

Gauge Fields, Fermions and Mass Gaps in 6D Brane Worlds

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Abstract

We study fluctuations about axisymmetric warped brane solutions in 6D minimal gauged supergravity. Much of our analysis is general and could be applied to other scenarios. We focus on bulk sectors that could give rise to Standard Model gauge fields and charged matter. We reduce the dynamics to Schroedinger type equations plus physical boundary conditions, and obtain exact solutions for the Kaluza-Klein wave functions and discrete mass spectra. The power-law warping, as opposed to exponential in 5D, means that zero mode wave functions can be peaked on negative tension branes, but only at the price of localizing the whole Kaluza-Klein tower there. However, remarkably, the codimension two defects allow the Kaluza-Klein mass gap to remain finite even in the infinite volume limit. In principle, not only gravity, but Standard Model fields could ‘feel’ the extent of large extra dimensions, and still be described by an effective 4D theory.

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1 Introduction

Minimal 6D gauged supergravity and its solutions have received much interest recently for several reasons. From the top down, the theory shares many features in common with 10D supergravity, whilst remaining relatively simple, and so it can be used as a toy model for 10D string theory compactifications [1]. From the bottom up, it provides a context in which to extend the well-trodden path of 5D brane world models to codimension two. Moreover, in 6D models, Supersymmetric Large Extra Dimensions (SLED) have shown some promise in addressing the two overshadowing fine-tuning problems of fundamental physics: the Gauge Hierarchy and the Cosmological Constant Problems [2].

In terms of the phenomenological study of brane worlds, one should ask what are the qualitative differences between 5D and 6D models. For example, in five dimensional Randall-Sundrum (RS) models [3] the warping of 4D spacetime slices is exponentially dependent on the proper radius of the extra dimension, whereas in the six dimensional models developed in Refs. [4, 5, 6] it is only power law dependent. Moreover, the singularities sourced by the branes are distinct, the codimension one case being a jump and the codimension two case being conical.

5D RS models with large (or infinite) extra dimension and the Standard Model (SM) confined to the brane were developed to explain the hierarchy in the Planck and Electroweak scales. Although the mass gap in the Kaluza-Klein spectrum goes to zero as usual in the infinite volume limit, 4D physics is retrieved thanks to the warp factor's localization of the zero mode graviton - and exponential suppression of higher modes - close to the brane with positive tension. Subsequently it was found that the step singularities in the geometry could also localize bulk fermions [7], in much the same way as previously achieved with scalar fields and kink topological defects [8]. Models were then developed in which SM fields all⁴ originate from the bulk as localized degrees of freedom [9].

A study of warped brane worlds in 6D supergravity was initiated in [4, 5, 6], where the focus was on the background solutions. As a result of supersymmetry [10], and anomaly cancellation [11], the 6D theory is highly constrained, and moreover fairly general classes of solutions have been obtained. In [4], a general solution with 4D Poincaré and 2D axial symmetry was derived, and it was shown that warping always leads to conical (or worse) singularities in the internal manifold, which can be naturally interpreted as 3-branes sources. It is certainly interesting to go beyond these background solutions, and study the dynamics of their fluctuations. The final objective would be to obtain the effective theory describing 4D physics, and an understanding of when this effective theory is valid.

Although in these constructions the SM is usually put in by hand, envisioned on a 3-brane source, the bulk theory is potentially rich enough to contain the SM gauge and matter fields. As hinted above, should the SM arise as Kaluza-Klein zero modes of bulk fields, there are two ways to hide the heavy modes and recover 4D physics. They may have a large mass gap, and thus be unattainable at the energy scales thus-far encountered in our observed universe. Or they may be light but very weakly coupled to the massless modes, for example if the massless modes are peaked near to the brane, and the massive modes are

⁴Although the Higgs field should be confined to the brane in order not to lose the gauge hierarchy.

not. In any case, whether or not one expects the bulk to give rise to the SM, one should study its degrees of freedom and determine under which conditions they are observable or out of sight. This could also prove useful for a deeper understanding of the self-tuning mechanism of SLED, and its quantum corrections.

A complete study of the linear perturbations is a very complicated problem, involving questions of gauge-fixings and a highly coupled system of dynamical equations. Some partial results have been obtained for the scalar perturbations in [12]. In this paper we consider sectors within which SM gauge and charged matter fields might be found. By some fortune, these also happen to be two of the least complicated ones. Much of our discussion is general, and could easily be applied or extended to other 6D models with axial symmetry. We follow the usual Kaluza-Klein procedure, and reduce the equations of motion to an equivalent non-relativistic quantum mechanics problem, which we are able to solve exactly. We consider carefully the boundary conditions that the physical modes must satisfy, and from these derive the wave function profiles and complete discrete mass spectra.

Our exact solutions enable us to analyze in detail the effects of the power-law warping and conical defects that arise in 6D brane worlds. We find that the warping cannot give rise to zero modes peaked at the brane, without also leading to peaked profiles for the entire Kaluza-Klein tower. On the other hand, the conical defects do break another standard lore of the classical Kaluza-Klein theory. Remarkably, even if the volume of the internal manifold goes to infinity, the mass gap does not necessarily go to zero. This decoupling between the mass gap and volume means that in principle SM fields, in addition to gravity, could ‘feel’ the extent of large extra dimensions, whilst still being accurately described by a 4D effective field theory.

The paper is organised as follows. We begin in Section 2 by reviewing the 6D supergravity theory, and its axially symmetric warped brane world solutions. Then, in Section 3 we analyze the gauge field fluctuations, deriving the wave functions and masses of the Kaluza-Klein spectrum. A similar analysis is presented in Section 4 for the fermions. Section 5 discusses the physical implications of the results found, and in particular whether they can be naturally applied to the Large Extra Dimension scenario. Finally we end in Section 6 with some conclusions and future directions.

In the appendices we give some results that are useful for the detailed calculations. Appendix A sets our conventions, and Appendix B explains how the conical defects manifest themselves in the metric ansatz. In Appendix C we show in detail how the boundary conditions are applied to obtain a discrete mass spectrum, and we give the complete fermionic mass spectrum thus derived in Appendix D.

2 6D Supergravity and Axisymmetric Solutions

Let us begin by presenting 6D $\mathcal{N} = 1$ gauged supergravity, and its warped brane world solutions, whose fluctuations we will then study. The field content consists of a supergravity-tensor multiplet $(G_{MN}, B_{MN}, \sigma, \psi_M, \chi)$ - with metric, anti-symmetric Kalb-Ramond field (with field strength H_{MNP}), dilaton, gravitino and dilatino - coupled to a number of gauge multiplets (\mathcal{A}_M, λ) - with gauge potentials and gauginos - and a number of hypermultiplets

(Φ, Ψ) - with hyperscalars and hyperinos. The fermions are all Weyl spinors, satisfying $\Gamma_7\psi_M = \psi_M$, $\Gamma_7\chi = -\chi$, $\Gamma_7\lambda = \lambda$ and $\Gamma_7\Psi = -\Psi$, and in general the theory has anomalies. However, for certain gauge groups and hypermultiplet representations these anomalies can be cancelled via a Green-Schwarz mechanism [11, 13]. We will consider a general matter content, with gauge group of the form $\mathcal{G} = \tilde{\mathcal{G}} \times U(1)_R$, where $U(1)_R$ is an R-symmetry, and $\tilde{\mathcal{G}}$ is in general a product of simple groups. For example we could take the anomaly free group $\mathcal{G} = E_6 \times E_7 \times U(1)_R$, under which the fermions are charged as follows: $\psi_M \sim (1, 1)_1$, $\chi \sim (1, 1)_1$, $\lambda \sim (78, 1)_1 + (1, 133)_1 + (1, 1)_1$, $\Psi \sim (1, 912)_0$ [11].

The bosonic action takes the form⁵ [10]

$$S_B = \int d^6 X \sqrt{-G} \left[\frac{1}{\kappa^2} R - \frac{1}{4} \partial_M \sigma \partial^M \sigma - \frac{1}{4} e^{\kappa\sigma/2} \text{Tr} \left(\frac{1}{\tilde{g}^2} \tilde{F}^2 + \frac{1}{g_1^2} F_1^2 \right) - \frac{1}{12} e^{\kappa\sigma} H_3^2 - G_{\alpha\beta}(\Phi) D_M \Phi^\alpha D^M \Phi^\beta - \frac{8}{\kappa^4} e^{-\kappa\sigma/2} v(\Phi) \right], \quad (2.1)$$

where κ represents the 6D Planck scale, \tilde{g} and g_1 are the $\tilde{\mathcal{G}}$ and $U(1)_R$ gauge coupling constants respectively⁶ and \tilde{F}_{MN} and F_{1MN} the corresponding field strengths. $G_{\alpha\beta}(\Phi)$ is the metric on the target manifold of the hyperscalars, and here the index α runs over all the hyperscalars. The dependence of the scalar potential on Φ^α is such that its minimum is at $\Phi^\alpha = 0$, where it takes a positive-definite value, $v(0) = g_1^2$, due to the R-symmetry gauging [14, 15]. We therefore fix $\Phi^\alpha = 0$. The remaining equations of motion (EOM) are

$$\begin{aligned} \frac{1}{\kappa^2} R_{MN} &= \frac{1}{4} \partial_M \sigma \partial_N \sigma + \frac{1}{2} e^{\kappa\sigma/2} \left[\frac{1}{\tilde{g}^2} \text{Tr} \left(\tilde{F}_{MP} \tilde{F}_N{}^P \right) + \frac{1}{g_1^2} \text{Tr} \left(F_{1MP} F_{1N}{}^P \right) \right] \\ &\quad + \frac{1}{4} e^{\kappa\sigma} H_{MPQ} H_N{}^{PQ} - \frac{1}{4\kappa} G_{MN} \square \sigma, \\ \frac{1}{\kappa} \square \sigma &= \frac{1}{4} e^{\kappa\sigma/2} \left[\frac{1}{\tilde{g}^2} \text{Tr} \tilde{F}^2 + \frac{1}{g_1^2} \text{Tr} F_1^2 \right] + \frac{1}{6} e^{\kappa\sigma} H_{MNP} H^{MNP} - \frac{8g_1^2}{\kappa^4} e^{-\kappa\sigma/2}, \\ D_M \left(e^{\kappa\sigma/2} \tilde{F}^{MN} \right) &= \frac{\kappa}{2} e^{\kappa\sigma} H^{NPQ} \tilde{F}_{PQ}, \quad D_M \left(e^{\kappa\sigma/2} F_1^{MN} \right) = \frac{\kappa}{2} e^{\kappa\sigma} H^{NPQ} F_{1PQ}, \\ D_M \left(e^{\kappa\sigma} H^{MNP} \right) &= 0, \end{aligned} \quad (2.2)$$

where D_M is the gauge and Lorentz covariant derivative.

We will consider a general class of warped solutions with 4D Poincaré symmetry, and axial symmetry in the transverse dimensions:

$$\begin{aligned} ds^2 = G_{MN} dX^M dX^N &= e^{A(\rho)} \eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + e^{B(\rho)} d\varphi^2, \\ \mathcal{A} &= \mathcal{A}_\varphi(\rho) Q d\varphi, \\ \sigma &= \sigma(\rho), \\ H_{MNP} &= 0, \end{aligned} \quad (2.3)$$

⁵We give conventions and notation in Appendix A. For fermionic terms see [10].

⁶In general, since $\tilde{\mathcal{G}}$ consists of several simple factors, \tilde{g} represents a collection of independent gauge couplings.

with $0 \leq \rho \leq \bar{\rho}$ and $0 \leq \varphi < 2\pi$. Here $\mu, \nu = 0, 1, 2, 3$, \mathcal{A} is a gauge field, and Q is a generator of a $U(1)$ subgroup of a simple factor of \mathcal{G} , satisfying $Tr(Q^2) = 1$.

In the following we shall also use the radial coordinate defined by

$$u(\rho) \equiv \int_0^\rho d\rho' e^{-A(\rho')/2}, \quad (2.4)$$

whose range is $0 \leq u \leq \bar{u}$. In this frame the metric reads

$$ds^2 = e^{A(u)} \left(\eta_{\mu\nu} dx^\mu dx^\nu + du^2 \right) + e^{B(u)} d\varphi^2. \quad (2.5)$$

Given the above ansatz, the general solution to the equations of motion (2.2) has been found by Gibbons, Güven and Pope (GGP) in [4]⁷. Although much of our formalism for the perturbation analysis can be applied to the general ansatz (2.3), we will focus on a subset of this general solution, namely that which contains singularities no worse than conical. Thus, in addition to the ansatz (2.3), we impose the following asymptotic behaviour for the metric:

$$e^A \xrightarrow{\rho \rightarrow 0} \text{constant} \neq 0, \quad e^A \xrightarrow{\rho \rightarrow \bar{\rho}} \text{constant} \neq 0,$$

and

$$e^B \xrightarrow{\rho \rightarrow 0} (1 - \delta/2\pi)^2 \rho^2, \quad e^B \xrightarrow{\rho \rightarrow \bar{\rho}} (1 - \bar{\delta}/2\pi)^2 (\bar{\rho} - \rho)^2, \quad (2.6)$$

that is we assume conical defects with deficit angle δ at $\rho = 0$ and $\bar{\delta}$ at $\rho = \bar{\rho}$, at which points the Ricci scalar contains delta-functions (see Appendix B). These can be interpreted as 3-brane sources with tensions $T = 2\delta/\kappa^2$ and $\bar{T} = 2\bar{\delta}/\kappa^2$ [16].

The explicit conical-GGP solution is then⁸ [4]:

$$\begin{aligned} e^A &= e^{\kappa\sigma/2} = \sqrt{\frac{f_1}{f_0}}, \quad e^B = \alpha^2 e^A \frac{r_0^2 \cot^2(u/r_0)}{f_1^2}, \\ \mathcal{A} &= -\frac{4\alpha g}{q\kappa f_1} Q d\varphi, \end{aligned} \quad (2.7)$$

where α is a real number, which we can take to be positive without loss of generality, q is a real number and g is the gauge coupling constant corresponding to the background gauge field. For example, if \mathcal{A} lies in $U(1)_R$, then $g = g_1$. Also,

$$f_0 \equiv 1 + \cot^2\left(\frac{u}{r_0}\right), \quad f_1 \equiv 1 + \frac{r_0^2}{r_1^2} \cot^2\left(\frac{u}{r_0}\right), \quad (2.8)$$

with $r_0^2 \equiv \kappa^2/(2g_1^2)$, $r_1^2 \equiv 8/q^2$. Moreover, $\bar{u} \equiv \pi r_0/2$ and the deficit angles are

$$\delta = 2\pi \left(1 - \alpha \frac{r_1^2}{r_0^2} \right), \quad (2.9)$$

$$\bar{\delta} = 2\pi(1 - \alpha). \quad (2.10)$$

⁷Relaxing the condition of 4D Poincaré symmetry to that of only 4D maximal symmetry should allow more general solutions to be found [17]. More general solutions also exist breaking axial symmetry [18], or having nontrivial VEVs for the hyperscalars [19].

⁸The coordinate u is related to the coordinate r in [4] by $r = r_0 \cot(u/r_0)$.

Notice then that a non-trivial warping enforces the presence of a 3-brane source. On the other hand, the parameter α is not fixed by the EOM and it represents a modulus.

The expression for the gauge field background in eq. (2.7) is well-defined in the limit $u \rightarrow 0$, but not as $u \rightarrow \bar{u}$. We should therefore use a different patch to describe the $u = \bar{u}$ brane, and this must be related to the patch including the $u = 0$ brane by a single-valued gauge transformation. This leads to a Dirac quantization condition, which for a field interacting with \mathcal{A} through a charge e gives

$$e \frac{4\alpha g}{\kappa q} = e \alpha \frac{r_1}{r_0} \frac{g}{g_1} = N, \quad (2.11)$$

where N is an integer. The charge e can be computed once we have selected the background gauge group, since it is an eigenvalue of the generator Q . Finally, note that the internal manifold corresponding to solution (2.7) has an S^2 topology (its Euler number equals 2).

We end this section by considering the various parameters in the model, and the phenomenological constraints which can arise when we give it a brane world interpretation. There are three free parameters in the 6D theory, which can be taken to be the gauge coupling \tilde{g} , and two out of the following three parameters: the 6D Planck scale, κ , the gauge coupling g_1 and the length-scale $r_0 = \kappa/\sqrt{2}g_1$. In the solution there are two free parameters, r_1 (or q) and α . However, one combination of all these parameters is constrained by the quantization condition (2.11).

The relation between the 6D Planck scale κ and our observed 4D Planck scale κ_4 is

$$\frac{1}{\kappa^2} V_2 = \frac{1}{\kappa_4^2}, \quad (2.12)$$

where the volume V_2 is given by

$$V_2 = \int d^2y \sqrt{-G} e^{-A} = 2\pi \int du e^{(3A+B)/2}. \quad (2.13)$$

For solution (2.7) we have

$$V_2 = 4\pi\alpha \left(\frac{r_0}{2}\right)^2. \quad (2.14)$$

Notice that this volume does not depend on r_1 , and so we can keep it fixed whilst varying the warp factor, namely e^A in (2.7). Moreover from (2.12) a phenomenological constraint follows between the bulk couplings and the brane tensions, which can be written:

$$\frac{g_1}{\sqrt{\alpha}} = \sqrt{\frac{\pi}{2}} \kappa_4. \quad (2.15)$$

This implies that $g_1/\sqrt{\alpha}$ is very small, of the order of the Planck length.

Now let us embed the Arkani Hamed-Dimopoulos-Dvali (ADD) scenario [20] into the present model, in order to try to explain the large hierarchy between the Electroweak scale and the Planck scale via the size of the extra dimensions. Thus identifying the 6D

fundamental scale with the Electroweak scale $\kappa \sim TeV^{-2}$ and constraining the observed 4D Planck scale $\kappa_4^2 \sim 10^{-30}TeV^{-2}$; the above relation translates to⁹:

$$\sqrt{\alpha}r_0 \sim 0.1mm. \quad (2.16)$$

Here, the large extra dimensions corresponds to tuning the bulk gauge coupling and brane tensions. However, we can also observe that (2.16) fixes just one parameter among α , r_0 and r_1 and we still have two independent parameters even if we require large extra dimensions. Later we will see that this novel feature proves to have interesting consequences for the mass spectrum of fluctuations.

3 Gauge Fields

Having established the brane world solution and its properties, we are now ready to examine the fluctuations about this background, which will represent the physical fields in our model. In this section our focus will be on the gauge field fluctuations.

Normalizable gauge field zero modes in axially symmetric codimension two branes are known to exist [21, 22, 23]. However, in these known examples there is no mass gap between the zero and non-zero modes which renders an effective 4-dimensional description somewhat problematic, especially in non-Abelian case [24]. In contrast to this for the axisymmetric solutions studied in this paper the presence of a mass gap will be automatic due to the compactness of the transverse space. In this section we shall give the full spectrum of zero and non-zero modes.

Given the symmetries of the problem, we can expect that the gauge fields in the low energy effective theory belong to $\mathcal{H} \times U(1)_{KK}$, where \mathcal{H} is the unbroken subgroup of \mathcal{G} that commutes with the $U(1) \subset \mathcal{G}$ in which the monopole lies¹⁰. The $U(1)_{KK}$ arises from the vector fluctuations of the metric, due to the axial symmetry of the internal manifold, and is promoted to $SU(2)_{KK}$ in the sphere limit of the background.

The non-Abelian sector of \mathcal{H} may be rich enough to contain the SM gauge group. For example, consider the anomaly free model of [11], with gauge group $\mathcal{G} = E_6 \times E_7 \times U(1)_R$, and the monopole background in E_6 . The surviving gauge group, $SO(10) \times E_7 \times U(1)_R \times U(1)_{KK}$, then contains the Grand Unified Group $SO(10)$, and the model also includes charged matter in the fundamental of $SO(10)$. Therefore our present interest will be in the fields belonging to various representations of \mathcal{H} . Specifically, we will consider gauge field fluctuations orthogonal to the monopole background. For the case $\mathcal{G} = E_6 \times E_7 \times U(1)_R$, the gauge field sectors that are covered by our analysis are given in Table 1.

3.1 Kaluza-Klein Modes

Using the background solution in the 6D action (2.1), we can identify the bilinear action for the fluctuations. This step requires some care, because to study the physical spectrum we

⁹We use the following conversion relation: $(TeV)^{-1} \sim 10^{-16}mm$.

¹⁰Due to the Chern-Simons coupling in supergravity, the $U(1)$ gauge field in the direction of the monopole eats the axion arising from the Kalb-Ramond field and acquires a mass [11, 1].

must first remove the gauge freedoms in the action due to 6D diffeomorphisms and gauge transformations. The problem has been studied in a general context in [23], where the authors choose a light-cone gauge fixing.

In the light-cone gauge, the action for the gauge field fluctuations, orthogonal to the monopole background, at the bilinear level reads [23]

$$S_G(V, V) \equiv - \int d^6 X \sqrt{-G} \frac{1}{2} e^\phi \left(\partial_\mu V_j \partial^\mu V^j + e^{-A} \partial_\rho V_j \partial_\rho V_j + D_\varphi V_j D^\varphi V^j \right), \quad (3.1)$$

where V_j is the gauge field fluctuation in the light cone gauge ($j=1,2$) and all the indices in (3.1) are raised and lowered with the ρ -dependent metric G_{MN} given in (2.3). Indeed, here and below G_{MN} represents the background metric. We have multiplied the formula of [23] by an overall e^ϕ , with $\phi \equiv \kappa\sigma/2$ and σ in the background, due to the presence of the dilaton in our theory¹¹. Notice that since we are looking at the sector orthogonal to the monopole background, the Chern-Simons term does not contribute, and the action takes a simple form.

In general, the covariant derivative $D_\varphi V_j$ includes the gauge field background

$$D_\varphi V_j = \partial_\varphi V_j + ie_V \mathcal{A}_\varphi V_j, \quad (3.2)$$

where again the charge e_V can be computed using group theory once the gauge group $\tilde{\mathcal{G}}$ is chosen. The value $e_V = 0$ corresponds to the gauge fields in the 4D low energy effective theory. However, since we can do so without much expense, we keep a generic value of e_V . Those fluctuations with $e_V \neq 0$ corresponds to vector fields in a non-trivial representation of the 4D effective theory gauge group. The Dirac quantization condition (2.11) then gives $e_V 4\alpha g/(\kappa q) = N_V$, where N_V is an integer.

Next we perform a Kaluza-Klein expansion of the 6D fields. Since our internal space is topologically S^2 , we require gauge fields to be periodic functions of φ :

$$V_j(X) = \sum_m V_{jm}(x) f_m(\rho) e^{im\varphi}, \quad (3.3)$$

where m is an integer.

If we put (3.3) in (3.1) we obtain kinetic terms for the 4D effective fields proportional to

$$\int d^4 x \sum_m \eta^{\mu\nu} \partial_\mu V_{jm}^\dagger \partial_\nu V_{jm} \int d\rho e^{\phi+B/2} |f_m|^2. \quad (3.4)$$

¹¹That the dilaton invokes only this simple change with respect to Ref. [23] can be seen as follows. First, notice that since we are considering fluctuations orthogonal to the $U(1)$ background, there are no mixings with other sectors, and the bilinear action is simply $S_G = -1/4 \int d^6 X \sqrt{-G} e^\phi G^{MN} G^{PQ} (F_{MP} F_{NQ})^{(2)}$. We emphasise that G_{MN} and ϕ now signify the background fields. Also, $(\)^{(2)}$ indicates the bilinear part in the fluctuations. Next, make the change of coordinates, $d\rho = e^{-\phi/2} d\tilde{\rho}$, and rewrite the background metric in (2.3) as $ds^2 = e^{-\phi} \left(e^{\tilde{A}} \eta_{\mu\nu} dx^\mu dx^\nu + d\tilde{\rho}^2 + e^{\tilde{B}} d\varphi^2 \right)$, with $\tilde{A} \equiv A + \phi$ and $\tilde{B} \equiv B + \phi$. In this way, the bilinear action, S_G , reduces to exactly the same form as that of Ref. [23], and we can proceed as they do to transform into light-cone coordinates, fix the light-cone gauge, and eliminate redundant degrees of freedom using their equations of motion.

Therefore physical fluctuations, having a finite kinetic energy, must satisfy the following normalizability condition (NC):

$$\int du |\psi|^2 < \infty, \quad (3.5)$$

where

$$\psi = e^{(2\phi+A+B)/4} f_m. \quad (3.6)$$

The quantity $|\psi|^2$ represents the probability density of finding a gauge field in $[u, u + du]$.

In fact, this is not the only condition that the physical fields must satisfy. If we want to derive the EOM from (3.1) through an action principle we have to impose¹² the following boundary condition [25, 26, 27]

$$\int d^6 X \partial_M \left(\sqrt{-G} e^{\phi-A} V_j \mathcal{D}^M V_j \right) = 0, \quad (3.7)$$

where \mathcal{D}_M is the gauge covariant derivative. Equation (3.7) represents conservation of current $J_M = e^{\phi-A} V_j \mathcal{D}_M V_j$. Moreover, since the fields are periodic functions of φ , (3.7) becomes

$$\left[\sqrt{-G} e^{\phi-A} V_j \partial_\rho V_j \right]_0^{\bar{\rho}} = 0. \quad (3.8)$$

The EOM can then be derived as:

$$\sqrt{-G} e^{\phi-2A} \eta^{\mu\nu} \partial_\mu \partial_\nu V_j = -\partial_\rho \left(\sqrt{-G} e^{\phi-A} \partial_\rho V_j \right) - \sqrt{-G} e^{\phi-A-B} D_\varphi^2 V_j. \quad (3.9)$$

By inserting (3.3) in (3.9) we obtain

$$-\frac{e^{-\phi+2A}}{\sqrt{-G}} \partial_\rho \left(\sqrt{-G} e^{\phi-A} \partial_\rho f_m \right) + e^{A-B} (m + e_V \mathcal{A}_\varphi)^2 f_m = M_{V,m}^2 f_m, \quad (3.10)$$

where $M_{V,m}^2$ are the eigenvalues of $\eta^{\mu\nu} \partial_\mu \partial_\nu$.

At this stage, we can already identify the massless fluctuation that is expected from symmetry arguments. For $e_V = 0$, when $m = 0$, a constant f_0 is a solution of (3.10) with $M_{V,0}^2 = 0$. This solution corresponds to 4D effective theory gauge fields. It has a finite kinetic energy, and trivially satisfies (3.8). The fact that such gauge fields have a constant transverse profile guarantees charge universality of fermions in the 4D effective theory (see below).

To find the massive mode solutions, we can express (3.10) in terms of u and ψ and obtain a Schrodinger equation:

$$\left(-\partial_u^2 + V \right) \psi = M_V^2 \psi, \quad (3.11)$$

where the ‘‘potential’’ is

$$V(u) = e^{A-B} (m + e_V \mathcal{A}_\varphi)^2 + e^{-(2\phi+A+B)/4} \partial_u^2 e^{(2\phi+A+B)/4}. \quad (3.12)$$

¹²Actually we impose that for every pair of fields V_j and V'_j the condition $\int d^6 X \partial_M \left(\sqrt{-G} e^{\phi-A} V_j \mathcal{D}^M V'_j \right) = 0$ is satisfied but in (3.7) the prime is understood.

We want to find the complete set of solutions to (3.11) satisfying the NC (3.5) and the boundary conditions (3.8), which can be written in terms of u and ψ as follows

$$\left(\lim_{u \rightarrow \bar{u}} - \lim_{u \rightarrow 0} \right) \left\{ \psi^* \left[-\partial_u + \frac{1}{4} (2\partial_u \phi + \partial_u A + \partial_u B) \right] \psi \right\} = 0. \quad (3.13)$$

In order for (3.13) to be satisfied, both the limits $u \rightarrow 0$ and $u \rightarrow \bar{u}$ must be finite. Condition (3.13) ensures that the Hamiltonian in the Schroedinger equation (3.11) is hermitian, and so has real eigenvalues and an orthonormal set of eigenfunctions. Therefore, we shall call it the hermiticity condition (HC).

So far our analysis has been valid for all axially symmetric solutions of the form (2.3). We will now use these results to determine the fluctuation spectrum about the conical-GGP solution (2.7). We observe that $V(u)$ then contains a delta-function contribution, arising from the second-order derivative of the conical metric function $\partial_u^2 B$ (see Appendix B and eq. (B.6)). However we can drop it because $\partial_u^2 B$ also contains stronger singularities at $u = 0$ and $u = \bar{u}$: respectively $1/u^2$ and $1/(\bar{u} - u)^2$. These singularities are also a consequence of the behaviour of e^B given in (2.6) and they imply that the behaviour of the wave functions close to $u = 0$ and $u = \bar{u}$ cannot depend on the mass. In particular, this immediately implies that if the wave functions of zero modes are peaked near to one of the branes, then the same will be true also for the infinite tower of non-zero modes. In other words, we cannot hope to dynamically generate a brane world scenario, in which zero modes are peaked on the brane, and massive modes are not, leading to weak coupling between the two sectors¹³. If we are to interpret the zero mode gauge fields as those of the SM, therefore, for the massive modes to have escaped detection they must have a large mass gap.

Meanwhile, we note that in contrast to the non-relativistic quantum mechanics problem, here we cannot deduce qualitative results about the mass spectrum from the shape of the potential. This is because the boundary conditions to be applied in the context of dimensional reduction are in general different to those in problems of quantum mechanics. In particular, the HC (3.13) is a non-linear condition, contrary to the less general linear boundary conditions usually encountered in quantum mechanics to ensure hermiticity of the Hamiltonian. We will be able to impose the more general case thanks to the universal asymptotic behaviour of the Kaluza-Klein tower.

Returning then to our explicit calculation of the Kaluza-Klein spectrum, we can write $V(u)$ as

$$V(u) = V_0 + v \cot^2 \left(\frac{u}{r_0} \right) + \bar{v} \tan^2 \left(\frac{u}{r_0} \right), \quad (3.14)$$

and

$$r_0^2 V_0 \equiv 2m\omega(m - N_V)\bar{\omega} - \frac{3}{2}, \quad r_0^2 v \equiv m^2 \omega^2 - \frac{1}{4}, \quad r_0^2 \bar{v} \equiv (m - N_V)^2 \bar{\omega}^2 - \frac{1}{4}. \quad (3.15)$$

Moreover in this case the expression (3.13) for the HC becomes

$$\lim_{u \rightarrow \bar{u}} \psi^* \left(-\partial_u + \frac{1}{2} \frac{1}{u - \bar{u}} \right) \psi - \lim_{u \rightarrow 0} \psi^* \left(-\partial_u + \frac{1}{2u} \right) \psi = 0. \quad (3.16)$$

¹³In fact, a similar singular behaviour for the potential in general arises for the general axisymmetric solutions given in [4] and studied in [6], where the hypothesis (2.6) is relaxed.

If we introduce z and y in the following way [28]

$$z = \cos^2\left(\frac{u}{r_0}\right), \quad \psi = z^\gamma (1-z)^\beta y(z), \quad (3.17)$$

equation (3.11) becomes

$$z(1-z)\partial_z^2 y + [c - (a+b+1)z]\partial_z y - aby = 0, \quad (3.18)$$

where

$$\begin{aligned} \gamma &\equiv \frac{1}{4}[1 + 2(m - N_V)\bar{\omega}], \quad \beta \equiv \frac{1}{4}(1 + 2m\omega), \quad c \equiv 1 + (m - N_V)\bar{\omega}, \\ a &\equiv \frac{1}{2} + \frac{m}{2}\omega + \frac{1}{2}(m - N_V)\bar{\omega} + \frac{1}{2}\sqrt{r_0^2 M_{V,m}^2 + 1 + [m\omega - (m - N_V)\bar{\omega}]^2}, \\ b &\equiv \frac{1}{2} + \frac{m}{2}\omega + \frac{1}{2}(m - N_V)\bar{\omega} - \frac{1}{2}\sqrt{r_0^2 M_{V,m}^2 + 1 + [m\omega - (m - N_V)\bar{\omega}]^2}, \end{aligned} \quad (3.19)$$

and

$$\omega \equiv (1 - \delta/2\pi)^{-1}, \quad \bar{\omega} \equiv (1 - \bar{\delta}/2\pi)^{-1}. \quad (3.20)$$

Equation (3.18) is the hypergeometric equation and its solutions are known. For $c \neq 1$ the general solution is a linear combination of the following functions:

$$y_1(z) \equiv F(a, b, c, z), \quad y_2(z) \equiv z^{1-c} F(a+1-c, b+1-c, 2-c, z), \quad (3.21)$$

where F is Gauss's hypergeometric function. So for $c \neq 1$ the general integral of the Schroedinger equation is

$$\psi = K_1 \psi_1 + K_2 \psi_2, \quad (3.22)$$

where

$$\psi_i \equiv z^\gamma (1-z)^\beta y_i. \quad (3.23)$$

and $K_{1,2}$ are integration constants. For $c = 1$ we have $\psi_1 = \psi_2$ but we can construct a linearly independent solution using the Wronskian method and the general solution reads

$$\psi = K_1 \psi_1 + K_2 \psi_1 \int^u \frac{du'}{\psi_1^2(u')}. \quad (3.24)$$

Now we must impose the NC (3.5) and HC (3.16), to select the physical modes. In Appendix C we give explicit calculations; the final result is that the NC and HC give the following discrete spectrum. The wave functions are

$$\psi \propto z^\gamma (1-z)^\beta F(a, b, c, z), \quad \text{for } m \geq N_V, \quad (3.25)$$

$$\psi \propto z^{\gamma+1-c} (1-z)^\beta F(a+1-c, b+1-c, 2-c, z), \quad \text{for } m < N_V. \quad (3.26)$$

and the squared masses are as follows:

- For $N_V \leq m < 0$

$$M_{V,n,m}^2 = \frac{4}{r_0^2} \left\{ n(n+1) + \left(\frac{1}{2} + n\right) [-m\omega + (m - N_V)\bar{\omega}] \right\} > 0. \quad (3.27)$$

- For $m \geq N_V$ and $m \geq 0$

$$M_{V,n,m}^2 = \frac{4}{r_0^2} \left\{ n(n+1) + \left(\frac{1}{2} + n \right) [m\omega + (m - N_V)\bar{\omega}] + m\omega(m - N_V)\bar{\omega} \right\} \geq 0. \quad (3.28)$$

- For $m < N_V$ and $m < 0$

$$M_{V,n,m}^2 = \frac{4}{r_0^2} \left\{ n(n+1) + \left(\frac{1}{2} + n \right) [-m\omega + (N_V - m)\bar{\omega}] - m\omega(N_V - m)\bar{\omega} \right\} > 0. \quad (3.29)$$

- For $0 \leq m < N_V$

$$M_{V,n,m}^2 = \frac{4}{r_0^2} \left\{ n(n+1) + \left(\frac{1}{2} + n \right) [m\omega + (N_V - m)\bar{\omega}] \right\} > 0. \quad (3.30)$$

The masses given in (3.27) and (3.28) correspond to the wave function (3.25) whereas the masses given in (3.29) and (3.30) correspond to the wave function (3.26). We observe that there are no tachyons and that the only zero mode is for $n = 0$, $m = 0$ and $N_V = 0$ ($e_V = 0$), corresponding to gauge fields in the 4D low energy effective theory.

As a check, we can consider the S^2 limit ($\omega, \bar{\omega} \rightarrow 1$), whose mass spectrum is well-known. Our spectrum (3.27)-(3.30) reduces to

$$a^2 M_V^2 = l(l+1) - \left(\frac{N_V}{2} \right)^2, \quad \text{multiplicity} = 2l + 1, \quad (3.31)$$

where¹⁴ $l = \lfloor \frac{N_V}{2} \rfloor + k$ and $k = 0, 1, 2, 3, \dots$. This is exactly the result that one finds by using the spherical harmonic expansion [29] from the beginning.

At this stage we can point towards a novel property of the final mass spectrum. Observe that in the large α (small $\bar{\omega}$) limit the volume V_2 given in equation (2.14) becomes large but the mass gap between two consecutive Kaluza-Klein states does not reduce to zero as in standard Kaluza-Klein theories¹⁵. This a consequence of the shape of our background manifold and in particular of the conical defects. Notice that the large α limit corresponds to a negative tension brane at $u = \bar{u}$, but not necessarily at $u = 0$.

In Section 4 we will show that the same effect appears also in the fermionic sector, and we will turn to a discussion of its implications in Section 5.

3.2 4D Effective Gauge Coupling

Let us end the discussion on gauge fields by briefly presenting the 4D effective gauge coupling. This can be obtained by dimensionally reducing the 6D gauge kinetic term. We

¹⁴The number l is defined in different ways in equations (3.27)-(3.30). For instance we have $l \equiv n + \lfloor N_V/2 \rfloor$ for (3.27).

¹⁵This is also true for the proper volume of the 2D internal manifold.

consider the zero mode fluctuations in \mathcal{H} , about the background (2.3), so that the 4D effective gauge kinetic term is:

$$\int d^6 X \sqrt{-G} \left\{ -\frac{1}{4g^2} e^{\kappa\sigma/2} \text{Tr} F_{MN} F^{MN} \right\} \rightarrow \int d^4 x \left\{ -\frac{1}{4g^2} \left[\int dud\varphi e^{(3A+B)/2} f_0^2 \right] \text{Tr} F_{\mu\nu} F^{\mu\nu} \right\}. \quad (3.32)$$

Recalling that $f_0 = \text{const}$ and normalizing it to one, we can read:

$$\frac{1}{g_{eff}^2} = \frac{1}{g^2} V_2. \quad (3.33)$$

4 Fermions

We will now consider fermionic perturbations, and in particular our interest will be in the sector charged under the 4D effective gauge group, \mathcal{H} , discussed above. These fields arise from the hyperinos and the \mathcal{H} gauginos, for which we also restrict ourselves to those orthogonal to the $U(1)_R$. Thus we are considering matter charged under the non-Abelian gauge symmetries of the 4D effective theory. For instance, for the anomaly free model $E_6 \times E_7 \times U(1)_R$, with the monopole embedded in the E_6 , the gauginos in the 78 of E_6 contain a $16 + \overline{16}$ fundamental representation of the grand unified gauge group $SO(10)$, and our analysis will be applicable to them. In Tabel 1, we give the complete list of fermion fields that are included in our study, for the said example.

We proceed in much the same way as for the gauge field sector of the previous section, transforming the dynamical equations and necessary boundary conditions into a Schroedinger-like problem, to obtain the physical modes and discrete mass spectrum.

The bilinear action for the fluctuations of interest takes a particularly simple form, comprising as it does of the standard Dirac action:

$$S_F = \int d^6 X \sqrt{-G} \bar{\lambda} \Gamma^M D_M \lambda, \quad (4.1)$$

where¹⁶

$$D_M \lambda = \left(\partial_M + \frac{1}{8} \Omega_M^{[A,B]} [\Gamma_A, \Gamma_B] + ie \mathcal{A}_M \right) \lambda. \quad (4.2)$$

Here e is the charge of λ under the $U(1)$ monopole, and G_{MN} , $\Omega_M^{[A,B]}$ and \mathcal{A}_M are the background metric, spin connection and gauge field corresponding to an axisymmetric solution (2.3). Analogously to the gauge field analysis, in order to derive the Dirac equation

$$\Gamma^M D_M \lambda = 0 \quad (4.3)$$

from (4.1) by using an action principle [30], we require conservation of fermionic current:

$$\int d^6 X \partial_M \left(\sqrt{-G} \bar{\lambda} \Gamma^M \lambda \right) = 0. \quad (4.4)$$

¹⁶Our conventions for Γ^A and $\Omega_M^{[A,B]}$ are given in Appendix A.

Equation (4.4) implies that the Dirac operator $\Gamma^M D_M$ is hermitian, and we shall again refer to it as the HC. Our aim is to find the complete fermionic spectrum, that is a complete set of normalizable solutions of (4.3) satisfying (4.4).

Some care is now needed when discussing the background felt by the fermionic sector in (4.2). As already mentioned, in order to have a correctly defined gauge connection, it is necessary to use two patches related by a single-valued gauge transformation. The same is true for the spin connection, which must be defined in such a way as to imply the conical defects in the geometry. Henceforth we focus on the patch including the $\rho = 0$ brane, chosen to be $0 \leq \rho < \bar{\rho}$. For this patch a good choice for the vielbein is

$$e_\mu^\alpha = e^{A/2} \delta_\mu^\alpha, \quad \{e_m^a\} = \begin{pmatrix} \cos \varphi & -e^{B/2} \sin \varphi \\ \sin \varphi & e^{B/2} \cos \varphi \end{pmatrix}, \quad (4.5)$$

where α is a 4D flat index, $a = 5, 6$ a 2D flat index and $m = \rho, \varphi$. The corresponding spin connection is

$$\begin{aligned} \Omega_\mu^{[\alpha,5]} &= \frac{1}{2} A' e^{A/2} \delta_\mu^\alpha \cos \varphi, & \Omega_\mu^{[\alpha,6]} &= \frac{1}{2} A' e^{A/2} \delta_\mu^\alpha \sin \varphi, \\ \Omega_\rho^{[5,6]} &= 0, & \Omega \equiv \Omega_\varphi^{[5,6]} &= \left(1 - \frac{1}{2} B' e^{B/2}\right), \end{aligned} \quad (4.6)$$

where $' \equiv \partial_\rho$. It can be checked that this gauge choice correctly reproduces Stokes' theorem for a small domain including the conical defect¹⁷.

We are now ready to study the Dirac equation (4.3) for 6D fluctuations, and write it in terms of 4D effective fields. Since λ is a 6D Weyl spinor we can represent it by

$$\lambda = \begin{pmatrix} \lambda_4 \\ 0 \end{pmatrix}, \quad (4.7)$$

where λ_4 is a 4D Dirac spinor: $\lambda_4 = \lambda_R + \lambda_L$, $\gamma^5 \lambda_R = \lambda_R$, $\gamma^5 \lambda_L = -\lambda_L$. By using the ansatz (2.3), the vielbein (4.5), the spin connection (4.6) and our conventions for Γ^A in Appendix A, the Dirac equation (4.3) becomes

$$e^{-A/2} \gamma^\mu \partial_\mu \lambda_L = e^{i\varphi} \left[-\partial_\rho - ie^{-B/2} (\partial_\varphi + ie\mathcal{A}_\varphi) - A' + \frac{1}{2} \Omega e^{-B/2} \right] \lambda_R, \quad (4.8)$$

$$e^{-A/2} \gamma^\mu \partial_\mu \lambda_R = e^{-i\varphi} \left[\partial_\rho - ie^{-B/2} (\partial_\varphi + ie\mathcal{A}_\varphi) + A' - \frac{1}{2} \Omega e^{-B/2} \right] \lambda_L. \quad (4.9)$$

Performing the Fourier mode decomposition:

$$\lambda_4(X) = \lambda_R(X) + \lambda_L(X) = \sum_m (\lambda_{R,m}(x) f_{R,m}(\rho) + \lambda_{L,m}(x) f_{L,m}(\rho)) e^{im\varphi}, \quad (4.10)$$

where m is an integer, and inserting into (4.8) and (4.9) we find:

$$e^{-A/2} \gamma^\mu \partial_\mu \lambda_{L,m+1} f_{L,m+1} = \left[-\partial_\rho + e^{-B/2} \left(m + \frac{1}{2} \Omega + e\mathcal{A}_\varphi \right) - A' \right] \lambda_{R,m} f_{R,m}, \quad (4.11)$$

$$e^{-A/2} \gamma^\mu \partial_\mu \lambda_{R,m-1} f_{R,m-1} = \left[\partial_\rho + e^{-B/2} \left(m - \frac{1}{2} \Omega + e\mathcal{A}_\varphi \right) + A' \right] \lambda_{L,m} f_{L,m}. \quad (4.12)$$

¹⁷See Appendix B for some steps in this calculation.

For the boundary conditions, analogously to the gauge fields, the NC can be found to be:

$$\int du |\psi|^2 < \infty \quad (4.13)$$

where

$$\psi \equiv e^{A+B/4} f_{Rm} \quad (4.14)$$

and a similar condition for left-handed spinors. Meanwhile, the HC (4.4) can be written:

$$\left[\sqrt{-G} f_{L,m+1} f_{R,m}^* \right]_0^{\bar{\rho}} = 0. \quad (4.15)$$

Having set up the dynamical equations and the relevant boundary conditions, we shall now use this information to study the complete fermionic spectrum in Subsections 4.1 and 4.2. In particular, we will focus on the questions of wave function localization, and the mass gap problem, crucial to the development of a phenomenological brane world model.

4.1 Zero Modes

We begin by finding the zero mode solutions, for which the problem simplifies considerably. Indeed, for the zero modes $\gamma^\mu \partial_\mu = 0$, and the equations for right- and left-handed modes (4.11) and (4.12) decouple:

$$\left[\partial_\rho - e^{-B/2}(m + e\mathcal{A}_\varphi) + A' - \frac{1}{2}\Omega e^{-B/2} \right] f_{R,m} = 0, \quad (4.16)$$

$$\left[\partial_\rho + e^{-B/2}(m + e\mathcal{A}_\varphi) + A' - \frac{1}{2}\Omega e^{-B/2} \right] f_{L,m} = 0. \quad (4.17)$$

By using the expression for Ω in equation (4.6), the solution of (4.16) is

$$f_{R,m}(\rho) \propto \exp \left[-A - \frac{1}{4}B + \int^\rho d\rho' e^{-B/2} \left(m + \frac{1}{2} + e\mathcal{A}_\varphi \right) \right], \quad (4.18)$$

whereas the solution of (4.17) can be obtained by replacing $m, e \rightarrow -m, -e$ in (4.18). The solution (4.18) for $e = 0$ was found in [31]. Here we give the expression for every e because we want to include charged fermions. We note that the zero mode solution (4.18) automatically satisfies the HC given in (4.15). From (4.18), (4.14) and (2.4) we obtain

$$\psi \propto \exp \left[\int^u du' e^{(A-B)/2} \left(m + \frac{1}{2} + e\mathcal{A}_\varphi \right) \right]. \quad (4.19)$$

For the conical-GGP background, (2.7), the explicit expression for ψ is

$$\psi \propto \sin \left(\frac{u}{r_0} \right)^{\omega(1/2+m)} \cos \left(\frac{u}{r_0} \right)^{\bar{\omega}(N-1/2-m)}, \quad (4.20)$$

where we used (2.11). The NC is satisfied when

$$\frac{\delta}{4\pi} - 1 < m < N - \frac{\bar{\delta}}{4\pi}. \quad (4.21)$$

From here we retrieve the result that for the sphere, which has $\delta = \bar{\delta} = 0$, there exist normalizable zero modes only for $e \neq 0$ ($N \neq 0$), that is for a non-vanishing monopole background [29]. Moreover, as found in [31], we see that the conical defects also make massless modes possible, provided that there is at least one negative deficit angle, even if $N = 0$. However, (4.21) implies that for positive tension branes, $\delta, \bar{\delta} > 0$, the adjoint of \mathcal{H} , which has $e = 0$, is projected out. If \mathcal{H} contains the SM gauge group, this is appealing since the fermions of the SM are not in adjoint representations. In any case, the number of families depends on $\delta, \bar{\delta}$ and N .

Let us now consider the wave function profiles, (4.20). Observe that ψ is peaked on the $u = 0$ brane (that is, $\psi \rightarrow \infty$ as $u \rightarrow 0$, and $\psi \rightarrow 0$ as $u \rightarrow \bar{u}$) when

$$m < -1/2, \quad \text{and} \quad m < -1/2 + N. \quad (4.22)$$

By comparing (4.21) and (4.22) we understand that we have normalizable and peaked ψ only for $\bar{\delta} < 0$ (that is for negative tension brane). If $N - \bar{\delta}/4\pi > 0$ we can have normalizable zero modes for $\delta > 0$ (positive tension brane) but the corresponding ψ are not peaked on the $u = 0$ brane (indeed, $\psi \rightarrow 0$ as $u \rightarrow 0$). On the other hand from (4.20) we see that ψ is peaked on the $u = \bar{u}$ brane when

$$m > -1/2, \quad \text{and} \quad m > -1/2 + N. \quad (4.23)$$

By comparing (4.21) and (4.23) we understand that we have normalizable and peaked ψ only for $\bar{\delta} < 0$ (that is for negative tension brane).

We can also analyze the chirality structure. Since the left handed wave functions can be obtained from (4.20) by replacing $m, N \rightarrow -m, -N$, in order for them to be peaked on the $u = 0$ brane we need

$$m > 1/2, \quad \text{and} \quad m > 1/2 + N, \quad (4.24)$$

whereas in order for the left handed wave functions to be peaked on the $u = \bar{u}$ brane we need

$$m < 1/2, \quad \text{and} \quad m < 1/2 + N. \quad (4.25)$$

So, if we were to require that ψ be peaked on a brane, we always have a chiral massless spectrum because the chirality index counts the difference of modes in $f_{R,m}$ and $f_{L,m}$ with given m , $N_R(m) - N_L(m)$ [31]. We should point out, however, that in fact peaked zero modes, ψ , may not be necessary in order to have an acceptable phenomenology. The answer to this question can be found only after constructing the complete spectrum, and studying the couplings between different 4D effective fields.

4.2 Massive Modes

We now move on to a study of the complete Kaluza-Klein tower for the fermions. We begin by establishing the corresponding Schroedinger problem. The two coupled first order ODEs,

equations (4.11) and (4.12), can be equivalently expressed as a single second order ODE and a constraint equation, as follows¹⁸

$$e^A \left(-\partial_\rho^2 + h\partial_\rho + g_m \right) f_{R,m} = M_{F,m}^2 f_{R,m}, \quad (4.26)$$

$$M_{F,m} f_{L,m+1} = e^{A/2} \left[-\partial_\rho - A' + \left(m + \frac{1}{2}\Omega + e\mathcal{A}_\varphi \right) e^{-B/2} \right] f_{R,m}, \quad (4.27)$$

where $M_{F,m}^2$ are the eigenvalues of $(\gamma^\mu \partial_\mu)^2$ and

$$\begin{aligned} h &\equiv -\frac{5}{2}A' + (\Omega - 1)e^{-B/2}, \\ g_m &\equiv \left[\frac{1}{2}\Omega' - \frac{1}{4}\Omega B' - \frac{m}{2}B' + \frac{5}{4}A'\Omega + \left(\frac{m}{2} - 1 \right) A' - \frac{e}{2}B'\mathcal{A}_\varphi + e\mathcal{A}'_\varphi + \frac{e}{2}A'\mathcal{A}_\varphi \right] e^{-B/2} \\ &\quad + \left[m(m+1) + \frac{1}{2}\Omega - \frac{\Omega^2}{4} + (2m+1)e\mathcal{A}_\varphi + e^2\mathcal{A}_\varphi^2 \right] e^{-B} - A'' - \frac{3}{2}(A')^2. \end{aligned} \quad (4.28)$$

Once $f_{R,m}$ is known we can compute $f_{L,m+1}$ by using (4.27), so we can focus on f_R and study the second order ODE (4.26). If we express this equation in terms of ψ and u we obtain the Schroedinger equation

$$\left(-\partial_u^2 + V \right) \psi = M_{F,m}^2 \psi, \quad (4.30)$$

where the ‘‘potential’’ V is given by

$$\begin{aligned} V(u) &= e\partial_u \mathcal{A}_\varphi e^{(A-B)/2} + \left(\frac{1}{2} + m + e\mathcal{A}_\varphi \right) \partial_u e^{(A-B)/2} \\ &\quad + \left[\frac{1}{4} + m + e\mathcal{A}_\varphi + (m + e\mathcal{A}_\varphi)^2 \right] e^{A-B}. \end{aligned} \quad (4.31)$$

We observe that transformation (4.14) exactly removes the delta-functions which appear in (4.29) through Ω' (see Appendix B). However, just as for the gauge fields, a singular behaviour is observed in the potential, so that the asymptotic behaviour of the wave functions does not depend on their mass.

Our problem is now reduced to solving equation (4.30) with the conditions NC (4.13) and HC (4.15). By using (4.27) for $M_F \neq 0$ and definitions (2.4) and (4.14) we can rewrite (4.15) as follows

$$\left(\lim_{u \rightarrow \bar{u}} - \lim_{u \rightarrow 0} \right) \psi^* \left[-\partial_u + \left(m + \frac{1}{2} + e\mathcal{A}_\varphi \right) e^{(A-B)/2} \right] \psi = 0. \quad (4.32)$$

We can now proceed in exactly the same way as for the gauge field sector. For the conical-GGP solution (2.7) the explicit expression for V has the form (3.14), but now with

$$r_0^2 V_0 \equiv \left(\frac{1}{2} + m \right) \left[\bar{\omega} - \omega + 2\omega\bar{\omega} \left(\frac{1}{2} + m - N \right) \right] - \bar{\omega}N, \quad (4.33)$$

¹⁸Eq.(4.26) can also be obtained by squaring the 6D Dirac operator, and using that $[D_M, D_N] = \frac{1}{4}R_{MN}{}^{AB}\Gamma_{AB} + ieF_{MN}$.

$$r_0^2 v \equiv \left(\frac{1}{2} + m\right) \left[-\omega + \omega^2 \left(\frac{1}{2} + m\right)\right], \quad (4.34)$$

$$r_0^2 \bar{v} \equiv \left(\frac{1}{2} + m\right) \left[\bar{\omega} + \bar{\omega}^2 \left(\frac{1}{2} + m - 2N\right)\right] + \bar{\omega}N(\bar{\omega}N - 1). \quad (4.35)$$

Moreover, in this case the explicit expression for the HC is

$$\lim_{u \rightarrow \bar{u}} \psi^* \left(-\partial_u + \bar{\omega} \frac{m + 1/2 - N}{\bar{u} - u}\right) \psi - \lim_{u \rightarrow 0} \psi^* \left(-\partial_u + \omega \frac{m + 1/2}{u}\right) \psi = 0. \quad (4.36)$$

As in the gauge fields sector we introduce z and y in the following way

$$z = \cos^2 \left(\frac{u}{r_0}\right), \quad \psi = z^\gamma (1 - z)^\beta y(z), \quad (4.37)$$

so that equation (3.14) becomes a hypergeometric equation (3.18), with parameters:

$$\begin{aligned} \gamma &\equiv \frac{1}{2} \left[1 + \bar{\omega} \left(\frac{1}{2} + m - N\right)\right], \quad \beta \equiv \frac{1}{2} \left[1 - \omega \left(\frac{1}{2} + m\right)\right], \quad c \equiv \frac{3}{2} + \bar{\omega} \left(\frac{1}{2} + m - N\right), \\ a &\equiv 1 + \frac{\bar{\omega}}{2} \left(\frac{1}{2} + m - N\right) - \frac{\omega}{2} \left(\frac{1}{2} + m\right) + \frac{1}{2} \sqrt{\Delta}, \\ b &\equiv 1 + \frac{\bar{\omega}}{2} \left(\frac{1}{2} + m - N\right) - \frac{\omega}{2} \left(\frac{1}{2} + m\right) - \frac{1}{2} \sqrt{\Delta}, \\ \Delta &\equiv r_0^2 M_{F,m}^2 + (\bar{\omega}N)^2 + \left(\frac{1}{2} + m\right) \left[\bar{\omega}(\bar{\omega} - 2\omega) \left(\frac{1}{2} + m - N\right) - \bar{\omega}^2 N + \omega^2 \left(\frac{1}{2} + m\right)\right]. \end{aligned}$$

We can construct two independent solutions ψ_1 and ψ_2 of the Schroedinger equation (3.14) as in Section 3, and impose the NC (4.13) and the HC (4.36) to obtain the physical modes. The resulting wave functions are:

$$\psi = K_1 \psi_1 + K_2 \psi_2, \quad (4.38)$$

where the integration constants, $K_{1,2}$, are fixed in Appendix D. We plot a few of the wave function profiles in Figure 1.

The complete discrete mass spectrum is also given in Appendix D. There it can be seen that the same finiteness of the mass gap in the large α (hence large volume) limit, found in the gauge field spectrum, can be observed here. Moreover, for $\alpha \sim 1$, the mass gap between the zero modes and the massive states now goes as:

$$M_{GAP}^2 \sim \frac{1}{r_0^2} + \frac{1}{r_1^2}. \quad (4.39)$$

Therefore, for the fermions, a finite mass gap in the large volume limit can also be obtained by taking $r_0 \rightarrow \infty$ and turning on δ , thus allowing r_1 to remain finite. This contrasting behaviour to the standard Kaluza-Klein picture is a consequence of the conical defects ($\bar{\omega} \neq 1$ and $\omega \neq 1$) in our internal manifold. Below we shall consider its implications for phenomenology.

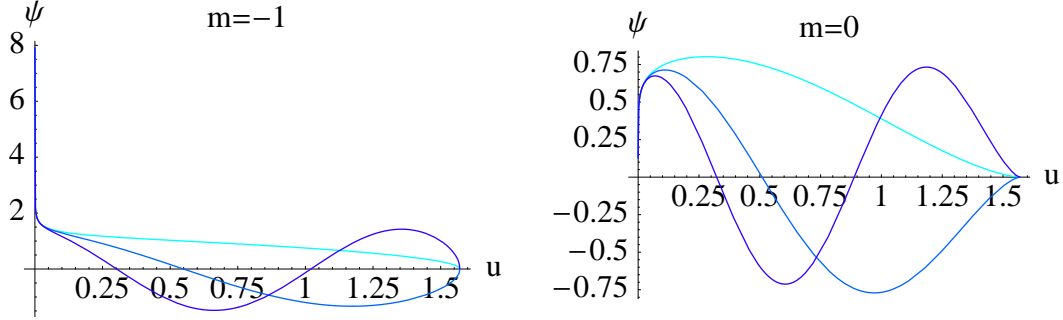


Figure 1: Fermion Wave Function Profiles: $n = 0, 1, 2$ modes plotted for angular momentum numbers $m = -1, 0$ (eqs (D.6) and (D.2) respectively). The parameters are chosen to be $(r_0, \omega, \bar{\omega}, e) = (1, 1/4, 1, 0)$, corresponding to a single negative tension brane at $u = 0$. Also the normalisation constant is set to 1. The number of intersections with the u -axis equals n , according to quantum mechanics. Notice that the $(m, n) = (-1, 0)$ mode is massless, and that given a localized massless mode, there is also an infinite Kaluza-Klein tower of localized massive modes.

4.3 4D Effective Fermion Charges

Let us first end this section on fermion fluctuations by obtaining their 4D effective gauge couplings. This can be calculated by going beyond their bilinear Lagrangian, and considering the interaction term:

$$\int d^6 X \sqrt{-G} \bar{\lambda} \Gamma^M D_M \lambda = \dots + \int d^6 X \sqrt{-G} \bar{\lambda} \Gamma^\mu (\partial_\mu + e V_\mu) \lambda + \dots \quad (4.40)$$

Using the results for the Kaluza-Klein decomposition found in the preceding sections

$$\begin{aligned} \lambda(X) &= \sum_{m,n} \lambda_{mn}(x) f_{mn}^{(\lambda)}(\rho) e^{im\varphi} = \sum_{m,n} \lambda_{mn}(x) \psi_{mn}^{(\lambda)}(u) e^{-A-B/4} e^{im\varphi}, \\ V_\mu(X) &= \sum_{m,n} V_{\mu mn}(x) f_{mn}^{(V)}(\rho) e^{im\varphi} = \sum_{m,n} V_{\mu mn}(x) \psi_{mn}^{(V)}(u) e^{-(3A+B)/4} e^{im\varphi}, \end{aligned} \quad (4.41)$$

and recalling that the gauge field zero mode is $f_{00}^{(V)} = 1$, a fermion mode λ_{mn} has the following coupling to the 4D effective gauge group:

$$\begin{aligned} e_{eff} &= \frac{e \int d\varphi du \overline{\psi_{mn}^{(\lambda)}} f_{00}^{(V)} \psi_{mn}^{(\lambda)}}{\int d\varphi du \overline{\psi_{mn}^{(\lambda)}} \psi_{mn}^{(\lambda)}} \\ &= e. \end{aligned} \quad (4.42)$$

Since the gauge field zero mode has a constant wave profile, the effective charges for the fermion modes are universal. This is a general result, and independent of any possible localization properties of the fermion modes: massless and massive fermion modes will always have the same coupling to the massless gauge fields. Again, we find that a large mass gap is required in the fermion spectrum in order to hide the Kaluza-Klein tower. We now consider this issue in more detail.

5 Large Volume Compactifications with a Large Mass Gap

In the previous two sections, we have calculated the complete Kaluza-Klein spectrum for the warped brane world compactification of 6D supergravity, for two interesting sectors of the gauge and matter fluctuations. We are now ready to consider the possible implications of our results.

6D brane world models have long been of interest in the context of Large Extra Dimensions, since these may help with the gauge hierarchy problem. In the conventional ADD picture [20], Standard Model particles must be confined to a 4D brane world, in order to explain why the large extra dimensions have escaped detection. It would certainly be of interest to develop a dynamical description of this localization, within the context of low energy effective field theory.

This could be achieved, for instance, if the zero mode wave profiles were peaked near to a brane, and the heavy modes suppressed there [24]. However, we have found that zero mode fermions can be peaked near to negative tension branes, only at the price of localizing the whole Kaluza-Klein tower (see Figure 1). Therefore, strong couplings are expected between light and heavy modes. If the zero mode bulk fermions are to be interpreted as matter in the SM, then apparently the only way to explain why we do not observe all the Kaluza-Klein modes is by insisting that their mass gap is larger than the 100GeV scale probed to date.

Usually, this would bring us back to the classical Kaluza-Klein scenario, with the extra dimensions required to be very small (at least $(100\text{GeV})^{-1}$ scale), and the generation of the gauge hierarchy lost. However, in our framework we have seen that a large mass gap can occur even if the volume V_2 , defined as the ratio κ^2/κ_4^2 , becomes large. In the fermionic sector this set up can be achieved with the conical defect associated with the warping ($\omega \neq 1$ and $\bar{\omega} = 1$), by taking the parameter r_1 to be small. Another way is turning on the other defect ($\bar{\omega} \neq 1$) and then taking the large α limit, that is small $\bar{\omega}$ limit, corresponding to a negative tension brane. The volume V_2 in (2.14) becomes large but, for both the fermions and gauge fields, the mass gap does not reduce to zero. We observe that the latter mechanism works also for $r_0 = r_1$, that is $\omega = \bar{\omega}$, which corresponds to the unwarped “rugby ball” solution with branes of equal tension at each of the two poles [32, 2].

The general idea of relaxing the phenomenological constraints on the size of the extra dimensions by deforming the shape of the internal space was proposed in [33]. There, it was shown that the presence of shape moduli can imply that there is no experimental limit on the size of the largest extra dimension. However, requiring a large Kaluza-Klein mass gap still constrained the overall volume of the extra dimensions. Here we give an explicit model which allows arbitrary large values for both V_2 and M_{GAP}^2 , at least for the fermions and vectors. This could have an interesting application in the large extra dimension scenario because we can have both $\kappa \sim \text{TeV}^{-2}$ and small effects from the massive modes by setting a large enough value of M_{GAP} .

In terms of hiding Large Extra Dimensions from our four dimensional universe, another possible approach is to interpret all the bulk fields that we have found (massless modes and massive ones) as a hidden sector, only gravitationally coupled to the SM. At this level, the SM must then be introduced by hand, confined on the delta-function brane. It seems that this is the approach to take if embedding the Supersymmetric Large Extra Dimensions

scenario in our calculations, proposed in [2], to resolve the Cosmological Constant Problem. This proposal relates the hierarchy in the Electroweak scale with that of the Cosmological Constant. The Electroweak scale is set by the size of the extra dimensions, r , and the Cosmological Constant is given by the Kaluza-Klein mass gap, here fixed by the same scale¹⁹ $1/r$. Both may have their observed values when the 6D fundamental scale is TeV , and $r \sim 0.1mm$. For the mass gap to be this small, SM particles must be localized to the brane.

Let us end by considering the tunings involved, when constructing a model with large volume (say, $\sqrt{V_2} \sim 0.1mm$) and large mass gap (say, $M_{GAP} \sim TeV^{-1}$). Consider first a large mass gap for the fermions. If we set $\alpha \sim 1$ and $r_1 \ll r_0$, then the Dirac quantization (2.11) implies:

$$e \frac{r_1 g}{r_0 g_1} \sim N \quad (5.43)$$

If we then assume $e \sim 1$ (which is natural from group theory) and $N \sim 1$ (which is required for a small number of families), the large volume - large mass gap condition requires a large hierarchy in the bulk gauge couplings:

$$\frac{g}{g_1} \sim 10^{15}. \quad (5.44)$$

Alternatively, we could set $r_1 \sim r_0$ and $\alpha \gg 1$, allowing a large mass gap for both fermions and gauge fields. In this case, requiring a large mass gap, $M_{GAP} \sim 1/r_0$, as well as large volume, $\sqrt{V_2} \sim \sqrt{\alpha}r_0$, requires $\alpha \sim 10^{30}$. Then, again, the Dirac quantization condition (2.11) reveals a large hierarchy in the bulk gauge couplings:

$$\frac{g}{g_1} \sim 10^{-30} \quad (5.45)$$

In both scenarios we cannot embed the background monopole in $U(1)_R$.

There are of course other combinations, for example with both $\alpha \gg 1$ and $r_1 \ll r_0$, in which these hierarchies may be relaxed. However, we should say that these tunings do not appear to be very natural or promising. For example, choosing the large dimensionless number $\alpha \gg 1$ corresponds to heavy negative tension branes, and deficit angles orders of magnitude less than zero. On the other hand, independently of trying to embed the Large Extra Dimension scenario into the present model, we have found an explicit example in which the Kaluza-Klein mass gap does not go to zero as the volume goes to infinity, contrary to standard lore.

6 Conclusions

In this paper we have analyzed an interesting subsector of gauge field and fermion fluctuations, in the warped brane world solutions of 6D minimal gauged supergravities. In

¹⁹However, the breakdown of SUSY in the bulk, as in the solutions studied here, may lead to a larger prediction for the Cosmological Constant, see [2].

Gauge Fields	Fermions
$V \sim (\mathbf{45}, \mathbf{1})_0$	$\lambda \sim (\mathbf{45}, \mathbf{1})_1$
$+(\mathbf{16}, \mathbf{1})_0$	$+(\mathbf{16} + \overline{\mathbf{16}}, \mathbf{1})_1$
$+(\mathbf{1}, \mathbf{133})_0$	$+(\mathbf{1}, \mathbf{133})_1$
$+(\mathbf{1}, \mathbf{1})_0$	$\Psi \sim (\mathbf{1}, \mathbf{912})_0$

Table 1: The gauge and fermion fields whose Kaluza-Klein spectrum is given by our work, for the illustrative example of the anomaly free model $E_6 \times E_7 \times U(1)_R$, when the monopole is embedded in E_6 . We give the quantum numbers under $\mathcal{H} = SO(10) \times E_7 \times U(1)_R$, which is the unbroken subgroup of the 6D gauge group.

particular, we have focused on bulk components which could give rise to SM or Grand Unified gauge and charged matter fields.

We performed a Fourier decomposition of 6D fields, and transformed the resulting field equations into a Schroedinger-like problem. We were then able to find the exact solutions for the Kaluza-Klein modes, in terms of hypergeometric functions. We considered in detail the boundary conditions that the physical modes must satisfy. In addition to the normalizability constraint, consistency also required a hermiticity condition, which can be interpreted as demanding current conservation. We were able to implement this in its general, quadratic form. Together, these conditions selected the physical modes, and gave rise to a discrete mass spectrum, which we presented in full. The discreteness of the spectrum is of course to be expected given the compact topology of the internal manifold, whose Euler number is two.

Our study can be applied to several sectors of the 6D supergravities. In Table 1 we summarise the 6D fields that are covered by our analysis, for the illustrative example of the anomaly free model $E_6 \times E_7 \times U(1)$. Moreover, the corresponding spectra for the non-supersymmetric model of [34] (at least for the unwarped 4D Poincaré invariant case) and [32], generalized to Einstein-Yang-Mills with fermions, can be straightforwardly extracted from those given above simply by setting the warp factor to one, that is $r_1 = r_0$.

The exact results presented in this paper enabled us to study the effects of the conical defects, sourced by codimension two branes, on the Kaluza-Klein wave profiles and mass gaps. As usual, the gauge fields have a zero mode with constant wave profile. For the fermions, we found that some zero modes can be peaked on a negative tension brane, but in this case the whole Kaluza-Klein tower is peaked there too. Therefore, in order to interpret the bulk zero modes as 4D effective fields of the SM, the mass gap must be large.

Intriguingly, this does not necessarily drive us to the conventional Kaluza-Klein picture, with small compact dimensions. It does not, because the conical defects allow a novel behaviour in the mass gap, which can be decoupled from the volume of the compactification, defined by κ^2/κ_4^2 . This continues to be observed in the unwarped limit, where the rugby ball model of [32, 2] is retrieved. Contrary to standard lore, a finite mass gap can be obtained, even if the volume goes to infinity.

For example, a large volume could be arranged in order to generate the Electroweak hierarchy, whilst maintaining a large mass gap between the zero modes and the Kaluza-Klein tower. This picture does not seem to provide a realisation of the Supersymmetric Large

Extra Dimension scenario, where the volume and mass gap should be related. However, in this way, SM fields could arise from bulk fields, and along with gravity propagate through the large extra dimensions, perfectly consistently with observation. Moreover, for better or worse, this picture seems to render the Large Extra Dimension scenario less falsifiable than previously thought, since we do not have to expect that the bulk Kaluza-Klein modes are accessible at TeV scales. However, arranging for both a large volume and large mass gap seems to require a large degree of fine tuning in bulk couplings. Furthermore, for a more complete idea, we would have to consider the Kaluza-Klein spectrum for the remaining bulk sectors, and in particular the gravitational fluctuations to know the effect of large extra dimensions on post-Newtonian tests.

Indeed, our analysis of 4D effective gauge fields and charged fermions is only a first step towards a complete analysis of fluctuations about the warped brane world background in 6D supergravity. A final objective would be to derive the full 4D effective field theory describing light fluctuations, and an understanding of when 6D physics comes into play. The sectors that we have studied here are the simplest ones, in terms of the mixings, but it should be possible to continue the project to other fields by extending our work.

Of the remaining sectors to be analyzed, the scalar perturbations have special importance, since they can contain Higgs fields and can have implications for the stability of the background solution. Indeed, in the round sphere limit of the model that we have studied, tachyons in general emerge from the internal components of the 6D gauge field orthogonal to the gauge field background [35]. A first step in the study of the scalar fluctuations has been presented in [12], and it would certainly be interesting if general results can be found. The difficulty may be in the mixings of different scalar fluctuations, which lead to a complicated system of coupled ODEs.

In another direction, much of our analysis was general, and could also be used to study other theories and other backgrounds with 4D Poincaré-2D axial symmetry.

Finally, it would be interesting to investigate whether there exist other mechanisms, which lead to the same decoupling between the mass gap and internal volume that we have found here. Indeed, in our set up we have been able to show that the decoupling arises due to the conical defects, but it may be possible to find other sources in different frameworks. In this way, our explicit example may be a realisation of a more general mechanism.

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Appendix

A Conventions and Notation

We choose the signature $-, +, +, +, \dots$ for the metric G_{MN} . The Riemann tensor is defined as follows

$$R_{MNS}^R = \partial_M \Gamma_{NS}^R - \partial_N \Gamma_{MS}^R + \Gamma_{MP}^R \Gamma_{NS}^P - \Gamma_{NP}^R \Gamma_{MS}^P, \quad (\text{A.1})$$

where the Γ 's are the Levi-Civita connection. Whereas the Ricci tensor and the Ricci scalar

$$R_{MN} = R_{PMN}^P, \quad R = G^{MN} R_{MN}. \quad (\text{A.2})$$

Here M, N, \dots run over all space-time dimensions.

Our choice for the 6D constant gamma matrices Γ^A , $A = 0, 1, 2, 3, 5, 6$, is

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix}, \quad \Gamma^6 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\text{A.3})$$

where the γ^μ are the 4D constant gamma matrices and γ^5 the 4D chirality matrix. Moreover the spin connection is

$$\Omega_M^{[A,B]} = \eta^{BC} \Omega_M^A{}_C = \eta^{BC} \left(e_N^A \Gamma_{MR}^N e_B^R + e_N^A \partial_M e_B^N \right), \quad (\text{A.4})$$

where e_M^A is the vielbein.

B Delta-Function Singularities

In this appendix we briefly review how the Ricci scalar acquires a delta-function contribution in the presence of a deficit angle, and examine what this implies for our choice of metric function e^B in (2.3).

Let us consider the ansatz (2.3,2.6), and illustrate the case for the deficit angle δ at $\rho = 0$. Near $\rho = 0$ the metric ds_2^2 of the 2D internal space can be written as follows

$$ds_2^2 = d\rho^2 + \left(1 - \frac{\delta}{2\pi}\right)^2 \rho^2 d\varphi^2. \quad (\text{B.1})$$

By using the change of coordinate $r^{1-\delta/2\pi}/(1-\delta/2\pi) = \rho$, this metric becomes

$$ds^2 = r^{-\delta/\pi} \left(dr^2 + r^2 d\varphi^2 \right). \quad (\text{B.2})$$

From (B.2) one can show [36]

$$R = 2\delta r^{\delta/\pi} \delta^{(2)}(\mathbf{y}) + \dots, \quad (\text{B.3})$$

where the 2D vector \mathbf{y} is defined by $\mathbf{y} = (r \cos \varphi, r \sin \varphi)$, $\delta^{(2)}$ is the 2D Dirac delta-function and the dots are the smooth contributions²⁰. On the other hand, near to $\rho = 0$ the Ricci scalar, R , can be expressed in terms of derivatives of B :

$$R = -B'' - \frac{1}{2}(B')^2, \quad (\text{B.4})$$

where $' \equiv \partial_\rho$, and from (B.3) and (B.4) it follows that

$$B'' = -2 \delta r^{\delta/\pi} \delta^{(2)}(\mathbf{y}) + \dots, \quad (\text{B.5})$$

That is, the metric function e^B contains a delta-function contribution in its second order derivative with respect to ρ . In coordinate system (2.4), (B.5) becomes

$$\partial_u^2 B = -2 \delta r^{\delta/\pi} \delta^{(2)}(\mathbf{y}) + \dots, \quad (\text{B.6})$$

The delta-function in the curvature also gives rise to a delta-function in the derivative of the spin connection (4.6). The Riemann tensor is defined in terms of the spin connection as:

$$R^A_B = d\Omega^A_B + \Omega^A_C \wedge \Omega^C_B. \quad (\text{B.7})$$

Near to the brane $\rho = 0$, relation (B.7) gives $R_{\rho\varphi}^5_6 = \partial_\rho \Omega_{\varphi 6}^5$, and from the expression for the spin-connection (4.6):

$$\Omega' = - \left(\frac{1}{2} B'' + \frac{1}{4} (B')^2 \right) e^{B/2}, \quad (\text{B.8})$$

thus leading to the delta-function behaviour from (B.5).

These results must be recalled when obtaining the Schroedinger-like equations that govern the fluctuations.

C Imposing Boundary Conditions

Here we study the implications of the NC and the HC for gauge field fluctuations. To impose the boundary conditions the following properties will be useful:

$$F(a, b, c, z) \xrightarrow{z \rightarrow 0} 1, \quad (\text{C.1})$$

$$\begin{aligned} F(a, b, c, z) &= \Gamma_1 F(a, b, a + b - c + 1, 1 - z) \\ &+ \Gamma_2 (1 - z)^{c-a-b} F(c - a, c - b, c - a - b + 1, 1 - z), \end{aligned} \quad (\text{C.2})$$

where

$$\Gamma_1 \equiv \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Gamma_2 \equiv \frac{\Gamma(c)\Gamma(-c+a+b)}{\Gamma(a)\Gamma(b)}, \quad (\text{C.3})$$

²⁰Eq. (B.3) describes the asymptotic behaviour in the vicinity of the brane [36]. It seems that for positive δ the Ricci scalar vanishes at the origin. However, the first term on the right hand side of (B.3) should be interpreted in a distributional sense. The effect of the Ricci scalar as a distribution on a scalar test function $f(\mathbf{y})$ is then $\int d^2\mathbf{y} \sqrt{g} R f(\mathbf{y}) = 2 \delta f(\mathbf{0}) + \dots$

and Γ is the Euler gamma function. The relation (C.2) is valid if $c - a - b$ is not an integer [37] and y_1 and y_2 in (3.21) are both well defined when c is not an integer. In general $c - a - b$ and c are not integers for generic ω and $\bar{\omega}$; so we can consider ω and $\bar{\omega}$ as regulators to use (3.21) and (C.2) and at the end we can take the limits in which $c - a - b$ and c go to an integer, which will turn out to be well defined.

We first consider the behaviour of ψ for $u \rightarrow \bar{u}$, that is $z \rightarrow 0$ because of the definition $z = \cos^2\left(\frac{u}{r_0}\right)$. For $c \neq 1$ we use the expression for ψ given in (3.22) and property (C.1) gives us

$$\psi \xrightarrow{u \rightarrow \bar{u}} K_1(\bar{u} - u)^{2\gamma} + K_2(\bar{u} - u)^{1-2\gamma}, \quad (\text{C.4})$$

where we used $c = 1/2 + 2\gamma$. So the NC (3.5) implies $K_1 = 0$ when $\gamma \leq -1/4$ and $K_2 = 0$ when $\gamma \geq 3/4$. On the other hand the HC (3.16) implies²¹

$$\lim_{u \rightarrow \bar{u}} \psi^* \left(-\partial_u + \frac{1}{2} \frac{1}{u - \bar{u}} \right) \psi < \infty \quad (\text{C.5})$$

and by using the behaviour (C.4) this limit becomes

$$\left(2\gamma - \frac{1}{2} \right) \lim_{u \rightarrow \bar{u}} \left[|K_1|^2 (\bar{u} - u)^{4\gamma-1} - K_1^* K_2 + K_1 K_2^* - |K_2|^2 (\bar{u} - u)^{-4\gamma+1} \right], \quad (\text{C.6})$$

so the HC implies $K_1 = 0$ when $\gamma < 1/4$ and $K_2 = 0$ when $\gamma > 1/4$. The case $\gamma = 1/4$ corresponds to $c = 1$ and so we have to use the expression of ψ given in (3.24). We have then

$$\psi \xrightarrow{u \rightarrow \bar{u}} K_1(\bar{u} - u)^{1/2} - K_2(\bar{u} - u)^{1/2} \ln(\bar{u} - u) \quad (\text{C.7})$$

for which (C.5) implies $K_2 = 0$. Therefore we obtain (3.25) and (3.26).

The discreteness of the spectrum emerges when we impose the NC and HC for $u \rightarrow 0$. For instance for $m \geq N_V$, up to an overall constant, the behaviour of ψ is given by properties (C.1) and (C.2):

$$\psi \xrightarrow{u \rightarrow 0} \Gamma_1 u^{2\beta} + \Gamma_2 u^{1-2\beta}, \quad (\text{C.8})$$

where $\Gamma_{1,2}$ are defined in (C.3) and we used $c - a - b = 1/2 - 2\beta$. Behaviour (C.8) is similar to (C.4) but γ is replaced by β . So, following the same steps as above, the NC and the HC imply that $\Gamma_1 = 0$ for $\beta < 1/4$ and $\Gamma_2 = 0$ for $\beta > 1/4$. Let us study the case $m \geq N_V$ and $\beta < 1/4$, that is

$$N_V \leq m < 0. \quad (\text{C.9})$$

We then have

$$0 = \Gamma_1 \equiv \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \quad (\text{C.10})$$

²¹In fact, for the gauge field sector, each term in the HC (3.16) is separately zero if one requires their finiteness. Therefore (C.5) and its counter-part for $u = 0$ are sufficient to ensure (3.16). On the other hand, for the fermions, there are some cases in which they are each finite and non-zero, and so (3.16) requires that they cancel.

Since the Euler gamma function never vanishes we require that $\Gamma(c-a) = \infty$ or $\Gamma(c-b) = \infty$ and this is possible only when $c-a = -n$ or $c-b = -n$, where $n = 0, 1, 2, 3, \dots$. By using the definitions (3.19) both conditions lead to the following squared masses

$$M_{V,n,m}^2 = \frac{4}{r_0^2} \left\{ n(n+1) + \left(\frac{1}{2} + n \right) [-m\omega + (m - N_V)\bar{\omega}] \right\} \quad (\text{C.11})$$

which are positive because of (C.9) and $n \geq 0$. When $m \geq N_V$ and²² $\beta \geq 1/4$, that is

$$m \geq N_V \quad \text{and} \quad m \geq 0, \quad (\text{C.12})$$

we have

$$0 = \Gamma_2 \equiv \frac{\Gamma(c)\Gamma(-c+a+b)}{\Gamma(a)\Gamma(b)} \quad (\text{C.13})$$

and this implies $a = -n$ or $b = -n$, where $n = 0, 1, 2, 3, \dots$. The corresponding squared masses are

$$M_{V,n,m}^2 = \frac{4}{r_0^2} \left\{ n(n+1) + \left(\frac{1}{2} + n \right) [m\omega + (m - N_V)\bar{\omega}] + m\omega(m - N_V)\bar{\omega} \right\} \quad (\text{C.14})$$

which are positive or vanishing. We can study the case $m < N_V$ in a similar way. The complete result for the gauge fields sector is given in equations (3.27)-(3.30). We observe that (3.16) is now automatically satisfied by every pair of wave functions ψ and ψ' , for a given quantum number m , since the asymptotic behaviour of the wave function cannot depend on the quantum number n : this is a consequence of the $1/u^2$ and $1/(\bar{u} - u)^2$ singularities of the potential V in (3.14).

D Complete Fermionic Mass Spectrum

In this appendix we give the complete fermionic spectrum which is also labeled by an integer quantum number $n = 0, 1, 2, \dots$. Although much longer, the calculation proceeds in exactly the same way as for the gauge field sector, outlined in the previous appendix²³.

$m \geq -\frac{1}{2} + N + \frac{1}{2\bar{\omega}}$: in this case $K_2 = 0$ and we obtain the following squared masses.

- For $m > -\frac{1}{2} + \frac{1}{2\bar{\omega}}$

$$M_{F,n,m}^2 = \frac{4}{r_0^2} \left[\frac{1}{2} + n + \omega \left(\frac{1}{2} + m \right) \right] \left[\frac{1}{2} + n + \bar{\omega} \left(\frac{1}{2} + m - N \right) \right] > 0. \quad (\text{D.1})$$

²²The $\beta = 1/4$ case is recovered by taking the limit $\omega \rightarrow 0$.

²³There is one additional subtlety. Here, for the values of m which allow a zero mode, we must impose a mixed HC between the massless mode and massive modes, in addition to the diagonal HC. In general the HC involving distinct wave functions ψ_{mn} and $\psi_{mn'}$ does not lead to additional constraints, because the asymptotic behaviour of the modes is independent of n . However, the massless modes are more strongly constrained than the massive ones, obeying as they do a decoupled Dirac equation in addition to the Schroedinger equation.

- For $-\frac{1}{2} - \frac{1}{2\omega} < m < -\frac{1}{2} + \frac{1}{2\omega}$

$$M_{Fn,m}^2 = \frac{4}{r_0^2} \left[\frac{1}{2} + n + \omega \left(\frac{1}{2} + m \right) \right] \left[\frac{1}{2} + n + \bar{\omega} \left(\frac{1}{2} + m - N \right) \right] > 0. \quad (\text{D.2})$$

or

$$M_{Fn,m}^2 = \frac{4}{r_0^2} (1+n) \left[1 + n + \bar{\omega} \left(\frac{1}{2} + m - N \right) - \omega \left(\frac{1}{2} + m \right) \right] > 0. \quad (\text{D.3})$$

- For $m \leq -\frac{1}{2} - \frac{1}{2\omega}$

$$M_{Fn,m}^2 = \frac{4}{r_0^2} (1+n) \left[1 + n + \bar{\omega} \left(\frac{1}{2} + m - N \right) - \omega \left(\frac{1}{2} + m \right) \right] > 0. \quad (\text{D.4})$$

$m \leq -\frac{1}{2} + N - \frac{1}{2\bar{\omega}}$: in this case $K_1 = 0$ and we obtain the following squared masses.

- For $m > -\frac{1}{2} + \frac{1}{2\omega}$

$$M_{Fn,m}^2 = \frac{4}{r_0^2} n \left[n - \bar{\omega} \left(\frac{1}{2} + m - N \right) + \omega \left(\frac{1}{2} + m \right) \right] \geq 0. \quad (\text{D.5})$$

- For $-\frac{1}{2} - \frac{1}{2\omega} < m < -\frac{1}{2} + \frac{1}{2\omega}$

$$M_{Fn,m}^2 = \frac{4}{r_0^2} n \left[n - \bar{\omega} \left(\frac{1}{2} + m - N \right) + \omega \left(\frac{1}{2} + m \right) \right] \geq 0. \quad (\text{D.6})$$

- For $m \leq -\frac{1}{2} - \frac{1}{2\omega}$

$$M_{Fn,m}^2 = \frac{4}{r_0^2} \left[\frac{1}{2} + n - \omega \left(\frac{1}{2} + m \right) \right] \left[\frac{1}{2} + n - \bar{\omega} \left(\frac{1}{2} + m - N \right) \right] > 0. \quad (\text{D.7})$$

$-\frac{1}{2} + N - \frac{1}{2\bar{\omega}} < m < -\frac{1}{2} + N + \frac{1}{2\bar{\omega}}$: this case is possible only when $\bar{\delta} < 0$.

- For $m > -\frac{1}{2} + \frac{1}{2\omega}$ we have $K_1 = 0$ and

$$M_{Fn,m}^2 = \frac{4}{r_0^2} n \left[n - \bar{\omega} \left(\frac{1}{2} + m - N \right) + \omega \left(\frac{1}{2} + m \right) \right] \geq 0. \quad (\text{D.8})$$

- For $m \leq -\frac{1}{2} - \frac{1}{2\omega}$ we have two possibilities. We have $K_2 = 0$ and

$$M_{Fn,m}^2 = \frac{4}{r_0^2} (1+n) \left[1 + n + \bar{\omega} \left(\frac{1}{2} + m - N \right) - \omega \left(\frac{1}{2} + m \right) \right] > 0 \quad (\text{D.9})$$

or $K_1 = 0$ and

$$M_{Fn,m}^2 = \frac{4}{r_0^2} \left[\frac{1}{2} + n - \omega \left(\frac{1}{2} + m \right) \right] \left[\frac{1}{2} + n - \bar{\omega} \left(\frac{1}{2} + m - N \right) \right] > 0 \quad (\text{D.10})$$

- $-\frac{1}{2} - \frac{1}{2\omega} < m < -\frac{1}{2} + \frac{1}{2\omega}$: this case is possible only when $\delta < 0$ and we obtain $K_1 = 0$ and

$$M_{F n,m}^2 = \frac{4}{r_0^2} n \left[n - \bar{\omega} \left(\frac{1}{2} + m - N \right) + \omega \left(\frac{1}{2} + m \right) \right] \geq 0. \quad (\text{D.11})$$

Again, we can perform a check of our results by considering the S^2 limit ($\omega \rightarrow 1, \bar{\omega} \rightarrow 1$). In this case, the mass spectrum (D.1)-(D.11) reduces correctly to

$$a^2 M_F^2 = \left(l + \frac{1+N}{2} \right) \left(l + \frac{1-N}{2} \right), \quad \text{multiplicity} = 2l + 1 \quad (\text{D.12})$$

where²⁴ $l = \frac{|N|-1}{2} + k$ and $k = 0, 1, 2, 3, \dots$

References

- [1] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, “SUSY breaking and moduli stabilization from fluxes in gauged 6D supergravity,” *JHEP* **0303** (2003) 032 [arXiv:hep-th/0212091].
- [2] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, “Towards a naturally small cosmological constant from branes in 6D supergravity,” *Nucl. Phys. B* **680** (2004) 389 [arXiv:hep-th/0304256].
C. P. Burgess, “Supersymmetric large extra dimensions and the cosmological constant problem,” [arXiv:hep-th/0510123].
- [3] L. Randall and R. Sundrum, “A large mass hierarchy from a small extra dimension,” *Phys. Rev. Lett.* **83**, 3370 (1999) [arXiv:hep-ph/9905221].
L. Randall and R. Sundrum, “An alternative to compactification,” *Phys. Rev. Lett.* **83** (1999) 4690 [arXiv:hep-th/9906064].
- [4] G. W. Gibbons, R. Guven and C. N. Pope, “3-branes and uniqueness of the Salam-Sezgin vacuum,” *Phys. Lett. B* **595** (2004) 498 [arXiv:hep-th/0307238].
- [5] Y. Aghababaie *et al.*, “Warped brane worlds in six dimensional supergravity,” *JHEP* **0309** (2003) 037 [arXiv:hep-th/0308064].
- [6] C. P. Burgess, F. Quevedo, G. Tasinato and I. Zavala, “General axisymmetric solutions and self-tuning in 6D chiral gauged supergravity,” *JHEP* **0411** (2004) 069 [arXiv:hep-th/0408109].
- [7] S. Chang, J. Hisano, H. Nakano, N. Okada and M. Yamaguchi, “Bulk standard model in the Randall-Sundrum background,” *Phys. Rev. D* **62** (2000) 084025 [arXiv:hep-ph/9912498].

²⁴The number l is defined in different ways in equations (D.1)-(D.11). For instance we have $l \equiv n + m + (1 - N)/2$ for (D.1) and $l \equiv 1/2 + n - N/2$ for (D.4).

- [8] V. A. Rubakov and M. E. Shaposhnikov, “Do We Live Inside A Domain Wall?,” *Phys. Lett. B* **125** (1983) 136.
- [9] *For some early work see:*
 W. D. Goldberger and M. B. Wise, “Bulk fields in the Randall-Sundrum compactification scenario,” *Phys. Rev. D* **60** (1999) 107505 [arXiv:hep-ph/9907218].
 H. Davoudiasl, J. L. Hewett and T. G. Rizzo, “Bulk gauge fields in the Randall-Sundrum model,” *Phys. Lett. B* **473** (2000) 43 [arXiv:hep-ph/9911262].
 T. Gherghetta and A. Pomarol, “Bulk fields and supersymmetry in a slice of AdS,” *Nucl. Phys. B* **586** (2000) 141 [arXiv:hep-ph/0003129].
- [10] H. Nishino and E. Sezgin, “Matter And Gauge Couplings Of N=2 Supergravity In Six-Dimensions,” *Phys. Lett. B* **144** (1984) 187.
- [11] S. Randjbar-Daemi, A. Salam, E. Sezgin and J. A. Strathdee, “An Anomaly Free Model In Six-Dimensions,” *Phys. Lett. B* **151** (1985) 351.
- [12] H. M. Lee and A. Papazoglou, “Scalar mode analysis of the warped Salam-Sezgin model,” *Nucl. Phys. B* **747**, 294 (2006) [arXiv:hep-th/0602208].
- [13] S. D. Avramis, A. Kehagias and S. Randjbar-Daemi, “A new anomaly free gauged supergravity in six dimensions,” *JHEP* **0505** (2005) 057 [arXiv:hep-th/0504033].
 S. D. Avramis and A. Kehagias, “A systematic search for anomaly free supergravities in six dimensions,” *JHEP* **0510** (2005) 052 [arXiv:hep-th/0508172].
 R. Suzuki and Y. Tachikawa, “More anomaly free models of six-dimensional gauged supergravity,” arXiv:hep-th/0512019.
- [14] H. Nishino and E. Sezgin, “The Complete N=2, D = 6 Supergravity With Matter And Yang-Mills Couplings,” *Nucl. Phys. B* **278** (1986) 353.
- [15] S. Randjbar-Daemi and E. Sezgin, “Scalar potential and dyonic strings in 6d gauged supergravity,” *Nucl. Phys. B* **692**, 346 (2004) [arXiv:hep-th/0402217].
- [16] J. W. Chen, M. A. Luty and E. Ponton, “A critical cosmological constant from millimeter extra dimensions,” *JHEP* **0009** (2000) 012 [arXiv:hep-th/0003067].
- [17] A. J. Tolley, C. P. Burgess, D. Hoover and Y. Aghababaie, “Bulk singularities and the effective cosmological constant for higher co-dimension branes,” *JHEP* **0603** (2006) 091 [arXiv:hep-th/0512218].
- [18] H. M. Lee and C. Ludeling, “The general warped solution with conical branes in six-dimensional supergravity,” *JHEP* **0601**, 062 (2006) [arXiv:hep-th/0510026].
- [19] S. L. Parameswaran, G. Tasinato and I. Zavala, “The 6D SuperSwirl,” *Nucl. Phys. B* **737**, 49 (2006) [arXiv:hep-th/0509061].
- [20] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, “The hierarchy problem and new dimensions at a millimeter,” *Phys. Lett. B* **429** (1998) 263 [arXiv:hep-ph/9803315].

- I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, “New dimensions at a millimeter to a Fermi and superstrings at a TeV,” *Phys. Lett. B* **436** (1998) 257 [arXiv:hep-ph/9804398].
- [21] M. Giovannini, “Graviphoton and graviscalars delocalization in brane world scenarios,” arXiv:hep-th/0111218.
- [22] M. Giovannini, J. V. Le Be and S. Riederer, “Zero modes of six-dimensional Abelian vortices,” *Class. Quant. Grav.* **19**, 3357 (2002) [arXiv:hep-th/0205222].
- [23] S. Randjbar-Daemi and M. Shaposhnikov, “A formalism to analyze the spectrum of brane world scenarios,” *Nucl. Phys. B* **645** (2002) 188 [arXiv:hep-th/0206016].
- [24] S. Randjbar-Daemi and M. Shaposhnikov, “QED from six-dimensional vortex and gauge anomalies,” *JHEP* **0304**, 016 (2003) [arXiv:hep-th/0303247].
- [25] H. Nicolai and C. Wetterich, “On The Spectrum Of Kaluza-Klein Theories With Non-compact Internal Spaces,” *Phys. Lett. B* **150** (1985) 347.
- [26] G. W. Gibbons and D. L. Wiltshire, “Space-Time As A Membrane In Higher Dimensions,” *Nucl. Phys. B* **287**, 717 (1987) [arXiv:hep-th/0109093].
- [27] A. Kehagias, “On non-compact compactifications with brane worlds,” arXiv:hep-th/9911134.
- [28] M. F. Manning, “Exact Solutions of the Schroedinger Equation,” *Phys. Rev.* **48** (1935) 161.
- [29] S. Randjbar-Daemi, A. Salam and J. A. Strathdee, “Spontaneous Compactification In Six-Dimensional Einstein-Maxwell Theory,” *Nucl. Phys. B* **214** (1983) 491.
- [30] C. Wetterich, “Chiral Fermions In Six-Dimensional Gravity,” *Nucl. Phys. B* **253** (1985) 366.
- [31] J. M. Schwindt and C. Wetterich, “Holographic branes,” *Phys. Lett. B* **578** (2004) 409 [arXiv:hep-th/0309065].
- [32] S. M. Carroll and M. M. Guica, “Sidestepping the cosmological constant with football-shaped extra dimensions,” arXiv:hep-th/0302067.
- [33] K. R. Dienes, “Shape versus volume: Making large flat extra dimensions invisible,” *Phys. Rev. Lett.* **88** (2002) 011601 [arXiv:hep-ph/0108115].
- [34] C. Wetterich, “The Cosmological Constant And Noncompact Internal Spaces In Kaluza-Klein Theories,” *Nucl. Phys. B* **255** (1985) 480.
- [35] S. Randjbar-Daemi, A. Salam and J. A. Strathdee, “Instability Of Higher Dimensional Yang-Mills Systems,” *Phys. Lett. B* **124**, 345 (1983) [Erratum-ibid. *B* **144**, 455 (1984)].

- [36] S. Randjbar-Daemi and V. A. Rubakov, “4d-flat compactifications with brane vorticities,” JHEP **0410** (2004) 054 [arXiv:hep-th/0407176].
- [37] E. T. Whittaker and G. N. Watson “A course for Modern Analysis,” 4th Ed. Cambridge, England: Cambridge University Press, 1990.