Solutions to the nonlinear Schrödinger equation carrying momentum along a curve. Part I: study of the limit set and approximate solutions

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ABSTRACT. We prove existence of a special class of solutions to the (elliptic) Nonlinear Schrödinger Equation $-\varepsilon^2\Delta\psi + V(x)\psi = |\psi|^{p-1}\psi$, on a manifold or in the Euclidean space. Here V represents the potential, p an exponent greater than 1 and ε a small parameter corresponding to the Planck constant. As ε tends to zero (namely in the semiclassical limit) we prove existence of complex-valued solutions which concentrate along closed curves, and whose phase is highly oscillatory. Physically, these solutions carry quantum-mechanical momentum along the limit curves. In this first part we provide the characterization of the limit set, with natural stationarity and non-degeneracy conditions. We then construct an approximate solution up to order ε^2, showing that these conditions appear naturally in a Taylor expansion of the equation in powers of ε. Based on these, an existence result will be proved in the second part [40].

Key Words: Nonlinear Schrödinger Equation, Singularly Perturbed Elliptic Problems, Local Inversion.

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1 Introduction

In this paper we are concerned with concentration phenomena for solutions of the singularly-perturbed elliptic equation

$$(NLS_\varepsilon) \qquad -\varepsilon^2\Delta_g\psi + V(x)\psi = |\psi|^{p-1}\psi \qquad \text{on } M,$$

where M is an n-dimensional compact manifold (or the flat Euclidean space $\mathbb{R}^n$), V a smooth positive function on M satisfying the properties

$$(1) \qquad 0 < V_1 \leq V \leq V_2; \qquad \|V\|_{C^3} \leq V_3,$$

(for some fixed constants V_1, V_2, V_3) ψ a complex-valued function, $\varepsilon > 0$ a small parameter and p an exponent greater than 1. Here Δ_g stands for the Laplace-Beltrami operator on (M, g).

(NLS_ε) arises from the study of the Nonlinear Schrödinger Equation

$$(2) \qquad i\hbar\frac{\partial\tilde{\psi}}{\partial t} = -\hbar^2\Delta\tilde{\psi} + V(x)\tilde{\psi} - |\tilde{\psi}|^{p-1}\tilde{\psi} \qquad \text{on } M \times [0, +\infty),$$

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where $\tilde{\psi} = \tilde{\psi}(x,t)$ is the *wave function*, $V(x)$ a potential, and $\hbar$ the *Planck constant.* A special class of solutions to (2) is constituted by the functions whose dependence on the variables x and t is of the form $\tilde{\psi}(x,t) = e^{-i\frac{\omega t}{\hbar}}\psi(x)$. Such solutions are called *standing waves* and up to substituting $V(x)$ with $V(x) - \omega$, they give rise to solutions of (NLS_ε), for $\varepsilon = \hbar$.

An interesting case is the *semiclassical limit* $\varepsilon \to 0$, where one should expect to recover the Newton law of classical mechanics. In particular, near stationary points of the potential, one is lead to search highly concentrated solutions, which could mimic point-particles at rest.

In recent years, a lot of attention has been devoted to the study of the above problem: one of the first results in this direction is due to Floer and Weinstein in [26], where the case of $M = \mathbb{R}$ and $p = 3$ is considered, and where existence of solutions highly concentrated near critical points of V has been proved. This result has then been extended by Oh, [51], to the case of $\mathbb{R}^n$ for arbitrary n, provided $1 < p < \frac{n+2}{n-2}$. The profile of these solutions is given by the *ground state* U_{x_0} (namely the solution with minimal energy, which is real-valued, everywhere positive and can be assumed radial) of the following *limit equation*

$$-\Delta u + V(x_0)u = u^p \qquad \text{in } \mathbb{R}^n, \tag{3}$$

where x_0 is the concentration point. The solutions u_ε obtained in the aforementioned papers behave qualitatively like $u_\varepsilon(x) \simeq U_{x_0}\left(\frac{x-x_0}{\varepsilon}\right)$ as ε tends to zero, and since U_{x_0} decays exponentially to zero at infinity, u_ε vanishes rapidly away from x_0.

Two comments are in order: first of all the criticality of V at x_0 is a *necessary condition* for such a behavior, as shown in [55]. Secondly, as pointed out in [17], also the upper bound $p < \frac{n+2}{n-2}$ is required for having solutions concentrating at points: indeed the well-known *Pohozaev's identity* imposes this restriction for having existence of solutions to (3) tending to zero at infinity.

The above existence results have been extended in several directions, including the construction of solutions with multiple peaks, the case of degenerate potentials, potentials tending to zero at infinity and more general nonlinearities. We refer the interested reader for example to the (incomplete) list of works [1], [2], [3], [6], [7], [8], [16], [22], [28], [33] and to the bibliographies therein.

We also mention the mathematical similarities between (NLS_ε) and the following problem

$$(P_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $p > 1$, and where ν denotes the exterior unit normal vector to $\partial\Omega$. Problem (P_ε) arises in the study of some biological models, see for example [47] and references therein, and as (NLS_ε) it exhibits concentration of solutions at some points of $\overline{\Omega}$. Since the last equation is homogeneous, the location of the concentration points is determined by the geometry of the domain: if it occurs at the boundary, these are critical points of the mean curvature while if it occurs at the interior these points are (roughly) singular points for the distance function from the boundary. About this topic, we refer the reader to [19], [23], [29], [30], [31], [32], [35], [36], [37], [48], [49], [50], [56].

More recently, new types of solutions to (NLS_ε) have been found, since when ε tends to zero they do not concentrate at points, but instead at sets of higher dimension. Before stating our main result, it is convenient to recall the progress on this topic and to illustrate the new phenomena involved. Some first results in this direction were given in [12], [14] in the case of radial symmetry, and later improved in [4] (see also [5] for the problem in bounded domains), where necessary and sufficient conditions for the location of the concentration set have been given. Differently from the previous case, the limit set is not anymore stationary for the potential V: indeed, from heuristic considerations, the *energy* of a solution concentrated near a sphere of radius r depends both on V and on its *volume*, which is proportional to εr^{n-1}. In [4] it was shown that the candidate radii of concentration are the critical points of the function $r^{n-1}V(r)^{\frac{p+1}{p-1}-\frac{1}{2}}$ (the power of V in this formula arises from some scaling argument, related to the dependence in $V(x_0)$ of the solutions to (3), see also Section 2). Furthermore, no upper bound on the exponent p is required: in fact the profile of these solutions is given by the solution of (3) in $\mathbb{R}^1$, and in one dimension there is no restriction for the existence of entire solutions.

Based on the above energy considerations, in [4] it is also stated a conjecture concerning concentration on k-dimensional manifolds, for $k = 1, \dots, n-1$: it is indeed expected that, under suitable non-degeneracy assumptions, the limit set should satisfy the equation

$$\theta_k \nabla^N V = V\mathbf{H}, \qquad \text{with} \quad \theta_k = \frac{p+1}{p-1} - \frac{1}{2}(n-k), \tag{4}$$

where ∇^N stands for the normal gradient, $\mathbf{H}$ the curvature vector, and the profile of the solutions at a point x_0 in the limit set should be asymptotic, in the normal directions, to the ground state of

$$-\Delta u + V(x_0)u = u^p \qquad \text{in } \mathbb{R}^{n-k}. \tag{5}$$

Since the Pohozaev identity implies $p < \frac{n-k+2}{n-k-2}$ for the existence of non trivial solutions, the latter condition is expected to be a natural one for dealing with this phenomenon.

Actually, concerning (P_ε) another conjecture has been previously stated, asserting existence of solutions concentrating at sets of positive dimension. Concerning the latter problem, starting from the paper [43], there has been some progress in the general setting (without symmetry assumptions), and after the works [39], [42], [44], existence in now known for arbitrary dimension and codimension. About problem (NLS_ε), the conjecture in [4] has been verified in [24] for $n = 2$ and $k = 1$. Some other (and related) results, under some reduced symmetry assumptions (as cylindrical or similar) have been given in [13], [20], [46], [52]. Specifically we mention the note [57]: instead of (2), it is considered there the nonlinear wave equation

$$-\frac{\partial^2 \tilde{\psi}}{\partial t^2} = -\Delta\tilde{\psi} + V(x)\tilde{\psi} - |\tilde{\psi}|^{p-1}\tilde{\psi} \qquad \text{on } M \times [0, +\infty) \tag{6}$$

for $p = 3$. It has been proved in [9] and [10] that

$$-\frac{\partial^2 \tilde{\psi}}{\partial t^2} = -\Delta\tilde{\psi} + V(x)\tilde{\psi}$$

has solutions which remain concentrated near elliptic closed geodesics in M for long periods of time, but which eventually drift away when $t \to \infty$. A. Weinstein in [57] proved that there do exist periodic solutions of (6) remaining concentrated for all times, whenever $p = 3$, $M = S^2$ and V is odd.

It is worth pointing out a major difference between the symmetric and the non-symmetric situation. In fact, since the ground states of (3) or (5) are of mountain-pass type (namely critical points of some Euler functional with Morse index equal to 1), equation (NLS_ε) becomes highly resonant. To explain the reason, we consider for example a real-valued function ψ in $\mathbb{R}^2$ with a radial potential. One can begin by finding approximate (radial) solutions of the form $u_{\overline{r},\varepsilon}(r) \simeq U_{\overline{r}}\left(\frac{r-\overline{r}}{\varepsilon}\right)$, where $U_{\overline{r}}$ is the solution of (3) for $n = 1$ corresponding to $V(\overline{r})$. Then, with a good choice of $\overline{r}$, one can try to linearize the equation and find true solutions via the implicit function theorem. The linearized equation, taking ψ real for simplicity, becomes

$$-\varepsilon^2\Delta\psi + V(r)\psi - pu_{\overline{r},\varepsilon}(r)^{p-1}\psi \qquad \text{in } \mathbb{R}^2.$$

Using polar coordinates (r, ϑ) and a Fourier decomposition of ψ with respect to ϑ, $\psi(r, \vartheta) = \sum_j e^{ij\vartheta}\psi_j(r)$, we see that on each component ψ_j acts the operator

$$\underbrace{-\varepsilon^2\psi_j'' - \varepsilon^2\frac{1}{r}\psi_j' + V(r)\psi_j - pu_{\overline{r},\varepsilon}(r)^{p-1}\psi_j}_{L_{1,\varepsilon}\psi_j} + \frac{1}{r^2}\varepsilon^2 j^2\psi_j \qquad \text{on } [0, +\infty), \tag{7}$$

where $L_{1,\varepsilon}$ (apart from the term $\varepsilon^2\frac{1}{r}\psi_j'$ which is not relevant to the next discussion) represents the linearized equation of (NLS_ε) in one dimension near a soliton. Since one expects to deal with functions which are highly concentrated near $r = \overline{r}$, the last term in the above formula naively *increases* the eigenvalues by a quantity of order $\frac{1}{\overline{r}^2}\varepsilon^2 j^2$ compared to those of $L_{1,\varepsilon}$.

The operator $L_{1,\varepsilon}$ possesses a negative eigenvalue η_ε lying between two negative constants independent of ε (since $U_{\overline{r}}$ is of mountain-pass type, as explained before) and a (nearly) zero eigenvalue σ_ε, by the translation invariance of (3) in $\mathbb{R}^1$. As a consequence, the operator in (7) will possess two sequences of eigenvalues qualitatively of the form $\eta_{j,\varepsilon} \simeq \eta_\varepsilon + \varepsilon^2 j^2$ and $\sigma_{j,\varepsilon} \simeq \sigma_\varepsilon + \varepsilon^2 j^2$. This might generate two kinds of resonances: for small values of j, when $\sigma_{j,\varepsilon} \simeq 0$, and for j of order $\frac{1}{\varepsilon}$, when $\eta_{j,\varepsilon}$ could be close to zero. A comment is in order on the corresponding eigenfunctions, which can be roughly studied with a separation of variables as before. The ones relative to $\sigma_{j,\varepsilon}$ (for j small) are slowly oscillating along the limit set, while the ones relative to the resonant $\eta_{j,\varepsilon}$'s are fast oscillating with a number of oscillations proportional to $k \simeq \frac{1}{\varepsilon}$.

The invertibility of the linearized operator will then be equivalent to having all the $\sigma_{j,\varepsilon}$'s and all the $\eta_{j,\varepsilon}$'s different from zero. A control on the resonant $\sigma_{j,\varepsilon}$'s can be obtained (via some careful expansions) from a suitable non-degeneracy condition involving the limit set and the potential V. In [4] for example, this can be achieved from the fact of having a non-degenerate critical point of the function $r^{n-1}V(r)^{\frac{p+1}{p-1}-\frac{1}{2}}$. On the other hand, the possible vanishing of some $\eta_{j,\varepsilon}$ is peculiar of this concentration behavior and more intrinsic, so invertibility can only be achieved by choosing suitable values of ε. It is interesting to compare this phenomenon (which is also present in (P_ε)) to a result in [18], asserting that if the Morse index of a family of solutions to (P_ε) stays bounded as $\varepsilon \to 0$, these must concentrate at a finite number of points.

These formal considerations can also apply to the case of concentration near a general manifold (without symmetry) in higher dimension or codimension. Instead of expanding in polar coordinates one can use (naively) a Fourier decomposition with respect to the eigenfunctions of the Laplace-Beltrami operator and the normal Laplacian on the limit manifold, see [39]. If the latter has dimension k then the term $\varepsilon^2 j^2$, by the Weyl's asymptotic formula, has to be substituted with a quantity behaving like $\varepsilon^2 j^{\frac{2}{k}}$. We notice that in this way the average distance between two consecutive $\eta_{j,\varepsilon}$'s (when they are close to zero) is of order ε^k so, even if we have invertibility, the distance of the spectrum to zero is (in the best cases) of order $\min\{\min_j |\eta_{j,\varepsilon}|, \min_j |\sigma_{j,\varepsilon}|\} \simeq \min\{\varepsilon^2, \varepsilon^k\}$. Therefore the inverse operator is always large in norm. By this reason, to apply the implicit function theorem we need first to find good approximate solutions, with a precision depending on k, and then prove indeed that the linearized operator is invertible for suitable values of ε. This is indeed a rather delicate issue: for reasons of brevity we do not discuss it here but we refer directly to [43] and [44]. Related phenomena appear in some geometric problems as well, dealing with the construction of surfaces with constant mean curvature, see [38], [45].

When Ω is a radially symmetric domain and the potential V is radially symmetric, the problem is simpler, since working in spaces of invariant functions avoids most of the above resonances. In this case only finitely-many eigenvalues (depending on the dimension and the codimension of the concentration set) can approach zero, and localization can be determined with a finite-dimensional reduction of the problem.

In this paper and in [40] we construct a new type of solutions, which concentrate along some curve γ, and which physically carry momentum along the limit set. Differently from the solutions discussed before, these are complex-valued and their profile near any point x_0 in the image of γ is asymptotic to a solution to (3) which decays exponentially to zero away from the x_n axis of $\mathbb{R}^n$ and is periodic in x_n. More precisely, we consider solutions of the form

$$\phi(x', x_n) = e^{-i\hat{f}x_n}\hat{U}(x'), \qquad x' = (x_1, \dots, x_{n-1}),$$

where $\hat{f}$ is some constant and $\hat{U}(x')$ a real function. With this choice of ϕ, the function $\hat{U}$ satisfies

$$-\Delta\hat{U} + \left(\hat{f}^2 + V(x_0)\right)\hat{U} = |\hat{U}|^{p-1}\hat{U} \qquad \text{in } \mathbb{R}^{n-1}, \tag{8}$$

and decays to zero at infinity. Solutions to (8) can be found by considering the (real) function U satisfying $-\Delta U + U = U^p$ in $\mathbb{R}^{n-1}$ (decaying to zero at infinity), and by using the scaling

$$\hat{U}(x') = \hat{h}U(\hat{k}x'), \qquad \hat{h} = \left(\hat{f}^2 + V(x_0)\right)^{\frac{1}{p-1}}, \quad \hat{k} = \left(\hat{f}^2 + V(x_0)\right)^{\frac{1}{2}}. \tag{9}$$

In the above formulas $\hat{f}$ can be taken arbitrarily, and $\hat{h}, \hat{k}$ have to be chosen accordingly, depending on $V(x_0)$. The constant $\hat{f}$ represents the speed of the phase oscillation, and is physically related to the velocity of the quantum-mechanical particle represented by the wave function.

We are aware of one result only in this direction, given in [21], where the case of an axially-symmetric potential is considered, and our goal here is to treat this phenomenon in a generic situation, without any symmetry restriction. Some of the difficulties of such an extension were naively summarized in the above discussion but some new ones arise, due to the fact that the standing waves are complex-valued, and due to their highly oscillatory phase. In this first part we determine the concentration set and show that its geometric characterization appears when we construct approximate solutions to (NLS_ε), while a full existence result will be given in [40].

Before stating our main result we discuss how to determine the limit set: if we look for a solution ψ to (NLS_ε) with the above profile, then it should qualitatively behave as

$$\psi(\overline{s}, \zeta) \simeq e^{-i\frac{f(\overline{s})}{\varepsilon}} h(\overline{s}) U\left(\frac{k(\overline{s})\zeta}{\varepsilon}\right), \tag{10}$$

where $\overline{s}$ stands for the arc-length parameter of γ, and ζ for a system of geodesic coordinates normal to γ. For having more flexibility, we chose the phase oscillation to depend on the point $\gamma(\overline{s})$, while $h(\overline{s}), k(\overline{s})$ should satisfy

$$h(\overline{s}) = \left((f'(\overline{s}))^2 + V(\gamma(\overline{s}))\right)^{\frac{1}{p-1}}, \qquad k(\overline{s}) = \left((f'(\overline{s}))^2 + V(\gamma(\overline{s}))\right)^{\frac{1}{2}}, \tag{11}$$

which is the counterpart of (9) for a variable potential.

The function $f(\overline{s})$ can be (heuristically) determined using an expansion of (NLS_ε) at order ε: a computation performed in Subsection 2.3 (see in particular formula (23)) shows that

$$f'(\overline{s}) \simeq \mathcal{A} h^\sigma(\overline{s}) \qquad \text{with} \quad \sigma = \frac{(n-1)(p-1)}{2} - 2, \tag{12}$$

where $\mathcal{A}$ is an arbitrary constant. At this point, only the curve γ should be determined. First of all, we notice that the phase should be a periodic function in the length of the curve, and therefore by (12) it is natural to work in the class of loops

$$\Gamma := \left\{\gamma : \mathbb{R} \to M \text{ periodic } : \mathcal{A} \int_\gamma h(\overline{s})^\sigma d\overline{s} = \text{constant}\right\}, \tag{13}$$

where $\overline{s}$ stands for the arc-length parameter on γ. Problem (NLS_ε) has a variational structure, with Euler-Lagrange functional given by

$$E_\varepsilon(\psi) = \frac{1}{2} \int_M \left(\varepsilon^2 |\nabla_g \psi|^2 + V(x)|\psi|^2\right) - \frac{1}{p+1} \int_M |\psi|^{p+1}.$$

For a function of the form (10), by a scaling argument (see (26)) one has

$$E_\varepsilon(\psi) \simeq \varepsilon^{n-1} \int_\gamma h(\overline{s})^\theta d\overline{s}, \qquad \text{with} \quad \theta = p + 1 - \frac{1}{2}(p-1)(n-1), \tag{14}$$

therefore a limit curve γ should be a critical point of the functional $\int_\gamma h(\overline{s})^\theta d\overline{s}$ in the class Γ. With a direct computation, see Subsection 2.4, one can check that the extremality condition is the following

$$\nabla^N V = \mathbf{H}\left(\frac{p-1}{\theta} h^{p-1} - 2\mathcal{A}^2 h^{2\sigma}\right) \tag{15}$$

where, as before, $\nabla^N V$ represents the normal gradient of V and $\mathbf{H}$ the curvature vector of γ. Similarly, via some long but straightforward calculation, one can find a natural non-degeneracy condition for stationary points, which is expressed by the invertibility of the operator in (36) acting on the normal sections to γ (we refer the reader to Section 2 for the notation used in the formula). We notice that, since formula (12) determines only the derivative of the phase, to obtain periodicity we need to introduce some nonlocal terms, see (29). After these preliminaries, we are in position to state our main result.

Theorem 1.1 *Let M be a compact n-dimensional manifold, let $V : M \to \mathbb{R}$ be a smooth positive function and let $1 < p < \frac{n+1}{n-3}$. Let γ be a simple closed curve in M: then there exists a positive constant $\mathcal{A}_0$, depending on $V|_\gamma$ and p for which the following holds. If $0 \leq \mathcal{A} < \mathcal{A}_0$, if γ satisfies (15) and the operator in (36) is invertible on the normal sections of γ, there is a sequence $\varepsilon_k \to 0$ such that problem (NLS_{ε_k}) possesses solutions ψ_{ε_k} having the asymptotics in (10), with f satisfying (12).*

Remark 1.2 *(a) The statement of Theorem 1.1 remains unchanged if we replace M by $\mathbb{R}^n$ (or with an open manifold asymptotically Euclidean at infinity) and we assume V to be bounded between two positive constants and for which $\|\nabla^l V\| \leq C_l$, $l = 1, 2, 3$ for some positive constants C_l.*

(b) The restriction on the exponent p is natural, since (8) admits solitons if and only if p is subcritical with respect to the dimension $n - 1$.

(c) The smallness requirement on $\mathcal{A}$ is technical and we believe this condition could be relaxed. Anyway, for $\frac{n+2}{n-2} \leq p < \frac{n+1}{n-3}$, $\mathcal{A}$ should have an upper bound depending on V, to have solvability of both (9) and (12). About this condition see Remark 2.2 and Remark 2.7 in [40].

(d) Apart from the assumption on $\mathcal{A}$, Theorem 1.1 improves the result in [21]. In fact, in addition to removing the symmetry condition (which is the main issue), the characterization of the limit set is explicit, the assumptions on V are purely local, and the upper bound on p is sharp.

(e) The existence of solutions to (NLS_ε) only for a suitable sequence $\varepsilon_k \to 0$ is related to the resonance phenomenon described above. The result can be extended to a sequence of intervals in the parameter ε approaching zero but, at least with our proof, we do not expect to find existence for all the epsilon's.

Taking $\mathcal{A} = 0$ (which implies $f' \equiv 0$), from (11) it follows that $V = h^{p-1}$ and that condition (15) is equivalent to (4), so as a consequence of our result we can prove the conjecture in [4] for $k = 1$, extending the result in [24].

Corollary 1.3 *Let M be a compact Riemannian n-dimensional manifold with metric g, let $V : M \to \mathbb{R}$ be a function satisfying (1) and let $1 < p < \frac{n+1}{n-3}$. Let γ be a simple closed curve which is a non-degenerate geodesic with respect to the weighted metric $V^{\frac{p+1}{p-1} - \frac{n-1}{2}} g$. Then there is a sequence $\varepsilon_k \to 0$ such that problem (NLS_{ε_k}) possesses real-valued solutions ψ_{ε_k} concentrating near γ as $j \to +\infty$ and having the asymptotic behavior*

$$\psi_{\varepsilon_k}(\overline{s}, \zeta) \simeq V(\gamma(\overline{s}))^{\frac{1}{p-1}} U \left(\frac{V(\gamma(\overline{s}))^{\frac{1}{2}}}{\varepsilon_k} \zeta \right),$$

where $\overline{s}$ stands for the arc-length parameter of γ, and ζ for a system geodesic coordinates normal to γ.

Corollary 1.3 gives also some criterion for the applicability of Theorem 1.1: in fact, starting from a non-degenerate geodesic in the weighted metric, via the implicit function theorem for $\mathcal{A}$ sufficiently small one obtains a curve for which (15) and the invertibility of (36) hold. In particular, when V is constant, one can start with non-degenerate close geodesics on M in the ordinary sense.

The full proof of Theorem 1.1 will be given in the second part of the paper, [40]. In this first part we derive some formal expansions of equation (NLS_ε) for ψ of the form (10), and check that the assumptions of the theorem (the stationarity and the non-degeneracy conditions) guarantee to find approximate solutions up to any powers of ε. *At a formal level*, we consider expansions with coefficients depending smoothly on the variable $\overline{s}$, and the only obstructions to an iterative solvability of (NLS_ε) are given by the presence of a kernel in the linearization of (8), which is generated by the functions $(\partial_{x_j} U)$, $j = 1, \dots, n-1$ and by iU: this kernel arises naturally from the invariance of (8) by translation in $\mathbb{R}^n$ and by complex rotation. However, we can guarantee approximate solvability up to any order provided γ is stationary and non-degenerate: precisely, we prove here the following weaker version of the above theorem.

Theorem 1.4 *Suppose the assumptions of Theorem 1.1 hold. Then for any $m \in \mathbb{N}$ and for ε small, there exists a function $\psi_{\varepsilon,m}$ with the profile (10) and which solves (2) up to order $o(\varepsilon^m)$.*

As discussed before, also some fast-oscillating functions (along γ) contribute to generate some resonance, but we do not discuss this aspect here. Postponing the description of the rigorous proof to the introduction of [40], here we limit ourselves to mention the main new difficulty compared to the results in [24], [39] [42] and [43]. In our case the solutions are complex-valued, and this causes an extra degeneracy in the equation, due to its invariance under multiplication by a phase factor. As a consequence, we have a further (infinite-dimensional) approximate kernel, corresponding roughly to factor of ψ_ε in the form $e^{-if_1(\bar{s})}$, for f_1 arbitrary. The correction in the phase can also be determined by a formal expansion in powers of ε and, as for f', we still obtain nonlocal terms. Also, when expanding formally the solutions in powers of ε, the highly oscillatory behavior of solutions generates an increasing number of derivatives in $\bar{s}$: anyway in the first part, where formal expansions are carried out, this aspect is not very relevant.

The plan of the paper is the following. In Section 2 we study the functional in (14) constrained to the class of curves Γ, and we determine the Euler-Lagrange equation together with the non-degeneracy condition. In Section 3 approximate solutions to (NLS_ε) up to order ε are found, and the error terms of order ε^2 are displayed. The functions we obtain are allowed to depend on a section Φ of the normal bundle to γ and on a scalar function f_1. These correspond to some tilting of the approximate solution perpendicularly to the limit set and to a variation of the phase, in order to have more flexibility. In Section 4 we consider the terms of order ε^2 and we choose f_1 and Φ so that even the terms of order ε^2 in the expansion vanish: we then arrive to the proof of Theorem 1.4. Finally in Section 5 we collect some technical material, namely some integral identities and the verification of (36).

The results in this paper and in [40], together with the main ideas of the proofs, are briefly summarized in the note [41].

Notation and conventions

Dealing with coordinates, capital letters like $A, B, \ldots$ will vary between 1 and n while indices like $j, l, \ldots$ will run between 2 and n. The symbol i will always stand for the imaginary unit.

For summations, we use the standard convention of summing terms where repeated indices appear.

We will choose coordinates $(x_1, \cdots, x_n)$ near a curve γ and we will parameterize γ by arc-length letting $x_1 = \bar{s}$. Its dilation $\gamma_\varepsilon := \frac{1}{\varepsilon}\gamma$ will be parameterized by $s = \frac{1}{\varepsilon}\bar{s}$. The length of γ is denoted by L.

For simplicity, a constant C is allowed to vary from one formula to another, also within the same line.

For a real positive variable r and an integer m, $O(r^m)$ (resp. $o(r^m)$) will denote a complex-valued quantity for which $\left|\frac{O(r^m)}{r^m}\right|$ remains bounded (resp. $\left|\frac{o(r^m)}{r^m}\right|$ tends to zero) when r tends to zero. We might also write $o_\varepsilon(1)$ for a quantity which tends to zero as ε tends to zero.

2 Study of the reduced functional

In this section we consider the functional in the right-hand side of (14) defined on the set Γ, representing the approximate energy E_ε of a function concentrated near γ with the profile (10). We first introduce a convenient set of coordinates near an arbitrary (smooth) closed curve in M. Then, using these coordinates we write the Euler equation and the second variation formula at a stationary point.

2.1 Geometric preliminaries

In this Subsection we discuss some preliminary geometric facts, referring for example to [25], [53]. Given an arbitrary simple closed curve γ in M, we choose coordinates $x_1 \ldots, x_n$ near γ, called *Fermi coordinates* in the following way. We let x_1 parameterize the curve γ by arc-length. At some point q in the image of γ we consider an orthonormal $(n-1)$-tuple $(Y_2, \dots, Y_n)$ which form a basis for $N_q\gamma$, the normal bundle of γ at q. We extend the Y_l's as vector fields along γ via parallel transport along the curve with respect to the normal connection ∇^N, namely by the condition $\nabla^N_{\dot\gamma} Y_l = 0$ for $l = 2, \dots, n$.

Next we parameterize a point near γ using the following coordinates $(\bar s, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$

$$(\bar s, y_2, \dots, y_n) \mapsto \exp_{\gamma(\bar s)}(y_2 Y_2 + \dots + y_n Y_n),$$

where $\exp_q$ is the exponential map in M through the point q. In this way, fixing $\bar s$, each curve $t \mapsto ty$, for $y \in \mathbb{R}^{n-1} \setminus \{0\}$ and t close to zero, is mapped into a geodesic in M passing through $\gamma(\bar s)$.

Let us now define the vector fields $E_1 = \frac{\partial}{\partial \bar s}$ and $E_l = \frac{\partial}{\partial y_l}$ for $l = 2, \dots, n$. We notice that on γ each E_l coincides with Y_l, while E_1 on γ is nothing but $\dot\gamma$. By our choice of coordinates it follows that $\nabla_E E = 0$ on γ for any vector field E which is a linear combination (with coefficients depending only on $\bar s$) of the E_j's, $j = 2, \dots, n$. In particular, for any $l, j = 2, \dots, n$, and for any $\alpha \in \mathbb{R}$ we have $\nabla_{E_l + \alpha E_j}(E_l + \alpha E_j) = 0$ on γ, which implies $\nabla_{E_l} E_j + \nabla_{E_j} E_l = 0$ for every $l, j = 2, \dots, n$. Using the fact that E_A's are coordinate vectors for $A = 1, \dots, n$ and in particular $\nabla_{E_A} E_B = \nabla_{E_B} E_A$ for all $A, B = 1, \dots, n$, we obtain that $\nabla_{E_l} E_j = 0$ for every $l, j = 2, \dots, n$. This immediately yields

$$\partial_m g_{lj} = E_m \langle E_l, E_j \rangle = \langle \nabla_{E_m} E_l, E_j \rangle + \langle E_l, \nabla_{E_m} E_j \rangle = 0 \qquad \text{on } \gamma, \qquad l, j, m = 2, \dots, n.$$

Moreover, still since the E_A's are coordinate vectors for $A = 1, \dots, n$, we obtain

$$\begin{aligned} \partial_m g_{1j} &= E_m \langle E_1, E_j \rangle = \langle \nabla_{E_m} E_1, E_j \rangle + \langle E_1, \nabla_{E_m} E_j \rangle \\ &= \langle \nabla_{E_1} E_m, E_j \rangle + \langle E_1, \nabla_{E_m} E_j \rangle = 0 \qquad \text{on } \gamma, \qquad m, j = 2, \dots, n. \end{aligned}$$

Here we used the fact that $\nabla^N_{E_1} E_m = 0$ on γ, namely that $\nabla_{E_1} E_m$ has zero normal components.

If $\mathbf{H} = H^m E_m$ is the curvature vector of γ (which is normal to the curve), then one has $\langle \nabla_{E_1} E_m, E_1 \rangle = -H^m$ on γ, so we easily deduce that

$$\partial_m g_{11} = E_m \langle E_1, E_1 \rangle = 2 \langle \nabla_{E_1} E_m, E_1 \rangle = -2H^m \qquad \text{on } \gamma. \tag{16}$$

One can also prove that the components R_{1m1j} of the curvature tensor are given by

$$R_{1m1j} = -\frac{1}{2} \partial^2_{jm} g_{11} + H^m H^j. \tag{17}$$

Indeed, we have

$$\begin{aligned} -R_{1m1j} &= \langle R(E_1, E_j) E_1, E_m \rangle = \langle \nabla_{E_1} \nabla_{E_j} E_1, E_m \rangle - \langle \nabla_{E_j} \nabla_{E_1} E_1, E_m \rangle \\ &= \langle \nabla_{E_1} \nabla_{E_j} E_1, E_m \rangle - E_j \langle \nabla_{E_1} E_1, E_m \rangle - \langle \nabla_{E_1} E_1, \nabla_{E_j} E_m \rangle \\ &= \langle \nabla_{E_1} \nabla_{E_j} E_1, E_m \rangle - E_j \langle \nabla_{E_1} E_1, E_m \rangle \\ &= \langle \nabla_{E_1} \nabla_{E_j} E_1, E_m \rangle - E_j E_1 \langle E_1, E_m \rangle + E_j \langle E_1, \nabla_{E_1} E_m \rangle \\ &= \langle \nabla_{E_1} \nabla_{E_j} E_1, E_m \rangle + E_j \langle E_1, \nabla_{E_m} E_1 \rangle \\ &= E_1 \langle \nabla_{E_j} E_1, E_m \rangle - \langle \nabla_{E_j} E_1, \nabla_{E_1} E_m \rangle + \frac{1}{2} E_j E_m \langle E_1, E_1 \rangle \\ &= \frac{1}{2} \partial^2_{jm} g_{11} - \frac{1}{4} \partial_m g_{11} \, \partial_j g_{11}, \end{aligned}$$

where here we have used the above properties and the fact that

$$\nabla_{E_j} E_1 = \nabla_{E_1} E_j = \frac{1}{2} \partial_j g_{11} E_1.$$

Using (16) and (17), the above discussion can be summarized in the following result.

Lemma 2.1 *In the coordinates $(\overline{s}, y)$, for y close to zero the metric coefficients satisfy*

$$
\begin{aligned}
g_{11}(y) &= 1 - 2\sum_{m=2}^{n} H^m y_m + \frac{1}{2}\sum_{m,l=2}^{n} (H^m H^j - R_{1m1j}|_\gamma) y_m y_l + O(|y|^3);\\
g_{1j}(y) &= \frac{1}{2}\sum_{m,l=2}^{n} \partial^2_{ml} g_{1j}|_\gamma y_m y_l + O(|y|^3);\\
g_{kj}(y) &= \delta_{kj} + \frac{1}{2}\sum_{m,l=2}^{n} \partial^2_{ml} g_{kj}|_\gamma y_m y_l + O(|y|^3).
\end{aligned}
$$

The second derivatives $\partial^2_{ml} g_{1j}$ and $\partial^2_{ml} g_{kj}$ could be expressed in terms of the curvature tensor and the curvature of γ reasoning as for (17). However for our purposes it is not necessary to have such a formula, so we leave the expansion of these coefficients in a generic form.

2.2 First and second variations of the length functional

We recall next the formulas for the variations of the length of a curve with respect to normal displacements. We start with a regular closed curve γ in M of length L, which we parameterize by arc-length, using a parameter $\overline{s} \in [0, L]$. Then we consider a two-parameter family of closed curves $\gamma_{t_1,t_2} : [0, L] \to M$, for t_1, t_2 in a neighborhood of 0 in $\mathbb{R}$, such that $\gamma_{0,0} \equiv \gamma$. The length $L(t_1, t_2)$ of γ_{t_1,t_2} is given by

$$
L(t_1, t_2) = \int_{\gamma_{t_1,t_2}} dl = \int_0^L \langle \dot{\gamma}_{t_1,t_2}, \dot{\gamma}_{t_1,t_2} \rangle^{\frac{1}{2}} d\overline{s},
$$

where dl is the arc-length parameter and $\dot{\gamma}_{t_1,t_2}$ stands for $\frac{d\gamma_{t_1,t_2}}{d\overline{s}}$. We also define the vector fields $\mathcal{V}, \mathcal{W}$ along γ_{t_1,t_2} as $\mathcal{V} = \frac{\partial \gamma_{t_1,t_2}}{\partial t_1}$ and $\mathcal{W} = \frac{\partial \gamma_{t_1,t_2}}{\partial t_2}$. In the above coordinates, the vector fields $\mathcal{V}$ and $\mathcal{W}$ along γ can be written as

$$
\mathcal{V} = \sum_{j=2}^{n} \mathcal{V}^j(\overline{s}) E_j; \qquad \mathcal{W} = \sum_{m=2}^{n} \mathcal{W}^m(\overline{s}) E_m.
$$

Differentiating $L(t_1, t_2)$ with respect to t_1 we find

$$
\frac{\partial L(t_1, t_2)}{\partial t_1} = -\int_0^L \frac{\langle \nabla_{\mathcal{V}} \dot{\gamma}_{t_1,t_2}, \dot{\gamma}_{t_1,t_2} \rangle}{\langle \dot{\gamma}_{t_1,t_2}, \dot{\gamma}_{t_1,t_2} \rangle^{\frac{1}{2}}} d\overline{s}. \tag{18}
$$

Using (16), at $(t_1, t_2) = (0, 0)$ we have

$$
\langle \nabla_{\mathcal{V}} \dot{\gamma}_{t_1,t_2}, \dot{\gamma}_{t_1,t_2} \rangle = -\mathcal{V}^m H^m,
$$

therefore we can write the variation of the length at γ in the following way

$$
\frac{\partial L(t_1, t_2)}{\partial t_1}|_{(t_1,t_2)=(0,0)} = -\int_0^L \mathcal{V}^m H^m d\overline{s} = -\int_0^L \langle \mathcal{V}, \mathbf{H} \rangle d\overline{s}. \tag{19}
$$

Using (18) we can evaluate the second variation of the length as

$$
\frac{\partial^2 L(t_1, t_2)}{\partial t_1 \partial t_2} = \int_0^L \left[\frac{\langle \nabla_{\mathcal{W}} \dot{\gamma}_{t_1,t_2}, \nabla_{\mathcal{V}} \dot{\gamma}_{t_1,t_2} \rangle + \langle \dot{\gamma}_{t_1,t_2}, \nabla_{\mathcal{W}} \nabla_{\mathcal{V}} \dot{\gamma}_{t_1,t_2} \rangle}{\langle \dot{\gamma}_{t_1,t_2}, \dot{\gamma}_{t_1,t_2} \rangle^{\frac{1}{2}}} - \frac{\langle \dot{\gamma}_{t_1,t_2}, \nabla_{\mathcal{V}} \dot{\gamma}_{t_1,t_2} \rangle \langle \dot{\gamma}_{t_1,t_2}, \nabla_{\mathcal{W}} \dot{\gamma}_{t_1,t_2} \rangle}{\langle \dot{\gamma}_{t_1,t_2}, \dot{\gamma}_{t_1,t_2} \rangle^{\frac{3}{2}}} \right] d\overline{s},
$$

so at $(t_1, t_2) = (0, 0)$ we find

$$
\frac{\partial^2 L(t_1, t_2)}{\partial t_1 \partial t_2}|_{(t_1,t_2)=(0,0)} = \int_0^L \left[\langle \nabla_{\mathcal{W}} \dot{\gamma}, \nabla_{\mathcal{V}} \dot{\gamma} \rangle + \langle \dot{\gamma}, \nabla_{\mathcal{W}} \nabla_{\mathcal{V}} \dot{\gamma} \rangle - \langle \dot{\gamma}, \nabla_{\mathcal{V}} \dot{\gamma} \rangle \langle \dot{\gamma}, \nabla_{\mathcal{W}} \dot{\gamma} \rangle \right] d\overline{s}.
$$

Using the definition of the Riemann tensor and the fact that $\mathcal{V}$ and $\mathcal{W}$ are coordinate vector fields (so that $[\mathcal{V}, \mathcal{W}] = 0$) the last formula yields

$$\begin{aligned}
\frac{\partial^2 L(t_1,t_2)}{\partial t_1 \partial t_2}|_{(t_1,t_2)=(0,0)} &= \int_0^L [\langle \nabla_{\dot\gamma}\mathcal{W}, \nabla_{\dot\gamma}\mathcal{V}\rangle + \langle \dot\gamma, \nabla_{\mathcal{W}}\nabla_{\dot\gamma}\mathcal{V}\rangle - \langle \dot\gamma, \nabla_{\dot\gamma}\mathcal{V}\rangle\langle \dot\gamma, \nabla_{\dot\gamma}\mathcal{W}\rangle]\, d\overline{s} \\
&= \int_0^L [\langle \nabla_{\dot\gamma}\mathcal{W}, \nabla_{\dot\gamma}\mathcal{V}\rangle + \langle R(\mathcal{W},\dot\gamma)\mathcal{V}, \dot\gamma\rangle - \langle \dot\gamma, \nabla_{\dot\gamma}\mathcal{V}\rangle\langle \dot\gamma, \nabla_{\dot\gamma}\mathcal{W}\rangle]\, d\overline{s} \\
&- \int_0^L \langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{\mathcal{W}}\mathcal{V}\rangle d\overline{s}.
\end{aligned}$$

Here, we have used the fact that $g(\dot\gamma, \nabla_{\mathcal{W}}\nabla_{\dot\gamma}\mathcal{V}) = \langle R(\mathcal{W},\dot\gamma)\mathcal{V}, \dot\gamma\rangle + \dot\gamma\langle \nabla_{\mathcal{W}}\mathcal{V}, \dot\gamma\rangle - \langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{\mathcal{W}}\mathcal{V}\rangle$ and $\int_0^L \dot\gamma\langle \nabla_{\mathcal{W}}\mathcal{V}, \dot\gamma\rangle d\overline{s} = 0$. Since $\nabla_{E_l}E_j = 0$ on γ for $l, j = 2, \dots, n$, we have

$$\nabla_{\mathcal{W}}\mathcal{V} = \sum_{j,m=2,\dots,n} \mathcal{W}^m \mathcal{V}^j \nabla_{E_m} E_j \qquad \Longrightarrow \qquad \int_0^L \langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{\mathcal{W}}\mathcal{V}\rangle d\overline{s} = 0.$$

Moreover, recalling (16) we obtain

$$\nabla_{\dot\gamma}\mathcal{V} = \sum_{j=2}^n \dot{\mathcal{V}}^j E_j + \sum_{j=2}^n \mathcal{V}^j \nabla_{E_1} E_j = \sum_{j=2}^n \dot{\mathcal{V}}^j E_j - \sum_{j=2}^n H^j \mathcal{V}^j E_1.$$

This implies, at γ

$$\langle \nabla_{\dot\gamma}\mathcal{W}, \nabla_{\dot\gamma}\mathcal{V}\rangle + \langle R(\mathcal{W},\dot\gamma)\mathcal{V}, \dot\gamma\rangle - \langle \dot\gamma, \nabla_{\dot\gamma}V\rangle\langle \dot\gamma, \nabla_{\dot\gamma}\mathcal{W}\rangle \;=\; \sum_{j=2}^n \dot{\mathcal{V}}^j \dot{\mathcal{W}}^j - \sum_{j,l=2}^n R_{1j1l}\mathcal{V}^j \mathcal{W}^l.$$

In this way the second variation of the length at γ becomes

$$\frac{\partial^2 L(t_1,t_2)}{\partial t_1 \partial t_2}|_{(t_1,t_2)=(0,0)} = \int_0^L \left(\sum_j^n \dot{\mathcal{V}}^j \dot{\mathcal{W}}^j - \sum_{j,l=2}^n R_{1j1l}\mathcal{V}^j \mathcal{W}^l \right) d\overline{s}. \tag{20}$$

2.3 Determining the phase factor

In this section we derive formally the asymptotic profile of the solutions to (NLS_ε) which concentrate near some curve γ, and we determine some necessary conditions satisfied by the limit curve. For doing this, using the coordinates $(\overline{s}, y)$ introduced in Subsection 2.1, we look for approximate solutions $\psi(\overline{s}, y)$ of (NLS_ε) making the following *ansatz*

$$\psi(\overline{s}, y) = e^{-i\frac{f(\overline{s})}{\varepsilon}} h(\overline{s}) U\left(\frac{k(\overline{s})y}{\varepsilon}\right), \qquad \overline{s} \in [0, L], \quad y \in \mathbb{R}^{n-1},$$

where the function U is the unique radial solution (see [15], [27], [34], [54]) of the problem

$$\begin{cases} -\Delta U + U = U^p & \text{in } \mathbb{R}^{n-1}; \\ U(y) \to 0 & \text{as } |y| \to +\infty; \\ U > 0 & \text{in } \mathbb{R}^{n-1}, \end{cases} \tag{21}$$

and where the functions f, h and k are periodic on $[0, L]$ and have to be determined. With some easy computations we obtain

$$\frac{\partial \psi}{\partial \overline{s}} = -\frac{i f'(\overline{s})}{\varepsilon} h(\overline{s}) U\left(\frac{k(\overline{s})y}{\varepsilon}\right) e^{-i\frac{f(\overline{s})}{\varepsilon}} + e^{-i\frac{f(\overline{s})}{\varepsilon}} h'(\overline{s}) U\left(\frac{k(\overline{s})y}{\varepsilon}\right) + e^{-i\frac{f(\overline{s})}{\varepsilon}} h(\overline{s}) k'(\overline{s}) \nabla_y U\left(\frac{k(\overline{s})y}{\varepsilon}\right) \cdot \frac{y}{\varepsilon};$$

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial\overline{s}^2} &= \Bigg[-i\frac{f''(\overline{s})}{\varepsilon}h(\overline{s})U\left(\frac{k(\overline{s})y}{\varepsilon}\right) - 2i\frac{f'(\overline{s})}{\varepsilon}h'(\overline{s})U\left(\frac{k(\overline{s})y}{\varepsilon}\right) - 2i\frac{f'(\overline{s})}{\varepsilon}h(\overline{s})k'(\overline{s})\nabla_y U\left(\frac{k(\overline{s})y}{\varepsilon}\right)\cdot\frac{y}{\varepsilon} \\
&- \frac{(f'(\overline{s}))^2}{\varepsilon^2}h(\overline{s})U\left(\frac{k(\overline{s})y}{\varepsilon}\right) + 2h'(\overline{s})k'(\overline{s})\nabla U\left(\frac{k(\overline{s})y}{\varepsilon}\right)\cdot\frac{y}{\varepsilon} + h(\overline{s})k''(\overline{s})\nabla_y U\left(\frac{k(\overline{s})y}{\varepsilon}\right)\cdot\frac{y}{\varepsilon} \\
&+ h(\overline{s})(k'(\overline{s}))^2\nabla_y^2 U\left(\frac{k(\overline{s})y}{\varepsilon}\right)\left[\frac{y}{\varepsilon},\frac{y}{\varepsilon}\right] + h''(\overline{s})U\left(\frac{k(\overline{s})y}{\varepsilon}\right)\Bigg]e^{-i\frac{f(\overline{s})}{\varepsilon}},
\end{aligned}
$$

and also

$$
\Delta_y\psi(\overline{s},y) = \frac{(k(\overline{s}))^2}{\varepsilon^2}\Delta_y U\left(\frac{k(\overline{s})y}{\varepsilon}\right)e^{-i\frac{f(\overline{s})}{\varepsilon}}h(\overline{s}).
$$

Since U decays to zero at infinity (exponentially indeed, by the results in [27]), and since the function ψ is scaled of order ε near the curve γ, in a first approximation we can assume the metric g of M to be flat in the coordinates $(\overline{s},y)$, see the expansions in Lemma 2.1. We look now at the leading terms in (NLS_ε), which are of order 1. Since $-\Delta_g\psi$ is multiplied by ε^2, we have to focus on the terms of order $\frac{1}{\varepsilon^2}$ in the Laplacian of ψ. In the above expressions of $\frac{\partial\psi}{\partial\overline{s}}$, $\frac{\partial^2\psi}{\partial\overline{s}^2}$ and $\Delta_y\psi$, we have that the function U and its derivatives are of order 1 when $|y| = O(\varepsilon)$, therefore when the variables y appear as factors in these expressions, we consider them to be of order ε. For example, $\nabla^2 U\left(\frac{k(\overline{s})y}{\varepsilon}\right)\left[\frac{y}{\varepsilon},\frac{y}{\varepsilon}\right]$ will be regarded as a term of order 1.

With these criteria, using the above computations and assumptions, imposing the leading terms in (NLS_ε) to vanish we obtain

$$
-k^2(\overline{s})h(\overline{s})\Delta_y U\left(\frac{k(\overline{s})y}{\varepsilon}\right) + h(\overline{s})\left[V(\overline{s}) + (f'(\overline{s}))^2\right]U\left(\frac{k(\overline{s})y}{\varepsilon}\right) = h(\overline{s})^p U\left(\frac{k(\overline{s})y}{\varepsilon}\right)^p.
$$

From (21), we have the two relations

$$
k^2(\overline{s}) = h(\overline{s})^{p-1}; \qquad \left[V(\overline{s}) + (f'(\overline{s}))^2\right] = k(\overline{s})^2 = h(\overline{s})^{p-1}. \tag{22}
$$

We next obtain an equation for f, which is derived looking at the next-order expansion of (NLS_ε). The next coefficient arises from the terms of order $\frac{1}{\varepsilon}$ in $-\Delta_g\psi$, which are given by

$$
i\left[\frac{f''(\overline{s})}{\varepsilon}h(\overline{s})U\left(\frac{k(\overline{s})y}{\varepsilon}\right) + 2\frac{f'(\overline{s})}{\varepsilon}h'(\overline{s})U\left(\frac{k(\overline{s})y}{\varepsilon}\right) + 2\frac{f'(\overline{s})}{\varepsilon}h(\overline{s})k'(\overline{s})\nabla_y U\left(\frac{k(\overline{s})y}{\varepsilon}\right)\cdot\frac{y}{\varepsilon}\right]e^{-i\frac{f(\overline{s})}{\varepsilon}}.
$$

Multiplying this expression by $U\left(\frac{k(\overline{s})y}{\varepsilon}\right)$ and integrating in $y \in \mathbb{R}^{n-1}$, imposing vanishing of this integral as well gives

$$
\begin{aligned}
0 &= f''(\overline{s})h(\overline{s})\int_{\mathbb{R}^{n-1}} U^2\left(\frac{k(\overline{s})y}{\varepsilon}\right)dy + 2h'(\overline{s})f'(\overline{s})\int_{\mathbb{R}^{n-1}} U^2\left(\frac{k(\overline{s})y}{\varepsilon}\right)dy \\
&+ 2f'(\overline{s})h(\overline{s})k'(\overline{s})\int_{\mathbb{R}^{n-1}} U\left(\frac{k(\overline{s})y}{\varepsilon}\right)\nabla_y U\left(\frac{k(\overline{s})y}{\varepsilon}\right)\cdot\frac{y}{\varepsilon}\,dy.
\end{aligned}
$$

Integrating by parts and reasoning as for the usual Pohozaev's identity we obtain that f must satisfy

$$
f''(\overline{s})h(\overline{s}) + 2f'(\overline{s})h'(\overline{s}) - (n-1)f'(\overline{s})h(\overline{s})\frac{k'(\overline{s})}{k(\overline{s})} = 0.
$$

This is solvable in $f'(\overline{s})$ and gives, for an arbitrary constant $\mathcal{A}$

$$
f'(\overline{s}) = \mathcal{A}k(\overline{s})^{n-1}h(\overline{s})^{-2} = \mathcal{A}h(\overline{s})^{\frac{(n-1)(p-1)}{2}-2}, \tag{23}
$$

where we have used the above equation (22) for k. Now we can solve the equation for $h(\overline{s})$ depending on the potential $V(\overline{s})$ and the above constant $\mathcal{A}$. In fact, we get that $h(\overline{s})$ should solve

$$
V(\overline{s}) + \mathcal{A}^2h(\overline{s})^{2\sigma} := V(\overline{s}) + \mathcal{A}^2h(\overline{s})^{(n-1)(p-1)-4} = h(\overline{s})^{p-1}, \tag{24}
$$

where we have set

$$\sigma = \frac{(n-1)(p-1)}{2} - 2. \tag{25}$$

Remark 2.2 *We notice that, assuming $\mathcal{A}$ to be small enough (depending on V and p), the above equation is always solvable in $h(\overline{s})$. More precisely, when $p < \frac{n+2}{n-2}$ (and hence when $2\sigma < p-1$), the solution is also unique. For $p \geq \frac{n+2}{n-2}$ there might be a second solution. In this case, we just consider the smallest one, which stays uniformly bounded (both from above and below) when $\mathcal{A}$ is small enough, see Figures 1 and 2 below.*

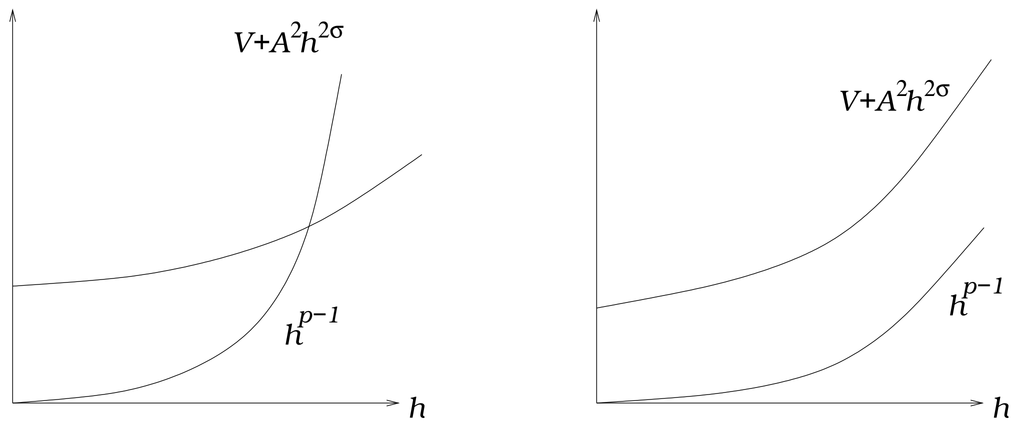

Figure 1: the graphs of $V + \mathcal{A}^2 h^{2\sigma}$ and h^{p-1} for $p < \frac{n+2}{n-2}$ and for $p = \frac{n+2}{n-2}$ with $\mathcal{A} < 1$

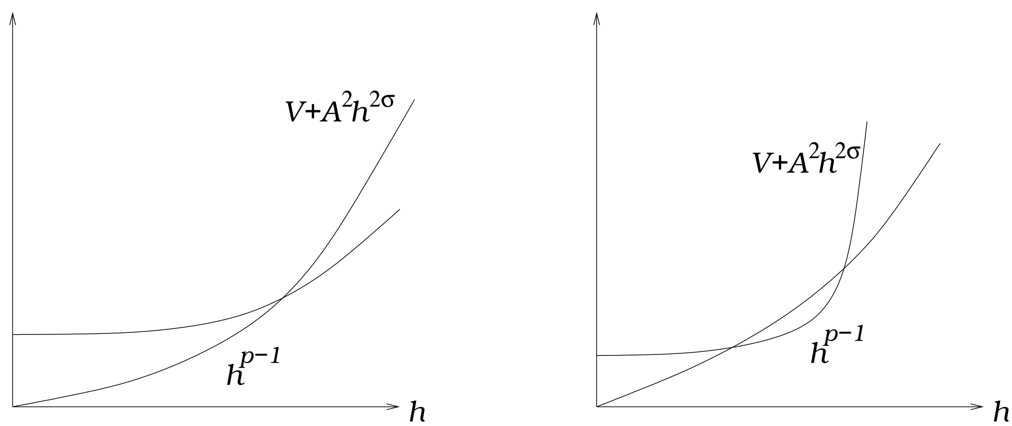

Figure 2: the graphs of $V + \mathcal{A}^2 h^{2\sigma}$ and h^{p-1} for $p = \frac{n+2}{n-2}$ with $\mathcal{A} \geq 1$ and for $p > \frac{n+2}{n-2}$ with $\mathcal{A}$ small

Remark 2.3 *In the above expansions, considering the terms of order ε, as already noticed, we considered the metric g to be flat near the curve γ, and we tacitely assumed the potential V to depend only on the variable $\overline{s}$. Indeed, expanding the Laplace-Beltrami operator and the potential V taking the variables y into account, we obtain an extra term of order ε which does not affect our computations since it turns out to be odd in y, so it vanishes once multiplied by $U\left(\frac{k(\overline{s})y}{\varepsilon}\right)$ and integrated over $\mathbb{R}^{n-1}$. For more details, we refer to Section 3, where precise estimates are worked out (in a system of coordinates scaled in ε).*

2.4 The Euler equation

Using the heuristic considerations of the previous subsection, we now compute the energy of an approximate solution ψ concentrated near a closed curve γ, and then find the γ's for which this energy is stationary. We let $\psi_{\gamma,\mathcal{A}}$ denote the function constructed in Subsection 2.3. In order for the function $\psi_{\gamma,\mathcal{A}}$ to be globally well defined, we need to impose one more restriction, namely that $\psi_{\gamma,\mathcal{A}}$ is periodic in $\overline{s}$ with period L. This is equivalent to require that $\int_0^L f'(\overline{s})d\overline{s}$ is an integer multiple of $2\pi\varepsilon$, since we have the phase factor $e^{-i\frac{f(\overline{s})}{\varepsilon}}$ in the expression of $\psi_{\gamma,\mathcal{A}}$. From (23), then we find that also $\int_0^L h(\overline{s})^\sigma d\overline{s}$ is an integer multiple of $2\pi\varepsilon$.

Multiplying (NLS_ε) by $\psi_{\gamma,\mathcal{A}}$ and integrating by parts, from the fact that $\psi_{\gamma,\mathcal{A}}$ is an approximate solution we find

$$\begin{aligned} E_\varepsilon(\psi_{\gamma,\mathcal{A}}) &= \frac{1}{2}\int_M \left(\varepsilon^2|\nabla_g\psi_{\gamma,\mathcal{A}}|^2 + V(x)|\psi_{\gamma,\mathcal{A}}|^2\right)dV_g - \frac{1}{p+1}\int_M |\psi_{\gamma,\mathcal{A}}|^{p+1} \\ &\simeq \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_M |\psi_{\gamma,\mathcal{A}}|^{p+1}dV_g. \end{aligned}$$

Since $\psi_{\gamma,\mathcal{A}}$ is highly concentrated near γ, using the coordinates $(\overline{s}, y)$ introduced in 2.1 we have that

$$\int_M |\psi_{\gamma,\mathcal{A}}|^{p+1}dV_g \simeq \int_0^L d\overline{s}\int_{\mathbb{R}^{n-1}} h(\overline{s})^{p+1}\left|U\left(\frac{k(\overline{s})y}{\varepsilon}\right)\right|^{p+1} dy.$$

Using a change of variables, the last two formulas and (22) we find that

$$E_\varepsilon(\psi_{\gamma,\mathcal{A}}) \simeq \overline{C}_0\varepsilon^{n-1}\int_\gamma h(\overline{s})^\theta d\overline{s}, \tag{26}$$

where

$$\overline{C}_0 = \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^{n-1}} |U(y)|^{p+1}dy,$$

and where we have set

$$\theta = p + 1 - \frac{1}{2}(p-1)(n-1) = p - \sigma - 1. \tag{27}$$

Consider now a one-parameter family of closed curves $\gamma_t : [0, L] \to M$, where t belongs to a neighborhood of 0 in $\mathbb{R}$ and where $\gamma_0 \equiv \gamma$. We compute next the approximate value of the derivative in t of the corresponding energy defined by (26).

As in Subsection 2.2 we let $\mathcal{V}_t$ denotes the vector field $\mathcal{V}_t(\overline{s}) = \frac{\partial\gamma_t}{\partial t}(\overline{s})$ and we assume that $\mathcal{V} := \mathcal{V}_0$ is normal to γ. For any t near zero, we let $k_t(\overline{s}), h_t(\overline{s}), f_t(\overline{s})$ be defined by (22) replacing γ by γ_t and $V(\overline{s})$ by $V_t(\overline{s}) := V(\gamma_t(\overline{s}))$. Since we require periodicity of each curve γ_t in the variable $\overline{s}$, we also allow the constant $\mathcal{A}$ given in (23) to depend on t. Denoting this by $\mathcal{A}_t$, by the above considerations we choose $\mathcal{A}_t$ so that the following condition holds for every value of t

$$\int_0^L \mathcal{A}_t h_t(\overline{s})^\sigma d\overline{s} = \int_0^L f_t'(\overline{s})d\overline{s} = const. \tag{28}$$

Below, we let $\mathcal{A}_t' = \frac{d}{dt}\mathcal{A}_t$ and we will consider $h_t(\overline{s})$ as a function of $\mathcal{A}_t$ while $V_t(\overline{s})$ as implicitly defined in (24). From (19) and $\frac{\partial V_t(\overline{s})}{\partial t}|_{t=0} = \langle\nabla^N V(\overline{s}), \mathcal{V}(\overline{s})\rangle$, differentiating (28) with respect to t at $t = 0$ we get

$$\int_0^L \mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial V}\langle\nabla^N V, \mathcal{V}\rangle d\overline{s} - \mathcal{A}\int_0^L h^\sigma\langle\mathcal{V}, \mathbf{H}\rangle d\overline{s} + \mathcal{A}\mathcal{A}'\sigma\int_0^L h^{\sigma-1}\frac{\partial h}{\partial\mathcal{A}}d\overline{s} + \mathcal{A}'\int_0^L h^\sigma d\overline{s} = 0,$$

where we have set $\mathcal{A}' = \mathcal{A}_0'$ and where $\nabla^N V$ stands for the component of ∇V normal to γ. From this formula we obtain the following expression of $\mathcal{A}'$

$$\mathcal{A}' = -\mathcal{A}\frac{\int_0^L\left(\sigma h^{\sigma-1}\frac{\partial h}{\partial V}\langle\nabla^N V, \mathcal{V}\rangle - h^\sigma\langle\mathcal{V}, \mathbf{H}\rangle\right)d\overline{s}}{\int_0^L\left(\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial\mathcal{A}} + h^\sigma\right)d\overline{s}}. \tag{29}$$

Similarly, computing the derivative of the (approximate) energy with respect to t we find

$$\frac{dE_\varepsilon(u_{\gamma_t,\mathcal{A}_t})}{dt}|_{t=0} = \int_0^L \left(\theta h^{\theta-1}\frac{\partial h}{\partial V}\langle\nabla^N V,\mathcal{V}\rangle - h^\theta\langle\mathcal{V},\mathbf{H}\rangle + \theta\mathcal{A}' h^{\theta-1}\frac{\partial h}{\partial\mathcal{A}}\right) d\overline{s}.$$

Using (29) we deduce that the variation is given by

$$\begin{aligned}\frac{dE_\varepsilon(u_{\gamma_t,\mathcal{A}_t})}{dt}|_{t=0} &= \int_0^L \frac{\partial h}{\partial V}\langle\nabla^N V,\mathcal{V}\rangle\left[\theta h^{\theta-1} - \frac{\mathcal{A}\sigma h^{\sigma-1}\int_0^L \theta h^{\theta-1}\frac{\partial h}{\partial\mathcal{A}}d\overline{s}}{\int_0^L\left(\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial\mathcal{A}} + h^\sigma\right)d\overline{s}}\right]d\overline{s}\\ &- \int_0^L\langle\mathcal{V},\mathbf{H}\rangle\left[h^\theta - \frac{\mathcal{A}h^\sigma\int_0^L\theta h^{\theta-1}\frac{\partial h}{\partial\mathcal{A}}d\overline{s}}{\int_0^L\left(\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial\mathcal{A}} + h^\sigma\right)d\overline{s}}\right]d\overline{s}.\end{aligned}$$

Differentiating (24) with respect to $\mathcal{A}$ and V we get

$$\frac{\partial h}{\partial\mathcal{A}} = \frac{2\mathcal{A}h^{2\sigma}}{(p-1)h^{p-2} - 2\sigma\mathcal{A}^2h^{2\sigma-1}} = 2\mathcal{A}h^{2\sigma}\frac{\partial h}{\partial V}, \tag{30}$$

so it follows that

$$\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial\mathcal{A}} + h^\sigma = \frac{(p-1)h^{p-1}}{(p-1)h^\theta - 2\sigma\mathcal{A}^2h^\sigma}. \tag{31}$$

Similarly, since $\theta = p-\sigma-1$, see (25) and (27), we deduce that

$$h^{\theta-1}\frac{\partial h}{\partial\mathcal{A}} = \frac{2\mathcal{A}h^{p+\sigma-2}}{(p-1)h^{p-2} - 2\sigma\mathcal{A}^2h^{2\sigma-1}}.$$

Therefore we find also

$$\frac{\theta h^{\theta-1}\frac{\partial h}{\partial\mathcal{A}}}{\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial\mathcal{A}} + h^\sigma} = \frac{2\mathcal{A}\theta}{p-1}. \tag{32}$$

Hence from the last formulas the variation of the energy becomes

$$\frac{dE_\varepsilon(\psi_{\gamma_t,\mathcal{A}_t})}{dt}|_{t=0} = \int_0^L\langle\nabla^N V,\mathcal{V}\rangle\frac{\partial h}{\partial V}\left[\theta h^{\theta-1} - \frac{2\mathcal{A}^2\sigma\theta}{p-1}h^{\sigma-1}\right]d\overline{s} - \int_0^L\langle\mathcal{V},\mathbf{H}\rangle\left[h^\theta - \frac{2\mathcal{A}^2\theta}{p-1}h^\sigma\right]d\overline{s}. \tag{33}$$

Also, from the second equality in (30), dividing by h^σ, multiplying by $\frac{\theta}{p-1}$ and using the identity $p-\sigma-2 = \theta-1$ we obtain

$$h^{-\sigma}\frac{\theta}{p-1} + 2\sigma\mathcal{A}^2h^{\sigma-1}\frac{\partial h}{\partial V}\frac{\theta}{p-1} = \theta h^{p-\sigma-2}\frac{\partial h}{\partial V} = \theta h^{\theta-1}\frac{\partial h}{\partial V}.$$

Using (33) and the last formula, we get the following simplified expression

$$\frac{dE_\varepsilon(\psi_{\gamma_t,\mathcal{A}_t})}{dt}|_{t=0} = \int_0^L\frac{\theta}{p-1}h^{-\sigma}\left[\langle\nabla^N V,\mathcal{V}\rangle - \langle\mathcal{V},\mathbf{H}\rangle\left(\frac{p-1}{\theta}h^{p-1} - 2\mathcal{A}^2h^{2\sigma}\right)\right]d\overline{s}.$$

Therefore the stationarity condition for the energy (under the constraint (28)) becomes $\langle\nabla^N V,\mathcal{V}\rangle = \langle\mathcal{V},\mathbf{H}\rangle\left(\frac{p-1}{\theta}h^{p-1} - 2\mathcal{A}^2h^{2\sigma}\right)$ for every normal section $\mathcal{V}$, namely

$$\nabla^N V = \mathbf{H}\left(\frac{p-1}{\theta}h^{p-1} - 2\mathcal{A}^2h^{2\sigma}\right). \tag{34}$$

We will see that this formula will be crucial later on to find approximate solutions.

Remark 2.1 *By* (31), *we have that*

$$\frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}h^\sigma\right) = \frac{(p-1)h^{p-1}}{(p-1)h^\theta - 2\sigma\mathcal{A}^2h^\sigma}.$$

If $\mathcal{A}$ is sufficiently small (depending on V and p), then we have $\frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}h^\sigma\right) > 0$. This will be used in the second part [40] when, for a fixed ε, we will adjust the value of the constant $\mathcal{A}$ for obtaining periodicity of the function f.

2.5 Second variation and non-degeneracy condition

We evaluate next the second variation of the Euler functional. As in Subsection 2.2 we consider a two-parameter family of closed curves γ_{t_1,t_2}, where t_1, t_2 are two real numbers belonging to a small neighborhood of zero in $\mathbb{R}$, and where $\gamma_{0,0} = \gamma$. As before, we require the constraint (28) along the whole two-dimensional family of curves, and we assume the functions f, h, k and the constant $\mathcal{A}$ to depend on t_1 and t_2, and we will use the notation $\mathcal{A}_{t_1,t_2}$, etc.. Keeping this in mind, we define the two vector fields $\mathcal{V}_{t_1,t_2} = \frac{\partial \gamma_{t_1,t_2}}{\partial t_1}$, $\mathcal{W}_{t_1,t_2} = \frac{\partial \gamma_{t_1,t_2}}{\partial t_2}$, and we can assume that $\mathcal{V} := \mathcal{V}_{0,0}, \mathcal{W} := \mathcal{W}_{0,0}$ are normal to the initial curve γ. With some computations, which are worked out in Section 5.2, one finds that, at $(t_1, t_2) = (0, 0)$

$$
\begin{aligned}
\frac{\partial^2 E_\varepsilon(u_{\psi_{t_1,t_2},\mathcal{A}_{t_1,t_2}})}{\partial t_1 \partial t_2} &= \int_0^L \left[h^\theta - \frac{2\mathcal{A}^2\theta}{p-1} h^\sigma \right] \left[\sum_j \dot{\mathcal{V}}^j \dot{\mathcal{W}}^j - \sum_{j,m} R_{1j1m} \mathcal{V}^j \mathcal{W}^m \right] d\bar{s} \\
&+ \frac{\theta}{p-1} \int_0^L \left\{ ((\nabla^N)^2 V)[\mathcal{V}, \mathcal{W}] - \langle \nabla^N V, \mathcal{V} \rangle \langle \mathbf{H}, \mathcal{W} \rangle - \langle \nabla^N V, \mathcal{W} \rangle \langle \mathbf{H}, \mathcal{V} \rangle \right\} h^{-\sigma} d\bar{s} \\
&- \frac{\sigma\theta}{p-1} \int_0^L h^{-\sigma-1} \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V} \rangle \langle \nabla^N V, \mathcal{W} \rangle d\bar{s} \\
&+ \mathcal{A}_1' \mathcal{A}_2' \frac{2\theta}{p-1} \int_0^L \left(\mathcal{A} \sigma h^{\sigma-1} \frac{\partial h}{\partial \mathcal{A}} + h^\sigma \right) d\bar{s}.
\end{aligned} \tag{35}
$$

Here, $\mathcal{A}_l'$ stands for $\frac{\partial \mathcal{A}_{t_1,t_2}}{\partial t_l}|_{(t_1,t_2)=(0,0)}$, $\dot{\mathcal{V}}^j = \frac{d\mathcal{V}^j}{d\bar{s}}$, and $\dot{\mathcal{W}}^j = \frac{d\mathcal{W}^j}{d\bar{s}}$, where the $\mathcal{V}^j$'s, $\mathcal{W}^j$'s are the components of $\mathcal{V}$ and $\mathcal{W}$ with respect to the basis $(E_j)_j$ introduced in Subsection 2.1.

Integrating by parts and using (29), from the last formula one derives that the non-degeneracy condition is equivalent to the invertibility of the linear operator $\mathfrak{J} : \chi(N\gamma) \to \chi(N\gamma)$ (from the family of smooth sections of the normal bundle to γ into itself) whose components are defined by

$$
\begin{aligned}
(\mathfrak{J}\mathcal{V})^m &= -\left(h^\theta - \frac{2\mathcal{A}^2\theta}{p-1} h^\sigma \right) \left[\ddot{\mathcal{V}}^m + \sum_j R_{1j1m} \mathcal{V}^j \right] - \theta \left(h^{\theta-1} - \frac{2\mathcal{A}^2\sigma}{p-1} h^{\sigma-1} \right) h' \dot{\mathcal{V}}^m \\
&+ \frac{\theta}{p-1} h^{-\sigma} \left\{ ((\nabla^N)^2 V)(\mathcal{V}, E_m) - H^m \langle \nabla^N V, \mathcal{V} \rangle - \langle \mathbf{H}, \mathcal{V} \rangle \langle \nabla^N V, E_m \rangle \right\} \\
&- \frac{\sigma\theta}{p-1} h^{-(\sigma+1)} \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V} \rangle \langle \nabla^N V, E_m \rangle - \frac{2\theta}{p-1} \mathcal{A}\mathcal{A}_1' \left(\sigma h^{\sigma-1} \frac{\partial h}{\partial V} \langle \nabla^N V, E_m \rangle - h^\sigma H^m \right),
\end{aligned}
$$

where $h' = \frac{dh(\bar{s})}{d\bar{s}}$. Recalling formula (17), using (34) and some other elementary computations we obtain

$$
\begin{aligned}
(\mathfrak{J}\mathcal{V})^m = &- \left(h^\theta - \frac{2\mathcal{A}^2\theta}{p-1} h^\sigma \right) \ddot{\mathcal{V}}^m - \theta \left(h^{\theta-1} - \frac{2\mathcal{A}^2\sigma}{p-1} h^{\sigma-1} \right) h' \dot{\mathcal{V}}^m + \frac{\theta}{p-1} h^{-\sigma} ((\nabla^N)^2 V)(\mathcal{V}, E_m) \\
&+ \frac{1}{2} \left(h^\theta - \frac{2\mathcal{A}^2\theta}{p-1} h^\sigma \right) \left(\sum_j \left(\partial^2_{jm} g_{11} \right) \mathcal{V}^j \right) - \frac{2\theta}{p-1} \mathcal{A}\mathcal{A}_1' \left(\sigma h^{\sigma-1} \frac{\partial h}{\partial V} \langle \nabla^N V, E_m \rangle - h^\sigma H^m \right) \\
&+ H^m \langle \mathbf{H}, \mathcal{V} \rangle \left(\frac{2\mathcal{A}^2\theta}{p-1} h^\sigma - h^\theta \right) \left[\frac{(p-1)\left(3 + \frac{\sigma}{\theta}\right) h^\theta - 8\sigma\mathcal{A}^2 h^\sigma}{(p-1)h^\theta - 2\mathcal{A}^2\sigma h^\sigma} \right].
\end{aligned}
$$

For future convenience, we expand the last product explicitly, finding

$$
H^m \langle \mathbf{H}, \mathcal{V} \rangle \left[\frac{-(p-1)\left(3 + \frac{\sigma}{\theta}\right) h^{2\theta} - \frac{16\sigma\theta\mathcal{A}^4}{p-1} h^{2\sigma} + 2\mathcal{A}^2(5\sigma + 3\theta) h^{\theta+\sigma}}{(p-1)h^\theta - 2\mathcal{A}^2\sigma h^\sigma} \right].
$$

We also notice that

$$
\sigma h^{\sigma-1} \langle \nabla^N V, E_m \rangle \frac{\partial h}{\partial V} - h^\sigma H^m = \frac{(p-1)(\theta - \sigma) h^{p-1}}{\theta \left[(p-1)h^\theta - 2\sigma\mathcal{A}^2 h^\sigma \right]} H^m.
$$

In conclusion the non-degeneracy condition is equivalent to the invertibility of the operator $\mathfrak{J} : \chi(N\gamma) \to \chi(N\gamma)$ given in components by

$$
\begin{aligned}
(\mathfrak{J}\mathcal{V})^m &= -\left(h^\theta - \frac{2\mathcal{A}^2\theta}{p-1}h^\sigma\right)\ddot{\mathcal{V}}^m - \theta\left(h^{\theta-1} - \frac{2\mathcal{A}^2\sigma}{p-1}h^{\sigma-1}\right)h'\dot{\mathcal{V}}^m + \frac{\theta}{p-1}h^{-\sigma}((\nabla^N)^2 V)[\mathcal{V}, E_m] \\
&+ \frac{1}{2}\left(h^\theta - \frac{2\mathcal{A}^2\theta}{p-1}h^\sigma\right)\left(\sum_j \left(\partial^2_{jm} g_{11}\right)\mathcal{V}^j\right) - 2\mathcal{A}\mathcal{A}_1'\frac{(\theta-\sigma)h^{p-1}}{[(p-1)h^\theta - 2\sigma\mathcal{A}^2 h^\sigma]}H^m \\
&+ H^m\langle \mathbf{H}, \mathcal{V}\rangle\left[\frac{-(p-1)\left(3+\frac{\sigma}{\theta}\right)h^{2\theta} - \frac{16\sigma\theta\mathcal{A}^4}{p-1}h^{2\sigma} + 2\mathcal{A}^2(5\sigma+3\theta)h^{\theta+\sigma}}{(p-1)h^\theta - 2\mathcal{A}^2\sigma h^\sigma}\right].
\end{aligned}
\tag{36}
$$

We summarize the results of this section in the following Proposition.

Proposition 2.4 *Consider the functional on curves $\int_\gamma h^\theta(\bar{s})\, d\bar{s}$ restricted to the set Γ in (13). Then the stationarity condition is (34) and the non-degeneracy of a critical point is equivalent to the invertibility of the operator $\mathfrak{J}$ in (36).*

3 Approximate solutions

Using a change of variables, equation (NLS_ε) is equivalent to the following

$$
-\Delta_{g_\varepsilon}\psi + V(\varepsilon x)\psi = |\psi|^{p-1}\psi \qquad \text{in } M_\varepsilon, \tag{37}
$$

where M_ε denotes the manifold M endowed with the scaled metric $g_\varepsilon = \frac{1}{\varepsilon^2}g$. With an abuse of notation we will often denote it through the scaling $M_\varepsilon = \frac{1}{\varepsilon}M$, and if $x \in M_\varepsilon$ we write εx to indicate the corresponding point on M.

In this section we find a family of approximate solutions to the scaled equation (37). We consider a simple closed curve γ which is stationary within the class Γ, namely satisfying (34). First, we introduce some convenient coordinates near the scaled curve $\gamma_\varepsilon = \frac{1}{\varepsilon}\gamma$, expanding the Laplace-Beltrami operator with respect to the scaled metric in powers of ε. Then, using these expansions, we construct the approximate solutions solving formally (37) up to order ε. Since in the second part [40] we will need to work out rigorous estimates, in order not to repeat later the expansions we will treat some terms carefully and not only at a formal level.

3.1 Choice of coordinates in M_ε and expansion of the metric coefficients

Using the coordinates $(\bar{s}, y)$ of Section 2 defined near γ, for some smooth normal section $\Phi(\bar{s})$ in $N\gamma$, we define the following new coordinates (s, z) (here and below we use the notation $\bar{s} = \varepsilon s$) near $\frac{1}{\varepsilon}\gamma$

$$
z = y - \Phi(\varepsilon s); \qquad\qquad z \in \mathbb{R}^{n-1}. \tag{38}
$$

In this choice we are motivated by the fact that in general we allow the approximate solutions to be *tilted* normally to γ_ε, where the tilting Φ depends (slowly) on the variable s: this allows some extra flexibility in the construction, as in [24], [39] and [42] . As we will see, the choice of Φ is irrelevant for solving (37) up to order ε; on the other hand, the non-degeneracy assumption will be necessary to guarantee solvability of the equation up to higher orders.

We denote by $\tilde{g}_{AB}$ the metric coefficients in the new coordinates (s, z). Since $y = z + \Phi(\varepsilon s)$, it follows

$$
\tilde{g}_{CD} = \sum_{AB} g_{AB}\left(\frac{\partial y_A}{\partial z_C}\right)\left(\frac{\partial y_B}{\partial z_D}\right).
$$

Explicitly, we then find

$$
\tilde{g}_{11} = g_{11}|_{z+\Phi} + 2\varepsilon\sum_j \Phi_j' g_{1j}|_{z+\Phi} + \varepsilon^2\sum_{j,m}\Phi_j'(\varepsilon s)\Phi_m'(\varepsilon s)g_{jm}|_{z+\Phi};
$$

$$\tilde g_{1j} = g_{1j}|_{z+\Phi} + \varepsilon \sum_m \Phi'_m(\varepsilon s) g_{jm}|_{z+\Phi}; \qquad \tilde g_{jm} = g_{jm}|_{z+\Phi}.$$

At this point, it is convenient to introduce some notation. For a positive integer q, we denote by $R_q(z)$, $R_q(z,\Phi)$ and $R_q(z,\Phi,\Phi')$ error terms which satisfies respectively the following bounds, for some positive constants C and d

$$|R_q(z)| \le C\varepsilon^q(1+|z|^d),$$

$$\begin{cases} |R_q(z,\Phi)| \le C\varepsilon^q(1+|z|^d); \\ |R_q(z,\Phi) - R_q(z,\tilde\Phi)| \le C\varepsilon^q(1+|z|^d)[\,|\Phi-\tilde\Phi|], \end{cases}$$

and

$$\begin{cases} |R_q(z,\Phi,\Phi')| \le C\varepsilon^q(1+|z|^d); \\ |R_q(z,\Phi,\Phi') - R_q(z,\tilde\Phi,\tilde\Phi')| \le C\varepsilon^q(1+|z|^d)[\,|\Phi-\tilde\Phi| + |\Phi'-\tilde\Phi'|]. \end{cases}$$

We also introduce error terms involving also second derivatives of Φ, $R_q(z,\Phi,\Phi',\Phi'')$ which satisfy

$$|R_q(z,\Phi,\Phi',\Phi'')| \le C\varepsilon^q(1+|z|^d) + C\varepsilon^{q+1}(1+|z|^d)|\Phi''|;$$

$$\begin{aligned} |R_q(z,\Phi,\Phi',\Phi'') - R_q(z,\tilde\Phi,\tilde\Phi',\tilde\Phi'')| \quad &\le \quad C\varepsilon^q(1+|z|^d)[|\Phi-\tilde\Phi| + |\Phi'-\tilde\Phi'|]\Big(1+\varepsilon(|\Phi''|+|\tilde\Phi''|)\Big) \\ &+ \quad C\varepsilon^{q+1}(1+|z|^d)|\Phi''-\tilde\Phi''|. \end{aligned}$$

Using the expansion of the metric coefficients g_{AB} in Lemma 2.1 and this notation, we then obtain

$$\begin{aligned} \tilde g_{11} \quad &= \quad 1 - 2\varepsilon\sum_{m=2}^n H^m(z_m+\Phi_m) + \frac12\varepsilon^2 \sum_{m,l=2}^n \partial^2_{ml} g_{11}|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) \\ &+ \quad \varepsilon^2|\Phi'|^2 + R_3(z,\Phi,\Phi'); \end{aligned} \tag{39}$$

$$\tilde g_{1j} \quad = \quad \varepsilon\Phi'_j + \frac12\varepsilon^2 \sum_{m,l=2}^n \partial^2_{ml} g_{1j}|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) + R_3(z,\Phi,\Phi');$$

$$\tilde g_{kj} \quad = \quad \delta_{kj} + \frac12\varepsilon^2 \sum_{m,l=2}^n \partial^2_{ml} g_{kj}|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) + R_3(z,\Phi,\Phi').$$

Next we compute the inverse metric coefficients. Recall that, given a formal expansion of a matrix as $M = 1 + \varepsilon A + \varepsilon^2 B$, we have

$$M^{-1} = 1 - \varepsilon A + \varepsilon^2 A^2 - \varepsilon^2 B.$$

In our specific case the matrix A is the following

$$A = \begin{pmatrix} -2\sum_{m=2}^n H^m(z_m+\Phi_m) & \Phi'_j \\ \Phi'_j & 0 \end{pmatrix}, \tag{40}$$

and the elements of its square are given by

$$(A^2)_{11} = 4\left(\sum_{m=2}^n H^m(z_m+\Phi_m)\right)^2 + \sum_j \left(\Phi'_j\right)^2;$$

$$(A^2)_{1j} = -2\left(\sum_{m=2}^n H^m(z_m+\Phi_m)\right)\left(\Phi'_j\right); \qquad \text{and} \qquad (A^2)_{lj} = \left(\Phi'_l\right)\left(\Phi'_j\right).$$

Therefore, using the above formula, the inverse coefficients are

$$\begin{aligned}\tilde{g}^{11} &= 1+2\varepsilon\sum_{m=2}^{n} H^m(z_m+\Phi_m) - \frac{1}{2}\varepsilon^2 \sum_{m,l=2}^{n} \partial^2_{ml} g_{11}|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) \\ &+ 4\varepsilon^2 \left(\sum_{m=2}^{n} H^m(z_m+\Phi_m)\right)^2 + R_3(z,\Phi,\Phi'),\end{aligned}$$

We also get

$$\begin{aligned}\tilde{g}^{1j} &= -\varepsilon\Phi'_j - \frac{1}{2}\varepsilon^2 \sum_{m,l=2}^{n} \partial^2_{ml} g_{1j}|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) - 2\varepsilon^2 \left(\sum_{m=2}^{n} H^m(z_m+\Phi_m)\right)\Phi'_j \\ &+ R_3(z,\Phi,\Phi').\end{aligned}$$

Moreover

$$\partial_j(\tilde{g}^{1j}) = -\varepsilon^2 \sum_{l=2}^{n} \partial^2_{lj} g_{1j}|_\gamma (z_l+\Phi_l) - 2\varepsilon^2 H^j \Phi'_j + R_3(z,\Phi,\Phi').$$

Similarly, with some simple calculations one also finds

$$\partial_1(\tilde{g}^{11}) = 2\varepsilon^2 \sum_{m=2}^{n} (H^m)'(z_m+\Phi_m) + 2\varepsilon^2 \sum_{m=2}^{n} H^m \Phi'_m + R_3(z,\Phi,\Phi',\Phi'').$$

Differentiating now $\tilde{g}^{1j}$ with respect to the first variable we obtain

$$\partial_1(\tilde{g}^{1j}) = -\varepsilon^2 \Phi''_j - 2\varepsilon^3 \left(\sum_{m=2}^{n} H^m(z_m+\Phi_m)\right)\Phi''_j + R_3(z,\Phi,\Phi',\Phi'').$$

Analogously, we get

$$\tilde{g}^{kj} = \delta_{kj} - \frac{1}{2}\varepsilon^2 \sum_{m,l=2}^{n} \partial^2_{ml} g_{kj}|_\gamma (z_m+\Phi_m)(z_l+\Phi_l) + \varepsilon^2 \Phi'_k \Phi'_j + R_3(z,\Phi,\Phi');$$

$$\partial_k(\tilde{g}^{kj}) = -\varepsilon^2 \sum_{l=2}^{n} \partial^2_{kl} g_{kj}|_\gamma (z_l+\Phi_l) + R_3(z,\Phi,\Phi').$$

Finally, using the formal expansion $\tilde{g}_{CD} = \delta_{CD} + \varepsilon A_{CD} + \varepsilon^2 B_{CD} + o(\varepsilon^2)$, analyzing carefully the error terms we obtain

$$\sqrt{\det \tilde{g}} = 1 + \frac{1}{2}\varepsilon tr(A) + \varepsilon^2 \left(\frac{1}{8}(tr(A))^2 - \frac{1}{4} tr(A^2)\right) + \frac{1}{2}\varepsilon^2 tr(B) + O(\varepsilon^3).$$

From the above expressions in (39), (40) we deduce that

$$\begin{aligned}\sqrt{\det \tilde{g}} &= 1 - \varepsilon \sum_m H^m(z_m+\Phi_m) \\ &+ \varepsilon^2 \left[\frac{1}{4}\sum_{m,l} \partial^2_{ml} g_{11}(z_m+\Phi_m)(z_l+\Phi_l) - \frac{1}{2}\left(\sum_{m=2}^{n} H^m(z_m+\Phi_m)\right)^2\right] \\ &+ R_3(z,\Phi,\Phi');\end{aligned}$$

$$\partial_m \sqrt{\det \tilde g} = -\varepsilon H^m + \varepsilon^2 \left[\frac{1}{2} \sum_l \partial^2_{ml} g_{11}(z_l + \Phi_l) - H^m \left(\sum_l H^l (z_l + \Phi_l) \right) \right] + R_3(z, \Phi, \Phi'),$$

moreover

$$\partial_1 \sqrt{\det \tilde g} = -\varepsilon^2 \sum_m (H^m)'(z_m + \Phi_m) - \varepsilon^2 \sum_m H^m \Phi'_m + R_3(z, \Phi, \Phi', \Phi'').$$

The Laplacian of a smooth function u in coordinates (s, z) has the following expression

$$-\Delta_{\tilde g} u = -\sum_{A,B} \tilde g^{AB} \partial^2_{AB} u - \sum_{A,B} \partial_B (\tilde g^{AB}) \partial_A u - \frac{1}{\sqrt{\det \tilde g}} \sum_{A,B} \tilde g^{AB} \left(\partial_B \sqrt{\det \tilde g} \right) \partial_A u.$$

We are going to expand next each of these terms. First, we consider the determinant of $\tilde g$. Recall that for a matrix of the form $1 + \varepsilon A + \varepsilon^2 B$ the square root of the determinant admits the formal expansion

$$\sqrt{\det g} = 1 + \frac{\varepsilon}{2} trA + \varepsilon^2 \left(\frac{1}{8}(tr\ A)^2 - \frac{1}{4} tr(A^2) + \frac{1}{2} trB \right) + o(\varepsilon^2). \tag{41}$$

Lemma 3.1 *Let u be a smooth function. Then in the above coordinates (s, z) we have that*

$$\begin{aligned}
\Delta_{\tilde g} u \;&= \partial^2_{ss} u + \Delta_z u - \varepsilon \sum_j H^j \partial_j u - 2\varepsilon \sum_j \Phi'_j \partial^2_{sj} u + 2\varepsilon \langle \mathbf{H}, z + \Phi \rangle \partial^2_{ss} u \\
- \;& \varepsilon^2 \langle \mathbf{H}, z + \Phi \rangle \sum_{m,j} H^j \partial_j u - \frac{1}{2} \varepsilon^2 \partial^2_{ml} g_{11} (z_m + \Phi_m)(z_l + \Phi_l) \partial^2_{ss} u + 4\varepsilon^2 \langle \mathbf{H}, z + \Phi \rangle^2 \partial^2_{ss} u \\
- \;& \varepsilon^2 \partial^2_{ml} g_{1j} (z_m + \Phi_m)(z_l + \Phi_l) \partial^2_{sj} u - 4\varepsilon^2 \langle \mathbf{H}, z + \Phi \rangle \sum_j \Phi'_j \partial^2_{sj} u + \varepsilon^2 \sum_{t,j} \Phi'_t \Phi'_j \partial^2_{tj} u \\
- \;& \frac{1}{2} \varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{tj} (z_m + \Phi_m)(z_l + \Phi_l) \partial^2_{tj} u + \varepsilon^2 \langle \mathbf{H}', z + \Phi \rangle \partial_s u - \varepsilon^2 \sum_{l,j} \partial^2_{lj} g_{1j} (z_l + \Phi_l) \partial_s u \\
- \;& \varepsilon^2 \sum_j \Phi''_j \partial_j u - \varepsilon^2 \sum_{t,j,l} \partial^2_{tl} g_{tj} (z_l + \Phi_l) \partial_j u - 2\varepsilon^3 \langle \mathbf{H}, z + \Phi \rangle \sum_j \Phi''_j \partial_j u \\
+ \;& R_3(z, \Phi, \Phi') \partial^2_{ss} u + R_3(z, \Phi, \Phi') \partial^2_{sj} u + R_3(z, \Phi, \Phi') \partial^2_{lj} u + R_3(z, \Phi, \Phi', \Phi'') \left(\partial_s u + \partial_j u \right).
\end{aligned}$$

Moreover, given two smooth normal sections Φ and $\tilde\Phi$ and defining the corresponding coordinates

$$(s, y - \Phi(\varepsilon s)) \qquad \textit{and} \qquad (s, y - \tilde\Phi(\varepsilon s))$$

and set $u_\Phi(s,y) := u(s, y-\Phi(\varepsilon s))$, $u_{\tilde\Phi}(s,y) := u(s, y-\tilde\Phi(\varepsilon s))$. *We then have*

$$
\begin{aligned}
\Delta_{\tilde g} u_\Phi - \Delta_{\tilde g} u_{\tilde\Phi} &= -2\varepsilon \sum_j (\Phi'_j - \tilde\Phi'_j)\partial^2_{sj} u + 2\varepsilon\langle \mathbf{H}, \Phi - \tilde\Phi\rangle \partial^2_{ss} u + \varepsilon^2 \sum_{t,j} (\Phi'_t \Phi'_j - \tilde\Phi'_t \tilde\Phi'_j)\partial^2_{tj} u \\
&- \frac{1}{2}\varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{tj} \Big[2 z_m (\Phi_l - \tilde\Phi_l) + \Phi_l(\Phi_m - \tilde\Phi_m) + \tilde\Phi_l(\Phi_m - \tilde\Phi_m)\Big] \partial^2_{tj} u \\
&- \varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{1j} \Big[2 z_m (\Phi_l - \tilde\Phi_l) + \Phi_l(\Phi_m - \tilde\Phi_m) + \tilde\Phi_l(\Phi_m - \tilde\Phi_m)\Big] \partial^2_{sj} u \\
&- \frac{1}{2}\varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{11} \Big[2 z_m (\Phi_l - \tilde\Phi_l) + \Phi_l(\Phi_m - \tilde\Phi_m) + \tilde\Phi_l(\Phi_m - \tilde\Phi_m)\Big] \partial^2_{ss} u \\
&- 2\varepsilon^2 \sum_l H^j \Big[z_l (\Phi'_l - \tilde\Phi'_l) + \Phi_l(\Phi'_l - \tilde\Phi'_l) + \tilde\Phi'_l(\Phi_l - \tilde\Phi_l)\Big] \partial^2_{sj} u \\
&+ 4\varepsilon^2 \sum_{m,l} H^m H^l \Big[2 z_m (\Phi_l - \tilde\Phi_l) + \Phi_l(\Phi_m - \tilde\Phi_m) + \tilde\Phi_l(\Phi_m - \tilde\Phi_m)\Big] \partial^2_{ss} u \\
&- \varepsilon^2 \sum_j (\Phi''_j - \tilde\Phi''_j)\partial_j u - \varepsilon^2 \sum_{t,j,l} \partial^2_{tl} g_{tj} (\Phi_l - \tilde\Phi_l)\partial_j u - \varepsilon^2 \langle \mathbf{H}, \Phi - \tilde\Phi\rangle \sum_j H^j \partial_j u \\
&+ \varepsilon^2 \langle \mathbf{H}', \Phi - \tilde\Phi\rangle \partial_s u - \varepsilon^2 \sum_{l,j} \partial^2_{lj} g_{1j} (\Phi_l - \tilde\Phi_l)\partial_s u \\
&- 2\varepsilon^3 \sum_{mj} H^m \Big[(z_m + \Phi_m)(\Phi''_j - \tilde\Phi''_j) + \tilde\Phi''_j (\Phi_m - \tilde\Phi_m)\Big] \partial_j u \\
&+ O(1+|z|^d) \Big[\varepsilon^4 (|\Phi - \tilde\Phi| + |\Phi' - \tilde\Phi'|)|\partial^2_{ss} u| + \varepsilon^3 (|\Phi - \tilde\Phi| + |\Phi' - \tilde\Phi'|)(|\partial^2_{sj} u| + |\partial^2_{lj} u)| \Big] \\
&+ O(1+|z|^d) \Big[\varepsilon^3 (|\Phi - \tilde\Phi| + |\Phi' - \tilde\Phi'|) + \varepsilon^4 (|\Phi''||\Phi - \tilde\Phi| + |\Phi'' - \tilde\Phi''|)\Big] (|\partial_s u| + |\partial_j u|).
\end{aligned}
$$

Proof. The proof is based on the Taylor expansion of the metric coefficients given above. We recall that the Laplace-Beltrami operator is given by

$$
\Delta_{\tilde g} = \sum_{A,B} \frac{1}{\sqrt{\det \tilde g}} \, \partial_A (\sqrt{\det \tilde g} \, (g_\varepsilon)^{AB} \, \partial_B \,),
$$

where indices A and B run between 1 and n. We can also write

$$
\Delta_{\tilde g} = \sum_{A,B} \Big(\tilde g^{AB} \, \partial^2_{AB} + \big(\partial_A \, \tilde g^{AB}\big) \, \partial_B + \frac{1}{\sqrt{\det \tilde g}} \tilde g^{AB} \, \Big(\partial_B \sqrt{\det \tilde g}\Big) \, \partial_A \Big).
$$

Using the expansion of the metric coefficients determined above and (41), one can easily prove that

$$
\begin{aligned}
\sum_{AB} \tilde g^{AB} \partial^2_{AB} u &= \Delta_z u + \partial^2_{ss} u - 2\varepsilon \sum_j \Phi'_j \partial^2_{sj} u + 2\varepsilon \langle \mathbf{H}, z + \Phi\rangle \partial^2_{ss} u + \varepsilon^2 \sum_{l,j} \Phi'_l \Phi'_j \partial^2_{lj} u \\
&+ 4\varepsilon^2 \langle \mathbf{H}, z + \Phi\rangle^2 \partial^2_{ss} u - \frac{1}{2}\varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{kj} (z_m + \Phi_m)(z_l + \Phi_l)\partial^2_{kj} u \\
&- \frac{1}{2}\varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{1j} (z_m + \Phi_m)(z_l + \Phi_l)\partial^2_{sj} u - 4\varepsilon^2 \langle \mathbf{H}, z + \Phi\rangle \Phi'_j \partial^2_{sj} u \\
&- \frac{1}{2}\varepsilon^2 \sum_{m,l} \partial^2_{ml} g_{11} (z_m + \Phi_m)(z_l + \Phi_l)\partial^2_{ss} u + R_3(z, \Phi, \Phi') \left(\partial^2_{ss} u + \partial^2_{sj} u + \partial^2_{lj} u\right)
\end{aligned}
$$

$$\sum_{A,B}\partial_A\tilde{g}^{AB}\partial_B u = -\varepsilon^2\sum_j \Phi_j''\partial_j u-\varepsilon^2\sum_{i,j,l}\partial^2_{kl}g_{kj}(z_l+\Phi_l)\partial_j u-2\varepsilon^3\langle \mathbf{H},z+\Phi\rangle\sum_j\Phi_j''\partial_j u$$
$$+\ 2\varepsilon^2\langle \mathbf{H}',z+\Phi\rangle\partial_s u-\varepsilon^2\sum_{l,j}\partial^2_{lj}g_{1j}(z_l+\Phi_l)\partial_s u+R_3(z,\Phi,\Phi',\Phi'')(\partial_s u+\partial_j u)$$

$$\sum_{A,B}\frac{1}{\sqrt{\det\tilde{g}}}\tilde{g}^{AB}\left(\partial_B\sqrt{\det\tilde{g}}\right)\partial_A u = -\varepsilon\sum_j H^j\partial_j u-\varepsilon^2\langle \mathbf{H},z+\Phi\rangle\sum_j H^j\partial_j u-\varepsilon^2\langle \mathbf{H}',z+\Phi\rangle\partial_s u$$
$$+\ \frac{1}{2}\sum_j\partial^2_{lj}g_{11}(z_l+\Phi_l)\partial_j u+R_3(z,\Phi,\Phi')(\partial_s u+\partial_j u)$$

The result then follows by collecting these three terms. ■

3.2 Expansion at first order in ε

In this subsection we solve formally equation (37) up to order ε, discarding the terms which turn out to be of order ε^2 and higher.

For the approximate solution as in (10), we make a more precise *ansatz* of the following form

$$\psi_{1,\varepsilon}(s,z)=e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\left\{h(\varepsilon s)U\left(k(\varepsilon s)z\right)+\varepsilon\left[w_r+iw_i\right]\right\},\qquad s\in[0,2\pi],y\in\mathbb{R}^{n-1},\tag{42}$$

where $\tilde{f}_0(\varepsilon s)=f(\varepsilon s)+\varepsilon f_1(\varepsilon s)$. By direct computation, the first and second derivatives of $\psi_{1,\varepsilon}$ satisfy

$$\partial_s\psi_{1,\varepsilon} = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\left[-i\tilde{f}_0'(\varepsilon s)h(\varepsilon s)U(k(\varepsilon s)z)+\varepsilon h'(\varepsilon s)U(k(\varepsilon s)z)+\varepsilon h(\varepsilon s)k'(\varepsilon s)\nabla U(k(\varepsilon s)z)\cdot z\right]$$
$$+\ e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\left[-i\varepsilon f'w_r+\varepsilon f'w_i\right]+O(\varepsilon^2);$$

$$\partial_i\psi_{1,\varepsilon}=e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\left[h(\varepsilon s)k(\varepsilon s)\partial_i U\left(k(\varepsilon s)z\right)+\varepsilon\partial_i w_r+i\varepsilon\partial_i w_i\right];$$

$$\partial^2_{ss}\psi_{1,\varepsilon} = -(\tilde{f}_0')^2hU(kz)e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}-i\varepsilon e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\left[f''hU(kz)+2f'h'U(kz)+2f'hk'\nabla U(kz)\cdot z\right]$$
$$-\ \varepsilon f'^2e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}[w_r+iw_i]+O(\varepsilon^2);$$

$$\partial^2_{lj}\psi_{1,\varepsilon}=e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\left[h(\varepsilon s)k^2(\varepsilon s)\partial^2_{lj}U\left(k(\varepsilon s)z\right)+\varepsilon\partial^2_{lj}w_r+i\varepsilon\partial^2_{lj}w_i\right];$$

$$\partial^2_{sj}\psi_{1,\varepsilon} = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\left[-i\tilde{f}_0'(\varepsilon s)h(\varepsilon s)+\varepsilon h'(\varepsilon s)\right]k(\varepsilon s)\partial_j U(kz)$$
$$+\ \varepsilon e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}h(\varepsilon s)k'(\varepsilon s)\left[k\sum_l\partial^2_{lj}U(kz)z_l+\partial_j U(kz)\right]$$
$$-\ i\varepsilon f'(\varepsilon s)\partial_j w_r(\varepsilon s,z)e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}+\varepsilon f'(\varepsilon s)\partial_j w_i(\varepsilon s,z)e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}+O(\varepsilon^2).$$

Similarly, the potential V satisfies

$$V(\varepsilon x)=V(\varepsilon s)+\varepsilon\langle\nabla^N V,z+\Phi\rangle+O(\varepsilon^2).$$

Expanding (37) in powers of ε, we obtain

$$e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\left(-\Delta_g\psi_{1,\varepsilon}+V(\varepsilon x)\psi_{1,\varepsilon}-|\psi_{1,\varepsilon}|^{p-1}\psi_{1,\varepsilon}\right)=\varepsilon\mathcal{R}_r+i\varepsilon\mathcal{R}_i+O(\varepsilon^2),$$

with

$$\mathcal{R}_r = \mathcal{L}_r w_r + 2f' f_1' hU + 2f'^2 hU(kz)\langle \mathbf{H}, z + \Phi\rangle + hk\langle \mathbf{H}, \nabla U(kz)\rangle + \langle \nabla^N V, z + \Phi\rangle hU(kz); \tag{43}$$

$$\mathcal{R}_i = \mathcal{L}_i w_i + [f'' hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z] - 2\sum_j [\Phi_j' f' hk\partial_j U(kz)], \tag{44}$$

and where we have defined the two operators $\mathcal{L}_r$ and $\mathcal{L}_i$ as

$$\mathcal{L}_r w = -\Delta_z w + (V + f'^2)w - ph^{p-1}U(kz)^{p-1}w;$$

$$\mathcal{L}_i w = -\Delta_z w + (V + f'^2)w - h^{p-1}U(kz)^{p-1}w.$$

It is well-known, see for example [51], that the kernel of $\mathcal{L}_r$ is generated by the functions $\partial_2 U(k\cdot), \dots, \partial_n U(k\cdot)$, while that of $\mathcal{L}_i$ is one-dimensional and generated by $U(k\cdot)$.

We choose the functions w_r and w_i in such a way that $\mathcal{R}_r$ and $\mathcal{R}_i$ vanish. Since $\mathcal{L}_r$ is Fredholm, the solvability condition for w_r is that the right-hand side of this equation is orthogonal in $L^2(\mathbb{R}^{n-1})$ to $\partial_2 U(k\cdot), \dots, \partial_n U(k\cdot)$. Therefore, to get solvability, we should multiply the right-hand side by each of these functions and get 0. The same holds true for w_i, but replacing the functions $\partial_{z_j} U(k\cdot)$ by $U(k\cdot)$. We discuss the solvability in w_i first. Writing this equation as $\mathcal{L}_i w_i = \mathfrak{f}$, we can multiply it by $U(k\cdot)$ and use the self-adjointness of $\mathcal{L}_i$ to get

$$0 = \int_{\mathbb{R}^{n-1}} w_i \mathcal{L}_i U(k\cdot) = \int_{\mathbb{R}^{n-1}} U(k\cdot)\mathcal{L}_i w_i = \int_{\mathbb{R}^{n-1}} \mathfrak{f} U(k\cdot).$$

Following the computations of Subsection 2.3, this condition yields

$$f'' hk^{-(n-1)} + 2f'h'k^{-(n-1)} = (n-1)hk'k^{-n} f',$$

which implies

$$f' = \mathcal{A}\frac{k^{n-1}}{h^2} = \mathcal{A}h^{\sigma}.$$

This equation is nothing but (23), and hence the solvability is guaranteed. Since $\mathcal{L}_i$ clearly preserves the parity in z, we can decompose w_i in its even and odd parts as

$$w_i = w_{i,e} + w_{i,o},$$

with $w_{i,e}$ and $w_{i,o}$ solving respectively the equations

$$\mathcal{L}_i w_{i,e} = -\left[f'' hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z\right]; \qquad \mathcal{L}_i w_{i,o} = 2\sum_j \left[\Phi_j' f' hk\partial_j U(kz)\right],$$

where the right-hand sides are respectively the even and odd parts of the datum in (44). We notice that, since the kernel of $\mathcal{L}_i$ consists of even functions, only the even part of the equation plays a role in the solvability, since the product with the odd part vanishes by oddness.

Indeed, (43) and (44) can be solved explicitly, and the solutions are given by

$$w_{i,e} = \frac{p-1}{4} f'h'|z|^2 U(kz); \qquad w_{i,o} = -\sum_j \Phi_j' f' hz_j U(kz).$$

In fact, as one can easily check, we have the following relations

$$\mathcal{L}_i(z_j U(kz)) = -2k\partial_j U(kz); \qquad \mathcal{L}_i(|z|^2 U(kz)) = -2(n-1)U(kz) - 4k\nabla U(kz)\cdot z,$$

which imply the above claim (here we also used (22) and some manipulations).

Turning to w_r, if we multiply by $\partial_j U$, we integrate by parts and use some scaling, we find that the following conditions holds true, for $j = 2, \dots, n$

$$2H^j \left((f')^2 \int_{\mathbb{R}^{n-1}} U(z)^2 dz - \frac{k^2}{n-1} \int_{\mathbb{R}^{n-1}} |\nabla U(z)|^2 dz \right) + \langle \nabla^N V, E_j \rangle \int_{\mathbb{R}^{n-1}} U(z)^2 dz = 0.$$

Using (22), we get equivalently

$$2H^j \left(\mathcal{A}^2 h^{2\sigma} \int_{\mathbb{R}^{n-1}} U(z)^2 dz - \frac{h^{p-1}}{n-1} \int_{\mathbb{R}^{n-1}} |\nabla U(z)|^2 dz \right) + \langle \nabla^N V, E_j \rangle \int_{\mathbb{R}^{n-1}} U(z)^2 dz = 0, \quad j = 2, \dots, n.$$

From a Pohozaev-type identity (playing with (21) and integrating by parts) one finds

$$\int_{\mathbb{R}^{n-1}} |\nabla U(z)|^2 dz = \frac{(n-1)(p-1)}{(3-n)(p+1)+2(n-1)} \int_{\mathbb{R}^{n-1}} U(z)^2 dz = \frac{(n-1)(p-1)}{2\theta} \int_{\mathbb{R}^{n-1}} U(z)^2 dz. \tag{45}$$

Using this formula the solvability condition then becomes

$$H^j \left((p-1) \frac{h^{p-1}}{\theta} - 2\mathcal{A}^2 h^{2\sigma} \right) = \langle \nabla^N V, E_j \rangle, \qquad j = 2, \dots, n,$$

which is nothing but the stationary condition (34). Therefore, since we are indeed assuming this condition, also the solvability for w_r is guaranteed. As for w_i, we can decompose w_r in its even and odd parts as

$$w_r = w_{r,e} + w_{r,o},$$

where $w_{r,e}$ and $w_{r,o}$ solve respectively

$$\mathcal{L}_r w_{r,e} = -2f' f_1' hU - 2(f')^2 hU(kz) \langle \mathbf{H}, \Phi \rangle - \langle \nabla^N V, \Phi \rangle hU(kz);$$

$$\mathcal{L}_r w_{r,o} = -2(f')^2 hU(kz) \langle \mathbf{H}, z \rangle - hk \sum_j H^j \partial_j U(kz) - \langle \nabla^N V, z \rangle hU(kz).$$

Using the Euler equation, one gets

$$\mathcal{L}_r w_{r,o} = -h \sum_j H^j \left(k \partial_j U + h^{p-1} \frac{p-1}{\theta} z_j U \right).$$

It is also convenient to have the explicit expression of w_r. We notice first that

$$\mathcal{L}_r \left(-\frac{1}{(p-1)h^{p-1}} U(kz) - \frac{1}{2k} \nabla U(kz) \cdot z \right) = U(kz).$$

Hence it follows

$$w_{r,e} = \left[h \langle \nabla^N V + 2(f')^2 \mathbf{H}, \Phi \rangle + 2f' f_1' h \right] \left(\frac{1}{(p-1)h^{p-1}} U(kz) + \frac{1}{2k} \nabla U(kz) \cdot z \right).$$

Using (34) we finally find

$$w_{r,e} = \left[\frac{p-1}{\theta} h^p \langle \mathbf{H}, \Phi \rangle + 2f' f_1' h \right] \left(\frac{1}{(p-1)h^{p-1}} U(kz) + \frac{1}{2k} \nabla U(kz) \cdot z \right).$$

By the above computations we obtain the following result.

Lemma 3.2 *Suppose $h(\overline{s})$ and $f(\overline{s})$ satisfy (11) and (12) for some $\mathcal{A} > 0$: assume also that the curve γ verifies (15). Then there exist two smooth functions $w_r(\overline{s}, z), w_i(\overline{s}, z)$ for which the terms $\mathcal{R}_r$ and $\mathcal{R}_i$ in (43)-(44) vanish identically. Therefore, the function $\psi_{1,\varepsilon}$ in (42) satisfies (37) up to an error $O(\varepsilon^2)$.*

3.3 Expansions at second order in ε

Next we compute the terms of order ε^2 in the above expression. Adding a correction $\varepsilon^2[v_r + iv_i]$ to the function in (42) we define an approximate solution of the form
(46)
$$\psi_{2,\varepsilon}(s,z) = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} \left\{ h(\varepsilon s)U\left(k(\varepsilon s)z\right) + \varepsilon\left[w_r + iw_i\right] + \varepsilon^2\left[v_r + iv_i\right] \right\}; \qquad s \in [0, 2\pi], y \in \mathbb{R}^{n-1},$$

where $\tilde{f}_0 = f(\varepsilon s) + \varepsilon f_1(\varepsilon s)$. The first and second derivatives of $\psi_{2,\varepsilon}$ are given by

$$\begin{aligned} e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} \partial_s \psi_{2,\varepsilon} &= \Big[-i\tilde{f}_0'(\varepsilon s)h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon h'(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon h(\varepsilon s)k'(\varepsilon s)\nabla U(k(\varepsilon s)z)\cdot z \Big] \\ &+ \Big[-i\varepsilon\tilde{f}_0' w_r + \varepsilon\tilde{f}_0' w_i \Big] + \Big[-i\varepsilon^2\tilde{f}_0' v_r + \varepsilon^2\tilde{f}_0' v_i \Big] + \varepsilon^2(\partial_s w_r + iw_i) + O(\varepsilon^3); \end{aligned}$$

$$e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} \partial_j \psi_{2,\varepsilon} = \left[h(\varepsilon s)k(\varepsilon s)\partial_j U\left(k(\varepsilon s)z\right) + \varepsilon\partial_j w_r + i\varepsilon\partial_j w_i + \varepsilon^2\partial_j v_r + i\varepsilon^2\partial_j v_i \right];$$

$$\begin{aligned} e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} \frac{\partial^2 \psi_{2,\varepsilon}}{\partial s^2} &= -(\tilde{f}_0')^2 hU(kz) - i\varepsilon\Big[\tilde{f}_0'' hU(kz) + 2\tilde{f}_0' h'U(kz) + 2\tilde{f}_0' hk'\nabla U(kz)\cdot z \Big] - \varepsilon\tilde{f}_0'^2[w_r + iw_i] \\ &- \varepsilon^2\tilde{f}_0'^2[v_r + iv_i] + \varepsilon^2\Big[2\tilde{f}_0'\partial_s w_i + h''U(kz) + 2h'k'\nabla U\cdot z + hk''\nabla U\cdot z \\ &+ hk'^2\nabla^2 U(kz)[z,z] + \tilde{f}_0'' w_i \Big] - i\varepsilon^2\Big[2\tilde{f}_0'\partial_s w_r + \tilde{f}_0'' w_r \Big] + O(\varepsilon^3); \end{aligned}$$

$$e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} \partial^2_{lj}\psi_{1,\varepsilon} = \left[h(\varepsilon s)k^2(\varepsilon s)\partial^2_{lj} U\left(k(\varepsilon s)z\right) + \varepsilon\partial^2_{lj} w_r + i\varepsilon\partial^2_{lj} w_i + \varepsilon^2\partial^2_{lj} v_r + i\varepsilon^2\partial^2_{lj} v_i \right];$$

$$\begin{aligned} e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} \partial^2_{sj}\psi_{2,\varepsilon} &= \Big[-i\tilde{f}_0'(\varepsilon s)h(\varepsilon s)k(\varepsilon s) + \varepsilon h'(\varepsilon s)k(\varepsilon s) + \varepsilon h(\varepsilon s)k'(\varepsilon s \Big]\partial_j U(kz) \\ &+ \varepsilon h(\varepsilon s)k(\varepsilon s)k'(\varepsilon s)\sum_l \partial^2_{lj} U(kz)z_l + \partial_j U(kz) - i\varepsilon\tilde{f}_0'\partial_j w_r \\ &+ \varepsilon\tilde{f}_0'(\varepsilon s)\partial_j w_i - i\varepsilon^2\tilde{f}_0'\partial_j v_r + \varepsilon^2\tilde{f}_0'(\varepsilon s)\partial_j v_i + \varepsilon^2\partial^2_{sj} w_r + i\varepsilon^2\partial^2_{sj} w_r. \end{aligned}$$

We also have the formal expansion

$$\begin{aligned} e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} |\psi_{2,\varepsilon}|^{p-1}\psi_{2,\varepsilon} &= h^p|U|^{p-1}U + p\varepsilon h^{p-1}|U|^{p-1}w_r + i\varepsilon h^{p-1}|U|^{p-1}w_i + \frac{1}{2}p(p-1)\varepsilon^2 h^{p-2}|U|^{p-3}Uw_r^2 \\ &+ \frac{1}{2}(p-1)\varepsilon^2 h^{p-2}|U|^{p-3}Uw_i^2 + i(p-1)\varepsilon^2 h^{p-2}|U|^{p-3}Uw_r w_i \\ &+ p\varepsilon^2 h^{p-1}|U|^{p-1}v_r + i\varepsilon^2 h^{p-1}|U|^{p-1}v_i + O(\varepsilon^3). \end{aligned}$$

Similarly, expanding V up to order ε^2, we have

$$V(\varepsilon x) = V(\varepsilon s) + \varepsilon\langle\nabla^N V, z + \Phi\rangle + \frac{1}{2}\varepsilon^2(\nabla^N)^2 V[z+\Phi, z+\Phi] + R_3(z,\Phi).$$

Using the expansions of Subsection 2.3, we obtain

$$\begin{aligned} e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} \Big(-\Delta_g\psi_{2,\varepsilon} + V(\varepsilon x)\psi_{2,\varepsilon} - |\psi_{2,\varepsilon}|^{p-1}\psi_{2,\varepsilon} \Big) &= \varepsilon^2(\tilde{\mathcal{R}}_r + i\tilde{\mathcal{R}}_i) \\ &= \varepsilon^2(\tilde{R}_{r,e} + \tilde{R}_{r,o}) + \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}_{i,o}) \\ &+ \varepsilon^2(\tilde{R}_{r,e,f_1} + \tilde{R}_{r,o,f_1}) + \varepsilon^2 i(\tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1}) \\ &+ \varepsilon^2\mathcal{L}_r v_r + \varepsilon^2 i\mathcal{L}_i v_i + O(\varepsilon^3), \end{aligned}$$

where

$$
\begin{aligned}
\tilde{R}_{r,e} &= -\frac{1}{2}(f')^2 hU(kz)\sum_{l,m}\partial^2_{lm}g_{11}(z_m z_l+\Phi_m\Phi_l)+2(f')^2\langle \mathbf{H}, w_{r,e}\Phi+w_{r,o}z\rangle \\
&+ 4(f')^2 hU(kz)\left[\langle \mathbf{H}, z\rangle^2+\langle \mathbf{H},\Phi\rangle^2\right] \\
&- \left[h''U(kz)+2h'k'\nabla U(kz)\cdot z+hk''\nabla U(kz)\cdot z+h(k')^2\nabla^2 U(kz)[z,z]\right] \\
&+ 2f'\partial_s w_{i,e}+f''w_{i,e}+2\Phi'_j f'\partial_j w_{i,o}+\left[\frac{1}{2}\sum_{l,m}\partial^2_{lm}g_{tj}(z_m z_l+\Phi_m\Phi_l)-\Phi'_t\Phi'_j\right]hk^2\partial^2_{tj}U(kz) \qquad (47)\\
&+ hk\sum_{l,m,j}\partial^2_{lm}g_{mj}z_l\partial_j U(kz)+\sum_m H^m\partial_m w_{r,o}+hk\langle \mathbf{H},z\rangle\sum_m H^m\partial_m U(kz) \\
&+ kh\sum_m\left[\langle \mathbf{H},z\rangle H^m-\frac{1}{2}\sum_l\partial^2_{ml}g_{11}z_l\right]\partial_m U(kz) \\
&- \frac{1}{2}p(p-1)h^{p-2}U(kz)^{p-2}(w^2_{r,e}+w^2_{r,o})-\frac{1}{2}(p-1)h^{p-2}U(kz)^{p-2}(w^2_{i,e}+w^2_{i,o}) \\
&+ \langle \nabla^N V, w_{r,o}z+w_{r,e}\Phi\rangle+\frac{1}{2}\sum_{m,j}\partial^2_{mj}V(z_m z_j+\Phi_m\Phi_j)hU(kz);
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}_{r,o} &= -(f')^2 hU(kz)\sum_{l,m}\partial^2_{lm}g_{11}z_m\Phi_l+8(f')^2hU(kz)\langle \mathbf{H},z\rangle\langle \mathbf{H},\Phi\rangle \\
&+ 2(f')^2\langle \mathbf{H}, w_{r,e}z+w_{r,o}\Phi\rangle-2f'\partial_s w_{i,o}-f''w_{i,o}+2h'k\sum_j\Phi'_j\partial_j U(kz) \\
&+ 2h'k\sum_j\Phi'_j\left[k\sum_l\partial^2_{lj}U(kz)z_l+\partial_j U(kz)\right]+2f'\sum_j\Phi'_j\partial_j w_{i,e}+hk\langle \mathbf{H},\Phi\rangle\sum_m H^m\partial_m U(kz) \\
&+ \left[\frac{1}{2}\sum_{l,m}\partial^2_{lm}g_{tj}(z_m\Phi_l+z_l\Phi_m)\right]hk^2\partial^2_{tj}U(kz)+hk\sum\Phi''_j\partial_j U(kz)+\left(\sum_{j,l,m}\partial^2_{lm}g_{mj}\Phi_l\right)hk\partial_j U(kz) \qquad (48)\\
&+ \sum_m H^m\partial_m w_{r,e}+hk\sum_m\left[\langle \mathbf{H},\Phi\rangle H^m-\frac{1}{2}\sum_l\partial^2_{ml}g_{11}\Phi_l\right]\partial_m U(kz) \\
&- p(p-1)h^{p-2}U(kz)^{p-2}w_{r,e}w_{r,o}-(p-1)h^{p-2}U(kz)^{p-2}w_{i,e}w_{i,o} \\
&+ \langle \nabla^N V, w_{r,e}z+w_{r,o}\Phi\rangle+\sum_{j,l}\partial^2_{jl}Vz_j\Phi_l hU(kz);
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}_{i,e} &= 2\left[f''hU(kz)+2f'h'U(kz)+2f'hk'\nabla U(kz)\cdot z\right]\langle \mathbf{H},\Phi\rangle \\
&+ 2(f')^2\langle \mathbf{H}, w_{i,e}\Phi+w_{i,o}z\rangle+2f'\partial_s w_{r,e}+f''w_{r,e}-2f'\sum_j\Phi'_j\partial_j w_{r,o} \\
&- 2f'hk\sum_j\partial_j U(kz)\left[2\langle \mathbf{H},z\rangle\Phi'_j+\frac{1}{2}\sum_{l,m}\partial^2_{lm}g_{1j}(z_m\Phi_l+z_l\Phi_m)\right] \qquad (49)\\
&- f'hU(kz)\left(\sum_m(\partial^2_{1m}g_{11}\Phi_m-2\langle \mathbf{H},\Phi'\rangle)\right)-f'h\left[2\langle \mathbf{H},\Phi'\rangle+\sum_{j,l}\partial^2_{lj}g_{1j}\Phi_l\right]U(kz) \\
&+ \frac{1}{2}f'h\left(\sum_l\partial_{1l}g_{11}\Phi_l\right)U(kz)+\sum_j H^j\partial_j w_{i,o} \\
&- (p-1)h^{p-2}U(kz)^{p-2}(w_{r,e}w_{i,e}+w_{r,o}w_{i,o})+\langle \nabla^N V, w_{i,o}z+w_{i,e}\Phi\rangle;
\end{aligned}
$$

$$\tilde{R}_{i,o} = 2\left[f''hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z\right]\langle \mathbf{H}, z\rangle + \sum_i H^j \partial_j w_{i,e}$$
$$+ \; 2(f')^2\langle \mathbf{H}, w_{i,e}z + w_{i,o}\Phi\rangle + 2f'\partial_s w_{r,o} + f'' w_{r,o} - 2f'\sum_j \Phi'_j \partial_j w_{r,e}$$
$$(50) \qquad - \; 2f'hk\sum_j \partial_j U(kz)\left[2\langle \mathbf{H}, \Phi\rangle \Phi'_j + \frac{1}{2}\sum_{l,m}\partial^2_{lm} g_{1j}(z_m z_l + \Phi_l \Phi_m)\right]$$
$$- \; f'hU(kz)\left(\sum_m \partial^2_{1m} g_{11} z_m\right) - f'h\left(\sum_{j,l}\partial^2_{lj} g_{1j} z_l\right)U(kz) + \frac{1}{2}f'h\left(\sum_l \partial^2_{1l} g_{11} z_l\right)U(kz)$$
$$- \; (p-1)h^{p-2}U(kz)^{p-2}(w_{r,e}w_{i,o} + w_{r,o}w_{i,e}) + \langle \nabla^N V, w_{i,e}z + w_{i,o}\Phi\rangle.$$

We used the notation $\tilde{R}_{r,e,f_1}$, $\tilde{R}_{r,o,f_1}$, $\tilde{R}_{i,e,f_1}$ and $\tilde{R}_{i,o,f_1}$ for the terms involving f_1, namely

$$\tilde{R}_{r,e,f_1} = (f'_1)^2 hU + 2f'f'_1 w_{r,e} + 4\langle \mathbf{H}, \Phi\rangle f'f'_1 hU - 2p(p-1)h^{p-2}|U|^{p-2}h^2 f'^2 f_1'^2 \tilde{U}^2$$
$$(51) \qquad + \; 2f'f'_1 h\langle \nabla^N V, \Phi\rangle \tilde{U} + 4\langle \mathbf{H}, \Phi\rangle (f')^3 f'_1 h\tilde{U} - 2p\frac{(p-1)^2}{\theta}h^{2p-1}f'f'_1\langle \mathbf{H}, \Phi\rangle U^{p-2}\tilde{U}^2;$$

$$\tilde{R}_{r,o,f_1} = 2f'f'_1 w_{r,o} + 4\langle \mathbf{H}, z\rangle f'f'_1 hU - 2p(p-1)f'f'_1 h^{p-1}U^{p-2}\tilde{U}w_{r,o}$$
$$(52) \qquad + \; 2f'f'_1 h\langle \nabla^N V, z\rangle \tilde{U} + 2H^j f'f'_1 hk\partial_j \tilde{U} + 4\langle \mathbf{H}, z\rangle (f')^2 f'f'_1 h\tilde{U};$$

$$\tilde{R}_{i,e,f_1} = 2h'f'_1 U + 2hf'_1 k'\nabla U\cdot z + 2f'f'_1 w_{i,e} + f''_1 hU + 4f'\partial_s(hf'f'_1\tilde{U})$$
$$(53) \qquad + \; 2f''hf'f'_1\tilde{U} - 2(p-1)h^{p-1}|U|^{p-2}f'f'_1\tilde{U}w_{i,e};$$

$$(54) \qquad \tilde{R}_{i,o,f_1} = 2f'f'_1 w_{i,o} - 2(p-1)h^{p-1}|U|^{p-2}f'f'_1\tilde{U}w_{i,o} - 4(f')^2 hkf'_1\Phi'_j\partial_j\tilde{U} - 2hkf'_1\Phi'_j\partial_j U,$$

where we wrote for brevity $\tilde{U} = \frac{1}{(p-1)h^{p-1}}U(kz) + \frac{1}{2k}\nabla U(kz)\cdot z$. Again, we collect the results of this section in one proposition.

Proposition 3.3 *Suppose Φ, f_1 are smooth functions on $[0, L]$, let z be the normal coordinates given in* (38) *and w_i, w_r be be as in Lemma 3.2. Then, if $\psi_{2,\varepsilon}$ is as in* (46)*, in the coordinates (s, z) we have*

$$-\Delta_{\tilde{g}_\varepsilon}\psi_{2,\varepsilon} + V(\varepsilon x)\psi_{2,\varepsilon} - |\psi_{2,\varepsilon}|^{p-1}\psi_{2,\varepsilon} = \varepsilon^2(\mathcal{L}_r v_r + i\mathcal{L}_i v_i)$$
$$(55) \qquad + \; \varepsilon^2(\tilde{R}_{r,e} + \tilde{R}_{r,o} + i\tilde{R}_{i,e} + i\tilde{R}_{i,o} + \tilde{R}_{r,e,f_1} + \tilde{R}_{r,o,f_1} + i\tilde{R}_{i,e,f_1} + i\tilde{R}_{i,o,f_1}) + o(\varepsilon^2),$$

where the above error terms are given respectively in (47)-(54).

4 Final expansions and proof of Theorem 1.4

In this Section we prove existence of approximate solutions to (NLS_ε) up to any power of ε. To do this, we first evaluate the projections of the error terms on the kernels of the operators $\mathcal{L}_i$ and $\mathcal{L}_r$, and adjust Φ and f_1 so that these projections vanish, in order to obtain solvability of (37) up to order ε^2. Then, with an iterative procedure, we turn to the general case.

4.1 Projection onto the kernel of $\mathcal{L}_i$

If one wants to find v_i so that the imaginary terms in (55) vanish, by Fredholm's alternative the imaginary $\tilde{R}$'s must be orthogonal for every s to the kernel of $\mathcal{L}_i$, which is given by $iU(k(\varepsilon s)\cdot)$. To compute this projection, by parity reasons, we need to multiply $\tilde{R}_{i,e}$ and $\tilde{R}_{i,e,f_1}$ by $U(k(\varepsilon s)\cdot)$ and to integrate over $\mathbb{R}^{n-1}$. We evaluate the two terms separately.

Contribution of $\tilde{R}_{i,e}$. After some manipulation we obtain

$$\tilde{R}_{i,e} = C_1 + C_2 + C_3 + C_4 + C_5,$$

where

$$\begin{aligned} C_1 &= 2\left[f''hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z\right]\langle \mathbf{H}, \Phi\rangle \\ &- f'hU(kz)\left(\sum_m (\partial^2_{1m} g_{11}\Phi_m - 2\langle \mathbf{H}, \Phi'\rangle)\right) + \frac{1}{2} f'h\left(\sum_l \partial_{1l} g_{11}\Phi_l\right) U(kz); \end{aligned}$$

$$C_2 = -2f'hk\sum_j \partial_j U(kz)\left(\sum_{l,m}\partial^2_{lm} g_{1j} z_m \Phi_l\right) - f'h\sum_{j,l}\partial^2_{lj} g_{1j}\Phi_l U(kz);$$

$$C_3 = -4\sum_j f'hk\partial_j U(kz)\langle \mathbf{H}, z\rangle \Phi'_j - 2f'h\sum_j H^j\Phi'_j U(kz);$$

$$\begin{aligned} C_4 &= -(f')^2\langle \mathbf{H}, w_{i,e}\Phi + w_{i,o}z\rangle + \sum_l H^l\partial_l w_{i,o} \\ &+ \langle \nabla^N V, w_{i,e}\Phi + w_{i,o}z\rangle - (p-1)h^{p-2}U(kz)^{p-2}(w_{r,e}w_{i,e} + w_{r,o}w_{i,o}); \end{aligned}$$

$$C_5 = 2f'\partial_s w_{r,e} + f''w_{r,e} - 2\sum_j \Phi'_j f'\partial_j w_{r,o}.$$

We now evaluate $\int C_1 U(kz)$. Arguing as for the derivation of (23) we find the following identity

$$\int_{\mathbb{R}^{n-1}} U(kz)\left[f''hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z\right] = 0,$$

which implies

$$\int_{\mathbb{R}^{n-1}} C_1 U(kz) = f'h\left[2\langle \mathbf{H}, \Phi'\rangle - \frac{1}{2}\left(\sum_l \partial^2_{1l} g_{11}\Phi_l\right)\right]\int_{\mathbb{R}^{n-1}} U(kz)^2.$$

On the other hand, using integration by parts, we obtain easily

$$\int_{\mathbb{R}^{n-1}} C_2 U(kz) = \int_{\mathbb{R}^{n-1}} C_3 U(kz) = 0.$$

Turning to C_4, we recall that

$$\mathcal{L}_r w_r = -2(f')^2 hU(kz)\langle \mathbf{H}, \Phi + z\rangle - h\langle \nabla^N V, \Phi + z\rangle U(kz) + \frac{1}{2}\sum_j \partial_j g_{11} hk\partial_j U(kz).$$

Therefore, after some computations we deduce

$$\int_{\mathbb{R}^{n-1}} C_4 U(kz) = -\frac{1}{h}\int_{\mathbb{R}^{n-1}} (\mathcal{L}_r w_r)w_i - 2k\sum_j H^j\int_{\mathbb{R}^{n-1}}\partial_j U(kz)w_i - (p-1)h^{p-2}\int_{\mathbb{R}^{n-1}} U(kz)^{p-1}w_r w_i.$$

Recalling the expressions of $\mathcal{L}_r$ and $\mathcal{L}_i$ and integrating by parts we then get

$$\int_{\mathbb{R}^{n-1}} C_4 U(kz) = -\frac{1}{h}\int_{\mathbb{R}^{n-1}} (\mathcal{L}_i w_i)w_r - 2k\sum_j H^j\int_{\mathbb{R}^{n-1}}\partial_j U(kz)w_i.$$

From the definition of w_i and using some cancelation we then find

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}(C_4+C_5)U(kz) &= \frac{1}{h}\partial_s\left[2hf'\int_{\mathbb{R}^{n-1}} w_1U(kz)\right]-2k\sum_j H^j\int_{\mathbb{R}^{n-1}}\partial_jU(kz)w_i\\ &= \frac{1}{h}\partial_s\left[2hf'\int_{\mathbb{R}^{n-1}} w_{r,e}U(kz)\right]-2k\sum_j H^j\int_{\mathbb{R}^{n-1}}\partial_jU(kz)w_i.\end{aligned}$$

To evaluate the last integral we need to use the explicit expression of $w_{r,e}$: recall that we have

$$w_{r,e}=h\left[\langle\nabla^N V,\Phi\rangle+2(f')^2\langle\mathbf{H},\Phi\rangle\right]\left(\frac{1}{(p-1)h^{p-1}}U(kz)+\frac{1}{2k}\nabla U(kz)\cdot z\right).$$

Therefore, adding all the terms and using some scaling we obtain

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}\tilde{R}_{i,e}U(kz) &= \frac{f'h}{k^{n-1}}\left[+2\langle\mathbf{H},\Phi'\rangle-\frac{1}{2}\left(\sum_l\partial_{1l}g_{11}\Phi_l\right)\right]\int_{\mathbb{R}^{n-1}}U^2-\frac{f'h}{k^{n-1}}\langle\mathbf{H},\Phi\rangle\int_{\mathbb{R}^{n-1}}U^2\\ &+ \frac{1}{h}\partial_s\left\{2h^2f'\left[\langle\nabla^N V,\Phi\rangle+2(f')^2\langle\mathbf{H},\Phi\rangle\right]\frac{1}{k^{n+1}}\left(\frac{1}{p-1}-\frac{n-1}{4}\right)\right\}\int_{\mathbb{R}^{n-1}}U^2.\end{aligned}$$

From the Euler equation (34) it follows that

$$\langle\nabla^N V,E_m\rangle+2(f')^2H^m=\frac{p-1}{\theta}h^{p-1}H^m.$$

Therefore after some manipulation we find

$$\int_{\mathbb{R}^{n-1}}\tilde{R}_{i,e}U(kz)=-\frac{2\mathcal{A}}{h}\left(\frac{p-1}{2\theta}-1\right)\int_{\mathbb{R}^{n-1}}U^2\partial_s\langle\mathbf{H},\Phi\rangle.$$

Contribution of $\tilde{R}_{i,e,f_1}$. Multiplying $\tilde{R}_{i,e,f_1}$ by $hU(kz)$ and integrating, recalling the expression of w_i determined in subsection 3.2 ($w_{i,o}=-\sum_j\Phi_j'f'hz_jU(kz)$ and $w_{i,e}=\frac{p-1}{4}f'h'|z|^2U(kz)$), we obtain

$$\begin{aligned}h\int_{\mathbb{R}^{n-1}}\tilde{R}_{i,e,f_1}U &= 2hh'f_1'\int_{\mathbb{R}^{n-1}}U^2+2h^2f_1'k'\int_{\mathbb{R}^{n-1}}\nabla U\cdot zU+\frac{p-1}{2}f'^2f_1'hh'\int_{\mathbb{R}^{n-1}}|z|^2U^2\\ &+ f_1''h^2\int_{\mathbb{R}^{n-1}}U^2+4h^2f'^2f_1''\int_{\mathbb{R}^{n-1}}U\tilde{U}+4hf'f_1'\int_{\mathbb{R}^{n-1}}U\partial_s(hf'\tilde{U})\\ &+ 2h^2f''f'f_1'\int_{\mathbb{R}^{n-1}}\tilde{U}U-\frac{(p-1)^2}{2}h^ph'f'^2f_1'\int_{\mathbb{R}^{n-1}}U^p|z|^2\tilde{U}\end{aligned}$$

We will need the following observations

$$\int_{\mathbb{R}^{n-1}}\nabla U(kz)\cdot zU(kz)dz=\frac{1}{k^n}\sum_j\int_{\mathbb{R}^n}\partial_jU(z)z_jU(z)dz=-\frac{n-1}{2k^n}\int_{\mathbb{R}^n}U^2(z)dz=-\frac{n-1}{2k^n}\int_{\mathbb{R}^n}U^2(z)dz;$$

which imply

$$(56)\quad \int_{\mathbb{R}^{n-1}}U\tilde{U}dz=\int_{\mathbb{R}^{n-1}}\frac{1}{(p-1)h^{p-1}}U^2(kz)+\frac{1}{2k}\nabla U(kz)\cdot zU(kz)dz=-\frac{\sigma}{2(p-1)k^{n+1}}\int_{\mathbb{R}^{n-1}}U^2(z)dz;$$

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}}|z|^2\nabla U(kz)\cdot zU^p(kz)dz=\frac{1}{k^{n+2}}\sum_{l,j}\int_{\mathbb{R}^n}z_l^2\partial_jU(z)z_jU^p(z)dz\\ &= -\frac{1}{k^{n+2}}\sum_j\int_{\mathbb{R}^n}(n+1)z_j^2U^{p+1}(z)dz-\frac{1}{k^{n+2}}\sum_{l,j}\int_{\mathbb{R}^n}pz_l^2\partial_jU(z)z_jU^p(z)dz\\ &= -\frac{n+1}{k}\int_{\mathbb{R}^n}|z|^2U^{p+1}(kz)dz-p\int_{\mathbb{R}^n}|z|^2\nabla U(kz)\cdot zU^p(kz)dz.\end{aligned}$$

From these we get

$$\int_{\mathbb{R}^{n-1}} |z|^2 \nabla U(kz)\cdot z U^p(kz)dz = -\frac{n+1}{(p+1)k}\int_{\mathbb{R}^n} |z|^2 U^{p+1}(kz)dz = -\frac{n+1}{(p+1)k^{n+1}}\int_{\mathbb{R}^n}|z|^2U^{p+1}(z)dz.$$

Moreover one can easily see that

$$\begin{aligned}
2hh'f_1'\int_{\mathbb{R}^{n-1}} U^2dz + 2h^2 f_1' k'\int_{\mathbb{R}^{n-1}} \nabla U\cdot zUdz + f_1''h^2\int_{\mathbb{R}^{n-1}} U^2 dz &= \partial_{\overline{s}}\left(h^2 f_1'\int_{\mathbb{R}^n} U^2(kz)\right)dz \\
&= \partial_{\overline{s}}\left(\frac{h^2 f_1'}{k^{n-1}}\int_{\mathbb{R}^n} U^2(z)dz\right),
\end{aligned}$$

and

$$\begin{aligned}
4hf'f_1'\int_{\mathbb{R}^{n-1}} U\partial_s(hf'\tilde{U})dz &= 4\partial_{\overline{s}}\left(h^2 f'^2 f_1'\int_{\mathbb{R}^{n-1}} U\tilde{U}\right)dz + 4f'h\partial_{\overline{s}}[hf'f_1']\int_{\mathbb{R}^{n-1}} U\tilde{U}dz \\
&+ 4f'^2h^2f_1'\int_{\mathbb{R}^{n-1}}\tilde{U}\partial_{\overline{s}}Udz \\
&= 4\partial_{\overline{s}}\left(h^2 f'^2 f_1'\int_{\mathbb{R}^{n-1}} U\tilde{U}\right)dz + 4f'h\partial_{\overline{s}}[hf'f_1']\int_{\mathbb{R}^{n-1}} U\tilde{U}dz \\
&+ 4f'^2h^2f_1'k'\frac{1}{(p-1)h^{p-1}}\int_{\mathbb{R}^{n-1}} U(kz)\nabla U(kz)\cdot zdz \\
&+ 2f'^2h^2f_1'\frac{k'}{k}\int_{\mathbb{R}^{n-1}}|z|^2|\nabla U(kz)|^2dz.
\end{aligned}$$

On the other hand, by (64) (see Section 5.1) we have

$$\begin{aligned}
\frac{p-1}{2}f'^2f_1'hh'\int_{\mathbb{R}^{n-1}}|z|^2U^2dz &= \frac{p-1}{p+1}f'^2f_1'hh'\int_{\mathbb{R}^{n-1}}|z|^2U^{p+1}(kz)dz \\
&- \frac{(n-5)(p-1)}{2(n+1)}f'^2f_1'hh'\int_{\mathbb{R}^{n-1}}|z|^2|\nabla U|^2dz.
\end{aligned}$$

Hence with some easy manipulations and by (65) one finds

$$\begin{aligned}
&\frac{p-1}{2}f'^2f_1'hh'\int_{\mathbb{R}^{n-1}}|z|^2U^2dz - \frac{(p-1)^2}{2}h^ph'f'^2f_1'\int_{\mathbb{R}^{n-1}}U^p|z|^2\tilde{U}dz + 2f'^2h^2f_1'\frac{k'}{k}\int_{\mathbb{R}^{n-1}}|z|^2|\nabla U(kz)|^2dz \\
&= -\frac{(p-1)(n-1)}{4}f'^2f_1'hh'\left[\frac{6}{n+1}\int_{\mathbb{R}^{n-1}}|z|^2|\nabla U(kz)|^2dz - \frac{p-1}{p+1}\int_{\mathbb{R}^{n-1}}|z|^2U^{p+1}(kz)dz\right] \\
&= -\frac{(p-1)(n-1)^2}{4k^{n+1}}f'^2f_1'hh'\int_{\mathbb{R}^{n-1}}U^2(z)dz.
\end{aligned}$$

Collecting the above expressions, and letting $C_0 = \int_{\mathbb{R}^{n-1}} U^2(z)dz$ one proves that

$$\begin{aligned}
\frac{h}{C_0}\int_{\mathbb{R}^{n-1}}\tilde{R}_{i,e,f_1}Udz &= \partial_{\overline{s}}\left(\frac{h^2f_1'}{k^{n-1}}\right) - 2\partial_{\overline{s}}\left(f'^2h^2f_1'\frac{\sigma}{(p-1)k^{n+1}}\right) - \frac{(p-1)(n-1)^2}{4k^{n+1}}f'^2hh'f_1' \\
&+ 2f'^2f_1'hh'\frac{\sigma}{(p-1)k^{n+1}} + (n-1)f'^2f_1'\frac{hh'}{k^{n+1}} + f'f''f_1'h^2\frac{\sigma}{(p-1)k^{n+1}} \\
&= \partial_{\overline{s}}\left(\frac{h^2f_1'}{k^{n-1}}\left[1-2f'^2\frac{\sigma}{(p-1)k^2}\right]\right) - \frac{\sigma}{(p-1)k^{n+1}}hf_1'\left(\sigma f'^2h' - f'f''\right);
\end{aligned}$$

using (23), we have that $\sigma f'^2 h' - f'f'' = 0$, which implies

$$\begin{aligned}
\frac{h}{C_0}\int_{\mathbb{R}^{n-1}}\tilde{R}_{i,e,f_1}U(kz) &= \partial_{\overline{s}}\left(\frac{h^2f_1'}{k^{n-1}}\left[1-2f'^2\frac{\sigma}{(p-1)k^2}\right]\right) \\
&= \partial_{\overline{s}}\left(\frac{h^2f_1'}{(p-1)k^{n+1}}\left[(p-1)h^{p-1} - 2\sigma\mathcal{A}^2h^{2\sigma}\right]\right).
\end{aligned}$$

Recalling the discussion at the beginning of this subsection, we want to find f_1 such that

$$\int_{\mathbb{R}^{n-1}} \tilde{R}_{i,e,f_1} U(kz) + \int_{\mathbb{R}^{n-1}} \tilde{R}_{i,e} U(kz) = 0.$$

This is equivalent to

$$Tf_1 := \partial_{\bar{s}} \left(\frac{h^2 f_1'}{(p-1)k^{n+1}} \left[(p-1)h^{p-1} - 2\sigma \mathcal{A}^2 h^{2\sigma} \right] \right) = 2\mathcal{A} \left(\frac{p-1}{2\theta} - 1 \right) \partial_{\bar{s}} \langle \mathbf{H}, \Phi \rangle. \tag{57}$$

Hence we get

$$f_1' = \frac{2\mathcal{A}(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma \mathcal{A}^2 h^{2\sigma+2}} \left(\frac{p-1}{2\theta} - 1 \right) \langle \mathbf{H}, \Phi \rangle + c \frac{(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma \mathcal{A}^2 h^{2\sigma+2}},$$

where c is a constant to be chosen so that $\int_0^L f_1' ds = 0$. Noticing that, by (31), we have

$$\frac{(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma \mathcal{A}^2 h^{2\sigma+2}} = A\sigma h^{\sigma-1} \frac{\partial h}{\partial \mathcal{A}} + h^{\sigma},$$

the required condition becomes

$$c = -\left(\frac{p-1}{2\theta} - 1 \right) 2\mathcal{A}(p-1) \frac{\int_0^L \frac{k^{n+1}}{(p-1)h^{p+1} - 2\sigma \mathcal{A}^2 h^{2\sigma+2}} \langle \mathbf{H}, \Phi \rangle ds}{\int_0^L A\sigma h^{\sigma-1} \frac{\partial h}{\partial \mathcal{A}} + h^{\sigma} ds}.$$

As one can easily check from (29) and (34), c coincides with $\mathcal{A}'$ and therefore we have in conclusion

$$f_1' = \frac{2\mathcal{A}(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma \mathcal{A}^2 h^{2\sigma+2}} \left(\frac{p-1}{2\theta} - 1 \right) \langle \mathbf{H}, \Phi \rangle + \mathcal{A}' \frac{(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma \mathcal{A}^2 h^{2\sigma+2}}. \tag{58}$$

4.2 Projection onto the kernel of $\mathcal{L}_r$

Similarly to the previous subsection, we need to annihilate the projection of the $\tilde{R}$'s in (55) onto the kernel of $\mathcal{L}_r$. This corresponds to multiplying the error terms by $\partial_m U(k\cdot)$, $m = 1, \dots, n-1$, integrating over $\mathbb{R}^{n-1}$ and taking the real part. As before, we are left to consider only two terms: $\tilde{R}_{r,o}$ and $\tilde{R}_{r,o,f_1}$.

We begin by multiplying $\tilde{R}_{r,o}$ by $\partial_m U$ and integrating. Recalling the expression of w_i determined in Subsection 3.2 ($w_{i,o} = -\sum_j \Phi_j' f' h z_j U(kz)$ and $w_{i,e} = \frac{p-1}{4} f' h' |z|^2 U(kz)$), we obtain

$$\int_{\mathbb{R}^{n-1}} (\tilde{R}_{r,o} + \tilde{R}_{r,o,f_1}) \partial_m U(kz) = A_1 + A_2 + A_3 + A_4 + A_5 + A_6,$$

where

$$A_1 = 2(f')^2 \Phi_m'' h \int_{\mathbb{R}^{n-1}} z_m U(kz) \partial_m U(kz) + \Phi_m'' hk \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2;$$

$$\begin{aligned} A_2 &= \Phi_m' \Big[2f'f''h \int_{\mathbb{R}^{n-1}} z_m U(kz) \partial_m U(kz) + 2(f')^2 h' \int_{\mathbb{R}^{n-1}} z_m U(kz) \partial_m U(kz) \\ &+ f'f''h \int_{\mathbb{R}^{n-1}} z_m U(kz) \partial_m U(kz) + 2h'k \int (\partial_m U(kz))^2 + 2hk' \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2 \\ &+ 2(f')^2 \frac{k'}{k} h \int_{\mathbb{R}^{n-1}} z_m U(kz) \partial_m U(kz) \Big]; \end{aligned}$$

$$\begin{aligned} A_3 &= \Phi_m' \Bigg[2(f')^2 hk' \int_{\mathbb{R}^{n-1}} z_m \partial_m U(kz) \nabla U(kz) \cdot z + 2hk'k \sum_l \int_{\mathbb{R}^{n-1}} z_l \partial_{ml}^2 U(kz) \partial_m U(kz) \\ &+ (f')^2 k' h \int_{\mathbb{R}^{n-1}} |z|^2 (\partial_m U(kz))^2 + \frac{1}{2}(p-1)(f')^2 h^p \frac{k'}{k} \int_{\mathbb{R}^{n-1}} |z|^2 (U(kz))^p z_m \partial_m U(kz) \Bigg]. \end{aligned}$$

Here we used the fact that $\frac{k'}{k} = \frac{p-1}{2}\frac{h'}{h}$. We have next

$$\begin{aligned} A_4 &= 2(f')^2 H^m \int_{\mathbb{R}^{n-1}} z_m \partial_m U(kz) w_{r,e} + 2(f')^2 \langle \mathbf{H}, \Phi \rangle \int_{\mathbb{R}^{n-1}} w_{r,o} \partial_m U(kz) \\ &+ H^m \int_{\mathbb{R}^{n-1}} \partial_m w_{r,e} \partial_m U(kz) - p(p-1)h^{p-2} \int_{\mathbb{R}^{n-1}} U(kz)^{p-2} w_{r,e} w_{r,o} \partial_m U(kz) \\ &+ \int_{\mathbb{R}^{n-1}} \langle \nabla^N V, w_{r,e} z + w_{r,o} \Phi \rangle \partial_m U(kz); \end{aligned}$$

$$\begin{aligned} A_5 &= -(f')^2 h \sum_l \Phi_l \partial^2_{lm} g_{11} \int_{\mathbb{R}^{n-1}} z_m U(kz) \partial_m U(kz) + 8(f')^2 h \langle \mathbf{H}, \Phi \rangle H^m \int_{\mathbb{R}^{n-1}} z_m U(kz) \partial_m U(kz) \\ &+ hk \langle \mathbf{H}, \Phi \rangle H^m \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2 + hk^2 \sum_{l,s,t,j} \partial^2_{ls} g_{tj} \Phi_l \int_{\mathbb{R}^{n-1}} \partial^2_{tj} U(kz) z_s \partial_m U(kz) \\ &+ hk \sum_{l,t} \partial^2_{lt} g_{tm} \Phi_l \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2 + hk \langle \mathbf{H}, \Phi \rangle H^m \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2 \\ &- \frac{1}{2} hk \left(\sum_l \partial^2_{ml} g_{11} \Phi_l \right) \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2 + \sum_j \partial^2_{mj} V \Phi_j h \int_{\mathbb{R}^{n-1}} z_m U(kz) \partial_m U(kz), \end{aligned}$$

and

$$\begin{aligned} A_6 &= 2f' f_1' \int_{\mathbb{R}^{n-1}} w_{r,o} \partial_m U(kz) + 4 f' f_1' h H^m \int_{\mathbb{R}^{n-1}} U z_m \partial_m U(kz) \\ &- 2p(p-1) f' f_1' h^{p-1} \int_{\mathbb{R}^{n-1}} U^{p-2} \tilde{U} w_{r,o} \partial_m U(kz) + 2 f' f_1' h (\nabla^N V)^m \int_{\mathbb{R}^{n-1}} z_m \tilde{U} \partial_m U(kz) \\ &+ 2 H^m f' f_1' h \int_{\mathbb{R}^{n-1}} \partial_m \tilde{U} \partial_m U(kz) + 4 H^m (f')^2 f' f_1' \int_{\mathbb{R}^{n-1}} z_m \partial_m U(kz) h \tilde{U}. \end{aligned}$$

Integrating by parts and using the above relations we obtain

$$A_1 = \Phi''_m \frac{h}{k^n} \left[k^2 \frac{p-1}{2\theta} - (f')^2 \right] \int_{\mathbb{R}^{n-1}} U^2(z) dz = \Phi''_m \frac{h}{k^n} \frac{p-1}{2\theta} h^\sigma \left[h^\theta - \frac{2\mathcal{A}^2 \theta}{p-1} h^\sigma \right] \int_{\mathbb{R}^{n-1}} U^2(z) dz.$$

Similarly, for A_2 after some integration by parts and some rescaling one finds

$$A_2 = \frac{\Phi'_m}{k^{n-1}} \left\{ \frac{2(hk)'}{n-1} \int_{\mathbb{R}^{n-1}} |\nabla U(z)|^2 dz - \frac{h}{k} \left[\left(\frac{3}{2} f' f'' + (f')^2 \frac{k'}{k} \right) + \frac{h'}{k} (f')^2 \right] \int_{\mathbb{R}^{n-1}} U^2(z) dz \right\}.$$

In particular, recalling the identity (45) one then gets

$$A_2 = \frac{\Phi'_m}{k^{n-1}} \left\{ \frac{p-1}{\theta} (hk)' - \left[\frac{h}{k} \left(\frac{3}{2} f' f'' + (f')^2 \frac{k'}{k} \right) + \frac{h'}{k} (f')^2 \right] \right\} \int_{\mathbb{R}^{n-1}} U^2(z) dz.$$

Now we turn to the term A_3. First of all we can write

$$\sum_l \int_{\mathbb{R}^{n-1}} z_l \partial^2_{ml} U(kz) \partial_m U(kz) = -\frac{n-1}{k} \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2 - \sum_l \int_{\mathbb{R}^{n-1}} z_l \partial^2_{ml} U(kz) \partial_m U(kz).$$

Hence, using a simple scaling and again formula (45) we obtain that

$$\sum_l \int_{\mathbb{R}^{n-1}} z_l \partial^2_{ml} U(kz) \partial_m U(kz) = -\frac{(n-1)(p-1)}{4\theta k^n} \int_{\mathbb{R}^{n-1}} U^2(z) dz.$$

Regarding the other terms in A_3, we write them in radial coordinates, obtaining

$$3\frac{(f')^2hk'}{(n-1)}\omega_{n-2}\int_0^\infty r^n(U'(kr))^2dr + \frac{1}{2}\frac{(p-1)(f')^2h^p}{(n-1)}\frac{k'}{k}\omega_{n-2}\int_0^\infty r^{n+1}U(kr)^pU'(kr)dr,$$

where ω_{n-2} is the volume of S^{n-2}. Using a change of variables and integrating by parts (recalling the relation $k^2 = h^{p-1}$) we then find

$$\frac{\omega_{n-2}(f')^2}{n-1}\frac{k'h}{k^{n+1}}\left[3\int_0^\infty r^n(U'(r))^2dr - \frac{1}{2}\frac{(n+1)(p-1)}{p+1}\int_0^\infty r^nU^{p+1}(r)dr\right].$$

Using the formula in the appendix we find that this expression becomes

$$\frac{n+1}{2}\frac{(f')^2k'h}{k^{n+1}}\int_{\mathbb{R}^{n-1}}U^2(z)dz.$$

Therefore we also obtain that

$$A_3 = \Phi_m'\left[\frac{n+1}{2}\frac{(f')^2k'h}{k^{n+1}}\int_{\mathbb{R}^{n-1}}U^2(z)dz - \frac{(n-1)(p-1)}{2\theta k^{n-1}}hk'\int_{\mathbb{R}^{n-1}}U^2(z)dz\right].$$

Finally, after some tedious but straightforward computation one also deduces

$$A_2 + A_3 = \Phi_m'\left[\frac{p-1}{2}\frac{h'}{k^{n-2}} - \mathcal{A}^2\sigma\frac{h^{2\sigma}h'}{k^n}\right]\int_{\mathbb{R}^{n-1}}U^2 = \Phi_m'\frac{p-1}{2}\frac{h'}{k^n}\left[h^{p-1} - \frac{2\mathcal{A}^2\sigma}{p-1}h^{2\sigma}\right]\int_{\mathbb{R}^{n-1}}U^2.$$

It remains now to evaluate the contribution of A_4 and of A_5. Regarding A_4 we recall that w_r satisfies

$$\mathcal{L}_rw_r = F,$$

where

$$F = (f')^2hU(kz)\left(\sum_m\partial_mg_{11}(z_m+\Phi_m)\right) + \frac{1}{2}\sum_j\partial_jg_{11}hk\partial_jU(kz) - \sum_m\frac{\partial V}{\partial z_m}(z_m+\Phi_m)hU(kz).$$

Differentiating this relation with respect to z_m we obtain

$$\mathcal{L}_r\partial_mw_r = \partial_mF + p(p-1)h^{p-1}kU(kz)^{p-2}\partial_mU(kz)w_r.$$

Multiplying by w_r, integrating and using the self-adjointness of $\mathcal{L}_r$ we find

$$\frac{1}{2}p(p-1)h^{p-2}\int_{\mathbb{R}^{n-1}}U(kz)^{p-2}\partial_mU(kz)w_r^2 - \frac{1}{2hk}\int_{\mathbb{R}^{n-1}}F\left(F\partial_mw_r - w_r\partial_mF\right).$$

We have clearly

$$0 = \int_{\mathbb{R}^{n-1}}\partial_m(Fw_r) = \int_{\mathbb{R}^{n-1}}\left(w_r\partial_mF + F\partial_mw_r\right).$$

Therefore it follows that

$$-\frac{1}{2}p(p-1)h^{p-2}\int_{\mathbb{R}^{n-1}}U(kz)^{p-2}\partial_mU(kz)w_r^2 = \frac{1}{hk}\int_{\mathbb{R}^{n-1}}w_r\partial_mF.$$

Hence A_4 can be written as

$$\begin{aligned}
A_4 &= -\int_{\mathbb{R}^{n-1}}w_r\left[k\sum_lH^l\partial^2_{ml}U(kz) + (f')^2\partial_mU(kz)\left(\sum_lH^l(z_l+\Phi_l)\right)\right] \\
&+ \frac{1}{k}\int_{\mathbb{R}^{n-1}}w_r\Big[-(f')^2U(kz)H^m - k\partial_mU(kz)\langle\nabla^NV, z+\Phi\rangle - hU(kz)\langle\nabla^NV, E_m\rangle\Big] \\
&+ \int_{\mathbb{R}^{n-1}}\partial_mU(kz)\left[w_r\langle\nabla^NV, z+\Phi\rangle + 2(f')^2\langle\mathbf{H}, z+\Phi\rangle w_r\right] \\
&+ h^2k^2\int_{\mathbb{R}^{n-1}}\partial_mU(kz)\sum_lH^l\partial_lw_r.
\end{aligned}$$

After some cancelations and some integration by parts we find the following formula

$$A_4 = -2k \sum_l H^l \int_{\mathbb{R}^{n-1}} w_r \partial^2_{ml} U(kz) + \frac{p-1}{2k\theta} k^2 \int_{\mathbb{R}^{n-1}} w_r U(kz).$$

Using the symmetries of the integrals we find

$$\begin{aligned} A_4 &= -2kH^m \left[\frac{1}{n-1} \int_{\mathbb{R}^{n-1}} w_r \Delta U(kz) + \frac{p-1}{2\theta} \int_{\mathbb{R}^{n-1}} w_r U(kz) \right] \\ &= -2kH^m \int_{\mathbb{R}^{n-1}} w_{r,e} \left[U(kz) \left(\frac{1}{n-1} + \frac{p-1}{2\theta} \right) - \frac{1}{n-1} U^p(kz) \right]. \end{aligned}$$

From the explicit expression of w_r and some integration by parts we obtain

$$\begin{aligned} A_4 &= 2kH^m \frac{(p-1)h}{\theta k^{n-1}} \langle \mathbf{H}, \Phi \rangle \\ &\times \left[\frac{1}{n-1} \left(\frac{1}{p-1} - \frac{n-1}{2(p+1)} \right) \int_{\mathbb{R}^{n-1}} U^{p+1} - \left(\frac{1}{p-1} - \frac{n-1}{4} \right) \left(\frac{1}{n-1} + \frac{p-1}{2\theta} \right) \int_{\mathbb{R}^{n-1}} U^2 \right]. \end{aligned}$$

Using (45) we find that $\int_{\mathbb{R}^{n-1}} U^{p+1} = \left(1 + \frac{(n-1)(p-1)}{2\theta}\right) \int_{\mathbb{R}^{n-1}} U^2$, and therefore it follows that

$$A_4 = kH^m \frac{(p-1)^2 h}{2\theta^2 k^{n-1}} \langle \mathbf{H}, \Phi \rangle \int_{\mathbb{R}^{n-1}} U^2.$$

Next we turn to A_5. First of all we consider the two terms

$$B_1 = hk^2 \sum_{l,s,t,j} \partial^2_{ls} g_{tj} \Phi_l \int_{\mathbb{R}^{n-1}} \partial^2_{tj} U(kz) z_s \partial_m U(kz) + hk \sum_{l,t} \partial^2_{lt} g_{tm} \Phi_l \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2.$$

Looking at the first one, by symmetry reasons the summands do not vanish if, either $s = m$ and $t = j$, if $t = m$ and $s = j$ or if $j = m$ and $s = t$. In the first case, we see appearing the second derivative of g_{tt}, which vanishes by our choice of the geodesic coordinates. Therefore we are left with the terms

$$B_2 = hk^2 \sum_{l,j} \partial^2_{lj} g_{mj} \Phi_l \int_{\mathbb{R}^{n-1}} \partial^2_{mj} U(kz) z_j \partial_m U(kz) + hk^2 \sum_{l,t} \partial^2_{lt} g_{tm} \Phi_l \int_{\mathbb{R}^{n-1}} \partial^2_{tm} U(kz) z_t \partial_m U(kz).$$

Integrating by parts one easily finds

$$B_2 = -\frac{1}{2} hk \sum_{l,j} \partial^2_{lj} g_{mj} \Phi_l \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2 - \frac{1}{2} hk \sum_{l,t} \partial^2_{lt} g_{tm} \Phi_l \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2,$$

therefore it follows that

$$B_1 = 0.$$

We turn now to the other terms. Integrating by parts and using (45) one deduces

$$\begin{aligned} &-(f')^2 h \sum_l \Phi_l \partial^2_{lm} g_{11} \int_{\mathbb{R}^{n-1}} z_m U(kz) \partial_m U(kz) - \frac{1}{2} hk \left(\sum_l \partial^2_{ml} g_{11} \Phi_l \right) \int_{\mathbb{R}^{n-1}} (\partial_m U(kz))^2 \\ &= \frac{1}{2} \left(\sum_l \Phi_l \partial^2_{lm} g_{11} \right) \frac{h}{k^n} \left[\mathcal{A}^2 h^{2\sigma} - h^{p-1} \frac{p-1}{2\theta} \right]. \end{aligned}$$

Hence after some integration one finds

$$\begin{aligned} A_5 &= \frac{1}{2} \left(\sum_l \Phi_l \partial^2_{lm} g_{11} \right) \frac{h}{k^n} \left[\mathcal{A}^2 h^{2\sigma} - h^{p-1} \frac{p-1}{2\theta} \right] \int_{\mathbb{R}^{n-1}} U^2 - \frac{h}{2k^n} \sum_j \partial^2_{mj} V \Phi_j \int_{\mathbb{R}^{n-1}} U^2 \\ &+ 4 \frac{h}{k^n} \langle \mathbf{H}, \Phi \rangle H^m \left[h^{p-1} \frac{p-1}{4\theta} - \mathcal{A}^2 h^{2\sigma} \right] \int_{\mathbb{R}^{n-1}} U^2. \end{aligned}$$

Finally we turn to A_6: we first evaluate the terms involving $w_{r,o}$, whose explicit expression is not known, but which can be handled via some integration by parts. Differentiating the equation $\mathcal{L}_r \tilde{U} = -U$ with respect to z_m we find that

$$\mathcal{L}_r(\partial_m \tilde{U}) = -\partial_m U + p(p-1)h^{p-1}U^{p-2}\tilde{U}\partial_m U.$$

Therefore, integrating by parts and recalling the definition of $w_{r,o}$ we obtain

$$\begin{aligned}
& 2f'f_1' \int_{\mathbb{R}^{n-1}} w_{r,o}\partial_m U(kz) - 2p(p-1)f'f_1'h^{p-1}\int_{\mathbb{R}^{n-1}} U^{p-2}\tilde{U}w_{r,o}\partial_m U(kz) \\
= \; & -2f'f_1' \int_{\mathbb{R}^{n-1}} w_{r,o}\mathcal{L}_r(\partial_m \tilde{U}) = -2f'f_1'\int_{\mathbb{R}^{n-1}} \partial_m \tilde{U}\mathcal{L}_r w_{r,0} \\
= \; & 2f'f_1'hH^m \int_{\mathbb{R}^{n-1}} \partial_1 \tilde{U}\left(k\partial_1 U + \frac{p-1}{\theta}h^{p-1}z_1 U\right).
\end{aligned}$$

Using (23) and (34), we can also combine the last term in the second row of A_6 with the last one in the third row to obtain

$$\begin{aligned}
& 2f'f_1'h(\nabla^N V)^m \int_{\mathbb{R}^{n-1}} z_1\tilde{U}\partial_1 U(kz) + 4H^m(f')^2 f'f_1'h\int_{\mathbb{R}^{n-1}} z_1\partial_1 U(kz)\tilde{U} \\
= \; & 2f'f_1'h\left[H^m\left(\frac{p-1}{\theta}h^{p-1} - 2\mathcal{A}^2h^{2\sigma}\right)\int_{\mathbb{R}^{n-1}} z_1\tilde{U}\partial_1 U(kz) + 2\mathcal{A}^2h^{2\sigma}H^m\int_{\mathbb{R}^{n-1}} z_1\partial_1 U(kz)\tilde{U}\right] \\
= \; & 2f'f_1'h\frac{p-1}{\theta}h^{p-1}H^m\int_{\mathbb{R}^{n-1}} z_1\tilde{U}\partial_1 U(kz).
\end{aligned}$$

With some manipulation, the sum of the last two formulas gives

$$2f'f_1'hH^m\left(\int_{\mathbb{R}^{n-1}} \partial_1\tilde{U}\partial_1 U - \frac{p-1}{\theta}\frac{h^{p-1}}{k}\int_{\mathbb{R}^{n-1}} U\tilde{U}\right).$$

Collecting all the terms in A_6 then we get

$$2f'f_1'hH^m\left(2k\int_{\mathbb{R}^{n-1}} \partial_1\tilde{U}\partial_1 U - \frac{p-1}{\theta}\frac{h^{p-1}}{k}\int_{\mathbb{R}^{n-1}} U\tilde{U} + 2\int_{\mathbb{R}^{n-1}} Uz_1\partial_1 U(kz)\right).$$

With some integration by parts, one finds that

$$\int_{\mathbb{R}^{n-1}} \partial_1\tilde{U}\partial_1 U = \frac{1}{n-1}\int_{\mathbb{R}^{n-1}} \tilde{U}(-\Delta U) = \frac{1}{n-1}\int_{\mathbb{R}^{n-1}} \tilde{U}(U^p - U),$$

and $\int_{\mathbb{R}^{n-1}} Uz_1\partial_1 U(kz) = -\frac{1}{2k}\int_{\mathbb{R}^{n-1}} U^2$, and therefore the last quantity becomes

$$2f'f_1'hH^m\left[-\left(\frac{2}{n-1} + \frac{p-1}{\theta}\right)\int_{\mathbb{R}^{n-1}} U\tilde{U} + \frac{2}{n-1}\int_{\mathbb{R}^{n-1}} \tilde{U}U^p - 1\right].$$

Using (45) one also finds

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \tilde{U}(kz)U^p(kz)dz \; &= \; \frac{1}{k^2}\left(\frac{1}{p-1} - \frac{n-1}{2(p+1)}\right)\int_{\mathbb{R}^{n-1}} U^{p+1}(kz)dz \\
&= \; \frac{1}{k^2}\frac{\theta}{(p+1)(p-1)}\frac{p+1}{\theta}\int_{\mathbb{R}^{n-1}} U^2(kz) = \frac{1}{k^2}\frac{1}{p-1}\int_{\mathbb{R}^{n-1}} U^2(kz).
\end{aligned}$$

From this formula, (56) and some manipulation we then get

$$\begin{aligned}
& 2f'f_1'hH^m\left(\frac{2\sigma(p+1)}{2(n-1)(p-1)\theta} + \frac{2}{(p-1)(n-1)} - 1\right)\int_{\mathbb{R}^{n-1}} U^2(kz) \\
= \; & 2f'f_1'\frac{h}{k^n}H^m\left(\frac{p-1}{2\theta} - 1\right)\int_{\mathbb{R}^{n-1}} U^2(z).
\end{aligned}$$

Now, recalling the expression of f_1' given by (58), the total projection onto the kernel of $\mathcal{L}_r$ is

$$\begin{aligned} & C_0 \Bigg\{ \frac{1}{2}\frac{h}{k^n}\left(\mathcal{A}^2 h^{2\sigma} - h^{p-1}\frac{p-1}{2\theta}\right)\sum_l \Phi_l \partial^2_{lm} g_{11} - \frac{h}{2k^n}\sum_j \partial^2_{mj} V \Phi_j \\ + \quad & 4\frac{h}{k^n}\langle \mathbf{H}, \Phi\rangle H^m \left[h^{p-1}\frac{p-1}{4\theta} - \mathcal{A}^2 h^{2\sigma}\right] + \frac{(p-1)^2 h}{2\theta^2 k^{n-2}} H^m \langle \mathbf{H}, \Phi\rangle \\ + \quad & \Phi_m'' \frac{h}{k^n}\frac{p-1}{2\theta} h^\sigma \left[h^\theta - \frac{2\mathcal{A}^2\theta}{p-1}h^\sigma\right] + \Phi_m' \frac{p-1}{2}\frac{h'}{k^n}\left[h^{p-1} - \frac{2\mathcal{A}^2\sigma}{p-1}h^{2\sigma}\right] \\ + \quad & \left(\frac{p-1}{2\theta} - 1\right)^2 \frac{h}{k^n}\frac{(2\mathcal{A})^2(p-1)h^{\sigma+p-1}}{(p-1)h^\theta - 2\sigma\mathcal{A}^2 h^\sigma}\langle \mathbf{H}, \Phi\rangle H^m - \left(\frac{\sigma-\theta}{2\theta}\right)\frac{h}{k^n}\frac{2\mathcal{A}\mathcal{A}'(p-1)h^{\sigma+p-1}}{(p-1)h^\theta - 2\sigma\mathcal{A}^2 h^\sigma} H^m \Bigg\}, \end{aligned}$$

where $C_0 = \int_{\mathbb{R}^{n-1}} U^2$. Now, recalling the definitions of σ and θ, choosing f_1 as in (58), with some calculation we find

$$\begin{aligned} & \int_{\mathbb{R}^{n-1}} (\tilde{R}_{r,o} + \tilde{R}_{r,o,f_1})\partial_m U(k(\varepsilon s)z)dz = \\ & -\frac{p-1}{2\theta}\frac{1}{hk}C_0 \Bigg\{ -\left(h^\theta - \frac{2\mathcal{A}^2\theta}{p-1}h^\sigma\right)\Phi_m'' - \theta\left(h^{\theta-1} - \frac{2\mathcal{A}^2\sigma}{p-1}h^{\sigma-1}\right)h'\Phi_m' + \frac{\theta}{p-1}h^{-\sigma}((\nabla^N)^2 V)\Phi_m \\ & \qquad + \frac{1}{2}\left(h^\theta - \frac{2\mathcal{A}^2\theta}{p-1}h^\sigma\right)\left(\sum_j \left(\partial^2_{jm} g_{11}\right)\Phi_j\right) - 2\mathcal{A}\mathcal{A}_1' \frac{(\theta-\sigma)h^{p-1}}{[(p-1)h^\theta - 2\sigma\mathcal{A}^2 h^\sigma]} H^m \\ & \qquad + H^m \langle \mathbf{H}, \Phi\rangle \left[\frac{-(p-1)\left(3+\frac{\sigma}{\theta}\right)h^{2\theta} - \frac{16\sigma\theta\mathcal{A}^4}{p-1}h^{2\sigma} + 2\mathcal{A}^2(5\sigma+3\theta)h^{\theta+\sigma}}{(p-1)h^\theta - 2\mathcal{A}^2\sigma h^\sigma}\right]\Bigg\}. \end{aligned} \tag{59}$$

We notice that the operator between brackets coincides precisely with the one in (36), corresponding to the second variation of the reduced functional which we determined in Subsection 2.5.

Remark 4.1 *According to the considerations in Subsection 2.4, to every normal variation of γ it corresponds some variation in the phase due to both the variation of position and the variation of the constant $\mathcal{A}$. Recall that the phase of the approximate solution is the following*

$$F_\varepsilon = \frac{1}{\varepsilon}f(\varepsilon s) = \frac{1}{\varepsilon}\int_0^{\varepsilon s} f' dl.$$

Differentiating with respect to a variation ν (see (12)) we obtain

$$\frac{\partial}{\partial \nu}F_\varepsilon = \frac{1}{\varepsilon}\int_0^{\varepsilon s}\mathcal{A}_\nu h^\sigma + \frac{1}{\varepsilon}\int_0^{\varepsilon s}\mathcal{A}\sigma h^{\sigma-1}\left(\frac{\partial h}{\partial \mathcal{A}}\mathcal{A}_\nu + \frac{\partial h}{\partial V}\frac{\partial V}{\partial \nu}\right) + \frac{1}{2}\frac{1}{\varepsilon}\int_0^{\varepsilon s}\partial_\nu g_{11}\mathcal{A}h^\sigma.$$

Recalling formula (32) we find

$$\frac{\partial}{\partial \nu}F_\varepsilon = \frac{1}{\varepsilon}\frac{p-1}{2\mathcal{A}}\mathcal{A}_\nu \int_0^{\varepsilon s} h^{\theta-1}\frac{\partial h}{\partial \mathcal{A}} + \frac{1}{\varepsilon}\int_0^{\varepsilon s}\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial V}\frac{\partial V}{\partial \nu} + \frac{1}{2}\frac{1}{\varepsilon}\int_0^{\varepsilon s}\partial_\nu g_{11}\mathcal{A}h^\sigma.$$

Therefore, when we take a variation ν_2 of γ, this also corresponds to a variation of the phase of $\frac{\partial}{\partial \nu_2}F_\varepsilon$. Notice that multiplying the horizontal part by $h\partial_m U$ corresponds to adding a variation of $-\frac{\nu_2}{k}$.

Hence integrating by parts we get

$$\mathcal{A}\left(\frac{p-1}{2\theta} - 1\right)\int\left(\sum_m \Phi_m \partial_m g_{11}\right)\left[\frac{p-1}{2\mathcal{A}}\mathcal{A}_{\nu_2}h^{\theta-1}\frac{\partial h}{\partial \mathcal{A}} - \mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial V}\frac{\partial V}{\partial \nu_2} - \frac{1}{2}\mathcal{A}h^\sigma \partial_{\nu_2}g_{11}\right].$$

4.3 Proof of Theorem 1.4

The proof of Theorem 1.4 can be deduced with an iterative procedure, adding higher order corrections (at any arbitrary order) to the above approximate solutions. This method has been used for other singularly perturbed equations, and is described in detail for example in [38], [39], [42], [44]: hence, we will limit ourselves to a formal proof, since rigorous estimates can be derived as in the aforementioned papers.

For $m \in \mathbb{N}$ we consider an approximate solution of the form

$$\psi_{m,\varepsilon} = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \left[h(\varepsilon s) U(k(\varepsilon s) z) + \sum_{j=1}^{m} \varepsilon^j (w_{r,j}(\varepsilon s, z) + i w_{i,j}(\varepsilon s, z)) \right]; \qquad z = y - \Phi(\varepsilon s).$$

Here y stands for normal Fermi coordinates as in Subsection 2.1, while we have set

$$\tilde{f}(\varepsilon s) = f + \sum_{j=1}^{m} \varepsilon^j f_j(\varepsilon s); \qquad \Phi(\varepsilon s) = \sum_{j=0}^{m} \varepsilon^j \Phi_j(\varepsilon s),$$

where f_j, Φ_j are smooth real functions and normal sections respectively. We then write

$$-\Delta_{\tilde{g}_\varepsilon} \psi_{m,\varepsilon} + V(\varepsilon x)\psi_{m,\varepsilon} - |\psi_{m,\varepsilon}|^{p-1}\psi_{m,\varepsilon} = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \left(\sum_{j=0}^{m} \varepsilon^j \mathcal{R}_j \right) + o(\varepsilon^m),$$

where $\mathcal{R}_j$ are error terms depending on M, V, p, Φ and $\tilde{f}$.

In Lemma 3.2 we showed how to choose $w_{r,1}$ and $w_{i,1}$ so that $\mathcal{R}_1$ vanishes identically. In the previous two subsections instead we proved that $\mathcal{R}_2$ can also be canceled provided f_1 satisfies (58) and Φ_0 (we are using the above notation) satisfies $\mathfrak{J}\Phi_0 = 0$. By the invertibility of $\mathfrak{J}$ it is indeed sufficient to take both f_1 and Φ_0 identically equal to zero.

Turning to higher order terms, we will find that the coefficient of ε^3 will be of the form (up to the phase factor)

$$\mathcal{L}_r w_{3,r} + i\mathcal{L}_i w_{3,i} + \mathcal{G}_3(\varepsilon s, z),$$

where $\mathcal{G}_3(\varepsilon s, z)$ is an expression depending on V, $w_{1,r}, w_{1,i}$, $w_{2,r}, w_{2,i}$, f, f_2 and Φ_1. As before, we will find that the above expression can be made vanish provided f_2 and Φ_1 satisfy are periodic solutions of a system of the form (see (57) and (59))

$$\begin{cases} T f_2 = 2\mathcal{A}\left(\frac{p-1}{2\theta} - 1\right) \partial_{\overline{s}} \langle \mathbf{H}, \Phi_1 \rangle + \mathcal{W}_{3,1} & \text{in } [0, L]; \\ \mathfrak{J}\Phi_1 = \mathcal{W}_{3,2} & \text{in } [0, L]. \end{cases}$$

Here $\mathcal{W}_{3,1}$ and $\mathcal{W}_{3,2}$ are smooth functions of $\overline{s}$ independent of f_2 and Φ_1. Again by our assumptions on γ the above system can be solved in f_2, Φ_1, and solvability up to order ε^3 can be guaranteed. For higher powers of ε one can proceed similarly.

Remark 4.2 *We stated Theorem 1.4 for general powers in ε for expository reasons. Indeed, for our purposes in [40] we will need approximate solutions up to order ε^3 only. However, an accurate analysis of the error terms will be needed.*

Remark 4.3 *If we multiply the operators $\mathfrak{J}$ and T (see* (36) *and* (57)*) by $h(\overline{s})k(\overline{s})$ and $h(\overline{s})$ respectively, they become self-adjoint. This fact will be used crucially in the second part [40], see in particular Subsection 2.3 there.*

5 Appendix

In this appendix we collect some technical results: some integral identities first, and then the derivation of the second variation formula (36).

5.1 Some generalized Pohozaev identities

In this Subsection we derive some identities which are useful in the above computations. They follow from a simple integration by parts, but the proof is rather involved so we give some explanation here. We recall that the function U in (21) is radial, and so it satisfies the following ordinary differential equation

$$- U'' - \frac{n-2}{r} U' + U = U^p \qquad \text{in } \mathbb{R}_+. \tag{60}$$

Differentiating this relation with respect to r we obtain

$$- U''' - \frac{n-2}{r} U'' + \frac{n-2}{r^2} U' + U' = p U^{p-1} U' \qquad \text{in } \mathbb{R}_+. \tag{61}$$

We multiply (60) by $r^4 U''$ and integrate by parts to get (recall that we are in $\mathbb{R}^{n-1}$)

$$\begin{aligned}
\int_0^\infty U' r^{n+2} U''' dr \quad &+ \quad (n+2) \int_0^\infty U' U'' r^{n+1} dr + (n-2) \int_0^\infty U' r^{n+1} U'' dr \\
&+ \quad (n-2)(n+1) \int_0^\infty (U')^2 r^n dr \\
&- \quad \int_0^\infty (U')^2 r^{n+2} dr - (n+2) \int_0^\infty U U' r^{n+1} dr \\
&= \quad -p \int_0^\infty (U')^2 U^{p-1} r^{n+2} dr - (n+2) \int_0^\infty U^p U' r^{n+1} dr.
\end{aligned}$$

Using (61) the above identity simplifies as follows

$$(n-5) \int_0^\infty r^n (U')^2 dr + (n+1) \int_0^\infty r^n U^2 dr = 2 \frac{n+1}{p+1} \int_0^\infty r^n U^{p+1} dr. \tag{62}$$

Similarly, if we multiply (60) by $r^3 U'$ and integrate again by parts we find

$$\begin{aligned}
&\int_0^\infty U r^{n+1} U''' dr + (n+1) \int_0^\infty U U'' r^n dr + (n-2) \int_0^\infty U r^n U'' dr + n(n-2) \int_0^\infty U U' r^{n-1} dr \\
&- \int_0^\infty U U' r^{n+1} dr - (n+1) \int_0^\infty U^2 r^n dr = -p \int_0^\infty U U' U^{p-1} r^{n+1} dr - (n+1) \int_0^\infty U^{p+1} r^n dr.
\end{aligned}$$

Using (61) the last identity simplifies as

$$(n-1) \int_0^\infty r^{n-2} (U)^2 dr = \int_0^\infty r^n (U')^2 dr + \int_0^\infty r^n U^2 dr - \int_0^\infty r^n U^{p+1} dr. \tag{63}$$

From (62) it then follows that

$$\int_0^\infty r^n U^2 dr = \frac{2}{p+1} \int_0^\infty r^n U^{p+1} dr - \frac{(n-5)}{n+1} \int_0^\infty r^n (U')^2 dr. \tag{64}$$

If we plug this identity into (63) we get

$$(n-1) \int_{\mathbb{R}^{n-1}} U^2 dz = \frac{6}{n+1} \int_{\mathbb{R}^{n-1}} |z|^2 |\nabla U|^2 dz - \frac{p-1}{p+1} \int_{\mathbb{R}^{n-1}} |z|^2 U^{p+1} dz. \tag{65}$$

5.2 Second variation

The aim of this Subsection is to prove formula (35) for the second variation of the length functional stated in the beginning of Subsection 2.5.

Proof of (35). Differentiating (28) twice with respect to t_1, t_2 (at $(t_1, t_2) = (0, 0)$), recalling our notations in Subsection 2.5, taking into account (19) and (20) we obtain

$$\Sigma_1 + \Sigma_2 = 0, \tag{66}$$

where

$$\begin{aligned}
\Sigma_1 &= \int_\gamma \mathcal{A}_2' \sigma h^{\sigma-1} \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V} \rangle d\overline{s} + \int_\gamma \mathcal{A}\sigma(\sigma-1) h^{\sigma-2} \left[\frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial h}{\partial \mathcal{A}} \mathcal{A}_2' \right] \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V} \rangle d\overline{s} \\
&+ \int_\gamma \mathcal{A}\sigma h^{\sigma-1} \left[\frac{\partial^2 h}{\partial V^2} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial^2 h}{\partial V \partial \mathcal{A}} \mathcal{A}_2' \right] \langle \nabla^N V, \mathcal{V} \rangle d\overline{s} + \int_\gamma \mathcal{A}\sigma h^{\sigma-1} \frac{\partial h}{\partial V} \left((\nabla^N)^2 V \right) [\mathcal{V}, \mathcal{W}] d\overline{s} \\
&- \mathcal{A}_2' \int_\gamma h^\sigma \langle \mathbf{H}, \mathcal{V} \rangle d\overline{s} - \int_\gamma \mathcal{A}\sigma h^{\sigma-1} \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V} \rangle \langle \mathbf{H}, \mathcal{W} \rangle d\overline{s} \\
&- \mathcal{A}\sigma \int_\gamma h^{\sigma-1} \left[\frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial h}{\partial \mathcal{A}} \mathcal{A}_2' \right] \langle \mathbf{H}, \mathcal{V} \rangle d\overline{s};
\end{aligned}$$

$$\begin{aligned}
\Sigma_2 &= \int_\gamma \mathcal{A} h^\sigma \left[\sum_j \dot{\mathcal{V}}^j \dot{\mathcal{W}}^j - \sum_{j,m} R_{1j1m} \mathcal{V}^j \mathcal{W}^m \right] d\overline{s} + \mathcal{A}_1' \mathcal{A}_2' \sigma \int_\gamma h^{\sigma-1} \frac{\partial h}{\partial \mathcal{A}} d\overline{s} \\
&+ \mathcal{A}\mathcal{A}_{12}'' \sigma \int_\gamma h^{\sigma-1} \frac{\partial h}{\partial \mathcal{A}} d\overline{s} + \mathcal{A}\mathcal{A}_1' \sigma(\sigma-1) \int_\gamma h^{\sigma-2} \left[\frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial h}{\partial \mathcal{A}} \mathcal{A}_2' \right] \frac{\partial h}{\partial \mathcal{A}} d\overline{s} \\
&+ \mathcal{A}\mathcal{A}_1' \sigma \int_\gamma h^{\sigma-1} \left[\frac{\partial^2 h}{\partial \mathcal{A} \partial V} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial^2 h}{\partial \mathcal{A}^2} \mathcal{A}_2' \right] d\overline{s} - \mathcal{A}\mathcal{A}_1' \sigma \int_\gamma h^{\sigma-1} \frac{\partial h}{\partial \mathcal{A}} \langle \mathbf{H}, \mathcal{W} \rangle d\overline{s} \\
&+ \mathcal{A}_1' \sigma \int_\gamma h^{\sigma-1} \left[\frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial h}{\partial \mathcal{A}} \mathcal{A}_2' \right] d\overline{s} - \mathcal{A}_1' \int_\gamma h^\sigma \langle \mathbf{H}, \mathcal{W} \rangle d\overline{s} + \mathcal{A}_{12}'' \int_\gamma h^\sigma d\overline{s}.
\end{aligned}$$

Then the second variation of the energy is given by the following formula at $(t_1, t_2) = (0, 0)$

$$\begin{aligned}
\frac{\partial^2 E_\varepsilon(u_{\psi_{t_1,t_2}, \mathcal{A}_{t_1,t_2}})}{\partial t_1 \partial t_2} &= \int_\gamma \theta(\theta-1) h^{\theta-2} \left[\frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial h}{\partial \mathcal{A}} \mathcal{A}_2' \right] \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V} \rangle d\overline{s} \\
&+ \int_0^L \theta h^{\theta-1} \frac{\partial h}{\partial V} \left((\nabla^N)^2 V \right) [\mathcal{V}, \mathcal{W}] d\overline{s} \\
&+ \int_0^L \theta h^{\theta-1} \left[\frac{\partial^2 h}{\partial V^2} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial^2 h}{\partial V \partial \mathcal{A}} \mathcal{A}_2' \right] \langle \nabla^N V, \mathcal{V} \rangle d\overline{s} \\
&- \int_0^L \theta h^{\theta-1} \left[\frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial h}{\partial \mathcal{A}} \mathcal{A}_2' \right] \langle \mathbf{H}, \mathcal{V} \rangle d\overline{s} \\
&+ \int_0^L h^\theta \left[\sum_j \dot{\mathcal{V}}^j \dot{\mathcal{W}}^j - \sum_{j,m} R_{1j1m} \mathcal{V}^j \mathcal{W}^m \right] d\overline{s} \\
&+ \mathcal{A}_1' \int_0^L \theta(\theta-1) h^{\theta-2} \left[\frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial h}{\partial \mathcal{A}} \mathcal{A}_2' \right] \frac{\partial h}{\partial \mathcal{A}} d\overline{s} \\
&+ \int_0^L \theta \mathcal{A}_1' h^{\theta-1} \left[\frac{\partial^2 h}{\partial \mathcal{A} \partial V} \langle \nabla^N V, \mathcal{W} \rangle + \frac{\partial^2 h}{\partial \mathcal{A}^2} \mathcal{A}_2' \right] d\overline{s} \\
&+ \int_0^L \theta \mathcal{A}_{12}'' h^{\theta-1} \frac{\partial h}{\partial \mathcal{A}} d\overline{s} - \int_0^L \left[\theta h^{\theta-1} \frac{\partial h}{\partial V} \langle \nabla^N V, \mathcal{V} \rangle + \theta \mathcal{A}_1' h^{\theta-1} \frac{\partial h}{\partial \mathcal{A}} \right] \langle \mathbf{H}, \mathcal{W} \rangle.
\end{aligned}$$

Now some cancelation will occur. We plug in the value of $\mathcal{A}_{12}''$ from (66) into the last equality to obtain

$$\frac{\partial^2 E_\varepsilon(u_{\psi_{t_1,t_2}, \mathcal{A}_{t_1,t_2}})}{\partial t_1 \partial t_2} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5 + \mathcal{E}_6 + \mathcal{E}_7 + \mathcal{E}_8 + \mathcal{E}_9,$$

where the terms $(\mathcal{E}_i)_i$ are given by

$$\mathcal{E}_1 = \int_0^L h^\theta \left[\sum_j \dot{\mathcal{V}}^j \dot{\mathcal{W}}^j - \sum_{j,m} R_{1j1m} \mathcal{V}^j \mathcal{W}^m\right] d\bar{s} - \frac{2\mathcal{A}^2\theta}{p-1}\int_0^L h^\sigma \left[\sum_j \dot{\mathcal{V}}^j \dot{\mathcal{W}}^j - \sum_{j,m} R_{1j1m} \mathcal{V}^j \mathcal{W}^m\right] d\bar{s};$$

$$\begin{aligned}
\mathcal{E}_2 &= \int_0^L \theta(\theta-1)h^{\theta-2}\frac{\partial h}{\partial V}\frac{\partial h}{\partial \mathcal{A}}\left[\langle \nabla^N V, \mathcal{V}\rangle \mathcal{A}_2' + \langle \nabla^N V, \mathcal{W}\rangle \mathcal{A}_1'\right] \\
&+ \int_0^L \theta h^{\theta-1}\frac{\partial^2 h}{\partial V \partial \mathcal{A}}\left[\mathcal{A}_2'\langle \nabla^N V, \mathcal{V}\rangle + \mathcal{A}_1'\langle \nabla^N V, \mathcal{W}\rangle\right] \\
&- \frac{2\mathcal{A}\theta}{p-1}\int_0^L \mathcal{A}\sigma(\sigma-1)h^{\sigma-2}\frac{\partial h}{\partial \mathcal{A}}\frac{\partial h}{\partial V}\left[\mathcal{A}_1'\langle \nabla^N V, \mathcal{W}\rangle + \mathcal{A}_2'\langle \nabla^N V, \mathcal{V}\rangle\right] \\
&- \frac{2\mathcal{A}\theta}{p-1}\int_0^L \mathcal{A}\sigma h^{\sigma-1}\frac{\partial^2 h}{\partial \mathcal{A}\partial V}\left[\mathcal{A}_1'\langle \nabla^N V, \mathcal{W}\rangle + \mathcal{A}_2'\langle \nabla^N V, \mathcal{V}\rangle\right] \\
&- \frac{2\mathcal{A}\theta}{p-1}\int_0^L \sigma h^{\sigma-1}\frac{\partial h}{\partial V}\left[\mathcal{A}_1'\langle \nabla^N V, \mathcal{W}\rangle + \mathcal{A}_2'\langle \nabla^N V, \mathcal{V}\rangle\right];
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_3 &= -\int_0^L \theta h^{\theta-1}\frac{\partial h}{\partial \mathcal{A}}\left[\mathcal{A}_1'\langle \mathbf{H}, \mathcal{W}\rangle + \mathcal{A}_2'\langle \mathbf{H}, \mathcal{V}\rangle\right] + \frac{2\mathcal{A}\theta}{p-1}\int_0^L h^\sigma\left[\mathcal{A}_1'\langle \mathbf{H}, \mathcal{W}\rangle + \mathcal{A}_2'\langle \mathbf{H}, \mathcal{V}\rangle\right] \\
&+ \frac{2\mathcal{A}\theta}{p-1}\mathcal{A}\int_\gamma \sigma h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}}\left[\mathcal{A}_1'\langle \mathbf{H}, \mathcal{W}\rangle + \mathcal{A}_2'\langle \mathbf{H}, \mathcal{V}\rangle\right];
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_4 &= \mathcal{A}_1'\mathcal{A}_2'\theta(\theta-1)\int_0^L h^{\theta-2}\left(\frac{\partial h}{\partial \mathcal{A}}\right)^2 + \int_0^L \theta\mathcal{A}_1'\mathcal{A}_2' h^{\theta-1}\frac{\partial^2 h}{\partial \mathcal{A}^2} \\
&- \frac{4\mathcal{A}\theta}{p-1}\mathcal{A}_1'\mathcal{A}_2'\sigma\int_0^L h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}} - \frac{2\mathcal{A}\theta}{p-1}\mathcal{A}\mathcal{A}_1'\sigma(\sigma-1)\int_0^L h^{\sigma-2}\left(\frac{\partial h}{\partial \mathcal{A}}\right)^2\mathcal{A}_2' \\
&- \frac{2\mathcal{A}\theta}{p-1}\mathcal{A}\mathcal{A}_1'\sigma\int_0^L h^{\sigma-1}\frac{\partial^2 h}{\partial \mathcal{A}^2}\mathcal{A}_2';
\end{aligned}$$

$$\mathcal{E}_5 = \int_0^L \theta h^{\theta-1}\frac{\partial h}{\partial V}((\nabla^N)^2 V)[\mathcal{V}, \mathcal{W}] - \frac{2\mathcal{A}\theta}{p-1}\mathcal{A}\sigma\int_0^L h^{\sigma-1}\frac{\partial h}{\partial V}((\nabla^N)^2 V)[\mathcal{V}, \mathcal{W}];$$

$$\mathcal{E}_6 = \int_0^L \theta h^{\theta-1}\left[\frac{\partial^2 h}{\partial V^2}\langle \nabla^N V, \mathcal{V}\rangle\langle \nabla^N V, \mathcal{W}\rangle\right] - \frac{2\mathcal{A}\theta}{p-1}\int_0^L \mathcal{A}\sigma h^{\sigma-1}\frac{\partial^2 h}{\partial V^2}\langle \nabla^N V, \mathcal{V}\rangle\langle \nabla^N V, \mathcal{W}\rangle;$$

$$\mathcal{E}_7 = -\int_0^L \theta h^{\theta-1}\frac{\partial h}{\partial V}\langle \nabla^N V, \mathcal{W}\rangle\langle \mathbf{H}, \mathcal{V}\rangle + \frac{2\mathcal{A}\theta}{p-1}\mathcal{A}\sigma\int_0^L h^{\sigma-1}\frac{\partial h}{\partial V}\langle \nabla^N V, \mathcal{W}\rangle\langle \mathbf{H}, \mathcal{V}\rangle;$$

$$\mathcal{E}_8 = -\int_0^L \theta h^{\theta-1}\frac{\partial h}{\partial V}\langle \nabla^N V, \mathcal{V}\rangle\langle \mathbf{H}, \mathcal{W}\rangle + \frac{2\mathcal{A}\theta}{p-1}\mathcal{A}\sigma\int_0^L h^{\sigma-1}\frac{\partial h}{\partial V}\langle \nabla^N V, \mathcal{V}\rangle\langle \mathbf{H}, \mathcal{W}\rangle;$$

$$\mathcal{E}_9 = \int_0^L \left[\theta(\theta-1)h^{\theta-2} - \frac{2\mathcal{A}^2\theta}{p-1}\sigma(\sigma-1)h^{\sigma-2}\right]\left(\frac{\partial h}{\partial V}\right)^2\langle \nabla^N V, \mathcal{V}\rangle\langle \nabla^N V, \mathcal{W}\rangle.$$

Now we will simplify each of these terms. We get immediately

$$\mathcal{E}_1 = \int_0^L \left[h^\theta - \frac{2\mathcal{A}^2\theta}{p-1}h^\sigma\right]\left[\sum_j \dot{V}^j\dot{W}^j - \sum_{j,m} R_{1j1m}V^jW^m\right] d\overline{s}.$$

Recall the identity (32)

$$h^{\theta-1}\frac{\partial h}{\partial \mathcal{A}} = \frac{2\mathcal{A}}{p-1}\left(h^\sigma + \mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}}\right):$$

differentiating with respect to V and $\mathcal{A}$ we get the two formulas

$$(\theta-1)h^{\theta-2}\frac{\partial h}{\partial V}\frac{\partial h}{\partial \mathcal{A}} + h^{\theta-1}\frac{\partial^2 h}{\partial \mathcal{A}\partial V} = \frac{2\mathcal{A}}{p-1}\left(\sigma h^{\sigma-1}\frac{\partial h}{\partial V} + \mathcal{A}\sigma(\sigma-1)h^{\sigma-2}\frac{\partial h}{\partial V}\frac{\partial h}{\partial \mathcal{A}} + \mathcal{A}\sigma h^{\sigma-1}\frac{\partial^2 h}{\partial \mathcal{A}\partial V}\right);$$

$$\begin{aligned}(\theta-1)h^{\theta-2}\left(\frac{\partial h}{\partial \mathcal{A}}\right)^2 + h^{\theta-1}\frac{\partial^2 h}{\partial \mathcal{A}^2} &= \frac{2\mathcal{A}}{p-1}\left(3\sigma h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}} + \mathcal{A}\sigma(\sigma-1)h^{\sigma-2}\left(\frac{\partial h}{\partial \mathcal{A}}\right)^2 + \mathcal{A}\sigma h^{\sigma-1}\frac{\partial^2 h}{\partial \mathcal{A}^2}\right)\\ &+ \frac{2}{p-1}h^\sigma.\end{aligned}$$

From these it follows immediately that

$$\mathcal{E}_2 = \mathcal{E}_3 = 0; \qquad \mathcal{E}_4 = \mathcal{A}_1'\mathcal{A}_2'\frac{2\theta}{p-1}\left(\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}} + h^\sigma\right).$$

By means of (31), the above identity (32) can also be written as

$$h^{\theta-1}2\mathcal{A}h^{2\sigma}\frac{\partial h}{\partial V} = \frac{2\mathcal{A}}{p-1}\left(\mathcal{A}\sigma h^{\sigma-1}2\mathcal{A}h^{2\sigma}\frac{\partial h}{\partial V} + h^\sigma\right),$$

from which it follows that

(67)
$$h^{\theta-1}\frac{\partial h}{\partial V} - \frac{2\mathcal{A}^2\sigma}{p-1}h^{\sigma-1}\frac{\partial h}{\partial V} = \frac{1}{p-1}h^{-\sigma}.$$

This formula implies

$$\mathcal{E}_5 = \frac{\theta}{p-1}\int_0^L ((\nabla^N)^2V)[\mathcal{V},\mathcal{W}]h^{-\sigma};$$

$$\mathcal{E}_7 = -\frac{\theta}{(p-1)}\int_0^L \langle\nabla^N V,\mathcal{W}\rangle\langle\mathbf{H},\mathcal{V}\rangle h^{-\sigma}; \qquad \mathcal{E}_8 = -\frac{\theta}{(p-1)}\int_0^L \langle\nabla^N V,\mathcal{V}\rangle\langle\mathbf{H},\mathcal{W}\rangle h^{-\sigma}.$$

Therefore in conclusion we get at $(t_1,t_2) = (0,0)$

$$\begin{aligned}\frac{\partial^2 E_\varepsilon(u_{\psi_{t_1,t_2},\mathcal{A}_{t_1,t_2}})}{\partial t_1\partial t_2} &= \int_0^L \left[h^\theta - \frac{2\mathcal{A}^2\theta}{p-1}h^\sigma\right]\left[\sum_j \dot{\mathcal{V}}^j\dot{\mathcal{W}}^j - \sum_{j,m} R_{1j1m}\mathcal{V}^j\mathcal{W}^m\right] d\overline{s}\\
&+ \int_0^L \left(\theta h^{\theta-1} - \frac{2\mathcal{A}^2\theta}{p-1}\sigma h^{\sigma-1}\right)\frac{\partial^2 h}{\partial V^2}\langle\nabla^N V,\mathcal{V}\rangle\langle\nabla^N V,\mathcal{W}\rangle\\
&+ \frac{\theta}{p-1}\int_0^L \Big\{((\nabla^N)^2V)[\mathcal{V},\mathcal{W}] - \langle\nabla^N V,\mathcal{W}\rangle\langle\mathbf{H},\mathcal{V}\rangle - \langle\nabla^N V,\mathcal{V}\rangle\langle\mathbf{H},\mathcal{W}\rangle\Big\}h^{-\sigma}\\
&+ \int_0^L \left[\theta(\theta-1)h^{\theta-2} - \frac{2\mathcal{A}^2\theta}{p-1}\sigma(\sigma-1)h^{\sigma-2}\right]\left(\frac{\partial h}{\partial V}\right)^2\langle\nabla^N V,\mathcal{V}\rangle\langle\nabla^N V,\mathcal{W}\rangle\\
&+ \mathcal{A}_1'\mathcal{A}_2'\frac{2\theta}{p-1}\int_0^L \left(\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial \mathcal{A}} + h^\sigma\right).\end{aligned}$$

Differentiating (67) with respect to V, we obtain

$$\left[(p-1)h^{\theta-1}-2\sigma\mathcal{A}^2h^{\sigma-1}\right]\frac{\partial^2 h}{\partial V^2}+\left[(p-1)(\theta-1)h^{\theta-2}-2\sigma(\sigma-1)\mathcal{A}^2h^{\sigma-2}\right]\left(\frac{\partial h}{\partial V}\right)^2=-\sigma h^{-\sigma-1}\frac{\partial h}{\partial V},$$

which yields

$$\begin{aligned}\int_0^L\left(\theta h^{\theta-1}-\frac{2\mathcal{A}^2\theta}{p-1}\sigma h^{\sigma-1}\right)\frac{\partial^2 h}{\partial V^2}\langle\nabla^N V,\mathcal{V}\rangle\langle\nabla^N V,\mathcal{W}\rangle &= -\frac{\theta\sigma}{p-1}\int_0^L\frac{\partial h}{\partial V}\langle\nabla^N V,\mathcal{V}\rangle\langle\nabla^N V,\mathcal{W}\rangle h^{-\sigma-1}\\ &+\int_0^L\left(\frac{2\sigma(\sigma-1)\mathcal{A}^2\theta}{p-1}h^{\sigma-2}-\theta(\theta-1)h^{\theta-2}\right)\left(\frac{\partial h}{\partial V}\right)^2\langle\nabla^N V,\mathcal{V}\rangle\langle\nabla^N V,\mathcal{W}\rangle.\end{aligned}$$

Collecting the above computation together with some further cancelation, one finds, at $(t_1,t_2)=(0,0)$

$$\begin{aligned}\frac{\partial^2 E_\varepsilon(u_{\psi_{t_1,t_2},\mathcal{A}_{t_1,t_2}})}{\partial t_1\partial t_2} &= \int_0^L\left[h^\theta-\frac{2\mathcal{A}^2\theta}{p-1}h^\sigma\right]\left[\sum_j\dot{\mathcal{V}}^j\dot{\mathcal{W}}^j-\sum_{j,m}R_{1j1m}\mathcal{V}^j\mathcal{W}^m\right]d\overline{s}\\ &+ \frac{\theta}{p-1}\int_0^L\left\{((\nabla^N)^2V)[\mathcal{V},\mathcal{W}]-\langle\nabla^N V,\mathcal{V}\rangle\langle\mathbf{H},\mathcal{W}\rangle-\langle\nabla^N V,\mathcal{W}\rangle\langle\mathbf{H},\mathcal{V}\rangle\right\}h^{-\sigma}d\overline{s}\\ &- \frac{\sigma\theta}{p-1}\int_0^L h^{-\sigma-1}\frac{\partial h}{\partial V}\langle\nabla^N V,\mathcal{V}\rangle\langle\nabla^N V,\mathcal{W}\rangle d\overline{s}\\ &+ \mathcal{A}_1'\mathcal{A}_2'\frac{2\theta}{p-1}\int_0^L\left(\mathcal{A}\sigma h^{\sigma-1}\frac{\partial h}{\partial\mathcal{A}}+h^\sigma\right)d\overline{s}.\end{aligned}$$

This proves formula (35). ■

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