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Twisted noncommutative equivariant cohomology: Weil and Cartan models

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Abstract

We propose Weil and Cartan models for the equivariant cohomology of covariant actions on toric deformation manifolds. The construction is based on the noncommutative Weil algebra of Alekseev and Meinrenken [1]; we show that one can implement a Drinfeld twist of their models in order to take into account the noncommutativity of the spaces we are acting on.

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Introduction

Equivariant cohomology is a useful tool for studying actions of Lie groups G (or, infinitesimally, Lie algebras $\mathfrak{g}$) on manifolds $\mathcal{M}$. In some sense, it replaces the cohomology of the orbit space when the latter is not well defined, due to the fact that the group action may not be free. A possible approach to equivariant cohomology (the Borel model) considers, for compact G, the ordinary cohomology of $(EG \times \mathcal{M})/G$, where EG is the total space of the universal G-bundle $EG \to BG$. Algebraic models for computing the equivariant cohomology involve the Weil algebra $W_G = Sym(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$, which represents differential forms over EG. The equivariant cohomology is defined as the cohomology of the basic subcomplex $(W_G \otimes \Omega(\mathcal{M}))^G_{hor}$ with respect to the total differential $\delta = d^W \otimes 1 + 1 \otimes d$ built with the Weil differential d^W and the De Rham differential d; this is the Weil model. Another construction, related to this by the so-called Kalkman map ϕ [14], is the Cartan complex $(Sym(\mathfrak{g}^*) \otimes \Omega(\mathcal{M}))^G$ with differential d_G induced from δ.

Both Weil and Cartan models consider the algebra $\Omega(\mathcal{M})$ and the induced G-action on it; in particular, we have a Lie derivative L and an interior derivative i which, together with the De Rham differential d and the commutation rules of these operators, realize what is called a $\tilde{\mathfrak{g}}$ differential algebra structure, represented by derivations. In general, one can define equivariant cohomology for every $\tilde{\mathfrak{g}}$-differential algebra $\mathcal{A}$, simply substituting $\Omega(\mathcal{M})$ by $\mathcal{A}$ in the definition of the Weil and Cartan models. For a beautiful review of these topics with references to the original works of Weil, Cartan and others see [13].

If one wishes to define the equivariant cohomology for a noncommutative $\tilde{\mathfrak{g}}$ differential algebra $\mathcal{B}$, the idea is to follow the same route by introducing an "appropriate" noncommutative Weil algebra $\mathcal{W}_G$ [1, 2]. In particular for $\mathfrak{g}$ quadratic (i.e. it admits an invariant non degenerate quadratic form), the algebra $\mathcal{W}_G$ may be realized as the enveloping algebra of a super Lie algebra $\tilde{\mathfrak{g}}$ defined using the quadratic form of $\mathfrak{g}$. Then the $\tilde{\mathfrak{g}}$ differential algebra structure, i.e. the action of a Lie derivative L and of an interior derivative i, is given by the adjoint actions with respect to even and odd generators, while the differential d^W is given by the commutator with a fixed element $\mathcal{D} \in \mathcal{W}_G$. Notice that in this construction the representation of $\tilde{\mathfrak{g}}$ via L and i on $\mathcal{W}_G$ and $\mathcal{B}$ is through derivations, as in the commutative case.

What can be said about the noncommutative algebra $\mathcal{B}$ representing a noncommutative space acted on by some Lie group G or Lie algebra $\mathfrak{g}$? If the noncommutative space is a toric deformation [8], we can think of it as coming from a Drinfeld twist [10, 11] in the enveloping algebra of the torus.

This link between toric deformations and Drinfeld twist is the starting point of our construction. We ask the action of $\mathfrak{g}$ on $\mathcal{B}$ to be covariant, i.e. to respect the (deformed) multiplicative structure of $\mathcal{B}$; this is equivalent to deform, with the same twist, the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ as well. Looking at the action of Lie derivative and interior derivative on $\mathcal{B}$, we find that they satisfy a twisted Leibniz rule on products; this is because of the twisted coproduct in $\mathfrak{U}(\mathfrak{g})$ after the deformation. In this sense we have a twisted $\tilde{\mathfrak{g}}$ differential algebra structure on $\mathcal{B}$ (the exterior differential, commuting with the action generating the twist, remains undeformed). Our final goal is to construct twisted noncommutative Weil and Cartan models for the equivariant cohomology of such algebras. We show that this is indeed possible, starting from the noncommutative Weil algebra of [1] provided we also twist the latter; for this we use the fact that for $\mathfrak{g}$ quadratic $\mathcal{W}_G$ is a super enveloping algebra. We realize its $\tilde{\mathfrak{g}}$ differential algebra structure again by means of the adjoint action, but this time with respect the twisted Hopf structure: we get twisted derivations. Once we have the Weil algebra, we define Weil and Cartan models as usual, except for some technicalities involving the definition of a Kalkman map. These models in principle work for every noncommutative geometry obtained using a Drinfeld twist, but here calculations are done by using an explicit expression of the twist element; this is known for a large class of interesting examples.

The search for an equivariant cohomology theory of noncommutative spaces was motivated by the use of equivariant localization techniques in the multi-istanton calculus of supersymmetric gauge theories. Under suitable assumptions, partition functions and others interesting physical (or geometric) quantities of the theory are expressed in terms of an integration (over the moduli space of instantons and spacetime) of equivariantly closed differential forms. Thus they can be computed using equivariant localization formulas [5], obtaining explicit results and bypassing a lot of technical difficulties inherent in the ADHM description [6]. The resolution of the moduli space of instantons may be interpreted as the moduli space of instantons over a noncommutative spacetime [18]; this means that if we have to integrate not only over the moduli space but also over spacetime, we need an equivariant localization formula which holds on $\mathbb{R}^4_\theta$. This was the starting point of our interest on noncommutative equivariant cohomology; for more on these applications, see also the brief discussion in the conclusions.

This paper is structured as follows: in Section 1 we recall the notion of toric deformation of a manifold, and how it relates to a Drinfeld twist of Hopf algebras; we then study covariant actions on these classes of noncommutative manifolds. We define the algebra of noncommutative differential forms, and realize on it Lie derivative and interior product as twisted derivations. In Section 2 we review the construction of [1, 2] of the noncommutative Weil algebra; we then define our twisted noncommutative Weil algebra $\mathcal{W}_G^{(\chi)}$ and its $\tilde{\mathfrak{g}}$ differential structure; we present explicit computations for the case of a quadratic Lie algebra $\mathfrak{g}$ with a toric twist element. In Section 3 we use $\mathcal{W}_G^{(\chi)}$ to build Weil and Cartan models for the equivariant cohomology of twisted noncommutative $\tilde{\mathfrak{g}}$ algebras, having in mind as a guiding example the algebra of noncommutative differential forms of a toric deformation. Finally, we outline in the conclusions possible applications for future work.

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1 Covariant actions on toric deformations

In this section we study some connections between toric deformations and Drinfeld twists, and explain how a Lie algebra $\mathfrak{g}$ can acts covariantly on such a noncommutative space. We introduce the algebra of noncommutative differential forms $\Omega^\bullet(\mathcal{M}_\theta)$ with a deformed wedge product, and we realize a $\tilde{\mathfrak{g}}$-differential algebra structure on it, i.e. we construct Lie derivative, interior product and De Rham differential. In the case of a non-trivial twist, this will be done via twisted derivations.

1.1 Toric deformations via a Drinfeld twist

The idea behind toric noncommutative manifolds consists in using a toric isometry of a smooth compact Riemannian manifold $\mathcal{M}$ in order to define an isospectral noncommutative deformation $\mathcal{M}_\theta$. One first defines a noncommutative product on $C^\infty(\mathcal{M})$ using Rieffel's strategy for actions of $\mathbb{R}^p$ [21]; with the noncommutative algebra $C^\infty(\mathcal{M}_\theta)$ one then constructs a spectral triple $(C^\infty(\mathcal{M}_\theta), D, \mathcal{H})$ with all the properties of a noncommutative spin geometry [8]. Since in this kind of spectral triples one is deforming the algebra of functions and its representation on $\mathcal{H}$, but not the Dirac operator D, they are also called isospectral deformations.

Suppose that a torus $\mathbb{T}^p$, with $p \geq 2$, acts on $\mathcal{M}$ by isometries. Since $\mathcal{M}$ is compact, we can decompose $C^\infty(\mathcal{M})$ in spectral subspaces labeled by weights r, such that every $f_r \in C^\infty_r(\mathcal{M})$ is an eigenfunction of the action. Representing elements of $\mathbb{T}^p$ as $e^{2\pi i t}$ with $t \in \mathbb{Z}^p$, the action σ on an eigenfunction f_r is given by a phase factor depending on r:

$$\sigma_t(f_r) = e^{2\pi i t\cdot r} f_r \qquad\qquad t \in \mathbb{Z}^p\,,\; r \in (\mathbb{Z}^p)^* \tag{1}$$

Taking for θ a real $p \times p$ skew-symmetric matrix we can define a deformed product between eigenfunctions

$$f_r \times_\theta g_s := \exp\,[\frac{i}{2}\,\theta^{kl} r_k s_l] \tag{2}$$

and by linearity extend it on the whole of $C^\infty(\mathcal{M})$. Clearly, $\mathbb{T}^p$-invariant functions form a commutative ideal in the noncommutative algebra

$$C^\infty(\mathcal{M}_\theta) := (C^\infty(\mathcal{M}), \times_\theta) \tag{3}$$

We will call (3) the algebra of functions of the noncommutative manifold $\mathcal{M}_\theta$.

Well known examples of this class of noncommutative geometries are the noncommutative torus, noncommutative spheres, and more general spherical manifolds [7]. Quite recently (involving some technical details due to noncompactness), also Moyal planes have been shown to be (non compact) spectral triples of isospectral deformations [12].

Now we want to study actions of some group G or its Lie algebra $\mathfrak{g}$ on such noncommutative manifolds $\mathcal{M}_\theta$, i.e. on the noncommutative algebra $C^\infty(\mathcal{M}_\theta)$. The main point will be to preserve the covariance of the action, and this will be achieved by deforming the Hopf algebra structure of $\mathfrak{U}(\mathfrak{g})$ in a compatible way with the deformation of the product in $C^\infty(\mathcal{M}_\theta)$. Indeed we can think of the isospectral deformation $\mathcal{M}_\theta$ as coming from a Drinfeld twist [10, 11] of the enveloping algebra of the torus acting on $\mathcal{M}$. By covariance, we shall realize the action of $\mathfrak{g}$ on $\mathcal{M}_\theta$ applying the same Drindeld twist to $\mathfrak{U}(\mathfrak{g})$.

We give here the basic definitions concerning Drinfeld twist and covariant actions. For more details, in addition to the above references to Drinfeld's original papers, one can see also [17, 4].

Definition 1.1 *Let $\mathcal{H} = (\mathcal{H}, \triangle, S, \epsilon)$ be a Hopf algebra, and $\chi \in \mathcal{H} \otimes \mathcal{H}$ an invertible counital two-cocycle, called the twist element. A new Hopf algebra $\mathcal{H}_\chi = (\mathcal{H}, \triangle_\chi, S_\chi, \epsilon)$, called the twisted Hopf algebra, is defined using the same underlying algebra $\mathcal{H}$ with*

· *coproduct:* $\triangle_\chi(h) := \chi(\triangle(h))\chi^{-1}$

· *antipode:* $S_\chi(h) := U(S(h))U^{-1}$ *whit* $U = \chi^{(1)}S(\chi^{(2)})$

· *counit: unchanged*

Remark: Here we write $\chi = \chi^{(1)} \otimes \chi^{(2)}$ with summation understood. The cocycle condition comes from a cohomology theory for Hopf algebras, see [17]. Explicitly one asks

$$(1 \otimes \chi)(id \otimes \triangle)\chi = (\chi \otimes 1)(\triangle \otimes id)\chi \tag{4}$$

This assures the twisted Hopf algebra to be coassociative; if this requirement is dropped, we get for $\mathcal{H}_\chi$ a quasi-Hopf algebra [10][11]. ■

The meaning of covariance and the link between deformation of the product in an $\mathcal{H}$-module algebra and a Drinfeld twist in $\mathcal{H}$ are given as follows.

Definition 1.2 *A Hopf algebra* $\mathcal{H} = (\mathcal{H}, \triangle, S, \epsilon)$ *acts in a covariant way on an unital associative algebra* $\mathcal{A}$ *if the action on the product is given by*

$$h \triangleright (a \cdot b) = \cdot\,(\triangle(h) \triangleright (a \otimes b)) \qquad \forall\, h \in \mathcal{H},\ a, b \in \mathcal{A} \tag{5}$$

and on the unit is given by

$$h \triangleright 1 = \epsilon(h) \qquad \forall\, h \in \mathcal{H} \tag{6}$$

We then say that $\mathcal{A}$ *is an* $\mathcal{H}$*-module algebra.*

In order to preserve covariance if we twist $\mathcal{H}$ then we have to deform the product in every $\mathcal{H}$-module algebra. The cocycle condition on the twist element χ assures the associativity of the deformed product [17]. Indeed it is not difficult to prove the following result:

Proposition 1.1 *Let* χ *be a twist element for a Hopf algebra* $\mathcal{H}$*. Then if* $\mathcal{A}$ *is an* $\mathcal{H}$*-module algebra with its natural product* $\cdot$*, the new product*

$$a \cdot_\chi b := \cdot\left(\chi^{-1} \triangleright (a \otimes b)\right) \qquad \forall\, a, b \in \mathcal{A} \tag{7}$$

defines a new algebra $\mathcal{A}_\chi$ *on which* $\mathcal{H}_\chi$ *acts covariantly:*

$$h \triangleright (a \cdot_\chi b) = \cdot_\chi\,(\triangle_\chi(h) \triangleright (a \otimes b)) \tag{8}$$

The idea is to regard the product (2) as generated by a Drinfeld twist in the formal quantum enveloping algebra of the torus $\mathbb{T}^n$: the twist element relevant for our toric deformations is the one given in [20]

$$\chi = \exp\{-\frac{i}{2}\,\theta^{kl} H_k \otimes H_l\} \qquad\qquad \chi \in (\mathfrak{U}(\mathfrak{t}^p) \otimes \mathfrak{U}(\mathfrak{t}^p))_{[[\theta]]} \tag{9}$$

where $\mathfrak{t}^p$ is the Lie algebra of the torus, H_i are its generators and θ a real matrix. The isomorphism classes of twists are in one-to-one correspondence with cohomology classes $[\chi]$, so that two twists elements which differ by a coboundary give the same twist structure. For this reason the matrix θ may always be taken skew-symmetric: a different choice would give different coproduct and antipode, but equivalent Hopf structure.

It is worth spending some words on (formal) deformations and their uniqueness properties. The theory of algebras and coalgebras deformations, and related cohomologies, is well defined in the setting of formal power series, and is presented for example in [22], where there are also references to the original works of Gerstenhaber, Schack, Shnider and Drinfeld. Several results proved by Drinfeld and Shnider, known as rigidity theorems, characterize the class of such deformations, up to isomorphism and/or twists. For example, in the case of enveloping algebras, whenever the first and second Hochschild cohomology groups of $\mathfrak{U}(\mathfrak{g})$ vanish (which is the case when $\mathfrak{g}$ is semisimple) every non trivial deformation $\mathfrak{U}_\theta(\mathfrak{g})$ which preserves the coassociativity is isomorphic to a Drinfeld twist of $\mathfrak{U}(\mathfrak{g})_{[[\theta]]}$. The problem is that usually it is difficult to obtain an explicit form for the twist element χ, or, equivalently, for the isomorphism which relates the two deformations; for isospectral deformations coming from the action of a torus we shall always use (9). A different approach is to avoid formal power series, and to realize directly a representation of the twisted universal envelping algebra as unitary operators on some Hilbert space; this can indeed be a better strategy if one want to study a covariant action in specific cases, as for example in [8, 16].

The idea to deform the product between functions applies also to differential forms, which are an $\mathfrak{U}(\mathfrak{t}^p)$-module algebra as well as $C^\infty(\mathcal{M})$. We assume the pull-back action on $\Omega^\bullet(\mathcal{M})$ commutes with the differential d and use the twist element (9).

Definition 1.3 *The algebra of noncommutative differential forms $\Omega^\bullet(\mathcal{M}_\theta)$ is* *the vector space* $\Omega^\bullet(\mathcal{M})$ *endowed with the deformed wedge product*

$$\begin{aligned}\omega \wedge_\theta \eta &= \wedge \left(\chi^{-1} \triangleright (\omega \otimes \eta)\right) \\ &= \wedge \left(exp\ \{-\frac{i}{2}\, \theta^{kl} H_k \otimes H_l\} \triangleright (\omega \otimes \eta)\right)\end{aligned} \tag{10}$$

for $\omega, \eta \in \Omega^\bullet(\mathcal{M})$.

The De Rham differential d is defined on $\Omega^\bullet(\mathcal{M}_\theta)$ as in the classical case, and it is easy to see that it is still a graded derivation of degree one with respect the new product $\wedge_\theta$. Thus it is possible to define cohomology in the standard way. Note that like for the functions, $\mathbb{T}^p$-invariant differential forms describe an ideal of $\Omega^\bullet(\mathcal{M}_\theta)$ where the product is not deformed.

The study of the $\mathfrak{U}_\chi(\mathfrak{g})$-algebra structure of $\mathcal{A}_\chi$-type algebras will be the topic of the next section. Of course, a crucial point is the possible triviality of the twist. A sufficient condition for that is that the generators of $\mathfrak{g}$ commute with the generators of the twist: in this case we have $\mathfrak{U}_\chi(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})_{[[\theta]]}$. But this is not a necessary condition: for example consider the rotation group S^1 (with generator J) acting on $\mathbb{R}^2_\theta$, where the generators of the twist (and of the noncommutativity of the Moyal plane) are the momenta P_μ. Considering the Lie algebra $\mathfrak{g}$ generated by P_μ and J, even if $[J, P_\mu] \neq 0$ we still have $\mathfrak{U}_\chi(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})_{[[\theta]]}$. In these situations most of what we are going to construct reduces trivially to the classic (i.e. $\theta = 0$) setting. On the contrary the interesting case takes place when the twist is not trivial; as a simple example, take $SO(5)$ acting on S^4_θ as in [16].

A last remark: one may ask what does $\mathfrak{U}_\chi(\mathfrak{g})$ mean if the generators of the twist do not belong to $\mathfrak{g}$. The relevant algebra in this case is the product between $\mathfrak{t}^p$ and $\mathfrak{g}$. If we call this algebra $\mathfrak{g}'$, we are actually twisting $\mathfrak{U}(\mathfrak{g}')_{[[\theta]]}$. This is exactly what we did when considering the example of S^1 acting on $\mathbb{R}^2_\theta$ mentioned above. In what follows, for simplicity we will assume that $\mathfrak{g}$ contains the generators of the twist.

1.2 The action of i and L on $\Omega^\bullet(\mathcal{M}_\theta)$: twisted derivations

We want to study the action of $\mathfrak{g}$ on $\Omega^\bullet(\mathcal{M}_\theta)$. What we shall present in this section applies to a generic Lie algebra $\mathfrak{g}$, while from the next section the construction will work for quadratic algebras $\mathfrak{g}$, i.e. we shall need the existence of a non degenerate quadratic form B on $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple, the Killing form provides such a B; in general, there are quadratic Lie algebras which are not semisimple [2](section 4.1). Explicit computations in the twisted enveloping algebra $\mathfrak{U}_\chi(\mathfrak{g})$ will be done under the assumption that $\mathfrak{g}$ is semisimple; this has the practical advantage that a decomposition of $\mathfrak{g}$ in Cartan and roots generators is available. When $\mathfrak{g}$ is quadratic but not semisimple from a theoretic point of view nothing changes, but explicit results are obtained from the specific structure of $\mathfrak{g}$.

We choose a basis in $\mathfrak{g}$ given by $\{H_i, E_r\}$ where $\{H_i\}$ are generators of the Cartan subalgebra (with $i = 1, \ldots, n = \text{rank}\, \mathfrak{g}$) and roots $\{E_r\}$ labeled by the n-dimensional root vectors $\mathbf{r} = (r_1, \ldots, r_n)$. Remember that we are assuming the generators of the twist to be Cartan generators, even if in general they need not to span the whole Cartan subalgebra (if $p < n$). The structure constants of $\mathfrak{g}$ are written as follows:

$$\begin{array}{rclrcl} [H_i, H_j] &=& 0 & [H_i, E_r] &=& r_i E_r \\ {[E_{-r}, E_r]} &=& \sum_i r_i H_i & [E_r, E_s] &=& N_{r,s} E_{r+s} \end{array} \tag{11}$$

where r are the root vectors and $N_{r,s}$, whose explicit form is not needed in what follows, vanishes if $r + s$ is not a root vector.

Next we consider the Hopf algebra structure of $\mathfrak{U}_\chi(\mathfrak{g})$; we start by calculating the coproduct of the generators of $\mathfrak{g}$. Recalling the definition of the twisted coproduct and (9) one easily finds

$$\triangle_\chi(H_i) = \triangle(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{12}$$

$$\triangle_\chi(E_r) = E_r \otimes \lambda_r^{-1} + \lambda_r \otimes E_r \tag{13}$$

where

$$\lambda_r = \exp\, \{\frac{i}{2}\, \theta^{kl} r_k H_l\} \tag{14}$$

is an untwisted group-like element (one for each root r):

$$\triangle_\chi\, (\lambda_r) = \triangle(\lambda_r) = \lambda_r \otimes \lambda_r^{-1} \tag{15}$$

Note that on the Cartan subalgebra, and in particular on the generators of the twist, the coproduct is not deformed. This means that these elements act on $\mathcal{A}_\chi$ or $\Omega^\bullet(\mathcal{M}_\theta)$ with the Leibniz rule, i.e. as classical derivations or vector fields. This is no more true for the E_r's.

We want to define on $\Omega^\bullet(\mathcal{M}_\theta)$ the action of Lie derivative and interior product; having in mind the definition of covarant action the following choice seems natural:

Definition 1.4 *The Lie derivative L_X for $X \in \mathfrak{g}$ on generators of $\Omega^\bullet(\mathcal{M}_\theta)$ is the classical one. When it acts on products of forms, by covariance we set, according to relations (12) and (13),*

$$
\begin{aligned}
L_{H_i}(\omega \wedge_\theta \eta) &= H_i \triangleright (\omega \wedge_\theta \eta) \\
&= \wedge_\theta \left(\triangle_\chi(H_i) \triangleright (\omega \otimes \eta)\right) \\
&= (H_i \triangleright \omega) \wedge_\theta \eta + \omega \wedge_\theta (H_i \triangleright \eta) \\
&= (L_{H_i}\omega) \wedge_\theta \eta + \omega \wedge_\theta (L_{H_i}\eta)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
L_{E_r}(\omega \wedge_\theta \eta) &= E_r \triangleright (\omega \wedge_\theta \eta) \\
&= \wedge_\theta \left(\triangle_\chi(E_r) \triangleright (\omega \otimes \eta)\right) \\
&= (E_r \triangleright \omega) \wedge_\theta (\lambda_r^{-1} \triangleright \eta) + (\lambda_r \triangleright \omega) \wedge_\theta (E_r \triangleright \eta) \\
&= (L_{E_r}\omega) \wedge_\theta (\lambda_r^{-1} \triangleright \eta) + (\lambda_r \triangleright \omega) \wedge_\theta (L_{E_r}\eta)
\end{aligned}
\tag{17}
$$

As we have already pointed out, $X \in \mathfrak{g}$ acts as a derivation on $\Omega^\bullet(\mathcal{M}_\theta)$ if and only if $\triangle(X) = X \otimes 1 + 1 \otimes X$. This is the case for Cartan generators H_i. On the contrary, looking at (17), we shall call L_{E_r} a twisted derivation. We also simplify the notation by writing L_i or $H_i\triangleright$ for L_{H_i}, L_r or $E_r\triangleright$ for L_{E_r}; with a small abuse of notation, we might also write L_{λ_r} for $\lambda_r\triangleright$. The same ideas apply to the interior product:

Definition 1.5 *On generators of $\Omega^\bullet(\mathcal{M}_\theta)$ the action of i_X for $X \in \mathfrak{g}$ is the classical one. On products, i_{H_i} is a true derivation of degree -1*

$$
i_{H_i}(\omega \wedge_\theta \eta) = (i_{H_i}\omega) \wedge_\theta \eta + (-1)^{|\omega|}\omega \wedge_\theta (i_{H_i}\eta) \tag{18}
$$

while for the roots E_r the twisted coproduct suggests us to put

$$
i_{E_r}(\omega \wedge_\theta \eta) = (i_{E_r}\omega) \wedge_\theta (L_{\lambda_r^{-1}}\eta) + (-1)^{|\omega|}(L_{\lambda_r}\omega) \wedge_\theta (i_{E_r}\eta) \tag{19}
$$

obtaining a twisted derivation of degree -1.

Let us once more stress that we have not changed the action of L and i on generators of $\Omega^\bullet(\mathcal{M}_\theta)$. By a purely algebraic computation one can check that the twisted derivation rule is compatible with the well-known relations among L, i and d. We then obtain a result completely analogous to the classical one.

Proposition 1.2 *The differential d, Lie derivative L and interior product i span a super Lie algebra*

$$
\tilde{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{20}
$$

with generators respectively i_a, L_a *and* d *(for* $\{e_a\}$ *a basis in* $\mathfrak{g}$*), having degrees* -1, 0 *and* 1 *and relations (compatible with degrees)*

$$\begin{array}{llll} [i_a, i_b] & = 0 & [L_a, L_b] & = f_{ab}^{c} L_c \\ [i_a, L_b] & = f_{ab}{}^{c} i_c & [i_a, d] & = L_a \\ [L_a, d] & = 0 & [d, d] & = 0 \end{array} \tag{21}$$

where we use the standard sign rules for commutators in super Lie algebras.

So the only new feature is that while usually $\tilde{\mathfrak{g}}$ is represented on $\Omega^\bullet(\mathcal{M})$ via derivations, now on $\Omega^\bullet(\mathcal{M}_\theta)$ we have a representation in twisted derivations in order to respect the structure of the deformed wedge product $\wedge_\theta$.

2 The twisted Weil algebra

In classical geometry, the construction of the Weil algebra W_G is the first step for defining equivariant cohomology for a G-manifold $\mathcal{M}$; we shall come back to this point in the next section. Here we start by giving a brief review of the construction of Alekseev and Meinrenken of a noncommutative Weil algebra $\mathcal{W}_G$ [1, 2], stressing the most important points for our future construction; $\mathcal{W}_G$ is the relevant object dealing with noncommutative $\tilde{\mathfrak{g}}$-differential algebras $\mathcal{B}$. Our purpose then is to take $\mathcal{B}$ as a twisted noncommutative $\tilde{\mathfrak{g}}$-differential algebra (the guiding example is $\Omega^\bullet(\mathcal{M}_\theta)$), and show that in this case the natural Weil algebra is obtained from $\mathcal{W}_G$ using a Drinfeld twist.

We will often compare our constructions and results with the classical Weil algebra (see for example [13] for a beautiful introduction to these topics) and the noncommutative Weil algebra, in order to point out similarities and novel features of our model.

2.1 The noncommutative Weil algebra $\mathcal{W}_G$

We write $\tilde{\mathfrak{g}}$-ds (resp. $\tilde{\mathfrak{g}}$-da) for $\tilde{\mathfrak{g}}$-differential space (resp. algebra).

Definition 2.1 *The classical Weil algebra* W_G *is the universal commutative locally free* $\tilde{\mathfrak{g}}$*-da.*

We recall that a $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ is called locally free if it admits an algebraic connection, i.e. a linear map $\varpi : \mathfrak{g}^* \to \mathcal{A}$ satisfying

$$i_X(\varpi(\mu)) = \mu(X) \qquad L_X(\varpi(\mu)) = -\varpi(L_X\mu) \qquad \forall\ X \in \mathfrak{g},\ \mu \in \mathfrak{g}^* \tag{22}$$

The universality of W_G means that for every commutative locally free $\tilde{\mathfrak{g}}$-da $\mathcal{A}$ there exists a unique $\tilde{\mathfrak{g}}$-da homomorphism $c^{\varpi} : W_G \to \mathcal{A}$ such that the following diagram commutes:

$$\begin{array}{ccc} W_G & \xrightarrow{c^{\varpi}} & \mathcal{A} \\ \uparrow & \nearrow_{\varpi} & \\ \mathfrak{g}^* & & \end{array} \tag{23}$$

An equivalent way to describe the algebraic connection is through the super vector space $E_{\mathfrak{g}^*} = (\mathfrak{g}^*)^{(0)} \oplus (\mathfrak{g}^*)^{(1)}$ equipped with a Koszul differential space structure (see [2] for more details). An algebraic connection on $\mathcal{A}$ is a $\tilde{\mathfrak{g}}$-ds homomorphism

$$c : E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c} \to \mathcal{A} \tag{24}$$

where $\mathfrak{c}$ is an even generator, which maps to the unity of $\mathcal{A}$, and $\mathbb{F}$ is the ground field. Since $\mathcal{A}$ is commutative, the homomorphism (24) can be lifted to the symmetric algebra $Sym(E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c})$; the Weil algebra W_G turns out to be the quotient

$$W_G = Sym(E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c})/\langle \mathfrak{c} - 1 \rangle \simeq Sym(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*) \tag{25}$$

This construction is readily generalized to a nc $\tilde{\mathfrak{g}}$-da $\mathcal{A}$; now the homomorphism (24) cannot be lifted to the super symmetric algebra but only to the tensor algebra. For this reason Alekseev and Meinrenken [2] give the following definition:

Definition 2.2 *The noncommutative Weil algebra $\mathcal{W}_G$ is the universal noncommutative locally free $\tilde{\mathfrak{g}}$-differential algebra*

They also exhibit an explicit expression for $\mathcal{W}_G$,

$$\mathcal{W}_G = T(E_{\mathfrak{g}^*} \oplus \mathbb{F}\mathfrak{c})/ < \mathfrak{c} - 1 > \tag{26}$$

and prove its universality. At this point, if we make the additional requirement that $\mathfrak{g}$ is quadratic (so that $\mathfrak{g}$ carries a non degenerate invariant quadratic form B which we use to identify $\mathfrak{g}^* \simeq \mathfrak{g}$), we can construct out of $\mathfrak{g}$ a super Lie algebra $\bar{\mathfrak{g}}$.

Definition 2.3 *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra. Fix a basis $\{e_a\}$ and let ${f_{ab}}^c$ be the structure constants on this basis. The super Lie algebra $\bar{\mathfrak{g}}$ is defined as the super vector space $\mathfrak{g}^{ev} \oplus \mathfrak{g}^{odd} \oplus \mathbb{F}\mathfrak{c}$, with basis given by even elements $\{e_a, \mathfrak{c}\}$ and odd ones $\{\xi_a\}$, and brackets given by*

$$\begin{array}{llllll} [e_a, e_b] & = {f_{ab}}^c e_c & [e_a, \xi_b] & = {f_{ab}}^c \xi_c & [\xi_a, \xi_b] & = B_{ab}\mathfrak{c} \\ [e_a, \mathfrak{c}] & = 0 & [\xi_a, \mathfrak{c}] & = 0 & & \end{array} \tag{27}$$

Identifying $\mathfrak{g} \simeq \mathfrak{g}^*$ we have then a (super) Lie algebra structure also on $E_{\mathfrak{g}^*} \oplus \mathbb{F}$; looking back at the homomorphism (24), we note that now it is a Lie algebra homomorphism (where on $\mathcal{A}$ we use the commutator w.r.t. the noncommutative product) and so it lifts to the (super) enveloping algebra $\mathfrak{U}(\bar{\mathfrak{g}})$. The noncommutative Weil algebra can then be written as

$$\mathcal{W}_G = \mathfrak{U}(\bar{\mathfrak{g}})/\langle \mathfrak{c} - 1 \rangle \simeq \mathfrak{U}(\mathfrak{g}) \otimes Cl(\mathfrak{g}). \tag{28}$$

From now on we shall consider $\mathcal{W}_G$ as a super enveloping algebra; formally we are working in $\mathfrak{U}(\bar{\mathfrak{g}})$ assuming implicitly every time $\mathfrak{c} = 1$. A remark is in order: if the algebra $\mathcal{A}$ is commutative, then its Lie algebra structure is trivial, (24) lifts to the symmetric algebra $Sym(\bar{\mathfrak{g}})$ and we come back to the classical Weil algebra W_G. The expression (28) is by the time being only true as vector space isomorphism; to become an algebra equality we have to pass to even generators which commute with odd ones: this will be done in a while.

We study now the $\tilde{\mathfrak{g}}$-da structure of $\mathcal{W}_G$. The main point here is that the action of L and i on $\mathcal{W}_G$ may be realized as the (super) adjoint action of even and odd generators.

Definition 2.4 *On a generic element $X \in \mathcal{W}_G$ the actions of L and i are given by*

$$L_a(X) := ad_{e_a}(X) \qquad\qquad i_a(X) := ad_{\xi_a}(X) \tag{29}$$

On generators one has

$$\begin{array}{ll} L_a(e_b) = f_{ab}{}^c e_c & \qquad i_a(e_b) = f_{ab}{}^c \xi_c \\ L_a(\xi_b) = f_{ab}{}^c \xi_c & \qquad i_a(\xi_b) = B_{ab} \end{array} \tag{30}$$

Then L_a and i_a are derivations, and their action coincides with the commutator in the enveloping algebra. To avoid confusion, let us stress that the commutator for a fixed element in an enveloping algebra $\mathfrak{U}(\mathfrak{g})$ (i.e. w.r.t. the associative product of $\mathfrak{U}(\mathfrak{g})$) is always a derivation, thanks to the Jacobi identity of the Lie algebra structure of $(\mathfrak{U}(\mathfrak{g}), [\,,\,])$. Instead the adjoint action ad_Y acts on products as $ad_Y(X_1X_2) = (ad_{(\triangle Y)_{(1)}} X_1)(ad_{(\triangle Y)_{(2)}} X_2)$, and so it is a derivation if and only if Y is primitive. When this happens one has $ad_Y(X) = [Y, X]$, while in general the two structures are different.

The differential d^W on the noncommutative Weil algebra is the Koszul differential

$$d^W(e_a) = 0 \qquad\qquad d^W(\xi_a) = e_a \tag{31}$$

so that $\mathcal{W}_G$ is acyclic. These definitions allow us to check quite easily that the operators (d^W, L, i) satisfy the relations in theorem (1.2), and so define a $\tilde{\mathfrak{g}}$-da structure on $\mathcal{W}_G$.

The set of generators $\{e_a, \xi_a\}$ of $\mathcal{W}_G$ will be called of the Koszul type. It is often more convenient to use another set of generators, where the even ones are horizontal (i.e. their are killed by i). This is obtained by the transformations

$$u_a := e_a + \frac{1}{2} f_a{}^{bc} \xi_b \xi_c \tag{32}$$

where we use B to raise and lower indexes. One can easily verify that $\{u_a, \xi_a\}$ is another set of generators for $\mathcal{W}_G$, with relations (compare with (27)):

$$[u_a, u_b] = f_{ab}{}^c u_c \qquad [u_a, \xi_b] = 0 \qquad [\xi_a, \xi_b] = B_{ab} \tag{33}$$

The $\tilde{\mathfrak{g}}$-da structure, still given by adjoint action of generators $\{e_a, \xi_a\}$, now on $\{u_a, \xi_a\}$ reads:

$$\begin{array}{ll} L_a(u_b) = f_{ab}{}^c u_c & L_a(\xi_b) = f_{ab}^c \xi_c \\ i_a(u_b) = 0 & i_a(\xi_b) = B_{ab} \\ d^W(u_a) = -f_a{}^{bc} \xi_b u_c & d^W(\xi_a) = u_a - \frac{1}{2} f_a{}^{bc} \xi_b \xi_c \end{array} \tag{34}$$

Notice that $\mathcal{W}_G$ is a filtered algebra, with associated graded differential algebra the classical Weil algebra W_G. The $\tilde{\mathfrak{g}}$-da structure of $\mathcal{W}_G$ agree with the classical one if we pass to $Gr(\mathcal{W}_G)$.

The operator d^W too may be expressed as an inner derivation: indeed it is given by the commutator with an element $\mathcal{D} \in (\mathcal{W}_G^{(3)})^G$. There are several ways (depending on the choice of generators used) one can write $\mathcal{D}$, and the simplest one for our calculations is

$$\mathcal{D} = \frac{1}{3} \xi^a e_a + \frac{2}{3} \xi^a u_a \tag{35}$$

For a generic element $X \in \mathcal{W}_G$ we can then write $d^W(X) = [\mathcal{D}, X]$.

As pointed out in [1], this element $\mathcal{D}$ may be viewed as a Dirac operator; it is also related to Kostant's cubic Dirac operator [15]. Recently it also appeared in [19], where Dirac operators on quantum groups are discussed. Our construction of twisted noncommutative Weil algebras may provide a natural framework where to look for these quantum Dirac operators.

2.2 The twist of $\mathcal{W}_G$

If we consider, instead of a generic noncommutative $\tilde{\mathfrak{g}}$-da $\mathcal{A}$, an algebra $\mathcal{A}_\chi$ where noncommutativity is realized via a Drinfel twist as in Proposition (1.1), its $\tilde{\mathfrak{g}}$-differential structure is given by twisted derivations like the ones defined for $\Omega^\bullet(\mathcal{M}_\theta)$ in the previous section. In what follows everything is realized having in mind this example, but the construction really applies to a generic twisted noncommutative algebra. Dealing with this kind of algebras, it seems natural to ask for a $\tilde{\mathfrak{g}}$-da twisted structure also on the Weil algebra $\mathcal{W}_G$. This can be done: we know how to twist an enveloping algebra, and how this twist modifies the adjoint action.

A remark before starting: we stress again the fact that the construction of twisted noncommutative Weil algebras applies to every quadratic Lie algebra, and it makes sense also in the case an explicit form of the twist element χ is unknown. All what we need is the existence of the twist, and in this we can get help from the above-mentioned rigidity and uniqueness theorems of Drinfeld and others; so it is a quite general procedure. The same holds true for the construction of the equivariant cohomology models of the next section. Obviously if one needs to deal with explicit expressions and computations, like the ones presented here, an explicit form of χ is crucial.

Definition 2.5 *Let $\mathfrak{g}$ be a quadratic Lie algebra, and $\mathcal{A}_\chi$ a noncommutative algebra associated to a Drinfeld twist χ. The twisted noncommutative Weil algebra $\mathcal{W}_G^{(\chi)}$ is defined as the Drinfeld twist of $\mathcal{W}_G$ by the same χ, now viewed as an element in $(\mathcal{W}_G \otimes \mathcal{W}_G)^{(ev)}_{[[\theta]]}$.*

Note that the twisted Hopf algebra $\mathfrak{U}_\chi(\mathfrak{g})$ introduced in section (1.3) is the even subalgebra of $\mathcal{W}_G^{(\chi)}$.

We then repeat the construction of the previous section in the twisted case: we start with even and odd generators $\{e_i, e_r, \xi_i, \xi_r\}$ distinguishing, as in section (1.3), between Cartan and roots generators of $\mathfrak{g}$. By direct computation and as it already happened for $\mathfrak{U}_\chi(\mathfrak{g})$, twisted coproducts of Cartan elements are unchanged while for roots elements we have

$$\begin{aligned} \triangle_\chi(e_r) &= e_r \otimes \lambda_r^{-1} + \lambda_r \otimes e_r \\ \triangle_\chi(\xi_r) &= \xi_r \otimes \lambda_r^{-1} + \lambda_r \otimes \xi_r \end{aligned} \tag{36}$$

It is interesting to note that for completely skew-symmetric twist elements χ, the twisted antipode S_χ (recall definition (1.1)) is not deformed. This because $U = \chi^{(1)} S(\chi^{(2)}) = 1$, as one can check directly.

By using these relations we can define the action of L and i on $\mathcal{W}_G^{(\chi)}$:

Definition 2.6 *The action of L and i on $\mathcal{W}_G^{(\chi)}$ is given by the adjoint action with respect to even and odd generators. In particular $L_i = ad_{e_i}$ and $i_i = ad_{\xi_i}$ are the same as in the untwisted case. On the contrary, due to (36), for roots elements the operators L_r and i_r are modified even on a single generator:*

$$\begin{aligned} L_r(X) &= ad_{e_r}(X) = e_r X \lambda_r - \lambda_r X e_r \\ i_r(X) &= ad_{\xi_r}(X) = \xi_r X \lambda_r + (-1)^{|X|} \lambda_r X \xi_r \end{aligned} \tag{37}$$

We are using the twisted coproduct and antipode on $\mathcal{W}_G^{(\chi)}$, and the definition of adjoint action on super Hopf algebra

$$ad_Y(X) = \sum (-1)^{|X||(Y)_{(2)}|} (Y)_{(1)} X (S(Y)_{(2)}) \tag{38}$$

Expressing explicitly this action on $\{e_a, \xi_a\}$ we have (one should compare with (30)):

$$\begin{array}{rlrl} L_j(e_a) = & f_{ja}{}^b \, e_b & L_j(\xi_a) = & f_{ja}{}^b \, \xi_b \\ L_r(e_i) = & e_r e_i \lambda_r - \lambda_r e_i e_r & L_r(\xi_i) = & e_r \xi_i \lambda_r - \lambda_r \xi_i e_r \\ = & -r_i \lambda_r e_r & = & -r_i \lambda_r \xi_r \\ L_r(e_s) = & e_r e_s \lambda_r - \lambda_r e_s e_r & L_r(\xi_s) = & e_r \xi_s \lambda_r - \lambda_r \xi_s e_r \end{array} \tag{39}$$

$$\begin{array}{rlrl} i_j(e_a) = & f_{ja}{}^b \, \xi_b & i_j(\xi_a) = & B_{ja} = \delta_{ja} \\ i_r(e_i) = & \xi_r e_i \lambda_r - \lambda_r e_i \xi_r & i_r(\xi_i) = & \xi_r \xi_i \lambda_r + \lambda_r \xi_i \xi_r \\ = & -r_i \lambda_r \xi_r & = & \lambda_r B_{ri} = 0 \\ i_r(e_s) = & \xi_r e_s \lambda_r - \lambda_r e_s \xi_r & i_r(\xi_s) = & \xi_r \xi_s \lambda_r + \lambda_r \xi_s \xi_r \end{array} \tag{40}$$

where we use i, j for Cartan indexes, r, s for roots indexes and a, b for generic indexes. On products one just applies the usual rule for the adjoint action

$$ad_Y(X_1 X_2) = (ad_{Y_{(1)}} X_1)(ad_{Y_{(2)}} X_2) \tag{41}$$

which shows that L_r and i_r are twisted derivations.

Due to the presence of the λ_r terms coming from twisted coproducts, the classical set of generators $\{e_a, \xi_a\}$ is no more closed under the action of L, i. This is not a big problem, but there is however another set of generators (which we will call quantum generators for their relation to quantum Lie algebras, see the remark below) which seems to be more natural when dealing with the twist.

Definition 2.7 *We take as new set of generators of $\mathcal{W}_G^{(\chi)}$ the elements*

$$X_a := \lambda_a e_a \qquad \eta_a := \lambda_a \xi_a \tag{42}$$

Recall from (14) that for $a = i$ we have $\lambda_i = 1$, so $X_i = e_i$. We define also coefficients

$$q_{rs} := exp\ \{\frac{i}{2}\theta^{kl} r_k s_l\} \tag{43}$$

with properties $q_{sr} = q_{rs}^{-1}$ and $q_{rs} = 1$ if $r = -s$; we may also set $q_{ab} = 1$ if at least one index is of Cartan type.

The following relations, easily proved by direct computation, will be very useful:

$$\begin{aligned} \lambda_r \lambda_s &= \lambda_{r+s} & \lambda_r \lambda_s &= \lambda_s \lambda_r \\ \lambda_r e_s &= q_{rs} e_s \lambda_r & \lambda_r \xi_s &= q_{rs} \xi_s \lambda_r \\ L_{\lambda_r} e_s &= q_{rs} e_s & L_{\lambda_r} \xi_s &= q_{rs} \xi_r \end{aligned} \tag{44}$$

and since all λ_r's commute with each other, the same equalities hold for X_r and η_r.

Proposition 2.1 *The action of L and i on quantum generators is*

$$\begin{aligned} L_a X_b &= f_{ab}{}^c X_c & i_a X_b &= f_{ab}{}^c \eta_c \\ L_a \eta_b &= f_{ab}{}^c \eta_c & i_a \eta_b &= B_{ab} \end{aligned} \tag{45}$$

Proof: By direct computation, using the definition of the adjoint action, the relations (44) and the commutation rules between $\{e_a, \xi_a\}$ in $\mathcal{W}_G^{(\chi)}$. ■

Remark: Note that this is exactly the same action we have in the classical case (30). The difference however is that we act on quantum generators with classical generators: $L_a X_b = ad_{e_a} X_b \neq ad_{X_a} X_b$.

The fact that the base of the Lie algebra $\{e_a, \xi_a\}$ is not closed under the (twisted) adjoint action is a typical feature of quantized enveloping algebras $\mathfrak{U}_q(\mathfrak{g})$, where the deformation (of the Drinfeld-Jimbo type) involves also the Lie algebra structure of $\mathfrak{g}$ (while with the Drinfeld twist we change only the Hopf-algebra structure on the enveloping algebra, but not the Lie bracket in $\mathfrak{g}$). Since $\mathfrak{g}$ can be viewed as the closed ad-submodule of $\mathfrak{U}(\mathfrak{g})$ where the adjoint action is given by Lie bracket, one can try to recover a Lie algebra inside $\mathfrak{U}_q(\mathfrak{g})$ by defining the quantum Lie algebra $\mathfrak{g}_q$ as a closed ad-submodule of $\mathfrak{U}_q(\mathfrak{g})$ with quantum Lie bracket given by the adjoint action of $\mathfrak{U}_q(\mathfrak{g})$. Linearity still holds, skew-symmetry becomes q-skew-symmetry and the Jacobi identity generalizes to a braided identity [9].

The deformation of the coproduct in $\mathfrak{U}_\chi(\mathfrak{g})$ leads to a deformation of the adjoint action, even if the brackets $[e_a, e_b]$ are unchanged; thus $ad_{e_r} e_s$ is no more equal to $[e_r, e_s]$. However $\{X_a\}$ are generators of a closed ad-submodule (see (44)), so we can define quantum Lie brackets $[\,,\,]_{(\chi)}$ using the twisted adjoint action, obtaining a quantum Lie algebra structure $\mathfrak{g}_\chi$:

$$\begin{array}{l}
[X_i, X_j]_{(\chi)} := ad_{X_i} X_j = 0 \\
[X_i, X_r]_{(\chi)} := ad_{X_i} X_r = r_i X_r = -[X_r, X_i]_{(\chi)} \\
[X_{-r}, X_r]_{(\chi)} := ad_{X_{-r}} X_r = \sum r_i X_i = [X_r, X_{-r}]_{(\chi)} \\
[X_r, X_s]_{(\chi)} := ad_{X_r} X_s = q_{rs} f_{rs}{}^{r+s} X_{r+s} \\
[X_s, X_r]_{(\chi)} := ad_{X_s} X_r = q_{sr} f_{sr}{}^{r+s} X_{r+s} = -(q_{rs})^{-1} f_{rs}{}^{r+s} X_{r+s}
\end{array} \tag{46}$$

The q-antisymmetry is explicit only in the $[X_r, X_s]_{(\chi)}$ brackets since $q_{ab} \neq 1$ iff both indexes are root type. The same result holds also for the odd part of $\bar{\mathfrak{g}}$, so we may consider $\{X_a, \eta_a, \mathfrak{c}\}$ a base for the quantum (super) Lie algebra inside $\mathfrak{U}_\chi(\bar{\mathfrak{g}})$.

The last observation is that $\triangle_\chi X_r = X_r \otimes 1 + \lambda_r^2 \otimes X_r$, so if we want $\mathfrak{g}_\chi$ to be closed also under the coproduct, we may consider mixed generators $\{\Lambda_j, X_r\}$ where the Cartan-type generators are defined as group-like elements $\Lambda_j := \exp\{\frac{i}{2}\theta^{jl} H_l\}$. Now $\{\Lambda_j, X_r, \mathfrak{c}\}$ describe a different quantum Lie algebra $\mathfrak{g}'_\chi$, due to the presence of group-like elements; the structure of $\mathfrak{g}_\chi$ is recovered taking the first order terms in θ of the commutators involving Λ_j's (this is a standard procedure in quantum enveloping algebras). ∎

We can pass to horizontal generators (remember eq. (32) and the terminology introduced there) by defining

$$K_a := \lambda_a u_a = \lambda_a (e_a + \frac{1}{2} f_a{}^{bc} \xi_b \xi_c) = X_a - \frac{1}{2} \eta^b ad_{X_b}(\eta_a) \tag{47}$$

Indeed the K_a are horizontal

$$i_a K_b = ad_{\xi_a}(\lambda_b u_b) = \xi_a \lambda_b u_b \lambda_a - \lambda_a \lambda_b u_b \xi_a = 0 \tag{48}$$

and their transformation under L_a is given by

$$L_a K_b = ad_{e_a}(\lambda_b u_b) = e_a \lambda_b u_b \lambda_a - \lambda_a \lambda_b u_b e_a = f_{ab}{}^c K_c \tag{49}$$

The last thing to describe is the action of the differential d^W. Recall that in $\mathcal{W}_G$ we had $d^W(X) = [\mathcal{D}, X]$, and this is still true in $\mathcal{W}_G^{(\chi)}$. In fact $\mathcal{D} = 1/3\, \xi^a e_a + 2/3\, \xi^a u_a = 1/3\, \eta^a X_a + 2/3\, \eta^a K_a$. Moreover d^W being a commutator, the Jacobi identity assures it is an untwisted derivation. This is not surprising: the twisted $\tilde{\mathfrak{g}}$-da structure of an algebra does not change the action of the differential. Note that $\eta^a = \lambda_a^{-1} \xi^a$ and $d^W \lambda_a = [\mathcal{D}, \lambda_a] = 0$. For even generators we have

$$d^W(K_a) = \lambda_a d_W(u_a) = -f_a{}^{bc} \lambda_a \xi_b u_c = -f_a{}^{bc} \lambda_b \lambda_c \xi_b u_c = -q_{ab} f_a{}^{bc} \eta_b K_c \tag{50}$$

where if we want to raise the index of η we have to take in account also the λ inside η

$$-q_{ab}f_a{}^{bc}\eta_b K_c = -q_{ba}f_{ab}{}^c\eta^b K_c \tag{51}$$

and for odd generators

$$d^W(\eta_a) = \lambda_a e_a = \lambda_a(u_a - \frac{1}{2}f_a{}^{bc}\xi_b\xi_c) = K_a - \frac{1}{2}q_{ba}f_{ab}{}^c\eta^b\eta_b \tag{52}$$

We have found all the relations which define a twisted $\tilde{\mathfrak{g}}$-da structure on $\mathcal{W}_G^{(\chi)}$. The difference with the untwisted case is that the elements $\{K_a, \eta_a\}$ generate the whole algebra $\mathcal{W}_G^{(\chi)}$, but $L_a = ad_{e_a} \neq ad_{K_a}$. We have thus proved:

Theorem 2.1 *The twisted $\tilde{\mathfrak{g}}$-da structure of $\mathcal{W}_G^{(\chi)}$ is given, on the set of generators $\{K_a, \eta_a\}$ of definition* (2.2.3)*, by equations* (*45*), (*48*), (*49*), (*50*), (*52*).

3 Equivariant cohomology

In this section we introduce the equivariant cohomology of a twisted $\tilde{\mathfrak{g}}$-da $\mathcal{B}_\chi$. Classically, equivariant cohomology is a tool for studying the geometry of a manifold acted on by a group (see for example [13] for a nice introduction to this topic). If the action is proper and free, the equivariant cohomology $H_G(\mathcal{M})$ is the ordinary cohomology of $\mathcal{M}/G$, or equivalently

$$H_G(\mathcal{M}) := ((\Omega^\bullet(\mathcal{M}))_{bas}, d) \tag{53}$$

In the case the action is not free the space of orbits has singularities, due to the fixed points of the action. For compact connected groups G the idea is then to take cohomology of a larger complex, homotopic to $\Omega^\bullet(\mathcal{M})_{bas}$, but where the G action is free. This is realized by means of the Weil algebra $W_G = Sym(\mathfrak{g}^*) \otimes \wedge\mathfrak{g}^*$:

$$H_G(\mathcal{M}) := \big((W_G \otimes \Omega(\mathcal{M}))_{bas}, \quad \delta = d^W \otimes 1 + 1 \otimes d\big) \tag{54}$$

where the $\tilde{\mathfrak{g}}$-da structures of both factors are used to define global operators $L = L \otimes 1 + 1 \otimes L$ and $i = i \otimes 1 + 1 \otimes i$ and their basic subcomplex. This is the Weil model for the equivariant cohomology of $\mathcal{M}$, or of the $\tilde{\mathfrak{g}}$-da $\Omega^\bullet(\mathcal{M})$. Geometrically W_G is a (finite-dimensional) model for the (infinite-dimensional) algebra of differential forms over the universal G-bundle EG. Since EG is contractible, one can think of $H_G(\mathcal{M})$ as the ordinary cohomology of $\mathcal{M} \times_G EG$ (this is called the Borel construction).

The Weil model construction is well defined also for the twisted noncommutative Weil algebra $\mathcal{W}_G^{(\chi)}$. This will provide the relevant complex for computing the cohomology of a generic twisted noncommutative $\tilde{\mathfrak{g}}$-da $\mathcal{B}_\chi$; the first example is given by the algebra of noncommutative differential forms $\mathcal{B}_\chi = \Omega^\bullet(\mathcal{M}_\theta)$ for an isospectral deformation realized with twist element χ.

Usually, another (simpler) model is used to compute equivariant cohomology: the Cartan model. We want to review the link between Weil and Cartan models in the classical setting, then in the noncommutative construction of [1] and finally to study it in our twisted noncommutative case.

3.1 Classical and noncommutative Cartan models

The simplest way to obtain the Cartan model of a commutative $\tilde{\mathfrak{g}}$-da $\mathcal{B}$ starting from the Weil model is via the Kalkman map [14]. In the Weil algebra $W_G = Sym(\mathfrak{g}^*) \otimes \wedge \mathfrak{g}^*$ we consider even and odd generators given by $\{v_a, y_a\}$. The Kalkman map is defined as

$$\phi = \exp\,\{y^a \otimes i_a\} : W_G \otimes \mathcal{B} \longrightarrow W_G \otimes \mathcal{B} \tag{55}$$

The role of the Kalkman map is elucidated by the following theorem [14].

Theorem 3.1 *The map ϕ realizes an algebra isomorphism*

$$(W_G \otimes \mathcal{B})_{bas} \overset{\phi}{\simeq} (Sym(\mathfrak{g}^*) \otimes \mathcal{B})^G \tag{56}$$

Proof: The operator $y^a \otimes i_a$ is nilpotent, so the exponential is defined as a finite sum and ϕ is clearly invertible with inverse $\phi^{-1} = \exp\{-y^a \otimes i_a\}$. This operator is also a derivation, so its exponential is an algebra homomorphism. Then ϕ is an automorphism of $W_G \otimes \mathcal{B}$. To see where the basic subcomplex is mapped by ϕ, let us show how L and i are intertwined:

$$\phi\, L_a\, \phi^{-1} = \sum_n \frac{1}{n!}\,[y^b \otimes i_b, [\ldots, [y^b \otimes i_b, L_a \otimes 1 + 1 \otimes L_a]]]$$

The zeroth order term is simply L_a. At the first order we have

$$\begin{aligned}[y^b \otimes i_b, L_a \otimes 1 + 1 \otimes L_a] &= y^b \otimes (f_{ba}{}^c i_c) - L_a(y^b) \otimes i_b \\ &= -f_{ab}{}^c y^b \otimes i_c - f_{ab}{}^c y_c \otimes i_b = 0\end{aligned}$$

and this show that L_a commutes with ϕ. Now the same calculation for i:

$$\phi\, i_a\, \phi^{-1} = \sum_n \frac{1}{n!}\,[y^b \otimes i_b, [\ldots, [y^b \otimes i_b, i_a \otimes 1 + 1 \otimes i_a]]]$$

At the first order we have

$$[y^b \otimes i_b, i_a \otimes 1 + 1 \otimes i_a] = -\,\delta^b_a \otimes i_b = -1 \otimes i_a$$

and the second order term vanishes

$$\frac{1}{2}\,[y^b \otimes i_b, -\,1 \otimes i_a] = -y^b \otimes [i_a, i_b] = 0$$

So the only contributions come from order zero and one, and we obtain

$$\phi\, i_a\, \phi^{-1} = 1 \otimes i_a + i_a \otimes 1 - 1 \otimes i_a = i_a \otimes 1$$

Remembering that $(W_G)_{hor} = Sym(\mathfrak{g}^*)$ we have hence showed that the basic subcomplex of $W_G \otimes \mathcal{B}$ is transformed by ϕ in $(Sym(\mathfrak{g}^*) \otimes \mathcal{B})^G$ and this completes the proof. ∎

The algebra $(Sym(\mathfrak{g}^*) \otimes \mathcal{B})^G$ appearing in the rhs of (56) will be called the Cartan algebra (or complex) for the equivariant cohomology of $\mathcal{B}$, and denoted by $C_G(\mathcal{B})$. The differential d_G on $C_G(\mathcal{B})$ is that induced by the Kalkman map: it is given by $d_G := \phi\, \delta_{|bas} \phi^{-1}$.

Theorem 3.2 *The Cartan differential d_G on $C_G(\mathcal{B})$ takes the form*

$$d_G = 1 \otimes d - v^a \otimes i_a \tag{57}$$

<u>Proof</u>: First of all note that $\phi_{|\,bas} = P_{hor} \otimes 1$ where P_{hor} is the projector of W_G onto the horizontal subalgebra $Sym(\mathfrak{g}^*) \otimes 1$. Indeed on basic elements we have (we sum over latin indexes a but *not* over greek indexes α)

$$\begin{aligned} \exp\{y^a \otimes i_a\} &= \prod_\alpha (1 + y^\alpha \otimes i_\alpha) = \prod_\alpha (1 - y^\alpha i_\alpha \otimes 1) \\ &= \prod_\alpha (i_\alpha y^\alpha \otimes 1) = P_{hor} \otimes 1 \end{aligned}$$

We want to compute $(P_{hor} \otimes 1)(d^W \otimes 1 + 1 \otimes d)_{|bas}(P_{hor} \otimes 1)^{-1}$. The operator $1 \otimes d$ commutes with $P_{hor} \otimes 1$. A better way to express the Weil differential is $d^W \otimes 1 = y^a L_a \otimes 1 + (v^a - \frac{1}{2} f^{abc} y_b y_c) i_a \otimes 1$; now all the terms involving y's are killed by $P_{hor} \otimes 1$, the surviving $v^a i_a \otimes 1$ on the basic complex (where $i_a \otimes 1 + 1 \otimes i_a = 0$) is equal to $-v^a \otimes i_a$ and so it commutes with $P_{hor} \otimes 1$. At the end we are left with (57). ∎

Remark: The Cartan model and its differential are related to the BRST model built up starting from $\mathcal{M}$ and G, as pointed out in [14]; we quickly recall the idea. Denote by A the differential algebra $W_G \otimes \Omega^\bullet(\mathcal{M})$ with $\delta = d^W \otimes 1 + 1 \otimes d$. On the same algebra it is possible to define another differential, the BRST operator: $\delta^{BRST} = \delta + y^a L_a^\Omega - v^a i_a^\Omega$. We call B the differential algebra $(W_G \otimes \Omega^\bullet(\mathcal{M}))$ with δ^{BRST} (for the physical interpretation of B see [14]). The Kalkman map is a da-isomorphism from A to B, i.e. it intertwines the two differential structures. When restricted to $A_{|bas}$ its image is the Cartan model, now seen as the G-invariant subcomplex of the BRST model B; then also the Cartan differential d_G is nothing but the restriction to the invariant subcomplex of the BRST differential δ^{BRST}. ■

Definition 3.1 *The Cartan model for the equivariant cohomology of a commutative $\tilde{\mathfrak{g}}$-da $\mathcal{B}$ is the cohomology of the Cartan complex $C_G(\mathcal{B})$:*

$$H_G(\mathcal{B}) = \big((Sym(\mathfrak{g}^*) \otimes \mathcal{B})^G,\, d_G = 1 \otimes d - v^a \otimes i_a\big) \tag{58}$$

Remark: Any homomorphism of $\tilde{\mathfrak{g}}$-da induces an homomorphism between the corresponding equivariant cohomologies; for every $\tilde{\mathfrak{g}}$-da $\mathcal{B}$ over a field $\mathbb{F}$ one has the natural homomorphism $\mathbb{F} \rightarrow \mathcal{B}$, which then induces an $H_G(\mathbb{F}) = (Sym(\mathfrak{g}^*))^G$ module structure on $H_G(\mathcal{B})$. The differential d_G commutes with this module structure. ■

It is possible to repeat this construction, with some changes, also for the noncommutative Weil model $\mathcal{W}_G \otimes \mathcal{B}$ of [1]. Here we review the main points. The bigger difference is that now $\mathcal{W}_G = \mathfrak{U}(\mathfrak{g}) \otimes Cl(\mathfrak{g})$ has generators $\{u_a, \xi_a\}$ subject to relations (33), and $\phi = \exp\{\xi^a \otimes i_a\}$ is no more an algebra homomorphism since $\xi^a \otimes i_a$ is no more a derivation. As vector spaces ϕ still gives an isomorphism between $(\mathcal{W}_G \otimes \mathcal{B})_{bas}$ and $(\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{B})^G$, but the natural algebra structure of $(\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{B})^G$ is different from the one induced by ϕ and it is not compatible with the induced differential $d_G = \phi \delta_{|bas} \phi^{-1}$.

We first comupte d_G. In [1] the authors use the relation between the Clifford product and the exterior product to factorize products $\xi_a \xi_b$ as a wedge product $\xi_a \wedge \xi_b$ multiplied by an extra term, and they are able to show that all the changes in d_G and in the product in the Cartan model come directly from this extra term. Here we prefer to implement a direct computation at the various order of commutators, even if this is a quite long and tedious route, because this seems to be the only possible strategy in the twisted case. More in detail, the extra term which links the Clifford and the wedge product involves powers of the interior product i on $\bigwedge \mathfrak{g}^*$; on the basic subcomplex $(\mathcal{W}_G \otimes \mathcal{B})_{bas}$ this term can be moved onto the $\mathcal{B}$ factor by paying a sign, and then it commutes with $\phi_{|\,bas} = P_{hor} \otimes 1$. In the twisted case the problem is that this extra term involves the untwisted interior product i on $\bigwedge \mathfrak{g}^*$ (since the Clifford multiplication is still the same), while the basic subcomplex is taken considering the twisted interior product i. So we cannot move this terms onto the $\mathcal{B}$ side of the tensor product, and the trick does not work.

Theorem 3.3 *The differential $d_G = \phi\, \delta_{|bas}\, \phi^{-1}$ on $(\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{B})^G$ is*

$$d_G = 1 \otimes d - u^a \otimes i_a - \frac{1}{2} B^{ab} 1 \otimes L_b i_a + \frac{1}{24} f^{abc} 1 \otimes i_a i_b i_c \tag{59}$$

<u>Proof</u>: We compute $\phi\delta\phi^{-1}$ at the various order of commutators with $\xi^a \otimes i_a$. We begin with the first order:

$$\begin{aligned}[\xi^a \otimes i_a, d^W \otimes 1 + 1 \otimes d] &= -d^W(\xi^a) \otimes i_a + \xi^a \otimes L_a \\ &= -u^a \otimes i_a + \frac{1}{2} f^{abc} \xi_b \xi_c \otimes i_a + \xi^a \otimes L_a\end{aligned} \tag{60}$$

The second order term is

$$[\xi^a \otimes i_a, -u^b \otimes i_b + \frac{1}{2} f^{bcd} \xi_c \xi_d \otimes i_b + \xi^b \otimes L_b] \tag{61}$$

We consider the three terms in (61) separately; for the first one we have

$$[\xi^a \otimes i_a, -u^b \otimes i_b] = -\xi^a u^b \otimes [i_a, i_b] = 0$$

For the second we have

$$\begin{aligned}[\xi^a \otimes i_a, \frac{1}{2} f^{bcd} \xi_c \xi_d \otimes i_b] &= \frac{1}{2} f^{bcd} [\xi^a, \xi_c \xi_d] \otimes i_a i_b \\ &= \frac{1}{2} f^{bcd} \delta^a_c \xi_d \otimes i_a i_b - \frac{1}{2} f^{bcd} \delta^d_a \xi_c \otimes i_a i_b \\ &= -f^{bcd} \xi_b \otimes i_c i_d\end{aligned}$$

And the last term is

$$\begin{aligned}[\xi^a \otimes i_a, \xi^b \otimes L_b] &= -\xi^a \xi^b \otimes i_a L_b + \xi^a \xi^b \otimes L_b i_a - B_{ab} \otimes L_b i_a \\ &= -f_{ab}{}^c \xi^a \xi^b \otimes i_c - B^{ab} \otimes L_b i_a\end{aligned}$$

Now the third order:

$$[\xi^a \otimes i_a, -f^{bcd} \xi_b \otimes i_c i_d - f_{bc}{}^d \xi^b \xi^c \otimes i_d - B^{bc} \otimes L_c i_b] \tag{62}$$

The first term in (62) is

$$\begin{aligned}[\xi^a \otimes i_a, -f^{bcd}\xi_b \otimes i_c i_d] &= f^{bcd}\xi^a\xi_b \otimes i_a i_c i_d + f^{bcd}\xi_b\xi^a \otimes i_c i_d i_a \\ &= f^{bcd} 1 \otimes i_b i_c i_d\end{aligned}$$

The second one is

$$\begin{aligned}[\xi^a \otimes i_a, -f_{bc}{}^d \xi^b\xi^c \otimes i_d] &= -f^{bcd}[\xi^a, \xi_b\xi_c] \otimes i_a i_d \\ &= -f^{bcd}(\delta^a_b \xi_c - \delta^a_c \xi_b) \otimes i_a i_d \\ &= 2f^{bcd}\xi_b \otimes i_c i_d\end{aligned}$$

The third and last one:

$$[\xi^a \otimes i_a, -B^{bc} \otimes L_c i_b] = -B^{bc}\xi^a \otimes [i_a, L_c i_b] = f^{abc}\xi_a \otimes i_b i_c$$

We continue with the fourth order:

$$[\xi^a \otimes i_a, f^{bcd} \otimes i_b i_c i_d + 3f^{bcd}\xi_b \otimes i_c i_d] \tag{63}$$

This time we have only two terms: the first one vanishes

$$[\xi^a \otimes i_a, f^{bcd} \otimes i_b i_c i_d] = f^{bcd}\xi^a \otimes i_a i_b i_c i_d + f^{bcd}\xi^a \otimes i_b i_c i_d i_a = 0$$

and the second one gives

$$\begin{aligned}[\xi^a \otimes i_a, 3f^{bcd}\xi_b \otimes i_c i_d] &= -3f^{bcd}(\xi^a\xi_b \otimes i_a i_c i_d + \xi_b\xi^a \otimes i_c i_d i_a) \\ &= -3f^{bcd} \otimes i_b i_c i_d\end{aligned}$$

So now the fifth order reduces to

$$[\xi^a \otimes i_a, -3f^{bcd} \otimes i_b i_c i_d] \tag{64}$$

and this behaves like the first term in (63), so it vanishes. The series stops here, and now we have just to add the contributions from each order:

$$\begin{aligned}\phi\,\delta\,\phi^{-1} = d \otimes 1 + 1 \otimes d - u^a \otimes i_a + \xi^a \otimes L_a + \\ -\frac{1}{2}B^{ab} \otimes L_b i_a + \frac{1}{24} f^{abc} \otimes i_a i_b i_c\end{aligned} \tag{65}$$

This could be interpreted as the noncommutative BRST differential, following the ideas of [14]. Restricting to the basic subcomplex (i.e. negletting terms involving ξ_a's) we obtain the Cartan differential d_G of eq. (59). ∎

Remark: The differential (59) is the same as the one of [1](Prop. 4.2); we just expressed the $(u_L^a + u_R^a) \otimes i_a$ terms of [1] by terms involving standard left multiplication for u_a and operators L_a. This could be easily verified on generators. ■

Now we find the multiplicative structure of $\mathcal{C}_G(\mathcal{B})$ induced by the Kalkman map: by definition for $u_1 \otimes \omega_1$ and $u_2 \otimes \omega_2$ in $\mathcal{C}_G(\mathcal{B})$ we have

$$(u_1 \otimes \omega_1) \cdot_{\mathcal{C}_G(\mathcal{B})} (u_2 \otimes \omega_2) = \phi \left(\phi^{-1}(u_1 \otimes \omega_1) \cdot \phi^{-1}(u_2 \otimes \omega_2) \right) \tag{66}$$

We can obtain an explicit formula for $\cdot_{\mathcal{C}_G(\mathcal{B})}$; as proved in [1] we have

Theorem 3.4 *The multiplicative structure in $\mathcal{C}_G(\mathcal{B})$ induced by the Kalkman map is*

$$(u_1 \otimes \omega_1) \cdot_{\mathcal{C}_G(\mathcal{B})} (u_2 \otimes \omega_1) = u_1 u_2 \otimes \cdot_{\mathcal{B}} \{ exp(-\frac{1}{2} B^{ab} i_a \otimes i_b)(\omega_1 \otimes \omega_2) \} \tag{67}$$

for generic elements $u_i \otimes \omega_i \in \mathcal{C}_G(\mathcal{B})$.

Proof: Again, in [1] the proof is based on the relation between Clifford and wedge product. For the same reasons given before, we prefer to check (67) directly on generators of $\mathcal{C}_G(\mathcal{B})$. Let us consider then ω_i of degree 1 (the case of degree 0 is even simpler), u_i horizontal generators of $\mathfrak{U}(\mathfrak{g})$, and apply (66); we begin with the computation of $\phi^{-1}(u_1 \otimes \omega_1) \cdot \phi^{-1}(u_2 \otimes \omega_2)$:

$$\begin{aligned}
&[\exp(-\xi^a \otimes i_a)(u_1 \otimes \omega_1)] \cdot \left[\exp(-\xi^b \otimes i_b)(u_2 \otimes \omega_2) \right] = \\
&\qquad = u_1 u_2 \otimes \omega_1 \omega_2 + u_1 \xi^b u_1 u_2 \otimes \omega_1 (i_b \omega_2) \\
&\qquad - \xi^a u_1 u_2 \otimes (i_a \omega_1) \omega_2 + \xi^a u_1 \xi^b u_2 \otimes (i_a \omega_1)(i_c \omega_2) \\
&\qquad = u_1 u_2 \otimes \omega_1 \omega_2 - \xi^a u_1 u_2 \otimes i_a(\omega_1 \omega_2) + \xi^a \xi^b u_1 u_2 \otimes \omega_1 \omega_2
\end{aligned}$$

We have used the fact that u_i and ξ^a commute. The last term may be rewritten using the identity (valid for ν_i of any degree)

$$\begin{aligned}
-\frac{1}{2} \xi^a \xi^b u_1 u_2 \otimes i_a i_b (\nu_1 \nu_2) = & -\frac{1}{2} \xi^a \xi^b u_1 u_2 \otimes (i_a i_b \nu_1) \nu_2 - \frac{1}{2} \xi^a \xi^b u_1 u_2 \otimes \nu_1 (i_b i_c \nu_2) \\
& + (-1)^{(|\nu_1|-1)} \xi^a \xi^b u_1 u_2 \otimes i_a(\nu_1)(i_b \nu_2) \\
& - \frac{1}{2} (-1)^{(|\nu_1|-1)} B^{ab} u_1 u_2 \otimes (i_a \nu_1)(i_b \nu_2)
\end{aligned}$$

We obtain then

$$\begin{aligned}
& u_1 u_2 \otimes \omega_1 \omega_2 - \xi^a u_1 u_2 \otimes i_a(\omega_1 \omega_2) \\
& \qquad - \frac{1}{2} \xi^a \xi^b u_1 u_2 \otimes i_a i_b (\omega_1 \omega_2) + \frac{1}{2} B^{ab} u_1 u_2 \otimes (i_a \omega_1)(i_b \omega_2)
\end{aligned}$$

which is nothing but the expansion of the exponential

$$\exp(\xi^a \otimes i_a - \frac{1}{2} B^{ab} \otimes i_a^{(1)} i_b^{(2)})(u_1 u_2 \otimes \omega_1 \omega_2)$$

Applying ϕ we find the expression (67). ∎

In this way we have found the noncommutative Cartan model of [1].

Definition 3.2 *The Cartan model for the noncommutative equivariant cohomology of a nc $\tilde{\mathfrak{g}}$-da $\mathcal{B}$ is the cohomology of the Cartan complex $\mathcal{C}_G(\mathcal{B})$:*

$$H_G(\mathcal{B}) = \big((\mathfrak{U}(\mathfrak{g}) \otimes \mathcal{B})^G,\, d_G \big) \tag{68}$$

The differential d_G is given in (59), while the multiplicative structure of $\mathcal{C}_G(\mathcal{B})$ is given in (67).

Remark: The noncommutative equivariant cohomology $H_G(\mathcal{B})$ is equipped with a module structure over $H_G(\mathbb{F})$ (compare with the remark after Definition 3.1); the only difference is that now $H_G(\mathbb{F}) = (\mathfrak{U}(\mathfrak{g}))^G$. The Cartan differential commutes again with this module structure. ∎

3.2 Twisted noncommutative models

Following the classical definition (54), and in a way similar to [1], we define the twisted noncommutative Weil model:

Definition 3.3 *The Weil model for the twisted noncommutative equivariant cohomology of a twisted nc $\tilde{\mathfrak{g}}$-da $\mathcal{B}_\chi$ is the cohomology of the complex*

$$\mathcal{H}_G(\mathcal{B}_\chi) = \Big((\mathcal{W}_G^{(\chi)} \otimes \mathcal{B}_\chi)_{bas},\, \delta = d_W \otimes 1 + 1 \otimes d \Big) \tag{69}$$

It is worth noting that the twisted Leibniz rule for L and i assures the basic subcomplex is still well defined, and that the commutation rules for L, i and δ are the classical ones.

The construction of the Cartan model via a Kalkman map ϕ^χ works also starting from the twisted noncommutative Weil model. The first goal will be to construct such a ϕ^χ; we shall find that also in this case it is no more an algebra isomorphism, so we shall again need to compute explicitly the differential and the multiplicative structure of the Cartan complex.

In the classical case we showed that $\phi_{|bas} = P_{hor} \otimes 1$. We then construct the horizontal projector for the Weil algebra $\mathcal{W}_G^{(\chi)}$.

Theorem 3.5 *The operator* $P_{hor} = \prod_\alpha (ad_{\eta_\alpha} \eta^\alpha)$ *is the projector on the horizontal subalgebra of* $\mathcal{W}_G^{(\chi)}$.

Proof: From $[ad_{\eta_\alpha}, \eta^\alpha] = 1$ we have $ad_{\eta_\alpha}\eta^\alpha = 1 - \eta^\alpha ad_{\eta_\alpha}$. The operators $\eta^\alpha ad_{\eta_\alpha}$ (remember we do not sum over greek indexes) satisfy, by direct computation

$$(ad_{\eta_\alpha}\eta^\alpha)^2 = ad_{\eta_\alpha}\eta^\alpha \ , \qquad (ad_{\eta_\alpha}\eta^\alpha)(\eta_\beta) = 0 \ , \qquad (ad_{\eta_\alpha}\eta^\alpha)(K_\beta) = K_\beta \quad (70)$$

These relations show that P_{hor} has the required properties to be the desired projector. ■

In order to define ϕ^χ note that on $(\mathcal{W}_G^{(\chi)} \otimes \mathcal{B}_\chi)_{bas}$ we have

$$\begin{aligned} P_{hor} \otimes 1 &= (\prod_\alpha ad_{\eta_\alpha}\eta^\alpha) \otimes 1 = \prod_\alpha (1 - \eta^\alpha ad_{\eta_\alpha} \otimes 1) \\ &= \prod_\alpha (1 + \eta^\alpha ad_{\lambda_\alpha} \otimes i_\alpha) = \exp\ (\eta^a ad_{\lambda_a} \otimes i_a) \end{aligned}$$

Definition 3.4 *We define the Kalkman map* ϕ^χ *on* $\mathcal{W}_G^{(\chi)} \otimes \mathcal{B}_\chi$ *as*

$$\phi^\chi = exp\ (\eta^a ad_{\lambda_a} \otimes i_a) \quad (71)$$

Next, we look for the action of ϕ on the invariant and horizontal subalgebras; we find the same results as in the classical case.

Theorem 3.6 *The Kalkman map commute with the action of L:*

$$\phi^\chi (L_b \otimes 1 + 1 \otimes L_b)(\phi^\chi)^{-1} = L_b \otimes 1 + 1 \otimes L_b \quad (72)$$

Proof: We compute the lhs of (72) at the various orders in the commutators. The first order is

$$\begin{aligned} &[\eta^a ad_{\lambda_a} \otimes i_a, ad_{e_b} \otimes 1 + 1 \otimes ad_{e_b}] = [\eta^a ad_{\lambda_a}, ad_{e_b}] \otimes i_a + \eta^a ad_{\lambda_a}[i_a, L_b] = \\ &(q_{ab}\eta^a ad_{e_b} ad_{\lambda_a} - ad_{e_b}\eta^a ad_{\lambda_a}) \otimes i_a + f_{ab}{}^c \eta^a ad_{\lambda_a} \otimes i_c = \\ &(q_{ab}\eta^a ad_{e_b} ad_{\lambda_a} - q_{ab}\eta^a ad_{e_b} ad_{\lambda_a} - f_b{}^{ac}\eta_c ad_{\lambda_b^{-1}} ad_{\lambda_a}) \otimes i_a + f_{ab}{}^c \eta^a ad_{\lambda_a} \otimes i_c = \\ &- f_b{}^a{}_c \eta^c ad_{\lambda_c} \otimes i_c + f_{ab}{}^c \eta^a ad_{\lambda_a} \otimes i_c = 0 \end{aligned}$$

so only the zero order term does not vanish, and ϕ commutes with L_a. ■

Theorem 3.7 *The Kalkman map ϕ intertwines the action of i in the following way:*

$$\phi^\chi(i_b \otimes 1 + 1 \otimes i_b)(\phi^\chi)^{-1} = i_b \otimes 1 \tag{73}$$

Proof: Again we compute the first order in commutators:

$$\begin{aligned}
&[\eta^a ad_{\lambda_a} \otimes i_a, i_b \otimes 1 + 1 \otimes i_b] = -[\eta^a ad_{\lambda_a}, i_b] \otimes i_a + \eta^a ad_\lambda \otimes [i_a, i_b] = \\
&-(\eta^a ad_{\lambda_a} i_b - q_{ab}\eta^a i_b ad_{\lambda_a} + \delta^b_a ad_{\lambda_b^{-1}} ad_{\lambda_a}) \otimes i_a = \\
&-(q_{ab}\eta^a i_b ad_{\lambda_a} - q_{ab}\eta^a i_b ad_{\lambda_a} + \delta^b_a) \otimes i_a = -1 \otimes i_b
\end{aligned}$$

so that the second order term is $[\eta^a ad_{\lambda_a} \otimes i_a, -1 \otimes i_b] = 0$ and we have the claimed result. ■

So again the Kalkman map is not an algebra homomorphism, but as vector spaces one has

$$(\mathcal{W}_G^{(\chi)} \otimes \mathcal{B}_\chi)_{bas} \overset{\phi^\chi}{\simeq} (\mathfrak{U}_\chi(\mathfrak{g}) \otimes \mathcal{B}_\chi)^G \tag{74}$$

(and note that $\mathcal{W}_G^{(\chi)} = \mathcal{W}_G$ and $\mathfrak{U}_\chi(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})$ as vector spaces). Let us compute the transformation of the differential under ϕ^χ.

Theorem 3.8 *The twisted noncommutative Cartan differential is defined by $d_G = \phi^\chi\, \delta_{|bas}\, (\phi^\chi)^{-1}$. On $(\mathfrak{U}_\chi(\mathfrak{g}) \otimes \mathcal{B}_\chi)^G$ it is explicitly given by*

$$d_G = 1 \otimes d - K^a ad_{\lambda_a} \otimes i_a - \frac{1}{2} B^{ab} \otimes L_b i_a + \frac{1}{24} f^{abc} \otimes i_a i_b i_c \tag{75}$$

Proof: Very similar to that of Theorem (3.3): we compute directly $\phi^\chi\, \delta (\phi^\chi)^{-1}$. At first order we have

$$\begin{aligned}
[\eta^a ad_{\lambda_a} \otimes i_a, d^W \otimes 1 + 1 \otimes d] &= -[d^W, \eta^a] ad_{\lambda_a} \otimes i_a + \eta^a ad_{\lambda_a} \otimes L_a \\
&= -K^a ad_{\lambda_a} \otimes i_a + \frac{1}{2} q_{ba} f^{abc} \eta_b \eta_c ad_{\lambda_a} \otimes i_a \\
&+ \eta^a ad_{\lambda_a} \otimes L_a
\end{aligned} \tag{76}$$

At second order

$$[\eta^a ad_{\lambda_a} \otimes i_a, -K^a ad_{\lambda_b} \otimes i_b + \frac{1}{2}\, q_{bc} f^{bcd} \eta_c \eta_d ad_{\lambda_b} \otimes i_b + \eta^a ad_{\lambda_b} \otimes L_b] \tag{77}$$

We conside one term at a time: the first is

$$\begin{aligned}
[\eta^a ad_{\lambda_a} \otimes i_a, -K^a ad_{\lambda_b} \otimes i_b] &= [\eta^a ad_{\lambda_a}, K^b ad_{\lambda_b}] \otimes i_b i_a \\
&= (q_{ba}\eta^a K^b ad_{\lambda_a} ad_{\lambda_b} - q_{ab} K^b \eta^a ad_{\lambda_b} ad_{\lambda_a}) \otimes i_b i_a \\
&= (\lambda_a^{-1}\lambda_b^{-1}\xi^a u^b - \lambda_b^{-1}\lambda_a^{-1} u^b \xi^a) ad_{\lambda_a} ad_{\lambda_b} \otimes i_b i_a = 0
\end{aligned}$$

The second term in (77) is

$$[\eta^a ad_{\lambda_a} \otimes i_a, \frac{1}{2}\, q_{bc} f^{bcd} \eta_c \eta_d ad_{\lambda_b} \otimes i_b] =$$

$$= \frac{1}{2} f^{bcd} [\eta^a ad_{\lambda_a}, \eta_c ad_{\lambda_c} \eta_d ad_{\lambda_d}] \otimes i_a i_b$$

$$= \frac{1}{2} f^{bcd} \delta^a_c \eta_d ad_{\lambda_d^{-1}} \otimes i_a i_b - \frac{1}{2} f^{bcd} \eta_c ad_{\lambda_c^{-1}} \delta^a_d \otimes i_a i_b$$

$$= -f^{bcd} \eta_b ad_{\lambda_b^{-1}} \otimes i_c i_d$$

And the third and last term is

$$[\eta^a ad_{\lambda_a} \otimes i_a, \eta^b ad_{\lambda_b} \otimes L_b] =$$

$$= \eta^a ad_{\lambda_a} \eta^b ad_{\lambda_b} \otimes i_a L_b - (-\eta^a ad_{\lambda_a} \eta^b ad_{\lambda_b} + B^{ab}) \otimes L_b i_a$$

$$= -\eta^a ad_{\lambda_a} \eta^b ad_{\lambda_b} \otimes [i_a, L_b] - B^{ab} \otimes L_b i_a$$

$$= -f_{ab}{}^c \eta^a ad_{\lambda_a} \eta^b ad_{\lambda_b} \otimes i_c - B^{ab} \otimes L_b i_a$$

Thus we have reached the third order term:

$$[\eta^a ad_{\lambda_a} \otimes i_a, -f^{bcd} \eta_b ad_{\lambda_b^{-1}} \otimes i_c i_d - f_{bc}{}^d \eta^b ad_{\lambda_b} \eta^c ad_{\lambda_c} \otimes i_d - B^{bc} \otimes L_c i_b] \quad (78)$$

The first term gives

$$[\eta^a ad_{\lambda_a} \otimes i_a, -f^{bcd} \eta_b ad_{\lambda_b^{-1}} \otimes i_c i_d] = f^{bcd} [\eta^a ad_{\lambda_a}, \eta_b ad_{\lambda_b^{-1}}] \otimes i_a i_c i_d$$

$$= f^{bcd} \otimes i_b i_c i_d$$

The second term reads

$$[\eta^a ad_{\lambda_a} \otimes i_a, -\, f_{bc}{}^d \eta^b ad_{\lambda_b} \eta^c ad_{\lambda_c} \otimes i_d] =$$

$$= -f_{bc}{}^d [\eta^a ad_{\lambda_a}, \eta^b ad_{\lambda_b} \eta^c ad_{\lambda_c}] \otimes i_a i_d$$

$$= 2 f_b{}^{cd} \eta^b ad_{\lambda_b} \otimes i_c i_d$$

Now at fourth order

$$[\eta^a ad_{\lambda_a} \otimes i_a, f^{bcd} 1 \otimes i_b i_c i_d + 3 f_b{}^{cd} \eta^b ad_{\lambda_b} \otimes i_c i_d] \quad (79)$$

The first term vanishes since involves only commutators of i_a's, and the second one is

$$[\eta^a ad_{\lambda_a} \otimes i_a, 3 f_b{}^{cd} \eta^b ad_{\lambda_b} \otimes i_c i_d] =$$

$$= -3 f_b{}^{cd} [\eta^a ad_{\lambda_a}, \eta^b ad_{\lambda_b}] \otimes i_a i_c i_d = -3 f^{bcd} \otimes i_b i_c i_d$$

We end with the fifth order, which is zero:

$$[\eta^a ad_{\lambda_a} \otimes i_a, -3 f^{bcd} \otimes i_b i_c i_d] = 0 \quad (80)$$

If we sum up all the contributions we obtain the transformation of the differential of the Weil model under the action of the Kalkman map; we could call it the twisted noncommutative BRST differential (cfr. with (65)):

$$\begin{aligned}\phi^\chi\,\delta(\phi^\chi)^{-1} = d^W\otimes 1 + 1\otimes d - K^a ad_{\lambda_a}\otimes i_a + \eta^a ad_{\lambda_a}\otimes L_a + \\ -\frac{1}{2}B^{bc}\otimes L_c i_b + \frac{1}{24}f^{bcd}\otimes i_b i_c i_d\end{aligned} \tag{81}$$

When we restrict it to the basic subcomplex we get (75). ■

To complete our construction we have to find the multiplicative structure induced on $\mathcal{C}_G^{(\chi)}(\mathcal{B}_\chi) = (\mathfrak{U}_{(\chi)}(\mathfrak{g})\otimes\mathcal{B}_\chi)^G$ by ϕ^χ. Again this is defined by

$$(K_1\otimes\nu_1)\cdot_{\mathcal{C}_G^{(\chi)}(\mathcal{B}_\chi)}(K_2\otimes\nu_2) := \phi^\chi\left((\phi^\chi)^{-1}(K_1\otimes\nu_1)\cdot(\phi^\chi)^{-1}(K_2\otimes\nu_2)\right) \tag{82}$$

for generic elements $K_i\otimes\nu_i\in\mathcal{C}_G^{(\chi)}(\mathcal{B}_\chi)$. The explicit realization is given by the following theorem.

Theorem 3.9 *The multiplicative structure of $\mathcal{C}_G^{(\chi)}(\mathcal{B}_\chi)$ defined in (82) is*

$$(K_1\otimes\nu_1)\cdot(K_2\otimes\nu_2) = K_1K_2\otimes\cdot_{\mathcal{B}_\chi}\{exp(-\frac{1}{2}B^{jk}(L_{\lambda_k}i_j\otimes L_{\lambda_j}i_k)(\nu_1\otimes\nu_2))\} \tag{83}$$

Proof: We will show that (82) and (83) agree on generators. For $|\nu_i|=0$ this is easy. It is very long but straightforward for $|\nu_i|=1$. Starting from (82):

$$\begin{aligned}&\phi^\chi\left\{(K_a\otimes\nu_1 - \eta^b ad_{\lambda_b}K_a\otimes i_b\eta_1)\cdot(K_b\otimes\nu_2 - \eta^a ad_{\lambda_a}K_b\otimes i_a\nu_2)\right\} = \\ &= \phi^\chi\{K_aK_b\otimes\nu_1\nu_2 + K_a\eta^c ad_{\lambda_c}K_b\otimes\nu_1(i_c\nu_2) \\ &- \eta^c ad_{\lambda_c}K_aK_b\otimes(i_c\nu_1)\nu_2 + (\eta^c ad_{\lambda_c}K_a)(\eta^d ad_{\lambda_d}K_b)\otimes(i_c\nu_1)(i_d\nu_2)\}\end{aligned} \tag{84}$$

ϕ^χ acts on these generators only up to order three: $\phi^\chi = 1 + \eta^a ad_{\lambda_a}\otimes i_a - \frac{1}{2}(\eta^a ad_{\lambda_a}\eta^b ad_{\lambda_b})\otimes i_a i_b$. We compute also all the actions of the terms ad_{λ_a}'s: remember that $ad_{\lambda_a}K_b = q_{ab}K_b$ and so on. Then (84) becomes

$$\begin{aligned}&K_aK_b\otimes\nu_1\nu_2 + q_{ca}q_{ab}\eta^c K_a\eta^d K_b\otimes(i_c\nu_1)(i_d\nu_2) \\ &\quad - q_{da}q_{cd}q_{db}q_{cb}\eta^d K_a\eta^c K_b\otimes(i_d\nu_1)(L_{\lambda_d^{-1}}i_c\nu_2) \\ &\quad + q_{cd}q_{da}q_{db}q_{ca}\eta^d\eta^c K_aK_b\otimes(L_{\lambda_d}i_c\nu_1)(i_d\nu_2) \\ &\quad - \frac{1}{2}(B^{cd} - 2\eta^d ad_{\lambda_d}\eta^c ad_{\lambda_c})K_aK_b\otimes(L_{\lambda_c}i_d\nu_1)(i_c L_{\lambda_d^{-1}}\nu_2)\end{aligned} \tag{85}$$

Now we use the G-invariance to produce the missing terms needed to have an action of the twisted derivation i on products of ν_i-terms. Consider for example the third term

$$q_{da}q_{cd}q_{db}q_{cb}\eta^d K_a \eta^c K_b \otimes (i_d\nu_1)(L_{\lambda_d^{-1}} i_c \nu_2)$$

We want to have $(L_{\lambda_c} i_d \nu_1)$ instead of $(i_d\nu_1)$, and the price to pay is an $ad_{\lambda_c^{-1}}$ acting on the corresponding K_a term, which gives a q_{ca}:

$$q_{da}q_{cd}q_{db}q_{cb}q_{ca}\eta^d K_a \eta^c K_b \otimes (L_{\lambda_c} i_d\nu_1)(L_{\lambda_d^{-1}} i_c \nu_2) \tag{86}$$

Now consider the last term of (85): developing it we get

$$-\frac{1}{2}B^{cd}K_aK_b \otimes (L_{\lambda_c} i_d \nu_1)(L_{\lambda_d^{-1}} i_c \nu_2) \\ + q_{cd}q_{da}q_{db}q_{ca}q_{cb}\eta^d\eta^c K_aK_b \otimes (L_{\lambda_c} i_d\nu_1)(i_c L_{\lambda_d^{-1}} \nu_2)$$

and the last term cancels with (86). We then move all the η on the left and the K on the right ($K_b\eta^a = q_{ab}^2\eta^a K_b$) and (85) now becomes

$$\begin{aligned} &K_aK_b \otimes \nu_1\nu_2 + q_{ca}q_{db}q_{da}^2\eta^c\eta^d K_aK_b \otimes (i_c\nu_1)(i_d\nu_2) \\ &+q_{cd}q_{da}q_{db}q_{ca}\eta^d\eta^c K_aK_b \otimes (L_{\lambda_d} i_c\nu_1)(i_d\nu_2) \\ &-\frac{1}{2}B^{cd}K_aK_b \otimes (L_{\lambda_c} i_d\nu_1)(L_{\lambda_d^{-1}} i_c\nu_2) \end{aligned} \tag{87}$$

The second term may be rewritten, again using the G-invariance trick

$$\eta^c ad_{\lambda_c} K_a \otimes i_c\nu_1 = ad_{\lambda_d^{-1}}(\eta^c ad_{\lambda_c} K_a) \otimes L_{\lambda_d} i_c \nu_1$$

as

$$q_{dc}q_{ad}q_{ca}q_{db}q_{da}^2\eta^c\eta^d K_aK_b \otimes (L_{\lambda_d} i_c\nu_1)(i_d\nu_2) \tag{88}$$

while changing the order of the η's in third term we get

$$q_{cd}q_{da}q_{ab}q_{ca}(-q_{dc}^2\eta^c\eta^d + B^{cd})K_aK_b \otimes (L_{\lambda_d} i_c\nu_1)(i_d\nu_2)$$

and after the cancellation with (88) only the B^{cd} term survives. We are then left with

$$\begin{aligned} &K_aK_b \otimes \nu_1\nu_2 + B^{cd}K_aK_d \otimes (L_{\lambda_d} i_c\nu_1)(L_{\lambda_c} i_d\nu_2) \\ &-\frac{1}{2}B^{cd}K_aK_b \otimes (L_{\lambda_d} i_c\nu_1)(L_{\lambda_c^{-1}} i_d\nu_2) \end{aligned} \tag{89}$$

Now we consider the last two terms: taking all the L_λ's to the left side of the tensor product we obtain

$$B^{cd}q_{ad}(q_{bc}-\frac{1}{2}q_{cb})K_aK_b\otimes(i_c\nu_1)(i_d\nu_2)$$

but $B^{cd}(q_{bc}-\frac{1}{2}q_{cb}) = \frac{1}{2}B^{cd}q_{bc}$ (one should split the sum over positive and negative roots and use properties of B and q) and so we have exactly the rhs of (83):

$$K_aK_b\otimes\nu_1\nu_2+\frac{1}{2}B^{cd}K_aK_b\otimes(L_{\lambda_d}i_c\nu_1)(L_{\lambda_c}i_d\nu_2)$$

This finishes the proof. ■

We have now all what is needed to define the Cartan model:

Definition 3.5 *The Cartan model for the twisted noncommutative equivariant cohomology of a twisted noncommutative $\tilde{\mathfrak{g}}$-da $\mathcal{B}_\chi$ is the cohomology of the Cartan complex $\mathcal{C}_G^{(\chi)}(\mathcal{B}_\chi)$:*

$$\mathcal{H}_G(\mathcal{B}_\chi)=\big((\mathfrak{U}_\chi(\mathfrak{g})\otimes\mathcal{B}_\chi)^G,d_G\big) \tag{90}$$

The differential d_G is given in (75), while the multiplicative structure of $\mathcal{C}_G^{(\chi)}(\mathcal{B}_\chi)$ is given in (83).

Remark: The module structure of $\mathcal{H}_G(\mathcal{B}_\chi)$ is, as usual, induced from the functorial property of equivariant cohomology and from the natural homomorphism $\mathbb{F}\to\mathcal{B}_\chi$; however here the twist does not play any role, since $\mathcal{H}_G(\mathbb{F})=(\mathfrak{U}_\chi(\mathfrak{g}))^G=(\mathfrak{U}(\mathfrak{g}))^G$. ■

Conclusions

In this paper we have provided a natural framework for the study of equivariant cohomology of covariant actions on toric deformations. We would like to spend a few words on some applications and possible future works.

As discussed in the introduction, our interest on noncommutative equivariant cohomology started with the use of equivariant localization techniques in the multi-instanton calculus of supersymmetric gauge theories; considering instantons over $\mathbb{R}^4$, the resolution of the moduli space turns out to describe instantons over $\mathbb{R}^4_\theta$ [18], hence we need to generalize localization formulas to noncommutative spaces. In the specific example of Moyal plane $\mathbb{R}^4_\theta$ it seems that one can still apply classical localization formulas, since the toric action (the relevant one for the localization) on $\mathbb{R}^4_\theta$ does not suffer any twist; more on this will be maybe discussed in a forthcoming paper. However, several interesting applications in this sense may come from genuine noncommutative instantons, for example over S^4_θ [16].

Besides possible applications, the theory seems to have interest on its own. Several questions naturally arise after the definition of Weil and Cartan models; the first one is how this construction behaves under the reduction to the maximal torus. Classically one has $H_G(\mathcal{M}) \simeq H_T(\mathcal{M})^W$, where $T \subset G$ is the maximal torus and W is the Weyl group; this can be proved algebraically by spectral sequences arguments. The existence of such a reduction property for our construction seems to be a nontrivial result, since our (toric) twists does not affect equivariant cohomologies of abelian groups. A better understanding of the quantization map of [1] and its role after the twist could yield hints. Another interesting direction to investigate, again related to the maximal torus reduction, would be (possible) transformations under the twist of the small Cartan model construction [3]. Finally, it is likely that an equivariant localization formula for the general (i.e., nontrivial twist) case could be found by starting from our models.

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