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STRENGTHENED CONVERGENCE OF MARGINALS TO THE CUBIC NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We rewrite a recent derivation of the cubic non-linear Schrödinger equation by Adami, Golse, and Teta in the more natural form of the asymptotic factorisation of marginals at any fixed time and in the trace norm. This is the standard form in which the emergence of the non-linear effective dynamics of a large system of interacting bosons is proved in the literature.

1. **Introduction.** In this work we revisit the derivation of the cubic defocusing non-linear Schrödinger (CNLS) equation that provides an effective description of the dynamics of a large system of bosons coupled with a repulsive interaction.

As we are going to recall later, the emergence of such an effective dynamics is customarily controlled in a natural topology for the reduced density matrix of the system.

Instead, in the recent work [1] by Adami, Golse, and Teta, fluctuations from the effective dynamics are controlled in rather indirect sense, averaged in time, and it is not apriori clear that their result can be restated in the standard form.

The present work answers this question affirmatively. We show that it is possible to rewrite the conclusion of [1] as the convergence, in the trace class norm at any fixed time, of the one-body marginal to the projection onto the limiting dynamics, i.e., onto the solution of the CNLS equation.

This is possible thanks to the strong apriori estimate that can be deduced from the assumptions of [1] (see also Remark 2). The role of such an apriori estimate is to prove regularity in time of the fluctuations from the limiting dynamics, which is at the basis of the rigorous derivation of the CNLS by Adami, Golse, and Teta. In this work we use the regularity in time to localise *at any fixed time* the convergence result given in [1].

The core of this work are Sections 2 and 3. In Section 2 we revisit the main result of [1] and the scheme of the derivation of the CNLS equation. In Section 3 we state, prove, and comment our results. Further remarks and conclusions are discussed in Section 4.

2. **A derivation of the CNLS equation in dimension one.** In this Section we revisit the main result of [1] and the scheme of the derivation of the CNLS equation.

We consider the collection $\{\mathcal{S}_N\}_{N=2}^{\infty}$ of quantum mechanical systems described as follows. Each $\mathcal{S}_N$ consists of N indistinguishable spinless nonrelativistic bosons

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of mass m in one space dimension. Particles interact via a two-body repulsive potential V_N given by

$$V_N(x) \;=\; N^{\beta-1}V(N^\beta x) \tag{1}$$

where $\beta \in (0,1)$ is a fixed parameter and V is a non-negative and even function in the Schwartz class $\mathcal{S}(\mathbb{R})$. Thus, the system $\mathcal{S}_N$ is governed by the Hamiltonian

$$H_N = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x^2} + \sum_{1\leqslant i<j\leqslant N} N^{\beta-1}V\big(N^\beta(x_i-x_j)\big) \tag{2}$$

acting on $L^2(\mathbb{R}^N)$. Particles being bosons, the physical Hilbert space is the bosonic sector $L^2_{\mathrm{sym}}(\mathbb{R}^N)$ only, that is, the closed subspace of $L^2(\mathbb{R}^N)$ consisting of functions that are symmetric under permutation of variables.

It is assumed that at time $t = 0$ the system $\mathcal{S}_N$ is prepared in a pure state Φ_N with no inter-particle correlations, i.e.,

$$\Phi_N(x_1,\dots,x_N) = \varphi(x_1)\cdots\varphi(x_N) \tag{3}$$

for some one-body wave function (or one-body 'orbital') φ. For the r.h.s. of (3) we will also use the notation $\varphi^{\otimes N}$. Note that φ is a fixed given function in $L^2(\mathbb{R})$, it does not scale with N. Hence, the initial state of each system $\mathcal{S}_N$ has the same spatial size, independent of N: the $\mathcal{S}_N$'s are more and more populated but their size is always 'of order one' in N.

Here the interest is in the time evolution of Φ_N under the dynamics generated by (2), that is, the state $\Phi_{N,t} := e^{-iH_N t}\Phi_N$ which is the unique solution of the N-body Schrödinger equation

$$i\partial_t \Phi_{N,t} \;=\; H_N \Phi_{N,t} \tag{4}$$

with the initial condition $\Phi_{N,t}\big|_{t=0} \equiv \Phi_N$. Due to the symmetry of H_N, $\Phi_{N,t}$ too is symmetric under permutation of variables, but of course the initial factorisation of Φ_N is lost at later times due to the interaction term in H_N.

On the other hand, one notes that $NV_N(x) \to \|V\|_1\delta(x)$ in $\mathcal{S}'(\mathbb{R})$ as $N \to \infty$. Thus, at large N the interaction V_N essentially behaves as $1/N$ times a δ-function which is effective at the spatial scale $N^{-\beta}$. In other words, the larger N the more the particles in $\mathcal{S}_N$ experience a short-scale repulsion, being essentially free whenever their separation exceeds the very short range of the interaction. At large N's this mimics the typical experimental regime of cold atom systems, where a large number of bosons (of the order of 10^2 to 10^{10}) is prepared at very low temperature and high dilution, [7]. In this regime it is reasonable to expect that the correlations certainly forming in $\Phi_{N,t}$ at later times do not affect too much the initial factorisation. An almost factorised structure $\Phi_{N,t}(x_1,\dots,x_N) \approx \varphi_t(x_1)\cdots\varphi_t(x_N)$ is therefore expected to persists, at least in some weak sense.

The factorisation in Φ_N is sometimes referred to as the 'chaos property' for Φ_N. In a more physical context, this condition is a particular case of 'Bose-Einstein condensation' (BEC). The persistence of an almost factorised structure $\Phi_{N,t}(x_1,\dots,x_N) \approx \varphi_t(x_1)\cdots\varphi_t(x_N)$ at later times goes under the name of 'propagation of chaos' or 'stability of condensation', depending on the context.

BEC is a regime where the particles of the many-body bosonic system behave as if they all were the same φ. Here the condensation is assumed to be exact at time $t = 0$. In the general case it still has the meaning of an approximate factorisation, i.e., absence of interparticle correlations, but in the weaker sense briefly discussed here below (see [9] for a general discussion).

First of all, the notion of marginal is needed. Given $\Psi_N \in L^2(\mathbb{R}^N)$, we denote by $\gamma_N^{(k)}$ the so-called k-body reduced density matrix (or marginal) associated with Ψ_N. It is a density matrix (i.e., a positive trace class operator with trace equal to one) acting on $L^2(\mathbb{R}^k)$, whose integral kernel is given by

$$\begin{aligned}&\gamma_N^{(k)}(x_1,\dots x_k;y_1,\dots,y_k) = \\ &= \int_{\mathbb{R}^{N-k}} \Psi_N(x_1,\dots,x_k,z_{k+1},\dots,z_N)\overline{\Psi_N(y_1,\dots,y_k,z_{k+1},\dots,z_N)}\,\mathrm{d}z_{k+1}\cdots\mathrm{d}z_N\,.\end{aligned} \tag{5}$$

This means that $\gamma_N^{(k)}$ is obtained by 'tracing out' $N-k$ degrees of freedom from the density matrix of the system $\mathcal{S}_N$, the projection $|\Psi_N\rangle\langle\Psi_N|$ (in the Dirac notation). Ψ_N being permutation symmetric, it is irrelevant which $N-k$ variables one integrates out, and the kernel of $\gamma_N^{(k)}$ is permutation symmetric in each set of variables. Also, it is clearly seen that $\mathrm{Tr}\gamma_N^{(k)} = \|\Psi_N\|_{L^2(\mathbb{R}^N)}$.

The k-marginal associated with Ψ_N is all what is needed to evaluate the expectation value in the state Φ_N of an observable on the N-body system which is not trivial on k-particles only. In particular, the k-marginal of the completely factorised state (3) has exactly the form $|\varphi^{\otimes k}\rangle\langle\varphi^{\otimes k}|$, the projection onto the product state of k copies of the same φ. Thus, if $\gamma_N^{(k)}$ is close in some sense to some $|\psi^{\otimes k}\rangle\langle\psi^{\otimes k}|$, this has the natural interpretation that the k-particle subsystem behaves almost as if it were an isolated system in the product state $\psi^{\otimes k}$. Furthermore, the condition $\gamma_N^{(k)} \approx |\psi^{\otimes k}\rangle\langle\psi^{\otimes k}|$ for *any* fixed k provides a reasonable and natural way of expressing the quasi-factorisation of Ψ_N.

Of course the fact that Ψ_N is almost factorised at the level of all marginals is a far weaker condition than the closeness of the two N-body states Ψ_N and $\psi^{\otimes N}$. On the other hand, measuring how far a given Ψ_N is from a product state $\psi^{\otimes N}$ using the norm distance in $L^2(\mathbb{R}^N)$ is not useful in practice, when one is interested in very large N's. It suffices that one particle out of N is in an orbital $\psi^\perp$ orthogonal to ψ to make the vectors $\psi^{\otimes N}$ and $(\psi^\perp\otimes\psi^{\otimes(N-1)})_{\mathrm{sym}}$ orthogonal in $L^2(\mathbb{R}^N)$. (Here the subscript 'sym' denotes the obvious symmetrisation of the vector.) In this sense the L^2-norm is a too strong and detailed control on the many-body wave-function.

We are thus concerned with comparing $\gamma_{N,t}^{(k)}$, the k-marginal associated with $\Phi_{N,t} = e^{-iH_N t}\Phi_N$, with some $|\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|$. We also want to identify the law for the time evolution of φ_t. Both $\gamma_{N,t}^{(k)}$ and $|\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|$ being density matrices on $L^k(\mathbb{R}^N)$, it is natural to study their distance in the trace norm, $\mathrm{Tr}\big|\gamma_{N,t}^{(k)} - |\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|\big|$. Also, as a consequence of the permutation symmetry, if the above distance is small for some k then it is small at any other level k' in the following quantitative sense:

$$\mathrm{Tr}\big|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\big| \;\leqslant\; \mathrm{Tr}\big|\gamma_{N,t}^{(k)} - |\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|\big| \;\leqslant\; \sqrt{k}\,\sqrt{8\,\mathrm{Tr}\big|\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|\big|} \tag{6}$$

(see the remark after Theorem 1 in [8] and Theorem 7.1.1 in [10] for details). Therefore it is enough to focus on the level $k=1$ only.

An analytic or numerical knowledge of the solution $\Phi_{N,t}$ of the Schrödinger equation $i\partial_t\Phi_{N,t} = H_N\Phi_{N,t}$, and consequently of the marginals of $\Phi_{N,t}$, is definitely out of reach when N is large. The reason why one considers the entire collection of "similar" systems $\{\mathcal{S}_N\}_{N=2}^\infty$ is to obtain some asymptotics in the limit $N\to\infty$ (this approach is discussed in some more detail in [11]).

The last ingredient to state Theorem 2.1 is to introduce a convenient Sobolev space for density matrices. Let us denote by $\mathcal{L}^1$ and $\mathcal{L}^2$ the spaces of trace class and of Hilbert-Schmidt operators on $L^2(\mathbb{R})$ respectively, [13]. Recall that $\mathcal{L}^1$ is a complex Banach space with the norm

$$\|A\|_{\mathcal{L}^1} \;=\; \mathrm{Tr}|A|\,,$$

$\mathcal{L}^2$ is a complex Hilbert space with the scalar product

$$(A,B)_{\mathcal{L}^2} \;=\; \mathrm{Tr}(A^*B)\,.$$

If $\gamma(x,y)$ is the kernel of a Hilbert-Schmidt operator γ then

$$\|\gamma\|^2_{\mathcal{L}^2} \;=\; \int_{\mathbb{R}\times\mathbb{R}} |\gamma(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y\,. \tag{7}$$

Inside $\mathcal{L}^2$ we identify the subspace

$$\mathcal{L}^2_{\mathrm{reg}} \;:=\; \big\{\gamma\in\mathcal{L}^2 \;:\; S\gamma S\in\mathcal{L}^2)\big\} \tag{8}$$

where $S:=\sqrt{1-\frac{\mathrm{d}^2}{\mathrm{d}x^2}}$. $\mathcal{L}^2_{\mathrm{reg}}$ too is a Hilbert space with the scalar product

$$(A,B)_{\mathcal{L}^2_{\mathrm{reg}}} \;=\; \big((SAS),(SBS)\big)_{\mathcal{L}^2} \;=\; \mathrm{Tr}\big((SA^*S)(SBS)\big) \tag{9}$$

(note that $S=S^*$). If $\gamma(x,y)$ is the kernel of an operator $\gamma\in\mathcal{L}^2_{\mathrm{reg}}$ then

$$\|\gamma\|^2_{\mathcal{L}^2_{\mathrm{reg}}} \;=\; \int_{\mathbb{R}\times\mathbb{R}} \Big|\Big(1-\frac{\partial^2}{\partial x^2}\Big)\Big(1-\frac{\partial^2}{\partial y^2}\Big)\,\gamma(x,y)\Big|^2\,\mathrm{d}x\,\mathrm{d}y\,. \tag{10}$$

It is in the framework discussed so far that Adami, Golse, and Teta proved the following.

Theorem 2.1 (AGT 2006, [1])**.** *Consider the Hamiltonian H_N in (2), the initial state (3) with $\varphi\in H^1(\mathbb{R})$ and $\|\varphi\|_{L^2(\mathbb{R})}=1$, its time evolution $\Phi_{N,t}:=e^{-iH_Nt}\Phi_N$, and the associated marginal $\gamma^{(1)}_{N,t}$. Let $b_1:=\|V\|_{L^1(\mathbb{R})}$. Assume further that Φ_N is such that any k-th moment of the energy per particle is uniformly bounded in N, i.e.,*

$$\exists\,\mathcal{E}>0\;:\;\forall k\geqslant 1\;\;\forall N\geqslant 2\quad\Big\langle\,\Phi_N,\Big(\frac{H_N}{N}\Big)^k\Phi_N\Big\rangle_{L^2(\mathbb{R}^N)}\;\leqslant\;\mathcal{E}^k\,. \tag{11}$$

Then

$$\gamma^{(1)}_{N,\,\cdot}\longrightarrow|\varphi_{\,\cdot}\rangle\langle\varphi_{\,\cdot}|\qquad\text{as }N\to\infty \tag{12}$$

weakly-$*$ in $L^\infty(\mathbb{R},\mathcal{L}^2_{\mathrm{reg}})$ where $\varphi_t\in H^1(\mathbb{R})$ is the solution of the initial value problem

$$\left\{\begin{array}{rcl} i\hbar\partial_t\varphi_t &=& -\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2}\varphi_t+b_1|\varphi_t|^2\varphi_t \\[2ex] \varphi_t\big|_{t=0} &\equiv& \varphi\,. \end{array}\right. \tag{13}$$

Remark 1. Convergence (12) in AGT theorem reads

$$\begin{gathered}\lim_{N\to\infty}\int_{-\infty}^{+\infty}\mathrm{d}t\,\big(\,\rho_t\,,\,\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big)_{\mathcal{L}^2_{\mathrm{reg}}}=0\\ \forall\,\rho_{\,\cdot}\in L^1(\mathbb{R},\mathcal{L}^2_{\mathrm{reg}})\,.\end{gathered} \tag{14}$$

Recall that

$$\begin{aligned}&\big(\,\rho_t\,,\,\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big)_{\mathcal{L}^2_{\mathrm{reg}}}\\ &\quad=\int_{\mathbb{R}\times\mathbb{R}}\mathrm{d}x\,\mathrm{d}y\,\rho_t(y,x)\Big(1-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big)\Big(1-\frac{\mathrm{d}^2}{\mathrm{d}y^2}\Big)\big(\gamma^{(1)}_{N,t}(x,y)-\varphi_t(x)\,\overline{\varphi_t(y)}\big)\,.\end{aligned}$$

Remark 2. It was Elgart, Erdős, Schlein, and Yau in [4] who first introduced the growth assumption (11) and recognised that it yields a key apriori estimate on $\gamma^{(1)}_{N,t}$, i.e., the fact that the sequence $\{\gamma^{(1)}_{N,t}\}_{N=2}^{\infty}$ is N-uniformly bounded in a convenient regularity (Sobolev) space for density matrices. We have already introduced the regularity space $\mathcal{L}^2_{\mathrm{reg}}$ for one-body density matrix. The generalisation to the regularity space $\mathcal{L}^2_{\mathrm{reg},k}$ for k-body density matrices is obvious (see [1] for details), and it follows from the growth assumption (11) that

$$\|\gamma^{(k)}_{N,t}\|_{\mathcal{L}^2_{\mathrm{reg},k}} \;<\; \frac{\langle \Phi_N, H_N^k \Phi_N\rangle}{N^k} \tag{15}$$

for all times t, all sufficiently large N, and all integers $k = 1, \dots, N$.

Remark 3. A necessary and sufficient condition on φ for the product state $\Phi_N = \varphi^{\otimes N}$ to fulfil (11) is unknown in general. If $\beta \in (0, \frac{1}{2})$ Adami, Golse, and Teta proved that it suffices φ to have a compactly supported Fourier transform. For general $\beta \in (0,1)$ an approximation procedure is needed, introduced first by Erdős, Schlein, and Yau in [5]: a high energy cut-off is taken in $\varphi^{\otimes N}$ so to get a smoothed state satisfying the growth condition (11), then the theorem is proved for such a smoothed initial state, and finally the cut-off is removed leaving the conclusion (12) unchanged.

AGT Theorem 2.1 is one among a number of analogous (and mostly recent) results for a variety of similar many-body models. Such results involve different settings (non-relativistic, semi-relativistic), dimensions (one, two, and three), more singular interactions among particles, different scalings in the interaction, and different techniques for the proof. The general structure of each of them is the diagram

$$\begin{array}{ccccc}
\Phi_N & \xrightarrow{\text{partial trace}} & \gamma^{(k)}_N & \xrightarrow{N\to\infty} & |\varphi\rangle\langle\varphi|^{\otimes k} \\
{\scriptstyle \text{many-body linear dynamics}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \text{NLS equation}} \\
\Phi_{N,t} & \xrightarrow{\text{partial trace}} & \gamma^{(k)}_{N,t} & \xrightarrow{N\to\infty} & |\varphi_t\rangle\langle\varphi_t|^{\otimes k}
\end{array} \tag{16}$$

We refer to the reviews [15, 16] and to the references therein for a general discussion, as well as to the more recent approach [6].

What we want to emphasise here is that typically the asymptotic factorisation of marginals is *not* proved in the form (12). It has instead the more natural form

$$\lim_{N\to\infty} \operatorname{Tr}\big|\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\big| \;=\; 0 \tag{17}$$

at any fixed time t (recall the relation (6)). In some cases (17) can be supplemented with a quantitative rate in N of the vanishing of the trace norm, as well as with a control of how the estimate deteriorates in t. Sometimes other indicators of convergence are used, they are in any case equivalent to the control in trace norm, [12].

It is then natural and interesting to ask whether AGT convergence (12) can be put in the standard form (17). In the next section we answer positively to this question.

To conclude this discussion on AGT Theorem, we present another result proved in [1]. We put it in evidence since we use it in the proof of our main result.

Lemma 2.2. *Assume the hypotheses of Theorem 2.1, i.e., the Hamiltonian* H_N *defined in* (2)*, the initial state* (3) *with* $\varphi \in H^1(\mathbb{R})$ *and* $\|\varphi\|_{L^2(\mathbb{R})} = 1$*, the growth condition* (11)*, and the time evolution* $\Phi_{N,t} := e^{-iH_N t}\Phi_N$*. Define*

$$\widetilde{\sigma}_{N,t}(x,y) \;:=\; \int_{\mathbb{R}} V(x-z)\,\gamma^{(2)}_{N,t}(x,z;y,z)\,\mathrm{d}z \tag{18}$$

where $\gamma^{(2)}_{N,t}$ *is the two-body marginal associated with* $\Phi_{N,t}$*. Then, for all* t *and for all sufficiently large* N*,* $\widetilde{\sigma}_{N,t}(x,y)$ *is the kernel of an operator* $\widetilde{\sigma}_{N,t} \in \mathcal{L}^2_{\mathrm{reg}}$ *with*

$$\|\widetilde{\sigma}_{N,t}\|_{\mathcal{L}^2_{\mathrm{reg}}} \;<\; 4\,\|V\|_{L^1(\mathbb{R})}\,\frac{\langle \Phi_N, H_N^2\Phi_N\rangle}{N^2}\,. \tag{19}$$

As a consequence, it follows by scaling that for all t *and for all sufficiently large* N

$$\sigma_{N,t}(x,y) \;:=\; (N-1)\int_{\mathbb{R}} \big[V_N(x-z) - V_N(y-z)\big]\,\gamma^{(2)}_{N,t}(x,z;y,z)\,\mathrm{d}z \tag{20}$$

is the kernel of an operator $\sigma_{N,t} \in \mathcal{L}^2_{\mathrm{reg}}$ *with*

$$\|\sigma_{N,t}\|_{\mathcal{L}^2_{\mathrm{reg}}} \;<\; 8\,\|V\|_{L^1(\mathbb{R})}\,\frac{\langle \Phi_N, H_N^2\Phi_N\rangle}{N^2}\,. \tag{21}$$

Proof. Proposition 2.4, Remark 2.5., Proposition 2.1, and Remark 2.2 of [1]. □

Remark 4. In [1], estimate (19) is used to show that there is a unique solution, precisely $\{|\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|\}_{k=1}^{\infty}$, of the infinite hierarchy of evolutionary equations satisfied by the limit of $\{\gamma^{(k)}_{N,t}\}_{k=1}^{N}$ as $N \to \infty$. This is done by means of an abstract version of the Cauchy-Kowalewski theorem by Nirenberg and Nisida (see Section 4 of [1] and Section 3 of [2] for details). It is also possible to show uniqueness in a direct way, due to the regularity of the interaction V, by expanding the solution of the infinite hierarchy in its Duhamel series and by controlling it term by term (see the discussion in Remarks 4.3 and 4.4 of [1]). Thus, strictly speaking, the bound (19) is not even necessary to prove Theorem 2.1.

Remark 5. We are going to use estimate (19) to obtain regularity in time for $\gamma^{(1)}_{N,t}$ from the regularity in space given by the apriori estimate (15). The link is given by the Schrödinger equation $i\partial_t\Phi_{N,t} = H_N\Phi_{N,t}$, more precisely by

$$\begin{aligned} i\hbar\,\partial_t\gamma^{(1)}_{N,t}(x,y) \;=\; & \frac{\hbar^2}{2m}(-\partial_x^2 + \partial_y^2)\,\gamma^{(1)}_{N,t}(x,y) + \\ & + (N-1)\int_{\mathbb{R}} \big[V_N(x-z) - V_N(y-z)\big]\,\gamma^{(2)}_{N,t}(x,z;y,z)\,\mathrm{d}z\,. \end{aligned} \tag{22}$$

This is the Schrödinger equation itself written after taking the partial trace (5). We mention that (22) is the first equation of the celebrated BBGKY hierarchy ([15]), a hierarchy of equations coupling $\partial_t\gamma^{(k)}_{N,t}$ with $\gamma^{(k+1)}_{N,t}$, $k = 1,\dots,N-1$, which is just a different form of the Schrödinger equation for $\Phi_{N,t}$.

Remark 6. While regularity of $\gamma^{(1)}_{N,t}$ follows from the growth assumption (11), regularity for its limit point $|\varphi_t\rangle\langle\varphi_t|$ is a consequence of the global well-posedness of the initial value problem (13). For future reference ([3]), let us recall that given $u_0 \in H^1(\mathbb{R})$ and $\alpha \in \mathbb{R}$, there exists a unique $\varphi_\cdot \in C(\mathbb{R}, H^1(\mathbb{R})) \cup C^1(\mathbb{R}, H^{-1}(\mathbb{R}))$ satisfying the CNLS $i\partial_t\varphi_t = -\frac{\partial^2}{\partial x^2}\varphi_t + \alpha|\varphi_t|^2\varphi_t$ in $H^{-1}(\mathbb{R})$ for all $t \in \mathbb{R}$, and such that $\varphi_t\big|_{t=0} = u_0$. Moreover, the L^2-norm is conserved and the H^1-norm is uniformly bounded in time for such a solution:

$$\|\varphi_t\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}\,, \qquad \|\varphi_t\|_{H^1(\mathbb{R})} \leqslant C \qquad \forall t \in \mathbb{R}\,. \tag{23}$$

3. **Strengthened convergence for reduced density matrices.** In this section we state and prove our main results and we make a number of comments on them. Here is our main theorem:

Theorem 3.1. *Consider the Hamiltonian H_N defined by (2) and the initial state (3) with $\varphi \in H^1(\mathbb{R})$ and $\|\varphi\|_{L^2(\mathbb{R})} = 1$. Assume the growth condition (11) for the initial state. Consider the Schrödinger evolution $\Phi_{N,t} := e^{-iH_N t}\Phi_N$ and let $\gamma^{(1)}_{N,t}$ be the associated one-body reduced density matrix. Then*

$$\lim_{N\to\infty} \mathrm{Tr}\,\big|\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\big| \;=\; 0 \tag{24}$$

at any fixed $t \in \mathbb{R}$.

As an intermediate step towards Theorem 3.1, we need to prove the following.

Lemma 3.2. *For any $t \in \mathbb{R}$, let $\{\gamma_{N,t}\}_{N=1}^{\infty}$ be a sequence of density matrices acting on $L^2(\mathbb{R})$ and φ_t be a function in $L^2(\mathbb{R})$ with $\|\varphi_t\|_{L^2(\mathbb{R})} = 1$. Assume further that*

$$\begin{gathered}\lim_{N\to\infty} \int_{-\infty}^{+\infty} \mathrm{d}t\, \big(\,\rho_t\,,\, \gamma_{N,t} - |\varphi_t\rangle\langle\varphi_t|\,\big)_{\mathcal{L}^2_{\mathrm{reg}}} = 0 \\ \forall\, \rho_{\cdot} \in L^1(\mathbb{R}, \mathcal{L}^2_{\mathrm{reg}})\,.\end{gathered} \tag{25}$$

Then, $\forall\, f \in L^1(\mathbb{R})$,

$$\lim_{N\to\infty} \int_{-\infty}^{+\infty} \mathrm{d}t\, f(t)\, \big\|\,\gamma_{N,t} - |\varphi_t\rangle\langle\varphi_t|\,\big\|^2_{\mathcal{L}^2} = 0 \tag{26}$$

and also

$$\lim_{N\to\infty} \int_{-\infty}^{+\infty} \mathrm{d}t\, f(t)\, \big\|\,\gamma_{N,t} - |\varphi_t\rangle\langle\varphi_t|\,\big\|_{\mathcal{L}^1} = 0\,. \tag{27}$$

Both (26), (27), and (24) provide a strengthened version of AGT convergence (25). Lemma 3.2 is an abstract result: it does not require $\gamma_{N,t}$ to be the one-body marginal of a many-body function Φ_N. Instead, trace norm convergence (25) is a consequence of the growth condition (11) on the initial state and of the apriori estimate of Lemma 2.2.

Proof of Lemma 3.2. It is enough to take $f \geqslant 0$ and $\|f\|_{L^1(\mathbb{R})} = 1$: indeed, once (26) and (27) are proved under this restriction, the general case follows writing any non zero $f \in L^1(\mathbb{R})$ as the linear combination

$$f = \|f_+\|_{L^1(\mathbb{R})} \frac{f_+}{\|f_+\|_{L^1(\mathbb{R})}} - \|f_-\|_{L^1(\mathbb{R})} \frac{f_-}{\|f_-\|_{L^1(\mathbb{R})}} \tag{28}$$

where $f_\pm(x) := \max\{\pm f(x), 0\}$. Define

$$\rho_t := f(t) S^{-2} |\varphi_t\rangle\langle\varphi_t| S^{-2}\,. \tag{29}$$

One has

$$\begin{aligned}\big\| S^{-2}|\varphi_t\rangle\langle\varphi_t|S^{-2}\big\|^2_{\mathcal{L}^2_{\mathrm{reg}}} \;&=\; \mathrm{Tr}\big[\,\big(SS^{-2}|\varphi_t\rangle\langle\varphi_t|S^{-2}S\big)^*\big(SS^{-2}|\varphi_t\rangle\langle\varphi_t|S^{-2}S\big)\,\big]\\ &=\; \big(\,\varphi_t\,,\,S^{-2}\varphi_t\,\big)^2_{L^2(\mathbb{R})} \;=\; \Big(\int_{-\infty}^{+\infty}\frac{1}{1+4\pi^2k^2}\,\big|\widehat{\varphi}_t(k)\big|^2\mathrm{d}k\Big)^2\\ &\leqslant\; \|\widehat{\varphi}_t\|^4_{L^2(\mathbb{R})}=\|\varphi_t\|^4_{L^2(\mathbb{R})}=1\end{aligned} \tag{30}$$

since L^2-norm of φ_t is conserved, (23). Then

$$\int_{-\infty}^{+\infty}\mathrm{d}t\,\big\|\,\rho_t\,\big\|_{\mathcal{L}^2_{\mathrm{reg}}}=\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\big\|\,S^{-2}|\varphi_t\rangle\langle\varphi_t|S^{-2}\,\big\|_{\mathcal{L}^2_{\mathrm{reg}}}\leqslant\;1\,, \tag{31}$$

that is, $\rho_{\cdot}\in L^1(\mathbb{R},\mathcal{L}^2_{\mathrm{reg}})$. Moreover,

$$\begin{aligned}\big(\,\rho_t\,,\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big)_{\mathcal{L}^2_{\mathrm{reg}}}\;&=\;\mathrm{Tr}\Big[\big(S\rho_tS\big)^*\big(S(\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|)S\big)\Big]\\ &=\;f(t)\,\mathrm{Tr}\big[\,|\varphi_t\rangle\langle\varphi_t|\,(\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|)\,\big]\,.\end{aligned} \tag{32}$$

Using that

$$\begin{aligned}\big\|\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big\|^2_{\mathcal{L}^2}\;&=\;\mathrm{Tr}\big[\,(\gamma_{N,t})^2-\gamma_{N,t}|\varphi_t\rangle\langle\varphi_t|-|\varphi_t\rangle\langle\varphi_t|\gamma_{N,t}+|\varphi_t\rangle\langle\varphi_t|\,\big]\\ &\leqslant\;2\,\mathrm{Tr}\big[\,|\varphi_t\rangle\langle\varphi_t|\,(\,|\varphi_t\rangle\langle\varphi_t|-\gamma_{N,t})\,\big]\,,\end{aligned} \tag{33}$$

one concludes that

$$\begin{aligned}\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\,\big\|\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big\|^2_{\mathcal{L}^2}\;&\leqslant 2\int_{-\infty}^{+\infty}\mathrm{d}t\,\big(\,\rho_t\,,\,|\varphi_t\rangle\langle\varphi_t|-\gamma_{N,t}\,\big)_{\mathcal{L}^2_{\mathrm{reg}}}\\ &\xrightarrow{N\to\infty}\;0\end{aligned} \tag{34}$$

by the assumption (25). This proves (26). To prove (27) we first split

$$\begin{aligned}\big\|\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big\|_{\mathcal{L}^1}\;&=\;\big\|\,\big(\,|\varphi_t\rangle\langle\varphi_t|+(\,\mathbb{1}-|\varphi_t\rangle\langle\varphi_t|\,)\big)\,(\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,)\,\big\|_{\mathcal{L}^1}\\ &\leqslant\;\big\|\,|\varphi_t\rangle\langle\varphi_t|\,(\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,)\,\big\|_{\mathcal{L}^1}+\big\|\,(\,\mathbb{1}-|\varphi_t\rangle\langle\varphi_t|\,)\,\gamma_{N,t}\,\big\|_{\mathcal{L}^1}\,.\end{aligned} \tag{35}$$

The first term in the r.h.s. of (35) is the trace norm of a rank-1 operator: then it is the same as its Hilbert-Schmidt norm. Thus,

$$\begin{aligned}&\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\,\big\|\,|\varphi_t\rangle\langle\varphi_t|\,(\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,)\,\big\|_{\mathcal{L}^1}=\\ &\qquad=\;\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\,\big\|\,|\varphi_t\rangle\langle\varphi_t|\,(\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,)\,\big\|_{\mathcal{L}^2}\\ &\qquad\leqslant\;\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\,\big\|\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big\|_{\mathcal{L}^2}\\ &\qquad=\;\int_{-\infty}^{+\infty}\mathrm{d}t\,\sqrt{f(t)}\cdot\sqrt{f(t)}\,\big\|\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big\|_{\mathcal{L}^2}\\ &\qquad\leqslant\;\Big(\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\Big)^{1/2}\Big(\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\,\big\|\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big\|^2_{\mathcal{L}^2}\Big)^{1/2}\\ &\qquad=\;\Big(\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\,\big\|\,\gamma_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big\|^2_{\mathcal{L}^2}\Big)^{1/2}\;\xrightarrow{N\to\infty}\;0\end{aligned} \tag{36}$$

by (26). The second term in the r.h.s. of (35) yields

$$
\begin{aligned}
\big\|\,(\,\mathbb{1}-|\varphi_t\rangle\langle\varphi_t|\,)\,\gamma_{N,t}\,\big\|_{\mathcal{L}^1} \;&\leqslant\; \big\|\,(\,\mathbb{1}-|\varphi_t\rangle\langle\varphi_t|\,)\,(\gamma_{N,t})^{1/2}\,\big\|_{\mathcal{L}^2}\cdot\big\|\,(\gamma_{N,t})^{1/2}\,\big\|_{\mathcal{L}^2} \\
&=\;\Big(\,\mathrm{Tr}\big[\,(\,\mathbb{1}-|\varphi_t\rangle\langle\varphi_t|\,)\,\gamma_{N,t}\,(\,\mathbb{1}-|\varphi_t\rangle\langle\varphi_t|\,)\,\big]\,\Big)^{1/2} \\
&=\;\Big(\,\mathrm{Tr}\big[\,|\varphi_t\rangle\langle\varphi_t|\,(\,|\varphi_t\rangle\langle\varphi_t|-\gamma_{N,t})\,\big]\,\Big)^{1/2}.
\end{aligned}
\tag{37}
$$

Note that the inequality above is just the $\mathcal{L}^1$-$\mathcal{L}^2$ Hölder inequality in the form $\|AB\|_{\mathcal{L}^1}\leqslant\|\,|A|^{1/2}\|_{\mathcal{L}^2}\|\,|A|^{1/2}B\|_{\mathcal{L}^2}$ (see, e.g., Chapter 2 in [17] for details), where in this case $A=\gamma_{N,t}\geqslant 0$ and $B=\mathbb{1}-|\varphi_t\rangle\langle\varphi_t|$. Then

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\,\big\|\,(\,\mathbb{1}-|\varphi_t\rangle\langle\varphi_t|\,)\,\gamma_{N,t}\,\big\|_{\mathcal{L}^1} \\
&\leqslant\;\Big(\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\Big)^{1/2}\Big(\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\,\mathrm{Tr}\big[\,|\varphi_t\rangle\langle\varphi_t|\,(\,|\varphi_t\rangle\langle\varphi_t|-\gamma_{N,t})\,\big]\Big)^{1/2} \\
&=\;\bigg(\int_{-\infty}^{+\infty}\mathrm{d}t\,\big(\,\rho_t\,,\,|\varphi_t\rangle\langle\varphi_t|-\gamma_{N,t}\,\big)_{\mathcal{L}^2_{\mathrm{reg}}}\bigg)^{1/2}\;\xrightarrow{N\to\infty}\;0
\end{aligned}
\tag{38}
$$

by (32) and by assumption (25). Now (27) follows from (35), (36), and (38). □

Remark 7. The only part of Lemma 3.2 that will actually enter our proof of Theorem 3.1 is (26). The reason is that the Hilbert-Schmidt norm is easy to handle in terms of kernels, see (7), which is not at all the case for the trace norm. We shall prove that (26) implies $\|\gamma_{N,t}^{(1)}-|\varphi_t\rangle\langle\varphi_t|\|_{\mathcal{L}^2}\to 0$ as $N\to\infty$, at any fixed time. Although in general $\|\cdot\|_{\mathcal{L}^2}\leqslant\|\cdot\|_{\mathcal{L}^1}$, in this case it is possible to lift the Hilbert-Schmidt to the trace norm convergence thanks to the following inequality:

$$
\|\gamma_{N,t}^{(1)}-|\varphi_t\rangle\langle\varphi_t|\|_{\mathcal{L}^1}\;\leqslant\;2\,\|\gamma_{N,t}^{(1)}-|\varphi_t\rangle\langle\varphi_t|\|_{\mathcal{L}^2}\,. \tag{39}
$$

Bounds like (39) hold at various levels of generality (see Chapter 2 in [17]). For a simple explanation in this case (see Remark 1.4 in [14]) note first that $|\varphi_t\rangle\langle\varphi_t|$ is a rank one projection, so that $\gamma_{N,t}^{(1)}-|\varphi_t\rangle\langle\varphi_t|$ can only have one negative eigenvalue λ_-. Moreover, from $\mathrm{Tr}\big(\gamma_{N,t}^{(1)}-|\varphi_t\rangle\langle\varphi_t|\big)=0$ it follows that $|\lambda_-|$ equals the sum of all positive eigenvalues. Then

$$
\|\gamma_{N,t}^{(1)}-|\varphi_t\rangle\langle\varphi_t|\|_{\mathcal{L}^1}\;=\;2|\lambda_-|\;=\;2\|\gamma_{N,t}^{(1)}-|\varphi_t\rangle\langle\varphi_t|\|_{\mathrm{op}}\;\leqslant\;2\|\gamma_{N,t}^{(1)}-|\varphi_t\rangle\langle\varphi_t|\|_{\mathcal{L}^2}
$$

($\|\cdot\|_{\mathrm{op}}$ denoting the operator norm), whence (39).

Proof of Theorem 3.1. By Remark 7, it is enough to prove, at any fixed time, that $G_N(t):=\|\gamma_{N,t}^{(1)}-|\varphi_t\rangle\langle\varphi_t|\|^2_{\mathcal{L}^2}\to 0$ as $N\to\infty$. We know from Lemma 3.2 that

$$
\begin{aligned}
0\;&=\;\lim_{N\to\infty}\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\,\big\|\,\gamma_{N,t}^{(1)}-|\varphi_t\rangle\langle\varphi_t|\,\big\|^2_{\mathcal{L}^2} \\
&=\;\lim_{N\to\infty}\int_{-\infty}^{+\infty}\mathrm{d}t\,f(t)\,G_N(t)\,,\qquad\forall\,f\in L^1(\mathbb{R})\,.
\end{aligned}
\tag{40}
$$

Let $\tilde{t}$ be any fixed time, and $I\ni\tilde{t}$ a finite measure interval in $\mathbb{R}$. Our strategy is to prove

$$
\|G_N\|^2_{H^1(I)}\;=\;\int_I\big(\,|G_N(t)|^2+|G_N'(t)|^2\,\big)\,\mathrm{d}t\;\leqslant\;\mathrm{const} \tag{41}
$$

uniformly in N, for N large enough. (Here and in the following 'const' or C will be denoting a positive constant, independent of N and t.) If (41) holds, then $\{G_N\}_N$ is a sequence of uniformly bounded functionals on $H^{-1}(I)$. Since $L^1(I)$ is dense in $H^{-1}(I)$, then (40) reads

$$\lim_{N\to\infty}\int_I \mathrm{d}t\, f(t)\, G_N(t) = 0\,, \qquad \forall\, f\in H^{-1}(I)\,. \tag{42}$$

Choosing $f(t)=\delta(t-\tilde t\,)$ one has $G_N(\tilde t\,)\to 0$ (note that $\delta\in H^{-\sigma}\ \forall\sigma>\frac12$). Since $\tilde t$ is arbitrary, the thesis follows.

To prove (41), the only non-trivial part is the derivative term, since

$$G_N(t) \;=\; \big\|\,\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big\|^2_{\mathcal{L}^2} \;\leqslant\; 2\,\big(\,\big\|\,\gamma^{(1)}_{N,t}\big\|^2_{\mathcal{L}^2}+\big\|\,|\varphi_t\rangle\langle\varphi_t|\,\big\|^2_{\mathcal{L}^2}\,\big) \;\leqslant\; 4 \tag{43}$$

and consequently

$$\int_I |G_N(t)|^2\,\mathrm{d}t \;\leqslant\; 16|I|\,. \tag{44}$$

Here, and in the following, $|I|$ denotes the measure of the interval I. So we are left with proving

$$\int_I |G'_N(t)|^2\,\mathrm{d}t \;\leqslant\; \mathrm{const}\,. \tag{45}$$

One has

$$\begin{aligned} G'_N(t) \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\big|\,\gamma^{(1)}_{N,t}(x,y)-\varphi_t(x)\overline{\varphi_t(y)}\,\big|^2\mathrm{d}x\,\mathrm{d}y \\ &=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\Big(\overline{\gamma^{(1)}_{N,t}(x,y)}\,\gamma^{(1)}_{N,t}(x,y)-\overline{\gamma^{(1)}_{N,t}(x,y)}\,\varphi_t(x)\overline{\varphi_t(y)} \\ &\qquad\qquad -\gamma^{(1)}_{N,t}(x,y)\overline{\varphi_t(x)}\varphi_t(y)+|\varphi_t(x)|^2|\varphi_t(y)|^2\Big)\,\mathrm{d}x\,\mathrm{d}y\,. \end{aligned} \tag{46}$$

The fourth summand in the r.h.s. of (46) does not contribute, because by (23) the L^2-norm of φ_t is conserved. We denote the others by

$$\begin{aligned} I_1(N,t) \;&:=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\overline{\gamma^{(1)}_{N,t}(x,y)}\,\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ I_2(N,t) \;&:=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\overline{\gamma^{(1)}_{N,t}(x,y)}\,\varphi_t(x)\overline{\varphi_t(y)}\,\mathrm{d}x\,\mathrm{d}y \\ I_3(N,t) \;&:=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\gamma^{(1)}_{N,t}(x,y)\overline{\varphi_t(x)}\varphi_t(y)\,\mathrm{d}x\,\mathrm{d}y \;=\; \overline{I_2(N,t)}\,. \end{aligned} \tag{47}$$

The first term gives

$$\begin{aligned} |I_1(N,t)| \;&=\; \Big|\int_{\mathbb{R}^2}\Big(\overline{\partial_t\gamma^{(1)}_{N,t}(x,y)}\;\gamma^{(1)}_{N,t}(x,y)+\overline{\gamma^{(1)}_{N,t}(x,y)}\;\partial_t\gamma^{(1)}_{N,t}(x,y)\Big)\mathrm{d}x\,\mathrm{d}y\,\Big| \\ &\leqslant\; 2\,\Big|\int_{\mathbb{R}^2}\overline{\gamma^{(1)}_{N,t}(x,y)}\;\partial_t\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\Big| \\ &=\; \frac{2}{\hbar}\Big|\int_{\mathbb{R}^2}\overline{\gamma^{(1)}_{N,t}(x,y)}\;\Big[\frac{\hbar^2}{2m}(-\partial_x^2+\partial_y^2)\,\gamma^{(1)}_{N,t}(x,y)+\sigma_{N,t}(x,y)\Big]\,\mathrm{d}x\,\mathrm{d}y\,\Big| \\ &\leqslant\; \frac{\hbar}{m}\int_{\mathbb{R}^2}\Big(\big|\partial_x\gamma^{(1)}_{N,t}(x,y)\big|^2+\big|\partial_y\gamma^{(1)}_{N,t}(x,y)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad\qquad +\frac{2}{\hbar}\Big|\int_{\mathbb{R}^2}\overline{\gamma^{(1)}_{N,t}(x,y)}\,\sigma_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\Big| \\ &\leqslant\; \frac{\hbar}{m}\,\|\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|^2_{H^1(\mathbb{R}^2)}+\frac{2}{\hbar}\|\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|_{L^2(\mathbb{R}^2)}\,\|\sigma_{N,t}(\,\cdot\,,\,\cdot\,)\|_{L^2(\mathbb{R}^2)} \end{aligned} \tag{48}$$

where equation (22) has been plugged in (recall the definition (20) of $\sigma_{N,t}$). We now use

$$\begin{aligned} \|\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|_{L^2(\mathbb{R}^2)} &= \|\gamma^{(1)}_{N,t}\|_{\mathcal{L}^2} \leqslant \|\gamma^{(1)}_{N,t}\|_{\mathcal{L}^1} = 1 \qquad (49)\\ \|\sigma_{N,t}(\,\cdot\,,\,\cdot\,)\|_{L^2(\mathbb{R}^2)} &= \|\sigma_{N,t}\|_{\mathcal{L}^2} \leqslant \|\sigma_{N,t}\|_{\mathcal{L}^2_{\mathrm{reg}}} \leqslant \text{const} \qquad (50)\\ \|\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|^2_{H^1(\mathbb{R}^2)} &\leqslant \|\gamma^{(1)}_{N,t}\|_{\mathcal{L}^2_{\mathrm{reg}}} \leqslant \text{const} \qquad (51) \end{aligned}$$

for any time and for any N large enough. Estimate (49) is trivial. (50) follows from (21) of Lemma 2.2. (51) follows from

$$\begin{aligned} \|\gamma^{(1)}_{N,t}\|_{\mathcal{L}^2_{\mathrm{reg}}} &= \int_{\mathbb{R}^2} \big|(1-\partial_x^2)^{1/2}(1-\partial_y^2)^{1/2}\gamma^{(1)}_{N,t}(x,y)\big|^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} \overline{\gamma^{(1)}_{N,t}(x,y)}\,(1-\partial_x^2)\,(1-\partial_y^2)\,\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant \int_{\mathbb{R}^2} \overline{\gamma^{(1)}_{N,t}(x,y)}\,(1-\partial_x^2-\partial_y^2)\,\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \|\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|^2_{H^1(\mathbb{R}^2)} \end{aligned} \qquad (52)$$

and from the apriori estimate (15). So

$$|I_1(N,t)| \leqslant \text{const}\,. \qquad (53)$$

The second and third terms give

$$\begin{aligned} |I_2(N,t)| &= |I_3(N,t)| = \\ &= \Big|\int_{\mathbb{R}^2}\Big(\overline{\partial_t\varphi_t(x)}\,\varphi_t(y)\,\gamma^{(1)}_{N,t}(x,y) + \overline{\varphi_t(x)}\,\partial_t\varphi_t(y)\,\gamma^{(1)}_{N,t}(x,y) \\ &\qquad + \overline{\varphi_t(x)}\,\varphi_t(y)\,\partial_t\gamma^{(1)}_{N,t}(x,y)\Big)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &\leqslant |J_1(N,t)| + |J_2(N,t)| + |J_3(N,t)| \end{aligned} \qquad (54)$$

with

$$\begin{aligned} J_1(N,t) &:= \int_{\mathbb{R}^2} \overline{\partial_t\varphi_t(x)}\,\varphi_t(y)\,\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ J_2(N,t) &:= \int_{\mathbb{R}^2} \overline{\varphi_t(x)}\,\partial_t\varphi_t(y)\,\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ J_3(N,t) &:= \int_{\mathbb{R}^2} \overline{\varphi_t(x)}\,\varphi_t(y)\,\partial_t\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,. \end{aligned} \qquad (55)$$

Pluggin the CNSE (13) into J_1 yields

$$\begin{aligned} |J_1(N,t)| &= \Big|\int_{\mathbb{R}^2} \overline{\partial_t\varphi_t(x)}\,\varphi_t(y)\,\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &\leqslant \frac{\hbar^2}{2m}\Big|\int_{\mathbb{R}^2} \overline{\partial_x^2\varphi_t(x)}\,\varphi_t(y)\,\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &\qquad + b_1\Big|\int_{\mathbb{R}^2} |\varphi_t(x)|^2\,\overline{\varphi_t(x)}\,\varphi_t(y)\,\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \end{aligned} \qquad (56)$$

Due to the conservation laws (23), (49), and (51),

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\overline{\partial_x^2\varphi_t(x)}\,\varphi_t(y)\,\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \;&=\; \Big|\int_{\mathbb{R}^2}\overline{\partial_x\varphi_t(x)}\,\varphi_t(y)\;\partial_x\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &\leqslant\; \|\partial_x\varphi_t\|_{L^2(\mathbb{R})}\,\|\varphi_t\|_{L^2(\mathbb{R})}\,\|\partial_x\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|_{L^2(\mathbb{R}^2)} \\ &\leqslant\; \|\varphi_t\|_{H^1(\mathbb{R})}\,\|\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|_{H^1(\mathbb{R}^2)} \\ &\leqslant\; \text{const}\end{aligned} \tag{57}$$

and

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}|\varphi_t(x)|^2\,\overline{\varphi_t(x)}\,\varphi_t(y)\,\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &\qquad\leqslant\; \|\varphi_t\|^2_{L^6(\mathbb{R})}\,\|\varphi_t\|_{L^2(\mathbb{R})}\,\|\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|_{L^2(\mathbb{R}^2)} \\ &\qquad\leqslant\; \|\varphi_t\|^2_{H^1(\mathbb{R})}\,\|\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|_{L^2(\mathbb{R}^2)} \\ &\qquad\leqslant\; \text{const}\end{aligned} \tag{58}$$

(where we also used the embedding $L^6(\mathbb{R})\subset H^1(\mathbb{R})$). The estimate of J_2 is completely analogous. So

$$\begin{aligned}|J_1(N,t)| &\leqslant\; \text{const} \\ |J_2(N,t)| &\leqslant\; \text{const}\,.\end{aligned} \tag{59}$$

We now estimate J_3:

$$\begin{aligned}|J_3(N,t)| \;&=\; \Big|\int_{\mathbb{R}^2}\overline{\varphi_t(x)}\;\varphi_t(y)\;\partial_t\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &=\; \frac{1}{\hbar}\Big|\int_{\mathbb{R}^2}\overline{\varphi_t(x)}\,\varphi_t(y)\,\Big[\frac{\hbar^2}{2m}(-\partial_x^2+\partial_y^2)\,\gamma^{(1)}_{N,t}(x,y)+\sigma_{N,t}(x,y)\Big]\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &\leqslant\; \frac{\hbar}{2m}\Big|\int_{\mathbb{R}^2}\overline{\partial_x\varphi_t(x)}\,\varphi_t(y)\;\partial_x\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &\qquad+\frac{\hbar}{2m}\Big|\int_{\mathbb{R}^2}\overline{\varphi_t(x)}\,\partial_y\varphi_t(y)\;\partial_y\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &\qquad+\frac{1}{\hbar}\Big|\int_{\mathbb{R}^2}\overline{\varphi_t(x)}\,\varphi_t(y)\;\sigma_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big|\end{aligned} \tag{60}$$

where we plugged equation (22) in. Due to the conservation laws (23) and (51),

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}\overline{\partial_x\varphi_t(x)}\,\varphi_t(y)\;\partial_x\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &\qquad\leqslant\; \|\partial_x\varphi_t\|_{L^2(\mathbb{R})}\,\|\varphi_t\|_{L^2(\mathbb{R})}\,\|\partial_x\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|_{L^2(\mathbb{R}^2)} \\ &\qquad\leqslant\; \|\varphi_t\|_{H^1(\mathbb{R})}\,\|\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\|_{H^1(\mathbb{R}^2)} \\ &\qquad\leqslant\; \text{const}\end{aligned} \tag{61}$$

and analogously

$$\Big|\int_{\mathbb{R}^2}\overline{\varphi_t(x)}\,\partial_y\varphi_t(y)\;\partial_y\gamma^{(1)}_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big|\;\leqslant\;\text{const}\,. \tag{62}$$

Due to the conservation laws (23) and (50),

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\overline{\varphi_t(x)}\,\varphi_t(y)\;\sigma_{N,t}(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \;&\leqslant\; \|\varphi_t\|^2_{L^2(\mathbb{R})}\,\|\sigma_{N,t}(\,\cdot\,,\,\cdot\,)\|_{L^2(\mathbb{R}^2)} \\ &\leqslant\; \text{const}\,.\end{aligned} \tag{63}$$

So

$$|J_3(N,t)| \;\leqslant\; \text{const}\,. \tag{64}$$

Plugging (59) and (64) into (54) yields

$$|I_2(N,t)| \;=\; |I_3(N,t)| \;\leqslant\; \text{const}\,. \tag{65}$$

We have thus obtained the bounds (53) and (65), which yield

$$|G_N'(t)| \;\leqslant\; |I_1(N,t)|+|I_2(N,t)|+|I_3(N,t)| \;\leqslant\; C \tag{66}$$

and consequently

$$\int_I |G_N'(t)|^2\,\mathrm{d}t \;\leqslant\; C^2|I|\,. \tag{67}$$

So (45) holds and this concludes the proof. □

Remark 8. The core of the proof was to show the boundedness of

$$\int_{\mathbb{R}^2} \partial_t\big|\,\gamma^{(1)}_{N,t}(x,y)-\varphi_t(x)\overline{\varphi_t(y)}\,\big|^2\,\mathrm{d}x\,\mathrm{d}y$$

uniformly in N and t. Note that $\partial_t\varphi_t$ is L^2-bounded too, uniformly in time. This follows from the CLSE (13), and from the 'higher regularity' of φ_t: indeed, for finite times φ_t can be proved to be uniformly bounded also in H^2 (Theorem 5.3.1 in [3]). However, the same cannot be said on $\partial_t\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)$, for this regularity in time would follow from the first BBGKY equation (22) if one knew that $\partial_t\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)$ is uniformly bounded in $H^2(\mathbb{R}^2)$, while only uniform $H^1(\mathbb{R}^2)$-boundedness is known under the present assumptions.

Remark 9. Uniform L^2-boundedness of $\partial_t\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)$ and $\partial_t\varphi_t(\,\cdot\,)$ would have shortened the proof, because

$$\begin{aligned}|G_N'(t)| \;&=\; \Big|\frac{\mathrm{d}}{\mathrm{d}t}\big\|\,\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big\|^2_{\mathcal{L}^2}\Big| \\ &\leqslant\; 4\,\big\|\,\partial_t\gamma^{(1)}_{N,t}(\,\cdot\,,\,\cdot\,)\,\big\|_{L^2(\mathbb{R}^2)}+4\,\big\|\,\partial_t\varphi_t(\,\cdot\,)\,\big\|_{L^2(\mathbb{R}^2)}\,.\end{aligned} \tag{68}$$

(Inequality (68) follows from

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\big\|\,\rho_{N,t}\,\big\|^2_{\mathcal{L}^2}\Big| \;&=\; \Big|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\big|\,\rho_{N,t}(x,y)\,\big|^2\,\mathrm{d}x\,\mathrm{d}y\,\Big| \\ &=\; \Big|\int_{\mathbb{R}^2}\Big(\overline{\partial_t\rho_{N,t}(x,y)}\;\rho_{N,t}(x,y)+\overline{\rho_{N,t}(x,y)}\;\partial_t\rho_{N,t}(x,y)\Big)\,\mathrm{d}x\,\mathrm{d}y\,\Big| \\ &\leqslant\; 2\,\big\|\,\partial_t\rho_{N,t}(\,\cdot\,,\,\cdot\,)\,\big\|_{L^2(\mathbb{R}^2)}\,\big\|\,\rho_{N,t}(\,\cdot\,,\,\cdot\,)\,\big\|_{L^2(\mathbb{R}^2)}\end{aligned}$$

where in this case $\rho_{N,t}=\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|$.) Instead of estimate (68), our strategy was to perform the full computation (46) and to estimate it term by term with the insertion of the CNSE and BBGKY equations.

Remark 10. An equivalent strategy to that discussed through (40) to (42) is the following. Once (44) and (45) are proved as above, one also has

$$G_N\in W^{1,1}(I)=\{u: u \text{ is absolutely continuous and } u,u'\in L_1(I)\} \tag{69}$$

uniformly in N, since $|I|$ is finite. Then $\{G_N\}_N$ is a sequence of uniformly bounded functional on $W^{1,1}(I)^*$. Moreover, $W^{1,1}(I)\subset C(I)$, and then

$$C(I)^*=\mathcal{M}(I)=\text{ the space of Radon measures }\subset W^{1,1}(I)^*. \tag{70}$$

Since $L^1(I)$ is a dense subspace of $\mathcal{M}(I)$, assumption (40) reads

$$\lim_{N\to\infty}\int_I \mathrm{d}t\, f(t)\, G_N(t) = 0\,, \qquad \forall\, f\in\mathcal{M}(I)\,. \tag{71}$$

Since $\delta\in\mathcal{M}(I)$, the theorem follows.

4. **Discussion.** This work puts the rigorous derivation of the CNLS in dimension one recently obtained by Adami, Golse, and Teta (Theorem 2.1 and [1]) into the standard form of the asymptotic factorisation of marginals in the trace norm at any fixed time (Theorem 3.1).

The model studied in [1] is a one-dimensional many-body system of N bosons with short-scale repulsive interaction V_N given by (1), and initially prepared in a completely factorised initial state $\Phi_N = \varphi^{\otimes N}$ such that all the moments of the energy are bounded, $\langle\Phi_N,(H_N)^k,\Phi_N\rangle\leqslant\mathcal{E}^k N^k$.

More in abstract, our result applies to any d-dimensional many-body Hamiltonian

$$H_N \;=\; -\frac{\hbar^2}{2m}\sum_{i=1}^N(-\Delta_{x_i}) \;+\sum_{1\leqslant i<j\leqslant N} V_N(x_i-x_j) \tag{72}$$

on $L^2(\mathbb{R}^{Nd})$, i.e., to any scaling in the interaction V_N, for which the apriori bounds

$$\|\gamma^{(1)}_{N,t}\|_{\mathcal{L}^2_{\mathrm{reg}}} \;\leqslant\; \mathrm{const}\,, \qquad \|\sigma_{N,t}\|_{\mathcal{L}^2_{\mathrm{reg}}} \;\leqslant\; \mathrm{const} \tag{73}$$

hold uniformly in time and for all sufficiently large N, where

$$\sigma_{N,t}(x,y) \;:=\; N\int_{\mathbb{R}} V_N(x-z)\,\gamma^{(2)}_{N,t}(x,z;y,z)\,\mathrm{d}z\,. \tag{74}$$

Assuming (73), our result says that if

$$\lim_{N\to\infty}\int_{-\infty}^{+\infty}\mathrm{d}t\,\big(\,\rho_t\,,\,\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big)_{\mathcal{L}^2_{\mathrm{reg}}}=0 \qquad \forall\,\rho_{\cdot}\in L^1(\mathbb{R},\mathcal{L}^2_{\mathrm{reg}}) \tag{75}$$

then

$$\lim_{N\to\infty}\mathrm{Tr}\,\big|\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\big| \;=\; 0 \qquad \forall t\in\mathbb{R}\,. \tag{76}$$

Let us comment on the converse implication, i.e., (75) $\Leftarrow$ (76), although it is less natural and less convenient in practice to express the propagation of chaos, or the stability of BEC, in the form (75). From

$$\Big|\int_{-\infty}^{+\infty}\mathrm{d}t\,\big(\,\rho_t\,,\,\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big)_{\mathcal{L}^2_{\mathrm{reg}}}\Big| \;\leqslant\; \int_{-\infty}^{+\infty}\mathrm{d}t\,\|\gamma^{(1)}_{N,t}-|\,\varphi_t\rangle\langle\varphi_t|\,\|_{\mathrm{op}}\,\mathrm{Tr}|S^2\rho_t S^2|$$

and

$$\Big|\int_{-\infty}^{+\infty}\mathrm{d}t\,\big(\,\rho_t\,,\,\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big)_{\mathcal{L}^2_{\mathrm{reg}}}\Big| \;\leqslant\; \int_{-\infty}^{+\infty}\mathrm{d}t\,\mathrm{Tr}\big|\gamma^{(1)}_{N,t}-|\,\varphi_t\rangle\langle\varphi_t|\big|\,\|S^2\rho_t S^2\|_{\mathrm{op}}$$

($\|\cdot\|_{\mathrm{op}}$ denoting the operator norm), one has (75) $\Leftarrow$ (76) only for classes of $\rho_{\cdot}$'s strictly contained in $L^1(\mathbb{R},\mathcal{L}^2_{\mathrm{reg}})$. Such classes are also *dense* in $L^1(\mathbb{R},\mathcal{L}^2_{\mathrm{reg}})$, nevertheless (75) cannot follow by a density argument because we do not know if in general $\gamma^{(1)}_{N,\cdot}-|\varphi_{\cdot}\rangle\langle\varphi_{\cdot}|\in L^\infty(\mathbb{R},\mathcal{L}^2_{\mathrm{reg}})$ uniformly in N. On the other hand, from

$$\Big|\big(\,\rho_t\,,\,\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\big)_{\mathcal{L}^2_{\mathrm{reg}}}\Big| \;\leqslant\; \|\rho_t\|_{\mathcal{L}^2_{\mathrm{reg}}}\,\|\,\gamma^{(1)}_{N,t}-|\varphi_t\rangle\langle\varphi_t|\,\|_{\mathcal{L}^2_{\mathrm{reg}}}$$

it is clear that (75) would follow if $\|\,\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\,\|_{\mathcal{L}^2_{\mathrm{reg}}} \to 0$ instead of $\|\,\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\,\|_{\mathcal{L}^2} \to 0$ as in (76).

Last, we want to underline that the original AGT result (75) was obtained as a weak convergence result via a compactness argument. Hence, (75) could not be supplemented with a quantitative rate of convergence in N. The same then holds for our modified version (76) of AGT Theorem.

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