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# Two-loop Calculation of Higgs Mass in Gauge-Higgs Unification:

## 5D Massless QED Compactified on $S^1$

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**Abstract**

We calculate the quantum corrections to the mass of the zero mode of the fifth component of the gauge field at two-loop level in a five dimensional massless QED compactified on $S^1$. We discuss in detail how the divergences are exactly canceled and the mass becomes finite. The key ingredients to obtain the result are the shift symmetry and the Ward-Takahashi identity. We also evaluate the finite part of corrections.

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## 1 Introduction

Gauge-Higgs unification [1] is considered to be one of the attractive frameworks since it provides a solution to the gauge hierarchy problem without supersymmetry [2, 3, 4, 5]. In this scenario, the Higgs filed is identified with extra components of the gauge field in higher dimensional gauge theories. A remarkable feature in the scenario is that quantum corrections to the Higgs mass become finite and are independent of the cuttoff scale of the theory thanks to the gauge invariance in the higher dimensions nevertheless we consider nonrenormalizable theories. The Higgs mass is generated through the dynamics of the Wilson line for an extra component of the gauge field. Noting that the dynamics is nonlocal, we find no counter term in the lagrangian, which is assumed to be local, to cancel the divergence if the Higgs mass diverges. This implies that the Higgs mass should be finite under quantum corrections at all order of the perturbations. Actually, its finiteness at one-loop level was discussed by several authors [2]. (In Gravity-Gauge-Higgs unification, the finiteness is guaranteed by the general coordinate invariance, see [6].)

Although the concept for the finiteness of the Higgs mass is very clear, there are subtleties if we consider higher loop corrections to the Higgs mass beyond one-loop level. For instance, generally there appear divergences in the subdiagrams even if we consider the gauge-Higgs unification scenario. These divergences should be subtracted by adding the counter terms determined by the lower loop calculations. After such a subtraction, the Higgs mass becomes finite at any order of perturbations without any additional counter terms. This means that the Higgs mass can be predicted even within nonrenormalizable theories. In fact, a Higgs mass at two-loop level are calculated in a five dimensional (5D) supersymmetric theory [7], where the linear divergences appear in the one-loop subdiagrams and are subtracted by adding one-loop counter terms.

It is also very important to calculate the Higgs mass beyond one-loop level from the phenomenological viewpoint. It is known that the physical Higgs mass and the Kaluza-Klein (KK) mass tend to be too small in the scenario. To get a large KK mass, or in other words to get a small vacuum expectation value (VEV) of the Higgs fields compared to the KK mass, we rely on a mild tuning to cancel the Higgs mass corrections among one-loop contributions [4]. A large KK mass helps to enhance the physical Higgs mass. However, if the KK mass is taken so large, two-loop contributions can be important. Thus, we can not make the KK mass larger than $\mathcal{O}(10\text{TeV})$ reliably if we do not know the two-loop corrections. In this case, the physical Higgs mass can not exceed the present bound [8] if the low energy effective theory is just the standard model [9]. On the other hand, if we control the two-loop corrections, the KK mass can be enlarged up to the scale where three-loop contributions become important, say $\mathcal{O}(100\text{TeV})$. Then, the physical Higgs mass can pass the experimental test without additional low energy degrees.

As far as we know, there seems no calculation of the Higgs mass beyond one-loop order in the context of gauge-Higgs unification. Therefore, it is worthwhile to check explicitly the finiteness of the Higgs mass for higher order loop corrections. In this paper, we explicitly calculate the two-loop quantum corrections to the mass of the zero mode of the fifth component of the gauge field in a 5D massless QED compactified on $S^1$. As expected from the general argument of the renormalization theory, the mass is shown to be finite. A key ingredient to show the finiteness is the shift symmetry Ward-Takahashi identity. Although there appear linearly divergent vertex corrections and the wave function renormalizations in subdiagrams, these divergences are exactly canceled as expected from Ward-Takahashi identity. In this simple model, there is no need to take into account counter terms. We will discuss in detail the structure of cancellation of the divergences and also evaluate the finite part of the corrections.

This paper is organized as follows. In the next section, we introduce our setup and derive Feynman rules. Section 3 is the main part of this paper. Before calculating the two-loop corrections, we calculate the one-loop wave function renormalization and the vertex corrections to observe that these contributions are linearly divergent and have the same magnitude but an opposite sign. Then, the two-loop corrections to the mass of the zero mode of the fifth component of the gauge field are calculated and the structure of canceling divergences is clarified. The last section is devoted to summarize this paper.

## 2 5D Massless QED Compactified on $S^1$

As an illustration, we consider a 5D massless QED compactified on $S^1$ and calculate the mass correction to the zero mode of the fifth component of the gauge field $A_5$ at two-loop level. The action is written as

$$S = \int \mathrm{d}^4x\mathrm{d}y \left[ -\frac{1}{4} F_{MN}F^{MN} + \bar{\Psi} i \not{D}_5 \Psi + \mathcal{L}_{GF} \right], \tag{2.1}$$

where $\not{D}_5 = \not{D} - i\gamma_5 D_5$, $\gamma_5^2 = 1$, $D_M = \partial_M - igA_M (M = 0, 1, 2, 3, 5)$ is the covariant derivative. $g$ is the 5D gauge coupling constant. We take the mostly minus metric $\eta_{MN} = \mathrm{diag}(+, -, -, -, -)$. We choose the gauge fixing term as

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} \left( \partial_\mu A^\mu - \xi \partial_5 A^5 \right)^2, \tag{2.2}$$

where $\mu = 0, 1, 2, 3$ and $\xi$ is a gauge parameter. Then, the gauge part of the action becomes

$$\begin{aligned} S_G \;=\; & \int \mathrm{d}^4x\mathrm{d}y \frac{1}{2} \left[ -\left( \partial_\mu A_\nu \right)^2 + \left( 1 - \xi^{-1} \right) \left( \partial_\nu A_\nu \right)^2 + \left( \partial_5 A_\nu \right)^2 \right. \\ & \left. + \left( \partial_\mu A_5 \right)^2 - \xi \left( \partial_5 A_5 \right)^2 \right]. \end{aligned} \tag{2.3}$$

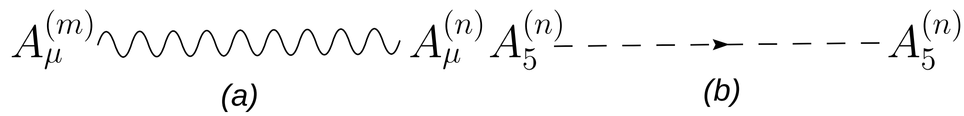


Figure 1: The propagators of the photon (a) and $A_5$ (b).

Expanding the gauge field in terms of the Kaluza-Klein modes,

$$A_\mu(x^\mu, y) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} A_\mu^{(n)}(x^\mu) \exp(2\pi i n \frac{y}{L}), \tag{2.4}$$

$$A_5(x^\mu, y) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} A_5^{(n)}(x^\mu) \exp(2\pi i n \frac{y}{L}), \tag{2.5}$$

where $A_M^{(n)*} = A_M^{(-n)}$ and $L = 2\pi R$ is the circumference of the $S^1$, it is written as

$$S_G = \int \mathrm{d}^4 x \sum_{n=-\infty}^{\infty} \frac{1}{2} \Big[ -\left|\partial_\mu A_\nu^{(n)}\right|^2 + \left(1 - \xi^{-1}\right) \left|\partial_\nu A_\nu^{(n)}\right|^2 + M_n^2 \left|A_\nu^{(n)}\right|^2 \\ + \left|\partial_\mu A_5^{(n)}\right|^2 - \xi M_n^2 \left|A_5^{(n)}\right|^2 \Big], \tag{2.6}$$

where $M_n = 2\pi n/L = n/R$ is the KK mass. This leads to the following propagator (see Fig. 1):

$$(a) = \delta_{mn} \left( \frac{\eta^{\mu\nu} - \frac{p_\mu p_\nu}{M_n^2}}{p^2 - M_n^2} + \frac{p_\mu p_\nu}{M_n^2} \frac{1}{p^2 - \xi M_n^2} \right), \tag{2.7}$$

$$(b) = \frac{-\delta_{mn}}{p^2 - \xi M_n^2}. \tag{2.8}$$

Next, expanding the fermion in terms of the KK modes,

$$\bar{\Psi}(x^\mu, y) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \bar{\Psi}^{(-n)}(x^\mu) \exp(i 2\pi n \frac{y}{L}), \tag{2.9}$$

$$\Psi(x^\mu, y) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \Psi^{(n)}(x^\mu) \exp(i 2\pi n \frac{y}{L}), \tag{2.10}$$

the fermion part is written as

$$S_m = \int \mathrm{d}^4 x \sum_{m,n} \bar{\Psi}^{(m)} \left( i\delta_{nm} \left(\not{\partial} + M_n \gamma_5\right) + \sum_l \delta_{m\ l+n} \left( g_4 \not{A}_\mu^{(l)} - i g_4 \gamma_5 A_5^{(l)} \right) \right) \Psi^{(n)} \tag{2.11}$$

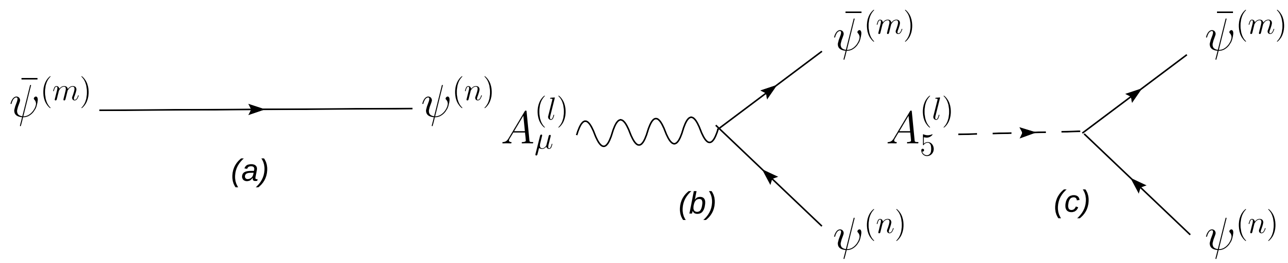


Figure 2: Feynman diagrams relevant for the fermion. (a), (b) and (c) are the fermion propagator, the gauge interaction vertex and the vertex of fermion-fermion-$A_5$, respectively.

where the 4D gauge coupling constant $g_4$ is defined as $g_4 = g/\sqrt{L}$. This leads to the following Feynman rule (see Fig. 2):

$$(a) = \frac{-\delta_{mn}}{\not{p} + iM_n\gamma_5} = -\delta_{mn}\frac{\not{p} + iM_n\gamma_5}{p^2 - M_n^2}, \tag{2.12}$$

$$(b) = g_4\delta_{m\ l+n}\gamma_\mu, \tag{2.13}$$

$$(c) = -ig_4\delta_{m\ l+n}\gamma_5. \tag{2.14}$$

# 3 Loop Calculations

## 3.1 One-loop

Before calculating two-loop corrections, we clarify the nature of divergences at one-loop level since the divergences appearing in the subdiagrams of two-loop diagrams have to be subtracted by adding the counter terms generally. The possible relevant counter terms at this order are those of the fermion propagator, the gauge-fermion-fermion vertex and the gauge propagator. The first one corresponds to that for fermion wave function renormalization. The second one corresponds to that for the gauge interaction vertex correction. The last one should correspond to the renormalization of the gauge coupling.

### 3.1.1 Fermion Wave Function Renormalization

The wave function renormalization of the fermion is calculated as

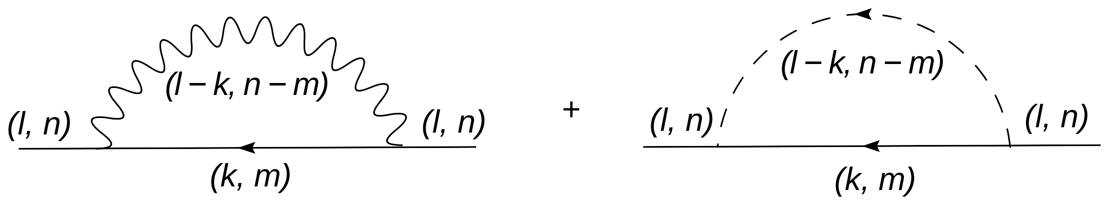


Figure 3: Wave function renormalization of the fermion. The corresponding 4D and KK momenta are denoted as $(k, m)$ for example.

$$
\begin{aligned}
\text{Fig. 3} &= \int \frac{\mathrm{d}^4 k}{i(2\pi)^4} \sum_m g_4^2 \left[ \gamma_\mu \frac{\not{k} + iM_m\gamma_5}{k^2 - M_m^2} \gamma_\nu \frac{\eta^{\mu\nu}}{(l-k)^2 - M_{n-m}^2} \right. \\
&\qquad \left. +(-i)\gamma_5 \frac{\not{k} + iM_m\gamma_5}{k^2 - M_m^2} (-i)\gamma_5 \frac{-1}{(l-k)^2 - M_{n-m}^2} \right] \\
&= \frac{g_4^2}{R} \int \frac{\mathrm{d}^4 k}{i(2\pi)^4} \sum_m \frac{-3\left(\not{k} + im\gamma_5\right)}{(k^2 - m^2)((l-k)^2 - (n-m)^2)},
\end{aligned}
\tag{3.15}
$$

where we normalized all the dimensionful parameters by $1/R$ in the last equation so that all the parameters become dimensionless. The gauge parameter is taken to be $\xi = 1$.

By using the Feynman integral

$$
\int_0^1 \mathrm{d}x \left[ \frac{1}{b + (a-b)x} \right]^2 = \frac{1}{ab}, \tag{3.16}
$$

the correction (3.15) is written as

$$
\begin{aligned}
&\frac{g_4^2}{R} \int \frac{\mathrm{d}^4 k}{i(2\pi)^4} \sum_m \int_0^1 \mathrm{d}x \frac{-3\left(\not{k} + im\gamma_5\right)}{\left((k - xl)^2 - (m - xn)^2 + (x - x^2)(l^2 - n^2)\right)^2} \\
=\ &\frac{g_4^2}{R} \int \frac{\mathrm{d}^4 k'}{i(2\pi)^4} \sum_m \int_0^1 \mathrm{d}x \frac{-3\left(x\not{l} + im\gamma_5\right)}{\left(k'^2 - (m - xn)^2 + (x - x^2)(l^2 - n^2)\right)^2}.
\end{aligned}
\tag{3.17}
$$

Here, we neglect the term that vanishes by the angular integration.

Now we carry out the infinite sum with respect to $m$. For this purpose, it is convenient to rewrite the summation by the contour integral in the complex plane,

$$
\sum_m f(m) \to \int_{C_0} \mathrm{d}z \frac{1}{1 - \exp(2\pi i z)} f(z) = \int_{C_0} \mathrm{d}z \left( 1 + \frac{1}{\exp(-2\pi i z) - 1} \right) f(z), \tag{3.18}
$$

where $C_0$ is a contour that encircle the real axis clockwise. If $\mathrm{Im}\, z \exp(-2\pi\, |\mathrm{Im}\, z|) f(z)$ vanishes at $|\mathrm{Im}\, z| \to \infty$ and $f(z)$ has no poles on the real axis but has poles $\{m_+^i\}$ in the upper half plane and poles $\{m_-^j\}$ in the lower half plane, the contour integral can be expressed by the summation of the residues at each pole and integration on the real axis:

$$\sum_i Res.\left\{\frac{2\pi i f(z)}{\exp(-2\pi i z)-1}; z=m_+^i\right\} + \sum_i Res.\left\{\frac{2\pi i f(z)}{1-\exp(2\pi i z)}; z=m_-^j\right\}$$
$$+\int_{-\infty}^{\infty} \mathrm{d}z f(z). \tag{3.19}$$

Note that if $f(z)$ is a real function, each $m_+^i$ has a counter part of $m_-^i = {m_+^i}^*$, which means that (3.19) can be reduced to

$$2\mathrm{Re}\left[\sum_i Res.\left\{\frac{2\pi i f(z)}{\exp(-2\pi i z)-1}; z=m_+^i\right\}\right] + \int_{-\infty}^{\infty} \mathrm{d}z f(z). \tag{3.20}$$

An important point is that the residues always contain the exponential suppression $\exp(-2\pi \mathrm{Im}\ m_+^i)$ for a large Im $m_+^i$, leading to finite contributions. Thus, as far as we concern the divergent contributions, it is enough to evaluate the integration on the real axis in (3.20). In other words, we can replace the summation with respect to $m$ by the integration on the real axis. Then, the correction (3.17) is written as

$$\begin{aligned}
&\frac{g_4^2}{R}\int \frac{\mathrm{d}^4k'}{i(2\pi)^4}\int_{-\infty}^{\infty}\mathrm{d}z_m\int_0^1\mathrm{d}x\frac{-3\left(x\not{l}+iz_m\gamma_5\right)}{\left(k'^2-(z_m-xn)^2+(x-x^2)(l^2-n^2)\right)^2}\\
=\ &\frac{g_4^2}{R}\int \frac{\mathrm{d}^4k'}{i(2\pi)^4}\int_{-\infty}^{\infty}\mathrm{d}z'_m\int_0^1\mathrm{d}x\frac{-3\left(x\not{l}+ixn\gamma_5\right)}{\left(k'^2-z'^2_m+(x-x^2)(l^2-n^2)\right)^2}.
\end{aligned}$$

This shows that the divergent parts of the wave function renormalization and the mass renormalization (times $R$) for the fermion mode with $(l,n)$ are commonly given by

$$\begin{aligned}
\delta_{Wf} &= g_4^2\int \frac{\mathrm{d}^4k'}{i(2\pi)^4}\int_{-\infty}^{\infty}\mathrm{d}z'_m\int_0^1\mathrm{d}x\frac{-3x}{\left(k'^2-z'^2_m+(x-x^2)(l^2-n^2)\right)^2}\\
&= g_4^2\int\frac{\mathrm{d}k'_E}{8\pi^2}\frac{-3\pi {k'_E}^2}{4k'^2_E+l_E^2+n^2}\\
&\to g_4^2\int\frac{\mathrm{d}k'_E}{8\pi^2}\left[\frac{-3\pi}{4}+\mathcal{O}(k_E^{-2})\right]\ (k'_E\to\infty),
\end{aligned} \tag{3.21}$$

where we use the same parameter $k'_E$ for denoting the absolute value of the Wick rotated vector $k'_E$[3]. We find that this correction is linearly divergent.

### 3.1.2 Vertex Correction

The correction to the gauge-fermion-fermion vertex is calculated as

[3] In the next subsection, we use another notation.

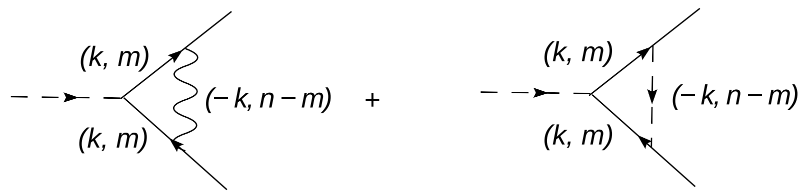


Figure 4: Vertex correction. The corresponding 4D and KK momenta are denoted as $(k, m)$ for example.

$$
\begin{aligned}
\text{Fig. 4} &= g_4^2 \int \frac{\mathrm{d}^4 k}{i(2\pi)^4} \sum_m \Bigg[ \gamma_\mu \frac{\not{k} + im\gamma_5}{k^2 - m^2} (-i)\gamma_5 \frac{\not{k} + im\gamma_5}{k^2 - m^2} \gamma_\nu \frac{\eta^{\mu\nu}}{k^2 - (n-m)^2} \\
&\qquad + (-i)\gamma_5 \frac{\not{k} + im\gamma_5}{k^2 - m^2} (-i)\gamma_5 \frac{\not{k} + im\gamma_5}{k^2 - m^2} (-i)\gamma_5 \frac{-1}{k^2 - (n-m)^2} \Bigg] \\
&= g_4^2 \int \frac{\mathrm{d}^4 k}{i(2\pi)^4} \sum_m (-i)\gamma_5 \frac{3\,(k^2 + m^2)}{(k^2 - m^2)^2 (k^2 - (n-m)^2)}
\end{aligned}
\tag{3.22}
$$

where we take the momenta of external lines to be zero.

Now concentrating on the divergence, we replace the summation with respect to $m$ by the integration on the real axis. By carrying out the Wick rotation and using the Feynman integral

$$
\int_0^1 \mathrm{d}x \frac{2!(1-x)}{((1-x)a + xb))^3} = \frac{1}{a^2 b},
\tag{3.23}
$$

the correction to the vertex $\delta_V$ becomes

$$
\begin{aligned}
\delta_V &= g_4^2 \int \frac{\mathrm{d}k_E k_E^3}{8\pi^2} \int_{-\infty}^{\infty} \mathrm{d}z'_m \int_0^1 \mathrm{d}x \frac{3\,(k_E^2 - (z'_m + xn)^2) \times 2!(1-x)}{(k_E^2 + {z'_m}^2 + (x - x^2)n^2)^3} \\
&= g_4^2 \int \frac{\mathrm{d}k_E}{8\pi^2} \frac{3\pi k_E^2\,(4k_E^2 - n^2)}{(4k_E^2 + n^2)^2} \\
&\to g_4^2 \int \frac{\mathrm{d}k_E}{8\pi^2} \left[ \frac{3\pi}{4} + \mathcal{O}(k_E^{-2}) \right] \ (k_E \to \infty).
\end{aligned}
\tag{3.24}
$$

We find that it is linearly divergent and is the same as the minus of that of $\delta_{Wf}$, as expected from Ward-Takahashi identity. This fact is very important to cancel divergences appearing in the subdiagrams, as will be seen in the next subsection.

### 3.1.3 Gauge Self Energy

In this subsubsection, we calculate the wave function renormalizations of $A_\mu$ and $A_5$. If we denote them as $Z_5$ and $Z_\mu$, respectively, these can be expressed at one-loop level

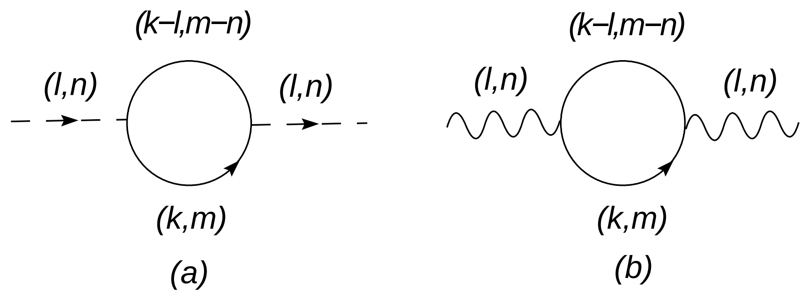


Figure 5: One-loop renormalizations for two-point function of $A_5$ (a) and the photon (b).

symbolically,

$$Z_5 = 1 + g_4^2(\Lambda + c) + \mathcal{O}(g_4^4), \quad (3.25)$$

$$Z_\mu = 1 + g_4^2(\Lambda + c') + \mathcal{O}(g_4^4) \quad (3.26)$$

where $\Lambda$ is a cutoff scale of the theory. These factors are linearly divergent. $c$ and $c'$ mean the physical renormalization factors after subtracted the divergence. Taking into account these renormalizations, the physical Higgs mass at two-loop level includes

$$\begin{aligned} m^2_{phys@2-loop} &= \frac{g_4^2}{Z_5} m^2_{H@1-loop} + \frac{g_4^4}{Z_5} m^2_{H@2-loop} + \mathcal{O}(g_4^6) \\ &= \frac{Z_\mu}{Z_5} g_R^2 m^2_{H@1-loop} + g_R^4 m^2_{H@2-loop} + \mathcal{O}(g_R^6) \\ &= \left[1 + g_R^2(c - c')\right] g_R^2 m^2_{H@1-loop} + g_R^4 m^2_{H@2-loop} + \mathcal{O}(g_R^6) \quad (3.27) \end{aligned}$$

where the renormalized gauge coupling $g_R$ is defined as $g_4^2 = g_R^2 Z_\mu$. $m^2_{H@1-loop}$ is a one-loop finite mass of the zero mode of $A_5$ arising from the diagram in Fig. 5 (a) with zero external momentum. Here we define such that $m^2_{H@1-loop}$ does not include the gauge coupling. $m^2_{H@2-loop}$ is a two-loop mass which we will evaluate in the next subsection. Note that the ultraviolet (UV) divergences appearing in (3.25) and (3.26) are guaranteed to be the same by the five dimensional Lorentz invariance. Below, we show it concretely. In addition, we will obtain, apart from $m^2_{H@2-loop}$, a finite mass of the zero mode of $A_5$ at two-loop level which is proportional to $m^2_{H@1-loop}$ and to the difference of the finite part, $c - c'$. Thus, we would like to evaluate also the finite parts of $Z_\mu$ and $Z_5$ and $m^2_{H@1-loop}$, not only the divergent part. However, note that this contribution should be discriminated from $m^2_{H@2-loop}$. This is because this contribution does not modify essentially the structure of the one-loop effective potential which is written in terms of $\cos(qgA_5)$, reflecting the phase structure of the Wilson line, where $q$ is a constant. In other words, the effect merely scales the effective potential in the horizontal direction and it is understood by replacing $g_R$ in the potential by $g_R^H = g_R\sqrt{Z_\mu/Z_5}$.

The wave function renormalizations of $A_5$ and $A_\mu$ are shown in Fig. 5 (a) and (b), and are calculated as

$$
\begin{aligned}
(a) &= (-1)\int \frac{d^4k}{(2\pi)^4 i}\sum_m \mathrm{tr}\left[(-ig_4\gamma_5)\frac{-(\not{k}+iM_m\gamma_5)}{k^2-M_m^2}(-ig_4\gamma_5)\frac{-(\not{k}-\not{l}+iM_{m-n}\gamma_5)}{(k-l)^2-M_{m-n}^2}\right] \\
&= -\frac{4g_4^2}{R^2}\int\frac{d^4k'_E}{(2\pi)^4}\sum_m\int_0^1 dx\frac{-{k'_E}^2-x(x-1){l_E}^2+m^2}{\left({k'_E}^2+m^2+(x-x^2){l_E}^2\right)^2} \qquad (3.28)
\end{aligned}
$$

and

$$
\begin{aligned}
(b) &= (-1)\int \frac{d^4k}{(2\pi)^4 i}\sum_m \mathrm{tr}\left[(g_4\gamma_\mu)\frac{-(\not{k}+iM_m\gamma_5)}{k^2-M_m^2}(g_4\gamma_\nu)\frac{-(\not{k}-\not{l}+iM_{m-n}\gamma_5)}{(k-l)^2-M_{m-n}^2}\right] \\
&= -\frac{4g_4^2}{R^2}\int\frac{d^4k'_E}{(2\pi)^4}\sum_m\int_0^1 dx\frac{N_{\mu\nu}}{\left({k'_E}^2+m^2+(x-x^2){l_E}^2\right)^2}, \qquad (3.29)
\end{aligned}
$$

respectively, where

$$
N_{\mu\nu} = -2k'_{E\mu}k'_{E\nu} + 2x(1-x)l_{E\mu}l_{E\nu} + g_{\mu\nu}[{k'_E}^2 + m^2 - x(1-x){l_E}^2].
$$

Here, we performed Wick rotation, omitted the terms that vanish after the angular integration of $k'_E$ and put $n=0$ since we are interested in the wave function of the zero modes. In the following, we consider only the term proportional to $l_{E\mu}l_{E\nu}$ and set $l_E^2=0$ to evaluate $Z_\mu$.

Let us show the divergent parts of (3.28) and (3.29), which are evaluated by replacing the summation to the integral as before, are the same. Carrying out the integration, we find

$$
\begin{aligned}
\int_0^\infty dz_m\frac{-{k'_E}^2-x(x-1)l_E^2+z_m^2}{({k'_E}^2+z_m^2+x(1-x)l_E^2)^2} &= -\pi\frac{x(1-x)l_E^2}{({k'_E}^2+x(1-x)l_E^2)^{3/2}}, \qquad (3.30) \\
\int_0^\infty dz_m\frac{2x(1-x)}{({k'_E}^2+m^2)^2} &= -\pi\frac{x(1-x)}{{k'_E}^3}. \qquad (3.31)
\end{aligned}
$$

From (3.30), we can see that this part does not contribute $m^2_{H@1-loop}$, and the contribution to the wave function renormalization is

$$
-\pi\frac{x(1-x)}{{k'_E}^3} \qquad (3.32)
$$

which, including the finite part, is exactly same as (3.31). Note that the integration over $z_m$ corresponds to the calculation in the case where the fifth momentum is continuous, *i.e.* the limit $R\to\infty$. In this decompactification limit, the 5D Lorentz symmetry, which is softly broken by the compactification, recovers. Therefore, the cancellation among these contribution is natural.

Next, we evaluate the residue parts which are free from UV divergences. As for the $Z_5$, we get

$$
\begin{aligned}
&2\mathrm{Re}\left[Res.\left\{\frac{2\pi i}{\exp(-2\pi i z_m)-1}\frac{-{k'_E}^2-x(x-1)l_E^2+z_m^2}{({k'_E}^2+z_m^2+x(1-x)l_E^2)^2}; z_m = i\sqrt{{k'_E}^2+x(1-x)l_E^2}\right\}\right] \\
= &-\pi\left[\frac{-2x(1-x)l_E^2}{({k'_E}^2+x(1-x)l_E^2)^{3/2}(e^{2\pi\sqrt{{k'_E}^2+x(1-x)l_E^2}}-1)^2}\right. \\
&\left.+\frac{4\pi {k'_E}^2 e^{2\pi\sqrt{{k'_E}^2+x(1-x)l_E^2}}}{({k'_E}^2+x(1-x)l_E^2)(e^{2\pi\sqrt{{k'_E}^2+x(1-x)l_E^2}}-1)^2}\right]. \qquad (3.33)
\end{aligned}
$$

We can find the one-loop correction $m^2_{H@1-loop}$ by setting $l_E^2 = 0$ as,

$$
g_4^2 m^2_{A_5@1-loop} = -\frac{4g_4^2}{R^2}\int\frac{d^4k'_E}{(2\pi)^4}\frac{-2\pi^2}{(-1+\cosh(2\pi k'_E))} = \frac{3g_4^2}{4\pi^4R^2}\zeta(3). \qquad (3.34)
$$

The wave function renormalization comes from the $l_E^2$ term, therefore we obtain by differentiating (3.33) with respect to $l_E^2$ and setting $l_E^2 = 0$,

$$
\begin{aligned}
-4g_4^2\int\frac{d^4k'_E}{(2\pi)^4}\int_0^1 dx(-\pi)&\left[\frac{2x(1-x)}{{k'_E}^3(e^{2\pi k'_E}-1)}+\frac{4\pi^2x(1-x)e^{2\pi k'_E}}{k'_E(e^{2\pi k'_E}-1)^2}\right. \\
&\left.-\frac{4\pi x(1-x)e^{2\pi k'_E}}{{k'_E}^2(e^{2\pi k'_E}-1)^2}+\frac{8\pi^2x(1-x)e^{4\pi k'_E}}{k'_E(e^{2\pi k'_E}-1)^3}\right]. \qquad (3.35)
\end{aligned}
$$

The overall factor $1/R^2$ disappears on the dimensional grounds in the differentiation. The contribution to $Z_\mu$ is calculated as

$$
\begin{aligned}
&2\mathrm{Re}\left[Res.\left\{\frac{2\pi i}{\exp(-2\pi i z_m)-1}\frac{2x(1-x)}{({k'_E}^2+z_m^2)^2}; z_m = ik'_E\right\}\right] \\
= &-4g_4^2\int\frac{d^4k'_E}{(2\pi)^4}\int_0^1 dx(-\pi)\left[\frac{2x(1-x)}{{k'_E}^3(e^{2\pi k'_E}-1)}-\frac{4\pi x(1-x)e^{2\pi k'_E}}{{k'_E}^2(e^{2\pi k'_E}-1)^2}\right]. \qquad (3.36)
\end{aligned}
$$

Note that these terms have the same for as the first term and the third term in (3.35).

From these results, we can obtain $Z_\mu/Z_5$ at one-loop level as

$$
\begin{aligned}
\left[\frac{Z_\mu}{Z_5}\right]_{\text{finite}} = &1-4g_R^2\int\frac{d^4k'_E}{(2\pi)^4}\int_0^1 dx(-\pi)\left[\frac{4\pi^2x(1-x)e^{2\pi k'_E}}{k'_E(e^{2\pi k'_E}-1)^2}+\frac{8\pi^2x(1-x)e^{4\pi k'_E}}{k'_E(e^{2\pi k'_E}-1)^3}\right] \\
&+\mathcal{O}(g_R^4), \qquad (3.37)
\end{aligned}
$$

which is in fact free from UV divergences but contains infrared (IR) divergences. This is because we consider exactly massless charged fermion for simplicity. However, we usually consider the case where $A_5$ which is identified as the Higgs field get non-vanishing VEV in the gauge-Higgs unification scenario. Then, the charged fermions

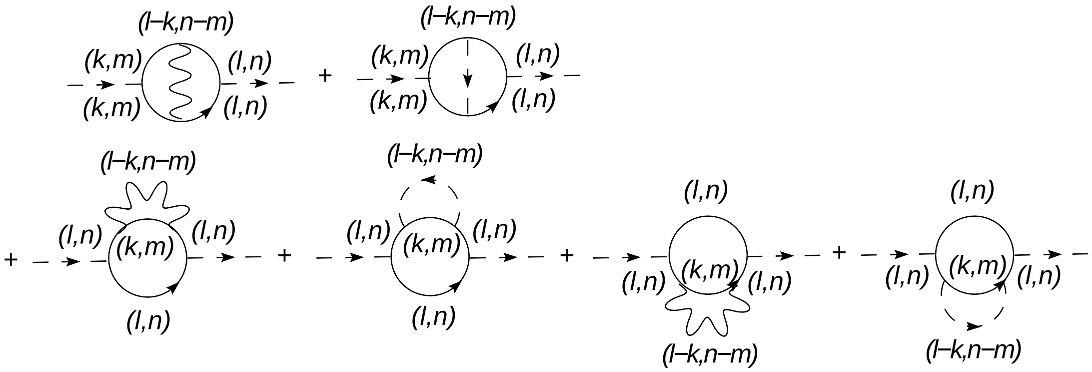


Figure 6: Two-loop diagrams for the mass of the zero mode of $A_5$. The corresponding 4D and KK momenta are denoted as $(k,m)$ for example.

acquires non-vanishing mass, and the IR divergences disappear. Thus, we recalculate $Z_\mu/Z_5$ under the non-trivial background, $\langle A_5\rangle = a/(gR)$, leading to

$$\begin{aligned}\left[\frac{Z_\mu}{Z_5}\right]_{\rm finite} &= 1-4g_R^2\int\frac{d^4k'_E}{(2\pi)^4}\frac{\pi^3\sinh(2\pi k'_E)(\cos^2(2\pi a)+\cos(2\pi a)\cosh(2\pi k'_E)-2)}{3k'_E(\cosh(2\pi k'_E)-\cos(2\pi a))^3}\\ &\quad+\mathcal{O}(g_R^4)\\ &= 1-\frac{g_R^2}{12}\ln(2\pi a)+\mathcal{O}(g_R^4), \end{aligned} \tag{3.38}$$

in the limit $a\to 0$.

## 3.2 Two-loop

In this subsection, we calculate two-loop corrections to the mass of the zero mode of $A_5$. In 5D massless QED, all the divergences at one-loop level are expected to cancel out. In fact, we have seen explicitly in the previous subsection that the divergences from the wave function renormalization and the vertex correction are exactly canceled as expected from Ward-Takahashi identity. Hence, we calculate two-loop diagrams without any counter terms. Straightforward calculation of Fig. 6 is given by

$$
\begin{aligned}
&\frac{g_4^4}{R^4}\int\frac{\mathrm{d}^4l\mathrm{d}^4k}{-(2\pi)^8}\sum_{n,m}(-1)\\
&\quad\left[\mathrm{tr}\left[(-i)\gamma_5\frac{\not{l}+in\gamma_5}{l^2-n^2}\gamma_\mu\frac{\not{k}+im\gamma_5}{k^2-m^2}(-i)\gamma_5\frac{\not{k}+im\gamma_5}{k^2-m^2}\gamma_\nu\frac{\not{l}+in\gamma_5}{l^2-n^2}\right]\frac{\eta^{\mu\nu}}{(l-k)^2-(n-m)^2}\right.\\
&\quad+\mathrm{tr}\left[\gamma_5\frac{\not{l}+in\gamma_5}{l^2-n^2}\gamma_5\frac{\not{k}+im\gamma_5}{k^2-m^2}\gamma_5\frac{\not{k}+im\gamma_5}{k^2-m^2}\gamma_5\frac{\not{l}+in\gamma_5}{l^2-n^2}\right]\frac{-1}{(l-k)^2-(n-m)^2}\\
&\quad+2\,\mathrm{tr}\left[(-i)\gamma_5\frac{\not{l}+in\gamma_5}{l^2-n^2}\gamma_\mu\frac{\not{k}+im\gamma_5}{k^2-m^2}\gamma_\nu\frac{\not{l}+in\gamma_5}{l^2-n^2}(-i)\gamma_5\frac{\not{l}+in\gamma_5}{l^2-n^2}\right]\frac{\eta^{\mu\nu}}{(l-k)^2-(n-m)^2}\\
&\quad\left.+2\,\mathrm{tr}\left[\gamma_5\frac{\not{l}+in\gamma_5}{l^2-n^2}\gamma_5\frac{\not{k}+im\gamma_5}{k^2-m^2}\gamma_5\frac{\not{l}+in\gamma_5}{l^2-n^2}\gamma_5\frac{\not{l}+in\gamma_5}{l^2-n^2}\right]\frac{-1}{(l-k)^2-(n-m)^2}\right] \qquad (3.39)\\
=\;&-12\frac{g_4^4}{R^2}\int\frac{\mathrm{d}^4l_E\mathrm{d}^4k_E}{(2\pi)^8}\sum_{n,m}\\
&\qquad\left[\frac{((l_E^2-n^2)(-k_E^2+m^2)-4nml_Ek_E)}{(l_E^2+n^2)^2(k_E^2+m^2)^2\left((l_E-k_E)^2+(n-m)^2\right)}\right.\\
&\qquad\left.+2\frac{((l_E^2-n^2)(k_El_E+nm)-2n^2k_El_E+2nml_E^2)}{(l_E^2+n^2)^3(k_E^2+m^2)\left((l_E-k_E)^2+(n-m)^2\right)}\right], \qquad (3.40)
\end{aligned}
$$

where we note that the contributions from the last two diagrams in Fig. 6 are the same as those from the third and the fourth diagrams. In the last equation, we carry out the Wick rotation. Now we perform the summations with respect to $n$ and $m$. For this purpose, we replace the summations by the integrations on the real axis and the summations of residues, as was done in the one-loop calculation. In other words, we decompose the summations of the function $f(m,n)$ to the following four parts:

$$
\begin{aligned}
\mathrm{I}\;&:\;\int\mathrm{d}z_n\mathrm{d}z_mf(z_m,z_n),\\
\mathrm{II}\;&:\;\int\mathrm{d}z_n2\mathrm{Re}\left[\sum_i Res.\left\{\frac{2\pi if(z_m,z_n)}{\exp(-2\pi iz_m)-1}\,;\,z_m=m_+^i\right\}\right],\\
\mathrm{III}\;&:\;2\mathrm{Re}\left[\sum_i Res.\left\{\frac{2\pi i}{\exp(-2\pi iz_n)-1}\int\mathrm{d}z_mf(z_m,z_n)\,;\,z_n=n_+^i\right\}\right],\\
\mathrm{IV}\;&:\;2\mathrm{Re}\left[\sum_i Res.\left\{\frac{2\pi i}{\exp(-2\pi iz_n)-1}2\mathrm{Re}\left[\sum_i Res.\left\{\frac{2\pi if(z_m,z_n)}{\exp(-2\pi iz_m)-1}\,;\,z_m=m_+^i\right\}\right]\right.\right.\\
&\qquad\left.\left.;\,z_n=n_+^i\right\}\right].
\end{aligned}
$$

Note that we always carry out the operation of $m$ before doing that of $n$. Let us discuss each part in order.

### 3.2.1 Part I

First, we evaluate the contribution from the first part, namely the summations are replaced by integrations on the real axis. This contribution is expected to correspond to the diagrams where both loops do not wind around $S^1$ and thus can be shrinked to a point. In general, such diagrams give the strongest divergences. However, in our case, the five dimensional gauge invariance will forbid such a contribution.

Before evaluating the contribution, we define a new vector from $l_E$ and $k_E$ as $\lambda_E \equiv k_E - l_E$, and we use the same parameters without the index $E$ to denote the absolute values of the vectors. Among these three vectors, we can choose any two vectors as the integral variables. Then the integrand of (3.40) is written as

$$\begin{aligned} I(m,n) \;\equiv\; & \frac{(l^2-n^2)(m^2-k^2)-2nm\,(l^2+k^2-\lambda^2)}{(l^2+n^2)^2(k^2+m^2)^2\,(\lambda^2+(n-m)^2)} \\ & +\frac{(l^2-3n^2)(l^2+k^2-\lambda^2)+2(3l^2-n^2)nm}{(l^2+n^2)^3(k^2+m^2)\,(\lambda^2+(n-m)^2)}. \end{aligned} \tag{3.41}$$

We can integrate over $z_m$ of (3.41) by adding the integration on the large half-circle in the upper half plane and evaluating the residues at the poles on the plane.

$$\begin{aligned} I(n) \;=\; & \int_{-\infty}^{\infty} \mathrm{d}z_m I(z_m,n) = Res.\left\{2\pi i I(z_m,n); z_m = ik, n+i\lambda\right\} \\ =\; & -\frac{\left(k(k+\lambda)^2 l^2 + \left((k-2\lambda)(k+\lambda)^2+(k+2\lambda)l^2\right)n^2 + kn^4\right)\pi}{(l^2+n^2)^2 k\lambda\left((k+\lambda)^2+n^2\right)^2} \\ & +\frac{\left((k+\lambda)l^2(k^2+l^2-\lambda^2) - 3(k-\lambda)\left((k+\lambda)^2-l^2\right)n^2 - 2kn^4\right)\pi}{(l^2+n^2)^3 k\lambda((k+\lambda)^2+n^2)} \end{aligned} \tag{3.42}$$

In a similar way, we can further perform the integration over $z_n$ of the above expression to find

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathrm{d}z_n I(z_n) = Res.\left\{2\pi i I(z_n); z_n = il, i(k+\lambda)\right\} \\ =\; & -\frac{(k+l-\lambda)\,\pi^2}{kl\lambda\,(k+l+\lambda)^2} + \frac{(k+l-\lambda)\,\pi^2}{kl\lambda\,(k+l+\lambda)^2} \quad = 0. \end{aligned} \tag{3.43}$$

As expected from the five dimensional gauge invariance, the contribution from this part vanishes although each term potentially gives divergent correction.

### 3.2.2 Part II

Next, we evaluate the contribution from the second part. This contribution is expected to correspond to the diagrams where one of the loops winds around $S^1$ while the other does not. Some examples are shown in Fig. 7. Because the latter loop can be shrinked to a point, generally this part gives a divergent contribution, even in the

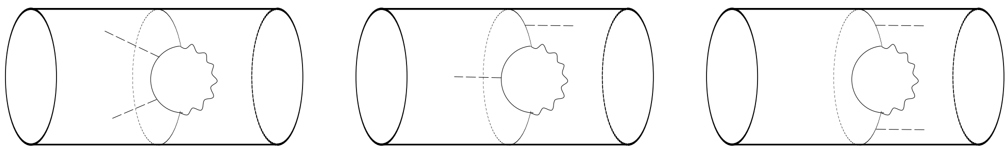


Figure 7: The diagrams where the fermion loop winds around $S^1$ but the photon loop does not. The cylinder denotes $S^1$. If the fermion-photon loop is shrinked to a point, these diagrams provide corrections of the 4-point vertex of the fermion-fermion-$A_5$-$A_5$ (a), the gauge interaction vertex (b) and the wave function renormalization (c), respectively.

gauge-Higgs unification scenario. However, as is well known, such divergences can be cancelled by the one-loop counter terms. In other words, after we remove all the divergences in the one-loop diagrams, the contribution from this part will be finite. Then, we obtain the finite mass at two-loop level without any additional counter terms. In [7], the Higgs mass at two-loop level are calculated in 5D supersymmetric theory, where supersymmetry is broken by Scherk-Schwarz mechanism. In fact, the linear divergences appear and are cancelled by the one-loop counter terms. In our particular case, Ward-Takahashi identity should make the divergence in this contributions same as the minus of the divergence in the contribution from the part Ⅲ. Thus, all the divergences are expected to cancel out with each other *without* any counter terms.

Now, let us evaluate the residues of the poles of $z_m$ in the upper half plane, *i.e.* $z_m = ik$ and $z_m = n + i\lambda$. We can interpret the former contribution as the one comes from the diagram where the fermion line with the momentum $k$ winds around $S^1$ (Fig. 7), while the latter as the one from the diagram where the photon line winds. The residue of the first term in (3.41) on the pole $z_m = ik$ is evaluated as

$$\begin{aligned} J_1(n) &\equiv 2\mathrm{Re}\left[Res.\left\{(\text{1st term of } I(z_m,n))\,\frac{2\pi i}{\exp(-2\pi i z_m)-1}; z_m = ik\right\}\right] \\ &= 4\pi\left[e_k\left(k^2l^2(k^2-\lambda^2)^2+(2k^6+k^4(l^2-3\lambda^2)-4k^2l^2\lambda^2-(l^2-\lambda^2)\lambda^4)n^2\right.\right. \\ &\qquad\left. +(4k^4-k^2(l^2+2\lambda^2)-2(l^2-\lambda^2)\lambda^2)n^4+(2k^2-l^2+\lambda^2)n^6\right) \\ &\qquad\left.+(1+e_k)k(k^2-\lambda^2+n^2)((k+\lambda)^2+n^2)((k-\lambda)^2+n^2)(l^2+n^2)\pi\right] \\ &\quad /\left[e_k^2k((k+\lambda)^2+n^2)^2((k-\lambda)^2+n^2)^2(l^2+n^2)^2\right] \end{aligned} \tag{3.44}$$

and that of the second term is given by

$$\begin{aligned}
J_2(n) &\equiv 2\mathrm{Re}\left[Res.\left\{(\text{2nd term of } I(z_m,n))\,\frac{2\pi i}{\exp(-2\pi i z_m)-1}; z_m = ik\right\}\right] \\
&= 2\pi\left[-(k^2-\lambda^2)^2(k^2+l^2-\lambda^2)^2 l^2\right. \\
&\quad \left.+(3k^4-2k^2(4l^2+3\lambda^2)+(l^2-\lambda^2)(l^2-3\lambda^2))n^2+(k^2-3l^2+3\lambda^2)n^4\right] \\
&\quad /\left[e_k k((k+\lambda)^2+n^2)((k-\lambda)^2+n^2)(l^2+n^2)^3\right], \qquad (3.45)
\end{aligned}$$

where $e_k \equiv \exp(2\pi k) - 1$. After the integration over $z_n$, we find these respectively become

$$\frac{2\pi^2\,(2(1+e_k)(k+l)((k+l)^2-\lambda^2)\pi + e_k(3(k+l)^2+\lambda^2))}{e_k^2 k l((k+l)^2-\lambda^2)^2}, \qquad (3.46)$$

$$-\frac{2\pi^2\,((3(k+l)^2+\lambda^2))}{e_k k l((k+l)^2-\lambda^2)^2} \qquad (3.47)$$

for $k > \lambda$ and

$$\frac{2\pi^2\,(2(1+e_k)k\lambda(k^2-(l+\lambda)^2)\pi + e_k(k^2(l+3\lambda)-(l-\lambda)(l+\lambda)^2))}{e_k^2 k l\lambda(k^2-(l+\lambda)^2)^2}, \qquad (3.48)$$

$$-\frac{2\pi^2\,(k^2(l+3\lambda)-(l-\lambda)(l+\lambda)^2)}{e_k k l\lambda((\lambda+l)^2-k^2)^2} \qquad (3.49)$$

for $k < \lambda$. Note that all terms vanish in the limit $e_k \to \infty$. This means that we do not have UV divergences in $k$ integration. On the other hand, we may encounter divergences in $l(\lambda)$ integration. In fact, we can see that the integration of (3.48) over $l_E$ is linearly divergent, while the one of (3.49) converges. These are consistent with the interpretation that these contributions correspond to the diagram where the fermion line with the momentum $k$ winds on $S^1$: (3.48) corresponds to the vertex correction (Fig. 7 (b)) while (3.49) corresponds to the correction of the four point vertex fermion-fermion-$A_5$-$A_5$ (Fig. 7 (a)).

In a similar way, contributions from the pole at $z_m = n + i\lambda$ is evaluated as

$$\begin{aligned}
J_3(n) &\equiv 2\mathrm{Re}\left[Res.\left\{(\text{1st term of } I(z_m,n))\,\frac{2\pi i}{\exp(-2\pi i z_m)-1}; z_m = n+i\lambda\right\}\right] \\
&= -2\pi\left[(k^2-\lambda^2)^2(k^2+\lambda^2)l^2+((k^2-\lambda^2)^2(k^2+5\lambda^2)\right. \\
&\quad +(3k^4-6k^2\lambda^2-5\lambda^4)l^2)n^2+(3k^4+k^2(3l^2+2\lambda^2)-5l^2\lambda^2+3\lambda^4)n^4 \\
&\quad \left.+(3k^2+l^2-\lambda^2)n^6+n^8\right]/\left[e_\lambda\lambda((k+\lambda)^2+n^2)^2((k-\lambda)^2+n^2)^2(l^2+n^2)^2\right] \\
&\qquad (3.50)
\end{aligned}$$

and that of the second term is given as

$$
\begin{aligned}
J_4(n) &\equiv 2\mathrm{Re}\left[Res.\left\{(\text{2nd term of } I(z_m,n))\frac{2\pi i}{\exp(-2\pi i z_m)-1}; z_m = n+i\lambda\right\}\right] \\
&= 2\pi\left[(k^2-\lambda^2)(k^2+l^2-\lambda^2)l^2+(-3(k^2-\lambda^2)^2+4(k^2+2\lambda^2)l^2+l^4)n^2\right. \\
&\quad \left.+(-5k^2+3l^2+\lambda^2)n^4-2n^6\right] / \left[e_\lambda\lambda((k+\lambda)^2+n^2)((k-\lambda)^2+n^2)(l^2+n^2)^3\right],
\end{aligned}
\tag{3.51}
$$

where $e_\lambda \equiv \exp(2\pi\lambda)-1$. After the integration over $z_n$, we find these terms give the same contributions with an opposite sign and the sum of these vanishes. This is also consistent with the interpretation that these contributions correspond to the diagram where the photon line winds around $S^1$. Namely, such contributions correspond to the correction of the four point vertex $A_M$-$A_M$-$A_5$-$A_5$. This is the correction to the $F_{MN}^4$ term which has vanishing contribution to the mass correction because its Feynman rule contains momenta of the four lines and our interest is zero external momenta case.

In summary, the contribution from this part is written as

$$
-\frac{4\pi^3(1+e_k)}{e_k^2 l((l+\lambda)^2-k^2)} - \frac{4\pi^3(1+e_k)}{e_k^2 k((k-\lambda)^2-l^2)}\theta(k-\lambda), \tag{3.52}
$$

where $\theta(x)$ is 0 for $x<0$ and 1 for $x>0$.

The first term is the linearly divergent term for $l$ momentum, which originated from the vertex correction. This divergence should be canceled by the term originated from the wave function renormalization. We will see in the next subsubsection that this is indeed the case. On the other hand, the second term is finite since this contribution exists only when the momentum $\lambda$ is smaller than the momentum $k$.

### 3.2.3 Part III

Now, we evaluate the contribution from the third part. The integration over $z_m$ is given in (3.42). It shows that there are two poles in the upper half plane: $z_n = il$ and $z_n = i(k+\lambda)$. The contribution from the pole at $z_n = il$ is calculated as

$$
\begin{aligned}
& 2\mathrm{Re}\left[Res.\left\{I(z_n)\frac{2\pi i}{\exp(-2\pi i z_n)-1}; z_n = il\right\}\right] \\
=\; & -\frac{4\pi^3(1+e_l)}{e_l^2 k((k+\lambda)^2-l^2)} - \frac{2\pi^2\left((k-\lambda)(k+\lambda)^2-(k+3\lambda)l^2\right)}{e_l k l\lambda((k+\lambda)^2-l^2)^2} \qquad (3.53) \\
& +\frac{8\pi^3(1+e_l)}{e_l^2 k((k+\lambda)^2-l^2)} + \frac{2\pi^2\left((k-\lambda)(k+\lambda)^2-(k+3\lambda)l^2\right)}{e_l k l\lambda((k+\lambda)^2-l^2)^2} \\
& +\frac{4\pi^4(1+e_l)(2+e_l)(k-\lambda)}{e_l^3 k l\lambda} \qquad (3.54)
\end{aligned}
$$

where $e_l \equiv \exp(2\pi l) - 1$. These terms vanish in the limit $e_l \to \infty$, and thus the $l$ integration is free from UV divergences. Note that if we choose $-l_E$ and $\lambda_E$ as the integral variables and rename them as $L_E$ and $K_E$, $k_E$ is written as $k_E = K_E - L_E \equiv \Lambda_E$. Then the three new momenta $(K_E, L_E, \Lambda_E)$ satisfy the same relation as $(k_E, l_E, \lambda_E)$. In addition, the integration measure under this rename is invariant. Thus, this means we can replace $k$ and $\lambda$ with each other. From this observation, it is clear that the 4D momentum integral of the last term in (3.54) does not contribute.

The first terms of (3.53) and (3.54) are linearly divergent with respect to $k$ integration. It is interesting to find that the divergence in (3.53) is the half of the one in (3.54) with the opposite sign, and is the same as the one in the part II. These results are again consistent with the interpretation that these contributions correspond to the diagram where the fermion line with the momentum $l$ winds around $S^1$: (3.53) corresponds to the vertex correction (Fig. 7 (b)) and (3.54) corresponds to the wave function correction of the fermion (Fig. 7 (c)). The second terms in (3.53) and (3.54) are canceled.

The contribution from the pole at $z_n = i(k+\lambda)$ is summarized as

$$2\mathrm{Re}\left[Res.\left\{I(z_n)\frac{2\pi i}{\exp(-2\pi i z_n) - 1}; z_n = i(k+\lambda)\right\}\right] = -\frac{4\pi^3(1+e_{k+\lambda})}{e_{k+\lambda}^2 k((k+\lambda)^2 - l^2))} \tag{3.55}$$

where $e_{k+\lambda} \equiv \exp(2\pi(k+\lambda)) - 1$. This term vanishes when $e_{k+\lambda} \to \infty$, and thus this contribution is finite under both $k$ and $l$ integrations. This contribution is interpreted as coming from the diagram where the fermion line with the momentum $k$ and the photon line wind around $S^1$.

In summary, the contribution from this part is written as

$$\frac{4\pi^3(1+e_l)}{e_l^2 k((k+\lambda)^2 - l^2)} - \frac{4\pi^3(1+e_{k+\lambda})}{e_{k+\lambda}^2 k((k+\lambda)^2 - l^2))}. \tag{3.56}$$

The first term is linearly divergent, which originated from two wave function renormalizations and a vertex correction, namely a wave function renormalization. One can see that this contribution and the first term in (3.52) are exactly canceled as expected from Ward-Takahashi identity.

### 3.2.4 Part IV

Finally we evaluate the contribution from the fourth part. The contribution of the part of the residues in $m$ is written in (3.44), (3.45), (3.50) and (3.51). They have poles on the upper half plane at $z_n = il, i(k+\lambda), i|k-\lambda|$. The first and the last two parts give the following contributions;

$$
\begin{aligned}
& 2\mathrm{Re}\left[Res.\left\{(J_1(z_n)+J_2(z_n))\frac{2\pi i}{\exp(-2\pi i z_n)-1}; z_n=il\right\}\right] \\
= \; & \frac{8\pi^3(1+e_k)(k^2-l^2-\lambda^2)}{e_k^2 e_l l(k+l+\lambda)(k+l-\lambda)(k-l+\lambda)(k-l-\lambda)} \\
& -[k\leftrightarrow l \text{ for the first term}]-\frac{8\pi^4(1+e_l)(2+e_l)}{e_k e_l^3 kl}, \qquad (3.57)\\
& 2\mathrm{Re}\left[Res.\left\{(J_3(z_n)+J_4(z_n))\frac{2\pi i}{\exp(-2\pi i z_n)-1}; z_n=il\right\}\right] \\
= \; & -\frac{16\pi^3(1+e_l)\lambda}{e_\lambda e_l^2(k+l+\lambda)(k+l-\lambda)(k-l+\lambda)(k-l-\lambda)}+\frac{8\pi^4(1+e_l)(2+e_l)}{e_\lambda e_l^3 l\lambda} \qquad (3.58)
\end{aligned}
$$

for the pole $z_n = il$,

$$
\begin{aligned}
& 2\mathrm{Re}\left[Res.\left\{(J_1(z_n)+J_2(z_n))\frac{2\pi i}{\exp(-2\pi i z_n)-1}; z_n=i(k+\lambda)\right\}\right] \\
= \; & -\frac{4\pi^3(e_k+e_{k+\lambda}+2e_k e_{k+\lambda})}{e_k^2 e_{k+\lambda}^2 k((k+\lambda)^2-l^2)}, \qquad (3.59)\\
& 2\mathrm{Re}\left[Res.\left\{(J_3(z_n)+J_4(z_n))\frac{2\pi i}{\exp(-2\pi i z_n)-1}; z_n=i(k+\lambda)\right\}\right] \\
= \; & -\frac{4\pi^3(1+e_{k+\lambda})}{e_\lambda e_{k+\lambda}^2 k((k+\lambda)^2-l^2)} \qquad (3.60)
\end{aligned}
$$

for the pole $z_n = i(k+\lambda)$, and

$$
\begin{aligned}
& 2\mathrm{Re}\left[Res.\left\{(J_1(z_n)+J_2(z_n))\frac{2\pi i}{\exp(-2\pi i z_n)-1}; z_n=i\,|k-\lambda|\right\}\right] \\
= \; & -\frac{4\pi^3(1+e_k)(2e_k+e_k^2-e_\lambda)}{e_k^2(e_k-e_\lambda)^2 k((k-\lambda)^2-l^2)}+\frac{4\pi^3(1+e_k)}{e_k^2 k((k-\lambda)^2-l^2)}\theta(k-\lambda), \qquad (3.61)\\
& 2\mathrm{Re}\left[Res.\left\{(J_3(z_n)+J_4(z_n))\frac{2\pi i}{\exp(-2\pi i z_n)-1}; z_n=i\,|k-\lambda|\right\}\right] \\
= \; & \frac{4\pi^3(1+e_k)(1+e_\lambda)}{e_\lambda(e_k-e_\lambda)^2 k((k-\lambda)^2-l^2)} \qquad (3.62)
\end{aligned}
$$

for the pole $z_n = i\,|k-\lambda|$.

In summary, the contribution of this part is written as

$$-\frac{16\pi^3(1+e_l)\lambda}{e_\lambda e_l^2(k+l+\lambda)(k+l-\lambda)(k-l+\lambda)(k-l-\lambda)}, \tag{3.63}$$

$$-\frac{4\pi^3(e_k+e_{k+\lambda}+2e_ke_{k+\lambda})}{e_k^2e_{k+\lambda}^2k((k+\lambda)^2-l^2)}-\frac{4\pi^3(1+e_{k+\lambda})}{e_\lambda e_{k+\lambda}^2k((k+\lambda)^2-l^2)}, \tag{3.64}$$

$$\frac{4\pi^3(1+e_k)}{e_k^2e_\lambda k((k-\lambda)^2-l^2)}+\frac{4\pi^3(1+e_k)}{e_k^2k((k-\lambda)^2-l^2)}\theta(k-\lambda). \tag{3.65}$$

Note that all terms above are finite because this part corresponds to the diagram where the fermion and the photon wind around $S^1$.

### 3.2.5 Summation

Now we sum up all the terms of (3.52), (3.56) and (3.63)-(3.65). Note that we can freely exchange $k$ and $l$ with each other keeping $\lambda$ unchanged, which is nothing but the rename of the integral variables $(k_E, l_E) \to (l_E, k_E)$. By using this freedom, we find that the summation becomes zero. This shows the finite part corrections vanish, apart from those due to the wave function renormalization of $A_5$. This cancellation seems to be accidental in our simple model because there is no clear physical reason to ensure such a cancellation. If we consider higher order loop corrections beyond two-loops even in 5D massless QED or calculate quantum corrections in more general models, the finite correction would be remained to be nonzero. This point would be clarified if we extend our analysis to the non-Abelian case, for example [10].

# 4 Summary

Even in gauge-Higgs unification, the Higgs mass diverges beyond one-loop level in general. The divergence arises from the subdiagrams and should be subtracted by adding lower loop counter terms. Then, we can obtain the finite Higgs mass at any order of perturbations without introducing any other counter terms.

In this paper, we have calculated quantum corrections to the mass of the zero mode of the gauge field at two-loop order in a five dimensional massless QED compactified on $S^1$. We have found that no counter terms are needed in this simple model and have discussed in detail how the possible divergences are canceled. The key ingredients to obtain such a cancellation are the fifth component of the 5D gauge symmetry (shift symmetry), and the fact that the (linear) divergences from the fermion wave function renormalization and the vertex correction are the same magnitude with an opposite sign. The latter feature is expected from Ward-Takahashi identity.

We also evaluated the finite part of corrections. We classified such corrections to two type: those come from the wavefunction renormalization of $A_5$ and those come from 1PI two-loop diagrams. The former keeps the structure of the one-loop effective potential essentially unchanged and is obtained from the product of the ratio of the wavefunction renormalization factors $Z_\mu/Z_5$ and one-loop finite Higgs mass. Although these wave function renormalization factors are linearly divergent, 5D Lorentz invariance ensures that these have same contributions. Therefore, the UV divergences are exactly canceled in $Z_\mu/Z_5$ while IR divergences appear. This is because we consider exactly massless charged fermion, and we introduce a small VEV of $A_5$ as an IR cutoff. As for the latter, we found that they cancel out among themselves in our calculation. This result seems to be accidental in our simple model because there is no clear physical reason to obtain such a result. If we consider higher order loop corrections beyond two-loops even in 5D massless QED or calculate quantum corrections in more general models, the finite correction would be remained to be nonzero.

We should note that the finite value itself may not be taken seriously because our regularization used in this paper does not have *4D gauge invariance*. Namely, the photon has a non-vanishing mass at one-loop level. However, we would like to emphasize that only the 5D Lorentz symmetry ($Z_\mu/Z_5$), the shift symmetry (Part I) and the relation expected from Ward-Takahashi identity (Part II and III) are important to cancel all possible divergences. In fact, the 4D gauge invariance is not so important for the finiteness of the mass of $A_5$ since the shift symmetry forbids the mass of $A_5$. Our regularization indeed preserves the shift symmetry by doing the summation of KK modes and the relation expected from Ward-Takahashi identity. We can conclude from these observations that the finiteness for the mass of $A_5$ is correct even in our regularization scheme. Off course, it is desirable to calculate the mass in a full 5D gauge invariant way to obtain a reliable finite mass. This subject is left for a future work.

Our discussion of obtaining the finite Higgs mass at any order of perturbations would be generic in any Gauge-Higgs unification models. Therefore, it would be very interesting to extend our analysis to non-Abelian case not only from the theoretical but also from the phenomenological viewpoints. This subject will be reported elsewhere [10].

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