Local Index Formula on the Equatorial Podleś Sphere

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Abstract

We discuss spectral properties of the equatorial Podleś sphere S^2_q. As a preparation we also study the 'degenerate' (i.e. $q = 0$) case (related to the quantum disk). Over S^2_q we consider two different spectral triples: one related to the Fock representation of the Toeplitz algebra and the isospectral one given in [7]. After the identification of the smooth pre-C^*-algebra we compute the dimension spectrum and residues. We check the nontriviality of the (noncommutative) Chern character of the associated Fredholm modules by computing the pairing with the fundamental projector of the C^*-algebra (the nontrivial generator of the K_0-group) as well as the pairing with the q-analogue of the Bott projector. Finally, we show that the local index formula is trivially satisfied.

1 Introduction.

The noncommutative differential (or *spectral*) geometry of quantum groups and q-deformed spaces has been recently intensively studied (see e.g. [6] for review). In particular the explicit computation of the local index formula for the total characteristic class, i.e. the Connes-Chern character, has been worked out on the quantum group $SU_q(2)$ in [4] for the 'singular' spectral triple of [13] and in [9] for the spectral triple of [8], with most of the results coinciding.

In this paper we present a systematic discussion of analogous spectral properties of another quantum space: the equatorial Podleś sphere S_q^2, originally defined for $q \neq 0$ as a homogeneous $SU_q(2)$-space. We also study separately the 'degenerate' (i.e. $q = 0$) case, that has a perfect meaning (though not as a homogeneous space).

Our main task is the analysis of the local index formula for the spectral triple on S_q^2 constructed in [7], which anticipated some of the interesting properties of that in [8]. We analyze also another spectral triple, related to the Fock representation of the Toeplitz algebra.

For that purpose it is convenient to study first the 'degenerate' (i.e. $q = 0$) case (related to the quantum disk), on which we consider two different spectral triples. Using these results, on S_q^2 we analyze two spectral triples: the one related to the Fock representation of the Toeplitz algebra (in Section 4) and the isopectral one given in [7] (in Section 5). After the identification of the pre-C^*-algebra of 'smooth' elements, we compute the dimension spectrum and residues. We check the nontriviality of the (noncommutative) Chern character of the associated Fredholm modules by computing the pairing with the fundamental projector of the C^*-algebra (the nontrivial generator of the K_0-group) as well as the pairing with the q-analogue of the Bott projector. Finally, we explicitely verify that the local index formula is trivially satisfied.

The relevance of such explicit calculations and results stems from their relative scarceness in the literature for interesting noncommutative spaces.

In the following we use the notation $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ and $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$.

2 Preliminaries about the equatorial Podleś sphere.

We use, with minor changes, notation of [7]. For $0 < q < 1$ the $*$-algebra (of polynomials) $\mathcal{A}(S_q^2)$ on the equatorial Podleś sphere is generated by a, a^* and $b = b^*$ with relations[1]

$$ba = q^2 ab \ , \qquad a^*a + b^2 = 1 \ , \qquad q^4 aa^* + b^2 = q^4 \ . \tag{1}$$

Its linear basis can be taken as $\{a^n b^m,\ (a^*)^{n+1} b^m,\ n, m \in \mathbb{N}\}$.

We denote by $C(S_q^2)$ the universal C^*-algebra of $\mathcal{A}(S_q^2)$. S_q^2 is known to be an embeddable $SU_q(2)$-homogeneous space, and carries a strongly continuous action of S^1. Being embeddable, the coaction of $SU_q(2)$ it carries is the restriction of the coproduct of $SU_q(2)$. One easily verifies that the subalgebra $\mathcal{A}(S_q^2) \subset \mathcal{A}(SU_q(2))$ is invariant with respect to the standard left action of $U_q(su(2))$.

The relations (1) make sense for any $q \in \mathbb{R}$ but the map $a \mapsto a^*$, $b \mapsto q^{-2}b$ extends to an isomorphism between $\mathcal{A}(S_q^2)$ and $\mathcal{A}(S_{q^{-1}}^2)$, while one has trivially $\mathcal{A}(S_q^2) \simeq \mathcal{A}(S_{-q}^2)$. So, without loss of generality, one can restrict to $0 \leq q \leq 1$. The two limiting cases are as follows:

[1]The original presentation of Podleś [14, eq. (7b)] corresponds to $A = q^{-2}b$, $B = a^*$ and $\mu = q$ (for $q < 1$).

1. If $q = 1$, this is just the 'polynomial' algebra on the (commutative) sphere S^2, while its C^*-algebra closure corresponds to continuous functions on S^2.
2. If $q = 0$, we have $b = 0$ and denoting $a = w$ we see that $\mathcal{A}(S_0^2)$ is the algebra of polynomial functions in the elements w and w^*, with relation $w^*w = 1$.

The associated universal C^*-algebra $C(S_0^2)$, being generated by one partial isometry, is known to be isometrically $*$-isomorphic to the algebra of Toeplitz operators $\mathcal{T}$ [2]. We can also think of $C(S_0^2) \simeq \mathcal{T}$ as the algebra $C(\mathcal{D}_q)$ of continuous 'functions' on a non-commutative disk $\mathcal{D}_q$, due to the short exact sequence[2]

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\sigma} C(S^1) \to 0 \ , \tag{2}$$

interpreted as the non-commutative analogue of the sequence

$$0 \to C_0(\text{open disk}) \to C(\text{closed disk}) \to C(S^1) \to 0 \ .$$

In eq. (2), we think of S^1 as the boundary of the non-commutative disk, σ as the evaluation on the boundary and the compact operators $\mathcal{K}$ as continuous 'functions' on $\mathcal{D}_q$ vanishing on the boundary.

In the sequel, unless stated otherwise we shall require that $0 < q < 1$.

From the last defining relation in (1) it easily follows that in all representations the operator norm of $q^{-2}b$ is ≤ 1. Hence the C^*-norm of b satisfies $||b|| < 1$. Therefore, $1 - b^2 = a^*a$ is invertible in the C^*-algebra and the element

$$p = 1 - a(a^*a)^{-1}a^* \in C(S_q^2) \tag{3}$$

is a projector (i.e. $p = p^* = p^2$). In Appendix B we show that the projective left $C(S_q^2)$-module $L^2(S_q^2)p$ is equivalent to the (graded) faithful representation given in [14] $\mu := \mu_+ \oplus \mu_- : C(S_q^2) \to \mathcal{B}(\ell^2(\mathbb{Z}_+) \oplus \ell^2(\mathbb{Z}_+))$, where

$$\mu_\pm(a) \, |n\rangle = \sqrt{1 - q^{4n}} \, |n+1\rangle \ , \qquad \mu_\pm(b) \, |n\rangle = \pm q^{2n} \, |n\rangle \ , \tag{4}$$

with $|n\rangle$ being the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$.

The representations $\mu_\pm$ are irreducible but not faithful; $\mu_+(b)$ has positive spectrum, while $\mu_-(b)$ has negative spectrum. For this reason, we interpret $C(S_q^2)/\ker \mu_\pm$ as a couple of closed (noncommutative) hemispheres composing the sphere.

We define $\mathcal{A}(\mathcal{D}_q) := \mu_+(\mathcal{A}(S_q^2)) = \mu_-(\mathcal{A}(S_q^2))$, i.e. as the polynomial $*$-algebra generated by $\mu_\pm(a)$ and $\mu_\pm(b)$, given by (4). (These are the noncommutative disks considered in [4, 9]). Notice that at the C^*-algebra level

$$C(S_q^2)/\ker \mu_\pm \simeq C(\mathcal{D}_q) = \mathcal{T} \simeq C(S_0^2) \ ,$$

even though $\mathcal{A}(\mathcal{D}_q) \neq \mathcal{A}(S_0^2)$.

[2]Here σ is the symbol map, sending the unilateral shift w to $e^{i\theta}$ and extended to a C^*-algebra morphism.

Identifying $C(S^2_q)$ with $\mu(C(S^2_q))$, in [15] it was shown that the map $f \mapsto (\mu_+(f), \mu_-(f))$ gives the isomorphism

$$C(S^2_q) \simeq \{(x,y) \in C(\mathcal{D}_q) \oplus C(\mathcal{D}_q) \,|\, \sigma(x) = \sigma(y)\} \ . \tag{5}$$

Thus, a 'function' on the equatorial Podleś sphere is given by a couple of 'functions' on the two hemispheres which coincide on the boundary $(\sigma(x) = \sigma(y))$. This allows an interpretation of S^2_q as a couple of noncommutative disks glued along the common boundary S^1.
In the degenerate case $q = 0$, one can think that the two disks 'collapse' one over the other, as $C(S^2_0) \simeq C(\mathcal{D}_q)$.

Notice that $C(S^2_q) \simeq C(S^2_{q'})$ for all $0 < q, q' < 1$. Furthermore the representation μ allows us to construct a regular even spectral triple over $\ell^2(\mathbb{Z}_+) \oplus \ell^2(\mathbb{Z}_+)$ with Dirac operator $(N \oplus N)F$, where F flips the two subspaces and

$$N\,|n\rangle = n\,|n\rangle$$

is the 'number' operator. This triple (modulo an irrelevant shift of N by 1) was studied in [13].

The kernel of the projection $C(S^2_q) \to \mathcal{T}$, $(x,y) \mapsto x$, is $\mathcal{K}$. The map $\mathcal{T} \to C(S^2_q)$, $x \mapsto (x,x)$, proves that the following sequence is split exact:

$$0 \to \mathcal{K} \to C(S^2_q) \leftrightarrow \mathcal{T} \to 0$$

and then $K_0(S^2_q) = \mathbb{Z} \oplus \mathbb{Z}$ and $K_1(S^2_q) = 0$ [16, cor. 8.2.2].

The C^*-algebra morphism $\rho : C(S^2_q) \to C(S^1)$, $\rho(x,y) := \sigma(x) = \sigma(y)$, gives rise to the exact sequence

$$0 \to \mathcal{K} \oplus \mathcal{K} \to C(S^2_q) \xrightarrow{\rho} C(S^1) \to 0 \ .$$

The map ρ is just the C^*-algebra morphism that extends the map $a \to e^{i\theta}$, $b \to 0$.

At the level of polynomial algebras, we have the isomorphism:

$$\mathcal{A}(S^2_q) \simeq \{(x,y) \in \mathcal{A}(\mathcal{D}_q) \oplus \mathcal{A}(\mathcal{D}_q) \,|\, \sigma(x) = \sigma(y)\} \ .$$

In Section. 3.1 a certain (Fréchet) pre-C^*-algebra $A^\infty \subset \mathcal{T}$ is introduced. It fits the exact sequence

$$0 \to \mathcal{S} \to A^\infty \to C^\infty(S^1) \to 0 \ ,$$

where $\mathcal{S}$ is isomorphic to the pre-C^*-algebra of rapid decay matrices on $\ell^2(\mathbb{N})$. There it is shown that A^∞ (the definition of which does not depend on q) contains both $\mathcal{A}(S^2_0)$ and $\mathcal{A}(\mathcal{D}_q)$ for any q, so it can be interpreted as the algebra of smooth elements (and used to construct a regular spectral triple) on both S^2_0 and $\mathcal{D}_q$.

Since S^2_q is obtained by gluing two copies of $\mathcal{D}_q$ along the boundary, we define the smooth 'functions' over S^2_q as

$$C^\infty(S^2_q) := \{(x,y) \in A^\infty \oplus A^\infty \,|\, \sigma(x) = \sigma(y)\} \subset C(S^2_q) \ . \tag{6}$$

It is a pre-C^*-algebra independent on q, and fits the exact sequence

$$0 \to \mathcal{S} \oplus \mathcal{S} \to C^\infty(S^2_q) \to C^\infty(S^1) \to 0 \ .$$

Let us describe now the spin representation of $C(S_q^2)$, as defined in [7]. For that we shall use the Hilbert space isomorphic to classical L^2-spinors over the round sphere S^2, $L^2(S^2) \otimes \mathbb{C}^2 = \mathcal{H}_+ \oplus \mathcal{H}_-$, with orthonormal basis of 'spinor harmonics' $|l,m\rangle_\pm$ labeled by $l \in \mathbb{N} + \frac{1}{2}$ and $m = -l, -l+1, \ldots, l$. In this basis, the Dirac operator (over S^2) is $D\,|l,m\rangle_\pm = (l+\frac{1}{2})\,|l,m\rangle_\mp$. This Hilbert space can be also viewed as a module over $\mathcal{A}(S_q^2)$ for any $0 < q < 1$, that naturally extends (by continuity) to a bounded representation of the C^*-algebra. The *chiral* representations of [7] are the faithful and irreducible representations $\pi_\pm : C(S_q^2) \to \mathcal{B}(\mathcal{H}_\pm)$ defined by

$$
\begin{aligned}
\pi_\pm(a)\,|l,m\rangle_\pm :=\; & q^{m-l-\frac{1}{2}} \frac{\sqrt{[l+m+1][l+m+2]}}{[2l+2]}\,|l+1,m+1\rangle_\pm \\
& - q^{m+l+\frac{1}{2}} \frac{\sqrt{[l-m-1][l-m]}}{[2l]}\,|l-1,m+1\rangle_\pm \\
& \pm \frac{(1+q^2)q^{m-\frac{1}{2}}}{[2l][2l+2]} \sqrt{[l+m+1][l-m]}\,|l,m+1\rangle_\pm \;, \\
\pi_\pm(b)\,|l,m\rangle_\pm :=\; & -q^{m+1} \frac{\sqrt{[l+m+1][l-m+1]}}{[2l+2]}\,|l+1,m\rangle_\pm \\
& - q^{m+1} \frac{\sqrt{[l+m][l-m]}}{[2l]}\,|l-1,m\rangle_\pm \\
& \pm \frac{[l-m+1][l+m] - q^2[l-m][l+m+1]}{[2l][2l+2]}\,|l,m\rangle_\pm \;,
\end{aligned}
$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$ is the q-analogue of $x \in \mathbb{C}$. The *spin representation* π over $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$ is the direct sum $\pi := \pi_+ \oplus \pi_-$.
When $q = 1$, π is just the representation of $C(S_q^2)$ defined by left multiplication on sections of the spin bundle over S^2.

We verify in Section 4 that $(C^\infty(S_q^2), \mathcal{H}, D)$ is an (even) regular spectral triple (isospectral, since $\mathcal{H}$ and D are the classical ones). Isospectrality also means that the noncommutative Sobolev spaces (defined as domains of $|D|^s$) are the ordinary ones over S^2; the same for the smooth domain of D and for smoothing operators.

Even though originally defined for $q \neq 0$, the representations $\mu_\pm$ and $\pi_\pm$ make sense also for $q = 0$. In this case, both $\mu_\pm$ reduces to the (irreducible faithful) Fock representation of the Toeplitz algebra, with w acting as the unilateral shift on $\ell^2(\mathbb{Z}_+)$

$$\mu_\pm(w)\,|n\rangle = |n+1\rangle \;. \tag{7}$$

Instead the representations $\pi_\pm$ of [7] become

$$\pi_\pm(w)\,|l,m\rangle = |l+1,m+1\rangle \;, \tag{8}$$

which when projected to a subspace with $l - m = k$ fixed, are equivalent to the Fock one.
Since in this degenerate $q = 0$ case, $\mu_+ = \mu_-$ and $\pi_+ = \pi_-$, the sign of the Dirac operator commutes with the algebra and is irrelevant. For this reason in the next section we consider only one copy of (chiral) representation and the absolute value of the Dirac operator.

3 Spectral triples over the quantum disk

We shall use the notation of Section 2.

3.1 Description of the algebra of "smooth" elements

We define $A^\infty \subset \mathcal{T}$ as the linear span of the elements

$$f = \sum_{n\in\mathbb{N}} (f_n w^n + f_{-n-1}(w^*)^{n+1}) + \sum_{j,k\in\mathbb{N}} f_{jk} w^j (1-ww^*)(w^*)^k \qquad \{f_n\} \in \mathcal{S}(\mathbb{Z}),\ \{f_{jk}\} \in \mathcal{S}$$

where $\mathcal{S}(\mathbb{Z})$ indicates rapid decay sequences and $\mathcal{S}$ rapid decay matrices on $\ell^2(\mathbb{N})$.

By direct calculation one checks that A^∞ is a $*$-algebra (associative, with unit) and the map $\sigma : A^\infty \to C^\infty(S^1)$, $f \mapsto \sigma(f) = \sum_{n\in\mathbb{Z}} f_n e^{in\theta}$ is a surjective $*$-algebra morphism (this follows from the simple observation that, via Fourier series, $\mathcal{S}(\mathbb{Z})$ equipped with convolution product is isomorphic to $C^\infty(S^1)$). We used the symbol σ because it is just the restriction to A^∞ of the symbol map in eq. (2), i.e. the C^*-algebra morphism defined by $\sigma(w) = e^{i\theta}$.

Furthermore, $\ker\sigma$ is a two-sided $*$-ideal in A^∞ isomorphic to $\mathcal{S}$. At the level of abstract algebras, this follows from the equality

$$w^j(1-ww^*)(w^*)^k w^{j'}(1-ww^*)(w^*)^{k'} = \delta_{j'k} w^j (1-ww^*)(w^*)^{k'}$$

and becomes more evident in the Fock representation, where

$$f\,|k\rangle = \sum\nolimits_{j\in\mathbb{Z}_+} f_{j-1,k-1}\,|j\rangle \ , \qquad \forall\, f = \sum\nolimits_{j,k\in\mathbb{N}} f_{jk} w^j (1-ww^*)(w^*)^k \in \ker\sigma \ .$$

Then, we have the short exact sequence anticipated in Section 2:

$$0 \to \mathcal{S} \to A^\infty \xrightarrow{\sigma} C^\infty(S^1) \to 0 \ . \tag{9}$$

Notice that the action of S^1 on $\mathcal{A}(S^2_0)$, given by $w \mapsto e^{i\theta}w$, defines a one-parameter group of automorphism implemented on $\ell^2(\mathbb{Z}_+)$ by the unitary operators $e^{i\theta N}$, i.e. for each $x \in \mathcal{A}(S^2_0)$ the action is $x \mapsto e^{i\theta N} x e^{-i\theta N}$. Being implemented by unitary operators, it extends to a (strongly continuous) action of S^1 on the C^*-algebra $\mathcal{T}$, and then on A^∞. The map σ commutes with this action (it is an S^1-module morphism), and elements of A^∞ are smooth for the action of S^1.

We conclude with the statement,

Proposition 3.1. *A^∞ is a Fréchet pre-C^*-algebra.*

This follows form the fact that as a vector space $\mathcal{A}^\infty$ is the cartesian product of two Fréchet spaces, hence it is a Fréchet space too. To prove that it is a pre-C^*-algebra it is sufficient to show that it contains the inverse of each element $f \in A^\infty$, whenever f is invertible in $\mathcal{T}$. To reach this conclusion, one can easily adapt the proof of proposition 1 in [4] to the present case.

Clearly, $A^\infty \supset \mathcal{A}(S^2_0)$ and it is dense in $\mathcal{T}$. But A^∞ contains also the algebra $\mathcal{A}(\mathcal{D}_q)$. Indeed,

$$q^{2N} = \sum\nolimits_{j,k\in\mathbb{N}} \delta_{jk} q^{2k} \mu(w)^{k+1} (1-ww^*) \mu(w^*)^{k+1}$$

is an element of A^∞. Now, A^∞ being closed under holomorphic functional calculus, also $\sqrt{1-q^{4N}} \in A^\infty$. Notice that $\mathcal{A}(\mathcal{D}_q)$ is generated by q^{2N} and $w\sqrt{1-q^{4N}}$. So, A^∞ contains the generators of $\mathcal{A}(\mathcal{D}_q)$ and then all the polynomial algebra.
For this reason, we can identify A^∞ with smooth "functions" over the quantum disk. (Notice that the isomorphism $C(S^2_0) = \mathcal{T} \xrightarrow{\sim} C(\mathcal{D}_q)$ is "smooth" but not polynomial).

3.2 Description of spectral triples for A^∞

We consider two spectral triples for A^∞. The first one is associated with the natural representation (7) of A^∞ on $\ell^2(\mathbb{Z}_+)$, with N taken as Dirac-type operator. The second one is associated with the representation (8) (i.e. the $q \to 0$ limit of the isospectral representation of [7]) together with $|D|$ (the absolute value of the classical Dirac operator D).

If $|n\rangle$ is the canonical basis of $\ell^2(\mathbb{Z}_+)$ and $|l,m\rangle$ the orthonormal basis of $\mathcal{H}_+ \simeq \bigoplus_{l+1/2\in\mathbb{Z}_+}\mathbb{C}^{2l+1}$ considered in [7], we are thus dealing respectively with the following representations

$$w\,|n\rangle = |n+1\rangle\ , \qquad w\,|l,m\rangle = |l+1,m+1\rangle\ , \qquad n, l+\tfrac{1}{2} \in \mathbb{Z}_+,\ m = -l,-l+1,\dots,l$$

and with the following 'Dirac' operators

$$N\,|n\rangle = n\,|n\rangle\ , \qquad |D|\,|l,m\rangle = (l+\tfrac{1}{2})\,|l,m\rangle$$

(we identify A^∞ with its representation). Since both these operators are positive, the associated index map is trivial.

From the equations $[N,w] = w$ and $[|D|,w] = w$ we deduce that A^∞ is invariant with respect to both the derivations $[N,\,.\,]$ and $[|D|,\,.\,]$, hence it is in their smooth domain. We have thus proved,

Proposition 3.2. *$(A^\infty, \ell^2(\mathbb{Z}_+), N)$ and $(A^\infty, \mathcal{H}_+, |D|)$ are regular spectral triples.*

Moreover

Proposition 3.3. *$\mathrm{Trace}_{\ell^2(\mathbb{Z}_+)}(fN^{-s})$ and $\mathrm{Trace}_{\mathcal{H}_+}(f|D|^{-s})$ are holomorphic functions on $\mathbb{C}$, for all $f \in \mathcal{S} = \ker\sigma$.*

Proof:
If $f = \sum_{j,k\in\mathbb{N}} f_{jk} w^j(1-ww^*)w^k$ is the generic element of $\ker\sigma$, then $f_{jj} \in \mathcal{S}(\mathbb{N})$.
In both the triples, $w^j(1-ww^*)(w^*)^k$ is off-diagonal if $j \neq k$, while if $j = k$ it is a projector on $|k+1\rangle$, respectively on $\mathbb{C}$-span of $\{|k+\tfrac{1}{2},m\rangle \mid m = -k-\tfrac{1}{2},\dots k+\tfrac{1}{2}\}$. Thus we have the equalities

$$\mathrm{Trace}_{\ell^2(\mathbb{Z}_+)}(fN^{-s}) = \sum\nolimits_{n\in\mathbb{Z}_+} n^{-s} f_{n-1,n-1}\ ,$$
$$\mathrm{Trace}_{\mathcal{H}_+}(f|D|^{-s}) = \sum\nolimits_{l+\frac{1}{2}\in\mathbb{Z}_+} (2l+1)(l+\tfrac{1}{2})^{-s} f_{l-\frac{1}{2},l-\frac{1}{2}}$$

and the series converge to holomorphic functions on $\mathbb{C}$ (Weierstrass theorem). □

Proposition 3.4. *The dimension spectrum is $\{1\}$ for the first triple and $\{2\}$ for the second. The residues are*

$$\mathrm{Res}_{s=1}\mathrm{Trace}_{\ell^2(\mathbb{Z}_+)}(fN^{-s}) = \frac{1}{2\pi}\int_{S^1}\sigma(f)\mathrm{d}\theta\ , \qquad \mathrm{Res}_{s=2}\mathrm{Trace}_{\mathcal{H}_+}(f|D|^{-s}) = \frac{1}{\pi}\int_{S^1}\sigma(f)\mathrm{d}\theta$$

for all $f \in A^\infty$, where σ is the map in (9).

Proof:
The algebra generated by A^∞ and the commutators with the Dirac operator is simply A^∞, in both cases. $\ker\sigma$ does not contribute to the dimension spectrum, due to prop. 3.3.
It remains to consider an element of the form $f = \sum_{n\in\mathbb{N}}(f_n w^n + f_{-n-1}(w^*)^{n+1}) \in A^\infty$. All the terms in f are off-diagonal, but the one proportional to $f_0 = \frac{1}{2\pi}\int_{S^1}\sigma(f)\mathrm{d}\theta$. Then $\mathrm{Trace}_{\ell^2(\mathbb{Z}_+)}(fN^{-s}) = f_0\,\zeta(s)$ and $\mathrm{Trace}_{\mathcal{H}_+}(f|D|^{-s}) = 2f_0\,\zeta(s-1)$, where $\zeta(s)$ is the Riemann zeta-function. This concludes the proof. □

Notice that 1 is not in the dimension spectrum of the second triple, even though S^1 is a classical subspace of S^2_0, which seems quite curious.

4 A spectral triple for $\mu(S^2_q)$.

Here we consider the spectral triple over S^2_q associated to the representation $\mu = \mu_+ \oplus \mu_-$ given by (4). The Hilbert space is naturally isomorphic to $\ell^2(\mathbb{Z}_+)\otimes\mathbb{C}^2$, and the representation reads

$$\mu(a) = w\sqrt{1-q^{4N}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \mu(b) = q^{2N} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

while as the Dirac operator D' we take

$$D' = N \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \text{ so } F := \mathrm{sign} D = id_{\ell^2(\mathbb{Z}_+)} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } |D'| = N \otimes id_{\mathbb{C}^2} .$$

The resulting spectral triple can be thought of as a particular limit of the one considered in [13, sec. 5][3].

Let A^∞ be the algebra defined in sec. 3.1, and recall that $\mu_\pm(a), \mu_\pm(b) \in A^\infty$. Then $\mu(a), \mu(b) \in A^\infty \oplus A^\infty$. Since $\sigma\mu_\pm(a) = e^{i\theta}, \sigma\mu_\pm(b) = 0$, if we identify $\mathcal{A}(S^2_q)$ with its representation μ, then

$$\mathcal{A}(S^2_q) \subset \{(x,y) \in A^\infty \oplus A^\infty \,|\, \sigma(x) = \sigma(y)\} =: C^\infty(S^2_q) , \tag{10}$$

where $C^\infty(S^2_q)$ is the algebra anticipated in eq. (6). From the direct sum of two copies of (9), passing to the diagonal ($\sigma(x) = \sigma(y)$) we obtain the exact sequence:

$$0 \to \mathcal{S} \oplus \mathcal{S} \to C^\infty(S^2_q) \xrightarrow{\rho} C^\infty(S^1) \to 0 , \tag{11}$$

where, as in the previous section, we call $\rho(x \oplus y) = \sigma(y)$, for $x \oplus y \in C^\infty(S^2_q)$. Then, ρ is the morphism defined by $\rho(a) = e^{i\theta}$, $\rho(b) = 0$.

Let B be the algebra generated by $C^\infty(S^2_q)$ and commutators with D'. Since $[D', \mu(a)] = \mu(a)F$ and $[D', \mu(b)] = 0$, it follows that B is isomorphic (as a vector space) to the direct sum of two copies of the algebra:

$$B \equiv C^\infty(S^2_q) + C^\infty(S^2_q)wF .$$

[3] The generators A, B used in [13] are related to ours generators through $a = \lim_{c\to\infty}(c^{-1/2}B^*)$ and $b = q^2\lim_{c\to\infty}(c^{-1/2}A)$, and the Dirac operator is shifted by 1, in order to remove 0 from the spectrum. A finite shift of D' results in a finite rank perturbation of the sign F, and so it does not affect dimension spectrum, residues and the index map.

Let $\delta' := [|D'|, .]$. From $\delta'\mu(a) = \mu(a)$ and $\delta'\mu(b) = 0$ it follows that B is δ'-invariant, and hence it is in the smooth domain of δ'. We have thus proved,

Proposition 4.1. *$(C^\infty(S^2_q), \ell^2(\mathbb{Z}_+) \oplus \ell^2(\mathbb{Z}_+), D')$ is a* regular *spectral triple.*

Let, for $s \in \mathbb{C}$ with sufficiently large real part, the 'zeta-type' function associated to $T \in B$ be given by

$$\zeta_T(s) := \mathrm{Trace}_{\ell^2(\mathbb{Z}_+)\oplus\ell^2(\mathbb{Z}_+)}(T|D'|^{-s}) \ .$$

If we define the following $*$-algebra morphism:

$$\begin{aligned} \tilde\rho : B &\to C^\infty(S^1) \otimes \mathrm{Mat}_2(\mathbb{C}) \ , \\ T = x + ywF &\mapsto \tilde\rho(T) = \rho(x) \cdot id_{\mathbb{C}^2} + \rho(y) e^{i\theta} F \ , \qquad \big(x, y \in C^\infty(S^2_q)\big) \end{aligned} \tag{12}$$

then,

Proposition 4.2. *The dimension spectrum is $\{1\}$ and the residue is*

$$\mathrm{Res}_{s=1}\zeta_T(s) = \frac{1}{2\pi}\int_{S^1} \mathrm{d}\theta \ \mathrm{Trace}_{\mathbb{C}^2}\tilde\rho(T)$$

for all $T \in B$, where $\tilde\rho$ is the map in (12).

Proof:
Let $T = x + ywF \in B$, $x, y \in C^\infty(S^2_q)$. Then $x = x_+ \oplus x_-$ for suitable $x_\pm \in A^\infty$.
Notice that $\zeta_{ywF}(s) = 0$ since ywF is off-diagonal due to the presence of F.
From the proof of the proposition 3.4 we derive

$$\begin{aligned} \zeta_T(s) = \zeta_x(s) &= \mathrm{Trace}_{\ell^2(\mathbb{Z}_+)}(x_+N^{-s}) + \mathrm{Trace}_{\ell^2(\mathbb{Z}_+)}(x_-N^{-s}) \\ &= \left(\frac{1}{2\pi}\int_{S^1}\sigma(x_+)\mathrm{d}\theta + \frac{1}{2\pi}\int_{S^1}\sigma(x_-)\mathrm{d}\theta\right)\zeta(s) = \left(\frac{1}{2\pi}\int_{S^1} 2\rho(x)\mathrm{d}\theta\right)\zeta(s) \end{aligned}$$

In the last step we used the definition $\sigma(x_\pm) = \rho(x)$. To conclude the proof is sufficient to notice that from the definition (12) it follows: $\mathrm{Trace}_{\mathbb{C}^2}\tilde\rho(T) \equiv 2\rho(x)$. □

4.1 Non-triviality of the Chern character

Consider the projector $|1\rangle\langle 1| \oplus 0$ over $\ell^2(\mathbb{Z}_+) \oplus \ell^2(\mathbb{Z}_+)$. It has finite rank, hence it is an element of $\mathcal{S} \oplus \mathcal{S}$ and its image in $C^\infty(S^2_q)$ via the morphism in (11) is a projector too. This projector generates, together with the trivial one, the K_0-group of S^2_q [12]. The index of the associated twisted Dirac operator (c.f. App. A) is easily computed as in [13, sec. 5], to be equal to 1, proving the non-triviality of the Chern character of the spectral triple.

Another interesting projector $p' \in \mathcal{A}(S^2_q) \otimes \mathrm{Mat}_2(\mathbb{C})$ is given by

$$p' = \frac{1}{2}\begin{pmatrix} 1+b & a^* \\ a & 1 - q^{-2}b \end{pmatrix} \tag{13}$$

(p' is associated to the $SU_q(2)$-principal bundle with base space S^2_q in [1][4]).

[4]Using the notations of [1], when $s = 1$ (the condition corresponding to the equatorial Podleś sphere), the elements (ξ, η, ζ) of [1] correspond to $(a^*, -a, -b)$ and their projector is $e_{s=1} \equiv p'$.

Note that $C(S_q^2)^{\oplus 2}p'$, completed with respect to a suitable inner product, is the analogue of the tautological line bundle. In fact when $q = 1$, $a \to 2z(1+z\bar{z})^{-1}$ and $1 - b \to 2(1+z\bar{z})^{-1}$, where $z = e^{i\varphi}\cot\theta/2$ is the stereographic coordinate on S^2. Thus when $q = 1$, the projector p' becomes the celebrated Bott projector

$$p_B(z) = \frac{1}{1+z\bar{z}} \begin{pmatrix} z\bar{z} & \bar{z} \\ z & 1 \end{pmatrix}, \quad p_B(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ,$$

which, together with [1] generates $K^0(S^2)$.
In the rest of this section we show that $\mathrm{ch}^F([p']) = -1$, proving that $1 - p'$ and $|1\rangle\langle 1| \oplus 0$ are equivalent projectors and that p' can be taken as non-trivial generator of $K_0(C(S_q^2))$.

Denote $V_\pm \simeq \ell^2(\mathbb{Z}_+)$ the two components of the Hilbert space with the canonical orthonormal basis $|n\rangle_\pm$. To deal with p' we need to lift the representation of the algebra and the sign of the Dirac operator to 2×2 matrices, so let $V'_\pm = \mathbb{C}^2 \otimes V_\pm$ and $F' = id_{\mathbb{C}^2} \otimes F$.

We choose the following (orthonormal) basis of $V'_\pm$

$$\begin{aligned}
|n\rangle^0_\pm &:= \sqrt{\frac{1-rq^{2n}}{2}} \binom{1}{0} \otimes |n\rangle_\pm - \sqrt{\frac{1+rq^{2n}}{2}} \binom{0}{1} \otimes |n+1\rangle_\pm \ , \qquad \forall\, r = \pm,\ n \in \mathbb{Z}_+ \ , \\
|n\rangle^1_\pm &:= \sqrt{\frac{1+rq^{2n}}{2}} \binom{1}{0} \otimes |n\rangle_\pm + \sqrt{\frac{1-rq^{2n}}{2}} \binom{0}{1} \otimes |n+1\rangle_\pm \ , \qquad \forall\, r = \pm,\ n \in \mathbb{Z}_+ \ , \\
|0\rangle_\pm &:= \binom{0}{1} \otimes |1\rangle_\pm \ .
\end{aligned}$$

A straightforward calculation shows that

$$\mu_\pm(p')\,|n\rangle^0_\pm = 0 \,, \qquad \mu_+(p')\,|0\rangle_+ = 0 \,, \qquad \mu_\pm(p')\,|n\rangle^1_\pm = |n\rangle^1_\pm \ , \qquad \mu_-(p')\,|0\rangle_- = |0\rangle_- \ .$$

From these we deduce that $\{|n\rangle^1_+\}$ is a basis for $\mu_+(p')V'_+$ and $\{|n\rangle^1_-, |0\rangle_-\}$ is a basis for $\mu_-(p')V'_-$.

Let $F_{p'} := \mu_-(p')F'\mu_+(p')$, it maps $\mu_+(p')V'_+$ to $\mu_-(p')V'_-$. Then

$$\begin{aligned}
F_{p'}\,|n\rangle^1_+ &= \sqrt{1-q^{4n}}\,|n\rangle^1_- \neq 0 \,, && \forall\, n \in \mathbb{Z}_+ \ , \\
F^*_{p'}\,|n\rangle^1_- &= \sqrt{1-q^{4n}}\,|n\rangle^1_+ \neq 0 \,, && \forall\, n \in \mathbb{Z}_+ \ , \\
F^*_{p'}\,|0\rangle_- &= 0 \ .
\end{aligned}$$

Hence $\ker F_{p'} = 0$, $\ker F^*_{p'} \simeq \mathbb{C}$ and $\mathrm{ch}^F([p']) = \mathrm{Index}(F_{p'}) = -1$.

4.2 Local index formula

Consider the $*$-linear map $\mu_0 := \mu_+ - \mu_- : C^\infty(S_q^2) \to \mathcal{B}(\ell^2(\mathbb{Z}_+))$. We claim that μ_0 has image in $\mathcal{S}$. In fact, the equality $\mu_0(xy) = \mu_0(x)\mu_+(y) + \mu_-(x)\mu_0(y)$ implies that $\mu_0(xy) \in \mathcal{S}$ if $\mu_0(x), \mu_0(y) \in \mathcal{S}$ ($\mathcal{S}$ being a two-sided $*$-ideal in A^∞). Moreover the generators of the algebra satisfy $\mu_0(a) = 0, \mu_0(b) = 2q^{2N} \in \mathcal{S}$. As a consequence the commutator

$$[F, \mu(x)] = \mu_0(x) \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is traceclass for all $x \in C^\infty(S_q^2)$ and the map ch_0^F in eq. (16) is well-defined.

The pairing between the cyclic cocycle defined by the map ch_0^F and the class $[p] \in K_0(\mathcal{A})$ of a projector p gives the index formula

$$\mathrm{Index}(pFp) = \tfrac{1}{2}\mathrm{Trace}_{\ell^2(\mathbb{Z}_+)\oplus\ell^2(\mathbb{Z}_+)}(\gamma F[F,\mu(p)]) = \mathrm{Trace}_{\ell^2(\mathbb{Z}_+)}\mu_0(p) \ ,$$

where $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the grading. Theorem A.2, applied to the present case, states that ch^F is cohomologous to the cocycle with only one component φ_0 given by

$$\varphi_0(x) = \mathrm{Res}_{s=0}s^{-1}\mathrm{Trace}_{\ell^2(\mathbb{Z}_+)\oplus\ell^2(\mathbb{Z}_+)}\big(\gamma\mu(x)|D'|^{-2s}\big) = \mathrm{Res}_{s=0}s^{-1}\mathrm{Trace}_{\ell^2(\mathbb{Z}_+)}\big(\mu_0(x)N^{-2s}\big) \ .$$

But $\psi(s) := \mathrm{Trace}_{\ell^2(\mathbb{Z}_+)}\big(\mu_0(x)N^{-2s}\big)$ is holomorphic since $\mu_0(x) \in \mathcal{S}$. Thus, $\mathrm{Res}_{s=0}s^{-1}\psi(s) = \psi(0)$ and $\varphi_0(p) \equiv \mathrm{ch}_0^F([p])$.

Interestingly, theorem A.2 is here trivially satisfied (i.e. the coboundary '$\mathrm{ch}^F - \varphi$' is zero). We shall encounter a similar situation in the next section.

5 A spectral triple for $\pi(S_q^2)$.

In this section we discuss the spectral triple of [7]. To simplify the story, we work with polynomial algebra $\mathcal{A}(S_q^2)$. Analogous results hold also for $C^\infty(S_q^2)$, but are more complicated. The Hilbert space is $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \simeq \hat{\mathcal{H}} \otimes \mathbb{C}^2$, where $|l,m\rangle$ is declared to be an orthonormal basis of $\hat{\mathcal{H}}$, $m = -l, -l+1, ..., l$.

We identify $\mathcal{H}_\pm$ with $\hat{\mathcal{H}}$ through the isometry $|l,m\rangle_\pm \to |l,m\rangle$ and consider the representations $\pi_\pm$ at page 5 as representations over $\hat{\mathcal{H}}$. Then, if we call

$$\rho_\pm := (\pi_+ \pm \pi_-)/2 : \mathcal{A}(S_q^2) \to \mathcal{B}(\hat{\mathcal{H}})$$

the spinorial representation $\pi = \pi_+ \oplus \pi_-$ can be rewritten as

$$\pi = \rho_+ \otimes id_{\mathbb{C}^2} + \rho_- \otimes \gamma \ ,$$

where $\gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the grading. As stated in the introduction, the Dirac operator is $D = |D| \otimes F$, where

$$|D|\,|l,m\rangle = (l+\tfrac{1}{2})\,|l,m\rangle \ , \qquad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Let us rewrite explicitly $\rho_\pm(a)$ and $\rho_\pm(b)$:

$$\rho_+(a)\,|l,m\rangle = \frac{\sqrt{1-q^{2(l+m+1)}}\sqrt{1-q^{2(l+m+2)}}}{1-q^{4(l+1)}}\,|l+1,m+1\rangle \tag{14a}$$

$$- \frac{\sqrt{q^{2(l+m)}-q^{4l}}\sqrt{q^{2(l+m+1)}-q^{4l}}}{1-q^{4l}}\,|l-1,m+1\rangle \ , \tag{14b}$$

$$\rho_+(b)\,|l,m\rangle = -\frac{\sqrt{1-q^{2(l+m+1)}}\sqrt{q^{2(l+m+2)}-q^{4l+6}}}{1-q^{4(l+1)}}\,|l+1,m\rangle \tag{14c}$$

$$- \frac{\sqrt{1-q^{2(l+m)}}\sqrt{q^{2(l+m+1)}-q^{4l+2}}}{1-q^{4l}}\,|l-1,m\rangle \ , \tag{14d}$$

$$\rho_-(a)\,|l,m\rangle = \frac{(1-q^4)q^{3l+m}}{(1-q^{2l})(1-q^{2l+2})}\sqrt{1-q^{2(l+m+1)}}\sqrt{1-q^{2(l-m)}}\,|l,m+1\rangle \ , \tag{14e}$$

$$\rho_-(b)\,|l,m\rangle = \frac{(1-q^2)q^{2l+1}}{(1-q^{2l})(1-q^{2l+2})}\Big\{1+q^{4l+2}-(1+q^2)q^{2(l+m)}\Big\}\,|l,m\rangle \ . \tag{14f}$$

Notice that $\rho_-(a)$ and $\rho_-(b)$ do not act on the index l, while $\rho_+(a)$ and $\rho_+(b)$ naturally decompose as the sum of two operators that shift the index l of ± 1. With obvious notations, call $\rho_+(a) = a_+ + a_-$ and $\rho_+(b) = b_+ + b_-$. Recall that $\delta := [|D|, \,.\,]$.

Proposition 5.1. *Let B be the $*$-algebra generated by $\pi(\mathcal{A}(S^2_q))$ and $[D,\pi(x)]$, $x \in \mathcal{A}(S^2_q)$. Then $B \subset \mathcal{B}(\mathcal{H})$ and $(\mathcal{A}(S^2_q),\mathcal{H},D)$ is a 2^+-dimensional regular spectral triple.*

Proof:
$\delta(a_\pm) = \pm a_\pm$ and $\delta(b_\pm) = \pm b_\pm$ are bounded and in the smooth domain of δ. Since

$$[D,\pi(a)] = (a_+ - a_-)\otimes F + 2\rho_-(a)D\otimes\gamma \qquad [D,\pi(b)] = (b_+ - b_-)\otimes F + 2\rho_-(b)D\otimes\gamma \ ,$$

then B is in the $*$-algebra generated by $\{1, a_\pm, b_\pm, F\}$, modulo terms in the kernel of δ, and hence B is in the smooth domain of δ. The metric dimension is 2 by isospectrality.
The boundedness of the terms linear in D is a consequence of the fact that ρ_- maps $\mathcal{A}(S^2_q)$ into rapid decay matrices. This will be proved in the next proposition. $\square$

Proposition 5.2. *ρ_- maps $\mathcal{A}(S^2_q)$ in $\mathcal{S}$. Therefore, $[F,\pi(x)] = 2\rho_-(x)\otimes F\gamma$ is traceclass for all $x \in \mathcal{A}(S^2_q)$ and the associated Fredholm module is finite summable.*

Proof:
Though ρ_- is not a representation, it satisfies the following identity

$$\rho_-(xy) = \pi_+(x)\rho_-(y) + \rho_-(x)\pi_-(y) \ .$$

Since $\mathcal{S}$ is a two-sided $*$-ideal for $\mathcal{A}(S^2_q)$, then $\rho_-(xy) \in \mathcal{S}$ if $\rho_-(x)$ and $\rho_-(y) \in \mathcal{S}$. So, to prove that ρ_- has image in $\mathcal{S}$ we need just to do the check for the generators of the algebra.
In (14e), $q^{l+m} \le 1$, the square roots are no greater than 1 and $(1-q^\alpha)^{-1} \le (1-q)^{-1}\ \forall\, \alpha \ge 1$. In (14f) the quantity in the big parenthesis is (in modulus) $\le 1 - q^{4l} \le 1$. Then, we deduce that the (nonzero) matrix coefficients of $\rho_-(a)$ and $\rho_-(b)$ satisfy

$$|\langle l,m+1|\rho_-(a)|l,m\rangle| \le (1-q)^{-2}q^{2l} \qquad |\langle l,m|\rho_-(b)|l,m\rangle| \le (1-q)^{-2}q^{2l}$$

and so they are rapid decay matrices. $\square$

(The Fredholm module (F,π) is 1-summable according to terminology of [3] and 0-summable according to [11].)

5.1 An approximate representation

To simplify the computations, it is useful to cut smoothing contributions.

For all $\alpha > 0$, $(1-q^{\alpha l})^{-1} - 1$ is a rapid decay sequence. For $0 \le u \le 1$, $|1-\sqrt{1-u}| \le u$. Then,

$$|q^{l+m} - \sqrt{q^{2(l+m)} - q^{4l}}| = q^{l+m}|1-\sqrt{1-q^{2(l-m)}}| \le q^{3l-m} \le q^{2l}$$

is a rapid decay sequence and the first square root in (14b) coincide with q^{l+m} modulo rapid decay sequences. Applying the same argument to the other square roots in (14), one prove that the operators $\lambda(a)$ and $\lambda(b)$, defined by

$$\lambda(a)\,|l,m\rangle := \sqrt{1-q^{2(l+m+1)}}\sqrt{1-q^{2(l+m+2)}}\,|l+1,m+1\rangle - q^{2(l+m)+1}\,|l-1,m+1\rangle \quad ,$$
$$\lambda(b)\,|l,m\rangle := -q^{l+m+2}\sqrt{1-q^{2(l+m+1)}}\,|l+1,m\rangle - q^{l+m+1}\sqrt{1-q^{2(l+m)}}\,|l-1,m\rangle \quad ,$$

differ from $\pi_\pm(a)$ and $\pi_\pm(b)$ by a rapid decay matrix.

The closure in the operator norm of $\mathcal{S}$ is the two-sided $*$-ideal $\mathcal{K}$. If $\tilde{\lambda}$ is the projection into the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$,

$$0 \to \mathcal{K} \to C(S^2_q) \xrightarrow{\tilde{\lambda}} C(S^2_q)/\mathcal{K} \to 0$$

then, $\lambda(a)$ and $\lambda(b)$ are representatives of $\tilde{\lambda}(a)$ and $\tilde{\lambda}(b)$, and the C^*-algebra they generate coincide with $C(S^2_q)$ modulo compact operators.

When considering $\mathcal{A}(S^2_q)$, the algebra of polynomials in $\lambda(a)$ and $\lambda(b)$ coincide with $\mathcal{A}(S^2_q)$ modulo rapid decay matrices, and the difference can be neglected when computing zeta-type functions.

The derivatives $\partial := [D,\,.\,]$ and $\delta := [|D|,\,.\,]$ send rapid decay matrices to rapid decay matrices, and then the operators $[D,\pi(a)] - [D,\lambda(a)\otimes id_{\mathbb{C}^2}]$, $[D,\pi(b)] - [D,\lambda(b)\otimes id_{\mathbb{C}^2}]$, $\delta\big(\pi(a)\big) - \delta\big(\lambda(a)\big)$, $\delta\big(\pi(b)\big) - \delta\big(\lambda(b)\big)$ are all rapid decay matrices and we have an "approximate representation" of all the algebra $\bigcup_{k\in\mathbb{N}}\delta^k(B)$ modulo $\mathcal{S}$.

5.2 The dimension spectrum

Let $U\,|l,m\rangle = q^{l+m}\,|l,m\rangle$ and $\mathcal{Q}_q$ be the (two-sided $*$-)ideal in $\mathcal{B}(\mathcal{H})$ generated by the two elements U and $V = 1-\sqrt{1-(qU)^2}$ (notice that they are not compact, $||U||_E = 1$ on each subspace $E\subset\hat{\mathcal{H}}$ of finite codimension).
Any operator in $\mathcal{Q}_q$ can be written in the form $T = xUy$ or $T = xVy$ for some bounded x,y, and so they satisfy (since $1-\sqrt{1-q^{2(l+m+1)}} \le q^{2(l+m+1)} \le q^{l+m}$)

$$|\,\langle l,m|T|l,m\rangle\,| \le ||x||\;||y||\,q^{l+m}\;.$$

Then, for all $T\in\mathcal{Q}_q$,

$$\begin{aligned}|\zeta_T(s)| &\le ||x||\;||y||\sum_{l+\frac{1}{2}\in\mathbb{Z}_+}(l+\tfrac{1}{2})^{-s}\sum_{l+m=0}^{2l}q^{l+m}\\ &= ||x||\;||y||\tfrac{1}{1-q}\zeta(s) + \text{holomorphic function}\;.\end{aligned}$$

So $\zeta_T(s)$ has a unique residue in $s=1$. This is not identically zero, since for $T=b^2$ we have

$$\begin{aligned}\zeta_{b^2}(s) &= \sum_{l+\frac{1}{2}\in\mathbb{Z}_+}(l+\tfrac{1}{2})^{-s}2q^2\sum_{k=0}^{2l}(q^{2k}-q^{4k})\\ &= \tfrac{2q^4}{1-q^2}\zeta(s) + \text{holomorphic function}\end{aligned}$$

and then

$$\mathrm{Res}_{s=1}\zeta_{b^2}(s) = \tfrac{2q^4}{1-q^2} \neq 0 \ .$$

Let $(\mathcal{S} \cup \mathcal{Q}_q)$ be the ideal generated by $\mathcal{S} \cup \mathcal{Q}_q$. Let $\nu : \mathcal{A}(S^2_q) \to \mathcal{B}(\mathcal{H})$ be the representation defined by

$$\nu(a)\,|l,m\rangle = |l+1,m+1\rangle \ , \qquad \nu(b) = 0 \ .$$

Then for $x \in \mathcal{A}(S^2_q)$, $\pi(x) - \nu(x) \in (\mathcal{S} \cup \mathcal{Q}_q)$, or with a slight abuse of notation ν is a projection from $\mathcal{A}(S^2_q)$ to $\mathcal{A}(S^2_q)/(\mathcal{S} \cup \mathcal{Q}_q)$.

The ideal $\mathcal{Q}_q$ is ∂-invariant and δ-invariant. Then, the computation of the dimension spectrum reduces to the computation of the dimension spectrum of $\nu(\mathcal{A}(S^2_q))$, i.e. of $\mathcal{A}(S^2_0)$ in the isospectral representation (8).

The contribution of $\mathcal{Q}_q$ gives a simple pole at $s = 1$, while the contribution of $\nu(\mathcal{A}(S^2_q))$ gives a simple pole at $s = 2$ (Proposition 4.2). Therefore we have

Proposition 5.3. *The dimension spectrum is* $\Sigma = \{1, 2\}$.

We compute now the residue at $s = 2$. Let $x \in \mathcal{A}(S^2_q)$ and call $T = \nu(x) \otimes id_{\mathbb{C}^2}$. From Proposition 3.4 we have

$$\mathrm{Res}_{s=2}\zeta_T(s) = 2\,\mathrm{Res}_{s=2}\mathrm{Trace}_{\hat{\mathcal{H}}}(T|D|^{-s}) = \frac{2}{\pi}\int_{S^1}\sigma(T)\mathrm{d}\theta \ .$$

The last residue is equal to the residue of $\zeta_x(s)$. Since $\sigma(\nu(a)) = e^{i\theta}$ and $\sigma(\nu(b)) = 0$, the final expression for the 'noncommutative integral' of $x \in \mathcal{A}(S^2_q)$ is

$$\mathrm{Res}_{s=2}\zeta_x(s) = \frac{2}{\pi}\int_{S^1}\rho(x)\mathrm{d}\theta \ , \tag{15}$$

where $\rho : \mathcal{A}(S^2_q) \to \mathcal{A}(S^1)$ is the $*$-algebra morphism defined by $\rho(a) = e^{i\theta}$, $\rho(b) = 0$.

5.3 Non-triviality of the Chern character

We show that the Chern character of the spectral triple is not trivial, computing the pairing with the class of the projector p' defined in eq. (13). From prop. 5.2, the commutator

$$[F, \pi(x)] = 2\rho_-(x) \otimes \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

is traceclass for all $x \in \mathcal{A}(S^2_q)$ and the map ch^F_0 in eq. (16) is well-defined,

$$\mathrm{ch}^F_0(x) = 2\mathrm{Trace}_{\hat{\mathcal{H}}}\rho_-(x) \ .$$

Then,

$$\mathrm{ch}^F([p']) = (1 - q^{-2})\,\mathrm{Trace}_{\hat{\mathcal{H}}}\rho_-(b) = -q^{-2}(1-q^2)^2\sum_{l,m}\frac{[l-m+1][l+m]}{[2l][2l+2]} \ .$$

Performing the sum in $m = -l, ..., l$ and setting $n = l + \frac{1}{2}$, we obtain

$$-\mathrm{ch}^F([p']) = \sum_{n=1}^{\infty}\frac{2n(1-x)^2(1+x^{2n}) - (1-x^2)(1-x^{2n})}{(1-x^{2n+1})(1-x^{2n-1})}x^{n-1} =: \sum_{n=1}^{\infty} f_n(x) = f(x)$$

where $x = q^2 \in]0,1[$, and $f(x)$ is an integer-valued function (being the index of a Fredholm operator) that we want to compute.

From the inequality $|f_n(x)| \leq (4n+2)x^{n-1}$ we deduce (Weierstrass M-test) that the series is absolutely (hence uniformly) convergent in each interval $[0,x_0] \subset [0,1[$. Then, it converges to a function $f(x)$ that is continuous in $[0,1[$. A continuous function $f :]0,1[\to \mathbb{Z}$ is constant. By continuity, $f(x)$ is constant in $[0,1[$ and can be computed setting $x=0$. Since $f_n(0) = \delta_{n,1}$, we deduce that $f(x) = f(0) = 1$ and so

$$\mathrm{ch}^F([p']) = -1 \ , \qquad \text{for all } 0 < q < 1 \, .$$

5.4 Local index formula

Thm. A.2, applied to the present case, states that ch^F is cohomologous to the cocycle with two components (φ_0, φ_2), given by

$$\varphi_0(a_0) = \mathrm{Res}_{s=0} s^{-1} \mathrm{Trace}(\gamma a_0 |D|^{-2s}) \ ,$$
$$\varphi_2(a_0,a_1,a_2) = \mathrm{Res}_{s=0} \mathrm{Trace}(\gamma a_0 [D,a_1][D,a_2]|D|^{-2(s+1)}) \ .$$

Neglecting rapid decay matrices, we have $[D,x] \sim \delta\rho_+(x)$ and then φ_2 is identically zero

$$\varphi_2(a_0,a_1,a_2) \equiv \mathrm{Res}_{s=0} \, \mathrm{Trace}(\rho_+(a_0)\, \delta\rho_+(a_1)\, \delta\rho_+(a_2)|D|^{-2(s+1)} \otimes \gamma) = 0 \ ,$$

since $\mathrm{Trace}_{\mathbb{C}^2}\gamma = 0$.

Moreover, $\psi(s) = \mathrm{Trace}(\gamma a_0 |D|^{-2s}) = 2\mathrm{Trace}_{\hat{\mathcal{H}}}\big(\rho_-(a_0)|D|^{-2s}\big)$ is holomorphic and

$$\varphi_0(a_0) := \mathrm{Res}_{s=0} s^{-1} \psi(s) = \psi(0) \equiv \mathrm{ch}_0^F(a_0) \, .$$

As in Section 4.2, also for the spinorial representation and the isospectral Dirac operator the theorem A.2 is trivially satisfied (the coboundary '$\mathrm{ch}^F - (\varphi_0, \varphi_2)$' is zero), though the cocycle φ could be expected to have higher dimensional components (as the metric dimension is 2).

A Generalities about spectral triples

In this appendix we recall some material from [3, 5, 4] (see also [10] and [11] for a comprehensive exposition).

Let $(A, \mathcal{H}, D)$ be a spectral triple and consider the following (unbounded) derivations on $\mathcal{B}(\mathcal{H})$:

$$\partial, \delta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \ , \qquad \partial T = [D,T] \ , \qquad \delta(T) = [|D|, T]$$

The triple is *regular* if

$$A \cup \partial A \subset \bigcap_{j \in \mathbb{N}} \mathrm{dom}\, \delta^j \ .$$

The operator D is *finite dimensional* if there exist a $d \in \mathbb{R}^+$ (called the *metric dimension*) such that the singular values of $|D|^{-1}$ (assume that D is invertible) are of order $n^{-1/d}$ when $n \to \infty$.

The 'zeta-type' function

$$\zeta_T(s) := \mathrm{Trace}_{\mathcal{H}}\big(T|D|^{-s}\big)$$

associated to $T \in V = \bigcup_{j\in\mathbb{N}} \delta^j(A \cup \partial A)$ is defined is defined (and holomorphic) for $s \in \mathbb{C}$ with $\mathrm{Re}\, s > d$.

For a finite-dimensional spectral triple, it makes sense the following definition:

Definition A.1. *A spectral triple has* dimension spectrum Σ *iff* $\Sigma \subset \mathbb{C}$ *is a countable set and for all* $T \in V$, $\zeta_T(s)$ *extends to a meromorphic function on* $\mathbb{C}$ *with poles in* Σ *as unique singularities.*

Residues of zeta-type functions are traces on the algebra, and are used to compute the pairing between K-theory and cyclic-cohomology.
Denoting $F = \mathrm{sign}\, D$, the bounded commutator condition implies that $[F, x]$ is compact for all $x \in A$, and then $(A, \mathcal{H}, F)$ is a Fredholm module. Recall that a Fredholm module is *finite summable* if for k sufficiently large, $[F, a_0] \dots [F, a_k]$ is traceclass for all $a_j \in A$.

From now on, we suppose the triple is even and denote by γ the grading. If the associated Fredholm module is finite summable, the map

$$\mathrm{ch}^F_n(a_0, \dots, a_n) = \frac{\Gamma(\frac{n}{2}+1)}{2n!}\mathrm{Trace}(\gamma F[F, a_0] \dots [F, a_n]) \tag{16}$$

defines a periodic cyclic cocycle ch^F (with all components equal to zero, but the nth that is ch^F_n) whose periodic cyclic cohomology class is independent of n, for all n even and sufficiently large. The pairing of ch^F with K-theory gives the index map.

If $\phi = (\phi_0, \phi_2, ...)$ is an element of the periodic cyclic cohomology group PHC^{ev} (only a finite number of components are different from zero) and $p \in \mathrm{Mat}_\infty(A)$ a projector, the explicit formula for the pairing is

$$\langle \phi, [p] \rangle := \phi_0(p) + \sum_{k\in\mathbb{Z}_+} (-1)^k \frac{(2k)!}{k!} \phi_{2k}(p - \tfrac{1}{2}, p, \dots, p)$$

and the index formula states that $\big\langle \mathrm{ch}^F, [p] \big\rangle$ is the index of the twisted Dirac operator pDp (or equivalently, pFp). In the case $\phi = \mathrm{ch}^F$, there is only one non-zero component.

There exists a general theorem relating the index to residues of zeta-type functions, which we quote in the case we are interested in:

Theorem A.2 (Connes-Moscovici [5, thm. II.3]). *Let* $(A, \mathcal{H}, D)$ *be a regular, even spectral triple (with finite metric dimension* d*), with dimension spectrum* Σ *made of simple poles. Then, the following formulas define a* (b, B)*-cocycle with the same cyclic cohomology class of the Chern character* ch^F *(*n *even* $\leq d$*):*

$$\varphi_0(a_0) = \mathrm{Res}_{s=0} s^{-1} \mathrm{Trace}(\gamma a_0 |D|^{-2s}) \ ,$$

$$\varphi_n(a_0, ..., a_n) = \sum_{k\in\mathbb{N}^{\times n}} \frac{(-1)^k}{k_1!...k_n!} \alpha_k \mathrm{Res}_{s=0} \mathrm{Trace}\Big(\gamma a_0 [D, a_1]^{(k_1)} ... [D, a_n]^{(k_n)} |D|^{-2(|k|+\frac{n}{2}+s)}\Big) \ .$$

Here $T^{(j+1)} = [D^2, T^{(j)}]\ \forall\, j \in \mathbb{N}$, $T^{(0)} = T$ *and* $\alpha_k^{-1} = (k_1 + 1)(k_1 + k_2 + 2)...(k_1 + ... + k_n + n)$.

If T is an order-zero operator, $T^{(j)}$ is of order j. Then, each residue of the previous theorem can be written as a residue in $s = |k| + n$ of the zeta-type function associated to a suitable order-zero operator. By definition, it is zero if $|k| + n \notin \Sigma$. The finiteness of the metric dimension guarantee that all terms in the sum with $|k| + n > d$ are zero, hence it is a finite sum. Moreover, if $0 \notin \Sigma$, then $\varphi_0(a_0) = \mathrm{Trace}(\gamma a_0 |D|^{-2s})\big|_{s=0}$.
All the terms φ_n with $n > 0$ are local (i.e. they don't care of traceclass contributions). The only non-local term is φ_0.

The theorem is particularly interesting when A is a pre-C^*-algebra, since in this case A has the same K-theory as its C^*-algebra completion.

B The projective module $L^2(S^2_q)p$

Let us return to the projector p in (3). The range of p is in the kernel of a^*: $a^*p = a^* - a^* = 0$. Notice also that $b^2 p = q^4(1 - aa^*)p = q^4 p$.

Let us construct the $C(S^2_q)$-module $C(S^2_q)p$. A linear basis is $\{a^n b^m p, (a^*)^{n+1} b^n p,\ n, m \in \mathbb{N}\}$, but $(a^*)^{n+1} b^n p \propto b^n (a^*)^{n+1} p = 0$. Moreover $a^n b^m p$ is proportional to $a^n p$ or $a^n bp$ depending on the parity of m (since $b^{2n} p = q^{4n} p$). Then a minimal linear basis can be taken as

$$|n\rangle_0 = a^{n-1} p \ , \qquad |n_1\rangle = a^{n-1} bp \ ,$$

with $n \in \mathbb{Z}_+$. It is immediate to compute:

$$\begin{cases} a\,|n\rangle_0 = |n+1\rangle_0 \\ a\,|n\rangle_1 = |n+1\rangle_1 \end{cases} , \qquad \begin{cases} a^*\,|n\rangle_0 = (1 - q^{4(n-1)})\,|n-1\rangle_0 \\ a^*\,|n\rangle_1 = (1 - q^{4(n-1)})\,|n-1\rangle_1 \end{cases} , \qquad \begin{cases} b\,|n\rangle_0 = q^{2(n-1)}\,|n\rangle_1 \\ b\,|n\rangle_1 = q^{2(n+1)}\,|n\rangle_0 \end{cases} .$$

We compute the inner product imposing that a and a^* are hermitian conjugates. It easily follows that $|n\rangle_s$ are orthogonal. We compute their norm imposing (omitting the subscript $0, 1$):

$$\delta_{m,n+1}\,\langle n+1|n+1\rangle = \langle m|a|n\rangle = (a^*\,|m\rangle)^\dagger\,|n\rangle = (1 - q^{4n})\delta_{m,n+1}\,\langle n|n\rangle \ .$$

So $c_n = \langle n|n\rangle$ satisfies $c_{n+1} = (1 - q^{4n})c_n$ and the (positive) solution is

$$c^{0,1}_{n+1} = (1 - q^{4n})(1 - q^{4(n-1)}) \dots (1 - q^4)c^{0,1}_0 \ .$$

Now we can complete the space in this norm and obtain an Hilbert space.

We define the basis:

$$|n\rangle_\pm = \frac{1}{\sqrt{2c^0_n}}\,|n\rangle_0 \pm \frac{1}{\sqrt{2c^1_n}}\,|n\rangle_1 \ , \qquad c^{0,1}_0 \in \mathbb{R}^+ \ .$$

If we fix $c^0_0 = 1$ and $c^1_0 = q^4$, in this (orthonormal) basis

$$a\,|n\rangle_\pm = \sqrt{1 - q^{4n}}\,|n+1\rangle_\pm \ , \qquad a^*\,|n\rangle_\pm = \sqrt{1 - q^{4(n-1)}}\,|n-1\rangle_\pm \ , \qquad b\,|n\rangle_\pm = \pm q^{2n}\,|n\rangle_\pm \ .$$

Then, the module associated with p is equivalent to the representation $\mu = \mu_+ \oplus \mu_-$ over $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, discussed in section 2. It has not a $q \to 1$ limit (there are no scalar projectors in $C(S^2)$).

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