \tag{4.26}$$

Extend v_k by 0, and $\bar{v}_k$ by v_0 to the whole $\mathbb{R}^N$. As $x_k \to 0$ it follows that $\mathcal{L}^N((Q-x_k)\triangle Q) \to 0$, and the equi-integrability of $\{|\nabla \bar{v}_k|^p\}$ together with the p-growth condition (3.2) implies

$$\int_{Q\triangle(x_k+Q)} \bar{f}\left(\frac{y}{\delta_k}, s_0+\rho_k v_k(y-x_k), \nabla \bar{v}_k(y-x_k)\right) dy \le \beta' \int_{(Q-x_k)\triangle Q} (1+|\nabla \bar{v}_k|^p)\, dy \to 0\,.$$

Hence (4.26) yields

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge \limsup_{k\to+\infty} \int_Q \bar{f}\left(\frac{y}{\delta_k}, s_0+\rho_k w_k, \nabla \bar{w}_k\right) dy\,, \tag{4.27}$$

where $w_k(y) := v_k(y-x_k)$ and $\bar{w}_k(y) := \bar{v}_k(y-x_k)$ converge to v_0 in $L^p(Q;\mathbb{R}^d)$, and $\{|\nabla \bar{w}_k|^p\}$ is equi-integrable as well.

Step 3. For $M>1$ and $k\in\mathbb{N}$, consider the set $E_{M,k} := \{x\in Q : |\nabla \bar{w}_k| \le M\}$. By Chebyschev inequality, (4.27) and (3.2), $\mathcal{L}^N(Q\setminus E_{M,k}) \le C/M^p$ for some constant $C>0$ independent of k and M.

By the Scorza-Dragoni Theorem (see [22], p. 235), for any $\eta>0$, we may find a compact set $K_\eta \subset \overline{Q}$ such that $\mathcal{L}^N(\overline{Q}\setminus K_\eta) < \eta$ and $f : K_\eta \times \mathbb{R}^{d\times N} \to [0,+\infty)$ is continuous. In particular the restriction of $\bar{f}(\cdot,s,\cdot)$ to $K_\eta \times B^{d\times N}(0,M)$ is uniformly continuous for every $s\in\mathcal{M}$. Therefore the function $\Psi_{\eta,M} : [0,+\infty) \to [0,+\infty)$ defined by

$$\Psi_{\eta,M}(t) = \sup\left\{ |f(y,\xi)-f(y,\xi')| \;:\; y\in K_\eta,\ \xi,\,\xi' \in B^{d\times N}(0,M)\,,\ |\xi-\xi'|\le t \right\},$$

is continuous, satisfies $\Psi_{\eta,M}(0)=0$, and is bounded. In view of (3.1), we have

$$|\bar{f}(y,s_1,\xi)-\bar{f}(y,s_2,\xi)| \le \Psi_{\eta,M}\big(M|\mathbf{P}_{s_1}-\mathbf{P}_{s_2}|\big) + C_M|\mathbf{P}_{s_1}-\mathbf{P}_{s_2}| =: \tilde{\Psi}_{\eta,M}\big(|\mathbf{P}_{s_1}-\mathbf{P}_{s_2}|\big)$$

for every $y\in K_\eta$, $s_1,s_2\in\mathcal{M}$ and $\xi\in B^{d\times N}(0,M)$, where the constant $C_M>0$ only depends on M and p. Define

$$K_\eta^{\rm per} := \bigcup_{\ell\in\mathbb{Z}^N} (\ell+K_\eta)\,.$$

Since $\bar{f}$ is 1-periodic in the first variable,

$$|\bar{f}(y,s_1,\xi)-\bar{f}(y,s_2,\xi)| \le \tilde{\Psi}_{\eta,M}\big(|\mathbf{P}_{s_1}-\mathbf{P}_{s_2}|\big) \tag{4.28}$$

for every $y\in K_\eta^{\rm per}$, $s_1,s_2\in\mathcal{M}$ and $\xi\in B^{d\times N}(0,M)$. From (4.27) and (4.28) it follows that

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &\ge \limsup_{k\to+\infty} \int_{E_{M,k}\cap(\delta_k K_\eta^{\rm per})} \bar{f}\left(\frac{y}{\delta_k}, s_0+\rho_k w_k, \nabla \bar{w}_k\right) dy\\ &\ge \limsup_{k\to+\infty} \int_{E_{M,k}\cap(\delta_k K_\eta^{\rm per})} \bar{f}\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy - \limsup_{k\to+\infty}\int_Q \tilde{\Psi}_{\eta,M}\big(|\mathbf{P}_{s_0+\rho_k w_k(y)}-\mathbf{P}_{s_0}|\big)\, dy\,.\end{aligned}$$

Since $\tilde{\Psi}_{\eta,M}$ is continuous and bounded, $\tilde{\Psi}_{\eta,M}(0)=0$, and (up to a subsequence) $|\mathbf{P}_{s_0+\rho_k w_k(y)}-\mathbf{P}_{s_0}| \to 0$ for a.e. $y\in Q$, we obtain by Dominated Convergence that

$$\lim_{k\to+\infty}\int_Q \tilde{\Psi}_{\eta,M}\big(|\mathbf{P}_{s_0+\rho_k w_k(y)}-\mathbf{P}_{s_0}|\big)\, dy = 0\,,$$

and thus

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge \limsup_{k\to+\infty} \int_{E_{M,k}\cap(\delta_k K_\eta^{\rm per})} \bar{f}\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy\,. \tag{4.29}$$

From the p-growth condition (3.2) and the Riemann-Lebesgue Lemma, we deduce that

$$\begin{aligned}\limsup_{k\to+\infty} \int_{E_{M,k}\setminus(\delta_k K_\eta^{\rm per})} \bar{f}\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy &\le \limsup_{k\to+\infty} \beta'(1+M^p)\mathcal{L}^N(Q\setminus(\delta_k K_\eta^{\rm per})) =\\ &= \beta'(1+M^p)\mathcal{L}^N(Q\setminus K_\eta) \le \beta'(1+M^p)\eta\,.\end{aligned}$$

Hence (4.29) yields

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge \limsup_{k\to+\infty} \int_{E_{M,k}} \bar{f}\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy - \beta'(1+M^p)\eta\,,$$

and sending $\eta \to 0$, we derive

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{k\to+\infty} \int_{E_{M,k}} \bar{f}\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy\,. \tag{4.30}$$

Since $\mathcal{L}^N(Q \setminus E_{M,k}) \to 0$ as $M \to +\infty$ (uniformly with respect to k), the equi-integrability of $\{|\nabla \bar{w}_k|^p\}$ and the p-growth condition (3.2) imply

$$\sup_{k\in\mathbb{N}} \int_{Q\setminus E_{M,k}} \bar{f}\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy \leq \beta' \sup_{k\in\mathbb{N}} \int_{Q\setminus E_{M,k}} (1 + |\nabla \bar{w}_k|^p)\, dy \longrightarrow 0 \quad \text{as } M \to +\infty\,.$$

Plugging this estimate in (4.30) leads to

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{k\to+\infty} \int_Q \bar{f}\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy\,.$$

Since $\bar{w}_k \to v_0$ in $L^p(Q;\mathbb{R}^d)$, we can invoke standard homogenization results (see, *e.g.,* [14, Theorem 14.5]) to infer that

$$\limsup_{k\to+\infty} \int_Q \bar{f}\left(\frac{y}{\delta_k}, s_0, \nabla \bar{w}_k\right) dy \geq \int_Q \bar{f}_{\rm hom}(s_0, \nabla v_0)\, dy = \bar{f}_{\rm hom}(s_0, \xi_0)\,.$$

In view of Proposition 3.1 we finally conclude

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \bar{f}_{\rm hom}(s_0, \xi_0) = Tf_{\rm hom}(s_0, \xi_0)\,,$$

and the proof is complete. □

4.3.2. *The case of linear growth*

We now treat the case $p = 1$ assuming that the function u belongs to $W^{1,1}(\Omega;\mathcal{M})$. The general case where $u \in BV(\Omega;\mathcal{M})$ will be discussed in the next section. In contrast with the case $p > 1$, there is no equi-integrability result as the Decomposition Lemma. We follow here the approach of [25].

Lemma 4.4. *Assume $p = 1$. Then $\mathcal{F}(u) \geq \mathcal{F}_{\rm hom}(u)$ for every $u \in W^{1,1}(\Omega;\mathcal{M})$.*

Proof. Let $u \in W^{1,1}(\Omega;\mathcal{M})$. By a standard diagonal argument, we first obtain a subsequence $\{\varepsilon_n\}$ (not relabeled) and $\{u_n\} \subset W^{1,1}(\Omega;\mathcal{M})$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$ and

$$\mathcal{F}(u) = \lim_{n\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) dx < +\infty\,.$$

Up to the extraction of a further subsequence, we may assume that there exists a nonnegative Radon measure $\mu \in \mathcal{M}(\Omega)$ such that

$$f\left(\frac{\cdot}{\varepsilon_n}, \nabla u_n\right) \mathcal{L}^N \llcorner\, \Omega \xrightharpoonup{*} \mu \quad \text{in } \mathcal{M}(\Omega)\,. \tag{4.31}$$

Hence it is enough to prove that $\mu(\Omega) \geq \mathcal{F}_{\rm hom}(u)$. As in the proof of Lemma 4.3, it suffices to show that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq Tf_{\rm hom}(u(x_0), \nabla u(x_0)) \quad \text{for } \mathcal{L}^N\text{-a.e. } x_0 \in \Omega\,.$$

The proof will be divided into three steps. We first apply the blow-up method which reduces the study to affine limiting functions. Then we reproduce the argument of [25] which enables us to replace the original sequence by a uniformly converging one without increasing the energy. We will conclude using a classical homogenization result.

Step 1. Select a point $x_0 \in \Omega$ which is a Lebesgue point of u and ∇u, a point of approximate differentiability of u (so that $u(x_0) \in \mathcal{M}$, $\nabla u(x_0) \in [T_{u(x_0)}(\mathcal{M})]^N$) and such that the Radon-Nikodým derivative of μ with respect to the Lebesgue measure $\mathcal{L}^N$ exists and is finite. Note that $\mathcal{L}^N$-almost every points x_0 in Ω satisfy these properties. We write $s_0 := u(x_0)$ and $\xi_0 := \nabla u(x_0)$.

Up to a subsequence, we may assume that there exists a nonnegative Radon measure $\lambda \in \mathcal{M}(\Omega)$ such that $(1+|\nabla u_n|)\mathcal{L}^N \llcorner \Omega \xrightharpoonup{*} \lambda$ in $\mathcal{M}(\Omega)$. Consider a sequence $\{\rho_k\} \searrow 0^+$ such that $Q(x_0, 2\rho_k) \subset \Omega$ and $\mu(\partial Q(x_0,\rho_k)) = \lambda(\partial Q(x_0,\rho_k)) = 0$ for each $k \in \mathbb{N}$. Then (4.31) yields

$$\mu(Q(x_0,\rho_k)) = \lim_{n\to+\infty} \int_{Q(x_0,\rho_k)} f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) dx\,. \tag{4.32}$$

Set $\tau_n := \varepsilon_n \lfloor x_0/\varepsilon_n \rfloor \in \varepsilon_n \mathbb{Z}^N$. Since $\tau_n \to x_0$, given $r \in (1,2)$ we have $Q(\tau_n,\rho_k) \subset Q(x_0, r\rho_k)$ whenever n is large enough, and we may define for $x \in Q(0,\rho_k)$, $v_n(x) := u_n(x+\tau_n)$. By continuity of the translation in L^1, we get that

$$\begin{aligned}\int_{Q(0,\rho_k)} |v_n(x) - u(x+x_0)|\,dx &= \int_{Q(\tau_n,\rho_k)} |u_n(x) - u(x+x_0-\tau_n)|\,dx \\ &\leq \int_{Q(x_0,r\rho_k)} |u_n(x) - u(x+x_0-\tau_n)|\,dx \xrightarrow[n\to+\infty]{} 0\,.\end{aligned} \tag{4.33}$$

Changing variable in (4.32) and using the periodicity condition (H_1) of $f(\cdot,\xi)$ and the growth condition (H_2), we are led to

$$\begin{aligned}\mu(Q(x_0,\rho_k)) &= \lim_{n\to+\infty} \int_{Q(x_0-\tau_n,\rho_k)} f\left(\frac{x+\tau_n}{\varepsilon_n}, \nabla u_n(x+\tau_n)\right) dx \\ &= \lim_{n\to+\infty} \int_{Q(x_0-\tau_n,\rho_k)} f\left(\frac{x}{\varepsilon_n}, \nabla v_n\right) dx \\ &\geq \limsup_{n\to+\infty} \int_{Q(0,\rho_k)} f\left(\frac{x}{\varepsilon_n}, \nabla v_n\right) dx - \beta \limsup_{n\to+\infty} \int_{Q(\tau_n,\rho_k)\setminus Q(x_0,\rho_k)} (1+|\nabla u_n|)\,dx\,.\end{aligned} \tag{4.34}$$

On the other hand, by our choice of ρ_k,

$$\begin{aligned}\limsup_{n\to+\infty} \int_{Q(\tau_n,\rho_k)\setminus Q(x_0,\rho_k)} (1+|\nabla u_n|)\,dx &\leq \limsup_{r\to1^+}\, \limsup_{n\to+\infty} \int_{Q(x_0,r\rho_k)\setminus Q(x_0,\rho_k)} (1+|\nabla u_n|)\,dx \\ &\leq \limsup_{r\to1^+} \lambda\left(\overline{Q(x_0,r\rho_k)\setminus Q(x_0,\rho_k)}\right) \\ &\leq \lambda(\partial Q(x_0,\rho_k)) = 0\,,\end{aligned}$$

so that the last term in (4.34) vanishes. Hence

$$\mu(Q(x_0,\rho_k)) \geq \limsup_{n\to+\infty} \int_{Q(0,\rho_k)} f\left(\frac{x}{\varepsilon_n}, \nabla v_n\right) dx\,,$$

where $\{v_n\} \subset W^{1,1}(Q(0,\rho_k);\mathcal{M})$ satisfies $v_n \to u(x_0+\cdot)$ in $L^1(Q(0,\rho_k);\mathbb{R}^d)$ by (4.33).

Now we consider for every n, a sequence $\{v_{n,j}\} \subset \mathcal{C}^\infty(\overline{Q(0,\rho_k)};\mathbb{R}^d)$ such that $v_{n,j} \to v_n$ in $W^{1,1}(Q(0,\rho_k);\mathbb{R}^d)$, $v_{n,j} \to v_n$ and $\nabla v_{n,j} \to \nabla v_n$ a.e. in $Q(0,\rho_k)$ as $j \to +\infty$ (we emphasize that in general, $v_{n,j}$ is not $\mathcal{M}$-valued). Considering the integrand g given by Lemma 3.1, one may check

$$\lim_{j\to+\infty} \int_{Q(0,\rho_k)} g\left(\frac{x}{\varepsilon_n}, v_{n,j}, \nabla v_{n,j}\right) dx = \int_{Q(0,\rho_k)} g\left(\frac{x}{\varepsilon_n}, v_n, \nabla v_n\right) dx = \int_{Q(0,\rho_k)} f\left(\frac{x}{\varepsilon_n}, \nabla v_n\right) dx\,,$$

so that we can find a diagonal sequence $\bar{v}_n := v_{n,j_n}$ satisfying $\bar{v}_n \to u(x_0+\cdot)$ in $L^1(Q(0,\rho_k);\mathbb{R}^d)$ and

$$\mu(Q(x_0,\rho_k)) \geq \limsup_{n\to+\infty} \int_{Q(0,\rho_k)} g\left(\frac{x}{\varepsilon_n}, \bar{v}_n, \nabla \bar{v}_n\right) dx\,. \tag{4.35}$$

Changing variable in (4.35) yields

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \limsup_{k\to+\infty}\, \limsup_{n\to+\infty} \int_Q g\left(\frac{\rho_k\, x}{\varepsilon_n}, \bar{v}_n(\rho_k\, x), \nabla\bar{v}_n(\rho_k\, x)\right) dx \\ &= \limsup_{k\to+\infty} \limsup_{n\to+\infty} \int_Q g\left(\frac{\rho_k\, x}{\varepsilon_n}, s_0 + \rho_k\, w_{n,k}, \nabla w_{n,k}\right) dx\,,\end{aligned} \tag{4.36}$$

where we have set $w_{n,k}(x) := \big[\bar v_n(\rho_k\, x) - s_0\big]/\rho_k$. Since x_0 is a point of approximate differentiability of u and $\bar v_n \to u(x_0+\cdot)$ in $L^1(Q(0,\rho_k);\mathbb{R}^d)$, we have

$$\lim_{k\to+\infty}\lim_{n\to+\infty}\int_Q |w_{n,k}(x)-\xi_0\, x|\,dx = \lim_{k\to+\infty}\int_{Q(x_0,\rho_k)} \frac{|u(y)-s_0-\xi_0\,(y-x_0)|}{\rho_k^{N+1}}\,dy = 0\,. \tag{4.37}$$

In view of (4.36) and (4.37), we can find a diagonal sequence $\varepsilon_{n_k} < \rho_k^2$ such that $w_k := w_{n_k,k} \to w_0$ in $L^1(Q;\mathbb{R}^d)$ with $w_0(x) := \xi_0\, x$, and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{k\to+\infty}\int_Q g\left(\frac{x}{\delta_k}, s_0+\rho_k\, w_k, \nabla w_k\right)dx\,, \tag{4.38}$$

where $\delta_k := \varepsilon_{n_k}/\rho_k \to 0$.

Step 2. We now argue as in Step 3 of the proof of [25, Theorem 2.1] to show that there exists a sequence $\{\overline{w}_k\}\subset W^{1,\infty}(Q;\mathbb{R}^d)$ such that $\overline{w}_k \to w_0$ in $L^\infty(Q;\mathbb{R}^d)$, $\{\overline{w}_k\}$ is uniformly bounded in $W^{1,1}(Q;\mathbb{R}^d)$ and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{k\to+\infty}\int_Q g\left(\frac{x}{\delta_k}, s_0+\rho_k\, \overline{w}_k, \nabla \overline{w}_k\right)dx\,. \tag{4.39}$$

Given $0<s<t$, let $\zeta_{s,t}\in\mathcal{C}_c^\infty(\mathbb{R};[0,1])$ be a cut-off function satisfying $\zeta_{s,t}(\tau)=1$ if $|\tau|\leq s$, $\zeta_{s,t}(\tau)=0$ if $|\tau|\geq t$ and $|\zeta'_{s,t}|\leq C/(t-s)$. Define

$$w^k_{s,t} := w_0 + \zeta_{s,t}(|w_k-w_0|)(w_k-w_0)\,.$$

Obviously,

$$\|w^k_{s,t}-w_0\|_{L^\infty(Q;\mathbb{R}^d)} \leq t\,, \tag{4.40}$$

and the Chain Rule formula gives

$$\nabla w^k_{s,t} = \nabla w_0 + \zeta_{s,t}\big(|w_k-w_0|\big)(\nabla w_k - \nabla w_0) + \zeta'_{s,t}\big(|w_k-w_0|\big)(w_k-w_0)\otimes\nabla|w_k-w_0|\,. \tag{4.41}$$

In particular,

$$\begin{aligned}\int_Q g\left(\frac{x}{\delta_k}, s_0+\rho_k\, w^k_{s,t}, \nabla w^k_{s,t}\right)dx &= \int_{\{|w_k-w_0|\leq s\}} g\left(\frac{x}{\delta_k}, s_0+\rho_k\, w_k, \nabla w_k\right)dx\,+\\ &+\int_{\{s<|w_k-w_0|\leq t\}} g\left(\frac{x}{\delta_k}, s_0+\rho_k\, w^k_{s,t}, \nabla w^k_{s,t}\right)dx + \int_{\{|w_k-w_0|>t\}} g\left(\frac{x}{\delta_k}, s_0+\rho_k\, w_0, \xi_0\right)dx\,.\end{aligned} \tag{4.42}$$

From the growth condition (3.7), we infer that

$$\int_{\{|w_k-w_0|>t\}} g\left(\frac{x}{\delta_k}, s_0+\rho_k\, w_0, \xi_0\right)dx \leq \beta'(1+|\xi_0|)\,\mathcal{L}^N(\{|w_k-w_0|>t\})\,, \tag{4.43}$$

and (4.41) yields

$$\begin{aligned}\int_{\{s<|w_k-w_0|\leq t\}} g\left(\frac{x}{\delta_k}, s_0+\rho_k\, w^k_{s,t}, \nabla w^k_{s,t}\right)dx &\leq C\int_{\{s<|w_k-w_0|\leq t\}}(1+|\nabla w_k-\xi_0|)\,dx\,+\\ &+\frac{C}{t-s}\int_{\{s<|w_k-w_0|\leq t\}}|w_k-w_0|\,\big|\nabla|w_k-w_0|\,\big|\,dx\,.\end{aligned} \tag{4.44}$$

Observe that for $\mathcal{L}^1$-a.e. $t>0$,

$$\lim_{s\to t^-}\int_{\{s<|w_k-w_0|\leq t\}}(1+|\nabla w_k-\xi_0|)\,dy \leq C_k\,\lim_{s\to t^-}\mathcal{L}^N\big(\{s<|w_k-w_0|\leq t\}\big) = 0\,, \tag{4.45}$$

and by the Coarea formula,

$$\begin{aligned}\lim_{s\to t^-}\frac{1}{t-s}\int_{\{s<|w_k-w_0|\leq t\}}|w_k-w_0|\,\big|\nabla|w_k-w_0|\,\big|\,dx &= \lim_{s\to t^-}\frac{1}{t-s}\int_s^t \tau\mathcal{H}^{N-1}(\{|w_k-w_0|=\tau\})\,d\tau\\ &= t\mathcal{H}^{N-1}(\{|w_k-w_0|=t\})\,.\end{aligned} \tag{4.46}$$

Moreover, in view of (3.7) and (4.38) we infer that

$$\int_Q \big|\nabla|w_k-w_0|\,\big|\,dx \leq C\int_Q(1+|\nabla w_k|)\,dy \leq C_0\,.$$

Applying [25, Lemma 2.6], there exists $t_k \in \left(\|w_k - w_0\|^{1/2}_{L^1(Q;\mathbb{R}^d)}, \|w_k - w_0\|^{1/3}_{L^1(Q;\mathbb{R}^d)}\right)$ such that (4.45) and (4.46) hold with $t = t_k$, and

$$t_k \mathcal{H}^{N-1}(\{|w_k - w_0| = t_k\}) \leq \frac{C_0}{\ln\left(\|w_k - w_0\|^{-1/6}_{L^1(Q;\mathbb{R}^d)}\right)} . \tag{4.47}$$

According to (4.45), (4.46) and (4.47), there exists $s_k \in \left(\|w_k - w_0\|^{1/2}_{L^1(Q;\mathbb{R}^d)}, t_k\right)$ such that

$$\int_{\{s_k < |w_k - w_0| \leq t_k\}} (1 + |\nabla w_k - \xi_0|)\, dx \leq \frac{1}{k} , \tag{4.48}$$

and

$$\frac{1}{t_k - s_k} \int_{\{s_k < |w_k - w_0| \leq t_k\}} |w_k - w_0|\, \big|\nabla |w_k - w_0|\, \big|\, dx \leq \frac{C_0}{\ln\left(\|w_k - w_0\|^{-1/6}_{L^1(Q;\mathbb{R}^d)}\right)} + \frac{1}{k} , \tag{4.49}$$

while (4.43) together with Chebyshev inequality yields

$$\int_{\{|w_k - w_0| > t_k\}} g\left(\frac{x}{\delta_k}, s_0 + \rho_k\, w_0, \xi_0\right) dy \leq C\|w_k - w_0\|^{1/2}_{L^1(Q;\mathbb{R}^d)} . \tag{4.50}$$

Define now $\overline{w}_k := w^k_{s_k,t_k}$ so that $\overline{w}_k \to w_0$ in $L^\infty(Q;\mathbb{R}^d)$ by (4.40). Moreover, gathering (4.42), (4.44), (4.48), (4.49) and (4.50), we deduce

$$\limsup_{k\to+\infty} \int_Q g\left(\frac{x}{\delta_k}, s_0 + \rho_k\, \overline{w}_k, \nabla \overline{w}_k\right) dx \leq \limsup_{k\to+\infty} \int_Q g\left(\frac{x}{\delta_k}, s_0 + \rho_k\, w_k, \nabla w_k\right) dx ,$$

which proves (4.39). The fact that $\{\nabla \overline{w}_k\}$ is uniformly bounded in $L^1(Q;\mathbb{R}^{d\times N})$ is a consequence of (4.39) and the coercivity condition (3.7).

Step 3. Since $\{\|\overline{w}_k\|_{L^\infty(Q;\mathbb{R}^d)}\}$ and $\{\|\nabla\overline{w}_k\|_{L^1(Q;\mathbb{R}^{d\times N})}\}$ are uniformly bounded, we derive from (3.8) that

$$\lim_{k\to+\infty} \int_Q \left| g\left(\frac{x}{\delta_k}, s_0 + \rho_k\, \overline{w}_k, \nabla \overline{w}_k\right) - g\left(\frac{x}{\delta_k}, s_0, \nabla \overline{w}_k\right)\right| dx = 0 .$$

In view of (4.39), it leads to

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \lim_{k\to+\infty} \int_Q g\left(\frac{x}{\delta_k}, s_0, \nabla \overline{w}_k\right) dx .$$

Using standard homogenization results (see *e.g.,* [14, Theorem 14.5]) together with (3.11), we finally conclude that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq g_{\rm hom}(s_0, \xi_0) = T f_{\rm hom}(s_0, \xi_0) ,$$

which completes the proof of the lemma. □

4.4. *Proof of Theorem 1.1 and Theorem 1.2*

Since $L^p(\Omega;\mathbb{R}^d)$ is separable ($1 \leq p < +\infty$), there exists a subsequence $\{\varepsilon_{n_k}\}$ such that $\mathcal{F}$ is the Γ-limit of $\{\mathcal{F}_{\varepsilon_{n_k}}\}$ for the strong $L^p(\Omega;\mathbb{R}^d)$-topology (see [19, Theorem 8.5]).

Case $p > 1$. In view of (H_2) and the closure of the pointwise constraint under strong L^p-convergence, we have $\mathcal{F}(u) < +\infty$ if and only if $u \in W^{1,p}(\Omega;\mathcal{M})$. Hence, as a consequence of Lemmas 4.2 and 4.3, the functionals $\{\mathcal{F}_{\varepsilon_{n_k}}\}$ Γ-converge to $\mathcal{F}_{\rm hom}$ in $L^p(\Omega;\mathbb{R}^d)$. Since the Γ-limit does not depend on the extracted subsequence, we get in light of [19, Proposition 8.3] that the whole sequence $\{\mathcal{F}_{\varepsilon_n}\}$ Γ-converges to $\mathcal{F}_{\rm hom}$.

Case $p = 1$. As a consequence of Lemmas 4.2 and 4.4, the functionals $\{\mathcal{F}_{\varepsilon_{n_k}}\}$ Γ-converge to $\mathcal{F}_{\rm hom}$ in $W^{1,1}(\Omega;\mathcal{M})$. Again, the Γ-limit does not depend on the extracted subsequence, so that the whole sequence $\{\mathcal{F}_{\varepsilon_n}\}$ Γ-converges to $\mathcal{F}_{\rm hom}$ in $W^{1,1}(\Omega;\mathcal{M})$. □

5. Homogenization in BV spaces

In this section we extend Theorem 1.2 to the BV setting. Indeed Theorem 1.2 only gives the description of the Γ-limit on $W^{1,1}(\Omega;\mathcal{M})$, although its entire domain is exactly the space $BV(\Omega;\mathcal{M})$.

5.1. *Localization*

As in the previous section, we consider an arbitrary given sequence $\{\varepsilon_n\} \searrow 0^+$ and we localize the functionals $\{\mathcal{F}_{\varepsilon_n}\}_{n\in\mathbb{N}}$ on the family $\mathcal{A}(\Omega)$, *i.e.*, for every $u \in L^1(\Omega;\mathbb{R}^d)$ and every $A \in \mathcal{A}(\Omega)$, we set

$$\mathcal{F}_{\varepsilon_n}(u,A) := \begin{cases} \displaystyle\int_A f\left(\frac{x}{\varepsilon_n}, \nabla u\right) dx & \text{if } u \in W^{1,1}(A;\mathcal{M})\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Next we define for $u \in L^1(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$,

$$\mathcal{F}(u,A) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} \mathcal{F}_{\varepsilon_n}(u_n,A) \,:\, u_n \to u \text{ in } L^1(A;\mathbb{R}^d) \right\}.$$

Note that $\mathcal{F}(u,\cdot)$ is an increasing set function for every $u \in L^1(\Omega;\mathbb{R}^d)$ and that $\mathcal{F}(\cdot,A)$ is lower semicontinuous with respect to the strong $L^1(A;\mathbb{R}^d)$-convergence for every $A \in \mathcal{A}(\Omega)$.

Since $L^1(A;\mathbb{R}^d)$ is separable, [19, Theorem 8.5] and a diagonalization argument bring the existence of a subsequence (still denoted $\{\varepsilon_n\}$) such that $\mathcal{F}(\cdot,A)$ is the Γ-limit of $\mathcal{F}_{\varepsilon_n}(\cdot,A)$ for the strong $L^1(A;\mathbb{R}^d)$-topology for every $A \in \mathcal{R}(\Omega)$ (or $A = \Omega$).

We have the following locality property of the Γ-limit which, in the BV setting, parallels Lemma 4.1.

Lemma 5.1. *For every $u \in BV(\Omega;\mathcal{M})$, the set function $\mathcal{F}(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure absolutely continuous with respect to $\mathcal{L}^N + |Du|$.*

Proof. Let $u \in BV(\Omega;\mathcal{M})$ and $A \in \mathcal{A}(\Omega)$. By Theorem 3.9 in [6], there exists a sequence $\{u_n\} \subset W^{1,1}(A;\mathbb{R}^d) \cap \mathcal{C}^\infty(A;\mathbb{R}^d)$ such that $u_n \to u$ in $L^1(A;\mathbb{R}^d)$ and $\int_A |\nabla u_n|\,dx \to |Du|(A)$. Moreover, this sequence is obtained by standard convolution arguments so that one may check that $u_n(x) \in \mathrm{co}(\mathcal{M})$ for a.e. $x \in A$ and every $n \in \mathbb{N}$. Applying Proposition 2.1 to u_n, we obtain a new sequence $\{w_n\} \subset W^{1,1}(A;\mathcal{M})$ satisfying

$$\int_A |\nabla w_n|\,dx \le C_\star \int_A |\nabla u_n|\,dx,$$

for some constant $C_\star > 0$ depending only on $\mathcal{M}$ and d. Then we easily infer from the construction of w_n that $w_n \to u$ in $L^1(A;\mathbb{R}^d)$. Taking $\{w_n\}$ as admissible sequence, we deduce in light of the growth condition (H_2) that

$$\mathcal{F}(u,A) \le \beta\big(\mathcal{L}^N(A) + C_\star|Du|(A)\big)\,.$$

We now prove that

$$\mathcal{F}(u,A) \le \mathcal{F}(u,B) + \mathcal{F}(u,A\setminus\overline{C})$$

for every A, B and $C \in \mathcal{A}(\Omega)$ satisfying $\overline{C} \subset B \subset A$. Then the measure property of $\mathcal{F}(u,\cdot)$ can be obtained as in the proof of Lemma 4.1 with minor modifications. For this reason, we shall omit it.

Let $R \in \mathcal{R}(\Omega)$ such that $C \subset\subset R \subset\subset B$ and consider $\{u_n\} \subset W^{1,1}(R;\mathcal{M})$ satisfying $u_n \to u$ in $L^1(R;\mathbb{R}^d)$ and

$$\lim_{n\to+\infty} \mathcal{F}_{\varepsilon_n}(u_n,R) = \mathcal{F}(u,R)\,. \tag{5.1}$$

Given $\eta > 0$ arbitrary, there exists a sequence $\{v_n\} \subset W^{1,1}(A\setminus\overline{C};\mathcal{M})$ such that $v_n \to u$ in $L^1(A\setminus\overline{C};\mathbb{R}^d)$ and

$$\liminf_{n\to+\infty} \mathcal{F}_{\varepsilon_n}(v_n,A\setminus\overline{C}) \le \mathcal{F}(u,A\setminus\overline{C}) + \eta\,. \tag{5.2}$$

By Theorem 2.1, we can assume without loss of generality that $u_n \in \mathcal{D}(R;\mathcal{M})$ and $v_n \in \mathcal{D}(A \setminus \overline{C};\mathcal{M})$. Let $L := \mathrm{dist}(C, \partial R)$ and define for every $i \in \{0, \dots, n\}$,

$$R_i := \left\{ x \in R : \, \mathrm{dist}(x, \partial R) > \frac{iL}{n} \right\}.$$

Given $i \in \{0, \dots, n-1\}$, let $S_i := R_i \setminus \overline{R_{i+1}}$ and consider a cut-off function $\zeta_i \in \mathcal{C}_c^\infty(\Omega;[0,1])$ satisfying $\zeta_i(x) = 1$ for $x \in R_{i+1}$, $\zeta_i(x) = 0$ for $x \in \Omega \setminus R_i$ and $|\nabla \zeta_i| \leq 2n/L$. Define

$$z_{n,i} := \zeta_i u_n + (1 - \zeta_i) v_n \in W^{1,1}(A;\mathbb{R}^d).$$

If $\Pi_1(\mathcal{M}) \neq 0$, $z_{n,i}$ is smooth in $A \setminus \Sigma_{n,i}$ with $\Sigma_{n,i} \in \mathcal{S}$, while $z_{n,i}$ is smooth in A if $\Pi_1(\mathcal{M}) = 0$. Observe that $z_{n,i}(x) \in \mathrm{co}(\mathcal{M})$ for a.e. $x \in A$ and actually, $z_{n,i}$ fails to be $\mathcal{M}$-valued exactly in the set S_i. To get an admissible sequence, we project $z_{n,i}$ on $\mathcal{M}$ using Proposition 2.1. It yields a sequence $\{w_{n,i}\} \subset W^{1,1}(A;\mathcal{M})$ satisfying $w_{n,i} = z_{n,i}$ a.e. in $A \setminus S_i$,

$$\int_A |w_{n,i} - u| \, dx \leq \int_A |z_{n,i} - u| \, dx + C\mathcal{L}^N(S_i), \tag{5.3}$$

for some constant $C > 0$ depending only on the diameter of $\mathrm{co}(\mathcal{M})$, and

$$\int_{S_i} |\nabla w_{n,i}| \, dx \leq C_\star \int_{S_i} |\nabla z_{n,i}| \, dx \leq C_\star \int_{S_i} \Big(|\nabla u_n| + |\nabla v_n| + \frac{n}{2L} |u_n - v_n| \Big) \, dx.$$

Arguing exactly as in the proof of Lemma 4.1, we now find an index $i_n \in \{0, \dots, n-1\}$ such that

$$\begin{aligned} \mathcal{F}_{\varepsilon_n}(w_{n,i_n}, A) \leq \mathcal{F}_{\varepsilon_n}(u_n, R) + \mathcal{F}_{\varepsilon_n}(v_n, A \setminus \overline{C}) + \\ + C_0 \int_{R \setminus \overline{C}} |u_n - v_n| \, dx + \frac{C_0}{n} \sup_{k \in \mathbb{N}} \int_{R \setminus \overline{C}} (1 + |\nabla u_k| + |\nabla v_k|) \, dx, \end{aligned} \tag{5.4}$$

for some constant C_0 independent of n.

Set $f(t) := \mathcal{H}^{N-1}(\{x \in R : \mathrm{dist}(x, \partial R) = t\})$. A well know consequence of the Coarea formula yields (see, *e.g.*, [23, Lemma 3.2.34]),

$$\int_0^L f(t) \, dt = \mathcal{L}^N\big(\{x \in \mathbb{R}^N : 0 < \mathrm{dist}(x, \partial R) < L\}\big) < +\infty,$$

since R is bounded. In particular, $f \in L^1(0, L)$, and since $\mathcal{L}^1\big((i_n L/n, (i_n + 1)L/n)\big) = L/n \to 0$, we infer that

$$\mathcal{L}^N(S_{i_n}) = \int_{i_n L/n}^{(i_n+1)L/n} f(t) \, dt \to 0 \quad \text{as } n \to +\infty. \tag{5.5}$$

As a consequence of (5.3) and (5.5), $w_{n,i_n} \to u$ in $L^1(A;\mathbb{R}^d)$. Taking the lim inf in (5.4) and using (5.1) together with (5.2), we derive

$$\mathcal{F}(u, A) \leq \mathcal{F}(u, R) + \mathcal{F}(u, A \setminus \overline{C}) + \eta \leq \mathcal{F}(u, B) + \mathcal{F}(u, A \setminus \overline{C}) + \eta.$$

The conclusion follows from the arbitrariness of η. □

Remark 5.1. In view of Lemma 5.1, for every $u \in BV(\Omega;\mathcal{M})$, the set function $\mathcal{F}(u, \cdot)$ can be uniquely extended to a Radon measure on Ω. Such a measure is given by

$$\mathcal{F}(u, B) := \inf\big\{\mathcal{F}(u, A) \, : \, A \in \mathcal{A}(\Omega), \, B \subset A\big\},$$

for every $B \in \mathcal{B}(\Omega)$ (see, *e.g.*, [6, Theorem 1.53]).

5.2. *Integral representation on partitions*

Besides the locality of $\mathcal{F}(u,\cdot)$, another key point of the analysis is to prove an abstract integral representation on partitions. To get it as precise as possible, we first prove the translation invariance of the Γ-limit. It is expressed in the following lemma.

Lemma 5.2. *For every $u \in BV(\Omega;\mathcal{M})$, every $A \in \mathcal{A}(\Omega)$ and every $y \in \mathbb{R}^N$ such that $y + A \subset \Omega$, we have*

$$\mathcal{F}(\tau_y u, y + A) = \mathcal{F}(u, A)\,,$$

where $(\tau_y u)(x) := u(x - y)$.

Proof. Let $B \in \mathcal{A}(\Omega)$ be such that $B \subset\subset A$, and find $R \in \mathcal{R}(\Omega)$ satisfying $B \subset\subset R \subset A$. Then consider a sequence $\{u_n\} \subset W^{1,1}(R;\mathcal{M})$ such that $u_n \to u$ in $L^1(R;\mathbb{R}^d)$ and

$$\mathcal{F}(u,R) = \lim_{n\to+\infty} \int_R f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) dx\,. \tag{5.6}$$

Set $y_n := \varepsilon_n[y/\varepsilon_n]$ and note that $y_n \to y$. Hence, for n large enough, $y - y_n + B \subset R$ and we may define $v_n := \tau_{y_n} u_n \in W^{1,1}(y + B;\mathcal{M})$. From the continuity of the translation in L^1, we infer that $v_n \to \tau_y u$ in $L^1(y + B;\mathbb{R}^d)$. Thus $\{v_n\}$ is an admissible sequence for $\mathcal{F}(\tau_y u, y + B)$. Thanks to the periodicity condition (H_1) and (5.6),

$$\mathcal{F}(\tau_y u, y+B) \le \liminf_{n\to+\infty} \int_{y+B} f\left(\frac{x}{\varepsilon_n}, \nabla v_n\right) dx = \liminf_{n\to+\infty} \int_{y-y_n+B} f\left(\frac{x+y_n}{\varepsilon_n}, \nabla u_n\right) dx \le$$
$$\le \lim_{n\to+\infty} \int_R f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) dx = \mathcal{F}(u,R) \le \mathcal{F}(u,A)\,.$$

From the arbitrariness of $B \subset\subset A$, we deduce that $\mathcal{F}(\tau_y u, y + A) \le \mathcal{F}(u, A)$ by inner regularity. Finally, we observe

$$\mathcal{F}(\tau_y u, y + A) \ge \mathcal{F}(\tau_{-y}(\tau_y u), -y + (y + A)) \ge \mathcal{F}(u, A)\,,$$

and the proof is complete. □

We are now in position to prove the integral representation of the Γ-limit on partitions. The proof is based on the general result [4, Theorem 3.1] and follows an argument of [15].

Proposition 5.1. *There exists a unique function $K : \mathcal{M} \times \mathcal{M} \times \mathbb{S}^{N-1} \to [0,+\infty)$ continuous in the last variable and such that*
(i) *$K(a, b, \nu) = K(b, a, -\nu)$ for every $(a, b, \nu) \in \mathcal{M} \times \mathcal{M} \times \mathbb{S}^{N-1}$,*
(ii) *for every finite subset T of $\mathcal{M}$,*

$$\mathcal{F}(u,S) = \int_S K(u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1}\,,$$

for every $u \in BV(\Omega;T)$ and every Borel subset S of $\Omega \cap S_u$.

Proof. Let T be a finite subset of $\mathcal{M}$. For every $u \in BV(\Omega;T)$ and $A \in \mathcal{A}(\Omega)$, we define

$$\mathcal{G}_T(u,A) := \mathcal{F}(u, A \cap S_u)\,.$$

We claim that
(i) $0 \le \mathcal{G}_T(u,A) \le C\mathcal{H}^{N-1}(A \cap S_u)$ for some constant C independent of u, A and T;
(ii) $\mathcal{G}_T(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;
(iii) $\mathcal{G}_T(u,A) = \mathcal{G}_T(v,A)$ whenever $u = v$ a.e. in A;
(iv) if $u_k \to u$ a.e. in A, then $\mathcal{G}_T(u,A) \le \liminf_{k\to+\infty} \mathcal{G}_T(u_k,A)$;
(v) for every $A \in \mathcal{A}(\Omega)$ and $y \in \mathbb{R}^N$ such that $y + A \subset \Omega$, we have $\mathcal{G}_T(\tau_y u, y + A) = \mathcal{G}_T(u,A)$.

Properties *(i)* and *(ii)* directly follow from Lemma 5.1 and the definition of $\mathcal{G}_T$. Then we easily see that $\mathcal{F}(u,A) = \mathcal{F}(v,A)$ whenever $u = v$ a.e. in A so Remark 5.1 yields *(iii)*. To prove *(iv)*, let $u_k \to u$ a.e. in A. Since u_k is uniformly bounded in $L^\infty(A;\mathbb{R}^d)$, the Dominated Convergence Theorem yields $u_k \to u$ in

$L^1(A;\mathbb{R}^d)$. Now consider an arbitrary open subset E of A satisfying $A \cap S_u \subset E$. By lower semicontinuity of $\mathcal{F}(\cdot, E)$,

$$\mathcal{F}(u,E) \leq \liminf_{k\to+\infty} \mathcal{F}(u_k,E)\,.$$

Since u takes its values in a finite set, $|Du|$ is absolutely continuous with respect to $\mathcal{H}^{N-1} \llcorner S_u$. Using Lemma 5.1 together with Remark 5.1, we infer that

$$\mathcal{F}(u_k,E) = \mathcal{F}(u_k,E\cap S_{u_k}) + \mathcal{F}(u_k,E\setminus S_{u_k}) \leq \mathcal{G}_T(u_k,A) + C\mathcal{L}^N(E)\,.$$

Therefore,

$$\mathcal{G}_T(u,A) = \mathcal{F}(u,S_u\cap A) \leq \mathcal{F}(u,E) \leq \liminf_{k\to+\infty} \mathcal{G}_T(u_k,A) + C\mathcal{L}^N(E)\,.$$

Taking the infimum over all such E's, we obtain the desired inequality. Finally, *(v)* is a consequence of Lemma 5.2 together with Remark 5.1.

We may now apply [4, Theorem 3.1] which yields the existence of a unique continuous function K_T : $\Omega\times T\times T\times\mathbb{S}^{N-1}\to[0,+\infty)$ such that $K_T(x,a,b,\nu) = K_T(x,b,a,-\nu)$ and

$$\mathcal{F}(u,A\cap S_u) = \mathcal{G}_T(u,A) = \int_{A\cap S_u} K_T(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}$$

for every $u\in BV(\Omega;T)$ and $A\in\mathcal{A}(\Omega)$. For $x_0\in\mathbb{R}^N$, $a,b\in\mathcal{M}$ and $\nu\in\mathbb{S}^{N-1}$, define

$$u^{a,b}_{x_0,\nu}(x) := \begin{cases} a & \text{if } (x-x_0)\cdot\nu \geq 0\,,\\ b & \text{if } (x-x_0)\cdot\nu < 0\,,\end{cases} \qquad \Pi_{x_0,\nu} := \big\{x\in\mathbb{R}^N : (x-x_0)\cdot\nu = 0\big\}\,. \tag{5.7}$$

Since K_T is continuous, we have

$$K_T(x_0,a,b,\nu) = \lim_{\rho\to0^+}\frac{\mathcal{F}\big(u^{a,b}_{x_0,\nu},Q_\nu(x_0,\rho)\cap\Pi_{x_0,\nu}\big)}{\mathcal{H}^{N-1}(Q_\nu(x_0,\rho)\cap\Pi_{x_0,\nu})}$$

for every $(x_0,a,b,\nu)\in\Omega\times T\times T\times\mathbb{S}^{N-1}$. Hence K_T can be replaced by a function K independent of T defined on $\Omega\times\mathcal{M}\times\mathcal{M}\times\mathbb{S}^{N-1}$. Moreover, in view of Lemma 5.2, we easily deduce that K is independent of x. Therefore

$$\mathcal{F}(u,A\cap S_u) = \int_{A\cap S_u} K(u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}$$

for every finite set $T\subset\mathcal{M}$, $A\in\mathcal{A}(\Omega)$ and $u\in BV(\Omega;T)$. Then the integral representation on Borel subsets of $\Omega\cap S_u$ follows by outer regularity noticing that $\mathcal{F}(u,\cdot)\llcorner S_u$ defines a Radon measure. □

5.3. *The upper bound*

We now adress the Γ-lim sup inequality. The upper bound on the diffuse part will be obtained using an extension of the relaxation result of [2] (see Theorem A.1 in the Appendix) together with the partial representation of the Γ-limit already established in $W^{1,1}$ (see Theorem 1.2). The estimate of the jump part relies on the integral representation on partitions in sets of finite perimeter stated in Proposition 5.1.

In view of the measure property of the Γ-limit, we may write for every $u\in BV(\Omega;\mathcal{M})$,

$$\mathcal{F}(u,\Omega) = \mathcal{F}(u,\Omega\setminus S_u) + \mathcal{F}(u,\Omega\cap S_u)\,. \tag{5.8}$$

Hence the desired upper bound $\mathcal{F}(u,\Omega)\leq\mathcal{F}_{\rm hom}(u)$ will follow estimating separately the two terms in the right handside of (5.8).

Lemma 5.3. *For every $u\in BV(\Omega;\mathcal{M})$, we have*

$$\mathcal{F}(u,\Omega\setminus S_u) \leq \int_\Omega Tf_{\rm hom}(u,\nabla u)\,dx + \int_\Omega Tf^\infty_{\rm hom}\left(\tilde{u},\frac{dD^cu}{d|D^cu|}\right)d|D^cu|\,.$$

Proof. Let $A \in \mathcal{A}(\Omega)$ and $\{u_n\} \subset W^{1,1}(A;\mathcal{M})$ be such that $u_n \to u$ in $L^1(A;\mathbb{R}^d)$. Since $\mathcal{F}(\cdot, A)$ is sequentially lower semicontinuous for the strong $L^1(A;\mathbb{R}^d)$ convergence, it follows from Theorem 1.2 that

$$\mathcal{F}(u,A) \leq \liminf_{n\to+\infty} \mathcal{F}(u_n, A) = \liminf_{n\to+\infty} \int_A Tf_{\rm hom}(u_n, \nabla u_n)\, dx\,.$$

Since the sequence $\{u_n\}$ is arbitrary, we deduce

$$\mathcal{F}(u,A) \leq \inf\left\{ \liminf_{n\to+\infty} \int_A Tf_{\rm hom}(u_n, \nabla u_n)\, dx : \{u_n\} \subset W^{1,1}(A;\mathcal{M}),\ u_n \to u \text{ in } L^1(A;\mathbb{R}^d)\right\}.$$

According to Propositions 3.1 and 3.2, the energy density $Tf_{\rm hom}$ is a continuous and tangentially quasiconvex function which fulfills the assumptions of Theorem A.1. Hence

$$\mathcal{F}(u,A) \leq \int_A Tf_{\rm hom}(u,\nabla u)\, dx + \int_A Tf^\infty_{\rm hom}\left(\tilde u, \frac{dD^c u}{d|D^c u|}\right) d|D^c u| + \int_{S_u\cap A} H(u^+,u^-,\nu_u)\, d\mathcal{H}^{N-1} \tag{5.9}$$

for some function $H : \mathcal{M}\times\mathcal{M}\times\mathbb{S}^{N-1} \to [0,+\infty)$. By outer regularity, (5.9) holds for every $A \in \mathcal{B}(\Omega)$. Taking $A = \Omega\setminus S_u$, we obtain

$$\mathcal{F}(u,\Omega\setminus S_u) \leq \int_\Omega Tf_{\rm hom}(u,\nabla u)\, dx + \int_\Omega Tf^\infty_{\rm hom}\left(\tilde u, \frac{dD^c u}{d|D^c u|}\right) d|D^c u|\,,$$

and the proof is complete. □

To prove the upper bound of the jump part, we first need to compare the energy density K obtained in Proposition 5.1 with the expected density $\vartheta_{\rm hom}$.

Lemma 5.4. *We have $K(a,b,\nu_1) \leq \vartheta_{\rm hom}(a,b,\nu_1)$ for every $(a,b,\nu_1) \in \mathcal{M}\times\mathcal{M}\times\mathbb{S}^{N-1}$.*

Proof. We will partially proceed as in the proof of Proposition 3.4 and we refer to it for the notation. Consider $\nu = (\nu_1,\dots,\nu_N)$ an orthonormal basis of $\mathbb{R}^N$. We shall prove that $K(a,b,\nu_1) \leq \vartheta_{\rm hom}(a,b,\nu_1)$. Since K and $\vartheta_{\rm hom}$ are continuous in the last variable, we may assume that ν is a rational basis, *i.e.*, for all $i\in\{1,\dots,N\}$ there exists $\gamma_i \in \mathbb{R}\setminus\{0\}$ such that $v_i := \gamma_i\nu_i \in \mathbb{Z}^N$, and the general case follows by density.

Given $0<\eta<1$ arbitrary, by Proposition 3.3 and (3.34) we can find $\varepsilon_0 > 0$, $u_0 \in \mathcal{B}_{\varepsilon_0}(a,b,\nu)$ and $\gamma_{\varepsilon_0} \in \mathcal{G}(a,b)$ such that $u_0(x) = \gamma_{\varepsilon_0}(x\cdot\nu_1/\varepsilon_0)$ and

$$\int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon_0}, \nabla u_0\right) dx \leq \vartheta_{\rm hom}(a,b,\nu_1) + \eta\,.$$

For every $\lambda = (\lambda_2,\dots,\lambda_N) \in \mathbb{Z}^{N-1}$, we set $x_n^{(\lambda)} := \varepsilon_n \sum_{i=2}^N \lambda_i v_i$ and $Q^{(\lambda)}_{\nu,n} := x_n^{(\lambda)} + (\varepsilon_n/\varepsilon_0)Q_\nu$. We define the set $\Lambda_n := \Lambda(\varepsilon_0,\varepsilon_n)$ as in (3.22) with $\nu' = \nu$. Next consider

$$u_n(x) = \begin{cases} u_0\left(\dfrac{\varepsilon_0(x - x_n^{(\lambda)})\cdot\nu_1}{\varepsilon_n}\right) & \text{if } x \in Q^{(\lambda)}_{\nu,n} \text{ for some } \lambda\in\Lambda_n\,,\\[2ex] \gamma_{\varepsilon_0}\left(\dfrac{x\cdot\nu_1}{\varepsilon_n}\right) & \text{otherwise}\,. \end{cases}$$

Note that $u_n \in W^{1,1}(Q_\nu;\mathcal{M})$, $\{\nabla u_n\}$ is bounded in $L^1(Q_\nu;\mathbb{R}^{d\times N})$, and $u_n \to u^{a,b}_{0,\nu_1}$ in $L^1(Q_\nu;\mathbb{R}^d)$ as $n\to+\infty$ with $u^{a,b}_{0,\nu_1}$ given by (5.7). Arguing as in the proof of Proposition 3.4, Step 2, we obtain that

$$\limsup_{n\to+\infty} \int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) dx \leq \int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon_0}, \nabla u_0\right) dx \leq \vartheta_{\rm hom}(a,b,\nu_1) + \eta\,. \tag{5.10}$$

For $\rho > 0$ define $A_\rho := Q_\nu \cap \{|x\cdot\nu_1| < \rho\}$. By construction the sequence $\{u_n\}$ is admissible for $\mathcal{F}(u^{a,b}_{0,\nu_1}, A_\eta)$ so that

$$\begin{aligned}\mathcal{F}\big(u^{a,b}_{0,\nu_1}, A_\eta\cap\Pi_{0,\nu_1}\big) &\leq \mathcal{F}(u^{a,b}_{0,\nu_1}, A_\eta) \leq \liminf_{n\to+\infty} \int_{A_\eta} f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) dx \leq \\ &\leq \beta\mathcal{L}^N(A_\eta) + \liminf_{n\to+\infty} \int_{A_{\varepsilon_n}} f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) dx \leq \liminf_{n\to+\infty} \int_{A_{\varepsilon_n}} f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) dx + \beta\eta\,, \end{aligned} \tag{5.11}$$

where we have used (H_2) and the fact that $\nabla u_n = 0$ outside A_{ε_n}. On the other hand, Proposition 5.1 yields

$$\mathcal{F}\big(u^{a,b}_{0,\nu_1}, A_\eta \cap \Pi_{0,\nu_1}\big) = \int_{A_\eta \cap \Pi_{0,\nu_1}} K(a,b,\nu_1)\, d\mathcal{H}^{N-1} = K(a,b,\nu_1)\,. \tag{5.12}$$

Using (H_4), the boundedness of $\{\nabla u_n\}$ in $L^1(Q_\nu;\mathbb{R}^{d\times N})$, the fact that $f^\infty(\cdot,0)\equiv 0$, and Hölder's inequality, we derive

$$\begin{aligned}\left|\int_{A_{\varepsilon_n}} f\left(\frac{x}{\varepsilon_n},\nabla u_n\right)dx - \int_{Q_\nu} f^\infty\left(\frac{x}{\varepsilon_n},\nabla u_n\right)dx\right| &\le C\int_{A_{\varepsilon_n}} (1+|\nabla u_n|^{1-q})\,dx\\ &\le C\big(\varepsilon_n + \varepsilon_n^q \|\nabla u_n\|^{1-q}_{L^1(Q_\nu;\mathbb{R}^{d\times N})}\big) \to 0 \end{aligned} \tag{5.13}$$

as $n\to\infty$. Gathering (5.10), (5.11), (5.12) and (5.13), we obtain $K(a,b,\nu_1) \le \vartheta_{\rm hom}(a,b,\nu_1) + (\beta+1)\eta$ and the conclusion follows from the arbitrariness of η. □

We are now in position to prove the upper bound on the jump part of the energy. The argument is based on Lemma 5.4 together with an approximation procedure of [7]. In view of Lemma 5.3 and (5.8), this will complete the proof of the upper bound $\mathcal{F}(u,\Omega) \le \mathcal{F}_{\rm hom}(u)$.

Corollary 5.1. *For every $u \in BV(\Omega;\mathcal{M})$, we have*

$$\mathcal{F}(u,\Omega\cap S_u) \le \int_{\Omega\cap S_u} \vartheta_{\rm hom}(u^+,u^-,\nu_u)\, d\mathcal{H}^{N-1}\,.$$

Proof. First assume that u takes a finite number of values, *i.e.*, $u \in BV(\Omega;T)$ for some finite subset $T\subset\mathcal{M}$. Then the conclusion directly follows from Proposition 5.1 together with Lemma 5.4.

Fix an arbitrary function $u \in BV(\Omega;\mathcal{M})$ and an open set $A\in\mathcal{A}(\Omega)$. For $\delta_0>0$ small enough, let $\mathcal{U} := \big\{s\in\mathbb{R}^d : \operatorname{dist}(s,\mathcal{M}) < \delta_0\big\}$ be the δ_0-neighborhood of $\mathcal{M}$ on which the nearest point projection $\Pi:\mathcal{U}\to\mathcal{M}$ is a well defined Lipschitz mapping. We extend $\vartheta_{\rm hom}$ to a function $\hat\vartheta_{\rm hom}$ defined in $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{N-1}$ by setting

$$\hat\vartheta_{\rm hom}(a,b,\nu) = \chi(a)\chi(b)\vartheta_{\rm hom}\Big(\Pi(a),\Pi(b),\nu\Big)\,,$$

for a cut-off function $\chi\in\mathcal{C}^\infty_c(\mathbb{R}^d;[0,1])$ satisfying $\chi(t)=1$ if $\operatorname{dist}(s,\mathcal{M})\le\delta_0/2$, and $\chi(s)=0$ if $\operatorname{dist}(s,\mathcal{M})\ge 3\delta_0/4$. In view of Proposition 3.5, we infer that $\hat\vartheta_{\rm hom}$ is continuous and satisfies

$$|\hat\vartheta_{\rm hom}(a_1,b_1,\nu) - \hat\vartheta_{\rm hom}(a_2,b_2,\nu)| \le C\big(|a_1-a_2|+|b_1-b_2|\big)\,,$$

and

$$\hat\vartheta_{\rm hom}(a_1,b_1,\nu) \le C|a_1-b_1|\,,$$

for every a_1, b_1, a_2, $b_2\in\mathbb{R}^d$, $\nu\in\mathbb{S}^{N-1}$, and some constant $C>0$. Therefore we can apply Step 2 in the proof of [7, Proposition 4.8] to obtain a sequence $\{v_n\}\subset BV(\Omega;\mathbb{R}^d)$ such that, for every $n\in\mathbb{N}$, $v_n\in BV(\Omega;T_n)$ for some finite set $T_n\subset\mathbb{R}^d$, $v_n\to u$ in $L^\infty(\Omega;\mathbb{R}^d)$ and

$$\begin{aligned}\limsup_{n\to+\infty}\int_{A\cap S_{v_n}} \hat\vartheta_{\rm hom}(v_n^+,v_n^-,\nu_{v_n})\,d\mathcal{H}^{N-1} &\le C|Du|(A\setminus S_u) + \int_{A\cap S_u}\hat\vartheta_{\rm hom}(u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}\\ &= C|Du|(A\setminus S_u) + \int_{A\cap S_u}\vartheta_{\rm hom}(u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}\,.\end{aligned}$$

Hence we may assume without loss of generality that for each $n\in\mathbb{N}$, $\|v_n-u\|_{L^\infty(\Omega;\mathbb{R}^d)} < \delta_0/2$, and thus $\operatorname{dist}(v_n^\pm(x),\mathcal{M}) \le |v_n^\pm(x)-u^\pm(x)| < \delta_0/2$ for $\mathcal{H}^{N-1}$-a.e. $x\in S_{v_n}$. In particular, we can define

$$u_n := \Pi(v_n)\,,$$

and then $u_n\in BV(\Omega;\mathcal{M})$, $u_n\to u$ in $L^1(\Omega;\mathbb{R}^d)$. Moreover, one may check that for each $n\in\mathbb{N}$, $S_{u_n}\subset S_{v_n}$ so that $\mathcal{H}^{N-1}\big(S_{u_n}\setminus(J_{u_n}\cap J_{v_n})\big) \le \mathcal{H}^{N-1}(S_{u_n}\setminus J_{u_n}) + \mathcal{H}^{N-1}(S_{v_n}\setminus J_{v_n}) = 0$, and

$$u_n^\pm(x) = \Pi(v_n^\pm(x)) \quad\text{and}\quad \nu_{u_n}(x) = \nu_{v_n}(x) \quad \text{for every } x\in J_{u_n}\cap J_{v_n}\,.$$

Consequently,

$$\limsup_{n\to+\infty}\int_{A\cap S_{u_n}}\vartheta_{\rm hom}(u_n^+,u_n^-,\nu_{u_n})\,d\mathcal{H}^{N-1}\leq \limsup_{n\to+\infty}\int_{A\cap S_{v_n}}\hat\vartheta_{\rm hom}(v_n^+,v_n^-,\nu_{v_n})\,d\mathcal{H}^{N-1}$$
$$\leq C|Du|(A\setminus S_u)+\int_{A\cap S_u}\vartheta_{\rm hom}(u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}\,. \tag{5.14}$$

Since u_n takes a finite number of values, we infer from Proposition 5.1 together with Lemma 5.4 that

$$\mathcal{F}(u_n,A\cap S_{u_n})\leq\int_{A\cap S_{u_n}}\vartheta_{\rm hom}(u_n^+,u_n^-,\nu_{u_n})\,d\mathcal{H}^{N-1}\,, \tag{5.15}$$

and, in view of Lemma 5.1,

$$\mathcal{F}(u_n,A\setminus S_{u_n})\leq C\mathcal{L}^N(A)\,. \tag{5.16}$$

Combining (5.14), (5.15) and (5.16), we deduce

$$\limsup_{n\to+\infty}\mathcal{F}(u_n,A)=\limsup_{n\to+\infty}\big(\mathcal{F}(u_n,A\setminus S_{u_n})+\mathcal{F}(u_n,A\cap S_{u_n})\big)$$
$$\leq\int_{A\cap S_u}\vartheta_{\rm hom}(u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}+C\big(\mathcal{L}^N(A)+|Du|(A\setminus S_u)\big)\,.$$

On the other hand, $\mathcal{F}(\cdot,A)$ is lower semicontinuous with respect to the strong $L^1(A;\mathbb{R}^d)$-convergence, and thus $\mathcal{F}(u,A)\leq\liminf\limits_{n\to+\infty}\mathcal{F}(u_n,A)$ which leads to

$$\mathcal{F}(u,A)\leq\int_{A\cap S_u}\vartheta_{\rm hom}(u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}+C\big(\mathcal{L}^N(A)+|Du|(A\setminus S_u)\big)\,.$$

Since A is arbitrary, the above inequality holds for any open set $A\in\mathcal{A}(\Omega)$ and, by Remark 5.1, it also holds if A is any Borel subset of Ω. Then taking $A=\Omega\cap S_u$ yields the desired inequality. □

5.4. *The lower bound*

We conclude this section with the Γ-lim inf inequality. Using the blow-up method, we follow the approach of [26], estimating separately the Cantor part and the jump part, while the bulk part is obtained exactly as in the $W^{1,1}$ analysis (see Lemma 4.4).

Lemma 5.5. *For every $u\in BV(\Omega;\mathcal{M})$, we have $\mathcal{F}(u,\Omega)\geq\mathcal{F}_{\rm hom}(u)$.*

Proof. Let $u\in BV(\Omega;\mathcal{M})$ and $\{u_n\}\subset W^{1,1}(\Omega;\mathcal{M})$ be such that

$$\mathcal{F}(u,\Omega)=\lim_{n\to+\infty}\int_\Omega f\left(\frac{x}{\varepsilon_n},\nabla u_n\right)dx\,.$$

Define the sequence of nonnegative Radon measures

$$\mu_n:=f\left(\frac{\cdot}{\varepsilon_n},\nabla u_n\right)\mathcal{L}^N\llcorner\,\Omega\,.$$

Up to the extraction of a subsequence, we can assume that there exists a nonnegative Radon measure $\mu\in\mathcal{M}(\Omega)$ such that $\mu_n\overset{*}{\rightharpoonup}\mu$ in $\mathcal{M}(\Omega)$. By the Besicovitch Differentiation Theorem, we can split μ into the sum of four mutually singular nonnegative measures $\mu=\mu^a+\mu^j+\mu^c+\mu^s$ where $\mu^a\ll\mathcal{L}^N$, $\mu^j\ll\mathcal{H}^{N-1}\llcorner\,S_u$ and $\mu^c\ll|D^cu|$. Since we have $\mu(\Omega)\leq\mathcal{F}(u,\Omega)$, it is enough to check that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0)\geq Tf_{\rm hom}(u(x_0),\nabla u(x_0))\qquad\text{for }\mathcal{L}^N\text{-a.e. }x_0\in\Omega\,, \tag{5.17}$$

$$\frac{d\mu}{d|D^cu|}(x_0)\geq Tf^\infty_{\rm hom}\left(\tilde u(x_0),\frac{dD^cu}{d|D^cu|}(x_0)\right)\qquad\text{for }|D^cu|\text{-a.e. }x_0\in\Omega\,, \tag{5.18}$$

and

$$\frac{d\mu}{d\mathcal{H}^{N-1}\llcorner\,S_u}(x_0)\geq\vartheta_{\rm hom}(u^+(x_0),u^-(x_0),\nu_u(x_0))\qquad\text{for }\mathcal{H}^{N-1}\text{-a.e. }x_0\in S_u\,. \tag{5.19}$$

The proof of (5.17) can be obtained exactly as in the proof of Lemma 4.4 and we shall omit it. The proofs of (5.18) and (5.19) are postponed to the remaining of this subsection. □

Proof of (5.18)**.** The lower bound on the density of the Cantor part will be achieved in three steps. We shall use the blow-up method to reduce the study to constant limits, and then a truncation argument as in the proof of Lemma 4.4, to replace the starting sequence by a uniformly converging one.

Step 1. Choose a point $x_0 \in \Omega$ such that

$$\lim_{\rho\to 0^+} \fint_{Q(x_0,\rho)} |u(x) - \tilde{u}(x_0)|\, dx = 0\,, \tag{5.20}$$

$$A(x_0) := \lim_{\rho\to 0^+} \frac{Du(Q(x_0,\rho))}{|Du|(Q(x_0,\rho))} \in [T_{\tilde{u}(x_0)}(\mathcal{M})]^N \ \text{ is a rank one matrix with } \ |A(x_0)| = 1\,, \tag{5.21}$$

$$\frac{d\mu}{d|D^c u|}(x_0) \quad \text{exists and is finite and} \quad \frac{d|Du|}{d|D^c u|}(x_0) = 1\,, \tag{5.22}$$

$$\lim_{\rho\to 0^+} \frac{|Du|(Q(x_0,\rho))}{\rho^{N-1}} = 0 \quad \text{and} \quad \lim_{\rho\to 0^+} \frac{|Du|(Q(x_0,\rho))}{\rho^N} = +\infty\,, \tag{5.23}$$

$$\liminf_{\rho\to 0^+} \frac{|Du|(Q(x_0,\rho)\setminus Q(x_0,\tau\rho))}{|Du|(Q(x_0,\rho))} \le 1-\tau^N \quad \text{for every } 0<\tau<1\,. \tag{5.24}$$

It turns out that $|D^c u|$-a.e. $x_0 \in \Omega$ satisfy these properties. Indeed (5.22) is immediate while (5.20) is a consequence of the fact that S_u is $|D^c u|$-negligible. Property (5.21) comes from Alberti Rank One Theorem together with Lemma 2.1, (5.23) from [6, Proposition 3.92 (a), (c)] and (5.24) from [26, Lemma 2.13]. Write $A(x_0) = a \otimes \nu$ for some $a \in \mathcal{M}$ and $\nu \in \mathbb{S}^{N-1}$. Upon rotating the coordinate axis, one may assume without loss of generality that $\nu = e_N$. To simplify the notations, we set $s_0 := \tilde{u}(x_0)$ and $A_0 := A(x_0)$.

Fix $t \in (0,1)$ arbitrarily close to 1, and in view of (5.24), find a sequence $\rho_k \searrow 0^+$ such that

$$\limsup_{k\to+\infty} \frac{|Du|(Q(x_0,\rho_k)\setminus Q(x_0,t\rho_k))}{|Du|(Q(x_0,\rho_k))} \le 1-t^N\,. \tag{5.25}$$

Now fix $t<\gamma<1$ and set $\gamma' := (1+\gamma)/2$. Using (5.22), we derive

$$\frac{d\mu}{d|D^c u|}(x_0) = \lim_{k\to+\infty} \frac{\mu(Q(x_0,\rho_k))}{|Du|(Q(x_0,\rho_k))} \ge \limsup_{k\to+\infty} \frac{\mu(\overline{Q(x_0,\gamma'\rho_k)})}{|Du|(Q(x_0,\rho_k))} \ge$$
$$\ge \limsup_{k\to+\infty}\limsup_{n\to+\infty} \frac{1}{|Du|(Q(x_0,\rho_k))} \int_{Q(x_0,\gamma'\rho_k)} f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) dx\,. \tag{5.26}$$

Arguing as in the proof of Lemma 4.4 with minor modifications, we construct a sequence $\{\bar{v}_n\} \subset W^{1,\infty}(Q(0,\rho_k);\mathbb{R}^d)$ satisfying $\bar{v}_n \to u(x_0+\cdot)$ in $L^1(Q(0,\rho_k);\mathbb{R}^d)$ and

$$\limsup_{n\to+\infty} \int_{Q(x_0,\gamma'\rho_k)} f\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) dx \ge \limsup_{n\to+\infty} \int_{Q(0,\gamma\rho_k)} g\left(\frac{x}{\varepsilon_n}, \bar{v}_n, \nabla \bar{v}_n\right) dx\,. \tag{5.27}$$

Setting $w_{n,k}(x) := \bar{v}_n(\rho_k\, x)$, a change of variable together with (5.26) and (5.27) yields

$$\frac{d\mu}{d|D^c u|}(x_0) \ge \limsup_{k\to+\infty}\limsup_{n\to+\infty} \frac{\rho_k^N}{|Du|(Q(x_0,\rho_k))} \int_{\gamma Q} g\left(\frac{\rho_k\, x}{\varepsilon_n}, w_{n,k}, \frac{1}{\rho_k}\nabla w_{n,k}\right) dx\,. \tag{5.28}$$

Then we infer from (5.20) that

$$\lim_{k\to+\infty}\lim_{n\to+\infty} \int_Q |w_{n,k} - s_0|\, dx = 0\,, \tag{5.29}$$

and

$$\lim_{k\to+\infty}\lim_{n\to+\infty} \frac{\rho_k^{N-1}}{|Du|(Q(x_0,\rho_k))} \int_Q \left| w_{n,k}(x) - u(x_0+\rho_k x) - \int_Q \big(w_{n,k}(y) - u(x_0+\rho_k y)\big)\, dy \right| dx = 0\,. \tag{5.30}$$

By (5.28), (5.29) and (5.30), we can extract a diagonal sequence $n_k \to +\infty$ such that $\delta_k := \varepsilon_{n_k}/\rho_k \to 0$, $w_k := w_{n_k,k} \to s_0$ in $L^1(Q;\mathbb{R}^d)$,

$$\frac{d\mu}{d|D^c u|}(x_0) \geq \limsup_{k\to+\infty} \frac{\rho_k^N}{|Du|(Q(x_0,\rho_k))} \int_{\gamma Q} g\left(\frac{x}{\delta_k}, w_k, \frac{1}{\rho_k}\nabla w_k\right) dx\,,$$

and

$$\lim_{k\to+\infty} \frac{\rho_k^{N-1}}{|Du|(Q(x_0,\rho_k))} \int_Q \left| w_k(x) - u(x_0+\rho_k\, x) - \int_Q \big(w_k(y) - u(x_0+\rho_k\, y)\big)\, dy \right| dx = 0\,. \tag{5.31}$$

Step 2. Now we reproduce the truncation argument used in Step 2 of the proof of Lemma 4.4 with minor modifications (make use of (5.23) and [26, Lemma 2.12] instead of [25, Lemma 2.6], see [26] for details). Setting $a_k := \int_Q w_k(y)\, dy$, it yields a sequence of cut-off functions $\{\zeta_k\} \subset \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ such that $\zeta_k(\tau) = 1$ if $|\tau| \leq s_k$, $\zeta_k(\tau) = 0$ is $|\tau| \geq t_k$ for some

$$\|w_k - a_k\|^{1/2}_{L^1(Q;\mathbb{R}^d)} < s_k < t_k < \|w_k - a_k\|^{1/3}_{L^1(Q;\mathbb{R}^d)}\,,$$

for which $\overline{w}_k := a_k + \zeta_k(|w_k - a_k|)(w_k - a_k) \in W^{1,1}(Q;\mathbb{R}^d)$ satisfies $\overline{w}_k \to s_0$ in $L^\infty(Q;\mathbb{R}^d)$ and

$$\frac{d\mu}{d|D^c u|}(x_0) \geq \limsup_{k\to+\infty} \frac{\rho_k^N}{|Du|(Q(x_0,\rho_k))} \int_{\gamma Q} g\left(\frac{x}{\delta_k}, \overline{w}_k, \frac{1}{\rho_k}\nabla \overline{w}_k\right) dx\,. \tag{5.32}$$

In view of the coercivity condition (3.7), (5.22) and (5.32),

$$\sup_{k\in\mathbb{N}} \frac{\rho_k^{N-1}}{|Du|(Q(x_0,\rho_k))} \int_{\gamma Q} |\nabla \overline{w}_k|\, dx < +\infty\,.$$

Therefore, (3.8), (5.32) and $\|\overline{w}_k - s_0\|_{L^\infty(Q;\mathbb{R}^d)} \to 0$ lead to

$$\frac{d\mu}{d|D^c u|}(x_0) \geq \limsup_{k\to+\infty} \frac{\rho_k^N}{|Du|(Q(x_0,\rho_k))} \int_{\gamma Q} g\left(\frac{x}{\delta_k}, s_0, \frac{1}{\rho_k}\nabla \overline{w}_k\right) dx\,.$$

Next we define the three following sequences for every $x \in Q$,

$$\begin{cases} \overline{u}_k(x) := \dfrac{\rho_k^{N-1}}{|Du|(Q(x_0,\rho_k))}\left(u(x_0+\rho_k\, x) - \displaystyle\int_Q u(x_0+\rho_k\, y)\, dy\right), \\ z_k(x) := \dfrac{\rho_k^{N-1}}{|Du|(Q(x_0,\rho_k))}\big(w_k(x) - a_k\big)\,, \\ \overline{z}_k(x) := \dfrac{\rho_k^{N-1}}{|Du|(Q(x_0,\rho_k))}\big(\overline{w}_k(x) - a_k\big)\,. \end{cases}$$

As a consequence of (5.31) we have $\|z_k - \overline{u}_k\|_{L^1(Q;\mathbb{R}^d)} \to 0$, and since

$$\int_Q \overline{u}_k(x)\, dx = 0 \quad \text{ and } \quad |D\overline{u}_k|(Q) = 1\,,$$

it follows that the sequence $\{\overline{u}_k\}$ is bounded in $BV(Q;\mathbb{R}^d)$ and thus relatively compact in $L^1(Q;\mathbb{R}^d)$. Hence $\{\overline{u}_k\}$ is equi-integrable, and consequently so is $\{z_k\}$. Up to a subsequence, $\overline{u}_k$ converges in $L^1(Q;\mathbb{R}^d)$ to some function $v \in BV(Q;\mathbb{R}^d)$, and then $z_k \to v$ in $L^1(Q;\mathbb{R}^d)$. By [6, Theorem 3.95] the limit v is representable by

$$v(x) = a\,\theta(x_N)$$

for some increasing function $\theta \in BV((-1/2,1/2);\mathbb{R})$ (recall that we assume $A_0 = a \otimes e_N$).

By construction, $\overline{w}_k$ coincides with w_k in the set $\{|w_k - a_k| \leq s_k\}$. Hence

$$\begin{aligned} \|\overline{z}_k - z_k\|_{L^1(Q;\mathbb{R}^d)} &= \frac{\rho_k^{N-1}}{|Du|(Q(x_0,\rho_k))} \int_{\{|w_k-a_k|>s_k\}} |w_k(x) - \overline{w}_k(x)|\, dx \leq \\ &\leq \frac{\rho_k^{N-1}}{|Du|(Q(x_0,\rho_k))} \int_{\{|w_k-a_k|>s_k\}} |w_k(x) - a_k|\, dx = \int_{\{|w_k-a_k|>s_k\}} |z_k(x)|\, dx\,. \end{aligned} \tag{5.33}$$

By Chebyshev inequality, we have

$$\mathcal{L}^N(\{|w_k - a_k| > s_k\}) \leq \frac{1}{s_k}\int_Q |w_k(x) - a_k|\,dx \leq \|w_k - a_k\|^{1/2}_{L^1(Q;\mathbb{R}^d)} \to 0\,, \tag{5.34}$$

and thus (5.33), (5.34) and the equi-integrability of $\{z_k\}$ imply $\|\overline{z}_k - z_k\|_{L^1(Q;\mathbb{R}^d)} \to 0$. Therefore $\overline{z}_k \to v$ in $L^1(Q;\mathbb{R}^d)$, and setting $\alpha_k := |Du|(Q(x_0,\rho_k))/\rho_k^N \to +\infty$,

$$\frac{d\mu}{d|D^c u|}(x_0) \geq \limsup_{k\to+\infty} \frac{1}{\alpha_k}\int_{\gamma Q} g\left(\frac{x}{\delta_k}, s_0, \alpha_k \nabla\overline{z}_k\right) dx\,. \tag{5.35}$$

Using (3.10) and the positive 1-homogeneity of the recession function $g^\infty(y,s,\cdot)$, we infer that

$$\begin{aligned}\int_{\gamma Q}\left|\frac{1}{\alpha_k} g\left(\frac{x}{\delta_k}, s_0, \alpha_k\nabla\overline{z}_k\right) - g^\infty\left(\frac{x}{\delta_k}, s_0, \nabla\overline{z}_k\right)\right| dx &\leq \frac{C}{\alpha_k}\int_{\gamma Q}(1+\alpha_k^{1-q}|\nabla\overline{z}_k|^{1-q})\,dx \\ &\leq C\big(\alpha_k^{-1} + \alpha_k^{-q}\|\nabla\overline{z}_k\|^{1-q}_{L^1(\gamma Q;\mathbb{R}^{d\times N})}\big) \to 0\,,\end{aligned}$$

where we have used Hölder's inequality and the boundedness of $\{\nabla\overline{z}_k\}$ in $L^1(\gamma Q;\mathbb{R}^{d\times N})$ (which follows from (3.7) and (5.35)). Consequently,

$$\frac{d\mu}{d|D^c u|}(x_0) \geq \limsup_{k\to+\infty}\int_{\gamma Q} g^\infty\left(\frac{x}{\delta_k}, s_0, \nabla\overline{z}_k\right) dx\,.$$

Step 3. Extend θ continuously to $\mathbb{R}$ by the values of its traces at $\pm 1/2$. Define $v_k(x) = v_k(x_N) := a\theta * \varrho_k(x_N)$ where ϱ_k is a sequence of (one dimensional) mollifiers. Then $v_k \to v$ in $L^1(Q;\mathbb{R}^d)$ and thus, since $\overline{u}_k - v_k \to 0$ in $L^1(Q;\mathbb{R}^d)$, it follows that (up to a subsequence)

$$D\overline{u}_k(\tau Q) - Dv_k(\tau Q) \to 0 \tag{5.36}$$

for $\mathcal{L}^1$-a.e. $\tau \in (0,1)$. Fix $\tau \in (t,\gamma)$ for which (5.36) holds. Since $\|\overline{z}_k - v_k\|_{L^1(Q;\mathbb{R}^d)} \to 0$, one can use a standard cut-off function argument (see [26, p. 29–30]) to modify the sequence $\{\overline{z}_k\}$ and produce a new sequence $\{\overline{\varphi}_k\} \subset W^{1,\infty}(\tau Q;\mathbb{R}^d)$ satisfying $\overline{\varphi}_k \to v$ in $L^1(\tau Q;\mathbb{R}^d)$, $\overline{\varphi}_k = v_k$ on a neighborhood of $\partial(\tau Q)$ and

$$\frac{d\mu}{d|D^c u|}(x_0) \geq \limsup_{k\to+\infty}\int_{\tau Q} g^\infty\left(\frac{x}{\delta_k}, s_0, \nabla\overline{\varphi}_k\right) dx\,. \tag{5.37}$$

A simple computation shows that

$$D\overline{u}_k(\tau Q) = \frac{Du(Q(x_0,\tau\rho_k))}{|Du|(Q(x_0,\rho_k))} \quad\text{and}\quad Dv_k(\tau Q) = \tau^N A_k\,, \tag{5.38}$$

where $A_k \in \mathbb{R}^{d\times N}$ is the matrix given by

$$A_k := a\otimes e_N \frac{\theta*\varrho_k(\tau/2) - \theta*\varrho_k(-\tau/2)}{\tau}\,.$$

We observe that A_k is bounded in k since θ has bounded variation.

Let $m_k := [\tau/\delta_k] + 1 \in \mathbb{N}$, and define for $x = (x', x_N) \in \delta_k m_k Q$,

$$\varphi_k(x) := \begin{cases} \overline{\varphi}_k(x) - A_k x & \text{if } x \in \tau Q\,, \\ v_k(x_N) - A_k\,x & \text{if } |x_N| \leq \tau/2 \text{ and } |x'| \geq \tau/2\,, \\ v_k(\tau/2) - A_k(x',\tau/2) & \text{if } x_N \geq \tau/2\,, \\ v_k(-\tau/2) - A_k(x',-\tau/2) & \text{if } x_N \leq -\tau/2\,. \end{cases}$$

One may check that $\varphi_k \in W^{1,\infty}(\delta_k m_k Q;\mathbb{R}^d)$, φ_k is $\delta_k m_k$-periodic, and that

$$\limsup_{k\to+\infty}\int_{\tau Q} g^\infty\left(\frac{x}{\delta_k}, s_0, \nabla\overline{\varphi}_k\right) dx = \limsup_{k\to+\infty}\int_{\delta_k m_k Q} g^\infty\left(\frac{x}{\delta_k}, s_0, A_k + \nabla\varphi_k\right) dx\,. \tag{5.39}$$

Setting $\phi_k(y) := \tau^N\delta_k^{-1}\varphi_k(\delta_k y)$ for $y \in m_k Q$, we have $\phi_k \in W^{1,\infty}_\#(m_k Q;\mathbb{R}^d)$, and a change of variables yields

$$\begin{aligned}\int_{\delta_k m_k Q} g^\infty\left(\frac{x}{\delta_k}, s_0, A_k + \nabla\varphi_k\right) dx &= \tau^{-N}\delta_k^N m_k^N ⨍_{m_k Q} g^\infty\left(y, s_0, \tau^N A_k + \nabla\phi_k\right) dy \\ &\geq \tau^{-N}\delta_k^N m_k^N (g^\infty)_{\mathrm{hom}}(s_0, \tau^N A_k)\,,\end{aligned} \tag{5.40}$$

since $(g^\infty)_{\rm hom}$ can be computed as follows (see Remark 3.4 and *e.g.,* [14, Remark 14.6]),

$$(g^\infty)_{\rm hom}(s,\xi)=\inf\left\{ \fint_{(0,m)^N} g^\infty(y,s,\xi+\nabla\phi(y))\,dy : m\in\mathbb{N},\ \phi\in W^{1,\infty}_{\#}((0,m)^N;\mathbb{R}^d)\right\}.$$

Gathering (5.37), (5.39) and (5.40), we derive

$$\frac{d\mu}{d|D^c u|}(x_0)\geq \limsup_{k\to+\infty}(g^\infty)_{\rm hom}(s_0,\tau^N A_k).$$

In view (5.36), (5.38), (5.25) and (5.21), we have

$$\limsup_{k\to+\infty}|\tau^N A_k-A_0|=\limsup_{k\to+\infty}|Dv_k(\tau Q)-A_0|=\limsup_{k\to+\infty}|D\overline{u}_k(\tau Q)-A_0|=$$
$$=\limsup_{k\to+\infty}\left|\frac{Du(Q(x_0,\tau\rho_k))}{|Du|(Q(x_0,\rho_k))}-A_0\right|=\limsup_{k\to+\infty}\frac{|Du|(Q(x_0,\rho_k)\setminus Q(x_0,\tau\rho_k))}{|Du|(Q(x_0,\rho_k))}\leq 1-t^N.$$

By Remark 3.4, $(g^\infty)_{\rm hom}(s_0,\cdot)$ is Lipschitz continuous, and consequently

$$\frac{d\mu}{d|D^c u|}(x_0)\geq (g^\infty)_{\rm hom}(s_0,A_0)-C(1-t^N).$$

From the arbitrariness of t, we finally infer that

$$\frac{d\mu}{d|D^c u|}(x_0)\geq (g^\infty)_{\rm hom}(s_0,A_0).$$

Since $s_0\in\mathcal{M}$ and $A_0\in[T_{s_0}(\mathcal{M})]^N$, Remark 3.4 and (3.17) yield $(g^\infty)_{\rm hom}(s_0,A_0)=T(f^\infty)_{\rm hom}(s_0,A_0)\geq Tf^\infty_{\rm hom}(s_0,A_0)$, and the proof is complete. □

Proof of (5.19)**.** The strategy used in that part follows the one already used for the bulk and Cantor parts. It still rests on the blow up method together with the projection argument in Proposition 2.1.

Step 1. Let $x_0\in S_u$ be such that

$$\lim_{\rho\to 0^+}\fint_{Q^\pm_{\nu_u(x_0)}(x_0,\rho)}|u(x)-u^\pm(x_0)|\,dx=0\,, \tag{5.41}$$

where $u^\pm(x_0)\in\mathcal{M}$,

$$\lim_{\rho\to 0^+}\frac{\mathcal{H}^{N-1}(S_u\cap Q_{\nu_u(x_0)}(x_0,\rho))}{\rho^{N-1}}=1\,, \tag{5.42}$$

and such that the Radon-Nikodým derivative of μ with respect to $\mathcal{H}^{N-1}\llcorner S_u$ exists and is finite. By Lemma 2.1, Theorem 3.78 and Theorem 2.83 (i) in [6] (with cubes instead of balls), it turns out that $\mathcal{H}^{N-1}$-a.e. $x_0\in S_u$ satisfy these properties. Set $s_0^\pm:=u^\pm(x_0)$, $\nu_0:=\nu_u(x_0)$.

Up to a further subsequence, we may assume that $(1+|\nabla u_n|)\mathcal{L}^N\llcorner\Omega\xrightarrow{*}\lambda$ in $\mathcal{M}(\Omega)$ for some nonnegative Radon measure $\lambda\in\mathcal{M}(\Omega)$. Consider a sequence $\rho_k\searrow 0^+$ satisfying $\mu(\partial Q_{\nu_0}(x_0,\rho_k))=\lambda(\partial Q_{\nu_0}(x_0,\rho_k))=0$ for each $k\in\mathbb{N}$. Using (5.42) we derive

$$\frac{d\mu}{d\mathcal{H}^{N-1}\llcorner S_u}(x_0)=\lim_{k\to+\infty}\frac{\mu(Q_{\nu_0}(x_0,\rho_k))}{\mathcal{H}^{N-1}(S_u\cap Q_{\nu_0}(x_0,\rho_k))}=\lim_{k\to+\infty}\frac{\mu(Q_{\nu_0}(x_0,\rho_k))}{\rho_k^{N-1}}=$$
$$=\lim_{k\to+\infty}\lim_{n\to+\infty}\frac{1}{\rho_k^{N-1}}\int_{Q_{\nu_0}(x_0,\rho_k)}f\left(\frac{x}{\varepsilon_n},\nabla u_n\right)dx.$$

Thanks to Theorem 2.1, one can assume without loss of generality that $u_n\in\mathcal{D}(\Omega;\mathcal{M})$ for each $n\in\mathbb{N}$. Arguing exactly as in Step 1 of the proof of Lemma 4.4 (with $Q_{\nu_0}(x_0,\rho_k)$ instead of $Q(x_0,\rho_k)$) we obtain a sequence $\{v_n\}\subset\mathcal{D}(Q_{\nu_0}(0,\rho_k);\mathcal{M})$ such that $v_n\to u(x_0+\cdot)$ in $L^1(Q_{\nu_0}(0,\rho_k);\mathbb{R}^d)$ as $n\to+\infty$, and

$$\frac{d\mu}{d\mathcal{H}^{N-1}\llcorner S_u}(x_0)\geq\limsup_{k\to+\infty}\limsup_{n\to+\infty}\frac{1}{\rho_k^{N-1}}\int_{Q_{\nu_0}(0,\rho_k)}f\left(\frac{x}{\varepsilon_n},\nabla v_n\right)dx$$

(note that the construction process to obtain v_n from u_n does not affect the manifold constraint). Changing variables and setting $w_{n,k}(x) = v_n(\rho_k x)$ lead to

$$\frac{d\mu}{d\mathcal{H}^{N-1} \llcorner S_u}(x_0) \geq \limsup_{k\to+\infty} \limsup_{n\to+\infty} \rho_k \int_{Q_{\nu_0}} f\left(\frac{\rho_k x}{\varepsilon_n}, \frac{1}{\rho_k}\nabla w_{n,k}\right) dx\,.$$

Defining

$$u_0(x) := \begin{cases} s_0^+ & \text{if } x\cdot\nu_0 > 0\,, \\ s_0^- & \text{if } x\cdot\nu_0 \leq 0\,, \end{cases}$$

we infer from (5.41) that

$$\lim_{k\to+\infty} \lim_{n\to+\infty} \int_{Q_{\nu_0}} |w_{n,k} - u_0|\, dx = 0\,.$$

By a standart diagonal argument, we find a sequence $n_k \nearrow +\infty$ such that $\delta_k := \varepsilon_{n_k}/\rho_k \to 0$, $w_k := w_{n_k,k} \in \mathcal{D}(Q_{\nu_0};\mathcal{M})$ converges to u_0 in $L^1(Q_{\nu_0};\mathbb{R}^d)$, and

$$\frac{d\mu}{d\mathcal{H}^{N-1} \llcorner S_u}(x_0) \geq \limsup_{k\to+\infty} \rho_k \int_{Q_{\nu_0}} f\left(\frac{x}{\delta_k}, \frac{1}{\rho_k}\nabla w_k\right) dx\,. \tag{5.43}$$

According to (H_4) and the positive 1-homogeneity of $f^\infty(y,\cdot)$, we have

$$\begin{aligned}\int_{Q_{\nu_0}} \left|\rho_k f\left(\frac{x}{\delta_k}, \frac{1}{\rho_k}\nabla w_k\right) - f^\infty\left(\frac{x}{\delta_k}, \nabla w_k\right)\right| dx &\leq C\rho_k \int_{Q_{\nu_0}} (1 + \rho_k^{q-1}|\nabla w_k|^{1-q})\, dx \\ &\leq C\left(\rho_k + \rho_k^q \|\nabla w_k\|^{1-q}_{L^1(Q_{\nu_0};\mathbb{R}^{d\times N})}\right)\,, \end{aligned} \tag{5.44}$$

where we have used Hölder's inequality and $0 < q < 1$. From (5.43) and the coercivity condition (H_2), it follows that $\{\nabla w_k\}$ is uniformly bounded in $L^1(Q_{\nu_0};\mathbb{R}^{d\times N})$. Gathering (5.43) and (5.44) yields

$$\frac{d\mu}{d\mathcal{H}^{N-1} \llcorner S_u}(x_0) \geq \limsup_{k\to+\infty} \int_{Q_{\nu_0}} f^\infty\left(\frac{x}{\delta_k}, \nabla w_k\right) dx\,. \tag{5.45}$$

Step 2. Now it remains to modify the value of w_k on a neighborhood of ∂Q_{ν_0} in order to get an admissible test function for the surface energy density. We argue as in [2, Lemma 5.2]. Using the notations of Subsection 3.2, we consider $\gamma \in \mathcal{G}(s_0^+, s_0^-)$, and set

$$\psi_k(x) := \gamma\left(\frac{x\cdot\nu_0}{\delta_k}\right)\,.$$

Using a De Giorgi's type slicing argument, we shall modify w_k in order to get a function which matches ψ_k on ∂Q_{ν_0}. To this end, define

$$r_k := \|w_k - \psi_k\|^{1/2}_{L^1(Q_{\nu_0};\mathbb{R}^d)}\,, \quad M_k := k[1 + \|w_k\|_{W^{1,1}(Q_{\nu_0};\mathbb{R}^d)} + \|\psi_k\|_{W^{1,1}(Q_{\nu_0};\mathbb{R}^d)}]\,, \quad \ell_k := \frac{r_k}{M_k}\,.$$

Since ψ_k and w_k converge to u_0 in $L^1(Q_{\nu_0};\mathbb{R}^d)$, we have $r_k \to 0$, and one may assume that $0 < r_k < 1$. Set

$$Q_k^{(i)} := (1 - r_k + i\,\ell_k)Q_{\nu_0} \quad \text{for } i = 0,\dots,M_k\,.$$

For every $i \in \{1,\dots,M_k\}$, consider a cut-off function $\varphi_k^{(i)} \in \mathcal{C}_c^\infty(Q_k^{(i)};[0,1])$ satisfying $\varphi_k^{(i)} = 1$ on $Q_k^{(i-1)}$ and $|\nabla\varphi_k^{(i)}| \leq c/\ell_k$. Define

$$z_k^{(i)} := \varphi_k^{(i)} w_k + (1 - \varphi_k^{(i)})\psi_k \in W^{1,1}(Q_{\nu_0};\mathbb{R}^d)\,,$$

so that $z_k^{(i)} = w_k$ in $Q_k^{(i-1)}$, and $z_k^{(i)} = \psi_k$ in $Q_{\nu_0} \setminus Q_k^{(i)}$. Since $z_k^{(i)}$ is smooth outside a finite union of sets contained in some $(N-2)$-dimensional submanifolds and $z_k^{(i)}(x) \in \mathrm{co}(\mathcal{M})$ for a.e. $x \in Q_{\nu_0}$, one can apply Proposition 2.1 to obtain new functions $\hat{z}_k^{(i)} \in W^{1,1}(Q_{\nu_0};\mathcal{M})$ such that $\hat{z}_k^{(i)} = z_k^{(i)}$ on $(Q_{\nu_0} \setminus Q_k^{(i)}) \cup Q_k^{(i-1)}$, and

$$\int_{Q_k^{(i)}\setminus Q_k^{(i-1)}} |\nabla \hat{z}_k^{(i)}|\, dx \leq C_\star \int_{Q_k^{(i)}\setminus Q_k^{(i-1)}} |\nabla z_k^{(i)}|\, dx \leq C_\star \int_{Q_k^{(i)}\setminus Q_k^{(i-1)}} \left(|\nabla w_k| + |\nabla\psi_k| + \frac{1}{\ell_k}|w_k - \psi_k|\right) dx\,.$$

In particular $\hat{z}_k^{(i)} \in \mathcal{B}_{\delta_k}(s_0^+, s_0^-, \nu_0)$, and by the growth condition (3.12),

$$\int_{Q_{\nu_0}} f^\infty\left(\frac{x}{\delta_k}, \nabla \hat{z}_k^{(i)}\right) dx \le \int_{Q_{\nu_0}} f^\infty\left(\frac{x}{\delta_k}, \nabla w_k\right) dx + C\int_{Q_{\nu_0}\setminus Q_k^{(i)}} |\nabla \psi_k|\, dx + \\ + C\int_{Q_k^{(i)}\setminus Q_k^{(i-1)}} \left(|\nabla w_k| + |\nabla \psi_k| + \frac{1}{\ell_k}|w_k - \psi_k|\right) dx\,.$$

Summing up over all $i = 1, \dots, M_k$ and dividing by M_k, we get that

$$\frac{1}{M_k}\sum_{i=1}^{M_k}\int_{Q_{\nu_0}} f^\infty\left(\frac{x}{\delta_k}, \nabla \hat{z}_k^{(i)}\right) dx \le \int_{Q_{\nu_0}} f^\infty\left(\frac{x}{\delta_k}, \nabla w_k\right) dx + \\ + C\int_{Q_{\nu_0}\setminus Q_k^{(0)}} |\nabla \psi_k|\, dx + \frac{C}{k} + C\|w_k - \psi_k\|^{1/2}_{L^1(Q_{\nu_0};\mathbb{R}^d)}\,.$$

Since

$$\int_{Q_{\nu_0}\setminus Q_k^{(0)}} |\nabla \psi_k|\, dx \le \mathbf{d}_{\mathcal{M}}(s_0^+, s_0^-)\mathcal{H}^{N-1}((Q_{\nu_0}\setminus Q_k^{(0)}) \cap \{x\cdot \nu_0 = 0\}) \to 0$$

as $k \to +\infty$, there exists a sequence $\eta_k \to 0^+$ such that

$$\frac{1}{M_k}\sum_{i=1}^{M_k}\int_{Q_{\nu_0}} f^\infty\left(\frac{x}{\delta_k}, \nabla \hat{z}_k^{(i)}\right) dx \le \int_{Q_{\nu_0}} f^\infty\left(\frac{x}{\delta_k}, \nabla w_k\right) dx + \eta_k\,.$$

Hence, for each $k \in \mathbb{N}$ we can find some index $i_k \in \{1, \dots, M_k\}$ satisfying

$$\int_{Q_{\nu_0}} f^\infty\left(\frac{x}{\delta_k}, \nabla \hat{z}_k^{(i_k)}\right) dx \le \int_{Q_{\nu_0}} f^\infty\left(\frac{x}{\delta_k}, \nabla w_k\right) dx + \eta_k\,. \tag{5.46}$$

Gathering (5.45) and (5.46), we obtain that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\llcorner S_u}(x_0) \ge \limsup_{k\to+\infty}\int_{Q_{\nu_0}} f^\infty\left(\frac{x}{\delta_k}, \nabla \hat{z}_k^{(i_k)}\right) dx\,.$$

Since $\hat{z}_k^{(i_k)} \in \mathcal{B}_{\delta_k}(s_0^+, s_0^-, \nu_0)$, we infer from Proposition 3.3, Proposition 3.4 and (3.34) that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\llcorner S_u}(x_0) \ge \vartheta_{\rm hom}(s_0^+, s_0^-, \nu_0)\,,$$

which completes the proof. □

5.5. *Proof of Theorem 1.3*

Proof of Theorem 1.3. In view of (H_2) and the closure of the pointwise constraint under strong L^1-convergence, $\mathcal{F}(u) < +\infty$ implies $u \in BV(\Omega; \mathcal{M})$. In view of (5.8), Lemma 5.3, Corollary 5.1 and Lemma 5.5, the subsequence $\{\mathcal{F}_{\varepsilon_n}\}$ Γ-converges to $\mathcal{F}_{\rm hom}$ in $L^1(\Omega; \mathbb{R}^d)$. Since the Γ-limit does not depend on the particular choice of the subsequence, we get in light of [19, Proposition 8.3] that the whole sequence Γ-converges. □

Appendix

We present in this appendix a relaxation result already proved in [2] for $\mathcal{M} = \mathbb{S}^{d-1}$, and in [35] for isotropic integrands. We shall omit the proof since it can be obtained repeating the one of [2, Theorem 3.1] with minor modifications. It suffices to replace the standard projection on the sphere (used in Lemma 5.2, Proposition 6.2 and Lemma 6.4 of [2]) by the projection on $\mathcal{M}$ of [33] as in Proposition 2.1.

Assume that $\mathcal{M}$ is a smooth, compact and connected submanifold of $\mathbb{R}^d$ without boundary, and let $h : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, +\infty)$ be a continous function satisfying:

(i) h is tangentially quasiconvex, *i.e.*, for all $x \in \Omega$, all $s \in \mathcal{M}$ and all $\xi \in [T_s(\mathcal{M})]^N$,

$$h(x, s, \xi) \le \int_Q h(x, s, \xi + \nabla\varphi(y))\, dy \quad \text{for every } \varphi \in W_0^{1,\infty}(Q; T_s(\mathcal{M}))\,;$$

(ii) there exist α and $\beta > 0$ such that

$$\alpha|\xi| \leq h(x,s,\xi) \leq \beta(1+|\xi|) \quad \text{for every } (x,s,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N};$$

(iii) for every compact set $K \subset \Omega$, there exists a continuous function $\omega : [0,+\infty) \to [0,+\infty)$ satisfying $\omega(0) = 0$ and

$$|h(x,s,\xi) - h(x',s',\xi)| \leq \omega(|x-x'| + |s-s'|)(1+|\xi|)$$

for every x, $x' \in \Omega$, s, $s' \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{d\times N}$;

(iv) there exist $C > 0$ and $q \in (0,1)$ such that

$$|h(x,s,\xi) - h^\infty(x,s,\xi)| \leq C(1+|\xi|^{1-q}), \quad \text{for every } (x,s,\xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N},$$

where $h^\infty : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0,+\infty)$ is the recession function of h defined by

$$h^\infty(x,s,\xi) := \limsup_{t\to+\infty} \frac{h(x,s,t\xi)}{t}.$$

Consider the functional $F : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ given by

$$F(u) := \begin{cases} \displaystyle\int_\Omega h(x,u,\nabla u)\,dx & \text{if } u \in W^{1,1}(\Omega;\mathcal{M}), \\ +\infty & \text{otherwise,} \end{cases}$$

and its relaxation for the strong $L^1(\Omega;\mathbb{R}^d)$-topology $\overline{F} : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined by

$$\overline{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} F(u_n) : u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d) \right\}.$$

Then the following integral representation result holds:

Theorem A.1. *For every $u \in L^1(\Omega;\mathbb{R}^d)$,*

$$\overline{F}(u) := \begin{cases} \displaystyle\int_\Omega h(x,u,\nabla u)\,dx + \int_{\Omega\cap S_u} H(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} \\ \displaystyle\qquad\qquad + \int_\Omega h^\infty\left(x,\tilde{u},\frac{dD^c u}{d|D^c u|}\right) d|D^c u| & \text{if } u \in BV(\Omega;\mathcal{M}), \\ +\infty & \text{otherwise}, \end{cases}$$

where for every $(x,a,b,\nu) \in \Omega \times \mathcal{M} \times \mathcal{M} \times \mathbb{S}^{N-1}$,

$$H(x,a,b,\nu) := \inf_\varphi \left\{ \int_{Q_\nu} h^\infty(x,\varphi(y),\nabla\varphi(y))\,dy : \varphi \in W^{1,1}(Q_\nu;\mathcal{M}),\ \varphi = a \text{ on } \{x\cdot\nu = 1/2\}, \right.$$
$$\left. \varphi = b \text{ on } \{x\cdot\nu = -1/2\} \text{ and } \varphi \text{ is 1-periodic in the } \nu_2,\dots,\nu_N \text{ directions} \right\},$$

$\{\nu,\nu_2,\dots,\nu_N\}$ forms an orthonormal basis of $\mathbb{R}^N$, and Q_ν stands for the open unit cube in $\mathbb{R}^N$ centered at the origin associated to this basis.

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