

Black Holes, Instanton Counting on Toric Singularities and q-Deformed Two-Dimensional Yang-Mills Theory

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ABSTRACT: We study the relationship between instanton counting in $\mathcal{N} = 4$ Yang-Mills theory on a generic four-dimensional toric orbifold and the semi-classical expansion of q-deformed Yang-Mills theory on the blowups of the minimal resolution of the orbifold singularity, with an eye to clarifying the recent proposal of using two-dimensional gauge theories to count microstates of black holes in four dimensions. We describe explicitly the instanton contributions to the counting of D-brane bound states which are captured by the two-dimensional gauge theory. We derive an intimate relationship between the two-dimensional Yang-Mills theory and Chern-Simons theory on generic Lens spaces, and use it to show that the correct instanton counting is only reproduced when the Chern-Simons contributions are treated as non-dynamical boundary conditions in the D4-brane gauge theory. We also use this correspondence to discuss the counting of instantons on higher genus ruled Riemann surfaces.

KEYWORDS: Black Holes, Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories, Chern-Simons Theories, Field Theories in Lower Dimensions.

Contents

1. Introduction	1
2. D-Brane Partition Function on Toric Orbifolds	3
3. q-Deformed Gauge Theory on Toric Singularities	7
3.1 Sewing Construction of the Partition Function	7
3.2 Semi-Classical Expansion	9
3.3 Emergence of Four-Dimensional Instantons	11
3.4 Example: Line Bundles over \mathbb{P}^1	13
3.5 Example: ALE Spaces	14
4. Chern-Simons Gauge Theory on Lens Spaces	15
4.1 Classical Solutions	15
4.2 Semi-Classical Expansion	18
5. Instantons on Higher Genus Ruled Surfaces	20
6. Conclusions	22
Appendices	22
A. Continued Fractions and the Cartan Matrix	22

1. Introduction

A renewed impetus into the description of BPS black hole microstates in four dimensions has been sparked by the OSV conjecture [1] which equates black hole entropy in Type IIA string theory compactified on a Calabi-Yau threefold \mathcal{M} to the modulus squared of the topological string partition function on \mathcal{M} . The black hole is constructed by wrapping D2-branes around arbitrary two-cycles of \mathcal{M} and D4-branes around a four-cycle which is a fixed ample divisor of \mathcal{M} . With respect to a fixed basis of two-cycles in $H_2(\mathcal{M}, \mathbb{Z})$, and a dual basis of four-cycles in $H_4(\mathcal{M}, \mathbb{Z})$, the D2 and D4 branes carry electric and magnetic charges $\mathbf{Q}_2, \mathbf{Q}_4 \in \mathbb{Z}^n$ where $n = h^{1,1}(\mathcal{M})$. We also specify the D0-brane charge $Q_0 \in \mathbb{Z}$ and turn off the D6-brane charge. The black hole partition function then takes the symbolic form

$$Z_{\text{BH}}(\mathbf{Q}_4, \varphi_2, \varphi_0) = \sum_{\mathbf{Q}_2 \in \mathbb{N}_0^n} \sum_{Q_0 \in \mathbb{N}_0} \Omega(\mathbf{Q}_4, \mathbf{Q}_2, Q_0) e^{-Q_0 \varphi_0 - \mathbf{Q}_2 \cdot \varphi_2} \quad (1.1)$$

where $\Omega(\mathbf{Q}_4, \mathbf{Q}_2, Q_0)$ is the indexed degeneracy of BPS states in spacetime with the specified charges, and φ_0, φ_2 are chemical potentials. The OSV conjecture then equates (1.1) for large black hole charges to the topological string amplitude $|Z_{\text{top}}(\mathbf{t}, g_s)|^2$, where $\mathbf{t} \in \mathbb{C}^n$ are the Kähler parameters of \mathcal{M} and the various moduli between the two partition functions are related by the attractor mechanism.

The index $\Omega(\mathbf{Q}_4, \mathbf{Q}_2, Q_0)$ can be computed by counting BPS states in the supersymmetric gauge theory on the D4-branes [2], where the D0-branes are interpreted as instantons and the D2-branes as sources of magnetic flux endowing the instantons with non-trivial first Chern class. Then (1.1) coincides with the partition function of $\mathcal{N} = 4$ topological Yang-Mills theory in four-dimensions, summed over all topological sectors and with the insertion of observables giving the inclusion of D0 and D2 brane charges. The structure of this gauge theory partition function has been recently argued in [3, 4] to simplify drastically in the case of a local Calabi-Yau space which is a rank 2 normal bundle over a compact Riemann surface Σ , and the D4-brane worldvolume is given by the total space of a non-trivial holomorphic line bundle over Σ . It is argued that the partition function (1.1) localizes onto field configurations which are invariant under the natural $U(1)$ action on the fibres of this line bundle, and the four-dimensional gauge theory reduces to a two-dimensional gauge theory on the base Σ called q-deformed Yang-Mills theory [5]–[7]. Various aspects of the black hole partition function on toric Calabi-Yau threefolds from this remarkable two-dimensional point of view are analysed in [3, 4, 8, 9].

In this paper we will analyse in detail the problem of computing the black hole partition function (1.1) using the sewing formalism of q-deformed Yang-Mills theory for the most general toric singularity $X(p, q)$ in four dimensions.¹ This construction extends the A_k ALE spaces which were considered in [8]. It also includes the four-manifolds $X(p, 1)$ which are the total spaces of the holomorphic line bundles $\mathcal{O}_{\mathbb{P}^1}(-p)$ and for which the relevant two-dimensional gauge theory is q-deformed Yang-Mills theory on the sphere which was studied in great detail in [4, 10]–[14]. Our results are in agreement with the recent analysis in [15] of the black hole partition function (1.1) using direct instanton calculations in the four-dimensional gauge theory.

One of our main computations is the modular inversion of the heat kernel representation of the q-deformed partition function which casts it as a sum over two-dimensional Yang-Mills instantons living on the blowups of the minimal resolution of the toric singularity. This resummation is necessary to match the topological expansion (1.1), and we immediately find problems with the black hole interpretation of the two-dimensional gauge theory. The semi-classical expansion of the two-dimensional gauge theory contains terms which cannot simply correspond to an indexed degeneracy Ω . We identify part of the expansion with the value of the Chern-Simons partition function on the boundary of

¹A word of caution about notation. In the literature on q-deformed gauge theory the symbol q is used to denote the q-deformation, which in the topological string setting is given by $q = e^{-g_s}$. In this paper we will only use q to denote the integer modulus of the toric four-manifold, always writing the q-deformation explicitly as e^{-g_s} with the usual identification $g_s = g_{\text{YM}}^2/2$ between the string and four-dimensional Yang-Mills coupling constants. Accordingly, we also avoid using the standard notation q for the arguments of modular forms which typically arise in four-dimensional instanton calculations.

the non-compact space $X(p, q)$, which is a generic three-dimensional Lens space $L(p, q)$. To match with the four-dimensional instanton computation we must follow the standard prescription of summing over the admissible *non-dynamical* boundary conditions on the gauge fields, whose asymptotic values are governed by Chern-Simons gauge theory. This amounts to identifying that part of the two-dimensional amplitude which corresponds to the perturbative expansion of the Chern-Simons theory about a given vacuum. We will find that, when these terms are stripped and only the classical Chern-Simons contributions are retained, the q-deformed gauge theory reproduces *exactly* the contributions given in [15] from “fractional” instantons which are stuck at the singularity of $X(p, q)$. In particular, it is not entirely clear exactly how the two-dimensional formalism can reproduce the remaining contributions, such as those coming from instantons which are free to propagate throughout the four-dimensional space.

In the course of this analysis we are faced with the derivation of the nonabelian localization formula for the Chern-Simons partition function on a generic Lens space $L(p, q)$ (we have not found a complete and general calculation in the literature). We carry out this computation in detail by using Seifert fibration techniques to evaluate the classical contributions and surgery methods to compute the fluctuation determinants. In particular, we derive the explicit mapping between Chern-Simons vacua and two-dimensional Yang-Mills connections, and hence with fractional instantons in four dimensions. We also briefly examine the problem of counting instantons on ruled Riemann surfaces for genus $g \geq 1$, which are non-toric four-manifolds for which little is known about the structure of instantons. We use the prescription given above to predict the structure of the full $U(1)$ partition function in four dimensions for any genus, and to predict the contributions from fractional instantons in the nonabelian case for genus $g = 1$. In particular, we conclude that the $U(N)$ partition function does not seem to factorize into a $U(1)^N$ contribution, as it does in the genus 0 cases.

The organisation of this paper is as follows. In Section 2 we review the structure of four-dimensional instanton partition functions on the toric manifolds $X(p, q)$ and in particular some of the results of [15]. In Section 3 we construct the pertinent q-deformed gauge theory amplitude and describe how to extract the four-dimensional instanton contributions. In Section 4 we work out the semi-classical expansion of Chern-Simons gauge theory on the Lens spaces $L(p, q)$. In Section 5 describe some analogous computations on the ruled Riemann surfaces. In Section 6 we summarize our findings. Finally, some technical details of our calculations are summarized in an appendix at the end of the paper.

2. D-Brane Partition Function on Toric Orbifolds

In this section we will study the partition function of a bound system of D0–D2–D4 branes where the D4-branes wrap a four-cycle X of a local Calabi-Yau threefold given by the total space of the canonical line bundle K_X . We take X to be a smooth four-dimensional manifold given by the minimal resolution of the quotient space \mathbb{C}^2/Γ , where $\Gamma \cong \mathbb{Z}_p$ is a generic finite cyclic group. The action of Γ on the coordinates (z, w) of \mathbb{C}^2 can be linearized

locally as

$$(z, w) \longmapsto (e^{2\pi i q/p} z, e^{2\pi i/p} w) \quad (2.1)$$

where (p, q) are coprime integers with $p > q > 0$. The orbifold action (2.1) generates an $A_{p,q}$ singularity at the origin of \mathbb{C}^2 , whose minimal resolution (known as the Hirzebruch-Jung resolution) gives rise to a smooth four-dimensional manifold $X(p, q)$ called a Hirzebruch-Jung space [16].

This space contains a chain of ℓ exceptional divisors at the origin given by projective lines \mathbb{P}^1 whose intersection numbers are summarized by the (generalized) Cartan matrix

$$C = - \begin{pmatrix} -e_1 & 1 & 0 & \cdots & 0 \\ 1 & -e_2 & 1 & \cdots & 0 \\ 0 & 1 & -e_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -e_\ell \end{pmatrix}. \quad (2.2)$$

The moduli of the self-intersection numbers $e_i \geq 2, i = 1, \dots, \ell$ are obtained by expanding the rational number $\frac{p}{q} > 1$ in a simple continued fraction

$$\frac{p}{q} = [e_1, \dots, e_\ell] := e_1 - \frac{1}{e_2 - \frac{1}{e_3 - \frac{1}{\ddots e_{\ell-1} - \frac{1}{e_\ell}}}} \quad (2.3)$$

with e_1 the smallest integer $> \frac{p}{q}$, and so on. For example, for $q = 1$ there is only one exceptional divisor with self-intersection number $-e_1 = -p$ and the manifold $X(p, 1)$ can be regarded as the total space of a holomorphic line bundle $\mathcal{O}_{\mathbb{P}^1}(-p)$ of degree p over \mathbb{P}^1 . The other limiting case $q = p - 1$ corresponds to an A_{p-1} ALE space, which contains a chain of $\ell = (p - 1)$ \mathbb{P}^1 's each with self-intersection number $-e_i = -2$, and in this case (2.2) coincides with the Cartan matrix of the A_{p-1} Dynkin diagram.

Standard arguments [17] show that the gauge theory living on N D4-branes wrapping $X(p, q)$ is a $U(N)$ Vafa-Witten topologically twisted $\mathcal{N} = 4$ Yang-Mills theory [18]. In this context, the D0-branes are interpreted as instantons of the four-dimensional gauge theory. These instantons can also have a non-vanishing first Chern class due to the presence of D2-branes wrapping the exceptional divisors which generate a non-trivial magnetic flux on the D4-brane worldvolume. Under suitable assumptions [18] the twisted $\mathcal{N} = 4$ Yang-Mills partition function computes the Euler number of the instanton moduli space. The powerful toric localization techniques developed in recent years [19]–[27] have enabled the computation of this partition function for ALE spaces [28] and for the total spaces of the $\mathcal{O}_{\mathbb{P}^1}(-p)$ bundles with $p = 1, 2$ [29, 30]. However, for generic $A_{p,q}$ singularities an explicit description of the instanton moduli space is not available at the moment and the direct evaluation of the complete instanton partition function on Hirzebruch-Jung spaces is still an open problem.

In this paper we will address this problem from a somewhat different perspective. According to the proposal of [4, 8], one can localize the four-dimensional path integral via the natural $U(1)$ action on the fibres of the normal bundles over the \mathbb{P}^1 's. In this way one reduces the D4-brane gauge theory to a q -deformed Yang-Mills theory living on the exceptional divisors which arise from the minimal resolution of the toric orbifold singularity. As we will see in the following, the two-dimensional computation gives results in agreement with the direct instanton counting presented recently in [15], where the instanton partition functions on $A_{p,q}$ toric orbifolds are described by assuming some factorization properties which we review below.

For ALE spaces an explicit description of the instanton moduli space in terms of ADHM data has been derived in [31] and reinterpreted in terms of D-brane bound states in [32]. Let us recall some features of this construction which will be useful in the following. The first and second Chern characters of a $U(N)$ instanton gauge bundle \mathcal{E} over an ALE space $X_p := X(p, p-1)$ are given by

$$\begin{aligned} \text{ch}_1(\mathcal{E}) &= \sum_{i=0}^{p-1} u_i \text{ch}_1(\mathcal{T}^i) , \\ \text{ch}_2(\mathcal{E}) &= \sum_{i=0}^{p-1} u_i \text{ch}_2(\mathcal{T}^i) - \frac{K}{p} \Omega_{X_p} \quad \text{with} \quad K = \sum_{i=0}^{p-1} k_i , \end{aligned} \quad (2.4)$$

where \mathcal{T}^i are principal $U(1)$ -bundles of degree i corresponding to the tautological bundles associated to each of the exceptional divisors [33]. In particular, \mathcal{T}^0 is the trivial line bundle. In eq. (2.4), $\text{ch}_1 = c_1$ and $\text{ch}_2 = \frac{1}{2} c_1^2 - c_2$, where c_1 and c_2 are the first and second Chern classes, respectively, and Ω_{X_p} is the unit volume form on X_p . The coefficients u_i , $i = 0, 1, \dots, p-1$ are given in terms of partitions $K = \sum_i k_i$ and $N = \sum_i N_i$ of the numbers of D0 and D4 branes, respectively, into the i -th irreducible representation of \mathbb{Z}_p as

$$u_i = -C_{ij} \int_{X_p} c_1(\mathcal{E}) \wedge c_1(\mathcal{T}^j) = N_i + k_{i+1} + k_{i-1} - 2k_i . \quad (2.5)$$

On the ALE space X_p one can distinguish between two classes of instantons, the *regular* and *fractional* instantons [34].

Regular instantons live in the regular representation $k_0 = k_1 = \dots = k_{p-1} = k$ of the orbifold group \mathbb{Z}_p . As such, they are free to move together with their orbifold images on the whole space X_p and their moduli space for gauge group $U(1)$ coincides with the Hilbert scheme $X_p^{[K]}$ of $K = kp$ points on X_p . The Poincaré polynomial of this space is well-known and is given by [35]

$$P(t | X_p^{[K]}) = \prod_{m=1}^{\infty} \frac{1}{(1 - e^{2\pi i m \tau} t^{2m})^{p-1} (1 - e^{2\pi i m \tau} t^{2m-2})} , \quad (2.6)$$

where $\tau = \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi}$ is the complexified gauge coupling. The expression (2.6) is a function of the usual Boltzmann weight of regular instantons in the supersymmetric Yang-Mills path

integral. By putting $t = 1$ in (2.6) one gets the Euler characteristic of the moduli space of regular $U(1)$ instantons given by

$$Z_{\text{reg}}^{U(1)} = \frac{1}{\hat{\eta}(\tau)^p} \quad \text{with} \quad \hat{\eta}(\tau) := \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau}) . \quad (2.7)$$

The generic $U(N)$ partition function is given by the N -th power of (2.7).

Fractional instantons are instead stuck at the orbifold singularity and have no moduli associated to their position in the four-dimensional space X_p . More precisely, they correspond to the non-trivial self-dual abelian gauge connections A^{frac} of the tautological bundle with curvature

$$F_A^{\text{frac}} = -2\pi i \sum_{i=0}^{p-1} u_i c_1(\mathcal{T}^i) . \quad (2.8)$$

From (2.4) we then immediately realize that their contribution to the path integral is weighted in terms of the intersection matrix

$$I^{ij} := \int_{X_p} c_1(\mathcal{T}^i) \wedge c_1(\mathcal{T}^j) = -(C^{-1})^{ij} , \quad (2.9)$$

where C^{-1} is the inverse of the Cartan matrix (2.2). Since the four-dimensional space X_p is non-compact, the Cartan matrix is not unimodular and so its inverse generally has rational-valued elements (see Appendix A). Thus fractional instantons indeed have a fractional charge. The contribution of fractional $U(1)$ instantons to the $\mathcal{N} = 4$ partition function on ALE spaces has been written elegantly in [36, 15] by rewriting the second Chern character in (2.4) as

$$\text{ch}_2(\mathcal{E}) = \sum_{i=0}^{p-1} (C^{-1})^{ii} u_i - \frac{K}{p} \Omega_{X_p} = \frac{1}{2} (C^{-1})^{ij} u_i u_j \quad (2.10)$$

to get

$$Z_{\text{frac}}^{U(1)} = \sum_{\mathbf{u} \in \mathbb{Z}^{p-1}} e^{\pi i \tau u_i (C^{-1})^{ij} u_j} e^{-u_i z^i} , \quad (2.11)$$

where $z^i = (C^{-1})^{ij} (\varphi_2)_j$ is the contribution of the magnetic fluxes associated to the D2-branes with chemical potentials $(\varphi_2)_i$. The result for general $U(N)$ gauge group can again be obtained by simply taking the N -th power of (2.11) [36, 15].

In [28] it was observed that the regular and fractional instanton contributions factorize in the evaluation of the $\mathcal{N} = 4$ partition function on ALE spaces. This result has been further developed and established on a firm mathematical basis in [36]. Thus the full partition function on ALE spaces is given simply by the N -th power of the product of (2.7) and (2.11). In [15] analogous formulas are proposed for the regular and fractional instantons on more general $A_{p,q}$ toric orbifolds. The results which follow in the next section indicate that an analogous factorization takes place as well for these more general four-manifolds, even though a more direct analysis is required to properly confirm this property.

3. q-Deformed Gauge Theory on Toric Singularities

In this section we will evaluate the q-deformed Yang-Mills partition function living on the minimal resolution of generic $A_{p,q}$ orbifold singularities. After carefully resolving some subtleties, our results will correctly reproduce the contributions of fractional instantons to the four-dimensional gauge theory partition function of the previous section. This is what one would naturally expect, since the fractional instantons are bounded to the exceptional divisors of the four-dimensional geometry. Moreover, it follows from eq. (2.8) that there is a one-to-one correspondence between fractional instantons on $X(p,q)$ and classical solutions of the q-deformed gauge theory which are obtained as configurations of magnetic monopoles on the \mathbb{P}^1 's (i.e. monopole connections on the tautological line bundles \mathcal{T}^i). We will describe this correspondence in more detail in Section 4.

3.1 Sewing Construction of the Partition Function

We will begin by computing the partition function of $\mathcal{N} = 4$ topologically twisted Yang-Mills theory on the Hirzebruch-Jung spaces $X(p,q)$ following the approach proposed in [4, 8]. This method was originally developed in [4] for the $q = 1$ cases and then later extended to the A_k ALE spaces in [8]. More general black hole microstate counting was also attempted in [8] using the same strategy, by considering theories derived from more complicated configurations of D4-branes on toric Calabi-Yau threefolds.

The main idea underlying the computation consists in cutting the four-manifold $X(p,q)$ into pieces where the theory is simple enough to solve explicitly. Then, thanks to the topological nature of the gauge theory, one glues the pieces back together using an appropriate set of rules. The Hirzebruch-Jung spaces can be obtained by patching together ℓ copies of \mathbb{C}^2 , suggesting that one should be able to derive the relevant Yang-Mills amplitudes by sewing topological amplitudes on \mathbb{C}^2 . Since both spaces \mathbb{C}^2 and $X(p,q)$ have \mathbb{T}^2 isometries, the four-dimensional gauge theory path integral should localize onto fixed points of these torus actions. Based on this observation, a simple set of local rules for constructing four-dimensional amplitudes in terms of q-deformed two-dimensional Yang-Mills theory was proposed in [8].

The important building block in the construction is the topological amplitude on \mathbb{C}^2 . By regarding \mathbb{C}^2 as a \mathbb{T}^2 fibration over \mathbb{R}^2 , it can be written as

$$\mathcal{Z}(U, V) = \sum_{R, Q} S_{R, Q} \text{Tr}_R(U) \text{Tr}_Q(V) \quad (3.1)$$

where U and V represent the holonomies of the four-dimensional gauge field along the boundaries of the two disks which are fixed by the torus action [8]. The sum runs over the irreducible representations R, Q of the $U(N)$ gauge group that label the boundary conditions on the gauge field through

$$\int_{M_1} F_a = \frac{1}{2} n_a(R) g_{\text{YM}}^2 \quad \text{and} \quad \int_{M_2} F_a = \frac{1}{2} n_a(Q) g_{\text{YM}}^2, \quad (3.2)$$

where $n_a(R)$ is the length of the a -th row in the Young tableau of R shifted by $\frac{1}{2}(N+1) - a$. The two-dimensional manifolds M_1 and M_2 are respectively the ‘‘fiber component’’ and the

“base component” of \mathbb{C}^2 regarded as a torus bundle. Finally, the quantity $S_{R,Q}$ is the basic correlator

$$S_{R,Q} = \left\langle \text{Tr}_R \exp\left(-i \int_{M_1} F\right) \text{Tr}_Q \exp\left(i \int_{M_2} F\right) \right\rangle \quad (3.3)$$

carrying the dynamical information of the topological Yang-Mills theory. Its explicit expression in terms of group theoretical data is given below.

The complete partition function on the toric four-manifold $X(p,q)$ is now gotten by appropriately gluing the patches together. Every time that two disks are glued together along their boundaries (with opposite orientation) a \mathbb{P}^1 appears, corresponding to a partial resolution of the orbifold singularity described in the previous section. Sewing the boundary holonomies is achieved by integrating over them as

$$\mathcal{Z}_{\mathbb{P}^1}(V, V') = \int_{U(N)} dU \mathcal{Z}(V, U) \mathcal{Z}(U^{-1}, V') \quad (3.4)$$

in the invariant Haar measure on the unitary group $U(N)$, and using standard orthogonality properties of the group characters $\text{Tr}_R(U)$. However, the amplitude (3.3) is expressed using coordinates in which \mathbb{C}^2 is a trivial fibration over both M_1 and M_2 . For the generic Hirzebruch-Jung spaces, the normal bundle to the i -th exceptional divisor is $\mathcal{O}_{\mathbb{P}^1}(-e_i)$ corresponding to its non-trivial self-intersection number in (2.2). It is argued in [4] (see also [3, 37]–[41]) that the dynamical effect of the non-trivial fibration $\mathcal{O}_{\mathbb{P}^1}(-e_i)$ is encoded in the term $T_R^{e_i}$ which accompanies the gluing operation creating the corresponding \mathbb{P}^1 , where

$$T_R = e^{-\frac{g_{\text{YM}}^2}{4} C_2(R)} \quad (3.5)$$

and $C_2(R)$ is the second Casimir invariant of the representation R . The presence of this term can also be interpreted [8] as an annulus insertion, within the general framework proposed by [42] to compute the relevant amplitudes using two-dimensional topological quantum field theory.

The resulting partition function on $X(p,q)$ is therefore a simple generalization of the partition function on A_k ALE spaces constructed in [8]. The difference here is that, in generating the chain of \mathbb{P}^1 's by gluing disks, the self-intersection moduli e_i are generically different from 2. At the “ends” of the chain we should turn off the gauge fields by taking trivial holonomies on the external disks, i.e. trivial representations $R = 0$. In this way the partition function on the Hirzebruch-Jung space takes the form

$$\begin{aligned} Z_{U(N)}^{q\text{YM}}(X(p,q), g_{\text{YM}}^2) &= \sum_{R_1, \dots, R_\ell} S_{0, R_1} S_{R_1, R_2} \cdots S_{R_{\ell-1}, R_\ell} S_{R_\ell, 0} T_{R_1}^{e_1} \cdots T_{R_\ell}^{e_\ell} \\ &\times e^{-i \sum_i \theta_i C_1(R_i)}. \end{aligned} \quad (3.6)$$

In (3.6) we have inserted one independent two-dimensional θ -angle θ_i , $i = 1, \dots, \ell$ for each exceptional divisor, owing to the fact that the divisors define independent homology two-cycles in $H_2(X(p,q), \mathbb{Z}) \cong \mathbb{Z}^\ell$. In the black hole context they are related to chemical potentials for D2-branes wrapping the divisors. We will see this explicitly in Section 3.3 below, but for the moment they simply weight here the $U(1)$ fluxes through the \mathbb{P}^1 's

represented by the first Casimir invariant $C_1(R) = \sum_a n_a(R)$ of the representation R [8]. Note that for $q = 1$ one has $\ell = 1$ and $e_1 = p$, and (3.6) reduces to the partition function of q -deformed Yang-Mills theory on the sphere [4].

It remains to write down explicit formulas for the amplitudes (3.3) and (3.5) above. Let $\hat{n}_a(R)$ be the weight vector classifying an irreducible representation R of the gauge group $U(N)$, where the index a spans the rows of the corresponding Young diagram, and let $r(R)$ denote the $U(1)$ charge of R . Then the second Casimir invariant of R can be conveniently written as

$$C_2(R) = \sum_{a=1}^N \left(\hat{n}_a(R) + r(R) - a - \frac{N-1}{2} \right)^2 = \begin{cases} \sum_{a=1}^N n_a(R)^2 & \text{for } N \text{ odd ,} \\ \sum_{a=1}^N \left(n_a(R) - \frac{1}{2} \right)^2 & \text{for } N \text{ even ,} \end{cases} \quad (3.7)$$

where in the second equality we have absorbed an irrelevant shift into the weight integers $\mathbf{n}(R) \in \mathbb{Z}^N$. (We have also dropped an overall factor depending only on N .) Note that the trivial representation $R = 0$ has weight $\mathbf{n}(0) = \mathbf{0}$. Throughout we will assume that the rank N is odd. This restriction is not necessary but it will simplify some of our analysis in the following. The correlators $S_{R_i, R_{i+1}}$ appearing in (3.6) arise from the gluing of disks and annuli to build the necklace of ℓ spheres, and they are given by

$$S_{R,Q} = \sum_{w \in S_N} \varepsilon(w) e^{-\frac{g_{\text{YM}}^2}{2} w(\mathbf{n}(R)+\boldsymbol{\rho}) \cdot (\mathbf{n}(Q)+\boldsymbol{\rho})} . \quad (3.8)$$

This operator is related to the modular S-matrix of the $U(N)$ WZW model in the Verlinde basis (see Section 4.2). Here $\boldsymbol{\rho}$ is the Weyl vector of $U(N)$ (the half sum of positive roots) whose components are given by

$$\rho_a = \frac{N-2a+1}{2} , \quad (3.9)$$

and the elements w of the Weyl group S_N of $U(N)$ act by permuting the entries of N -vectors with sign $\varepsilon(w)$.

3.2 Semi-Classical Expansion

The Poisson resummation of (3.6) has a natural interpretation as an expansion of the q -deformed gauge theory into a sum over classical solutions [10]–[12]. After some trivial manipulations and dropping of an overall irrelevant normalization, we can recast the partition function in the form

$$Z_{U(N)}^{q\text{YM}}(X(p, q), g_{\text{YM}}^2) = \sum_{w \in S_N} \varepsilon(w) \sum_{\mathbf{n}_1, \dots, \mathbf{n}_\ell \in \mathbb{Z}^N} e^{-\frac{g_{\text{YM}}^2}{4} \sum_i e_i \mathbf{n}_i^2 - \frac{g_{\text{YM}}^2}{2} \sum_{i < j} \mathbf{n}_i \cdot \mathbf{n}_j + i \sum_i \mathbf{f}_i \cdot \mathbf{n}_i} \\ \times e^{-\frac{g_{\text{YM}}^2}{2} (\boldsymbol{\rho} \cdot \mathbf{n}_1 - w(\boldsymbol{\rho}) \cdot \mathbf{n}_\ell)} , \quad (3.10)$$

where $\mathbf{f}_i = \theta_i (1, 1, \dots, 1)$. Each integer vector \mathbf{n}_i , $i = 1, \dots, \ell$ classifies one of the original irreducible $U(N)$ representations R_i appearing in (3.6). The quadratic form in the exponent of (3.10) can be succinctly rewritten in terms of the Cartan matrix (2.2) to get

$$Z_{U(N)}^{q\text{YM}}(X(p, q), g_{\text{YM}}^2) = \sum_{w \in S_N} \varepsilon(w) \sum_{\mathbf{n}_1, \dots, \mathbf{n}_\ell \in \mathbb{Z}^N} e^{-\frac{g_{\text{YM}}^2}{4} C_{ij} \mathbf{n}_i \cdot \mathbf{n}_j + i \mathbf{f}_i \cdot \mathbf{n}_i} e^{-\frac{g_{\text{YM}}^2}{2} (\boldsymbol{\rho} \cdot \mathbf{n}_1 - w(\boldsymbol{\rho}) \cdot \mathbf{n}_\ell)} \quad (3.11)$$

with an implicit sum over repeated indices.

The desired modular inversion is now realized through an elementary gaussian integration and one finds

$$\begin{aligned} Z_{U(N)}^{q\text{YM}}(X(p, q), g_{\text{YM}}^2) &= \left(\frac{4\pi}{g_{\text{YM}}^2} \right)^{N \ell / 2} \frac{1}{\det(C)^{N/2}} \\ &\times \sum_{\mathbf{m}_1, \dots, \mathbf{m}_\ell \in \mathbb{Z}^N} e^{-\frac{4\pi^2}{g_{\text{YM}}^2} (C^{-1})^{ij} (\mathbf{m}_i + \frac{\mathbf{f}_i}{2\pi}) \cdot (\mathbf{m}_j + \frac{\mathbf{f}_j}{2\pi})} \\ &\times \sum_{w \in S_N} \varepsilon(w) e^{2\pi i (\mathbf{m}_i + \frac{\mathbf{f}_i}{2\pi}) \cdot ((C^{-1})^{1i} \boldsymbol{\rho} + (C^{-1})^{i\ell} w(\boldsymbol{\rho}))} \\ &\times e^{\frac{g_{\text{YM}}^2}{4} ((C^{-1})^{11} \boldsymbol{\rho}^2 + 2(C^{-1})^{1\ell} w(\boldsymbol{\rho}) \cdot \boldsymbol{\rho} + (C^{-1})^{\ell\ell} w(\boldsymbol{\rho})^2)}. \end{aligned} \quad (3.12)$$

This expression can be simplified by exploiting the explicit form for the inverse of the Cartan matrix provided in Appendix A, where the definitions of the integers q_i and p_i appearing below may be found. We obtain

$$\begin{aligned} Z_{U(N)}^{q\text{YM}}(X(p, q), g_{\text{YM}}^2) &= \mathcal{N} \sum_{\mathbf{m}_1, \dots, \mathbf{m}_\ell \in \mathbb{Z}^N} e^{-\frac{4\pi^2}{g_{\text{YM}}^2} (C^{-1})^{ij} \mathbf{m}_i \cdot \mathbf{m}_j - \frac{4\pi}{g_{\text{YM}}^2} \mathbf{m}_i \cdot \mathbf{f}_i} \\ &\times \sum_{w \in S_N} \varepsilon(w) e^{\frac{2\pi i}{p} q_i \mathbf{m}_i \cdot (q \boldsymbol{\rho} + w(\boldsymbol{\rho})) + \frac{g_{\text{YM}}^2}{2p} w(\boldsymbol{\rho}) \cdot \boldsymbol{\rho}} \end{aligned} \quad (3.13)$$

where

$$\mathcal{N} := \left(\frac{4\pi}{g_{\text{YM}}^2} \right)^{N \ell / 2} p^{-N/2} e^{\frac{iN}{p} (p_i + q_i) \theta_i + \frac{g_{\text{YM}}^2}{24p} (N^3 - N) (q + q') - \frac{N}{g_{\text{YM}}^2} (C^{-1})^{ij} \theta_i \theta_j}. \quad (3.14)$$

The sum over permutations w in (3.13) does not depend on the instanton numbers \mathbf{m}_i individually, but rather only on the linear combination $q_i \mathbf{m}_i$. This special dependence suggests the change of variables

$$\mathbf{s}_1 = q_i \mathbf{m}_i \quad \text{and} \quad \mathbf{s}_j = \mathbf{m}_j \quad \text{for } j = 2, \dots, \ell \quad (3.15)$$

with $q_1 = 1$ (see Appendix A) and

$$\mathbf{m}_1 = \mathbf{s}_1 - \sum_{i=2}^{\ell} q_i \mathbf{s}_i. \quad (3.16)$$

Then the partition function (3.13) takes the form

$$\begin{aligned}
Z_{U(N)}^{q\text{YM}}(X(p, q), g_{\text{YM}}^2) &= \mathcal{N} \sum_{\mathbf{s}_1, \dots, \mathbf{s}_\ell \in \mathbb{Z}^N} e^{-\frac{4\pi^2}{g_{\text{YM}}^2} q_j h_i \mathbf{s}_i \cdot \mathbf{s}_j - \frac{4}{g_{\text{YM}}^2 p} \sum_{j=2}^k q_j \mathbf{s}_1 \cdot \mathbf{f}_j} e^{-\frac{4\pi^2}{g_{\text{YM}}^2} \frac{q \mathbf{s}_1^2}{p}} \\
&\times \sum_{w \in S_N} \varepsilon(w) e^{\frac{2\pi i}{p} \mathbf{s}_1 \cdot (q \boldsymbol{\rho} + w(\boldsymbol{\rho})) + \frac{g_{\text{YM}}^2}{2p} w(\boldsymbol{\rho}) \cdot \boldsymbol{\rho}}, \quad (3.17)
\end{aligned}$$

where the integers h_i , $i = 1, \dots, \ell$ are defined in Appendix A. The sum over permutations in (3.17) now depends only on the single integer \mathbf{s}_1 . Since the Weyl vector $\boldsymbol{\rho}$ is integer-valued for N odd, the dependence on \mathbf{s}_1 is periodic with period p . It is natural then to decompose the sum over \mathbf{s}_1 in two separate steps. First we sum over \mathbf{s}_1 modulo p , and then we sum over all integer multiples of p . This is achieved through the change of variable

$$\mathbf{s}_1 \longrightarrow \mathbf{m} + p \mathbf{s}_1 \quad (3.18)$$

with $\mathbf{m} \in \mathbb{Z}_p^N$ and $\mathbf{s}_1 \in \mathbb{Z}^N$.

After this final change of variable, we can recast the partition function (3.17) in the form

$$\begin{aligned}
Z_{U(N)}^{q\text{YM}}(X(p, q), g_{\text{YM}}^2) &= \mathcal{N} \sum_{\mathbf{s}_1, \dots, \mathbf{s}_\ell \in \mathbb{Z}^N} \sum_{\mathbf{m} \in \mathbb{Z}_p^N} e^{-\frac{4\pi^2}{g_{\text{YM}}^2} G_{ij} \mathbf{s}_i \cdot \mathbf{s}_j - \frac{4}{g_{\text{YM}}^2 p} \sum_{j=2}^k q_j \mathbf{f}_j \cdot (p \mathbf{s}_1 + \mathbf{m})} \\
&\times e^{-\frac{8\pi^2}{g_{\text{YM}}^2} q \mathbf{s}_1 \cdot \mathbf{m}} Z_{U(N)}^{\text{CS}}(L(p, q), \mathbf{m}), \quad (3.19)
\end{aligned}$$

where the symmetric $\ell \times \ell$ integer-valued matrix G_{ij} is defined in Appendix A and

$$Z_{U(N)}^{\text{CS}}(L(p, q), \mathbf{m}) = e^{-\frac{4\pi^2}{g_{\text{YM}}^2} \frac{q \mathbf{m}^2}{p}} \sum_{w \in S_N} \varepsilon(w) e^{\frac{2\pi i}{p} \mathbf{m} \cdot (q \boldsymbol{\rho} + w(\boldsymbol{\rho})) + \frac{g_{\text{YM}}^2}{2p} w(\boldsymbol{\rho}) \cdot \boldsymbol{\rho}}. \quad (3.20)$$

If we now set

$$g_{\text{YM}}^2 = \frac{4\pi i}{k + N} \quad (3.21)$$

then (3.20) can be identified as the partition function of $U(N)$ Chern-Simons gauge theory with level $k \in \mathbb{N}_0$ on the Lens space $L(p, q)$ in the background of the flat connection defined by the torsion vector $\mathbf{m} \in \mathbb{Z}_p^N$. We will derive this explicitly in Section 4. The relationship between q -deformed Yang-Mills theory and Chern-Simons theory is not surprising and was anticipated by [4, 37, 43]. In this paper we extend this correspondence very explicitly to the generic chain of exceptional divisors of the Hirzebruch-Jung spaces $X(p, q)$ whose boundaries are the more general Lens spaces $L(p, q)$.

3.3 Emergence of Four-Dimensional Instantons

In [3, 4, 8] the expression (3.6) is conjectured to be the partition function of the topologically twisted $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on the four-dimensional toric manifold $X(p, q)$ obtained by blowing up the $A_{p,q}$ singularity. However, as is manifest from the form (3.19), this interpretation is *a priori* difficult. Although there is a sum over ℓ vectors

$\mathbf{s}_i \in \mathbb{Z}^N$ playing the putative role of instanton numbers, this contribution is accompanied in (3.20) by a second sum over permutations w of N elements which is *perturbative* in its dependence on the Yang-Mills coupling constant. Such terms are interpreted as fluctuations around the given instanton background. However, in the topologically twisted $\mathcal{N} = 4$ gauge theory, contributions of this sort are absent because the partition function simply computes the Euler characteristic of the instanton moduli space. Similar problems with interpreting two-dimensional gauge theory partition functions as generating functions for instanton counting were noticed in [8, 41].

It is clear from eq. (3.19) that the two-dimensional gauge theory treats the asymptotic values of the gauge fields on the boundary $L(p, q)$ of $X(p, q)$ as dynamical quantities, whose evolution is governed by the Chern-Simons action on the Lens space $L(p, q)$. This is in marked contrast with what happens in a typical instanton computation, whereby the gauge fields on the boundary are fixed quantities and the partition function is simply obtained by summing over the admissible boundary conditions. In other words, in order to make contact with the four-dimensional instanton computations, it appears natural to drop the perturbative contribution to the Chern-Simons partition function (3.20) from (3.19) and keep only its classical part. This modifies (3.19) to the partition function

$$\begin{aligned} \mathcal{Z}_{X(p,q)}^{U(N)} = \mathcal{N} \sum_{\mathbf{s}_1, \dots, \mathbf{s}_\ell \in \mathbb{Z}^N} \sum_{\mathbf{m} \in \mathbb{Z}_p^N} e^{-\frac{4\pi^2}{g_{\text{YM}}^2} G_{ij} \mathbf{s}_i \cdot \mathbf{s}_j - \frac{4}{g_{\text{YM}}^2 p} \sum_{j=2}^{\ell} q_j \mathbf{f}_j \cdot (p \mathbf{s}_1 + \mathbf{m})} \\ \times e^{-\frac{4\pi^2}{g_{\text{YM}}^2} \frac{q \mathbf{m}^2}{p} - \frac{8\pi^2}{g_{\text{YM}}^2} q \mathbf{s}_1 \cdot \mathbf{m}} . \end{aligned} \quad (3.22)$$

After evaluating the lattice Gauss sum over $\mathbf{m} \in \mathbb{Z}_p^N$, we obtain finally the result

$$\mathcal{Z}_{X(p,q)}^{U(N)} = \mathcal{N} \sum_{\mathbf{u}_1, \dots, \mathbf{u}_\ell \in \mathbb{Z}^N} e^{-\frac{4\pi^2}{g_{\text{YM}}^2} (C^{-1})^{ij} \mathbf{u}_i \cdot \mathbf{u}_j - \mathbf{z}^i \cdot \mathbf{u}_i} \quad (3.23)$$

where we have identified $\mathbf{z}^i = 4(C^{-1})^{ij} \mathbf{f}_j / g_{\text{YM}}^2$ in terms of the chemical potentials for D2-branes wrapped on the exceptional divisors of $X(p, q)$.

The expression (3.23) has a nice interpretation in terms of instanton counting in the $\mathcal{N} = 4$ topologically twisted Yang-Mills theory. Apart from the trivial overall normalization factor, it corresponds precisely to the contribution of fractional instantons to the $\mathcal{N} = 4$ partition function, as provided explicitly in [15] for the family $X(p, q)$ of toric four-manifolds. This identification can also be made by recognizing that the exponent in (3.23) is exactly the classical action for fractional instantons, as we showed in Section 2 (see eq. (2.11)). The fact that the two-dimensional gauge theory naturally captures the contributions of fractional instantons can be understood by noting that they are the pullbacks to $X(p, q)$ of the two-dimensional classical solutions. There are in fact bijective correspondences between two-dimensional instantons on a certain orbifold of \mathbb{P}^1 , flat three-dimensional connections on the boundary $L(p, q)$, and fractional instantons on the four-manifold $X(p, q)$ as illustrated by the discussion of Section 2 and shown in detail in Section 4 below. Moreover, the expression (3.22) yields an even more refined formula providing the contributions of each

topological sector of fractional instantons with fixed holonomy $\mathbf{m} \in \mathbb{Z}_p^N$ at infinity from the finite action requirement that the gauge fields be asymptotically flat.

The contributions from regular instantons are more elusive, since they can move freely on the whole non-compact four-dimensional space, and require in general some sort of regularization procedure. In the q -deformed gauge theory, these ambiguities are most evident on the toric manifold \mathbb{C}^2 , where the fractional instanton contribution is absent. In this case, the gluing rules of Section 3.1 above yield a divergent amplitude which must be suitably regularized. To elucidate further the structure of our result and compare with existing results in the literature, in the remainder of this section we will look more closely at the two extreme cases $\mathcal{O}_{\mathbb{P}^1}(-p)$ ($q = 1$) and ALE spaces ($q = p - 1$).

3.4 Example: Line Bundles over \mathbb{P}^1

The limiting case $X(p, 1)$ is the total space of the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^1}(-p)$ of degree p over \mathbb{P}^1 . In this case the Cartan matrix (2.2) has just one element $e_1 = p$. The partition function (3.6) is that of q -deformed Yang-Mills theory on the sphere, whose instanton expansion was worked out explicitly in [12] and written for $\theta_1 = 0$ in the compact form

$$Z_{U(N)}^{q\text{YM}}(\mathcal{O}_{\mathbb{P}^1}(-p), g_{\text{YM}}^2) = \sum_{\substack{\mathbf{N} \in \mathbb{N}_0^p \\ \sum_k N_k = N}} \prod_{k=0}^{p-1} \frac{\theta_3\left(\frac{4\pi i p}{g_{\text{YM}}^2} \middle| \frac{4\pi i k}{g_{\text{YM}}^2}\right)^{N_k}}{N_k!} Z_{U(N)}^{\text{CS}}(L(p, 1), \mathbf{N}), \quad (3.24)$$

where

$$\theta_3(\tau|z) = \sum_{m \in \mathbb{Z}} e^{\pi i \tau m^2 + 2\pi i m z} \quad (3.25)$$

is a Jacobi-Erdelyi theta-function. Here

$$Z_{U(N)}^{\text{CS}}(L(p, 1), \mathbf{N}) = \exp\left(-\frac{4\pi^2}{g_{\text{YM}}^2 p} \sum_{m=0}^{p-1} N_m m^2\right) \mathcal{W}_p^{\text{inst}}(\underbrace{0, \dots, 0}_{N_0}, \dots, \underbrace{p-1, \dots, p-1}_{N_{p-1}}), \quad (3.26)$$

with

$$\mathcal{W}_p^{\text{inst}}(\mathbf{s}) = \left(\frac{4\pi}{g_{\text{YM}}^2 p}\right)^{N/2} e^{-\frac{g_{\text{YM}}^2(N^3-N)}{12p}} \sum_{w \in S_N} \varepsilon(w) e^{\frac{2\pi i}{p} \mathbf{s} \cdot (\boldsymbol{\rho} + w(\boldsymbol{\rho})) + \frac{g_{\text{YM}}^2}{4p} (\boldsymbol{\rho}^2 + 2w(\boldsymbol{\rho}) \cdot \boldsymbol{\rho})}, \quad (3.27)$$

is simply the partition function of $U(N)$ Chern-Simons gauge theory on the Lens space $L(p, 1)$, the boundary of the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-p)$, for the vacuum contribution corresponding to the p -component partition $\mathbf{N} \in \mathbb{N}_0^p$ of N , as computed in [44, 45] and in Section 4 below.

In this case, the only surviving sum in (3.24) is carried over ordered partitions \mathbf{N} of N into p parts. This ensures that we are summing over gauge inequivalent flat connections on the boundary of $\mathcal{O}_{\mathbb{P}^1}(-p)$. If we now drop the perturbative contribution in (3.27), we

immediately find

$$\begin{aligned}
Z_{\mathcal{O}_{\mathbb{P}^1}(-p)}^{U(N)} &= \mathcal{N} \sum_{\substack{N \in \mathbb{N}_0^p \\ \sum_k N_k = N}} \exp\left(-\frac{4\pi^2}{g_{\text{YM}}^2} \sum_{m=0}^{p-1} N_m m^2\right) \prod_{k=0}^{p-1} \frac{\theta_3\left(\frac{4\pi i p}{g_{\text{YM}}^2} \middle| \frac{4\pi i k}{g_{\text{YM}}^2}\right)^{N_k}}{N_k!} \\
&= \frac{1}{N!} \mathcal{N} \left[\sum_{k=0}^{p-1} e^{-\frac{4\pi^2 k^2}{g_{\text{YM}}^2}} \theta_3\left(\frac{4\pi i p}{g_{\text{YM}}^2} \middle| \frac{4\pi i k}{g_{\text{YM}}^2}\right) \right]^N \\
&= \frac{1}{N!} \mathcal{N} \left[\sum_{k=0}^{p-1} \sum_{m \in \mathbb{Z}} e^{-\frac{4\pi^2}{g_{\text{YM}}^2} (k+pm)^2} \right]^N = \frac{1}{N!} \mathcal{N} \theta_3\left(\frac{4\pi i}{g_{\text{YM}}^2} \middle| 0\right)^N. \quad (3.28)
\end{aligned}$$

This coincides with the contribution, derived in [15], of fractional instantons to the partition function of $\mathcal{N} = 4$ gauge theory on $\mathcal{O}_{\mathbb{P}^1}(-p)$ in the absence of D2-branes.

3.5 Example: ALE Spaces

The partition function of q-deformed Yang-Mills theory on the ALE spaces A_k was first computed in [8] by embedding this space into the local Calabi-Yau threefold $A_k \times \mathbb{C}$. This threefold can be thought of as the limit of the usual ALE fibration over \mathbb{P}^1 as the area of the base \mathbb{P}^1 becomes infinite. Setting $p = k + 1$ and $q = k$, the instanton representation of the partition function is given by

$$\begin{aligned}
Z_{U(N)}^{q\text{YM}}(A_k, g_{\text{YM}}^2) &= \left(\frac{4\pi}{g_{\text{YM}}^2}\right)^{Nk/2} \frac{1}{(k+1)^{N/2}} \\
&\times \sum_{\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{k-1} \in \mathbb{Z}^N} \sum_{\mathbf{m} \in \mathbb{Z}_p^N} e^{-\frac{4\pi^2}{g_{\text{YM}}^2} \mathcal{A}^{ij} \mathbf{s}_i \cdot \mathbf{s}_j - \frac{8\pi^2}{g_{\text{YM}}^2} \sum_j (k-j) \mathbf{s}_j \cdot \mathbf{m}} \\
&\times e^{-\frac{4\pi^2}{g_{\text{YM}}^2} \frac{k}{k+1} \mathbf{m}^2} \sum_{w \in S_N} \varepsilon(w) e^{\frac{2\pi i k}{k+1} \mathbf{m} \cdot (w(\boldsymbol{\rho}) - \boldsymbol{\rho}) + \frac{g_{\text{YM}}^2}{2(k+1)} (k \boldsymbol{\rho}^2 + w(\boldsymbol{\rho}) \cdot \boldsymbol{\rho})}
\end{aligned} \quad (3.29)$$

where the symmetric $k \times k$ matrix elements \mathcal{A}^{ij} for $i, j = 1, \dots, k-1$ are given by

$$\mathcal{A}^{ij} = (C^{-1})^{ij} + \frac{1}{2(k+1)} \left[(k+1-i)(k(k+1-j)-2j) + (k+1-j)(k(k+1-i)-2i) \right], \quad (3.30)$$

while $\mathcal{A}^{0j} = (k+1)(k-j)$ for any $j = 0, 1, \dots, k-1$. To avoid an overly cumbersome expression we have set all θ -angles $\theta_i = 0$. We have also used the fact that the Cartan matrix for ALE spaces coincides with the Cartan matrix for the A_k Dynkin diagram

$$C = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}. \quad (3.31)$$

Dropping the perturbative contribution again we arrive at

$$\mathcal{Z}_{A_k}^{U(N)} = \mathcal{N} \sum_{\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{Z}^N} e^{-\frac{4\pi^2}{g_{\text{YM}}^2} (C^{-1})^{ij} \mathbf{u}_i \cdot \mathbf{u}_j}, \quad (3.32)$$

which is exactly the contribution of fractional instantons to the $\mathcal{N} = 4$ Yang-Mills partition function, derived in [15] via explicit instanton computations, given by the N -th power of (2.11) for $z^i = 0$.

4. Chern-Simons Gauge Theory on Lens Spaces

In this section we will describe in some detail the nonabelian localization of $U(N)$ Chern-Simons gauge theory on the generic three-dimensional Lens spaces $L(p, q)$, and thereby prove some of the assertions made in the previous section. The quantum gauge theory is defined by the path integral

$$\mathcal{Z}_{U(N)}^{\text{CS}}(L(p, q), k) = \int \text{DA} e^{-S_{U(N)}^{\text{CS}}(A)} \quad (4.1)$$

where

$$S_{U(N)}^{\text{CS}}(A) = \frac{ik}{4\pi} \int_{L(p, q)} \text{Tr} \left(A \wedge \text{d}A + \frac{2}{3} A \wedge A \wedge A \right) \quad (4.2)$$

is the Chern-Simons action evaluated on a connection A of a principal $U(N)$ bundle over $L(p, q)$. As is well-known [43], the partition function (4.1) is given exactly by its semi-classical approximation. For this, one takes into account the one-loop radiative correction $k \rightarrow k + N$ and sums over all critical points of the Chern-Simons action (4.2), which are simply the flat $U(N)$ gauge connections on $L(p, q)$. The purpose of carrying out this calculation explicitly is two-fold. Firstly, we will correctly identify for the first time the individual flat connection contributions to the semi-classical expansion of (4.1), generalizing the results of [44, 45] to the generic Lens spaces $L(p, q)$ for all $1 \leq q < p$ and justifying our identification (3.20). Secondly, in the course of this calculation we will construct the explicit mapping between two-dimensional Yang-Mills instantons on a \mathbb{P}^1 orbifold, flat connections on $L(p, q)$, and fractional instantons on $X(p, q)$, which is the crux of the final results of the previous section.

4.1 Classical Solutions

We begin by constructing the flat connections on $L(p, q)$ explicitly. For this, it is convenient to realize the Lens space $L(p, q)$ as a Seifert fibration over the two-sphere [46] (see also [43]). The base is described by a projective line \mathbb{P}^1 with an arbitrarily chosen marked point at which the coordinate neighbourhood is modelled on \mathbb{C}/\mathbb{Z}_p , with the cyclic group acting on the local chart coordinate z as $z \mapsto e^{2\pi i/p} z$. We construct a line V-bundle $\mathcal{L}(p, q)$ over this \mathbb{P}^1 orbifold such that the local trivialization over the orbifold point is modelled by $\mathbb{C}^2/\mathbb{Z}_p$, where \mathbb{Z}_p acts on the local coordinates (z, w) of the base and fibre exactly as in (2.1). The Lens space $L(p, q)$ may then be described as the total space of the associated unit circle bundle $\mathcal{S}(\mathcal{L}(p, q))$. Since p and q are relatively prime, this construction realizes $L(p, q)$ as

the quotient of the three-sphere \mathbb{S}^3 by the free \mathbb{Z}_p -action (2.1), where \mathbb{S}^3 is regarded as the unit sphere in \mathbb{C}^2 . Since $\pi_1(\mathbb{S}^3) = 0$, it follows that the fundamental group of the Lens space is simply

$$\pi_1(L(p, q)) = \pi_0(\mathbb{Z}_p) = \mathbb{Z}_p \quad (4.3)$$

and it is generated by the noncontractible loop encircling the orbifold point on the base \mathbb{P}^1 . Moreover, the Chern class of the line V-bundle over \mathbb{P}^1 describing $L(p, q)$ is

$$c_1(\mathcal{L}(p, q)) = \frac{q}{p} . \quad (4.4)$$

This class cancels the local delta-function curvature at the marked point of \mathbb{P}^1 to ensure that the total degree of the fibration is 0.

Gauge equivalence classes of flat $U(N)$ connections on $L(p, q)$ are in one-to-one correspondence with conjugacy classes of homomorphisms ρ from the fundamental group (4.3) to $U(N)$, i.e. with N -dimensional unitary representations of \mathbb{Z}_p . The image of ρ in $U(N)$ decomposes into N_m copies of the m -th one-dimensional irreducible representation of \mathbb{Z}_p , where $m = 0, 1, \dots, p-1$, and any representation ρ lives in the maximal torus $U(1)^N \subset U(N)$ with

$$N = \sum_{m=0}^{p-1} N_m . \quad (4.5)$$

It follows that there is a one-to-one correspondence between flat $U(N)$ gauge connections on $L(p, q)$ and p -component partitions $\mathbf{N} \in \mathbb{N}_0^p$ of the rank N . Moreover, any such connection defines a central element of the Lie algebra of $U(N)$.

The isomorphism class $[\mathcal{T}]$ of the tautological line bundle over \mathbb{P}^1 is the generator of $H^1(\mathbb{P}^1, U(1)) \cong H^2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z}$. Since $H^1(\mathbb{P}^1, \mathbb{Z}) = 0$, it follows from the Thom-Gysin exact sequence for circle bundles [46] that $H^2(L(p, q), \mathbb{Z}) = H^2(\mathbb{P}^1, \mathbb{Z}) / \langle [\mathcal{T}^p] \rangle \cong \mathbb{Z}_p$. This means that all unitary vector bundles over $L(p, q)$ have p -torsion magnetic charges (Chern classes) m , and that all such torsion bundles over $L(p, q)$ are pullbacks of ordinary bundles over \mathbb{P}^1 under the bundle projection $\pi : \mathcal{S}(\mathcal{L}(p, q)) \rightarrow \mathbb{P}^1$. As we now explicitly demonstrate, this implies that every flat connection on $L(p, q)$ is the pullback of a configuration of Dirac monopoles on the sphere \mathbb{P}^1 . Extending the pullback to the bulk $X(p, q)$ is then in agreement with the construction of fractional instantons given in Section 2.

The critical points of the Yang-Mills action functional $\frac{1}{2g^2} \int_{\mathbb{P}^1} \text{Tr}(F_a \wedge *F_a)$ are the $U(N)$ gauge connections a satisfying $d_a *F_a = 0$. These are the connections with constant central curvature. On the two-sphere every constant curvature bundle is (up to isomorphism) a sum of line bundles. There is thus a one-to-one correspondence between Yang-Mills connections of a principal $U(N)$ -bundle \mathcal{P} of degree m over \mathbb{P}^1 and non-increasing sequences of integers $\mathbf{m} \in \mathbb{Z}^r$, of respective multiplicities $\mathbf{N} \in \mathbb{N}_0^r$, with

$$m = \sum_{i=1}^r m_i \quad \text{and} \quad N = \sum_{i=1}^r N_i . \quad (4.6)$$

On the sphere \mathbb{P}^1 , each such connection is gauge equivalent to the connection $a_0(\mathbf{m}, \mathbf{N}) = \bigoplus_i a^{(m_i)} \mathbb{1}_{N_i}$, where $a^{(m_i)}$ is the monopole potential of magnetic charge m_i and the i -th

block is an abelian connection on the bundle $(\mathcal{T}^{m_i})^{\oplus N_i}$. The curvature of this connection is given by

$$F_{a_0}(\mathbf{m}, \mathbf{N}) = da_0(\mathbf{m}, \mathbf{N}) = \bigoplus_{i=1}^r 2\pi m_i \mathbb{1}_{N_i} \otimes \omega_{\mathbb{P}^1} , \quad (4.7)$$

where $\omega_{\mathbb{P}^1}$ is the symplectic two-form on \mathbb{P}^1 normalized to unit volume

$$\int_{\mathbb{P}^1} \omega_{\mathbb{P}^1} = 1 . \quad (4.8)$$

The monopole connection of course has trivial monodromies around arbitrary smooth points on \mathbb{P}^1 . To take into account the orbifolding of \mathbb{P}^1 required to define the Lens space as a Seifert manifold, we need a connection which has non-trivial monodromy $e^{2\pi i q/p}$ about the given fixed marked point on \mathbb{P}^1 . Since the choice of orbifold point is arbitrary, we may thus define

$$a(\mathbf{m}, \mathbf{N}) := \frac{q}{p} a_0(\mathbf{m}, \mathbf{N}) \quad \text{and} \quad F_a(\mathbf{m}, \mathbf{N}) = da(\mathbf{m}, \mathbf{N}) = \frac{q}{p} F_{a_0}(\mathbf{m}, \mathbf{N}) \quad (4.9)$$

with the requisite monodromy. In particular, the Chern class (4.4) has a Chern-Weil description in terms of smooth curvature in the bulk of the \mathbb{P}^1 orbifold as $c_1(\mathcal{L}(p, q)) = \frac{1}{2\pi} \int_{\mathbb{P}^1} F_a(1, 1)$. The holonomy of this abelian gauge connection depends only on the values of the monopole numbers $m_i \bmod p$. By a trivial rearrangement, we will denote by $0 \leq N_m \leq N$ the multiplicity of the degree m monopole bundle \mathcal{T}^m for the torsion magnetic charges $m = 0, 1, \dots, p-1$ alluded to earlier and hence drop the labels \mathbf{m} from the notation above.

To describe the pullback of these gauge fields to $L(p, q)$, we use (4.4) to introduce a connection κ on the principal $U(1)$ -bundle $\pi : \mathcal{S}(\mathcal{L}(p, q)) \rightarrow \mathbb{P}^1$ whose curvature is given by

$$d\kappa = \frac{q}{p} \pi^*(\omega_{\mathbb{P}^1}) . \quad (4.10)$$

The integral of κ over a generic fibre of the Seifert fibration is given by [37, 43]

$$\oint_{\mathbb{S}^1} \kappa = 1 , \quad (4.11)$$

while from (4.8), (4.10) and (4.11) it follows that the Chern class (4.4) of the line V-bundle $\mathcal{L}(p, q)$ can be computed from the integral

$$\int_{L(p, q)} \kappa \wedge d\kappa = \frac{q}{p} . \quad (4.12)$$

We can now compute the pullback of the curvature in (4.9) as

$$F_A(\mathbf{N}) := \pi^*(F_a(\mathbf{N})) = \bigoplus_{m=0}^{p-1} \frac{2\pi m q}{p} \mathbb{1}_{N_m} \otimes \pi^*(\omega_{\mathbb{P}^1}) = \bigoplus_{m=0}^{p-1} 2\pi m \mathbb{1}_{N_m} \otimes d\kappa , \quad (4.13)$$

from which we may identify the pullback of the Yang-Mills instanton on the \mathbb{P}^1 orbifold up to gauge transformation as

$$A(\mathbf{N}) = \bigoplus_{m=0}^{p-1} 2\pi m \mathbb{1}_{N_m} \otimes \kappa . \quad (4.14)$$

Note that from (4.11) it follows that the connection (4.14) has trivial holonomy along any \mathbb{S}^1 fibre of the Seifert manifold, $\exp(i \oint_{\mathbb{S}^1} A(\mathbf{N})) = \mathbb{1}_N$, as required since all fibre loops are contractible in $L(p, q)$ and the only non-trivial elements of (4.3) arise from loops which wind around the marked point of the base \mathbb{P}^1 . Moreover, if $\mathcal{P} \rightarrow \mathbb{P}^1$ is the irreducible $U(N)$ bundle of degree m on which the two-dimensional gauge theory is defined, then the corresponding flat gauge bundle over $L(p, q)$ is [46] $\pi^*(\mathcal{P}) \otimes \pi^*(\mathcal{T}^{-m})$.

Finally, we can compute the value of the Chern-Simons action (4.2) on a generic classical solution on $L(p, q)$ by using the fact that the connection (4.14) has constant central curvature. After taking into account the quantum shift of the Chern-Simons level $k \rightarrow k + N$, one finds

$$\begin{aligned} S_{U(N)}^{\text{CS}}(\mathbf{N}) &:= S_{U(N)}^{\text{CS}}(A(\mathbf{N})) = \frac{i(k+N)}{4\pi} \int_{L(p,q)} \text{Tr} (A(\mathbf{N}) \wedge dA(\mathbf{N})) \\ &= \frac{i(k+N)}{4\pi} \sum_{m=0}^{p-1} (2\pi m)^2 N_m \int_{L(p,q)} \kappa \wedge d\kappa . \end{aligned} \quad (4.15)$$

By using eq. (4.12) we arrive at the final form

$$S_{U(N)}^{\text{CS}}(\mathbf{N}) = \frac{\pi i(k+N)q}{p} \sum_{m=0}^{p-1} N_m m^2 . \quad (4.16)$$

This result confirms the conjectured formula [47] for the set of values of the Chern-Simons action functional of flat G -connections on $L(p, q)$ in the case of gauge group $G = U(N)$. It also agrees with the classical part of the partition function (3.20), upon using the identification (3.21).

4.2 Semi-Classical Expansion

We will now describe how to compute the one-loop fluctuation determinants needed to write down the localization of the partition function (4.1) onto a sum over the classical solutions constructed in Section 4.1 above. For this, we use the well-known surgery construction of the Lens space $L(p, q)$ [48, 49]. Choose a pair of integers r, s which satisfy the Diophantine equation $sq - rp = 1$. Then the Seifert manifold $L(p, q)$ is obtained from $\mathbb{P}^1 \times \mathbb{S}^1$ by removing a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ (with disk $\mathbb{D}^2 \subset \mathbb{P}^1$) and gluing it back by twisting its torus boundary by the $SL(2, \mathbb{Z})$ modular transformation

$$\mathbf{M} = \begin{pmatrix} q & r \\ p & s \end{pmatrix} . \quad (4.17)$$

The basis element $(q, p) \in H_1(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z}) \cong \mathbb{Z}^2$ specifies the slope of the meridian of the boundary torus, while (r, s) gives the slope of the longitude. With the continued fraction expansion (2.3), the gluing matrix (4.17) can be cast in the form

$$\mathbf{M} = \mathbf{S} \mathbf{T}^{e_1} \mathbf{S} \dots \mathbf{S} \mathbf{T}^{e_\ell} \mathbf{S} , \quad (4.18)$$

where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (4.19)$$

are the standard generators of $SL(2, \mathbb{Z})$ obeying the relations $S^2 = (S T)^3 = \mathbb{1}$. In particular, the matrix (2.2) in this context gives the linking matrix of the framed surgery link with framings of components specified by the integers e_i .

According to the gluing rules of topological quantum field theory [50], the partition function (4.1) of Chern-Simons theory in the canonical two-framing of $L(p, q)$ may thus be computed (up to normalization) as the matrix element [48, 49]

$$Z_{U(N)}^{\text{CS}}(L(p, q), k) = \mathcal{R}(\mathbf{M})_{0,0}, \quad (4.20)$$

where \mathcal{R} is the representation of the mapping class group on the finite-dimensional quantum Hilbert space of Chern-Simons gauge theory on $\mathbb{P}^1 \times \mathbb{S}^1$. In the Verlinde basis of level k integrable representations R of the $U(N)$ WZW model, the generators (4.19) are represented as [49]

$$\mathcal{R}(S)_{R,Q} = S_{R,Q} \quad \text{and} \quad \mathcal{R}(T)_{R,Q} = \delta_{R,Q} T_R \quad (4.21)$$

in terms of the amplitudes (3.8) and (3.5) with the identification (3.21). We can thus write (4.20) as

$$Z_{U(N)}^{\text{CS}}(L(p, q), k) = \sum_{R_1, \dots, R_\ell} S_{0, R_1} S_{R_1, R_2} \cdots S_{R_{\ell-1}, R_\ell} S_{R_\ell, 0} T_{R_1}^{e_1} \cdots T_{R_\ell}^{e_\ell}. \quad (4.22)$$

Although formally identical to the q -deformed gauge theory partition function (3.6) with $\theta_i = 0$, in (4.20) one quantizes the Yang-Mills coupling as in (3.21) and restricts the sum to *integrable* representations of the $U(N)$ gauge group at level $k \in \mathbb{N}_0$. Similar correspondences between q -deformed Yang-Mills theory and Chern-Simons theory on circle bundles have been noted in [37].

The sums over integrable representations in (4.22) can be written in terms of weight vectors $\mathbf{n}(R_i)$ with $0 \leq n_a(R_i) \leq N + k - 1$. Using the explicit matrix elements in (3.8) and (3.5), this writes the Chern-Simons partition function as a lattice Gauss sum. To cast (4.22) as a sum over classical solutions, one uses the reciprocity formula for Gauss sums to resum the expansion over weight vectors. This calculation was first performed for the case of an $SU(2)$ gauge group in [49], and more recently extended in [47] to arbitrary simple Lie groups G . We will not enter into the intricate details of this calculation here, which are analogous to the Poisson resummation carried out in Section 3.2. Dropping irrelevant overall normalization factors, for $G = U(N)$ one finds [47]

$$Z_{U(N)}^{\text{CS}}(L(p, q), k) = \sum_{\mathbf{m} \in \mathbb{Z}_p^N} e^{-\frac{\pi i (k+N) q}{p} \mathbf{m}^2} \mathcal{W}_{U(N)}^{\text{fluct}}(p, q; \mathbf{m}) \quad (4.23)$$

where

$$\mathcal{W}_{U(N)}^{\text{fluct}}(p, q; \mathbf{m}) = \sum_{w \in S_N} \varepsilon(w) e^{-\frac{2\pi i}{(k+N)p} w(\boldsymbol{\rho}) \cdot \boldsymbol{\rho}} e^{\frac{2\pi i}{p} \mathbf{m} \cdot (q \boldsymbol{\rho} - w(\boldsymbol{\rho}))}. \quad (4.24)$$

In the first exponential factor of (4.23) we recognize the Boltzmann weights of the classical Chern-Simons action (4.16) evaluated on the set of critical points. The Weyl group sums (4.24) thereby represent the one-loop quantum fluctuation determinants about the classical solutions. This justifies the identification made in (3.20), and also the analysis of Section 3.3, after the reflections $(p, q) \rightarrow (-p, -q)$. This defines an orientation-reversing isometry of the four-manifold $X(p, q)$ under which the topologically twisted $\mathcal{N} = 4$ Yang-Mills theory is invariant, but under which the Chern-Simons and q -deformed gauge theories are not. The remarkable feature of the calculation performed in [47, 49] proceeding from (4.22) to (4.23) is that the final form depends only on the integers p and q which uniquely determine the Seifert space $L(p, q)$ up to isomorphism, and not on the continued fraction expansion (2.3). This is expected, since the surgery construction of the Chern-Simons partition function (4.20) is independent of the framing integers e_i [48]. Moreover, while the expansion (2.3) is not unique, any two such decompositions are related by an $SL(2, \mathbb{Z})$ transformation, and the Chern-Simons partition function is invariant under the action of the mapping class group. In marked contrast, the geometry of the Hirzebruch-Jung space $X(p, q)$ depends crucially on the continued expansion of $\frac{p}{q} \pmod{SL(2, \mathbb{Z})}$ and the corresponding gauge theory amplitudes reflect this dependence.

5. Instantons on Higher Genus Ruled Surfaces

In this section we will address the problem of counting instantons on the ruled Riemann surfaces [16], which can be described as the total space of a holomorphic line bundle $\mathcal{O}_{\Sigma_g}(-p)$ of degree p over a compact Riemann surface Σ_g of genus $g \geq 1$. This non-toric manifold can be viewed as a non-compact four-cycle in the local Calabi-Yau threefold which is the total space of the holomorphic rank 2 vector bundle $\mathcal{O}_{\Sigma_g}(-p) \oplus \mathcal{O}_{\Sigma_g}(2g - 2 + p)$, as considered by [3, 4] for the problem of counting BPS black hole microstates in four dimensions. In this case, the direct instanton counting in four dimensions is a difficult problem, and the two-dimensional gauge theory could thus provide valuable insight. In [4] the q -deformed gauge theory on Σ_g is proposed to compute the relevant Euler characteristic of the instanton moduli space.

In the case of gauge group $U(1)$, one can give a prediction for the partition function of $\mathcal{N} = 4$ gauge theory on $\mathcal{O}_{\Sigma_g}(-p)$ for any genus g . From the known partition function of $U(1)$ gauge theory on Σ_g [51], one can follow the prescription of Section 3.3 to read off the fractional instanton contribution directly as

$$Z_{\text{frac}}^{U(1)}(\Sigma_g) = \sqrt{\frac{4\pi}{g_{\text{YM}}^2 p}} \sum_{m \in \mathbb{Z}} e^{-\frac{4\pi^2}{g_{\text{YM}}^2 p} m^2 - z m}. \quad (5.1)$$

On the other hand, the moduli space of n regular instantons of rank 1 is given by the Hilbert scheme $\mathcal{O}_{\Sigma_g}(-p)^{[n]}$ of n points on the total space of the bundle $\mathcal{O}_{\Sigma_g}(-p)$ [35]. The generating function for the corresponding Poincaré polynomials is given by

$$\sum_{n=0}^{\infty} P(t | \mathcal{O}_{\Sigma_g}(-p)^{[n]}) e^{2\pi i n \tau} = \prod_{m=1}^{\infty} \frac{(1 + t^{2m-1} e^{2\pi i m \tau})^{2g}}{(1 - e^{2\pi i m \tau} t^{2m})^2 (1 - e^{2\pi i m \tau} t^{2m-2})}. \quad (5.2)$$

By setting $t = 1$ in (5.2) we get the contribution of regular instantons to the four-dimensional partition function. By assuming the factorization property between regular and fractional instanton contributions we finally get the total partition function (dropping irrelevant overall normalizations)

$$Z^{U(1)}(\Sigma_g) = \prod_{m=1}^{\infty} \frac{(1 + e^{2\pi i m \tau})^{2g}}{(1 - e^{2\pi i m \tau})^2} \sum_{n \in \mathbb{Z}} e^{\frac{\pi i \tau n^2}{p} - z n} = \frac{\hat{\eta}(2\tau)^{2g}}{\hat{\eta}(\tau)^{2g+2}} \theta_3\left(\frac{\tau}{p} \middle| \frac{i z}{2\pi}\right). \quad (5.3)$$

Compared to the genus 0 cases considered in the previous sections, the computation for higher rank gauge groups is much more involved in this case. In particular, the $U(N)$ instanton partition function does not trivially factorize into a $U(1)^N$ contribution. Let us illustrate this point in the genus 1 case, wherein a complete analysis of the two-dimensional Yang-Mills partition function and of its relation with Chern-Simons theory has been recently carried out in [41]. Starting from these results, it is possible to repeat the procedure of Section 3.3 to extract the contributions of fractional instantons in four dimensions for nonabelian gauge group. For example, for $U(2)$ gauge group and vanishing θ -angle the instanton expansion of the two-dimensional Yang-Mills partition function on Σ_1 is given by

$$\begin{aligned} Z_{U(2)}^{q\text{YM}}(\mathcal{O}_{\Sigma_1}(-p), g_{\text{YM}}^2) &= \sum_{m_1, m_2 \in \mathbb{Z}} \left((-1)^{m_1+m_2} \frac{4\pi}{g_{\text{YM}}^2 p} + \delta_{m_1, m_2} \frac{1}{\sqrt{2}} \sqrt{\frac{4\pi}{g_{\text{YM}}^2 p}} \right) \\ &\quad \times e^{-\frac{4\pi^2}{g_{\text{YM}}^2 p} (m_1^2 + m_2^2)} \\ &+ \sum_{m_0 \in \mathbb{Z}} \frac{(-1)^{2m_0+1}}{\sqrt{2}} \sqrt{\frac{4\pi}{g_{\text{YM}}^2 p}} e^{-\frac{2\pi^2}{g_{\text{YM}}^2 p} (2m_0+1)^2}, \end{aligned} \quad (5.4)$$

where the coefficients of the exponentials can be identified with the one-loop fluctuations in Chern-Simons gauge theory on a torus bundle over the circle [41]. It follows that the $U(2)$ partition function for fractional instantons on the total space of $\mathcal{O}_{\Sigma_1}(-p)$ is given by

$$\begin{aligned} Z_{\text{frac}}^{U(2)}(\Sigma_1) &= \sum_{m_1, m_2 \in \mathbb{Z}} e^{-\frac{4\pi^2}{g_{\text{YM}}^2 p} (m_1^2 + m_2^2)} + \sum_{m_0 \in \mathbb{Z}} e^{-\frac{8\pi^2}{g_{\text{YM}}^2 p} (m_0 + \frac{1}{2})^2} \\ &= \theta_3\left(\frac{\tau}{p} \middle| 0\right)^2 + \theta_2\left(\frac{2\tau}{p} \middle| 0\right). \end{aligned} \quad (5.5)$$

Due to the presence of the last term on the right-hand side of (5.5), the partition function for $U(2)$ gauge group cannot be written as the square of that for $U(1)$. This extra term is due to the appearance of singular fixed points in the nonabelian localization prescription on higher genus surfaces, arising from irreducible connections of the two-dimensional gauge theory. Thus, in order to provide a general formula in the nonabelian case for the class of non-toric four-manifolds modelled on $\mathcal{O}_{\Sigma_g}(-p)$, a more careful analysis is required.

6. Conclusions

In this paper we have shown how instanton counting on the most general four-dimensional toric singularities $A_{p,q}$ can be carried out by studying the classical solutions of a suitable two-dimensional gauge theory living on the necklace of \mathbb{P}^1 's arising in their minimal resolutions $X(p,q)$. We have found that the two-dimensional gauge theory captures the contributions of instantons which are stacked at the singularity. These instantons can be recovered from pullback of the classical solutions of two-dimensional Yang-Mills theory. Identical results have been obtained by a direct four-dimensional analysis in [15], where the contributions of instantons which are free to move in the non-compact directions of $X(p,q)$ have also been investigated. The appearance of these latter configurations is more elusive in the two-dimensional gauge theory and require suitable regularization. Due to the lack of an explicit construction of their moduli space, a complete evaluation of their contribution to the D0–D2–D4 brane partition function is not yet available except for ALE spaces [28, 36, 15], and the $\mathcal{O}_{\mathbb{P}^1}(-p)$ spaces for $p = 1$ [29, 52] and $p = 2$ [30].

In contrast to the four-dimensional case, the two-dimensional gauge theory description contains perturbative contributions coming from the fluctuations of flat connections at the boundary of $X(p,q)$. As shown in [10]–[12, 14] in the case of the space $X(p,1)$, these fluctuations are a crucial ingredient in reproducing the large N factorization of q-deformed Yang-Mills theory into holomorphic and antiholomorphic topological string amplitudes, in accordance with the OSV conjecture [1]. It would be interesting to better understand the meaning of these perturbative corrections from the perspective of counting black hole microstates and D-brane bound states.

The two-dimensional gauge theory can also be applied to more general non-toric manifolds such as the ruled Riemann surfaces studied in Section 5, which are four-cycles of the local Calabi-Yau threefold given by the total space of the bundle $\mathcal{O}_{\Sigma_g}(-p) \oplus \mathcal{O}_{\Sigma_g}(2g-2+p)$. In this case the pertinent two-dimensional gauge theory is still exactly solvable. Its large N chiral expansion in the case $p = 0$ has been carried out in [53]. Some results for $U(1)$ gauge group at any genus g and gauge group $U(2)$ at genus $g = 1$ are derived in Section 5. It would be gratifying to corroborate these expectations with a direct evaluation in four dimensions.

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A. Continued Fractions and the Cartan Matrix

In this appendix we collect some useful properties of the inverse of the intersection matrix (2.2) based on the continued fraction expansion (2.3) of the rational number $\frac{p}{q}$. These

properties are used extensively in Section 3 to simplify the final result of the Poisson resummation of the q -deformed Yang-Mills partition function. Given eq. (2.3), we construct two sequences of integers $\{p_j\}$ and $\{q_j\}$ defined by the continued fraction expansions

$$\frac{p_{j-1}}{p_j} = [e_j, e_{j+1}, \dots, e_\ell] \quad \text{and} \quad \frac{q_{j+1}}{q_j} = [e_j, e_{j-1}, \dots, e_1] \quad (\text{A.1})$$

for $1 \leq j \leq \ell$, together with the initial conditions $q_0 = 0$ and $p_\ell = 1$. They satisfy the Diophantine relations

$$q_i p_j - p_i q_j = p n_{ij} \quad (\text{A.2})$$

where

$$n_{ij} = \begin{cases} 0 & \text{for } i = j, \\ 1 & \text{for } i = j + 1, \\ \text{num}[e_{j+1}, \dots, e_{i-1}] & \text{for } i > j + 1, \end{cases} \quad (\text{A.3})$$

and num stands for the numerator of the continued fraction expansion. We have the obvious values

$$p_1 = q, \quad q_1 = 1 \quad \text{and} \quad q_\ell = q' \quad (\text{A.4})$$

where the integer q' is defined by $q q' \equiv 1 \pmod{p}$.

In terms of these sequences of integers, the inverse of the Cartan matrix admits the elegant form [54]

$$(C^{-1})_{ij} = \begin{cases} \frac{1}{p} q_i p_j & \text{for } 1 \leq i \leq j \leq \ell, \\ \frac{1}{p} p_i q_j & \text{for } 1 \leq j \leq i \leq \ell. \end{cases} \quad (\text{A.5})$$

A simple application of the form (A.5) of the Cartan matrix and of the relations (A.1)–(A.4) shows that

$$(C^{-1})_{1i} - q (C^{-1})_{i\ell} = \frac{1}{p} (q_1 p_i - p_1 q_i p_\ell) = \frac{1}{p} (q_1 p_i - p_1 q_i) = -n_{1i}. \quad (\text{A.6})$$

One can also show [54] from these relations that p is the determinant of the Cartan matrix C while q is the determinant of the first minor of C . In the main text we make use of the short-hand notation

$$h_i = n_{i1}, \quad (\text{A.7})$$

and define the symmetric $\ell \times \ell$ matrix G_{ij} by

$$G_{11} = p q, \quad G_{1s} = G_{s1} = h_s \quad \text{and} \quad G_{s_1 s_2} = h_{s_1} q_{s_2} \quad (\text{A.8})$$

for $s, s_1, s_2 = 2, \dots, \ell$ and $s_1 \leq s_2$.

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