Normal bundles to Laufer rational curves in local Calabi-Yau threefolds

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Abstract

We prove a conjecture by F. Ferrari. Let X be the total space of a nonlinear deformation of a rank 2 holomorphic vector bundle on a smooth rational curve, such that X has trivial canonical bundle and has sections. Then the normal bundle to such sections is computed in terms of the rank of the Hessian of a suitably defined superpotential at its critical points.

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Introduction. In this paper we consider particular embeddings of smooth rational curves in local Calabi-Yau threefolds, called Laufer curves [6, 5]. (We use the definition of the physics community, calling Calabi-Yau a quasi-projective threefold with trivial canonical bundle; the term "local" refers to non-compactness.) These geometries have shown to be very useful to understand several features of string theories and supersymmetric gauge theories.

In particular they are relevant for brane dynamics and geometric transition/large N dualities. Geometric transition interprets the resummation of the open string sector of an open-closed string theory as a transition in the target space geometry, connecting two different components of a moduli space of Calabi-Yau threefolds. The local Calabi-Yau that we consider represents the open string side of conjectured geometric transitions. In particular, open topological B-type strings in these geometry reduces to matrix models in which the parameters of the complex structure are the coupling constants.

This is directly connected (via F-terms) to the possibility of geometrically engineering supersymmetric gauge theories in Type IIB string theory. Let $\mathbb{R}^4 \times X$ be the target space of the theory, where X is a Calabi-Yau threefold, and $\mathcal{C}$ a rational curve in X, with normal bundle V and N D5 branes wrapped on it. The effective field theory is a $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $U(N)$. The space of vacua of this gauge theory, given by the critical points of the effective superpotential, is locally described by the versal deformation space of the curve in X. For a given vacua, there are h^0 massless chiral superfields in the adjoint representation of the gauge group, where $h^0 := \dim H^0(\mathcal{C}, V)$. On the other hand, the number of massless chiral multiplets is equal to the corank of the Hessian of the superpotential at this vacuum. This relation led [3] to conjecture the result expressed in our Proposition 2.

For an account of these aspects, see also [2, 7, 1] and references therein.

From a strictly mathematical viewpoint, the problem is the following. Let V be a rank-2 holomorphic vector bundle on a rational curve $\mathcal{C}$ such that its total space has trivial canonical bundle, and assume that V has a global section. We deform V to a nonlinear fibration X in such a way that X still has trivial canonical bundle and the fibration has sections. The normal bundle to such a section of course splits as a direct sum of two line bundles in view of Grothendieck's classification of vector bundles on curves of genus zero [4]. The problem is to compute these line bundles. The solution is obtained in terms of a superpotential W than one associates with the deformations of V: the sections of X are given by the critical points of W, and the degrees of the above mentioned line bundles are given, in accordance with a conjecture by Ferrari [3], by the rank of the Hessian of W at those critical points.

Definition of X. Let $\mathcal{C} \simeq \mathbb{P}^1$ be a smooth rational curve and $V \to \mathcal{C}$ a rank-2 holomorphic vector bundle on $\mathcal{C}$, with $\det V \simeq K_{\mathcal{C}} \simeq \mathcal{O}(-2)$, so that the total space of the bundle V has trivial canonical bundle. Then $V \simeq \mathcal{O}(-n-2) \oplus \mathcal{O}(n)$ for some n. We consider deformations of V given in terms of transition functions in the standard atlas $\mathcal{U} = \{U_0, U_1\}$ of $\mathbb{P}^1$ as

$$\begin{cases} z' &= 1/z \\ \omega_1' &= z^{-n}\omega_1 \\ \omega_2' &= z^{n+2}\left(\omega_2 + \partial_{\omega_1} B(z,\omega_1)\right) \ . \end{cases} \tag{1}$$

Note that the complex manifold X defined as the total space of this fibration has again trivial canonical bundle. The term $B(z,\omega)$ is a holomorphic function on $(U_0 \cap U_1) \times \mathbb{C}$ and is called the *geometric potential.* If we expand the function B in its second variable

$$B(z,\omega_1) = \sum_{d=1}^{\infty} \sigma_d(z)\, \omega_1^d \tag{2}$$

each coefficient σ_d may be regarded as a cocycle defining an element in the group

$$H^1(\mathbb{P}^1, \mathcal{O}(-2-dn)) \simeq H^0(\mathbb{P}^1, \mathcal{O}(nd))^* \,. \tag{3}$$

The superpotential. If we consider $\mathcal{C}$ as embedded in X as its zero section, and consider the problem of deforming the pair $(X\,\mathcal{C})$, the space of versal deformations can be conveniently described by a superpotential [5]. In the case at hand the superpotential W can be defined as the function of $n+1$ complex variables given by

$$W(x_0, \ldots, x_n) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{C}_0} B\left(z, \omega_1(z)\right) \mathrm{d}z \tag{4}$$

where z and z' are local coordinates on U_0 and U_1, and the parameters $x_0, \ldots, x_n$ define sections of the line bundle $\mathcal{O}(n)$ by letting

$$\omega_1(z) = \sum_{i=0}^{n} x_i z^i \ , \qquad \omega_1'(z') = \sum_{i=0}^{n} x_i (z')^{n-i} \,. \tag{5}$$

One should note that the superpotential W can be obtained by applying to the function B, regarded as an element in $H^0(\mathbb{P}^1, \mathcal{O}(nd))^*$, the dual of the multiplication morphism

$$H^0(\mathbb{P}^1, \mathcal{O}(n))^{\otimes d} \to H^0(\mathbb{P}^1, \mathcal{O}(nd)) \tag{6}$$

(here one should regard the dual of $H^0(\mathbb{P}^1, \mathcal{O}(nd))$ as a space of Laurent tails).

The key to the result we want to prove is the relationship occuring between the superpotential W and the sections of the fibration $X \to \mathcal{C}$ (cf. [5, 3]).

Lemma 1. *The holomorphic sections of the fibration $X \to \mathcal{C}$ are in a one-to-one correspondence with the critical points of the superpotential,* i.e., *with the solutions of the equations*

$$\frac{\partial W}{\partial x_i} = 0 \ , \quad i = 0, \ldots, n \ . \tag{7}$$

Proof. This can be verified by explicit calculations [3] after representing the sections of X as

$$\begin{aligned} \omega_2(z) &= -\frac{1}{2\mathrm{i}\pi} \oint_{C_z} \frac{\partial_\omega B(u, \omega_1(u))}{u - z} \, \mathrm{d}u \\ \omega_2'(z') &= \frac{1}{2\mathrm{i}\pi} \oint_{C_{z'}} \frac{\partial_\omega B(1/u, \omega_1(1/u))}{u^{n+2}(u - z)} \, \mathrm{d}u \end{aligned} \tag{8}$$

where the contour C_z (resp. $C_{z'}$) encircles the points 0 and z (resp z'). So (5) and (8) yield a rational curve $\Sigma \subset X$ for each critical point $(x_0, \ldots, x_n)$ of W. □

Ferrari's Conjecture. Now we state and prove Ferrari's conjecture.

Proposition 2. *The normal bundle to the section Σ of X determined by a critical point $(x_0, \ldots, x_n)$ of W is $\mathcal{O}_\Sigma(-r - 1) \oplus \mathcal{O}_\Sigma(r - 1)$ where r is the corank of the Hessian of W at that point.*

To calculate the normal bundle to Σ we first need to linearize the transition functions around the given section. Defining new coordinates $\delta_i =$

$\omega_i - \omega_i(z)$, $\delta'_i = \omega'_i - \omega'_i(z)$, we obtain

$$\delta'_2 = z^{n+2} \left(\delta_2 + h(z)\delta_1 + g(z) \right) \tag{9}$$

where

$$g(z) = \partial_\omega B(z, \omega_1(z)) \ , \qquad h(z) = \partial^2_\omega B(z, \omega_1(z)) \tag{10}$$

and at a critical point of W we have $g(z) = 0$ using relation (22) in the appendix. Furthermore, again from (22), for $h(z)$ we have

$$h(z) = \sum_{i \leq j = 0}^{n} \partial_i \partial_j W_d^{(k)} z^{-(i+j)-1} \tag{11}$$

up to terms that can be can be readsorbed by holomorphic change of coordinates (see the Appendix).

Now we need the following. Let us consider an extension of vector bundles on $\mathbb{P}^1$ of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-n-2) \longrightarrow \Phi \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n) \longrightarrow 0 \tag{12}$$

parametrized by a cocycle $\sigma \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2n-2))$. With respect to the two standard charts U_0, U_1 and in the coordinate z of U_0, σ can be written as

$$\sigma(z) = \sum_{k=0}^{2n} \widetilde{t}_k z^{-k-1} \ . \tag{13}$$

Let us define a quadratic form (quadratic superpotential) on the global sections of the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$:

$$H(x_0, \ldots, x_n) = \sum_{k=0}^{2n} \widetilde{t}_k \sum_{\substack{i,j=0 \\ i+j=k}}^{n} x_i x_j = \sum_{i,j=0}^{n} H_{ij} x_i x_j \ . \tag{14}$$

Lemma 3. *The vector bundle Φ is $\mathcal{O}_{\mathbb{P}^1}(r-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-r-1)$, where r is the corank of the quadratic form H.*

Proof. By Lemma 1 the sections of the bundle Φ correspond to the critical points of H, *i.e.,* to the solutions of the linear system

$$\sum_{j=0}^{n} H_{ij} x_j = 0 \ . \tag{15}$$

The dimension of this space is r, the corank of H. The only rank two vector bundle over $\mathbb{P}^1$ with determinant $\mathcal{O}_{\mathbb{P}^1}(-2)$ and r linearly indipendent holomorphic sections is $\mathcal{O}_{\mathbb{P}^1}(r-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-r-1)$. □

The proof of Proposition 2 is now complete: in fact, by (11) the quadratic form H corresponds to the Hessian of the superpotential W at its critical points.

A Some formulas for the potentials

We group here some formulas that turn out to be useful in checking the computations involved in the results presented in this paper.

The geometric potential. The geometric potential (deformation term) $B(z, \omega_1)$ is holomorphic on $\mathbb{C}^* \times \mathbb{C}$ and can be cast in the form

$$B(z,\omega) = \sum_{d=0}^{\infty} \sum_{k=0}^{dn} t_d^{(k)} B_d^{(k)}(z,\omega) \tag{16}$$

where

$$B_d^{(k)}(z,\omega) = z^{-k-1}\omega^d \qquad k = 0, \ldots, dn \ . \tag{17}$$

The terms with $k < 0$ or $k > dn$ can be readsorbed by a holomorphic change of coordinates. For $l := -k-1 \geq 0$, we define $\widetilde{\omega}_2 := \omega_2 + dz^l \omega_1^{d-1}$, and for $m := k - dn - 1 \geq 0$, we define

$$\widetilde{\omega}_2' := \omega_2' - (z')^m (\omega_1')^{d-1} \ . \tag{18}$$

The superpotential The superpotential that corresponds to $B_d^{(k)}$, given by (4), is

$$W_d^{(k)}(x_0, \dots, x_n) = \sum_{\substack{i_1,\dots,i_d=0 \\ i_1+\dots+i_d=k}}^{n} x_{i_1} \dots x_{i_d} \; . \tag{19}$$

We can obtain simple relations for the derivatives of these polynomials:

$$\begin{aligned} \frac{\partial W_d^{(k)}}{\partial x_j} &= \sum_{\substack{i_1,\dots,i_d=0 \\ i_1+\dots+i_d=k}}^{n} d \left(\frac{\partial x_{i_1}}{\partial x_j} x_{i_2} \dots x_{i_d} \right) \\ &= d \sum_{\substack{i_1,\dots,i_{d-1}=0 \\ i_1+\dots+i_{d-1}=k-j}}^{n} x_{i_1} \dots x_{i_{d-1}} \; = \; d W_{d-1}^{(k-j)} \end{aligned} \tag{20}$$

and in general we have

$$\frac{\partial}{\partial x_{j_1}} \dots \frac{\partial}{\partial x_{j_l}} W_d^{(k)} = d(d-1) \dots (d-l+1) W_{d-l}^{(k-j_1 \dots - j_l)} \tag{21}$$

Relations between the derivatives of the potentials Given a section $\omega_1(z)$, we have

$$\begin{aligned} \partial_\omega B(z, \omega_1(z)) &= \sum_{j=0}^{n} \frac{\partial W}{\partial x_j} z^{-j-1} + \text{trivial terms} \\ \partial_\omega^2 B(z, \omega_1(z)) &= \sum_{i \leq j=0}^{n} \partial_i \partial_j W z^{-(i+j)-1} + \text{trivial terms} \end{aligned} \tag{22}$$

where the “trivial terms” can be readsorbed by a holomorphic change of coordinates. We can obtain these results from (16) and (19). We have

$$\partial_\omega B_d^{(k)}(z, \omega_1(z)) = d \sum_{i_1,\dots,i_{d-1}=0}^{n} x_{i_1} \dots x_{i_{d-1}} z^{i_1+\dots+i_{d-1}-k-1} \tag{23}$$

and the only non-trivial terms are such that $0 \leq -(i_1 + \dots + i_{d-1} - k) \leq n$. In the same way, for the second derivatives we have

$$\partial_\omega^2 B_d^{(k)}(z, \omega_1(z)) = d(d-1) \sum_{i_1,\dots,i_{d-2}=0}^{n} x_{i_1} \dots x_{i_{d-2}} z^{i_1+\dots+i_{d-2}-k-1} \tag{24}$$

The relevant terms are those with $0 \leq -(i_1 + \dots + i_{d-1} - k) \leq 2n$.

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