Vanishing viscosity solutions of a 2×2 triangular hyperbolic system with Dirichlet conditions on two boundaries

Laura V. Spinolo

S.I.S.S.A., Via Beirut 4, Trieste 34014, Italy

e-mail: spinolo@sissa.it

Abstract

We consider the 2×2 parabolic systems

$$u^\varepsilon_t + A(u^\varepsilon) u^\varepsilon_x = \varepsilon u^\varepsilon_{xx}$$

on a domain $(t, x) \in]0, +\infty[\times]0, l[$ with Dirichlet boundary conditions imposed at $x = 0$ and at $x = l$. The matrix A is assumed to be in triangular form and strictly hyperbolic, and the boundary is not characteristic, i.e. the eigenvalues of A are different from 0.

We show that, if the initial and boundary data have sufficiently small total variation, then the solution u^ε exists for all $t \geq 0$ and depends Lipschitz continuously in L^1 on the initial and boundary data.

Moreover, as $\varepsilon \to 0^+$, the solutions $u^\varepsilon(t)$ converge in L^1 to a unique limit $u(t)$, which can be seen as the *vanishing viscosity solution* of the quasilinear hyperbolic system

$$u_t + A(u) u_x = 0, \quad x \in]0, l[.$$

This solution $u(t)$ depends Lipschitz continuously in L^1 w.r.t the initial and boundary data. We also characterize precisely in which sense the boundary data are assumed by the solution of the hyperbolic system.

2000 Mathematics Subject Classification: 35L65.

Key words: Hyperbolic systems, conservation laws, initial boundary value problems, viscous approximations.

1 Introduction

This paper deals with the initial-two-boundaries value problem

$$\begin{cases} u_t + A(u) u_x = 0, \quad x \in]0, l[, \ \ t \in]0, +\infty[\\ u(0, x) = \bar{u}_0(x), \\ u(t, 0) = \bar{u}_{b\,0}(t), \qquad u(t, l) = \bar{u}_{bl}(t). \end{cases} \tag{1.1}$$

The crucial hypotheses we assume are that the matrix A is strictly hyperbolic with eigenvalues different from 0 and that the initial and boundary data are small in BV norm and close to a constant state u^*.

An existence result for hyperbolic boundary value problems was proved in [21] using an adaptation of the Glimm scheme introduced in [20]. Improvements of Goodman's result ([21]) have been obtained by a wave-front tracking technique introduced in [9] and later used in a series of papers ([10, 12, 13, 17, 15, 14, 16]) to establish the well posedness of the Cauchy problem. Such a wave-front tracking technique was adapted to the initial-boundary value problem in [1], where a substantial improvement of Goodman's results ([21]) was achieved. The well posedness of the initial-boundary value problem was then proved in [19] relying on the wave-front tracking technique described in [1].

All the results quoted so far deal with conservative systems; a comprehensive account of the stability and uniqueness results for the Cauchy problem for a systems of conservation laws can be found in [11].

In [4, 5, 6] and [7] a different problem was dealt with: let u^ε be a family of solutions to the parabolic systems

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = \varepsilon u^\varepsilon_{xx}.$$

One expects that as $\varepsilon \to 0^+$ the solution u^ε converges in some sense to a solution of the corresponding hyperbolic system

$$u_t + A(u)u_x = 0.$$

The mathematical proof of this convergence was obtained via a suitable decomposition of the gradient of the solution u^ε along travelling waves. We refer to [7] for an account of the proof of the convergence of the vanishing viscosity approximation and of the uniqueness and the stability of the vanishing viscosity limit: it is important to underline, however, that in the paper the systems considered are not necessarily conservative.

Vanishing viscosity solutions to initial-one-boundary value problems were studied in [3]: it is proved the convergence of the approximated solutions and the stability and the uniqueness of the limit. In [3] the boundary characteristic case is allowed (i.e. one characteristic field is allowed to have speed close to that of the boundary) and the crucial tool in the proof of the convergence and the stability is the introduction of a suitable decomposition of the gradient of the vanishing viscosity solution.

In the present paper we will consider the vanishing viscosity approximation for the initial-two-boundaries value problem:

$$\begin{cases} u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = \varepsilon u^\varepsilon_{xx}, \quad x \in]0,\, l[, \quad t \in]0, +\infty[\\ u^\varepsilon(0,x) = \bar{u}_0(x), \\ u^\varepsilon(t,0) = \bar{u}_{b\,0}(t), \qquad u^\varepsilon(t,l) = \bar{u}_{bl}(t). \end{cases} \tag{1.2}$$

We will assume that A is in triangular form, i.e.

$$A(u) = \begin{pmatrix} \lambda_1(u_1) & 0 \\ g(u_1,\, u_2) & \lambda_2(u_1,\, u_2) \end{pmatrix}, \tag{1.3}$$

and sufficiently smooth in a compact neighborhood K of a fixed point u^*. Moreover, we assume A to be uniformly strictly hyperbolic, i.e. there exists a constant $c > 0$ ($2c$ is then the "separation speed"), such that

$$\lambda_1(u) < -c < 0 < c < \lambda_2(u) \quad \forall\, u \in K. \tag{1.4}$$

The above condition means that the speed of the boundary (in our case 0) is strictly different from the characteristic speeds of the two families of waves.

We denote with $r_1(u)$ the first eigenvector of $A(u)$, corresponding to the eigenvalues $\lambda_1(u)$, and with r_2 the second one. Due to the particular structure of A, we normalize r_1 and r_2 as

$$\langle (1,0), r_1(u) \rangle = 1, \quad r_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.5}$$

The dual base of $(r_1(u), r_2)$ is denoted by (ℓ_1, ℓ_2).

We will assume that the initial data $\bar{u}_0$ and boundary data $\bar{u}_{b0}$, $\bar{u}_{bl}$ have sufficiently small total variation, i.e.

$$\text{Tot Var}(\bar{u}_0),\ \text{Tot Var}(\bar{u}_{b\,0}),\ \text{Tot Var}(\bar{u}_{b\,l}) \leq \delta_1 \tag{1.6}$$

for a suitable $\delta_1 << 1$. Moreover, since we will study boundary layers with small total variation, we assume that there exists a value u^* such that

$$\|\bar{u}_0 - u^*\|_\infty \leq \delta_1 \quad \|\bar{u}_{b\,0} - u^*\|_\infty \leq \delta_1 \quad \|\bar{u}_{b\,l} - u^*\|_\infty \leq \delta_1. \tag{1.7}$$

For technical reasons, we will also assume some stronger regularity: the boundary and initial data will be sufficiently smooth and will satisfy

$$\|d^j \bar{u}_0/dx^j\|_{L^1},\ \|d^j \bar{u}_{b\,0}/dt^j\|_{L^1},\ \|d^j \bar{u}_{b\,l}/dt^j\|_{L^1} \leq M < +\infty \quad j = 2, \dots n, \tag{1.8}$$

for some $n \in \mathbb{N}$ and large constant M. Some observations about the extension of our results to the case of boundary and initial data with weaker regularity will be made in Remark 1.1.

We will denote by $\mathcal{U}_0$, $\mathcal{U}_b$ the set of functions u_0, u_b satisfying (1.6), (1.7), (1.8) in $]0, l[$ or $]0, +\infty[$, respectively. We also define the sets $\mathcal{D}_0 \subseteq L^1(0, l)$, $\mathcal{D}_b \subseteq L^1_{loc}(0, +\infty)$ of functions such that

$$\text{Tot Var}\Big\{\bar{u}_0\Big\} \leq \delta_1, \quad \text{Tot Var}\Big\{\bar{u}_b\Big\} \leq \delta_1, \tag{1.9}$$

respectively.

The first theorem concerns the existence of a solution to the parabolic problem (1.2); moreover, it ensures that such a solution satisfies stability estimates independent on ε.

Theorem 1.1. *Suppose $\bar{u}_0 \in \mathcal{U}_0$, $\bar{u}_{b\,0}$, $\bar{u}_{b\,l} \in \mathcal{U}_b$ and A is of the form (1.3) and satisfies (1.4). Then, for any $\varepsilon > 0$, the system (1.2) has a unique solution $u^\varepsilon(t)$ defined for all $t \geq 0$.*

This solution depends Lipschitz continuously in L^1 on the initial and boundary data: indeed, let $\bar{v}_0 \in \mathcal{U}_0$, $\bar{v}_{b\,0}$, $\bar{v}_{b\,l} \in \mathcal{U}_b$ be the initial and boundary data of a solution $v^\varepsilon(t)$ of (1.2). Then for some constants L_1 and L_2, depending only on the matrix A and the bound on the initial and boundary data δ_1, the following holds:

$$\begin{aligned} \|v^\varepsilon(t) - u^\varepsilon(t)\|_{L^1} \leq L_1 \Big(&\|\bar{v}_0 - \bar{u}_0\|_{L^1(0,\,l)} + \|\bar{v}_{b0} - \bar{u}_{b0}\|_{L^1(0,\,+\infty)} + \|\bar{v}_{bl} - \bar{u}_{bl}\|_{L^1(0,\,+\infty)} \Big) \\ &+ L_2 \Big(|t - s| + |\sqrt{t} - \sqrt{s}| \Big). \end{aligned} \tag{1.10}$$

The second theorem concerns the limit as $\varepsilon \to 0^+$. Since we have a uniform bound on the total variation, by Helly's theorem there is a subsequence of u^ε converging in L^1 to a limit function $u(t)$ on a countable dense set of times t_n. By the stability estimate (1.10), the convergence is on the whole $\mathbb{R}^+$.

However, different subsequences could a priori converge to different limits: we will actually prove that the limit is unique and that moreover the semigroup property holds.

Theorem 1.2. *As $\varepsilon \to 0^+$, the sequence $u^\varepsilon(t)$ of solutions of (1.2) converges to a unique function $u(t)$ for all $t \geq 0$: we denote such a limit by*

$$u(t) = p_t[\bar{u}_0,\, \bar{u}_{b\,0},\, \bar{u}_{b\,l}].$$

This convergence defines a unique semigroup

$$\begin{aligned} S : [0,\, +\infty] \times \mathcal{U}_0 \times \mathcal{U}_b \times \mathcal{U}_b &\to \mathcal{D}_0 \times \mathcal{U}_b \times \mathcal{U}_b \\ (t,\, u_0,\, u_{b\,0},\, u_{b\,l}) \qquad &\mapsto \Big(p_t[u_0,\, u_{b\,0},\, u_{b\,l}],\, u_{b\,0}(\,\cdot\, + t),\, u_{b\,l}(\,\cdot\, + t) \Big) \end{aligned} \tag{1.11}$$

which satisfies the following stability estimates in $L^1(0, l)$:

$$\begin{aligned} \Big\| p_t[\bar{u}_0,\, \bar{u}_{b\,0},\, \bar{u}_{b\,l}] - p_s[\bar{v}_0,\, \bar{v}_{b\,0},\, \bar{v}_{b\,l}] \Big\|_{L^1} \leq & L_1 \Big(\|\bar{v}_0 - \bar{u}_0\|_{L^1(0,\,l)} + \|\bar{v}_{b0} - \bar{u}_{b0}\|_{L^1(0,\,+\infty)} \\ & + \|\bar{v}_{bl} - \bar{u}_{bl}\|_{L^1(0,\,+\infty)} \Big) + L_2 |t - s|, \end{aligned} \tag{1.12}$$

for some constant L_1, L_2 depending only on A and on δ_1.

Remark 1.1. By the stability estimate (1.12) the semigroup S defined by (1.11) can be extended to initial and boundary data that satisfy much weaker regularity assumptions, i.e. $\bar{u}_0 \in \mathcal{D}_0$ and $\bar{u}_{b\,0},\ \bar{u}_{b\,l} \in \mathcal{D}_b$. Indeed, let $\{\rho_k\}$ be a sequence of regularizing kernels and let $\bar{u}_0 \in \mathcal{D}_0$. Then $\rho_k * \bar{u}_0$, $\rho_k * \bar{u}_{b\,0}$ and $\rho_k * \bar{u}_{b\,l}$ are initial and boundary data that satisfy the hypothesis (1.8): they are smooth and

$$\|d(\rho_k * \bar{u}_0)/dx\|_{L^1} \leq \mathrm{Tot\,Var}\big\{\bar{u}_0\big\} \leq \delta_1 \qquad \|d(\rho_k * \bar{u}_{b\,0})/dx\|_{L^1} \leq \delta_1 \qquad \|d(\rho_k * \bar{u}_{b\,l})/dx\|_{L^1} \leq \delta_1$$

$$\Big\|d^j(\rho_k * \bar{u}_0)/dx^j\Big\|_{L^1} = \Big\|d\Big((d^{j-1}\rho_k/dx^{j-1}) * \bar{u}_0\Big)/dx\Big\|_{L^1} \leq M(k,\,j)\delta_1 \qquad j = 1, \dots n$$

$$\Big\|d^j\Big(\rho_k * \bar{u}_{b\,0}\Big)/dx^j\Big\|_{L^1} \leq M(k,\,j)\delta_1 \qquad \Big\|d^j\Big(\rho_k * \bar{u}_0)\Big)/dx^j\Big\|_{L^1} \leq M(k,\,j)\delta_1 \qquad j = 1, \dots n.$$

The last estimates ensures that, for any fixed k, the L^1 norm of the derivatives is finite: the bound is not uniform with respect to k but, since the constant L_1 in (1.12) does not depend on the bound M in (1.8), it is enough to prove the extendibility of the semigroup to the whole domain $\mathcal{D}_0$. Indeed, let u^ε_k the sequence of solutions to the systems

$$\begin{cases} \Big(u^\varepsilon_k\Big)_t + A(u^\varepsilon_k)\Big(u^\varepsilon_k\Big)_x = \varepsilon\Big(u^\varepsilon_k\Big)_{xx} \\ u^\varepsilon_k(0,\,x) = \rho_k * \bar{u}_0 \\ u^\varepsilon_k(t,\,0) = \rho_k * \bar{u}_{b\,0} \quad u^\varepsilon_k(t,\,l) = \rho_k * \bar{u}_{b\,l} \end{cases}$$

Theorem 1.2 ensures that, for any $k \in \mathbb{N}$ and for any $t \geq 0$, the sequence $u^\varepsilon_k(t)$ converges as $\varepsilon \to 0^+$ to some limit function we will call $u_k(t)$. Then $u_k(t)$ is a Cauchy sequence since by (1.12)

$$\|u_k(t) - u_h(t)\|_{L^1(0,\,l)} \leq L_1\Big(\|(\rho_k - \rho_h) * \bar{u}_0\|_{L^1(0,\,l)} + \|(\rho_k - \rho_h) * \bar{u}_{b\,0}\|_{L^1(0,\,+\infty)} + \|(\rho_k - \rho_h) * \bar{u}_{b\,l}\|_{L^1(0,\,+\infty)}\Big)$$

The same estimate (1.12) ensures that the limit $\lim_{k \to +\infty} u_k(t)$ does not depend on the choice of the sequence ρ_k and therefore the extension

$$p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}] = \lim_{k \to +\infty} u^k(t)$$

is well defined.

For simplicity, in the following we won't prove that, if $(\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l})$ belongs to $\mathcal{D}_0 \times \mathcal{D}_b \times \mathcal{D}_b$ but not to $\mathcal{U}_0 \times \mathcal{U}_b \times \mathcal{U}_b$, then the solution of the system (1.2) converges as $\varepsilon_n \to 0^+$ to $p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}]$. However, we will exploit the extendibility property described before, in particular in Section 6.1 we will consider the *vanishing viscosity solution* of the Riemann and of the boundary Riemann problem, actually meaning the *extension of the semigroup of the vanishing viscosity solution* to piecewise constant initial and boundary data.

The function $u(t) = p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}]$ is the *vanishing viscosity solution* to

$$u_t + A(u)u_x = 0. \tag{1.13}$$

Note that it is not a weak solution, unless the system is conservative, but one can prove that it is a *viscosity solution*, in the sense of [2]. In particular, we obtain that, for a.e. t, the limits

$$\lim_{x \to 0^+} u(t,\,x) = u(t,\,0^+), \qquad \lim_{x \to l^-} u(t,\,x) = u(t,\,l^-) \tag{1.14}$$

and the boundary data $\bar{u}_{b\,0}(t)$, $\bar{u}_{b\,l}(t)$ can be connected by boundary profiles, i.e. there exists a solution of the boundary value problem

$$\begin{cases} A(v)v_x = v_{xx}, & x \in [0,\,+\infty[\\ v(0) = \bar{u}_{b\,0}(t), & \lim_{x \to +\infty} v(x) = u(t,\,0^+) \end{cases} \qquad \text{and} \qquad \begin{cases} A(v)v_x = v_{xx}, & x \in]-\infty,\,0] \\ v(0) = \bar{u}_{b\,l}(t), & \lim_{x \to -\infty} v(x) = u(t,\,l^-) \end{cases}$$

respectively. This means that the boundary datum $\bar{u}_{b\,0}$ lies on the stable manifold of $u(t,\,0+)$, and the boundary datum $\bar{u}_{b\,l}$ lies on the unstable manifold of $u(t,\,l^-)$.

The paper is organized as follows.

First of all we make a change of variables in (1.2): let $u(x,\,t) := u^\varepsilon(x/\varepsilon,\,t/\varepsilon)$. Then (1.2) is equivalent to the system

$$\begin{cases} u_t + A(u)u_x = u_{xx}, \quad x \in]0,\,L[,\ t \in]0, +\infty[\\ u(0,x) = u_0(x), \\ u(t,0) = u_{b\,0}(t), \qquad u(t,L) = u_{bL}(t) \end{cases} \tag{1.15}$$

where $L = l/\varepsilon,\ u_{b\,0}(t) = \bar{u}_{b\,0}(t/\varepsilon),\ u_{b\,L}(t) = \bar{u}_{b\,l}(t/\varepsilon),\ u_0(x) = \bar{u}_0(x/\varepsilon)$. One can easily check that

$$\mathrm{Tot\,Var}\big\{\bar{u}_{b\,0}\big\} = \mathrm{Tot\,Var}\big\{u_{b\,0}\big\} \le \delta_1 \qquad \mathrm{Tot\,Var}\big\{\bar{u}_{b\,l}\big\} = \mathrm{Tot\,Var}(u_{b\,L}) \le \delta_1$$
$$\mathrm{Tot\,Var}\big\{\bar{u}_0\big\} = \mathrm{Tot\,Var}\big\{u_0\big\} \le \delta_1.$$

Moreover, the derivatives of the boundary and initial data satisfy

$$\|d^j u_0/dx^j\|_{L^1},\ \|d^j u_{b\,0}/dt^j\|_{L^1},\ \|d^j u_{b\,L}/dt^j\|_{L^1} \le M\varepsilon^{j-1} < \delta_1 \quad j = 2, \ldots n \tag{1.16}$$

for ε small enough.

The crucial tool in the proof of the convergence of the solution of (1.15) as the scaling parameter $\varepsilon \to 0^+$ is Helly's theorem. One needs therefore to prove a uniform bound on the total variation, independent on length of the interval L and the L^1 norm of the boundary and initial data.

In Section 2 we prove a priori bounds on the solution of (1.15) that ensure the local existence and smoothness of solution. Moreover, we will show that, as long as the total variation of the solution remains small, the L^1 norm of u_{xx} is small too and the solution itself can be prolonged in time. The proof is based on the following observation: (1.15) can be seen as a perturbed heat equation and therefore one is led to introduce suitable convolution kernels. Since the technique used in this section does not depend on the dimension of the solution u, we perform the computations for the $n \times n$ system.

In Section 3 we introduce the crucial tool in the proof of the BV estimates: a suitable decomposition of the gradient of the solution. In the boundary free case [7], the gradient u_x is decomposed along a suitable set of unit vectors $\tilde{r}_i$, $i = 1, \ldots, n$, which correspond to the tangent vectors of the travelling wave profiles of

$$u_t + A(u)u_x = u_{xx}.$$

In the single boundary case [3], instead, the gradient u_x is decomposed along n travelling wave profiles (the same as in the boundary free case) and along a boundary profile, i.e. a solution to the stationary system

$$u_{xx} = A(u)u_x.$$

Such a boundary profile lays on a manifold whose dimension is related to the number of negative eigenvalues of $A(u)$, i.e. to the number of characteristic fields that leave the domain $x > 0$.

In our case, the basic idea is to split the part of the gradient due to the presence of the initial datum from the part due to the boundary data: the first part will be decomposed along the same tangent vectors $\tilde{r}_1$, $\tilde{r}_2$ to travelling wave profiles introduced in [7]. Moreover, following the same ideas as in [3], in order to decompose the part of the gradient due to the boundary data we use double boundary profiles, i.e. suitable solutions of the stationary system

$$\begin{cases} u_x = p, \\ p_x = A(u)p. \end{cases} \tag{1.17}$$

In the linear case the two components of the system (1.17) are decoupled and one can show that there is a solution of the boundary value problem

$$\begin{cases} u_x = p, \\ p_x = A(u)p, \\ u(0) = U_{b\,0}, \quad u(L) = U_{b\,L} \end{cases} \tag{1.18}$$

with total variation uniformly bounded with respect to L.

In the general case, the idea is to emulate the linear case, using the center-stable manifold theorem coupled with a contraction mapping argument: one finds that, provided the difference $|U_{b\,0} - U_{b\,L}|$ is small, there is a solution of (1.18) with uniformly bounded total variation. Such a solution can be seen as the sum of two components, one exponentially decreasing as $x \to +\infty$, the other as $x \to -\infty$: we will denote by $\hat{r}_1$ and $\hat{r}_2$ the tangent vectors to the first and the second part respectively. It is important to underline, however, that in the non linear case the two components are coupled: indeed, one finds that $\hat{\lambda}_1$, the speed of exponential decay of the first component, depends also on the second component, and viceversa $\hat{\lambda}_2$ depends on the first component. The introduction of the generalized eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ allows the equations satisfied by the components of the decomposition to be exactly in conservation form.

The decomposition of the gradient along travelling waves profiles and double boundary layers takes the form

$$u_x = v_1 \tilde{r}_1 + v_2 \tilde{r}_2 + p_1 \hat{r}_1 + p_2 \hat{r}_2. \tag{1.19}$$

In Section 3.1 we will show that, because of the triangular structure of the matrix A, the vector $\tilde{r}_2$ and $\hat{r}_2$ can chosen to be identically equal to $r_2 = (0,\, 1)$ and $\hat{\lambda}_1$ is identically equal to λ_1.

Note that (1.19) is a system of 2 equations in 4 unknowns: this allows some freedom in choosing in the most suitable way the boundary and initial conditions. The precise expression of all the boundary and Cauchy data we will impose on v_1, v_2, p_1 and p_2 can be found in Section 3.3, in the following however we will sketch the crucial ideas involved in the choice of those conditions.

Since p_1 and p_2 are the component of u_x along double boundary profiles, we don't want them to be influenced by the initial datum, and hence we impose

$$p_1(0,\, x) \equiv 0 \qquad p_2(0,\, x) \equiv 0.$$

Moreover, p_1 is the exponential decreasing component of the boundary profile and hence it should not be affected too much by the datum on the boundary $x = L$: more precisely, since the goal is to establish a uniform bound on the L^1 norm of p_1, it seems reasonable to look for some boundary condition that minimizes the increment of $\|p_1\|_{L^1(0,\,L)}$ due to the datum on the boundary $x = L$. An integration by parts ensures that

$$\frac{d}{dt} \int_0^L |p_1(t,\, x)| \leq |p_{1x} - \lambda_1 p_1|(t,\, L) + |p_{1x} - \lambda_1 p_1|(t,\, 0)$$

and therefore we will impose

$$|p_{1x} - \lambda_1 p_1|(t,\, L) \equiv 0$$

and, by analogous considerations,

$$|p_{2x} - \hat{\lambda}_2 p_2|(t,\, 0) \equiv 0.$$

On the other hand, v_1 and v_2 are the components of u_x along travelling profiles and therefore we don't want them to be strongly influenced by the presence of the boundary data. We observe that, in the hyperbolic limit

$$u_t + A(u) u_x = 0,$$

the waves of the first family go out from the domain through the boundary $x = 0$: we would like to emulate such a behavior in the parabolic approximation. More precisely, since the aim is to show a uniform bound on the L^1 norm of v_1, we look for some boundary condition that ensures that the derivative of the wave in the parabolic approximation crosses the boundary, as in the hyperbolic limit. To make the situation clearer, it is useful to consider the simple examples that follow: consider the linear scalar equation

$$z_t + \lambda_1^* z_x - z_{xx} = 0 \tag{1.20}$$

with some Dirichlet condition imposed on the boundaries $x = 0$ and $x = L$, for example

$$z(t,\, 0) \equiv 0, \qquad z(t,\, L) \equiv 1. \tag{1.21}$$

Moreover, let $z^D(t, x)$ be a solution of (1.20) and (1.21): the initial condition is not important at the moment, but suppose for simplicity that $\mathrm{Tot\,Var}\big\{z^D(t, 0)\big\} = 1$. For sure $\mathrm{Tot\,Var}\big\{z^D(t)\big\} \geq 1$ and hence the derivative of z^D cannot cross the boundary $x = 0$, or at least the loss of total variation that occur at $x = 0$ has to be compensated by an increase at $x = 0$.

On the other hand, let $z^N(t, x)$ be a solution of (1.20) that satisfy a homogeneous Neumann condition at $x = 0$, for example

$$z^N_x(t, 0) \equiv 0, \qquad z^N(t, L) \equiv 1,$$

then an integration by parts ensures that

$$\frac{d}{dt}\int_0^L |z^N_x(t, x)|dx \leq -|u^N_{xx}(t, 0)|,$$

and hence the total variation of z^N is flowing out from the domain through the boundary $x = 0$.

Hence we are are led by the previous considerations to impose on the boundary $x = 0$ a homogeneous Dirichlet condition on the function v_1, which corresponds to the derivative of a travelling wave of the first family:

$$v_1(t, 0) \equiv 0.$$

The considerations that motivate the choice

$$v_2(t, L) \equiv 0$$

are completely analogous.

Besides that in the choice of the boundary conditions, some freedom is also allowed in the attribution of the source terms: indeed, if one inserts (1.19) in the system

$$u_t + A(u)u_x - u_{xx} = 0$$

obtains the equations

$$\begin{aligned} &v_{1t} + (\lambda_1 v_1)_x - v_{1xx} + p_{1t} + (\lambda_1 p_1)_x - p_{1xx} = 0 \\ &v_{2t} + (\lambda_2 v_2)_x - v_{2xx} + p_{2t} + (\hat{\lambda}_2 p_2)_x - p_{2xx} = \tilde{s}_1(t, x). \end{aligned}$$

for some function $\tilde{s}_1$ whose exact expression can be found in the Appendix A.2.1 and is not important at the moment: however, it is crucial to observe that it is identically zero when the solution is exactly a travelling wave or a double boundary profile. Moreover, in general such a source term is spread on the whole interval $]0, L[$: since p_2, the part of the double boundary layer exponentially decaying as $x \to -\infty$, should be affected only by the datum in $x = L$, it seems reasonable to impose

$$\begin{aligned} &v_{1t} + (\lambda_1 v_1)_x - v_{1xx} = 0 \qquad && p_{1t} + (\lambda_1 p_1)_x - p_{1xx} = 0 \\ &v_{2t} + (\lambda_2 v_2)_x - v_{2xx} = \tilde{s}_1(t, x) \qquad && p_{2t} + (\hat{\lambda}_2 p_2)_x - p_{2xx} = 0 \end{aligned} \tag{1.22}$$

In Section 4 we exploit the decomposition (1.19) to prove that the total variation is uniformly bounded by $\mathcal{O}(1)\delta_1$. As we will see, the crucial point is to prove that, if $\mathrm{Tot\,Var}\big\{u_x(\sigma)\big\} \leq \mathcal{O}(1)\delta_1$ for all $t \leq t$, it holds an estimate of order two on the integrals of the source term:

$$\int_0^t \int_0^L |\tilde{s}_1(\sigma, x)|dxd\sigma \leq \mathcal{O}(1)\delta_1^2. \tag{1.23}$$

To show (1.23) we will basically deal with each of the term that appear in the expression of $\tilde{s}_1$ separately. Some of the estimates are based on the same techniques described in [7]: in particular we will use the interaction, area and length functional introduced in the boundary free case. Some estimates, on the other hand, require quite long computations and can be found in the appendix.

In Section 5 we will prove the stability of the vanishing viscosity approximation with respect to L^1 perturbations. More precisely, let $u_0,\ u_{b\,0},\ u_{b\,L}$ and $v_0,\ v_{b\,0},\ v_{b\,L}$ be the initial and boundary data of two solutions u and v of problem (1.15): we will show that there exists a constant L_1 such that

$$\|u(t)-v(t)\|_{L^1(0,L)} \leq L_1\Big(\|u_0-v_0\|_{L^1(0,\,L)}+\|u_{b0}-v_{b0}\|_{L^1(0,\,t)}+\|u_{bL}-v_{bL}\|_{L^1(0,\,t)}\Big).$$

Moreover, one has also stability with respect to time: if u is a solution to (1.15) then

$$\|u(t)-u(s)\|_{L^1} \leq L_2\big(|t-s|+|\sqrt{t}-\sqrt{s}|\big)$$

for a suitable constant L_2. We will see that the constants L_1 and L_2 depend uniquely on the matrix A and on the bound δ_1 on the total variation of the initial and boundary data. We will actually give just a sketch of the proof of the stability, since we will show that one can employ the same tools used to prove the BV estimates and repeat with minor changes the computations of Section 4.

One can then get back to the solution u^ε of the original problem (1.2) and obtain that for all $\varepsilon>0$ it satisfies

$$\begin{aligned}
&\mathrm{Tot\,Var}\big\{u^\varepsilon(t)\big\}\leq \mathcal{O}(1)\delta_1 \quad \forall\, t>0 \qquad \|u^\varepsilon(t)-u^*\|_\infty \leq \mathcal{O}(1)\delta_1 \quad \forall\, t>0\\
&\|u^\varepsilon(t)-v^\varepsilon(t)\|_{L^1(0,\,L)} \leq L_1(\|\bar{u}_0-\bar{v}_0\|_{L^1(0,\,L)}+\|\bar{u}_{b0}-\bar{v}_{b0}\|_{L^1(0,\,t)}+\|\bar{u}_{b0}-\bar{v}_{bL}\|_{L^1(0,\,t)})\\
&\|u^\varepsilon(t)-u^\varepsilon(s)\|_{L^1} \leq L_2\big(|t-s|+\sqrt{\varepsilon}\,|\sqrt{t}-\sqrt{s}|\big).
\end{aligned} \tag{1.24}$$

In the last estimate, $\bar{u}_0,\ \bar{u}_{b0}\ \bar{u}_{bL}$ and $\bar{v}_0,\ \bar{v}_{b0}\ \bar{v}_{bL}$ are the initial and boundary data for two solutions u^ε and v^ε of (1.2).

The uniform bound on the total variation of the solutions u^ε of (1.2) ensures that for any $(\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l})\in\mathcal{U}_0\times\mathcal{U}_b\times\mathcal{U}_b$, for any $t>0$ and $\varepsilon_n\to 0^+$ there is a subsequence ε_{n_k} such that $u^{\varepsilon_{n_k}}(t)$ converges in $L^1(0,\,l)$ to some limit function we will denote by $p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}]$. Letting $\varepsilon\to 0^+$ in (1.24) one finds that the limit satisfies the stability estimate

$$\begin{aligned}
\Big\|p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}]-p_s[\bar{v}_0,\,\bar{v}_{b\,0},\,\bar{v}_{b\,l}]\Big\|_{L^1} \leq & L_1\bigg(\|\bar{v}_0-\bar{u}_0\|_{L^1(0,\,l)}+\|\bar{v}_{b0}-\bar{u}_{b0}\|_{L^1(0,\,+\infty)}+\\
&+\|\bar{v}_{bl}-\bar{u}_{bl}\|_{L^1(0,\,+\infty)}\bigg)+L_2|t-s|.
\end{aligned} \tag{1.25}$$

By a standard diagonalization procedure one can show that there is a subsequence that converges for any rational time t and for any $(\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l})$ in a countable dense set of $\mathcal{U}_0\times\mathcal{U}_b\times\mathcal{U}_b$; the density is here intended in the L^1 norm. Then by the estimates (1.25) $p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}]$ must be defined on close sets of times and boundary and initial data. Hence $p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}]$ is defined for any $t\geq 0$ and for all $(\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l})\in\mathcal{U}_0\times\mathcal{U}_b\times\mathcal{U}_b$.

One can actually check that the operator

$$\begin{aligned}
S:[0,\,+\infty]\times\mathcal{U}_0\times\mathcal{U}_b\times\mathcal{U}_b &\to \mathcal{D}_0\times\mathcal{U}_b\times\mathcal{U}_b\\
(t,\,\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}) &\mapsto \Big(p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}],\,\bar{u}_{b\,0}(\,\cdot\,+t),\,\bar{u}_{b\,l}(\,\cdot\,+t)\Big)
\end{aligned}$$

satisfies the semigroup property

To complete the proof of Theorem 1.2 one is therefore left to show the uniqueness of the semigroup of vanishing viscosity solutions: indeed, different sequences $u^{\varepsilon_n}(t)$, $u^{\nu_n}(t)$ could a priori converge to different limits.

The proof of the uniqueness of the vanishing viscosity limit can be found in Section 6.4 and, following the same ideas as in [7], the crucial step will be to show that the semigroup defined via vanishing viscosity approximation is actually a *viscosity solution* in the sense of [2].

We refer to Section 6.4 for the precise statement, here however we underline that the definition of viscosity solution is based on local estimates that ensure, roughly speaking, a "good behavior" in comparison with the solutions of a suitable Riemann problem and of a suitable linear problem.

The notion of viscosity solution was first described in the conservative boundary free case in [10] and was strictly connected to the definition of Standard Riemann Semigroup (SRS) that was introduced in the same paper. For completeness, we recall here that a SRS is Lipschitz continuous with respect to the L^1 norm and in the case of piecewise constant initial data locally coincides with the standard Riemann solver defined by Lax in [22]. In [10] it is proved that if a SRS semigroup exists, then it necessarily coincides with the wave-front tracking limit and with the viscosity solution. One of the main advantages one gains introducing the notion of viscosity solution is therefore the characterization of global behaviors through local ones.

The definition of SRS semigroup and of viscosity solution was extended to conservative boundary value problems in [2]. Moreover, in the same paper it was proved that, also in the boundary case, if a SRS exist then it necessarily coincides with the wave-front tracking limit and with the viscosity solution. Hence the uniqueness of the SRS semigroup comes from the uniqueness of the wave-front tracking limit, proved in [19].

From the previous works it is clear that a crucial step in the definition of viscosity solution is the description of the Riemann solver and of the boundary Riemann solver.
As mentioned before, a solution of the Riemann problem in the boundary free case was introduced by Lax ([22]) for conservative systems in the case of linearly degenerate or genuinely non linear fields. Such a definition was then extended by Liu ([23]) to very general conservative systems. The characterization of the Riemann solver for non conservative system was introduced in [7], where it was also proved the effective convergence of the vanishing viscous solutions and it was extended in the natural way the notion of SRS and of viscosity solution. Finally, the Riemann solver for boundary value problems non necessarily in conservation form was first described in [3]; in this paper it was also extended in the natural way the notion of SRS and of viscosity solution.

In Section 6.1 we will describe the Riemann solver and the boundary Riemann solver defined by the vanishing viscosity limit, which however have an interest in their own. The problem dealt with is actually a particular case of the one solved in [3], where also the characteristic case was considered, but since the reduction to our case is not completely trivial, we will describe it explicitly.

In particular, we will consider the vanishing viscosity solution of the boundary Riemann problem

$$\begin{cases} u_t + A(u)u_x = 0 \\ u(t,\, 0) = \bar{u}_b \quad u(0,\, x) = \bar{u}_0. \end{cases}$$

Let $u(0^+) = \lim_{x\to 0^+} u(t,\, x)$ be the trace of the solution on the axis $x = 0$, which does not depend on time since the solution u is self-similar. We will show that there exists a solution of the ODE

$$A(U)U_x = U_{xx} \tag{1.26}$$

such that

$$U(0) = \bar{u}_b, \qquad \lim_{x\to+\infty} U(x) = u(0^+).$$

In other words, the boundary datum $\bar{u}_b$ does not necessarily coincides with the trace $u(0^+)$, but it certainly lays on the stable manifold of $u(0^+)$ with respect to the ODE (1.26).

Remark 1.2. The fact that the bounds on the total variation are uniform with respect to the length L of the interval implies that, for any fixed $\varepsilon > 0$, one can let $L \to +\infty$ in (1.15). Hence, coming back to the original system (1.2) one finds that also the solutions of

$$\begin{cases} u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = \varepsilon u^\varepsilon_{xx}, \quad x \in]0,\, +\infty[, \quad t \in]0, +\infty[\\ \\ u^\varepsilon(0, x) = \bar{u}_0, (x) \\ \\ u^\varepsilon(t, 0) = \bar{u}_{b\,0}(t). \end{cases}$$

have total variation uniformly bounded with respect to ε.

Hence the analysis of the vanishing viscosity approximations of the initial-one-boundary value problem can be deduced as a limit case from the study of the two boundaries case.

2 Parabolic estimates

In this section we will find a representation formula for the solution to (1.15)

$$\begin{cases} u_t + A(u)u_x = u_{xx}, \quad x \in]0, L[, \ t \in]0, +\infty[\\ u(0, x) = u_0(x), \\ u(t, 0) = u_{b\,0}(t), \qquad u(t, L) = u_{bL}(t) \end{cases} \tag{2.1}$$

with initial and boundary data satisfying (1.6), (1.7) and (1.8). The aim is to prove that the solution of (2.1) is regular and that the L^1 norm of the second derivative $\|u_{xx}(t)\|_{L^1(0,L)}$ is bounded, as soon as the total variation of $u(t)$ remains small. We will regard (2.1) as a perturbation of the linear parabolic system with constant coefficients

$$u_t + A^* u_x - u_{xx} = 0. \tag{2.2}$$

Here and in the following we will assume $A^* = A(u^*)$ and $\lambda_i^* = \lambda_i(u^*)$.

2.1 The convolution kernels

The fundamental step is to study the equation (4.1) in the scalar case, because the Green kernel for the general vector case (2.2) follows by using the base of eigenvectors of A^*. Thanks to the linearity, we split the Green kernel of the equation

$$z_t + \lambda_i^* z_x - z_{xx} = 0 \tag{2.3}$$

into 3 parts:

1. $\Delta^{\lambda_i^*}(t, x, y)$ is the solution of (2.3) with zero boundary conditions and initial condition

   $$\Delta^{\lambda_i^*}(0, x, y) = \delta_y \quad y \in]0, L[\,.$$

   This function is given by

   $$\Delta^{\lambda_i^*}(t, x, y) = \Bigg(\sum_{m=-\infty}^{m=+\infty} G(t, x + 2mL - y) - G(t, x + 2mL + y) \Bigg) \phi^{\lambda_i^*}(t, x, y), \tag{2.4}$$

   where $G(t, x) = \big(e^{-x^2/4t}\big)/2\sqrt{\pi t}$ is the standard heat kernel and

   $$\phi^{\lambda_i^*}(t, x, y) = \exp\left(\frac{\lambda_i^*}{2}(x - y) - \frac{(\lambda_i^*)^2}{4} t \right).$$

2. $J^{\lambda_i^*\,0}(t, x)$ is the solution of (2.3)) with zero initial datum and boundary conditions

   $$J^{\lambda_i^* 0}(t, 0) = 1 \quad J^{\lambda_i^* 0}(t, L) = 0. \tag{2.5}$$

   It follows that

   $$J^{\lambda_i^*\,0}(t, x) = A \exp\big(\lambda_i^* x\big) + B - \int_0^L \Delta^{\lambda_i^*}(t, x, y) \Big(A \exp\big(\lambda_i^* y\big) + B \Big) dy, \tag{2.6}$$

   with

   $$A = -\frac{1}{e^{\lambda_i^* L} - 1} \qquad B = \frac{e^{\lambda_i^* L}}{e^{\lambda^* L} - 1}.$$

3. $J^{\lambda_i^* L}(t,x)$ is the solution of (2.3) with zero Cauchy datum and boundary conditions

$$J^{\lambda_i^* L}(t,0) = 0 \quad J^{\lambda_i^* L}(t,L) = 1 \tag{2.7}$$

and it is given by

$$J^{\lambda_i^* L}(t,x) = C\exp\left(\lambda_i^* x\right) + D - \int_0^L \Delta^{\lambda*}(t,x,y)\Big(C\exp\left(\lambda_i^* y\right) + D\Big)dy, \tag{2.8}$$

where

$$C = \frac{1}{e^{\lambda^* L} - 1} \qquad D = A = -\frac{1}{e^{\lambda^* L} - 1}.$$

Note that all the coefficients A, B, C, D remain bounded as $L \to +\infty$. Moreover, one can apply the maximum principle and, via a comparison with the constant solutions, finds that $0 \leq J^{\lambda_i^* 0}(t,x),\ J^{\lambda_i^* L}(t,x) \leq 1$. Hence the integrals

$$\int_0^T J^{\lambda_i^* 0}(t,x)v'(t)dt \qquad \int_0^T J^{\lambda_i^* 0}(t,x)v'(t)dt$$

are well defined for every function $v(t) \in BV(0, +\infty)$ and for every T.

In the following, we will also need a further convolution kernel $\tilde{\Delta}^{\lambda_i^*}(t,\, x,\, y)$ such that

$$\tilde{\Delta}_y^{\lambda_i^*}(t,\, x,\, y) + \Delta_x^{\lambda_i^*}(t,\, x,\, y) = 0,$$

i.e.

$$\tilde{\Delta}^{\lambda_i^*}(t,\, x,\, y) = \int_y^L \Delta_x^{\lambda_i^*}(t,\, x,\, z)dz. \tag{2.9}$$

To get the previous formula we have arbitrarily imposed $\tilde{\Delta}^{\lambda_i^*}(t,\, x,\, L) = 0$.

Note that $\tilde{\Delta}^{\lambda_i^*}(t,\, x,\, 0)$ is the derivative with respect to x of a function $z(t,\, x)$ which satisfies

$$z(t,\, x) + J^{\lambda_i^* 0}(t,\, x) + J^{\lambda_i^* L}(t,\, x) = 1.$$

Hence,

$$\tilde{\Delta}^{\lambda_i^*}(t,\, x,\, 0) + J_x^{\lambda_i^* 0}(t,\, x) + J_x^{\lambda_i^* L}(t,\, x) = 0. \tag{2.10}$$

The following proposition provides some basic estimates on the convolution kernels we will need later.

Proposition 2.1. *The convolution kernel $\Delta^{\lambda_i^*}$ satisfies*

$$\|\Delta^{\lambda_i^*}(t,\, y)\|_{L^1} \leq \mathcal{O}(1) \quad \|\Delta_x^{\lambda_i^*}(t,\, y)\|_{L^1} \leq \mathcal{O}(1)/\sqrt{t} \qquad \forall\, t < 1,\ y \in]0,\, L[\,. \tag{2.11}$$

The following estimates hold for the boundary kernels $J^{\lambda_i^* 0}$, $J^{\lambda_i^* L}$:

$$0 \leq J^{\lambda_i^* 0}(t,x),\ J^{\lambda_i^* L}(t,x) \leq 1 \qquad\qquad \forall\, t \geq 0,\ x \in]0,L[$$

$$\|J_x^{\lambda_i^* 0}(t)\|_{L^1}, \|J_x^{\lambda_i^* L}(t)\|_{L^1} \leq \mathcal{O}(1) \quad \forall\ \ 0 < t < 1, \tag{2.12}$$

$$\|J_{xx}^{\lambda_i^* 0}(t)\|_{L^1}, \|J_{xx}^{\lambda_i^* L}(t)\|_{L^1} \leq \mathcal{O}(1)/\sqrt{t} \quad \forall\ \ 0 < t < 1.$$

The auxiliary convolution kernel $\tilde{\Delta}^{\lambda_i^*}$ satisfies estimates analogous to those of $\Delta^{\lambda_i^*}$:

$$\|\tilde{\Delta}^{\lambda_i^*}(t,\, y)\|_{L^1} \leq \mathcal{O}(1) \quad \|\tilde{\Delta}_x^{\lambda_i^*}(t,\, y)\|_{L^1} \leq \mathcal{O}(1)/\sqrt{t} \quad \forall\, 0 < t < 1,\ y \in]0,\, L[\,. \tag{2.13}$$

The proof of the proposition can be found in the Appendix A.1.1.

Now we are ready to deal with the vector case. Let $r_i^*,\ l_i^*\ i = 1,2$ be respectively the left and the right eigenvectors of $A^* = A(u^*)$. We define the matrix kernels

$$\begin{aligned} \Delta^* &:= \textstyle\sum_{i=1}^2 \Delta^{\lambda_i^*} r_i^* \otimes l_i^*, \qquad \tilde{\Delta}^* := \textstyle\sum_{i=1}^2 \tilde{\Delta}^{\lambda_i^*} r_i^* \otimes l_i^*, \\ J^{*\,0} &:= \textstyle\sum_{i=1}^2 J^{\lambda_i^* 0} r_i^* \otimes l_i^*, \quad J^{*\,L} := \textstyle\sum_{i=1}^2 J^{\lambda_i^* L} r_i^* \otimes l_i^*. \end{aligned} \tag{2.14}$$

By construction these are the matrix kernels for the initial data corresponding to the cases 1, 2 and 3 considered above (equations (2.4), (2.5) and (2.7) respectively).

2.2 Parabolic estimates

The solution of equation (2.1) can be written as

$$u(t,\,x) = \int_0^L \Delta^*(t,x,y)u_0(y)dy + u_0(0)J^{*\,0}(t,\,x) + \int_0^t J^{*\,0}(t-s,x)u'_{b0}(s)ds + u_0(L)J^{*\,L}(t,\,x) + \int_0^t J^{*\,L}(t-s,x)u'_{bL}(s)ds + \int_0^t\int_0^L \Delta^*(t-s,x,y)\big(A^*-A(u)\big)u_y(s,y)dyds, \tag{2.15}$$

and therefore, recalling (2.10) and integrating by parts,

$$\begin{aligned} u_x(t,\,x) = &\int_0^L \tilde{\Delta}^*(t,x,y)u'_0(y)dy + \int_0^t J^{*\,0}_x(t-s,\,x)u'_{b0}(s)ds + \int_0^t J^{*\,L}_x(t-s,\,x)u'_{bL}(s)ds \\ &+ \int_0^t\int_0^L \tilde{\Delta}^*(t-s,\,x,\,y)\Big(\Big(A^*-A(u)\Big)u_{yy} - DA(u)\Big(u_y\otimes u_y\Big)\Big)(s,\,y)dyds \\ &+ J^{*\,L}_x(t,\,x)\big(u_0(L)-u_0(0)\big) - \int_0^t \big(J^{*\,0}_x + J^{*\,L}_x\big)(t-s,\,x)(A^*-A(u))u_x(s,\,0)ds. \end{aligned} \tag{2.16}$$

From the previous expression we immediately have that, as long as it can be prolonged, the solution is regular. Moreover, the local existence of a solution of equation (2.1) follows from the representation formulas (2.15) and (2.16) via the contraction map theorem.

We can now use the representation (2.16) to prove the following proposition.

Proposition 2.2. *If $\|u_x(t)\|_{L^1} \le \mathcal{O}(1)\delta_1$ for all $t\in[0,\,1]$, then*

$$\|u_{xx}(t)\|_{L^1} \le \frac{\mathcal{O}(1)\delta_1}{\sqrt{t}} \qquad \forall\, t\in[0,\,1].$$

Proof. From (2.16) we get

$$\begin{aligned} u_{xx}(t,\,x) = &\int_0^L \tilde{\Delta}^*_x(t,\,x,\,y)u'_0(y)dy + \int_0^t J^{*\,0}_{xx}(t-s,x)u'_{b0}(s)ds + \int_0^t J^{*\,L}_{xx}(t-s,x)u'_{bL}(s)ds \\ &+ \int_0^t\int_0^L \tilde{\Delta}^*_x(t-s,\,x,\,y)\Big(\Big(A^*-A(u)\Big)u_{yy} - DA(u)\Big(u_y\otimes u_y\Big)\Big)(s,y)dyds \\ &+ J^{*\,L}_{xx}(t,\,x)\big(u_0(L)-u_0(0)\big) - \int_0^t \big(J^{*\,0}_{xx} + J^{*\,L}_{xx}\big)(t-s,\,x)\Big(A^*-A(u)\Big)u_x(s,0)ds. \end{aligned} \tag{2.17}$$

The previous representation formula shows that the function $t\mapsto \|u_{xx}(t)\|_{L^1}$ is continuous.

We claim that there is a constant C independent from L such that

$$\|u_{xx}(t)\|_{L^1} \le \frac{C\delta_1}{\sqrt{t}} \qquad \forall\, t<1.$$

Indeed, for a fixed large constant C, define

$$\tau = \inf\Big\{t:\ \|u_{xx}(t)\|_{L^1} \ge \frac{C}{\sqrt{t}}\delta_1\Big\}.$$

The time τ is strictly bigger than 0 if C is sufficiently large, since by hypothesis $\|u''_0\|_{L^1}$ is finite. Moreover, one has $\|u_{xx}(\tau)\|_{L^1} = C\delta_1/\sqrt{\tau}$ thanks to the continuity of the map $t\mapsto\|u_{xx}(t)\|_{L^1}$.

From (2.17) it follows that

$$\begin{aligned} \|u_{xx}(\tau)\|_{L^1} = \frac{C}{\sqrt{\tau}}\delta_1 &\le \|\Delta^*_x(\tau)\|_{L^1}\,\|u'_0\|_{L^1} + \mathcal{O}(1)\delta_1\int_0^\tau \|u_{yy}(s)\|_{L^1}\,\|\Delta^*_x(\tau-s)\|_{L^1}ds + 2\delta_1\int_0^\tau \frac{\mathcal{O}(1)}{\sqrt{\tau-s}}\,ds \\ &\quad + \frac{\mathcal{O}(1)}{\sqrt{\tau}}\delta_1 + 2\delta_1^2\int_0^t \frac{\mathcal{O}(1)\,C}{\sqrt{s(\tau-s)}}\,ds \\ &\le \frac{2\mathcal{O}(1)\delta_1}{\sqrt{\tau}} + 2\mathcal{O}(1)\,C\delta_1^2 + 2\mathcal{O}(1)\,\sqrt{\tau}\delta_1, \end{aligned}$$

which is a contradiction if C is large enough and δ_1 sufficiently small. In the previous estimate we have used the bounds

$$\|u'_{b\,0}\|_{L^\infty} \leq \|u''_{b\,0}\|_{L^1} \leq \delta_1 \qquad\qquad \int_0^\tau \frac{1}{\sqrt{s(\tau - s)}}\, ds = \pi.$$

□

If $t > 1$ and $\|u_x(s)\|_{L^1} \leq \mathcal{O}(1)\delta_1$ for any $s \in [0,\, t]$, we can apply the previous proposition to the interval $[t-1,\, t]$ and obtain

$$\|u_{xx}(t)\|_{L^1} \leq O(1)\delta_1 \quad t \geq 1.$$

Since the derivative u_x is regular, this implies in particular that, if $\|u_x(s)\|_{L^1} \leq \mathcal{O}(1)\delta_1$ for any $s \leq t$, then $\|u_x(t)\|_{L^\infty} \leq \mathcal{O}(1)\delta_1$ if $t \geq 1$: in other words, as long as u_x remains small in the L^1 norm, it remains small in the L^∞ norm too.

3 Gradient decomposition

3.1 Double boundary layers and travelling waves

In this section we will introduce a suitable decomposition of the gradient of the solution to (1.15),

$$\begin{cases} u_t + A(u)u_x = u_{xx}, \quad x \in]0,\, L[, \quad t \in]0, +\infty[\\ u(0,x) = u_0(x), \\ u(t,0) = u_{b\,0}(t), \qquad u(t,L) = u_{bL}(t). \end{cases}$$

We will employ a decomposition in the form

$$u_x = v_1\tilde{r}_1 + v_2\tilde{r}_2 + p_1\hat{r}_1 + p_2\hat{r}_2, \tag{3.1}$$

where the first two terms correspond to derivatives of travelling waves and the last two correspond to the derivative of a double boundary profile. More precisely, p_1 is the part of the double boundary profile exponentially decaying as $x \to +\infty$, p_2 is the part exponentially decaying as $x \to -\infty$.

The principal results of this section are the construction of the vectors $\hat{r}_1$, $\hat{r}_2$, the description of a decomposition of u_x in the form (3.1), the computations of the equations for the 4 components v_1, v_2, p_1, p_2 and finally the choice of the boundary conditions for the same components. In the description of the decomposition we will focus mainly on the construction of the double boundary profiles, because the construction of the travelling wave profiles follows the same steps as in [7].

The construction of the double boundary profile is based on the following idea: in the linear case, one finds that there is a solution of the boundary value problem

$$\begin{cases} u_x = p, \\ p_x = A(u)p, \\ u(0) = U_{b\,0}, \quad u(L) = U_{b\,L} \end{cases} \tag{3.2}$$

and such a solution is the sum two components: one exponentially decaying as $x \to +\infty$, the other as $x \to -\infty$. Moreover, when the length L is very large the solution has the behavior illustrated in figure 2 (on the left): it is very steep near the boundary $x = 0$ because of the presence of the exponentially decreasing component, then it is almost horizontal in a large interval and then it is steep again near the boundary $x = L$ because of the presence of the exponential decreasing part.

The idea is to try to simulate such a spatial behavior also in the non linear case: in this way, when L is large enough the derivative of the double boundary profile is concentrated near the boundaries $x = 0$ and $x = L$ and therefore there is essentially no interaction with the travelling wave profiles inside the domain. This behavior is the same one observes in the hyperbolic limit, where in $]0, L[$ the solution is generated only by travelling wave profiles. We will find out that, if $|U_{b\,0} - U_{b\,L}|$ is small enough, then there exists indeed a solution of the boundary value problem (3.2) with the behavior illustrated in figure 2.

In this way, we construct the functions $p_1\hat{r}_1(u,\, p_1)$ and $p_2\hat{r}_2(u,\, p_2)$: however, since the decomposition (3.1) is a 2-dimensional vector equation in 4 scalar unknowns, we have some freedom in assigning the initial and boundary data for v_1, v_2, p_1 and p_2. The detailed description of the boundary conditions can be found in Section 3.3, but the crucial idea is to impose some conditions that allow the component p_1 and p_2 to behave like the derivative of a double boundary layer, and thus to be independent from the choice of the initial datum and to be concentrated near the boundary $x = 0$ or $x = L$, respectively. On the other hand, we want to impose some conditions on the components v_1 and v_2 that forces them to behave like the derivative of waves in the hyperbolic limit, thus flowing out from the domain through the boundary $x = 0$ (waves of the first family) or through the boundary $x = L$ (waves of the second family).

Moreover, we have also some freedom in assigning the source terms, as it will be clear in Section 3.2: again the basic idea we will follow is that p_2, which corresponds to the component of the double boundary profile exponential decaying as $x \to +\infty$, should be affected only by the datum in $x = L$. Since in general the source term are spread on the whole interval $]0,\, L[$, we will impose that the equation for p_2 has no source term.

3.1.1 Double boundary profiles

As a first step, we characterize the solutions of the system

$$\begin{cases} u_x = p \\ p_x = A(u)p \end{cases} \tag{3.3}$$

that converge with exponential decay to some value $(\bar{u},\, 0)$ with $\bar{u}$ in a small enough neighborhood of the value u^* defined by the relation (1.7). Since $(u^*, 0)$ is an equilibrium point, we can consider the linearized system, whose center and stable subspaces are given by

$$V^c = \{p = 0\}, \qquad V^s = \mathrm{span}\langle r_1(u^*)\rangle, \qquad V^u = \mathrm{span}\langle r_2(u^*)\rangle.$$

Let $(p_1,\, p_2)$ be the coordinates of p with respect to the base defined by the eigenvalues $r_1(u^*)$ and $r_2(u^*)$ of $A(u^*)$: thanks to the center-stable manifold theorem, there exists a regular function

$$\begin{array}{rcl} \phi : \{(u, p_1) : |u - u^*|, |p_1| \le \varepsilon\} \subseteq V^c \oplus V^s & \to & \mathbb{R} \\ (u, p_1) & \mapsto & p_2 = \phi(u, p_1), \end{array}$$

which parameterizes the solutions of (3.3) that do not blow up exponentially for $x \to +\infty$. In our case, one can see that this manifold is made by the orbits which converge for $x \to +\infty$ to an equilibrium $(\bar{u},\, 0)$, with $\bar{u}$ close to u^* (figure 1). In particular this manifold is unique.

The dimension of this manifold is $dim\, V^c + dim\, V^s$, i.e. 3 in our case. Since $p_1 = 0$ implies $p_2 = \phi(u, p_1) = 0$, we can set $\phi(u, p_1) = p_1 h(u, p_1)$ and $\mathcal{M}^{cs}$ can be described by the following condition:

$$p = p_1 r_1(u^*) + p_1 h(u, p_1) r_2(u^*) = p_1 \begin{pmatrix} 1 \\ f(u, p_1) \end{pmatrix} := p_1 \hat{r}_1(u, p_1).$$

Inserting the previous expression in the system (3.3), one obtains

$$A(u)p_1\hat{r}_1 = \big(p_1\hat{r}_1\big)_x = p_{1x}\hat{r}_1 + (p_1)^2 \mathrm{D}\hat{r}_1\hat{r}_1 + p_1 p_{1x} \hat{r}_{1\,p}.$$

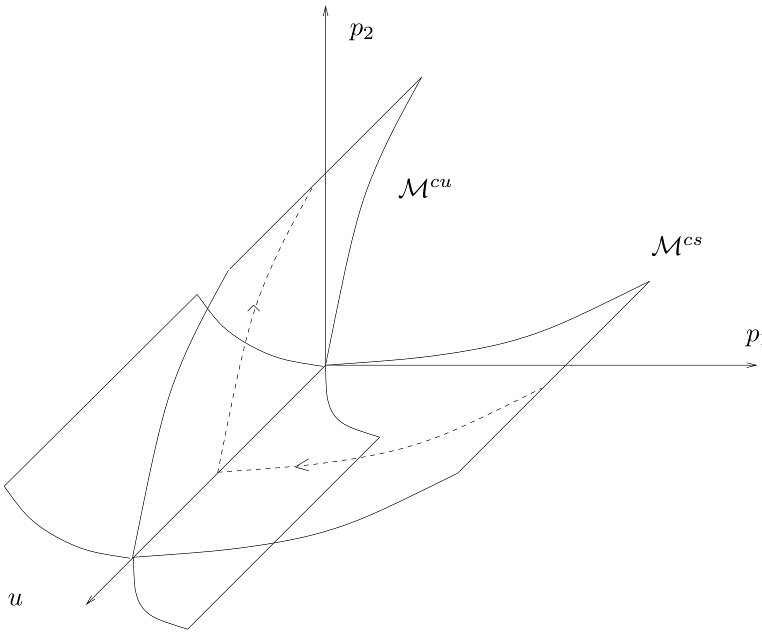
Let $\ell_1 = (1,\, 0)$: if we multiply the previous expression by ℓ_1 we obtain, since A is triangular,

$$\lambda_1 p_1 = p_{1x},$$

and hence

$$A(u)p_1\hat{r}_1 = \lambda_1 p_1 \hat{r}_1 + (p_1)^2 \mathrm{D}\hat{r}_1\hat{r}_1 + \lambda_1 p_1^2 \hat{r}_{1\,p}. \tag{3.4}$$

Figure 1: the center-stable manifold $\mathcal{M}^{cs}$ and the center-unstable manifold $\mathcal{M}^{cu}$ with orbits exponentially decaying to an equilibrium point as $x \to +\infty$ or $x \to -\infty$, respectively



It follows that

$$\hat{r}_1(u,\, 0) = r_1(u) \quad \forall\, u,$$

and therefore

$$|\hat{r}_1(u,\, p_1) - r_1(u)| \leq \mathcal{O}(1)|p_1|.$$

In a similar way one can also define a regular, 3-dimensional center-unstable manifold $\mathcal{M}^{cu}$ containing all the orbits that as $x \to -\infty$ converge with exponential decay to some point $(\bar{u},\, 0)$ with $\bar{u}$ close to u^*. The manifold is parameterized by $V^c \oplus V^u$; moreover, since the matrix A is triangular, one can choose

$$\hat{r}_2 \equiv r_2(u) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The manifold $\mathcal{M}^{cu}$ is thus described by the relation $p = p_2 r_2$.

As a second step, we show that the functions $p_1 \hat{r}_1$ and $p_2 r_2$ indeed allow us to construct a solution of the two-boundaries value problem

$$\begin{cases} z_{xx} = A(z) z_x, \\ z(0) = U_{b\,0} \quad z(L) = U_{b\,L} \end{cases} \tag{3.5}$$

Decomposing z_x as

$$z_x = p_1 \hat{r}_1(z,\, p_1) + p_2 r_2$$

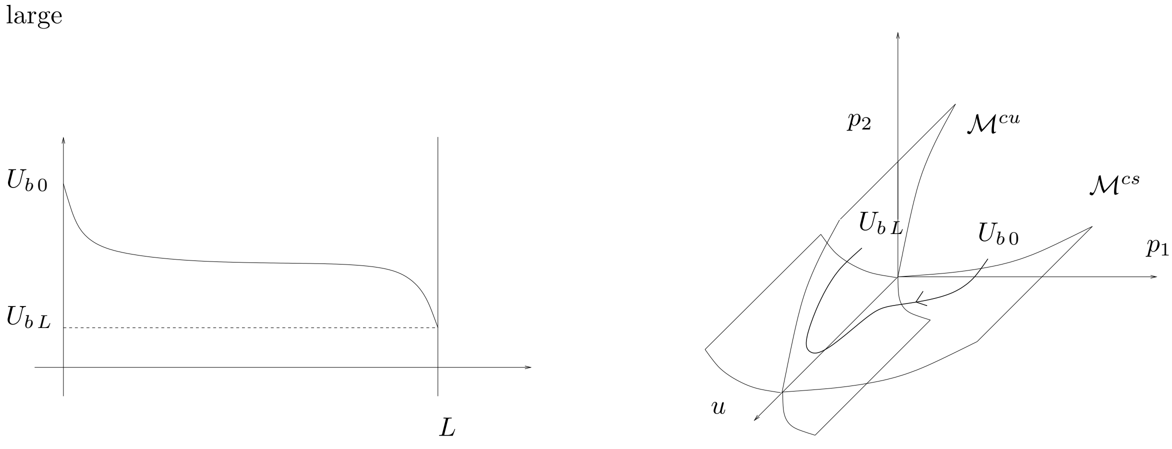
and using the relation (3.4), we obtain the system

$$\begin{cases} z_x = p_1 \hat{r}_1(z,\, p_1) + p_2 r_2, \\ p_{1x} = \lambda_1(z) p_1, \\ p_{2x} = \hat{\lambda}_2(z,\, p_1) p_2 \end{cases} \tag{3.6}$$

where we have defined

$$\hat{\lambda}_2(u,\, p_1) := \lambda_2(u) - p_1 \langle \hat{\ell}_2,\, \mathrm{D}\hat{r}_1 r_2 \rangle, \tag{3.7}$$

Figure 2: the graphic and the orbit of a double boundary layer when the length L of the interval is large



where the vector $\hat{\ell}_2$ satisfies $\langle \hat{\ell}_2,\, \hat{r}_1 \rangle = 0$ and $\langle \hat{\ell}_2,\, r_2 \rangle = 1$. Hence, while in the linear case the two components of the solution of the system (3.5) are decoupled, in the general case there is a coupling in the equation of z, and in the choice of $\hat{\lambda}_2$, which is in some sense the effective eigenvalue for p_2. Note that

$$\big|\hat{\lambda}_2(u, p_1) - \lambda_2(u)\big| \le \mathcal{O}(1) p_1. \tag{3.8}$$

An application of contraction principle ensures that, if $|U_{b\,0} - U_{b\,L}| \le \delta_1$ for a small enough δ_1, then the above system with boundary data $z(0) = U_{b\,0}$, $z(L) = U_{b\,L}$ has a unique solution. Moreover, one also finds that $\big|\hat{\lambda}_2(u,\, p_1) - \lambda(u)\big| \le \mathcal{O}(1)\delta_1$.

Since $\lambda_1 < 0$, $\hat{\lambda}_2 > 0$ for $\delta_1 \ll 1$, we obtain that p_1 is exponentially decaying, while p_2 is exponentially increasing. We can thus figure the double boundary profile as follows (figure 2): when the length L of the interval is very large, the solution will be steep near zero, because in that region p_1 varies exponentially fast. Then it will be almost horizontal for a long interval and becomes again very steep in a left neighborhood of $x = L$, because p_2 increases exponentially.

3.1.2 Travelling waves

We refer to [7] for an exhaustive account of the analysis that allow the definition of the decomposition along travelling waves: here we will only recall for completeness the crucial steps.

Consider the system

$$\begin{cases} u_x = p \\ p_x = \big(A(u) - \sigma I\big) p \\ \sigma_x = 0 \end{cases} \tag{3.9}$$

and an equilibrium point $(u^*,\, 0,\, \lambda_i(u^*))$. The center manifold theorem ensures that the center space $V^c = \big\{p = 0\big\}$ parameterizes a center manifold $\mathcal{M}^c$. This manifold contains all the solutions of (3.9) that do not diverge exponentially neither as $x \to -\infty$ nor as $x \to +\infty$.

It can be shown that the center manifold $\mathcal{M}^c$ around the equilibrium $(u^*,\, 0,\, \lambda_i(u^*))$ is described by a function $v_i \tilde{r}_i(u,\, v_i,\, \sigma_i)$. Since A is triangular, one can take

$$\tilde{r}_1(u,\, v_1,\, \sigma_1) = \begin{pmatrix} 1 \\ m(u,\, v_1,\, \sigma_1) \end{pmatrix}, \qquad \tilde{r}_2(u,\, v_2,\, \sigma_2) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

for some suitable function m (in general different from the function f in the vector $\hat{r}_1$). One can moreover show that the following equations hold:

$$A(u)\tilde{r}_1 = \lambda_1 \tilde{r}_1 + v_1 \mathrm{D}\tilde{r}_1 \tilde{r}_1 + v_1(\lambda_1 - \sigma_1)\tilde{r}_{1v},$$
$$\tilde{r}_1(u,\, 0,\, \sigma_1) = r_1(u) \quad \forall\, u,\, \sigma_1, \quad |\tilde{r}(u,\, v_1,\, \sigma_1) - r_1(u)| = \mathcal{O}(1) v_1, \quad \tilde{r}_{1\,\sigma} = \mathcal{O}(1) v_1.$$

Here and in the following we will denote by $(\ell_1,\; \tilde{\ell}_2)$ the dual base of $(\tilde{r}_1,\; r_2)$.

3.1.3 Gradient decomposition

We set

$$\begin{cases} u_x = v_1\tilde{r}_1(u,\, v_1,\, \sigma_1) + v_2 r_2 + p_1\hat{r}_1(u, p_1) + p_2 r_2 \\ \\ u_t = w_1\tilde{r}_1(u,\, v_1,\, \sigma_1) + w_2 r_2 \end{cases} \qquad \sigma_1 = \lambda_1(u^*) - \theta\Big(\frac{w_1}{v_1} + \lambda_1(u^*)\Big). \quad (3.10)$$

The function θ is here and in the following an odd cutoff such that

$$\theta(s) = \begin{cases} s & \text{if } |s| \leq \hat{\delta} \\ 0 & \text{if } |s| \geq 3\hat{\delta} \\ \text{smooth connection if} & \hat{\delta} \leq s \leq 3\hat{\delta} \end{cases} \qquad \delta_1 << \hat{\delta} \leq \frac{1}{3}. \quad (3.11)$$

The choice of the speed σ follows from the analysis of the boundary free case, [7].

Note that (3.10) is a system of 4 equations in 6 unknowns: as we underlined in the introduction, this will allow some freedom in choosing the boundary conditions for $v_i,\ i = 1, 2$ and $p_i,\ i = 1, 2$. More precisely, we will proceed as follows.

1. We will insert (3.10) in the parabolic equation (1.15). This will generate a system of 4 equations in 6 unknown.

2. We will obtain the equations for $v_i,\ w_i,\ p_i,\ i = 1,\ 2$ by assigning in a suitable way the terms obtained.

3. We will impose boundary and initial conditions on each of the 6 equations obtained. This procedure selects one and only one solution for each of those equations.

The decomposition (3.10) is thus complete. We observe that the idea is to let the equations to choose the components in the decomposition, by only imposing reasonable initial-boundary conditions and by assigning carefully the terms obtained by inserting (3.10) in the system (1.15).

3.2 The equations satisfied by $v_i,\ p_i,\ w_i\ \ i = 1,\ 2$

These equations are obtained via the computations in Appendix A.2.1: inserting the components $v_i,\ p_i\ w_i,\ i = 1,\ 2$ in the equation

$$u_t + A(u)u_x - u_{xx} = 0,$$

we find

$$\begin{aligned} &v_{1t} + (\lambda_1 v_1)_x - v_{1xx} + p_{1t} + (\lambda_1 p_1)_x - p_{1xx} = 0 \\ &v_{2t} + (\lambda_2 v_2)_x - v_{2xx} + p_{2t} + (\hat{\lambda}_2 p_2)_x - p_{2xx} = \tilde{s}_1(t,\, x) \\ &w_{1t} + (\lambda_1 w_1)_x - w_{1xx} = 0 \\ &w_{2t} + (\lambda_2 w_2)_x - w_{2xx} = \tilde{s}_2(t,\, x) \end{aligned}$$

for some function $\tilde{s}_i(t,\, x)\ i = 1,\ 2$ whose explicit expression can be found in the appendix. Moreover, as it is shown in the Appendix A.2.1, from the equation

$$u_t = u_{xx} - A(u)u_x$$

one gets the relations

$$\begin{aligned} w_1 &= v_{1x} - \lambda_1 v_1 + p_{1x} - \lambda_1 p_1 \\ w_2 &= v_{2x} - \lambda_2 v_2 + p_{2x} - \hat{\lambda}_2 p_2 + e(t,\, x) \end{aligned} \quad (3.12)$$

for a suitable error term $e(t,\, x)$. The following Proposition (whose proof can be found in the Appendix A.2.2) gives the form of the source terms:

Proposition 3.1. *The following estimate holds:*

$$
\begin{aligned}
|\tilde{s}_1(t,\,x)|,\ |\tilde{s}_2(t,\,x)|,\ |e(t,\,x)| \le \mathcal{O}(1)\Bigg\{ & \sum_{i\neq j} \Big[|v_i|\Big(|v_j|+|v_{jx}|+|w_j|+|w_{jx}|\Big)+|w_i|\Big(|w_j|+|v_{jx}|\Big)\Big] \\
& + \sum_{i,\,j}\Big(|p_i|+|p_{ix}|\Big)\Big(|v_j|+|v_{jx}|+|w_j|+|w_{jx}|\Big)+|p_{1x}-\lambda_1 p_1|\Big(|p_{1x}|+|p_2|\Big) \\
& + \Big|w_1 v_{1x}-v_1 w_{1x}\Big| + v_1^2\left|\left(\frac{w_1}{v_1}\right)_x\right|^2 \chi_{\{|w_1|\le \delta_1|v_1|\}} + |w_1+\sigma_1 v_1|\Big(|v_1|+|v_{1x}|+|w_1|+|w_{1x}|\Big)\Bigg\}.
\end{aligned}
\tag{3.13}
$$

Following the denomination of [3], we will denote the above terms as follows:

1. interaction between waves of family 1 and family 2
$$\sum_{i\neq j}|v_i|\Big(|v_j|+|v_{jx}|+|w_j|+|w_{jx}|\Big)+|w_i|\Big(|w_j|+|v_{jx}|\Big);$$
2. interaction of travelling waves with boundary profiles
$$\sum_{i,\,j}\Big(|p_i|+|p_{ix}|\Big)\Big(|v_j|+|v_{jx}|+|w_j|+|w_{jx}|\Big);$$
3. interaction among boundary profiles
$$|p_{1x}-\lambda_1 p_1|\Big(|p_{1x}|+|p_2|\Big);$$
4. σ_1 is not constant
$$|w_1 v_{1x}-v_1 w_{1x}| + v_1^2\left|\left(\frac{w_1}{v_1}\right)_x\right|^2 \chi_{\{|w_1|\le \delta_1|v_1|\}};$$
5. the cutoff function θ is active
$$|w_1+\sigma_1 v_1|\Big(|v_1|+|v_{1x}|+|w_1|+|w_{1x}|\Big).$$

Since the component p_2 of the boundary profile should remain close to the boundary $x = L$, and the source $\tilde{s}_1$ is in general spread in the whole interval $[0, L]$, we split the previous expression as follows:

$$
\begin{aligned}
& v_{1t} + (\lambda_1 v_1)_x - v_{1xx} = 0 && p_{1t} + (\lambda_1 p_1)_x - p_{1xx} = 0 \\
& v_{2t} + (\lambda_2 v_2)_x - v_{2xx} = \tilde{s}_1(t,\,x) && p_{2t} + (\hat{\lambda}_2 p_2)_x - p_{2xx} = 0
\end{aligned}
$$

3.3 Boundary conditions

To conclude the characterization of the equations satisfied by $v_i,\ p_i,\ w_i$, we have to assign the boundary conditions. The basic idea is that each component $v_i,\ p_i,\ i = 1$, should behave like a travelling wave or a boundary profile, respectively. More precisely, we can make the following observations:

1) In order to behave like a double boundary profile, p_1 and p_2 should be independent from the initial datum, hence we are led to impose

$$p_1(0,\,x) \equiv 0, \ \ p_2(0,\,x) \equiv 0.$$

It follows that the initial data for v_1 and v_2 are given by

$$v_1(0,\,x) = \langle \ell_1,\ u_0'(x)\rangle \qquad v_2(0,\,x) = \langle \tilde{\ell}_2,\ u_0'(x)\rangle.$$

2) To emulate the behavior observed in the hyperbolic limit, the waves of the first family should disappear when hitting the boundary $x = 0$, and the waves of the second family should disappear at $x = L$. To understand what kind of boundary condition it is convenient to impose, one can observe that an integration by parts leads to

$$\frac{d}{dt}\int_0^L |v_1(t,\, x)|dx = \int_0^L \mathrm{sign} v_1 \Big(v_{1x} - \lambda_1 v_1 \Big)_x dx$$
$$= \int_0^L \delta_{v=0}(v_{1x} - \lambda_1 v_1)dx + \Big[\mathrm{sign} v_1 (v_{1x} - \lambda_1 v_1) \Big]_0^L \leq \Big[\mathrm{sign} v_1 (v_{1x} - \lambda_1 v_1) \Big]_0^L,$$
$$\frac{d}{dt}\int_0^L |v_2(t,\, x)|dx \leq \int_0^t \int_0^L |\tilde{s}_1(s,\, x)| ds dx + \Big[\mathrm{sign} v_2 (v_{2x} - \lambda_2 v_2) \Big]_0^L.$$

(we have used the inequality $\delta_{v=0} v_x \leq 0$). To minimize the increment of $\|v_i(t)\|_{L^1}$ due to the interactions with the boundary we impose

$$v_1(t,\, 0) \equiv 0, \qquad v_2(t,\, L) \equiv 0,$$

and integrating with respect to t the previous equations we get

$$\begin{aligned} &\int_0^L |v_1(t,\, x)|dx \leq \int_0^L |v_1(0,\, x)|dx + \int_0^t |v_{1x} - \lambda_1 v_1|(s,\, L)ds, \\ &\int_0^L |v_2(t,\, x)|dx \leq \int_0^L |v_2(0,\, x)|dx + \int_0^t \int_0^L |\tilde{s}_1(s,\, x)| ds dx + \int_0^t |v_{2x} - \lambda_2 v_2|(s,\, 0)ds. \end{aligned} \tag{3.14}$$

We have used the following observations:

$$\begin{aligned} v_1(0) = 0 &\implies \lim_{x \to 0^+} \mathrm{sign}(v_1) v_{1x}(x) \geq 0 \\ v_2(L) = 0 &\implies \lim_{x \to L^-} \mathrm{sign}(v_2) v_{2x}(x) \leq 0. \end{aligned} \tag{3.15}$$

If one inserts the previous Dirichlet condition on v_i $i = 1,\, 2$ in the decomposition (3.10), obtains the followings boundary conditions for p_i:

$$p_1(t,\, 0) = \langle \ell_1,\, u_x(t,\, 0) \rangle, \qquad p_2(t,\, L) = \langle \tilde{\ell}_2,\, u_x(t,\, L) \rangle - p_1 \langle \tilde{\ell}_2,\, \hat{r}_1 \rangle. \tag{3.16}$$

3) Since p_1 should be located near $x = 0$, and p_2 near $x = L$, we would like to impose that the increment of $\|p_1\|_{L^1}$ due to the datum at $x = L$ is minimal, and similarly that the increment of $\|p_2\|_{L^1}$ caused by the boundary datum in $x = 0$ is as low as possible. Since the values $p_1(t,\, 0)$ and $p_2(t,\, L)$ are already determined, we will impose on p_1 some condition at $x = L$ and on p_2 at $x = 0$. We observe that an integration by parts like the ones performed before leads to

$$\int_0^L |p_1(t,\, x)| \leq \int_0^t |p_{1x} - \lambda_1 p_1|(s,\, 0)ds + \int_0^t |p_{1x} - \lambda_1 p_1|(s,\, L)ds.$$

Hence we are led by the previous considerations to impose

$$(p_{1x} - \lambda_1 p_1)(t,\, L) \equiv 0. \tag{3.17}$$

Similarly, we impose

$$(p_{2x} - \hat{\lambda}_2 p_2)(t,\, 0) \equiv 0. \tag{3.18}$$

From these two equations we obtain the boundary conditions for v_1, v_2: indeed, we have

$$\Big(v_{1,x} - \lambda_1 v_1 \Big)(t,\, L) = \langle \ell_1,\, u_t(t,\, L) \rangle$$

and

$$\Big(v_{2,x} - \lambda_2 v_2 \Big)(t,\, 0) = \langle \tilde{\ell}_2,\, u_t(t,\, 0) \rangle - e(t,\, 0).$$

At this point, the initial-boundary data are perfectly determined for all the components v_i, p_i, $i = 1, 2$, and thus the decomposition is complete.

4 BV estimates

Aim of this section is to prove the following theorem, which constitutes the first part of Theorem 1.1.

Theorem 4.1. *Let $u(t,\, x)$ be the local in time solution of the 2×2 system*

$$\begin{cases} u_t + A(u)u_x = u_{xx} \\ u(0, x) = u_0(x) \\ u(t, 0) = u_{b\,0}(t) \qquad u(t, L) = u_{bL}(t) \end{cases} \tag{4.1}$$

and suppose that the boundary and initial conditions are regular and satisfy

$$\left\| \frac{d^k u_0}{dx^k} \right\|_{L^1(0,\, L)}, \left\| \frac{d^k u_{b\,0}}{dt^k} \right\|_{L^1(0,\, +\infty)}, \left\| \frac{d^k u_{b\,L}}{dt^k} \right\|_{L^1(0,\, +\infty)} \leq \delta_1 \quad k = 1, \dots n,$$

for some δ_1 sufficiently small.

Then $u(t,\, x)$ is defined $\forall\, t > 0$ and its total variation is uniformly bounded:

$$\|u_x(t)\|_{L^1(0,\, L)} \leq C\delta_1 \tag{4.2}$$

for some constant C which does not depend on L.

It is enough to prove that there is a constant δ_0 such that $k\delta_1 \leq \delta_0 << 1$ with k small enough and such that the following holds: if δ_1 is small enough and $\|u_x(s)\|_{L^1} \leq C\delta_1$ $\forall s \in [0,\, t]$ then

$$\begin{aligned} &\int_0^t \int_0^L |\tilde{s}_1(\sigma,\, x)| dx d\sigma \leq \mathcal{O}(1)\delta_0^2, && \int_0^t \int_0^L |\tilde{s}_2(\sigma,\, x)| dx d\sigma \leq \mathcal{O}(1)\delta_0^2, \\ &\int_0^t |v_{2x} - \lambda_2 v_2|(\sigma,\, 0) d\sigma \leq m\delta_1, && \int_0^t |v_{1x} - \lambda_1 v_1|(\sigma,\, L) d\sigma \leq m\delta_1, \\ &\int_0^t |p_{1x} - \lambda_1 p_1|(\sigma,\, 0) d\sigma \leq m\delta_1, && \int_0^t |p_{2x} - \hat{\lambda}_2 p_2|(\sigma,\, 0) d\sigma \leq m\delta_1, \end{aligned} \tag{4.3}$$

for some constant m that does not depend on C.

Indeed, suppose the previous implication holds. From the representation formula (2.16) it immediately follows that the function $t \mapsto \|u_x(t)\|_{L^1}$ is continuous: hence, it will satisfy $\|u_x(t)\|_{L^1} < C\delta_1$ if t is small enough, since the total variation of the initial datum is bounded by δ_1.

Suppose by contradiction that τ is the first time such that $\|u_x(\tau)\|_{L^1} = C\delta_1$. Then we use the equations

$$\begin{aligned} &v_{1t} + (\lambda_1 v_1)_x - v_{1xx} = 0 && p_{1t} + (\hat{\lambda}_1 p_1)_x - p_{1xx} = 0 \\ &v_{2t} + (\lambda_2 v_2)_x - v_{2xx} = \tilde{s}_1(t,\, x) && p_{2t} + (\hat{\lambda}_2 p_2)_x - p_{2xx} = 0 \end{aligned}$$

and the boundary conditions described in Section 3.3 and, integrating by parts, we get

$$\begin{aligned} \int_0^L |u_x(\tau,\, x)| dx \leq & \sum_{i=1}^2 \int_0^L |v_i(\tau,\, x)| + \int_0^L |p_i(\tau,\, x)| dx \leq \sum_{i=1}^2 \int_0^L |v_i(0,\, x)| + \int_0^\tau \int_0^L |\tilde{s}_1(\sigma,\, x)| dx d\sigma \\ & + \int_0^\tau |v_{2x} - \lambda_2 v_2|(\sigma,\, 0) d\sigma + \int_0^\tau |v_{1x} - \lambda_1 v_1|(\sigma,\, L) d\sigma + \int_0^\tau |p_{1x} - \lambda_1 p_1|(\sigma,\, 0) d\sigma \\ & + \int_0^\tau |p_{2x} - \hat{\lambda}_2 p_2|(\sigma,\, L) d\sigma \leq (4m+2)\delta_1 + \mathcal{O}(1)\delta_0^2 < C\delta_1, \end{aligned}$$

if C is large enough: this contradicts the assumption $\|u_x(\tau)\|_{L^1} = C\delta_1$.

Note that since all the functions in the right hand side of (3.13) are continuous (and hence bounded on $[0, L]$), we have that

$$\int_0^s \int_0^L |\tilde{s}_i(\sigma, x)| dx d\sigma \leq \mathcal{O}(1)\delta_1 \quad i = 1, 2, \tag{4.4}$$

for s small enough. Hence to prove (4.3) we can suppose that (4.4) holds for any $s \in [0, t]$: since we will show that actually

$$\int_0^t \int_0^L |\tilde{s}_i(\sigma, x)| dx d\sigma \leq \mathcal{O}(1)\delta_0^2, \quad i = 1, 2,$$

the assumption will be a posteriori justified since $k\delta_1 \leq \delta_0 << 1$.

We will proceed as follows: in Section 4.1 we will show some elementary estimates, while in Section 4.2 we will introduce suitable functionals that allow the estimates

$$\int_0^t \int_0^L \sum_{i \neq j} \big(|v_i|(|v_j| + |v_{jx}| + |w_j| + |w_{jx}|) + |w_i|(|w_j| + |v_{jx}|) \big)(\sigma\, x) d\sigma dx \leq \mathcal{O}(1)\delta_1^2,$$

$$\int_0^t \int_0^L |w_1 v_{1x} - v_1 w_{1x}|(\sigma, x) d\sigma \leq \mathcal{O}(1)\delta_1^2,$$

$$\int_0^t \int_0^L \left| v_1^2 \left(\frac{w_1}{v_1} \right)_x \right|^2 \chi_{\{|w_1| \leq \delta_1 |v_1|\}}(\sigma, x) d\sigma dx \leq \mathcal{O}(1)\delta_1^2.$$

In Section 4.3 we will consider the term

$$\int_0^t \int_0^L |w_1 + \sigma_1 v_1|(|v_1| + |v_{1x}| + |w_1|)(\sigma, x) d\sigma dx,$$

and prove a bound of order δ_1^2.

4.1 Elementary estimates

This section is devoted to the estimates which can be obtained by elementary techniques, like the maximum principle. We will in particular show that the components p_i, $i = 1, 2$ are exponentially decaying as one moves far away from the boundary, and that their decay exponent does not depend on the interval length L. Moreover, by introducing various functional, we estimate the boundary data assigned to the components v_1, v_2 and prove that the functions v_i are integrable along all vertical lines $\{x = \text{const}\}$. This means that, as in the boundary free case, the profiles of travelling waves just cross the vertical lines.

4.1.1 Estimates via maximum principle

We will first deal with p_1. The results in Section 2.2 ensures that

$$\|u_x(t)\|_{L^\infty} \leq \|u_{xx}(t)\|_{L^1} \leq \mathcal{O}(1)\delta_1.$$

Hence it follows that

$$|p_1(t, 0)| = |\langle l_1, u_x(t, 0) \rangle| \leq k\delta_1,$$

for some k large enough.

The equation satisfied by p_1 is

$$p_{1t} + \lambda_1(u) p_{1x} + \lambda_{1x}(u) p_1 - p_{1xx} = 0.$$

This is a linear equation, with coefficients depending on the solution $u(t, x)$. Let $2c$ be the separation speed defined in (1.4) and

$$q(x) = k\delta_1 \exp\big(-cx/2\big).$$

Since $|\lambda_{1x}| \leq \mathcal{O}(1)\delta_1$ and $\delta_1 << 1$, q satisfies

$$q_t + \lambda_1 q_x + \lambda_{1x} q - q_{xx} > 0.$$

Hence the difference $(q - p_1)$ satisfies

$$\begin{cases} (q-p_1)_t + \lambda_1 (q-p_1)_x + \lambda_{1x}(q-p_1) - (q_{xx} - p_{1xx}) > 0 \\ (q-p_1)(t,\,0) \geq 0 \\ \Big((q-p_1) - \lambda_1 (q-p_1)_x \Big)(t,\,L) > 0. \end{cases}$$

By standard techniques it follows that $(q - p_1)(t,\, x) \geq 0$ for any $t,\ x$ and hence

$$|p_1(t,\,x)| \leq k\delta_1 \exp(-c\,x/2). \tag{4.5}$$

The boundary condition on p_2 satisfies the following bound:

$$|p_2(t,\,L)| = |\langle \hat{l}_2,\, u_x(t,\,L)\rangle - p_1 \langle \tilde{l}_2,\, \hat{r}_1\rangle| \leq \mathcal{O}(1)\delta_1, \qquad \forall\, t,\ x.$$

Since $|p_1(t,\,x)| \leq k\delta_1$, then from (3.8) it follows that $|\lambda_2 - \hat{\lambda}_2| \leq \mathcal{O}(1)\delta_1$ and hence in the same way as before one can prove

$$|p_2(t,\,x)| \leq \mathcal{O}(1)\delta_1 \exp(c(x-L)/2), \qquad \forall\, t,\ x. \tag{4.6}$$

From (4.5) it follows

$$\|p_1(t)\|_{L^1} \leq \mathcal{O}(1)\delta_1, \qquad \|v_1(t)\|_{L^1} \leq \mathcal{O}(1)\delta_1$$

and, since $\|u_x\|_{L^\infty} \leq \mathcal{O}(1)\delta_1$,

$$\|v_1\|_{L^\infty} \leq \mathcal{O}(1)\delta_1.$$

Analogously, from (4.6) it follow

$$\|p_2(t)\|_{L^1} \leq \mathcal{O}(1)\delta_1, \qquad \|v_2(t)\|_{L^1} \leq \mathcal{O}(1)\delta_1, \qquad \|v_2\|_{L^\infty} \leq \mathcal{O}(1)\delta_1.$$

The following proposition summarizes the results obtained in this paragraph:

Proposition 4.1. *Let $p_i,\ v_i$ be the solutions of* (1.22) *with the boundary conditions described in Section 3.3. Then*

$$|p_1(t,\,x)| \leq \mathcal{O}(1)\delta_1 \exp(-cx/2), \qquad |p_2(t,\,x)| \leq \mathcal{O}(1)\delta_1 \exp(c(x-L)/2),$$

where $2c$ is the separation speed defined by (1.4).

The previous estimates imply

$$\|p_i(t)\|_{L^1} \leq \mathcal{O}(1)\delta_1, \qquad \|v_i(t)\|_{L^1} \leq \mathcal{O}(1)\delta_1, \qquad \|v_i(t)\|_\infty \leq \mathcal{O}(1)\delta_1, \quad i = 1,\,2.$$

Remark 4.1. The estimate of $\|v_i(t)\|_{L^1}$ can also be obtained directly from (3.14): indeed, since

$$(p_{1x} - \lambda_1 p_1)(t,\,L) \equiv 0$$

and the total variation of $u_{b\,L}$ is bounded by δ_1, from (3.12) one gets

$$\int_0^t |v_{1x} - \lambda_1 v_1|(s,\,L) ds \leq \delta_1,$$

and hence $\|v_1(t)\|_{L^1} \leq 2\delta_1$.

To obtain the estimate on v_2 from (3.14) one has to start supposing

$$\int_0^t |e(s,\,0)| ds \leq \delta_1. \tag{4.7}$$

With the same computations as before one gets $\|v_2(t)\|_{L^1} \leq \mathcal{O}(1)\delta_1$. As it will be clear from the next sections, the assumption (4.7) actually leads to the estimate

$$\int_0^t |e(s,\,0)| ds \leq \mathcal{O}(1)\delta_1^2,$$

and therefore it is a posteriori well justified.

4.1.2 Integrability with respect to time

The following lemma, which can be proved by a simple integration by parts, introduces a useful estimate we will widely use in the following.

Lemma 4.1. *Let $P(x)$ be a non negative $\mathcal{C}^2$ function defined on $\mathbb{R}$ and let q be a solution of*

$$q_t + (\lambda q)_x - q_{xx} = s(t,\, x).$$

Then the following estimate holds:

$$\begin{aligned}\frac{d}{dt}\int_0^L |q(t,\, x)|P(x)dx \le \int_0^L |s(t,\, x)|P(x)dx + \int_0^L |q(t,\, x)|(\lambda P' + P'')(x)dx \\ - \Big[P'|q(t)|\Big]_{x=0}^{x=L} + \Big[P\,\mathrm{sign}(q)\,(q_x - \lambda q)(t)\Big]_{x=0}^{x=L}.\end{aligned}$$

Before applying the previous lemma, we recall that the boundary data of the scaled problem (1.15) belongs to $BV(0,\, +\infty)$ and that the L^1 norms of $u'_{b\,0}$ and $u'_{b\,L}$ are bounded by δ_1. From the decomposition $u_t = w_1 \tilde{r}_1 + w_2 r_2$, we immediately have

$$\|w_i(x=0)\|_{L^1(0,\,+\infty)} \le \delta_1 \qquad \|w_i(x=L)\|_{L^1(0,\,+\infty)} \le \delta_1 \quad i = 1,\, 2.$$

Moreover, in Section 3.2 we found that w_i $i = 1,\, 2$ can be decomposed as follows:

$$\begin{aligned} w_1 &= p_{1x} - \lambda_1 p_1 + v_{1x} - \lambda_1 v_1 \\ w_2 &= p_{2x} - \hat{\lambda}_2 p_2 + v_{2x} - \lambda_2 v_2 + e(t,\, x), \end{aligned} \tag{4.8}$$

where the error term $e(t,\, x)$ satisfies the estimate (3.13). As we anticipated in Remark 4.1, we will suppose

$$\int_0^t |e(s,\, x)|ds \le \delta_1 \quad \forall\, x \in [0,\, L].$$

Since we will obtain an estimate of order $\delta_1^2 \le \delta_1$, this assumption is a posteriori well justified.

From the boundary condition (3.18) $(p_{2x} - \hat{\lambda}_2 p_2)(t,\, 0) \equiv 0$ and from the decomposition (4.8) we get

$$\int_0^t |v_{2x} - \lambda_2 v_2|(s,\, 0)ds \le 2\delta_1.$$

Similarly, one obtains that

$$\int_0^t |v_{1x} - \lambda_1 v_1|(s,\, L)ds \le \delta_1.$$

An application of Lemma 4.1 with $P \equiv 1$ and $q = v_2$ leads by observation (3.15) to

$$\begin{aligned}\int_0^t |v_{2x}(s,\, L)|ds &\le \int_0^t\int_0^L |\tilde{s}_2(s,\, x)|dxds + \int_0^t |v_{2x} - \lambda_2 v_2|(s,\, 0)ds + \int_0^L |v_2(0,\, x)|dx \\ &\le \mathcal{O}(1)\delta_1 + 2\delta_1 + \mathcal{O}(1)\delta_1 \le \mathcal{O}(1)\delta_1,\end{aligned}$$

and similarly

$$\int_0^t |v_{1x}(s,\, 0)|ds \le \mathcal{O}(1)\delta_1.$$

Let $2c$ be the separation speed defined by (1.4): the application of Lemma 4.1 with $q(t,\, x) = v_2(t,\, x)$ and

$$P(x) = P_y(x) = \begin{cases} 1/c & x \le y \\ \exp\Big(c(y-x)\Big)/c & x > y \end{cases} \qquad y \in [0,\, L[$$

leads to the estimate

$$\begin{aligned}\int_0^t |v_2(s,\,y)|ds &\le \int_0^L |v_2(0,\,x)|dx + \frac{1}{c}\int_0^t\int_0^L |\tilde{s}_1(s,\,x)|dsdx \\ &\quad + P_y(0)\int_0^L |v_{2x}-\lambda_2 v_2|(s,\,0)ds + P_y(L)\int_0^t |v_{2x}(s,\,L)|ds \\ &\le \mathcal{O}(1)\delta_1 + \mathcal{O}(1)\delta_1 \le \mathcal{O}(1)\delta_1 \qquad \forall\, y \in [0,\,L[.\end{aligned}$$

Analogously, we get

$$\int_0^t |v_1(s,\,y)|ds \le \mathcal{O}(1)\delta_1 \quad \forall\, y \in]0,\,L].$$

The following proposition summarizes what we have proved so far:

Proposition 4.2. *Let* $v_i,\ p_i\ \ i=1,\,2$ *be the solutions to the equations* (1.22) *with the boundary conditions described in Section 3.3. Then it holds*

$$\int_0^t |v_{2x}-\lambda_2 v_2|(s,\,0)ds \le 2\delta_1, \qquad \int_0^t |v_{1x}-\lambda_1 v_1|(s,\,L)ds \le \delta_1,$$
$$\int_0^t |v_{1x}(s,\,0)|ds \le \mathcal{O}(1)\delta_1, \qquad \int_0^t |v_{2x}(s,\,L)|ds \le \mathcal{O}(1)\delta_1,$$

and

$$\int_0^t |v_i(s,\,y)|ds \le \mathcal{O}(1)\delta_1, \quad \forall\, y \in [0,\,L] \quad i=1,\,2.$$

Further computations (Appendix A.3.1) ensure that

$$|p_{1x}(t,\,x)| \le \mathcal{O}(1)\delta_1 \exp(-cx/2), \qquad |p_{2x}(t,\,x)| \le \mathcal{O}(1)\delta_1 \exp\big(c(x-L)/2\big). \tag{4.9}$$

The following proposition deals with other estimates of integrals with respect to time: the proof is quite long and requires the introduction of new convolution kernels. It can be found in the Appendix A.3.2.

Proposition 4.3. *In the same hypothesis of Proposition 4.2 it holds*

$$\int_0^t |v_{ix}(s,\,y)|ds \le \mathcal{O}(1)\delta_1 \quad \forall\, y \in [0,\,L] \quad i=1,\,2$$

and

$$\int_0^t |w_i(s,\,y)|ds \le \mathcal{O}(1)\delta_1 \quad \forall\, y \in [0,\,L] \quad i=1,\,2.$$

We also have

$$\int_0^t |w_{ix}(s,\,y)|ds \le \mathcal{O}(1)\delta_1 \quad \forall\, y \in [0,\,L] \quad i=1,\,2.$$

In the previous proposition the functions w_i are of course defined by relation $u_t = w_1\tilde{r}_1 + w_2 r_2$. Putting together Proposition 4.2 and 4.3 and the decomposition (4.8) one gets

$$\int_0^t |p_{1x}-\lambda_1 p_1|(s,\,y)ds \le \mathcal{O}(1)\delta_1, \qquad \int_0^t |p_{2x}-\hat{\lambda}_2 p_2|(s,\,y)ds \le \mathcal{O}(1)\delta_1, \quad \forall\, y \in [0,\,L],$$

and

$$\int_0^t |p_{1x}-\lambda_1 p_1|(s,\,0)ds \le m\delta_1, \qquad \int_0^t |p_{2x}-\hat{\lambda}_2 p_2|(s,\,L)ds \le m\delta_1,$$

where the constant m satisfies the hypothesis stated in Section 4.

The estimates obtained so far will be widely used in next sections and moreover allow to prove a bound of order $\mathcal{O}(1)\delta_1^2$ on some of the terms that appear on the right hand side of (3.13):

$$\begin{aligned}&\int_0^t\int_0^L \sum_{i,\,j}(|p_i|+|p_{ix}|)(|v_j|+|v_{jx}|+|w_j|+|w_{jx}|)(s,\,x)dsdx\\&\qquad\le \mathcal{O}(1)\delta_1\int_0^L (e^{-cx}+e^{c(x-L)})\int_0^t (|v_j|+|v_{jx}|+|w_j|+|w_{jx}|)(s,\,x)dsdx\le \mathcal{O}(1)\delta_1^2\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}&\int_0^t\int_0^L \sum_i |p_{1x}-\lambda_1 p_1|(|p_i|+|p_{ix}|)(s,\,x)dxds\\&\qquad\le \mathcal{O}(1)\delta_1\int_0^L e^{-cx}+e^{c(x-L)}\int_0^t |p_{1x}-\lambda_1 p_1|(s,\,x)dsdx\le \mathcal{O}(1)\delta_1^2.\end{aligned}\tag{4.11}$$

4.2 Interaction functionals

In this section we introduce three nonlinear functionals and we use them to bound those terms in the right hand side of (3.1) due to interaction between waves of different families and those due to the fact that the speed σ_1 is not constant. The form of the functionals is exactly the same considered in [7], with some more technicalities due to the presence of the boundary.

4.2.1 Interaction among waves of different families

We claim that the condition

$$\int_0^t\int_0^L |\tilde{s}_1(s,\,x)|dsdx\le \mathcal{O}(1)\delta_1\qquad \int_0^t\int_0^L |\tilde{s}_2(s,\,x)|dsdx\le \mathcal{O}(1)\delta_1$$

implies

$$\int_0^t\int_0^L \sum_{i\neq j}\Big(|v_i|\Big(|v_j|+|w_j|\Big)+|w_i w_j|\Big)(s,\,x)dsdx\le \mathcal{O}(1)\delta_1^2.\tag{4.12}$$

We will prove only that

$$\int_0^t\int_0^L |v_1 v_2|(s,\,x)dsdx\le \mathcal{O}(1)\delta_1^2,\tag{4.13}$$

because the other terms in (4.12) can be dealt with analogously: see for example [3].

Let $2c$ be the separation speed introduced in (1.4) and let $P(\xi)$ be defined as follows:

$$P(\xi):=\begin{cases} e^{c\xi}/2c & \xi<0\\ 1/2c & \xi\ge 0\end{cases}$$

One gets

$$\begin{aligned}\frac{d}{ds}\Big(\int_0^L\int_0^L P(x-y)|v_1(s,\,x)|\,|v_2(s,\,y)|dxdy\Big)&\le \int_0^L |v_2(s,\,y)|\Big[P(x-y)\mathrm{sign}v_1(v_{1x}-\lambda_1 v_1)(s,\,x)\Big]_{x=0}^{x=L}dy\\&+\int_0^L |v_1(s,\,x)|\Big[P(x-y)\mathrm{sign}v_2(v_{2x}-\lambda_2 v_2)(s,\,y)\Big]_{y=0}^{y=L}dx-\int_0^L |v_2(s,\,y)|\Big[P'(x-y)|v_1(s,\,x)|\Big]_{x=0}^{x=L}dy\\&+\int_0^L |v_1(s,\,x)|\Big[P'(x-y)|v_2(s,\,y)|\Big]_{y=0}^{y=L}+\int_0^L |v_1(s,\,x)|\int_0^L P(x-y)|\tilde{s}_1(s,\,y)|dy\\&+\int_0^L\int_0^L\Big(P'(x-y)\Big(\lambda_1(s,\,x)-\lambda_2(s,\,y)\Big)+2P''(x-y)\Big)|v_1(s,\,x)|\,|v_2(s,\,y)|dxdy.\end{aligned}$$

One has

$$P'(\lambda_1-\lambda_2)+2P''\le 2(-cP'+P'')=-\delta_{s=0},\quad 0\le P(s)\le\frac{1}{2c},\qquad 0\le P'(s)\le\frac{1}{2}$$

and moreover from the estimates of Proposition 4.1 and 4.2 it follows that

$$\int_0^t |v_{1x}-\lambda_1 v_1|(s,\,L)\int_0^L |v_2(s,\,y)|dyds\le\mathcal{O}(1)\delta_1^2\qquad \int_0^t |v_{2x}-\lambda_2 v_2|(s,\,0)\int_0^L |v_2(s,\,x)|dxds\le\mathcal{O}(1)\delta_1^2$$

$$\int_0^t\int_0^L |\tilde{s}_1(s,\,y)|\int_0^L |v_1(s,\,x)|dxdyds\le\mathcal{O}(1)\delta_1^2:$$

this completes the proof of the estimate (4.13).

With some technical computations, in Appendix A.3.3 it is proved

$$\int_0^t\int_0^L\sum_{i\neq j}\Big(|v_i|\Big(|v_{jx}|+|w_{jx}|\Big)+|w_i v_{jx}|\Big)(s,\,x)dsdx\le\mathcal{O}(1)\delta_1^2,\tag{4.14}$$

which completes the proof of the estimate

$$\int_0^t\int_0^L\sum_{i\neq j}\Big(|v_i|\Big(|v_j|+|v_{jx}|+|w_j|+|w_{jx}|\Big)+|w_i|(|w_j|+|v_{jx}|)\Big)(s,\,x)dsdx\le\mathcal{O}(1)\delta_1^2.$$

4.2.2 Length and area functionals

To prove the estimate

$$\int_0^t\int_0^L |v_{1x}w_1-v_1w_{1x}|(s,\,x)dxds\le\mathcal{O}(1)\delta_1^2,$$

we introduce the curve

$$\gamma(x)=\begin{pmatrix} v_1(x)\\ w_1(x)\end{pmatrix}\tag{4.15}$$

and the related area functional

$$\mathcal{A}(\gamma)(s)=\frac{1}{2}\int\int_{y\le x}|\gamma_x\wedge\gamma_y|dxdy=\frac{1}{2}\int_0^L\int_0^x |v_1(s,\,x)w_1(s,\,y)-v_1(s,\,y)w_1(s,\,x)|dxdy.$$

The curve γ_x satisfies

$$\gamma_{xt}+(\lambda_1\gamma_x)_x=\gamma_{xxx}$$

and moreover one has

$$\begin{aligned}\frac{d\mathcal{A}(s)}{ds}=&\frac{1}{2}\int_0^L\int_y^L \text{sign}\Big(v_1(s,\,x)w_1(s,\,y)-v_1(s,\,y)w_1(s,\,x)\Big)\Big(v_1(s,\,x)w_1(s,\,y)-v_1(s,\,y)w_1(s,\,x)\Big)_{xx}\\ &-\frac{1}{2}\int_0^L\int_y^L \text{sign}\Big(v_1(s,\,x)w_1(s,\,y)-v_1(s,\,y)w_1(s,\,x)\Big)\Big(\lambda_1(s,\,x)\Big(v_1(s,\,x)w_1(s,\,y)-v_1(s,\,y)w_1(s,\,x)\Big)\Big)_x\\ &+\frac{1}{2}\int_0^L\int_0^x \text{sign}\Big(v_1(s,\,x)w_1(s,\,y)-v_1(s,\,y)w_1(s,\,x)\Big)\Big(v_1(s,\,x)w_1(s,\,y)-v_1(s,\,y)w_1(s,\,x)\Big)_{yy}\\ &-\frac{1}{2}\int_0^L\int_0^x \text{sign}\Big(v_1(s,\,x)w_1(s,\,y)-v_1(s,\,y)w_1(s,\,x)\Big)\Big(\lambda_1(s,\,y)\Big(v_1(s,\,x)w_1(s,\,y)-v_1(s,\,y)w_1(s,\,x)\Big)\Big)_y\end{aligned}$$

and hence

$$\begin{aligned}\frac{d\mathcal{A}}{ds}\le&\frac{1}{2}\int_0^L\Big|v_{1y}(s,\,L)w_1(s,\,y)-v_1(s,\,y)w_{1y}(s,\,L)\Big|dy-\frac{1}{2}\int_0^L\Big|v_{1y}(s,\,y)w_1(s,\,y)-w_{1y}(s,\,y)v_1(s,\,y)\Big|dy\\ &-\frac{1}{2}\int_0^L\lambda_1(s,\,L)\Big|v_1(s,\,L)w_1(s,\,y)-v_1(s,\,y)w_1(s,\,L)\Big|dy\\ &-\frac{1}{2}\int_0^L\Big|v_1(s,\,x)w_{1x}(s,\,x)-w_1(s,\,x)v_{1x}(s,\,x)\Big|dx+\frac{1}{2}\int_0^L\Big|v_1(s,\,x)w_{1x}(s,\,0)-v_{1x}(s,\,0)w_1(s,\,x)\Big|dx\end{aligned}$$

Since $A(\gamma)(0) \leq \mathcal{O}(1)\delta_1^2$, one obtains, using the estimates in Propositions 4.2 and 4.3,

$$\int_0^t \int_0^L \big|v_1(s,\,x)w_{1x}(s,\,x) - v_{1x}(s,\,x)w_1(s,\,x)\big|dx \leq -\int_0^t \frac{d\mathcal{A}}{ds}ds + \mathcal{O}(1)\delta_1^2 \leq \mathcal{O}(1)\delta_1^2.$$

The length functional of the curve (4.15) is defined as

$$\mathcal{L}(\gamma)(s) = \int_0^L |\gamma_x|dx = \int_0^L \sqrt{v_1^2 + w_1^2}dx,$$

and will be used to prove the estimate

$$\int_0^t \int_0^L v_1^2 \left[\left(\frac{w_1}{v_1} \right)_x \right]^2 \chi dxds \leq \mathcal{O}(1)\delta_1^2, \tag{4.16}$$

where χ is the characteristic function of the set

$$\left\{ x : \left| \frac{w_1}{v_1}(x) - \lambda_1^* \right| \leq 3\hat{\delta} \right\}$$

(see Section 3.1.3 for the definition of $\hat{\delta}$).

We preliminary observe that the following equalities hold:

$$\begin{aligned}
|v_1| \left[\left(\frac{w_1}{v_1} \right)_x \right]^2 &= \frac{w_{1x}^2 v_1^2 + v_{1x}^2 w_1^2 - 2v_{1x}w_{1x}v_1w_1}{|v_1^3|} \leq C\frac{|\gamma_{xx}|^2|\gamma_x|^2 - \langle \gamma_x,\, \gamma_{xx}\rangle^2}{|\gamma_x|^3}, \\
|\lambda_1\gamma_x|_x &= \frac{\langle \lambda_1\gamma_x,\, (\lambda_1\gamma_x)_x\rangle}{|\lambda_1\gamma_x|} = -\frac{\langle \gamma_x,\, (\lambda_1\gamma_x)_x\rangle}{|\gamma_x|}, \\
|\gamma_x|_{xx} &= \left(\frac{\langle \gamma_x,\, \gamma_{xx}\rangle}{|\gamma_x|} \right)_x = \frac{\langle \gamma_x,\, \gamma_{xx}\rangle^2}{-|\gamma_x|^3} + \frac{\langle \gamma_x,\, \gamma_{xxx}\rangle}{|\gamma_x|} + \frac{|\gamma_{xx}|^2}{|\gamma_x|}.
\end{aligned}$$

From $\gamma_{xt} + (\lambda_1\gamma_x)_x = \gamma_{xxx}$, one gets integrating by parts

$$\begin{aligned}
\frac{d\mathcal{L}}{ds} &= \int_0^L \frac{\langle \gamma_{xxx},\, \gamma_x\rangle}{|\gamma_x|} - \int_0^L \frac{\langle (\lambda_1\gamma_x)_x,\, \gamma_x\rangle}{|\gamma_x|} \\
&= \int_0^L |\gamma_x|_{xx} + \int_0^L \frac{\langle \gamma_x,\, \gamma_{xx}\rangle^2}{|\gamma_x|^3} - \int_0^L \frac{|\gamma_{xx}|^2}{|\gamma_x|} + \int_0^L |\lambda_1\gamma_x|_x.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{C}\int_0^t \int_0^L |v_1| \left[\left(\frac{w_1}{v_1} \right)_x \right]^2 \chi dxds &\leq \int_0^t \int_0^L \frac{|\gamma_{xx}|^2|\gamma_x|^2 - \langle \gamma_x,\, \gamma_{xx}\rangle^2}{|\gamma_x|^3}dxds \\
&\leq -\int_0^t \frac{d\mathcal{L}}{ds}ds + \int_0^t \Big[|\gamma_x|_x(s,\,x) \Big]_{x=0}^{x=L} + \int_0^T \Big[|\lambda_1\gamma_x|(s,\,x) \Big]_{x=0}^{x=L} ds \leq \mathcal{O}(1)\delta_1.
\end{aligned}$$

In the previous estimate we have used the fact that $v_1,\ w_1,\ v_{1x},\ w_{1x}$ are integrable with respect to time and that their integrals are bounded by $\mathcal{O}(1)\delta_1$ (Propositions 4.2 and 4.3). Since $\|v_1\|_\infty$ is bounded by $\mathcal{O}(1)\delta_1$, the previous estimate complete the proof of (4.16).

4.3 Estimate on the error in choosing the speed

The final estimate is the source term due to the cutoff function θ. Also this computation is similar to the one performed in [7], taking into account the fact that here we have a double boundary. In Appendix A.3.4 one can find the proof of the estimates

$$\int_0^t \int_0^L \Big(|v_1| + |w_1| + |v_{1x}|\Big)\Big(|w_1 + \sigma_1 v_1|\Big)(s,x)dxds \leq \mathcal{O}(1)\delta_1^2. \tag{4.17}$$

This ends the proof of the estimate

$$\int_0^t \int_0^L |\tilde{s}_i(s, x)| ds dx \leq \mathcal{O}(1)\delta_1^2 \quad i = 1, 2,$$

and hence of Theorem 4.1.

5 Stability estimates

In this section we prove the second part of Theorem 1.1, completing the proof. Since the ideas are essentially the same as in the boundary free case, we will only sketch the line of the proof, paying more attention to the choice of the boundary conditions (which is the new element in this paper). The result of this section is thus:

Theorem 5.1. *There exist constants L_1, L_2 s.t. the following holds: let u^1, u^2 be two solutions of the parabolic system*

$$u_t + A(u)u_x - u_{xx} = 0, \tag{5.1}$$

with initial and boundary data u_0^1, u_{b0}^1, u_{bL}^1 and u_0^2, u_{b0}^2, u_{bL}^2 respectively. Then

$$\begin{aligned} \|u^1(t) - u^2(s)\|_{L^1(0,\,L)} \leq L_1 \Big(\|u_0^1 - u_0^2\|_{L^1(0,\,L)} + \|u_{b0}^1 - u_{b0}^2\|_{L^1(0,\,+\infty)} + \|u_{b0}^1 - u_{bL}^2\|_{L^1(0,\,+\infty)} \Big) \\ + L_2 \Big(|t - s| + |\sqrt{t} - \sqrt{s}\,| \Big). \end{aligned} \tag{5.2}$$

5.1 Stability with respect to initial and boundary data

We will prove that, in the hypothesis of Theorem 5.1,

$$\|u^1(t) - u^2(t)\|_{L^1(0,\,L)} \leq L_1 \Big(\|u_0^1 - u_0^2\|_{L^1(0,\,L)} + \|u_{b0}^1 - u_{b0}^2\|_{L^1(0,\,+\infty)} + \|u_{b0}^1 - u_{bL}^2\|_{L^1(0,\,+\infty)} \Big) \tag{5.3}$$

Let $z(t, x)$ be a first order perturbation of a solution $u(t, x)$ of (5.1). By straightforward computations one gets that z satisfies

$$z_t + \big(A(u)z\big)_x - z_{xx} = \big(DA(u)u_x\big)z - \big(DA(u)z\big)u_x. \tag{5.4}$$

To prove Theorem 5.1, it is enough to prove that any first order perturbation $z(t, x)$ satisfies the bound

$$\|z(t)\|_{L^1(0,\,L)} \leq L_1 \Big(\|z(t=0)\|_{L^1(0,\,L)} + \|z(x=0)\|_{L^1(0,\,+\infty)} + \|z(x=L)\|_{L^1(0,\,+\infty)} \Big). \tag{5.5}$$

Indeed, provided (5.5) holds, a homotopy argument which can be found in [8, 4] gives then the Lipschitz estimate (5.3).

To prove (5.5) it is convenient to introduce the auxiliary variable

$$\Upsilon = z_x - A(u)z,$$

which satisfies the equation

$$\begin{aligned} \Upsilon_t + (A(u)\Upsilon)_x - \Upsilon_{xx} = & \Big[DA(u)(u_x \otimes z - z \otimes u_x) \Big]_x - A(u) \Big[DA(u)\big(u_x \otimes z - z \otimes u_x\big) \Big] \\ & + DA(u)\big(u_x \otimes \Upsilon\big) - DA(u)\big(u_t \otimes z\big). \end{aligned} \tag{5.6}$$

Let $z_0(x)$, $z_{b\,0}(t)$ and $z_{b\,L}(t)$ be the initial and boundary conditions we impose on z: since the final goal is to apply (5.5) in the homotopy argument, it is not restrictive to suppose that $z_0(x)$, $z_{b\,0}(t)$ and $z_{b\,L}(t)$ satisfy the same regularity hypothesis as u. Indeed, the solution z of (5.4) that is used in the homotopy argument is on the boundaries and at $t = 0$ just the difference of the solutions u^1 and u^2 of (5.1).

Hence we will suppose that $z_0(x)$, $z_{b\,0}(t)$ and $z_{b\,L}(t)$ are regular and that $d^k z_0/dx^k$, $d^k z_{b\,0}/dt^k$ and $d^k z_{b\,L}/dt^k$, $k = 1, \dots n$ are integrable and have a small L^1 norm. Moreover, if $\|u_0^1 - u_0^2\|_{L^1(0,\,L)}$, $\|u_{b\,0}^1 - u_{b\,0}^2\|_{L^1(0,\,+\infty)}$ or $\|u_{b\,L}^1 - u_{b\,L}^2\|_{L^1(0,\,+\infty)}$ are infinite, then (5.3) holds trivially, and therefore we can suppose that $z_0 \in L^1(0,\,L)$, $z_{b\,0},\ z_{b\,L} \in L^1(0,\,+\infty)$.

From the hypothesis on z_0 it immediately follows that $\Upsilon(t = 0)$ is regular and small in L^1 and sup norm.

As in the proof of the BV bounds on the solution u, the crucial step to show (5.5) is the introduction of a suitable decomposition along travelling waves and double boundary layers: note, moreover, that u_x satisfies equation (5.4). Hence, it seems promising to decompose z along the same vectors $\tilde{r}_i(u,\,v_i,\,\sigma_i)$ and $\hat{r}_i(u,\,p_i)$ that appear in the decomposition (3.10) of u_x. This choice actually leads to non integrable source terms. We will therefore allow the vectors employed in the decomposition of z to depend not only on the solution u, but also on the perturbation z itself:

$$\begin{cases} z = z_1 \tilde{r}_1(u,\,v_1,\,\tau_1) + z_2 r_2 + q_1 \hat{r}_1(u,\,p_1) + q_2 r_2 \\ \Upsilon = \iota_1 \tilde{r}_1(u,\,v_1,\,\tau_1) + \iota_2 r_2. \end{cases}$$

In the previous expression the speed of the travelling waves described by the vector $\tilde{r}_1$ is not σ_1, but

$$\tau_1 = \theta\left(\lambda_1^* - \frac{z_1}{v_1}\right) - \lambda_1^*.$$

The function θ is the cutoff

$$\theta(s) = \begin{cases} s & \text{if } |s| \leq \hat{\delta} \\ 0 & \text{if } |s| \geq 3\hat{\delta} \\ \text{smooth connection if} & \hat{\delta} \leq s \leq 3\hat{\delta} \end{cases} \qquad \hat{\delta} \leq \frac{1}{3}.$$

The proof of (5.5) is from now on very similar to that of the BV bounds: one inserts the previous decomposition in the equations (5.4) and (5.6) and obtains the equations:

$$\begin{array}{ll} z_{1t} + (\lambda_1 z_1)_x - z_{1xx} = 0 & z_{2t} + (\lambda_2 z_2)_x - z_{2xx} = \underline{s}_1(t,\,x) \\ q_{1t} + (\lambda_1 q_1)_x - q_{1xx} = 0 & q_{2t} + (\hat{\lambda}_2 q_2)_x - q_{2xx} = 0 \\ \iota_{1t} + (\lambda_1 \iota_1)_x - \iota_{1xx} = \underline{s}_3(t,\,x) & \iota_{2t} + (\lambda_2 \iota_2)_x - \iota_{2xx} = \underline{s}_2(t,\,x) \end{array} \tag{5.7}$$

As in the proof of the BV bounds, to prove (5.5) it is sufficient to show that the condition

$$\|z(s)\|_{L^1(0,\,L)} \leq C\delta_1 \quad \forall\, s \in [0,\,t]$$

implies

$$\int_0^t \int_0^L |\underline{s}_i(s,\,x)| dx ds \leq \mathcal{O}(1)\delta_1^2 \quad i = 1,\,2,\,3$$

and suitable bounds on the boundary terms. Moreover, in the proof of the previous implication it is not restrictive to assume

$$\int_0^t \int_0^L |\underline{s}_i(s,\,x)| dx ds \leq \mathcal{O}(1)\delta_1 \quad i = 1,\,2,\,3,$$

because a posteriori one finds a bound of order δ_1^2.

Actually, one could observe that while the equations for u_x and u_t have no source term (see Appendix A.2.1 for details), the equations (5.4) and (5.6) have nontrivial source terms. However, one can show that both the source terms in (5.4) and (5.6) and the other terms that contribute to $\underline{s}_i$, $i = 1,\,2,\,3$ can be bounded by an expression analogous to the one that appears on the right side of (3.13). The computations that ensure such an estimate are quite similar to those performed in the proof of Section 3.1.

The proof of (5.5) can therefore be completed with the same tools described in Paragraph 4, hence we will skip all the details.

5.2 Stability with respect to time

Let $u(t, x)$ be a solution of (5.1): from Proposition 2.2 and the observations that follow one gets

$$\|u_{xx}(t)\|_{L^1} \leq \begin{cases} \mathcal{O}(1)\delta_1/\sqrt{t} & t \leq 1 \\ \mathcal{O}(1)\delta_1 & t > 1. \end{cases}$$

Let $t_1 \leq t_2$: the estimate above implies

$$\begin{aligned} \|u(t_1) - u(t_2)\|_{L^1(0,\,L)} &\leq \int_{t_1}^{t_2} \left\| \frac{\partial u}{\partial t}(t,\,x) \right\|_{L^1} dt \leq \int_{t_1}^{t_2} (\mathcal{O}(1)\|u_x(t,\,x)\|_{L^1} + \|u_{xx}(t,\,x))\|_{L^1})dt \\ &\leq \mathcal{O}(1) \int_{t_1}^{t_2} (\delta_1 + \delta_1/\sqrt{t})dt \leq \mathcal{O}(1)\delta_1|t_1 - t_2| + \mathcal{O}(1)\delta_1|\sqrt{t_1} - \sqrt{t_2}\,| \qquad (5.8) \\ &\leq L_2\Big(|t_1 - t_2| + |\sqrt{t_1} - \sqrt{t_2}\,|\Big). \end{aligned}$$

This completes the proof of Theorem 5.1 and hence of Theorem 1.1.

6 The vanishing viscosity limit

In this section we prove Theorem 1.2. The proof proceeds in two steps: first, by using the results of Theorem 1.1, we obtain that there exists a subsequence of solutions u^ϵ to the problem

$$\begin{cases} u_t + A(u)u_x = 0, & x \in]0,\,l[\;\; t \in]0, +\infty[\\ u(0,x) = \bar{u}_0(x) & \\ u(t,0) = \bar{u}_{b\,0}(t) & u(t,l) = \bar{u}_{bl}(t) \end{cases}$$

which converges to a Lipschitz semigroup. Then we use the machinery of viscosity solutions to complete the proof, showing the uniqueness of the limit. In particular, we exhibit explicitly the boundary Riemann solver.

Let $p_t^\varepsilon[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}]$ the solution of the system (1.2): from Theorem 4.1 one gets that the total variation of the solution of system (1.15) is uniformly bounded with respect to time and hence, by a change of variables, $p_t^\varepsilon[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}]$ satisfies

$$\mathrm{Tot\,Var}\big\{p_t^\varepsilon[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}]\big\},\ \Big|p_t^\varepsilon[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}](x)\Big| \leq \mathcal{O}(1)\delta_1 \quad \forall\, t > 0,\ x \in\ [0,\,l],\ \varepsilon > 0$$

and for any $\bar{u}_0 \in \mathcal{U}_0$ $\bar{u}_{b\,0},\,\bar{u}_{b\,l} \in \mathcal{U}_b$. By Helly's theorem, for every sequence $\varepsilon_n \to 0^+$ and for any $t \geq 0$ there exists a subsequence, which we still call ε_n for simplicity, such that $p_t^{\varepsilon_n}[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}]$ converges in $L^1(0,\,l)$. The stability with respect to time and to initial and boundary data ensures that, by a standard diagonalization procedure, one can find a function

$$\begin{array}{rcccc} p & : & [0,\,+\infty[\,\times\,\mathcal{U}_0\times\mathcal{U}_b\times\mathcal{U}_b & \to & \mathcal{D}_0 \\ & & (t,\,\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}) & \mapsto & p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}] \end{array}$$

such that, up to subsequences,

$$p_t^{\varepsilon_n}(t)[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}] \to p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}] \quad L^1(0,\,l) \quad \forall\, t \geq 0,\ \bar{u}_0 \in \mathcal{U}_0,\ \bar{u}_{b\,0},\,\bar{u}_{b\,l} \in \mathcal{U}_b.$$

Moreover, one can verify that the function

$$\begin{array}{rcccc} S & : & [0,\,+\infty[\,\times\,\mathcal{U}_0\times\mathcal{U}_b\times\mathcal{U}_b & \to & \mathcal{D}_0\times\mathcal{U}_b\times\mathcal{U}_b \\ & & (t,\,\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}) & \mapsto & \Big(p_t[\bar{u}_0,\,\bar{u}_{b\,0},\,\bar{u}_{b\,l}],\,\bar{u}_0(\,\cdot\,+t),\,\bar{u}_{b\,0}(\,\cdot\,+t),\,\bar{u}_{b\,l}(\,\cdot\,+t)\Big) \end{array} \qquad (6.1)$$

satisfies the semigroup properties, together with the Lipschitz estimate

$$\begin{aligned}\Big\|p_t[\bar u_0,\, \bar u_{b\,0},\, \bar u_{b\,l}] - p_s[\bar v_0,\, \bar v_{b\,0},\, \bar v_{b\,l}]\Big\|_{L^1} \le & L_1\Big(\|\bar v_0 - \bar u_0\|_{L^1(0,\,l)} + \|\bar v_{b0} - \bar u_{b0}\|_{L^1(0,\,+\infty)} \\ & + \|\bar v_{bl} - \bar u_{bl}\|_{L^1(0,\,+\infty)}\Big) + L_2|t-s|,\end{aligned} \tag{6.2}$$

We now make use of the tool of viscosity solution, which was first introduced in [10].

6.1 The Riemann solver and the boundary Riemann solver

A crucial step in the proof of the uniqueness of the vanishing viscosity limit is the local description of the vanishing viscosity solution in case of piecewise constant data, which however has an interest in its own. The aim of this section is to characterize the limit as $\varepsilon_n \to 0^+$ of the solution of

$$\begin{cases} u_t + A(u)u_x = \varepsilon_n u_{xx} \\ u(0,\, x) = \begin{cases} u^+ & x > 0 \\ u^- & x < 0 \end{cases} \\ u(t,\, 0) \equiv u_{b\,0} \qquad u(t,\, l) \equiv u_{b\,l} \end{cases} \tag{6.3}$$

where u^+, u^-, $u_{b\,0}$ and $u_{b\,l}$ are constants. In the following, we will write "solution to the Riemann problem" meaning "vanishing viscosity solution to the Riemann problem".

In [3, 7] it is shown that the solution of (6.3) is defined locally: to solve (6.3) it is therefore sufficient to characterize the vanishing viscous solutions in the following three cases:

1. the Cauchy problem with datum

$$u_0(x) = \begin{cases} u^- & x < 0 \\ u^+ & x > 0 \end{cases}$$

2. the boundary problem at $x = 0$

$$\begin{cases} u(0,\, x) \equiv u_0 \\ u(t,\, 0) \equiv u_{b\,0} \end{cases}$$

3. the boundary problem at $x = l$

$$\begin{cases} u(0,\, x) \equiv u_0 \\ u(t,\, l) \equiv u_{b\,l} \end{cases}$$

The second and the third case are clearly analogous, and therefore in Section 6.3 we will deal only with the second one. In the following section, instead, we will recall for completeness the essential steps of the construction of the solution in case 1: we refer to [7] for an exhaustive account.

In any of the three cases the crucial step is the definition of two families of admissible states, as it will be clearer in the following.

6.2 The non conservative Riemann solver

Since in this case the construction of the first and the second curve of admissible states is the same, we will describe only the construction of the first curve $T^1 u_r$ of the states that can be connected by waves of the first family to a right state u_r. For a general reference, see [7].

Consider the family $\Upsilon \subset \mathcal{C}^0([0,\, s]; \mathbb{R}^n \times \mathbb{R} \times \mathbb{R})$ of curves

$$\tau \mapsto (u(\tau),\, v_1(\tau),\, \sigma_1(\tau)), \quad \tau \in [0,\, s]$$

with

$$|u(\tau) - u^*| \le \varepsilon, \quad |v_1| \le \varepsilon, \quad |\lambda_1^* - \sigma_1(\tau)| \le \varepsilon.$$

The function $f_1(\tau)$ related to the curve $\gamma \in \Upsilon$ is defined as

$$f_1(\tau) = \int_0^\tau \lambda_1\big(u(\varsigma)\big)d\varsigma.$$

Let $\tilde{r}_1$ be the generalized eigenvector of the travelling waves of first family (see Section 3.1.2 for the proper definition of $\tilde{r}_1$). By the contraction map principle, one can show that if s is small enough then for any $\tau \in [0,\, s]$ there is a solution $(\hat{u},\, \hat{v}_1,\, \hat{\sigma}_1)$ of the following system:

$$\begin{cases} \hat{u}(\tau) = u_r + \displaystyle\int_0^\tau \tilde{r}_1\big(\hat{u}(\varsigma),\, \hat{v}_1(\varsigma),\, \hat{\sigma}_1(\varsigma)d\varsigma \\ \hat{v}_1(\tau) = \text{conc}_{[0,\, s]} f_1(\tau) - f_1(\tau) \\ \hat{\sigma}_1(\tau) = \dfrac{d\text{conc}_{[0,\, s]} f_1}{d\tau}. \end{cases}$$

We indicate with $\text{conc}_{[0,\, s]} f_1$ the concave envelope of f_1 in the interval $[0,\, s]$.

The curve of admissible states passing through u_r is defined as $T^1_s u_r = \hat{u}(s)$. Indeed, let

$$\tilde{u}(x/t) = \begin{cases} T^1_s u_r & x/t < \sigma_1(s) \\ \hat{u}(\tau) & \sigma_1(\tau) = x/t \\ u_r & x/t > \sigma_1(0) : \end{cases}$$

one can show that any sequence of vanishing viscosity solution of the Riemann problem with data $(u_r,\, T^1_s u_r)$ converges to $\tilde{u}$. Moreover, the curve $T^1_s u_r$ is Lipchitz continuous.

6.3 The boundary Riemann solver

In this paragraph we will construct the vanishing viscosity solution in case 2. We will proceed as follows: we will construct two curves of admissible states Z^1 and Z^2 and given a right state u_0 and a left state $u_{b\,0}$, we will show that there is a couple $(s_1,\, s_2)$ such that

$$Z^1_{s_1} \circ Z^2_{s_1} u_0 = u_{b\,0}.$$

The waves of the second family are entering the domain: it is therefore quite reasonable to suppose that they are not influenced by the presence of the boundary and therefore the second admissible curve will be the one defined in the previous paragraph, $Z^2_s u_0 = T^2_s u_0$. Let $\bar{u} = Z^2_{s_2} u_0$ be the value reached throughout the waves of the first family.

The waves of the first family are leaving the domain and are therefore affected by the boundary datum. To understand their behavior, it is convenient to focus the attention on the boundary layers of the first family, i.e. on the solution of

$$u_{xx} = A(u)u_x \tag{6.4}$$

that are exponentially decreasing to an equilibrium as $x \to +\infty$. One can now go back to the problem

$$A(u^\varepsilon)u^\varepsilon_x = \varepsilon u^\varepsilon_{xx}$$

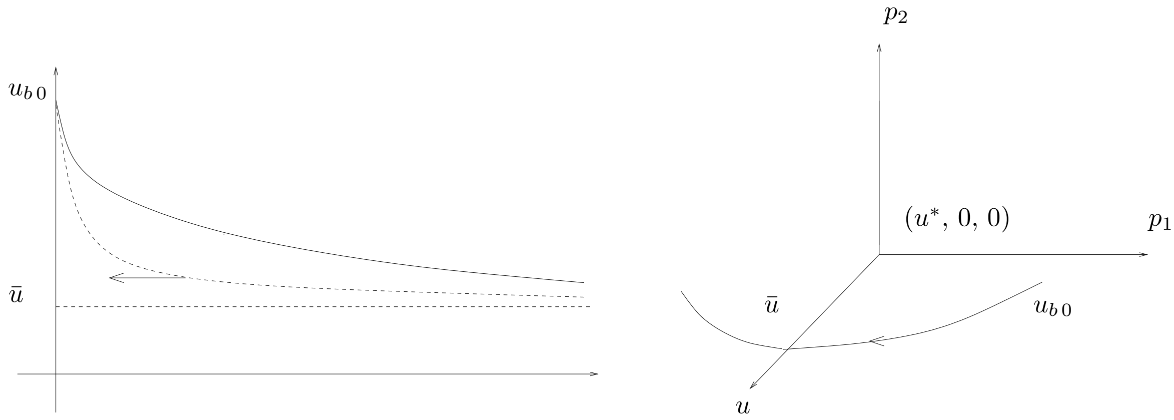
and let $\varepsilon \to 0^+$. Since $u^\varepsilon(x) = u(x/\varepsilon)$, we get

$$\lim_{\varepsilon \to 0^+} u^\varepsilon(0^+) = \lim_{x \to +\infty} u(x). \tag{6.5}$$

Such a behavior is illustrated in figure 3.

The value $\lim_{\varepsilon \to 0^+} u^\varepsilon(0^+)$ is the state reached throughout the waves of the second family: we called it $\bar{u}$. It also represents the trace of the hyperbolic limit on the boundary $x = 0$. From (6.5) it follows that the states which can be connected to $\bar{u}$ by boundary layers are the initial points of orbits that decrease exponentially to $\bar{u}$, i.e. that lay on the stable manifold throughout $\bar{u}$.

Figure 3: the graphic and the orbit of a boundary layer of the first family connecting $u_{b\,0}$ to $\bar{u}$: when $\varepsilon \to 0^+$ the graphic is pressed against the axis $x = 0$



The stable manifold at the equilibrium point $(\bar{u}, 0)$ of the system

$$\begin{cases} u_x = p \\ p_x = A(u)p \end{cases} \tag{6.6}$$

is parameterized by the projection p_1 of p on the stable space. Passages analogous to those in Section 3.1 ensure that the stable manifold is characterized by the relation $p = p_1 \check{r}_1(p_1)$ for a suitable vector function

$$\check{r}_1 = \begin{pmatrix} 1 \\ f(p_1) \end{pmatrix}.$$

One imposes $u_1(+\infty) = \langle l_1, \bar{u} \rangle$ and from the second equation gets

$$u_1(0) = \langle l_1, \bar{u} \rangle - p_1(0) \exp\left(\int_0^{+\infty} \lambda_1 \Big(u_1\big(p_1(0), x\big) \Big) dx \right).$$

Since $\lambda_1 \leq -c < 0$, the previous map is invertible and one can express $p_1(0)$ as a function of $u_1(0)$. The inverse map is clearly regular.

We parameterize the stable manifold by $s_1 := u_1 - < l_1, \bar{u} >$ and obtain (for some suitable regular function z) the map

$$Z^1_{s_1} \bar{u} = \begin{pmatrix} \langle l_1, \bar{u} \rangle + s_1 \\ z(s_1) \end{pmatrix},$$

defined on a small enough interval $[0, s]$.

The vanishing viscosity solution of

$$\begin{cases} u_t + A(u)u_x = 0 \\ u(t, 0) \equiv u_0 \qquad u(0, x) \equiv u_{b\,0} \end{cases} \tag{6.7}$$

can be constructed patching together the curve described so far. Let

$$u_{b\,0} = Z^1_{s_1} \circ T^2_{s_2} u_0 :$$

thanks to a version of the implicit function theorem valid for Lipschitz maps (see [18]), one can reconstruct from u_0 and $u_{b\,0}$ the couple (s_1, s_2). The vanishing viscosity solution of (6.7) is then given by

$$u(t, x) = \begin{cases} T^2_{s_2} u_0 & x/t < \sigma_2(s_2) \\ \hat{u}(\tau) & \sigma_2(\tau) = x/t \\ u_0 & x/t > \sigma_2(0). \end{cases}$$

One gets in particular that the trace of the solution at $x = 0$ is not necessarily the boundary value $u_{b\,0}$, but it is the intermediate state $T^2_{s_2} u_0$.

6.4 Viscosity solutions

Before giving the definition of viscosity solution we have to introduce some preliminary notation; moreover, in the following we will use the spaces $\mathcal{U}_0, \mathcal{U}_b, \mathcal{D}_0$ that have been defined in the introduction (equation (1.9) and previous lines).

Let $u(t, x)$ be a function such that, for any t, $u(t) \in \mathcal{D}_0$: given a point $(\tau, \xi) \in]0, l[\times[0, +\infty[$, let $A^\flat = A\big(u(\tau, \xi)\big)$ and let $U^\flat_{(u,\,\tau,\,\xi)}$ be the solution of the linear Cauchy problem

$$w_t + A^\flat w_x = 0 \qquad w(0, x) = u(\tau, x).$$

Viceversa, let $U^\sharp_{(u,\,\tau,\,\xi)}$ be the solution (defined in Section 6.2) of the Riemann problem

$$\begin{aligned} &u_t + A(u)u_x = 0 \\ &u(0, x) = \begin{cases} u(\tau, \xi^-) & x < 0 \\ u(\tau, \xi^+) & x > 0 \end{cases} \end{aligned}$$

The previous limits are well defined, since $u(\tau) \in BV(0, l)$. Given a function $\bar{u}_{b\,0} \in \mathcal{U}_b$, the definition of $U^\sharp_{(u,\,\tau,\,\xi)}$ can be extended naturally to the case $\xi = 0$: it is enough to define $U^\sharp_{(u,\,\bar{u}_{b0},\,\tau)}$ as the solution (described in Section 6.3) of the boundary Riemann problem

$$\begin{cases} u_t + A(u)u_x = 0 \\ u(0, x) \equiv u(\tau, 0^+) \qquad u(t, 0) \equiv \bar{u}_{b\,0}(\tau^+). \end{cases}$$

Given a function $\bar{u}_{b\,l} \in \mathcal{U}_b$, the definition of $U^\sharp_{(u,\,\bar{u}_{bl},\,\tau)}$ is clearly analogous.

Definition 6.1. Let $u(t, x)$ such that for any t, $u(t) \in \mathcal{D}_0$ and such that the function $t \mapsto u(t, \cdot)$ is continuous in L^1_{loc} and let $\bar{u}_{b\,0}, \ \bar{u}_{b\,l} \in \mathcal{U}_b$ and $\bar{u}_0 \in \mathcal{U}_0$.

Then u is a viscosity solution of the system

$$\begin{cases} u_t + A(u)u_x = 0, \quad x \in]0, l[, \ t \in]0, +\infty[\\ u(0, x) = \bar{u}_0(x) \\ u(t, 0) = \bar{u}_{b\,0}(t) \qquad u(t, l) = \bar{u}_{b\,l}(t) \end{cases} \tag{6.8}$$

if and only if the followings hold:
(i) $u(0) = \bar{u}_0$
(ii) for every $\beta > 0$ and for every point (τ, ξ) with $\xi \neq 0, \ l$

$$\lim_{h \to 0^+} \frac{1}{h} \int_{\max\{0,\,\xi - \beta h\}}^{\min\{l,\,\xi + \beta h\}} \big| u(\tau + h, x) - U^\sharp_{(u,\,\tau,\,\xi)}(h, x - \xi) \big| dx = 0$$

(iii) for every $\beta > 0$ and for every $\tau > 0$

$$\lim_{h \to 0^+} \frac{1}{h} \int_0^{\min\{l,\,\beta h\}} \big| u(\tau + h, x) - U^\sharp_{(u,\,\bar{u}_{b0},\,\tau)}(h, x) \big| dx = 0;$$

and

$$\lim_{h \to 0^+} \frac{1}{h} \int_{\max\{0,\,l - \beta h\}}^{l} \big| u(\tau + h, x) - U^\sharp_{(u,\,\bar{u}_{bl},\,\tau)}(h, x) \big| dx = 0$$

(iv) there exist constants C and β' such that for every point (τ, ξ) with $\xi \neq 0, \ l$ and for every $\rho > 0$ small enough

$$\limsup_{h \to 0^+} \frac{1}{h} \int_{\max\{0,\,\xi - \rho + \beta' h\}}^{\min\{l,\,\xi + \rho - \beta' h\}} \big| u(\tau + h, x) - U^\flat_{(u,\,\tau,\,\xi)}(h, x - \xi) \big| dx \leq C \Big(\text{Tot Var}\big(u(\tau), \,]\xi - \rho, \xi + \rho[\big) \Big)^2.$$

The previous definition may appear a bit complex: note, however, that, since ρ and h can be arbitrarily small, it is not restrictive to suppose

$$\begin{array}{ll} \max\{0,\,\xi-\beta h\}=\xi-\beta h & \min\{l,\,\xi+\beta h\}=\xi+\beta h \\ \max\{0,\,\xi-\rho+\beta' h\}=\xi-\rho+\beta' h & \min\{l,\,\xi+\rho-\beta' h\}=\xi+\rho-\beta' h \\ \max\{0,\,l-h\beta\}=l-h\beta & \min\{l,h\beta\}=h\beta \end{array}$$

The definition of viscosity solution ensures, roughly speaking, that a function is well approximated by the solution of a suitable linear problem and of a suitable Riemann problem.

The following proposition ensures that viscosity solutions coincide indeed with vanishing viscosity limits. The proof is very simile to that of the analogous property stated in [7] (Lemma 15.2, page 308) and will be therefore omitted.

Proposition 6.1. *Let $\bar u_0 \in \mathcal{U}_0$ and $\bar u_{b\,0},\ \bar u_{b\,l} \in \mathcal{U}_b$. Let $p_t(\bar u_0,\,\bar u_{b\,0},\,\bar u_{b\,l})$ be a vanishing viscosity solution of the system (6.8): then $p_t(\bar u_0,\,\bar u_{b\,0},\,\bar u_{b\,l})$ is a viscosity solution of the same system. Viceversa, if $u(t,\,x)$ is a viscosity solution of the problem (6.8) then*

$$u(t)=p_t(\bar u_0,\,\bar u_{b\,0},\,\bar u_{b\,l}) \quad \forall\, t\geq 0.$$

From the previous result it immediately follows the uniqueness of the semigroup: indeed, let by contradiction $p^1_t(\bar u_0,\,\bar u_{b\,0},\,\bar u_{b\,l})$ and $p^2_t(\bar u_0,\,\bar u_{b\,0},\,\bar u_{b\,l})$ be two different vanishing viscosity solutions. The function $p^1_t(\bar u_0,\,\bar u_{b\,0},\,\bar u_{b\,l})$ is hence a viscosity solution of problem (1.1) by the first part of Proposition 6.1. Then $p^1_t(\bar u_0,\,\bar u_{b\,0},\,\bar u_{b\,l}) = p^2_t(\bar u_0,\,\bar u_{b\,0},\,\bar u_{b\,l})$ for any $t\geq 0$ by the second part of the proposition.

A Appendix

A.1 Appendix to Section 2

A.1.1 Proof of Proposition 2.1

In the following, for simplicity we will suppose $\lambda^*_i=\lambda^*_2>0$, since the case $\lambda^*_i=\lambda^*_1<0$ is analogous. We denote by

$$\Gamma^{\lambda^*_2}(t,x,y)=(1-e^{-xy/t})G(t,\,x-y-\lambda^*_2 t)$$

the solution of the equation

$$z_t+\lambda^*_2 z_x - z_{xx}=0 \tag{A.1}$$

in the first quadrant with zero boundary datum and Cauchy datum δ_y. The following estimates have been proved in [7]:

$$\begin{array}{lll} \|\Gamma^{\lambda^*_2}(t,\,y)\|_{L^1(0,+\infty)}\leq \mathcal{O}(1) & \left|\displaystyle\int_0^{+\infty}\left|\int_y^{+\infty}\Gamma^{\lambda^*_2}_x(t,\,x,\,\xi)d\xi\right|dx\right|\leq\mathcal{O}(1) & \forall\, t\in\mathbb{R}^+ \\ \|\Gamma^{\lambda^*_2}_x(t,\,y)\|_{L^1(0,+\infty)}\leq \dfrac{\mathcal{O}(1)}{\sqrt t} & \left|\displaystyle\int_0^{+\infty}\left|\int_y^{+\infty}\Gamma^{\lambda^*_2}_{xx}(t,\,x,\,\xi)d\xi\right|dx\right|\leq\dfrac{\mathcal{O}(1)}{\sqrt t} & \forall\, t\in(0,1) \end{array} \tag{A.2}$$

Let $G(t,\,x)=\exp(-x^2/4t)/2\sqrt{\pi t}$: we will use the notation

$$G(t,\,x-\lambda^*_2 t)=G^{\lambda^*_2}(t,\,x).$$

The estimate on the L^1 norm of $\Delta^{\lambda^*_2}$ in Proposition 2.1 can be obtained via the maximum principle applied to equation (A.1): indeed,

$$0\leq\Delta^{\lambda^*_2}(t,\,x,\,y)\leq G^{\lambda^*_2}(t,\,x-y)\quad\forall\, t\geq 0\quad x,\,y\in]0,\,L[,$$

and therefore $\|\Delta^{\lambda^*_2}(t,\,y)\|_{L^1}\leq 1$.

To prove the estimate on the L^1 norm of $\Delta_x^{\lambda_2^*}$ it is convenient to write $\Delta^{\lambda_2^*}$ as

$$\Delta^{\lambda_2^*}(t,\,x,\,y) = \Gamma^{\lambda_2^*}(t,\,x,\,y) + \phi^{\lambda_2^*}(t,\,x,\,y)\sum_{m\neq 0}\Big[G(t,x+2mL-y)-G(t,x+2mL+y)\Big],$$

with

$$\phi^{\lambda_2^*}(t,\,x,\,y) = \exp\left(\frac{\lambda_2^*}{2}(x-y)-\frac{(\lambda_2^*)^2}{4}t\right).$$

Since $\lambda_2^* > 0$, for $m > 0$ it holds

$$\begin{aligned}&\left|\frac{\partial}{\partial x}\Big(\phi^{\lambda_2^*}(t,\,x,\,y)G(t,\,x-y+2mL)\Big)\right|\\ &\qquad = \phi^{\lambda_2^*}(t,\,x,\,y)\left|\frac{\lambda_2^*}{2}G(t,\,x-y+2mL)+G_x(t,\,x-y+2mL)\right| \le \big|G_x^{\lambda_2^*}(t,\,x-y+2mL)\big|,\end{aligned}$$

and similarly

$$\left|\frac{\partial}{\partial x}\left(\exp\left(\frac{\lambda_2^*}{2}(x-y)-\frac{\lambda^{*\,2}}{4}t\right)G(t,\,x+y+2mL)\right)\right| \le \big|G_x^{\lambda_2^*}(t,\,x+y+2mL)\big|.$$

The terms of the series corresponding to $m < 0$ can be estimated as follows: let $n := -m$ then

$$\begin{aligned}&\left|\frac{\partial}{\partial x}\Big(\phi(t,\,x,\,y)G(t,\,x-y-2nL)\Big)\right|\\ &\quad = \phi^{\lambda_2^*}(t,\,x,\,y)\left|-G_x(t,\,2nL-x+y)-\frac{\lambda_2^*}{2}G(t,\,2nL-x+y)+\lambda_2^*G(t,\,2nL-x+y)\right|\\ &\quad \le \left|G_x^{\lambda_2^*}(t,\,2nL-x+y)\right| + \lambda_2^*\left|G^{\lambda_2^*}(t,\,2nL-x+y)\right|\end{aligned}$$

and similarly

$$\left|\frac{\partial}{\partial x}\Big(\phi(t,\,x-y)G_x(t,\,x+y-2nL)\Big)\right| \le \left|G_x^{\lambda_2^*}(t,\,2nL-x-y)\right| + \lambda_2^*\left|G^{\lambda_2^*}(t,\,2nL-x-y)\right|.$$

Since $\|G_x^{\lambda_2^*}(t)\|_{L^1} \le \mathcal{O}(1)/\sqrt{t}$, one obtains

$$\begin{aligned}\|\Delta_x^{\lambda_2^*}(t,\,y)\|_{L^1(0,\,L)} &\le \int_0^L |\Gamma_x^{\lambda_2^*}(t,\,x,\,y)|dx + \int_{2L}^{+\infty}\Big(\big|G_z^{\lambda_2^*}(t,\,z+y)\big| + \big|G_z^{\lambda_2^*}(t,\,z-y)\big|\Big)dz\\ &+ \int_L^{+\infty}\Big(\big|G_z^{\lambda_2^*}(t,\,z+y)\big| + \big|G_z^{\lambda_2^*}(t,\,z-y)\big|\Big)dz + \lambda_2^*\int_L^{+\infty}\Big(\big|G_z^{\lambda_2^*}(t,\,z+y)\big| + \big|G_z^{\lambda_2^*}(t,\,z-y)\big|\Big)dz \le \frac{\mathcal{O}(1)}{\sqrt{t}}.\end{aligned}$$

In the following estimates, we will suppose $y < L/2$: by symmetry this is not restrictive. Observe that, for $y < L/2$

$$x+y-2L < -L/2 < 0 \quad \forall x \in [0,\,L]. \tag{A.3}$$

This assumption corresponds to the fact that the most singular part in Δ^{λ^*} is collected in Γ^{λ^*}, i.e. it is given by $G(t,x+y)-G(t,x-y)$. If $y > L/2$, then the most singular part would be given by $G(x-y)-G(x+y-2L)$.

One has

$$\tilde{\Delta}^{\lambda_2^*}(t,\,x,\,y) = \int_y^L \Gamma_x^{\lambda_2^*}(t,\,x,\,\xi)d\xi + \int_y^L \phi_x(t,\,x,\,\xi)\sum_{m\neq 0}\Big[G(t,\,x-\xi+2mL) - G(t,\,x+\xi+2mL)\Big]d\xi$$

$$+ \int_y^L \phi(t,\,x,\,\xi)\sum_{m\neq 0}\Big[G_x(t,\,x-\xi+2mL) - G_x(t,\,x+\xi+2mL)\Big]d\xi$$

$$= \int_y^L \Gamma_x^{\lambda_2^*}(t,\,x,\,\xi)d\xi - \int_y^L \Big\{\phi_x(t,\,x,\,\xi)\sum_{m\neq 0} G(t,\,x+\xi+2mL) - \phi(t,\,x,\,\xi)\sum_{m\neq 0} G_x(t,\,x+\xi+2mL)\Big\}d\xi$$

$$+ \int_y^L \Big\{\phi_x(t,\,x,\,\xi)\sum_{m\neq 0} G(t,\,x-\xi+2mL) + \phi(t,\,x,\,\xi)\sum_{m\neq 0} G_x(t,\,x-\xi+2mL)\Big\}d\xi$$

$$= \int_y^L \Gamma_x^{\lambda_2^*}(t,\,x,\,\xi)d\xi + \sum_{m\neq 0}\phi(t,\,x,\,y)G(t,\,x+y+2mL) - \sum_{m\neq 0}\phi(t,\,x,\,L)G(t,\,x+L+2mL)$$

$$- \int_y^L \sum_{m\neq 0}\lambda_2^*\phi(t,\,x,\,\xi)G(t,\,x+\xi+2mL)d\xi + \sum_{m\neq 0}\phi(t,\,x,\,y)G(t,\,x-y+2mL)$$

$$- \sum_{m\neq 0}\phi(t,\,x,\,L)G(t,\,x-L+2mL).$$

The integrability of the first term follows from (A.2), the other terms are clearly integrable because of the quadratic exponential decay of the heat kernel G: hence $\|\tilde{\Delta}^{\lambda_2^*}(t,y)\|_{L^1} \leq \mathcal{O}(1)$.

The function $\tilde{\Delta}_x^{\lambda_2^*}$ can be written as follows:

$$\tilde{\Delta}_x^{\lambda_2^*}(t,\,x,\,y) = \int_y^L \Gamma_{xx}^{\lambda_2^*}(t,\,x,\,\xi)d\xi + \sum_{m\neq 0}\frac{\lambda_2^*}{2}\phi(t,\,x,\,y)G(t,\,x+y+2mL)$$

$$+ \sum_{m\neq 0}\phi(t,\,x,\,y)G_x(t,\,x+y+2mL) - \sum_{m\neq 0}\frac{\lambda_2^*}{2}\phi(t,\,x,\,L)G(t,\,x+L+2mL)$$

$$- \sum_{m\neq 0}\phi(t,\,x,\,L)G_x(t,\,x+L+2mL) - \int_y^L \sum_{m\neq 0}\frac{(\lambda_2^*)^2}{2}\phi(t,\,x,\,\xi)G(t,\,x+\xi+2mL)d\xi$$

$$- \int_y^L \sum_{m\neq 0}\lambda_2^*\phi(t,\,x,\,\xi)G_x(t,\,x+\xi+2mL)d\xi + \sum_{m\neq 0}\frac{\lambda_2^*}{2}\phi(t,\,x,\,y)G(t,\,x,-y+2mL)$$

$$+ \sum_{m\neq 0}\phi(t,\,x,\,y)G_x(t,\,x,-y+2mL) - \sum_{m\neq 0}\frac{\lambda_2^*}{2}\phi(t,\,x,\,L)G(t,\,x-L+2mL)$$

$$- \sum_{m\neq 0}\phi(t,\,x,\,L)G_x(t,\,x-L+2mL),$$

and hence with computations similar to those performed before one gets

$$\|\tilde{\Delta}_x^{\lambda_2^*}(t,\,y)\|_{L^1} \leq \frac{\mathcal{O}(1)}{\sqrt{t}} \quad \forall\, t\leq 1 \;\; y\in]0,\,L[.$$

If one derives the explicit formula (2.6) for $J^{\lambda\, L}$ and then integrate by parts gets

$$\int_0^L |J_x^{\lambda_2^*\, L}(t,\,x)|dx = \int_0^L \left|\lambda_2^* C e^{\lambda_2^* x}dx - \lambda_2^* C\int_0^L \tilde{\Delta}^{\lambda_2^*}(t,\,x,\,y)e^{\lambda_2^* y}dy\right| dx \leq \mathcal{O}(1),$$

where we have used the estimate $\|\tilde{\Delta}^{\lambda_2^*}\|_{L^1} \leq \mathcal{O}(1)$. By symmetry it follows $\|J_x^{\lambda_2^*\, 0}\|_{L^1} \leq \mathcal{O}(1)$.

Deriving $J_x^{\lambda_2^*\, L}$ one obtains

$$\|J_{xx}^{\lambda_2^*\, L}(t)\|_{L^1} \leq \int_0^L |C(\lambda_2^*)^2 e^{\lambda_2^* x}|dx + C\lambda_2^*\int_0^L \left|\int_0^L \tilde{\Delta}_x^{\lambda_2^*}(t,\,x,\,y)e^{\lambda_2^* y}dy\right| dx \leq \frac{\mathcal{O}(1)}{\sqrt{t}},$$

thanks to the estimate on $\|\tilde{\Delta}_x^{\lambda_2^*}\|$. By symmetry one gets $\|J_{xx}^{\lambda_2^*\,0}(t)\|_{L^1} \le \mathcal{O}(1)/\sqrt{t}$: this concludes the proof of Proposition 2.1.

A.2 Appendix to Section 3

A.2.1 Explicit source terms

We want to find the equations satisfied by the quantities $v_1,\, v_2,\, p_1,\, p_2,\, w_1,\, w_2$: we will use the decomposition

$$\begin{cases} u_x = v_1 \tilde{r}_1(u,\, v_1,\, \sigma_1) + v_2 r_2 + p_1 \hat{r}_1(u, p_1) + p_2 r_2 \\ u_t = w_1 \tilde{r}_1(u,\, v_1,\, \sigma_1) + w_2 r_2 \end{cases} \qquad \sigma_1 = \lambda_1(u^*) - \theta\bigg(\frac{w_1}{v_1} + \lambda_1(u^*)\bigg).$$

and insert it in the parabolic equation

$$u_t + A(u) u_x - u_{xx} = 0.$$

A derivation w.r.t. x gives

$$\begin{aligned} \tilde{r}_{1x} &= \mathrm{D}\tilde{r}_1 (v_1 \tilde{r}_1 + v_2 r_2 + p_1 \hat{r}_1 + p_2 r_2) + v_{1x} \tilde{r}_{1v} + \sigma_{1x} \tilde{r}_{1\sigma} \\ \hat{r}_{1x} &= \mathrm{D}\hat{r}_1 (v_1 \tilde{r}_1 + v_2 r_2 + p_1 \hat{r}_1 + p_2 r_2) + p_{1x} \hat{r}_{1p}. \end{aligned}$$

Recalling that

$$\begin{aligned} &\hat{\lambda}_2 := \lambda_2 - p_1 \langle \hat{\ell}_2,\, \mathrm{D}\hat{r}_1 r_2 \rangle \\ &A(u)\tilde{r}_1 = v_1 \mathrm{D}\tilde{r}_1 \tilde{r}_1 + \lambda_1 \tilde{r}_1 + v_1 (\lambda_1 - \sigma_1) \tilde{r}_{1v} \\ &A(u)\hat{r}_1 = p_1 \mathrm{D}\hat{r}_1 \hat{r}_1 + \lambda_1 \hat{r}_1 + p_1 \lambda_1 \hat{r}_{1p} \end{aligned}$$

one gets

$$\begin{aligned} u_t =\; & u_{xx} - A(u) u_x \\ =\; & v_{1x} \tilde{r}_1 + v_1 \tilde{r}_{1x} + p_{1x} \hat{r}_1 + p_1 \hat{r}_{1x} + v_{2x} r_2 + p_{2x} r_2 - v_1 A(u) \tilde{r}_1 - p_1 A(u) \hat{r}_1 - \lambda_2 v_2 r_2 - \lambda_2 p_2 r_2 \\ =\; & (v_{1x} - \lambda_1 v_1)(\tilde{r}_1 + v_1 \tilde{r}_{1v}) + v_1^2 \sigma_1 \tilde{r}_{1v} + v_1 v_2 \mathrm{D}\tilde{r}_1 r_2 + v_1 p_1 \mathrm{D}\tilde{r}_1 \hat{r}_1 + v_1 p_2 \mathrm{D}\tilde{r}_1 r_2 + v_1 \sigma_{1x} \tilde{r}_{1\sigma} \\ & + (p_{1x} - \lambda_1 p_1)(\hat{r}_1 + p_1 \hat{r}_{1p}) + v_1 p_1 \mathrm{D}\hat{r}_1 \tilde{r}_1 + v_2 p_1 \mathrm{D}\hat{r}_1 r_2 + (v_{2x} - \lambda_2 v_2) r_2 + (p_{2x} - \hat{\lambda}_2 p_2) r_2. \end{aligned} \tag{A.4}$$

We multiply the previous expressions by ℓ_1 and by $\tilde{\ell}_2$, the vectors of the dual basis of $(\tilde{r}_1,\, r_2)$: we obtain

$$\begin{aligned} w_1 &= v_{1x} - \lambda_1 v_1 + p_{1x} - \lambda_1 p_1 \\ w_2 &= v_{2x} - \lambda_2 v_2 + p_{2x} - \hat{\lambda}_2 p_2 + e(t,\, x), \end{aligned} \tag{A.5}$$

where the error term $e(t,\, x)$ satisfies the estimate (3.13) in Paragraph 8.2.2.

Deriving (A.4), one obtains

$$
\begin{aligned}
u_{tx} =& \Big(v_{1xx} - (\lambda_1 v_1)_x\Big)\tilde r_1 + \Big(v_1\big(v_{1xx} - (\lambda_1 v_1)_x\big) + 2v_{1x}\big(v_{1x} - \lambda_1 v_1\big) + (v_1^2\sigma_1)_x\Big)\tilde r_{1v} \\
& + \Big(v_1\big(v_{1x} - \lambda_1 v_1\big)\Big)\mathrm{D}\tilde r_1 \tilde r_1 + \Big(v_2\big(v_{1x} - \lambda_1 v_1\big) + \big(v_1 v_2\big)_x\Big)\mathrm{D}\tilde r_1 r_2 \\
& + \Big(p_1\big(v_{1x} - \lambda_1 v_1\big) + (v_1 p_1)_x\Big)\mathrm{D}\tilde r_1 \hat r_1 + \Big(p_2\big(v_{1x} - \lambda_1 v_1\big) + (v_1 p_2)_x\Big)\mathrm{D}\tilde r_1 r_2 \\
& + \Big(\sigma_{1x}(v_{1x} - \lambda_1 v_1) + (v_1\sigma_{1x})_x\Big)\tilde r_{1\sigma} + \Big(v_1\big(v_{1x} - \lambda_1 v_1\big) + v_1^2\sigma_1\Big)(\tilde r_{1v})_x \\
& + v_1 v_2(\mathrm{D}\tilde r_1 r_2)_x + v_1 p_1(\mathrm{D}\tilde r_1 \hat r_1)_x + v_1 p_2(\mathrm{D}\tilde r_1 r_2)_x + v_1\sigma_{1x}(\tilde r_{1\sigma})_x \\
& + \Big(p_{1xx} - (\lambda_1 p_1)_x\Big)\hat r_1 + \Big(p_1\big(p_{1xx} - (\lambda_1 p_1)_x\big) + 2p_{1x}\big(p_{1x} - \lambda_1 p_1\big)\Big)\hat r_{1p} \\
& + \Big(v_1\big(p_{1x} - \lambda_1 p_1\big) + (v_1 p_1)_x\Big)\mathrm{D}\hat r_1 \tilde r_1 + \Big(v_2\big(p_{1x} - \lambda_1 p_1\big) + (v_2 p_1)_x\Big)\mathrm{D}\hat r_1 r_2 \\
& + \Big(p_1\big(p_{1x} - \lambda_1 p_1\big)\Big)\mathrm{D}\hat r_1 \hat r_1 + \Big(p_2\big(p_{1x} - \lambda_1 p_1\big)\Big)\mathrm{D}\hat r_1 r_2 + \Big(p_1\big(p_{1x} - \lambda_1 p_1\big)\Big)(\hat r_{1p})_x \\
& + v_1 p_1(\mathrm{D}\hat r_1 \tilde r_1)_x + v_2 p_1(\mathrm{D}\hat r_1 r_2) + \Big(v_{2xx} - (\lambda_2 v_2)_x\Big) r_2 + \Big(p_{2xx} - (\lambda_2 p_2)_x\Big) r_2.
\end{aligned}
\tag{A.6}
$$

On the other hand,

$$
u_{xt} = v_{1t}\tilde r_1 + v_1 \tilde r_{1t} + v_{2t} r_2 + p_{1t}\hat r_1 + p_1 \hat r_{1t} + p_{2t} r_2, \tag{A.7}
$$

where

$$
\begin{aligned}
\tilde r_{1t} &= \mathrm{D}\tilde r_1(w_1\tilde r_1 + w_2 r_2) + v_{1t}\tilde r_{1v} + \sigma_{1t}\tilde r_{1\sigma} \\
\hat r_{1t} &= \mathrm{D}\hat r_1(w_1\tilde r_1 + w_2 r_2) + p_{1t}\hat r_{1p}.
\end{aligned}
\tag{A.8}
$$

We equal (A.6) and (A.7) and we use (A.8), obtaining

$$
\begin{aligned}
0 =& \; u_{tx} - u_{xt} \\
=& \Big(v_{1xx} - (\lambda_1 v_1)_x - v_{1t}\Big)\tilde r_1 + \Big(v_1\big(v_{1xx} - (\lambda_1 v_1)_x - v_{1t}\big) + 2v_{1x}\big(v_{1x} - \lambda_1 v_1\big) + (v_1^2\sigma_1)_x\Big)\tilde r_{1v} \\
& + \Big(v_1(v_{1x} - \lambda_1 v_1) - v_1 w_1\Big)\mathrm{D}\tilde r_1 \tilde r_1 + \Big(v_2(v_{1x} - \lambda_1 v_1) + (v_1 v_2)_x - v_1 w_2\Big)\mathrm{D}\tilde r_1 r_2 \\
& + \Big(p_1\big(v_{1x} - \lambda_1 v_1\big) + (v_1 p_1)_x\Big)\mathrm{D}\tilde r_1 \hat r_1 + \Big(p_2\big(v_{1x} - \lambda_1 v_1\big) + (v_1 p_2)_x\Big)\mathrm{D}\tilde r_1 r_2 \\
& + \Big(\sigma_{1x}\big(v_{1x} - \lambda_1 v_1\big) + (v_1\sigma_{1x})_x - \sigma_{1t} v_1\Big)\tilde r_{1\sigma} + \Big(v_{1x} v_1 + v_1^2(\sigma_1 - \lambda_1)\Big)(\tilde r_{1v})_x + v_1 v_2(\mathrm{D}\tilde r_1 r_2)_x \\
& + v_1 p_1(\mathrm{D}\tilde r_1 \hat r_1)_x + v_1 p_2(\mathrm{D}\tilde r_1 r_2)_x + v_1\sigma_{1x}(\tilde r_{1\sigma})_x + \Big(p_{1xx} - (\lambda_1 p_1)_x - p_{1t}\Big)\hat r_1 \\
& + \Big(p_1\big(p_{1xx} - (\lambda_1 p_1)_x - p_{1t}\big) + 2p_{1x}\big(p_{1x} - \lambda_1 p_1\big)\Big)\hat r_{1p} + \Big(v_1\big(p_{1x} - \lambda_1 p_1\big) + (v_1 p_1)_x - w_1 p_1\Big)\mathrm{D}\hat r_1 \tilde r_1 \\
& + \Big(v_2(p_{1x} - \lambda_1 p_1) + (v_2 p_1)_x - w_2 p_1\Big)\mathrm{D}\hat r_1 r_2 + p_1\Big(p_{1x} - \lambda_1 p_1\Big)\mathrm{D}\hat r_1 \hat r_1 + \Big(p_2\big(p_{1x} - \lambda_1 p_1\big)\Big)\mathrm{D}\hat r_1 r_2 \\
& + \Big(p_1(p_{1x} - \lambda_1 p_1)\Big)(\hat r_{1p})_x + v_1 p_1(\mathrm{D}\hat r_1 \tilde r_1)_x + v_2 p_1(\mathrm{D}\hat r_1 r_2)_x + \Big(v_{2xx} - (\lambda_2 v_2)_x - v_{2t}\Big) r_2 \\
& + \Big(p_{2xx} - (\hat\lambda_2 p_2)_x - p_{2t}\Big) r_2 \\
=& \Big(v_{1xx} - (\lambda_1 v_1)_x - v_{1t}\Big)\tilde r_1 + \Big(p_{1xx} - (\lambda_1 p_1)_x - p_{1t}\Big)\hat r_1 \\
& + \Big(v_{2xx} - (\lambda_2 v_2)_x - v_{2t}\Big) r_2 + \Big(p_{2xx} - (\hat\lambda_2 p_2)_x - p_{2t}\Big) r_2 + s_1(t,x).
\end{aligned}
$$

We can therefore impose the conditions

$$
\begin{aligned}
& v_{1t} + (\lambda_1 v_1)_x - v_{1xx} = 0 \\
& p_{1t} + (\lambda_1 p_1)_x - p_{1xx} = 0 \\
& v_{2t} + (\lambda_2 v_2)_x - v_{2xx} = \langle \tilde\ell_2(t,x), s_1(t,x)\rangle = \tilde s_1(t,x) \\
& p_{2t} + (\hat\lambda_2 p_2)_x - p_{2xx} = 0,
\end{aligned}
$$

where $(\ell_1, \tilde{\ell}_2)$ is the dual basis of $(\tilde{r}_1, r_2)$.

To compute the equations satisfied by $w_1,\ w_2$ we will use

$$\begin{aligned} u_{tt} &= u_{xx\,t} - \big(A(u)u_x\big)_t \\ &= u_{t\,xx} - \big(A(u)u_t\big)_x + DA(u)\big(u_x \otimes u_t - u_t \otimes u_x\big), \end{aligned}$$

which follows from

$$\begin{aligned} \big(A(u)u_x\big)_t &= DA(u)(u_t \otimes u_x) + A(u)u_{xt} \\ \big(A(u)u_t\big)_x &= DA(u)(u_x \otimes u_t) + A(u)u_{tx}. \end{aligned}$$

Straightforward computations ensures that

$$\begin{aligned} u_{xt} - A(u)u_t =& (w_{1x} - \lambda_1 w_1)\tilde{r}_1 + w_1(v_{1x} - \lambda_1 v_1)\tilde{r}_{1v} \\ &+ w_1 v_1 \sigma_1 \tilde{r}_{1v} + w_1 v_2 \mathrm{D}\tilde{r}_1 r_2 + w_1 p_1 \mathrm{D}\tilde{r}_1 \hat{r}_1 + w_1 p_2 \mathrm{D}\tilde{r}_1 r_2 \\ &+ w_1 \sigma_{1x} \tilde{r}_{1\sigma} + (w_{2x} - \lambda_2 w_2) r_2 \end{aligned}$$

and

$$\begin{aligned} DA(u)\bigg(u_x \otimes u_t - u_t \otimes u_x\bigg) =& v_1 w_2 \tilde{r}_1 \otimes r_2 + v_2 w_1 r_2 \otimes \tilde{r}_1 + p_1 w_1 \hat{r}_1 \otimes \tilde{r}_1 + p_1 w_2 \hat{r}_1 \otimes r_2 + p_2 w_1 r_2 \otimes \tilde{r}_1 \\ &- w_1 v_2 \tilde{r}_1 \otimes r_2 - w_1 p_1 \tilde{r}_1 \otimes \hat{r}_1 - w_1 p_2 \tilde{r}_1 \otimes r_2 - w_2 v_1 r_2 \otimes \tilde{r}_1 - w_2 p_1 r_2 \otimes \hat{r}_1. \end{aligned} \tag{A.9}$$

Hence

$$\begin{aligned} u_{tt} &= w_{1t}\tilde{r}_1 + w_{2t} r_2 \\ &= -w_1^2 \mathrm{D}\tilde{r}_1 \tilde{r}_1 - w_1 w_2 \mathrm{D}\tilde{r}_1 r_2 - w_1 v_{1t} \tilde{r}_{1v} - w_1 \sigma_{1t} \tilde{r}_{1\sigma} + \Big(w_{1xx} - (\lambda_1 w_1)_x\Big)\tilde{r}_1 + \Big(v_1\big(w_{1x} - \lambda_1 w_1\big)\Big)\mathrm{D}\tilde{r}_1 \tilde{r}_1 \\ &\quad + \Big(v_2(w_{1x} - \lambda_1 w_1)\Big)\mathrm{D}\tilde{r}_1 r_2 + \Big(p_1(w_{1x} - \lambda_1 w_1)\Big)\mathrm{D}\tilde{r}_1 \hat{r}_1 + \Big(p_2(w_{1x} - \lambda_1 w_1)\Big)\mathrm{D}\tilde{r}_1 r_2 + \Big(v_{1x}(w_{1x} - \lambda_1 w_1)\Big)\tilde{r}_{1v} \\ &\quad + \Big(\sigma_{1x}(w_{1x} - \lambda_1 w_1)\Big)\tilde{r}_{1\sigma} + \Big(w_{1x}(v_{1x} - \lambda_1 v_1)\Big)\tilde{r}_{1v} + \Big(w_1(v_{1x} - \lambda_1 v_1)\Big)(\tilde{r}_{1v})_x + w_1 v_1 \sigma_1 (\tilde{r}_{1v})_x \\ &\quad + (w_1 v_1 \sigma_1)_x \tilde{r}_{1v} + (w_1 v_2)_x \mathrm{D}\tilde{r}_1 r_2 + w_1 v_2 (\mathrm{D}\tilde{r}_1 r_2)_x + (w_1 p_1)_x \mathrm{D}\tilde{r}_1 \hat{r}_1 + w_1 p_1 (\mathrm{D}\tilde{r}_1 \hat{r}_1)_x \\ &\quad + (w_1 p_2)_x \mathrm{D}\tilde{r}_1 r_2 + w_1 p_2 (\mathrm{D}\tilde{r}_1 r_2)_x + (w_1 \sigma_{1x})_x \tilde{r}_{1\sigma} + w_1 \sigma_{1x} (\tilde{r}_{1\sigma})_x + \Big(w_{2xx} - (\lambda_2 w_2)_x\Big) r_2 \\ &\quad + (p_1 w_2)_x \mathrm{D}\hat{r}_1 r_2 + p_1 w_2 (\mathrm{D}\hat{r}_1 r_2)_x \\ &\quad + DA(u)\Big(v_1 w_2 \tilde{r}_1 \otimes r_2 + v_2 w_1 r_2 \otimes \tilde{r}_1 + p_1 w_1 \hat{r}_1 \otimes \tilde{r}_1 + p_1 w_2 \hat{r}_1 \otimes r_2 + p_2 w_1 r_2 \otimes \tilde{r}_1 \\ &\quad - w_1 v_2 \tilde{r}_1 \otimes r_2 - w_1 p_1 \tilde{r}_1 \otimes \hat{r}_1 - w_1 p_2 \tilde{r}_1 \otimes r_2 - w_2 v_1 r_2 \otimes \tilde{r}_1 - w_2 p_1 r_2 \otimes \hat{r}_1\Big) \\ &= \Big(w_{1xx} - (\lambda_1 w_1)_x\Big)\tilde{r}_1 + \Big(w_{2xx} - (\lambda_2 w_2)_x\Big) r_2 + s_2(t,\, x). \end{aligned}$$

One can check that, since A is triangular,

$$\langle \ell_1,\, DA(u)\big(u_x \otimes u_t - u_t \otimes u_x\big)\rangle = 0$$

and therefore the equations satisfied by $w_i\ i = 1,\, 2$ are

$$\begin{aligned} & w_{1t} + (\lambda_1 w_1)_x - w_{1xx} = 0 \\ & w_{2t} + (\lambda_2 w_2)_x - w_{2xx} = \langle \tilde{\ell}_2(t,\, x),\, s_2(t,\, x)\rangle = \tilde{s}_2(t,\, x). \end{aligned} \tag{A.10}$$

A.2.2 Proof of Proposition 3.1

Equation (3.10) and (3.11) ensure that, since,

$$\sigma_1 = \lambda_1^* - \theta\left(\frac{w_1}{v_1} + \lambda_1^*\right),$$

then

$$\sigma_{1x} = -\theta'\left(\frac{w_1}{v_1} + \lambda_i^*\right)\left(\frac{w_1}{v_1}\right)_x = -\left(\frac{w_{1x}v_1 - v_{1x}w_1}{v_1^2}\right)\theta',$$

$$|v_1^2\sigma_{1x}| = \mathcal{O}(1)|w_{1x}v_1 - v_{1x}w_1|,$$

$$\sigma_{1x} \neq 0 \iff \left|\frac{w_1}{v_1} - \lambda_1^*\right| \leq 3\hat{\delta}.$$

Most of the terms in $\tilde{s}_i(t, x)$ $i = 1, 2$ and $e(t, x)$ can be directly reduced to those in Proposition 3.1. The terms which requires some technicalities are:

1. $$|p_{1x} - \lambda_1 p_1|\,|\langle\tilde{\ell}_2(u, v_1, \sigma_1),\ \hat{r}_1(u, p_1)\rangle| \leq \mathcal{O}(1)(|p_1| + |v_1|)|p_{1x} - \lambda_1 p_1|.$$

   Indeed,

   $$|\langle\tilde{\ell}_2(u, v_1, \sigma_1),\ \hat{r}_1(u, p_1)\rangle| \leq |\langle\tilde{\ell}_2,\ \hat{r}_1 - r_1^*\rangle| + |\langle\tilde{\ell}_2 - \ell_2^*,\ r_1^*\rangle| \leq \mathcal{O}(1)(|p_1| + |v_1|).$$

   We have denoted by r_1^* the first eigenvector of the matrix $A(u^*)$ and by (ℓ_1, ℓ_2^*) the dual base of (r_1^*, r_2).

2. $$\left|2v_{1x}(v_{1x} - \lambda_1 v_1) + (v_1^2\sigma_1)_x\right| = \left|2v_{1x}\Big(w_1 - (p_{1x} - \lambda_1 p_1)\Big) + 2v_1 v_{1x}\sigma_1 + v_1^2\sigma_{1x}\right|$$
   $$\leq \left|2v_{1x}(w_1 + v_1\sigma_1)\right| + \left|2v_{1x}(\lambda_1 p_1 - p_{1x})\right| + \mathcal{O}(1)\left|v_{1x}w_1 - v_1 w_{1x}\right|.$$

3. $\left|\sigma_{1x}(v_{1x} - \lambda_1 v_1) + (v_1\sigma_{1x})_x - \sigma_{1t}v_1\right||\tilde{r}_{1\sigma}|$: some computations ensures that

   $$\left(\frac{w_1}{v_1}\right)_x(v_{1x} - \lambda_1 v_1) + v_{1x}\left(\frac{w_1}{v_1}\right)_x + v_1\left(\frac{w_1}{v_1}\right)_{xx} - \left(\frac{w_1}{v_1}\right)_t v_1 = 0.$$

   Hence, since $|\tilde{r}_{1\sigma}| = \mathcal{O}(1)|v_1|$, one gets

   $$\left|\sigma_{1x}(v_{1x} - \lambda_1 v_1) + (v_1\sigma_{1x})_x - \sigma_{1t}v_1\right|\,|\tilde{r}_{1\sigma}|\ \leq \mathcal{O}(1)\chi_{\{|\lambda_1^* - w_1/v_1| \leq 3\hat{\delta}\}}v_1^2\left|\left(\frac{w_1}{v_1}\right)_x\right|^2.$$

4. $|-w_1\sigma_{1t} + \sigma_{1x}(w_{1x} - \lambda_1 w_1) + (w_1\sigma_{1x})_x|\,|\tilde{r}_{1\sigma}|$:
   since

   $$-w_1\theta'\left(\frac{w_1}{v_1}\right)_t + w_{1x}\theta'\left(\frac{w_1}{v_1}\right)_x - \lambda_1 w_1\theta'\left(\frac{w_1}{v_1}\right)_x + w_{1x}\theta'\left(\frac{w_1}{v_1}\right)_x + w_1\theta'\left(\frac{w_1}{v_1}\right)_{xx} = 0,$$

   one is left to the estimate

   $$\left|\theta''\left(\frac{w_1}{v_1}\right)_x\right|^2|w_1 v_1| \leq \mathcal{O}(1)v_1^2\chi_{\{|\lambda_1^* - w_1/v_1| \leq 3\hat{\delta}\}}\left|\left(\frac{w_1}{v_1}\right)_x\right|^2.$$

5. $|v_1\sigma_{1x}(\tilde r_{1\sigma})_x|$: first of all, we observe that that $\theta'(s)\neq 0$ implies $|w_1|\leq \mathcal{O}(1)|v_1|$ and therefore

$$|v_1\sigma_{1x}| = |v_1\theta'|\Big|\Big(\frac{w_1}{v_1}\Big)_x\Big| = \Big|\frac{w_{1x}v_1 - v_{1x}w_1}{v_1^2}\Big||v_1\theta'| \leq \mathcal{O}(1)(|w_{1x}|+|v_{1x}|).$$

We develop

$$|(\tilde r_{1\sigma})_x| = |(\tilde r_{1x})_\sigma| \leq \mathcal{O}(1)\bigg(|v_1|+|v_2|+|p_1|+|p_2|+|v_{1x}|\bigg) + \mathcal{O}(1)|v_1\sigma_{1x}|.$$

Since

$$\theta'\neq 0 \quad\Rightarrow\quad |w_1| = |v_{1x}-\lambda_1 v_1 + p_{1x} - \lambda_1 p_1| \leq \mathcal{O}(1)|v_1| \quad\Rightarrow\quad |v_{1x}|\leq \mathcal{O}(1)|v_1| + |p_{1x}-\lambda_1 p_1|,$$

one has

$$\begin{aligned}|v_{1x}\sigma_{1x}v_1| &= \Big|\frac{w_{1x}v_1 - v_{1x}w_1}{v_1^2}\Big||\theta' v_1 v_{1x}|\\ &\leq \mathcal{O}(1)|w_{1x}v_1 - w_1 v_{1x}| + \mathcal{O}(1)\bigg(|w_{1x}| + \mathcal{O}(1)|v_{1x}|\bigg)|p_{1x}-\lambda_1 p_1|.\end{aligned}$$

Using the previous estimates, we get

$$\begin{aligned}&|v_1\sigma_{1x}(\tilde r_{1\sigma})_x| \leq \mathcal{O}(1)|w_{1x}v_1 - v_{1x}w_1| + \mathcal{O}(1)\big(|v_1|+|w_{1x}|\big)\,\big(|v_2|+|p_1|+|p_2|\big)\\ &\quad + \mathcal{O}(1)|w_{1x}v_1 - w_1v_{1x}| + \mathcal{O}(1)\bigg(|w_{1x}|+\mathcal{O}(1)|v_{1x}|\bigg)|p_{1x}-\lambda_1p_1| + \mathcal{O}(1)v_1^2\chi_{\{|\lambda_1^*-w_1/v_1|\leq 3\hat\delta\}}\Big(\frac{w_1}{v_1}\Big)_x^2.\end{aligned}$$

6.

$$\begin{aligned}|v_1(w_{1x}-\lambda_1w_1) - w_1^2| &= |v_1w_{1x} - v_{1x}w_1 + v_{1x}w_1 - \lambda_1v_1w_1 - w_1^2|\\ &\leq |v_1w_{1x} - v_{1x}w_1| + |w_1(v_{1x}-\lambda_1v_1 - w_1)|\\ &\leq |v_1w_{1x} - v_{1x}w_1| + |w_1(p_{1x}-\lambda_1p_1)|.\end{aligned}$$

7.

$$\begin{aligned}&|w_{1x}(v_{1x}-\lambda_1v_1) + (w_1v_1\sigma_1)_x + (w_{1x}-\lambda_1w_1)v_{1x}|\\ &\qquad = |2w_{1x}(v_{1x}-\lambda_1v_1-w_1) - w_{1x}v_{1x} + \lambda_1w_{1x}v_1 + 2w_{1x}w_1\\ &\qquad\quad + w_{1x}v_1\sigma_1 + w_1v_{1x}\sigma_1 + w_1v_1\sigma_{1x} + w_{1x}v_{1x} - \lambda_1w_1v_{1x}|\\ &\qquad = |2w_{1x}(p_{1x}-\lambda_1p_1) + 2w_{1x}(w_1+\sigma_1v_1) - \sigma_1w_{1x}v_1\\ &\qquad\quad + \lambda_1w_{1x}v_1 + \sigma_1w_1v_{1x} + (w_{1x}v_1 - w_1v_{1x})\theta'(w_1/v_1) - \lambda_1w_1v_{1x}|\\ &\qquad \leq 2|w_{1x}(p_{1x}-\lambda_1p_1)| + 2|w_{1x}(w_1+\sigma_1v_1)|\\ &\qquad\quad + |(\lambda_1-\sigma_1)(w_{1x}v_1 - w_1v_{1x})| + (w_{1x}v_1 - w_1v_{1x})\theta'(w_1/v_1)|\end{aligned}$$

8.

$$|w_1(w_1+\sigma_1v_1 - p_{1x} + \lambda_1p_1)| = |w_1(w_1+\sigma_1v_1) + w_1(p_{1x}-\lambda_1p_1)|$$

9.

$$|w_1\sigma_{1x}(\tilde r_{1\sigma})_x| \leq |v_1\sigma_{1x}(\tilde r_{1\sigma})_x|,$$

and therefore one comes back to case (5).

This completes the proof of the estimate (3.1).

A.3 Appendix to Paragraph 4

A.3.1 Proof of the estimate (4.9)

It is convenient to introduce a representation formula for p_i, $i = 1, 2$. To this end, two new convolution kernels are needed: let $I^{\lambda_i^*\,0}(t, s, x)$ be the solution of the equation

$$I_t^{\lambda_i^*\,0} + \lambda_i^* I_x^{\lambda_i^*\,0} - I_{xx}^{\lambda_i^*} = 0,$$

with boundary and initial data

$$I^{\lambda_i^*\,0}(0, s, x) \equiv 0, \quad I^{\lambda_i^*\,0}(t, s, 0) = \delta_{t=s}, \quad I^{\lambda_i^*\,0}(t, s, L) \equiv 0.$$

Without specifying the explicit expression of $I^{\lambda_i^*}$, we observe that

$$\int_0^{+\infty} I^{\lambda_i^*\,0}(t, s, x)ds = J^{\lambda_i^*\,0}(t, x)$$

(see equation (2.6) for the definition of $J^{\lambda_i^*\,0}$). Analogously, let $I^{\lambda_i^*\,L}(t, x)$ be the solution of the equation

$$I_t^{\lambda_i^*\,L} + \lambda_i^* I_x^{\lambda_i^*\,L} - I_{xx}^{\lambda_i^*\,L} = 0,$$

with boundary and initial data:

$$I^{\lambda_i^*\,0}(0, s, x) \equiv 0, \quad I^{\lambda_i^*\,0}(t, s, 0) \equiv 0, \quad I^{\lambda_i^*\,0}(t, s, L) = \delta_{t=s}.$$

By construction, it satisfies

$$\int_0^{+\infty} I^{\lambda_i^*\,L}(t, s, x)ds = J^{\lambda_i^*\,L}(t, x)$$

(see equation (2.8) for the definition of $J^{\lambda_i^*\,L}$). If $t \leq 1$ the function p_1 admits the following representation formula:

$$\begin{aligned} p_1(t, x) = & \int_0^{+\infty} I^{\lambda_1^*\,0}(t, s, x)p_1(s, 0)ds + \int_0^{+\infty} I^{\lambda_1^*\,L}(t, s, x)p_1(s, L)ds + \\ & + \int_0^t \int_0^L \Delta^{\lambda_1^*}(t-s, x, y)\Big((\lambda_1^* - \lambda_1)p_{1y} - \lambda_{1y}p_1\Big)(s, y)dyds, \end{aligned}$$

and hence

$$\begin{aligned} p_{1x}(t, x) = & \int_0^{+\infty} I_x^{\lambda_1^*\,0}(t, s, x)p_1(s, 0)ds + \int_0^{+\infty} I_x^{\lambda_1^*\,L}(t, s, x)p_1(s, L)ds \\ & + \int_0^t \int_0^L \Delta_x^{\lambda_1^*}(t-s, x, y)\Big((\lambda_1^* - \lambda_1)p_{1y} - \lambda_{1y}p_1\Big)(s, y)dyds. \end{aligned}$$

From the expression of $\Delta^{\lambda_1^*}$, given by formula (2.4), it follows that

$$\Big\| \Delta_x^{\lambda_1^*}(t, \cdot, y) \exp(c(\cdot - y)/2) \Big\|_{L^1} \leq \frac{\mathcal{O}(1)}{\sqrt{t}},$$

and from the previous observations

$$\left| \exp(cx/2) \int_0^{\infty} I_x^{\lambda_i^*\,0}(t, s, x)ds \right| = \left| \exp(cx/2) J_x^{\lambda_i^*\,0}(t, x) \right| \leq \mathcal{O}(1)$$

$$\left| \int_0^{+\infty} I_x^{\lambda_i^*\,L}(t, s, x)ds \right| = \left| J_x^{\lambda_i^*\,L}(t, x) \right|.$$

Hence

$$\big|\exp(cx/2)p_1(t,\,x)\big| = \left|\exp(cx/2)\int_0^{+\infty} I_x^{\lambda_1^*\,0}(t,\,s,\,x)p_1(s,\,0)ds\right| + \left|\exp(cx/2)\int_0^{+\infty} I_x^{\lambda_1^*\,L}(t,\,s,\,x)p_1(s,\,L)ds\right|$$
$$+\left|\exp(cx/2)\int_0^t\int_0^L \Delta_x^{\lambda_1^*}(t-s,\,x,\,y)\Big((\lambda_1^*-\lambda_1)p_{1y}-\lambda_{1y}p_1\Big)(s,\,y)dyds\right|$$
$$\leq \mathcal{O}(1)|p(x=0)|_\infty + \mathcal{O}(1)|p(x=L)|_\infty$$
$$+\mathcal{O}(1)\delta_1\left|\int_0^t\left(\sup_y p_{1y}(s,\,y)\exp(cy/2)\right)\int_0^L \Delta_x^{\lambda_1^*}(t-s,\,x,\,y)\exp\big(c(x-y)/2\big)dsdy\right| + \mathcal{O}(1)\delta_1^2$$

and therefore

$$|\sup_x p_{1x}(t,\,x)\exp(cx/2)| \leq \mathcal{O}(1)\delta_1 \quad \forall\, t\leq 1.$$

The estimate

$$\sup_x \big|p_{2x}(t,\,x)\exp\big(c(L-x)/2\big)\big| \leq \mathcal{O}(1)\delta_1 \quad \forall\, t\leq 1$$

follows by symmetry.

If $t>1$ the following representation formula holds:

$$p_{1x}(t,\,x) = \int_0^L p_1(t-1,\ ,y)\Delta^{\lambda_1^*}(1,\,x,\,y)dy + \int_{t-1}^{+\infty} I_x^{\lambda_1^*\,0}(1,s,\,x)p_1(s,\,0)ds$$
$$+\int_{t-1}^{+\infty} I_x^{\lambda_1^*\,L}(1,\,s,\,x)p_1(s,\,L)ds$$
$$+\int_0^1\int_0^L \Delta_x^{\lambda_1^*}(1-s,\,x,\,y)\Big((\lambda_1^*-\lambda_1)p_{1y}-\lambda_{1y}p_1\Big)(t-1+s,\,y)dyds.$$

It follows that

$$|\sup_x p_{1x}(t,\,x)\exp(cx/2)| \leq \mathcal{O}(1)\delta_1 \quad \forall\, t>1,$$

and hence by symmetry

$$\big|\sup_x p_{2x}(t,\,x)\exp\big(c(L-x)/2\big)\big| \leq \mathcal{O}(1)\delta_1 \quad \forall\, t>1.$$

This concludes the proof of (4.9).

A.3.2 Proof of Proposition 4.3

We will perform the computations only for $v_2,\ w_2$ and w_{2x}, since those for $v_1,\ w_1$ and w_{1x} follow by symmetry.

Three new convolution kernels: the solution of equation

$$Q_t + \lambda_2^* Q_x - Q_{xx} = 0 \tag{A.11}$$

with boundary conditions

$$Q(0,\,x)=\delta_y, \qquad Q(t,\,0)=0, \qquad Q_x(t,\,L)=0$$

is

$$Q(t,\,x) = \Theta^{\lambda_2^*}(t,\,x,\,y) := \int_0^x \phi(t,\,z,\,y)\bigg(\sum_m G_z(z+2mL-y)+G_z(z+2mL+y)\bigg)dz \tag{A.12}$$

As in Section 2, we use the notation

$$\phi^{\lambda_2^*}(t,x,y) = \exp\left(\frac{\lambda_2^*}{2}(x-y) - \frac{(\lambda_2^*)^2}{4}t\right).$$

and $G(t,\,x) = \exp(-x^2/4t)/2\sqrt{\pi t}$. Note that, by construction,

$$\Theta_x^{\lambda_2^*}(t,\,0,\,y) \equiv 0 \quad \forall\, t \geq 0, \quad y \in]0,\,L[\,. \tag{A.13}$$

Moreover, an argument similar to that used in Section 4.1.1 ensures that a maximum principle holds for equation (A.11), in other words if

$$Q(0,\,x) \leq 0, \qquad Q(t,\,0) \leq 0, \qquad Q_x(t,\,L) \leq 0,$$

then $Q(t,\,x) \leq 0 \;\forall\, t,\; x$.

The solution of (A.11) with boundary conditions

$$Q(0,\,x) = 0, \qquad Q(t,\,0) = 1, \qquad Q_x(t,\,L) = 0$$

is

$$B^{\lambda_2^*}(t,\,x) = 1 - \int_0^L \Theta^{\lambda_2^*}(t,\,x,\,y)dy. \tag{A.14}$$

In the following, we will need another convolution kernel, $\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,y)$, such that

$$\tilde{\Theta}_y^{\lambda_2^*}(t,\,x,\,y) = -\Theta_x^{\lambda_2^*}(t,\,x,\,y). \tag{A.15}$$

We arbitrarily impose $\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,L) \equiv 0 \;\forall\, t,\; x$ and define

$$\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,y) := \int_y^L \Theta_x^{\lambda_2^*}(t,\,x,\,\xi)d\xi.$$

Recalling (A.13), we observe that $\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,y)$ is the derivative with respect to x of a function z such that

$$\begin{gathered} z_x(t,\,0,\,y) \equiv 0 \quad z_x(t,\,L,\,y) \equiv 0 \quad z(0,\,x,\,y) = \begin{cases} 0 & 0 < x \leq y \\ 1 & y \leq x < L \end{cases} \\ z_t + \lambda_2^* z_x - z_{xx} = 0. \end{gathered} \tag{A.16}$$

It follows that $\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,y)$ satisfies

$$\tilde{\Theta}^{\lambda_2^*}(t,\,0,\,y) \equiv 0 \quad \tilde{\Theta}^{\lambda_2^*}(t,\,L,\,y) \equiv 0 \quad \tilde{\Theta}^{\lambda_2^*}(0,\,x,\,y) = \delta_y$$

and hence actually

$$\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,y) \equiv \Delta^{\lambda_2^*}(t,\,x,\,y), \tag{A.17}$$

where $\Delta^{\lambda_2^*}$ is the convolution kernel defined by (2.4). In the following, however, for sake of clearness we will write $\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,y)$ when we want to underline that the relation (A.15) holds. From the identity (A.17) and the estimates (2.11) it follows

$$\|\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,y)\|_{L^1} \leq \mathcal{O}(1) \quad \|\tilde{\Theta}_x^{\lambda_2^*}(t,\,x,\,y)\|_{L^1} \leq \frac{\mathcal{O}(1)}{\sqrt{t}} \quad \forall\, t \leq 1. \tag{A.18}$$

Moreover, let z be as in (A.16) and let $B^{\lambda_2^*}$ be defined by (A.14), then $z(t,\,x,\,0) + B^{\lambda_2^*}(t,\,x) \equiv 1$ and hence

$$\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,0) + B_x^{\lambda_2^*}(t,\,x) = 0. \tag{A.19}$$

Such an identity, together with (A.18), implies

$$\|B_x^{\lambda_2^*}(t,\,x)\|_{L^1} \leq \mathcal{O}(1) \quad \|B_{xx}^{\lambda_2^*}(t,\,x)\|_{L^1} \leq \frac{\mathcal{O}(1)}{\sqrt{t}} \quad t \leq 1. \tag{A.20}$$

Since the kernels introduced so far will be used to prove the integrability of v_{2x} with respect to time, one has to prove that they are integrable on small time intervals.

- $$\int_0^1 |\tilde\Theta_x^{\lambda_2^*}(t,\,x,\,y)|dt = \int_0^1 |\Delta_x^{\lambda_2^*}(t,\,x,\,y)|dt \le \mathcal{O}(1) \quad \forall x \in\, [0,\,L], \quad \forall\, y \in]0,\,L[\tag{A.21}$$

Proof. One can check that

$$\int_0^1 |G_x^{\lambda_2^*}(t,\,x-y)|dt \le \mathcal{O}(1) \qquad \int_0^1 |G^{\lambda_2^*}(t,\,x-y)|dt \le \mathcal{O}(1) \quad \forall\, x,\,y \in \mathbb{R}. \tag{A.22}$$

Since

$$\Delta_x^{\lambda_2^*}(t,\,x,\,y) = \Big(\phi(t,\,x,\,y)\sum_{m\ge 0} G(t,\,x-y+2mL)\Big)_x - \Big(\phi(t,\,x,\,y)\sum_{m\ge 0} G(t,\,x+y+2mL)\Big)_x + \Big(\phi(t,\,x,\,y)\sum_{n>0} G(t,\,x-y-2nL)\Big)_x - \Big(\phi(t,\,x,\,y)\sum_{n>0} G(t,\,x+y-2nL)\Big)_x,$$

one gets

$$\begin{aligned}|\Delta_x^{\lambda_2^*}(t,\,x,\,y)| \le& \sum_{m\ge 0}|G_x^{\lambda_2^*}(t,\,x-y+2mL)| + \sum_{m\ge 0}|G_x^{\lambda_2^*}(t,\,x+y+2mL)|\\ &+ \sum_{n>0}|G_x^{\lambda_2^*}(t,\,2nL+y-x)| + \lambda_2^*\sum_{n>0}|G^{\lambda_2^*}(t,\,2nL+y-x)|\\ &+ \sum_{n>0}|G_x^{\lambda_2^*}(t,\,2nL-y-x)| + \lambda_2^*\sum_{n>0}|G^{\lambda_2^*}(t,\,2nL-y-x)|.\end{aligned}$$

Since

$$|G_x^{\lambda_2^*}(t,\,z+2mL)| \le e^{-mL}|G_x^{\lambda_2^*}(t,\,z)| \qquad |G^{\lambda_2^*}(t,\,z+2mL)| \le e^{-mL}|G^{\lambda_2^*}(t,\,z)|$$

if $m \ge 0$, $t \le 1$ and z is large enough, from the previous estimates and from (A.22) one deduces (A.21). □

- From equation (A.19) and the previous estimate it follows

$$\int_0^1 |B_{xx}^{\lambda_2^*}(t,\,x)|dt \le \mathcal{O}(1) \quad \forall\, x \in [0,\,L]. \tag{A.23}$$

A representation formula for v_2 : it is convenient to introduce the auxiliary function

$$V_2(t,\,x) = \int_0^x v_2(t,\,\xi)d\xi,$$

which satisfies the equation

$$V_{2t} + \lambda_2 V_{2x} - V_{2xx} = \tilde S_1(t,\,x),$$

where

$$\tilde S_1(t,\,x) = \int_0^x \tilde s_1(t,\,\xi)d\xi.$$

The boundary and initial conditions of $V_2(t,\,x)$ are

$$V_2(0,\,x) = \int_0^x v_2(0,\,\xi)d\xi, \qquad V_2(t,\,0) = \int_0^t (v_{2x} - \lambda_2 v_2)(s,\,0)ds, \qquad V_{2x}(t,\,L) = 0.$$

The convolution kernels (A.12) and (A.14) provide the representation formula

$$\begin{aligned} V_2(t,\,x) =& \int_0^L \Theta^{\lambda_2^*}(t,\,x,\,y)V_2(0,\,y)dy + \int_0^t B(t-s,\,x)(v_{2x}-\lambda_2 v_2)(s,\,0)ds \\ &+ \int_0^t\int_0^L \Theta^{\lambda_2^*}(t-s,\,x,\,y)\Big(\big(\lambda_2^*-\lambda_2\big)v_2\Big)(s,\,y)dyds \\ &+ \int_0^t\int_0^L \Theta^{\lambda_2^*}(t-s,\,x,\,y)\tilde{S}_1(s,\,y)dyds. \end{aligned} \tag{A.24}$$

Since

$$\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,0) + B_x^{\lambda_2^*}(t,\,x) \equiv 0, \qquad \tilde{S}_1(t,\,0)\equiv 0,$$

from (A.24) one gets

$$\begin{aligned} V_{2x}(t,\,x) =v_2(t,\,x) =& \int_0^L \tilde{\Theta}^{\lambda_2^*}(t,\,x,\,y)v_2(0,\,y)dy + \int_0^t B_x^{\lambda_2^*}(t-s,\,x)\Big(v_{2x}-\lambda_2^* v_2\Big)(s,\,0)ds \\ &+ \int_0^t\int_0^L \tilde{\Theta}^{\lambda_2^*}(t-s,\,x,\,y)\tilde{s}_1(s,\,y)dyds + \int_0^t\int_0^L \tilde{\Theta}^{\lambda_2^*}(t-s,\,x,\,y)\Big(\big(\lambda_2^*-\lambda_2\big)v_2\Big)_y(s,\,y)dyds \end{aligned}$$

and

$$\begin{aligned} v_{2x}(t,\,x) =& \int_0^L \tilde{\Theta}_x^{\lambda_2^*}(t,\,x,\,y)v_2(0,\,y)dy + \int_0^t B_{xx}^{\lambda_2^*}(t-s,\,x)\Big(v_{2x}-\lambda_2^* v_2\Big)(s,\,0)ds \\ &+ \int_0^t\int_0^L \tilde{\Theta}_x^{\lambda_2^*}(t-s,\,x,\,y)\tilde{s}_1(s,\,y)dyds \\ &+ \int_0^t\int_0^L \tilde{\Theta}_x^{\lambda_2^*}(t-s,\,x,\,y)\Big(\big(\lambda_2^*-\lambda_2\big)v_{2y}-\lambda_{2y}v_2\Big)(s,\,y)dyds. \end{aligned}$$

From the estimate (A.18), (A.21) and (A.23) on the convolution kernels it follows

$$\begin{aligned} \int_0^1 |v_{2x}(t,\,x)|dt \le& \|v_2(0)\|_{L^1}\sup_{x,\,y}\int_0^1 |\tilde{\Theta}^{\lambda_2^*}(t,\,x,\,y)|dt \\ &+ \mathcal{O}(1)\bigg(\int_0^1 \Big\{(v_{2x}-\lambda_2 v_2)(s,\,0) + (\lambda_2^*-\lambda_2)v_2(s,\,0)ds\Big\}ds\bigg) \\ &+ \left(\int_0^1 |\tilde{s}_1(s)|_\infty ds\right)\left(\int_0^1 \frac{\mathcal{O}(1)}{\sqrt{t}}\,dt\right) + \left(\int_0^1 \frac{\mathcal{O}(1)}{\sqrt{s}}\,ds\right)\left(\delta_1 \sup_y \int_0^1 |v_{2y}|(s,\,y)ds + \delta_1^2\right) \\ \le& \mathcal{O}(1)\delta_1, \end{aligned}$$

for all $x\in[0,L]$. If $t>1$ we can use for v_{2x} the expression

$$\begin{aligned} v_{2x}(t,\,x) =& \int_0^L \tilde{\Theta}_x^{\lambda_2^*}(1,\,x,\,y)v_2(t-1,\,y)dy + \int_0^1 B_{xx}^{\lambda_2^*}(1-s,\,x)\big(v_{2x}-\lambda_2^* v_2\big)(t-1+s,\,0)ds \\ &+ \int_0^1\int_0^L \tilde{\Theta}_x^{\lambda_2^*}(1-s,\,x,\,y)\tilde{s}_1(t-1+s,\,y)dyds \\ &+ \int_0^1\int_0^L \tilde{\Theta}_x^{\lambda_2^*}(1-s,\,x,\,y)\Big(\big(\lambda_2^*-\lambda_2\big)v_{2y}-\lambda_{2y}v_2\Big)(t-1+s,\,y)dyds. \end{aligned} \tag{A.25}$$

Computations analogous to the previous ones lead to

$$\int_1^T |v_{2x}(s,\,x)|ds \le \mathcal{O}(1)\delta_1.$$

Hence

$$\int_0^T |v_{2x}(s,\,x)|ds \le \mathcal{O}(1)\delta_1 \quad \forall T>0,\;\; x\in[0,\,L].$$

The integrability of w_2 with respect to time: it holds

$$\int_0^t |w_2(s,\,y)|ds \le \mathcal{O}(1)\delta_1 \quad \forall\, t>0, \quad \forall\, y\,\in\,[0,\,L]. \tag{A.26}$$

Proof. We preliminary observe that

$$w_2(0,\,x) = \langle \tilde{\ell}_2,\, u_t(0,\,x)\rangle, \qquad w_2(t,\,0) = \langle \tilde{\ell}_2,\, u'_{b\,0}(t)\rangle, \qquad w_2(t,\,L) = \langle \tilde{\ell}_2,\, u'_{b\,L}(t)\rangle,$$

where $\tilde{\ell}_2$ satisfies $\langle \tilde{\ell}_2,\, r_2\rangle \equiv 1$ and $\langle \tilde{\ell}_2,\, \tilde{r}_1\rangle \equiv 0$. Hence

$$\|w_2(t=0)\|_{L^1(0,\,L)} \le \mathcal{O}(1)\delta_1, \qquad \|w_2(x=0)\|_{L^1(0,\,+\infty)} \le \delta_1, \qquad \|w_2(x=L)\|_{L^1(0,\,+\infty)} \le \delta_1.$$

Let $2c$ be the separation speed defined by (1.4), let K be a compact neighborhood of the value u^* defined by (1.7) and let $C>0$ satisfy

$$0 < c \le \lambda_2(u) \le C \quad \forall\, u \in K.$$

If $y \in]0,\,L[$, the estimate (A.26) can be obtained applying Lemma 4.1 to the functional

$$P_y(x) = \begin{cases} a\big(1-e^{-Cx}\big) & x \le y \\ b\big(e^{-cx}-e^{-cL}\big) & x > y, \end{cases} \tag{A.27}$$

where a and b satisfy

$$\begin{cases} a\big(1-e^{-Cy}\big) = b\big(e^{-cy}-e^{-cL}\big) \\ aCe^{-Cy} + bce^{-cy} = 1. \end{cases} \tag{A.28}$$

By straightforward computations, from (A.28) one gets that the functional P_y satisfies

$$P_y(0) = P_y(L) = 0, \qquad 0 \le P_y(x) \le P_y(y) \le \mathcal{O}(1), \quad P'_y(0) \le \mathcal{O}(1), \quad -P'_y(L) \le \mathcal{O}(1), \quad \forall\, L >> 1$$
$$P''_y(x) + \lambda_2 P'_y(x) \le -\delta_{x=y}.$$

Since w_2 satisfies

$$w_{2t} + (\lambda_2 w_2)_x - w_{2xx} = \tilde{s}_2(t,\,x),$$

Lemma 4.1 ensures that

$$\begin{aligned} \int_0^t |w_2(s,\,y)|ds \le{}& \mathcal{O}(1)\int_0^L |w_2(0,\,x)|dx + \mathcal{O}(1)\int_0^t\int_0^L |\tilde{s}_2(s,\,x)|dxds \\ &+ \mathcal{O}(1)\int_0^t |w_2(s,\,0)|ds + \mathcal{O}(1)\int_0^t |w_2(s,\,L)|ds \\ \le{}& \mathcal{O}(1)\delta_1 \quad \forall\, y \in]0,\,L[. \end{aligned}$$

□

Integrability of w_{2x} with respect to time: it holds

$$\int_0^t |w_{2x}(s,\,x)|ds \le \mathcal{O}(1)\delta_1 \quad \forall\, t>0. \tag{A.29}$$

Proof. From the representation

$$\begin{aligned} w_{2x}(t,\,x) ={}& \int_0^L \Delta_x^{\lambda_2^*}(t,\,x,\,y)w_2(0,\,y)dy + \int_0^t\int_0^L \Delta_x^{\lambda_2^*}(t-s,\,x,\,y)\tilde{s}_2(s,\,y)dyds \\ &+ \int_0^t\int_0^L \Delta_x^{\lambda_2^*}(t-s,\,x,\,y)\Big((\lambda_2^*-\lambda_2)w_{2y} - \lambda_{2y}w_2\Big)(s,\,y)dsdy + w_2(0,\,L)J_x^{\lambda_2^*\,L}(t,\,x) \\ &+ w_2(0,\,0)J_x^{\lambda_2^*\,0}(t,\,x) + \int_0^t J_x^{\lambda_2^*\,0}(t-s,\,x)w'_2(s,\,0)ds + \int_0^t J_x^{\lambda_2^*\,L}(t-s,\,x)w'_2(s,\,L)ds \end{aligned}$$

it follows

$$\int_0^1 |w_{2x}|(t,\, x)dx \le \mathcal{O}(1)\delta_1.$$

If $t \ge 1$ one can write

$$\begin{aligned} w_{2x}(t,\, x) = &\int_0^L \Delta_x^{\lambda_2^*}(1,\, x,\, y) w_2(t-1,\, y)dy + \int_0^1 \int_0^L \Delta_x^{\lambda_2^*}(1-s,\, x,\, y)\tilde{s}_2(t-1+s,\, y)dyds \\ &+ \int_0^1 \int_0^L \Delta_x^{\lambda_2^*}(1-s,\, x,\, y)\Big((\lambda_2^* - \lambda_2)w_{2y} - \lambda_{2y}w_2\Big)(t-1+s,\, y)dsdy \\ &+ w_2(t-1,\, L)J_x^{\lambda_2^*\, L}(1,\, x) + w_2(t-1,\, 0)J_x^{\lambda_2^*\, 0}(1,\, x) \\ &+ \int_0^1 J_x^{\lambda_2^*\, 0}(1-s,\, x)w_2'(t-1+s,\, 0)ds + \int_0^1 J_x^{\lambda_2^*\, L}(1-s,\, x)w_2'(t-1+s,\, L)ds \end{aligned}$$

and obtains

$$\int_1^T |w_{2x}|(t,\, x)dt \le \mathcal{O}(1)\delta_1.$$

This concludes the proof of (A.29). □

A.3.3 Proof of the estimate (4.14)

We need three preliminary results:
• For any $t \le 1$, the following holds:

$$|\tilde{\Theta}_x^{\lambda_2^*}(t,\, x,\, y)| \le a(t,\, x-y) + b(t,\, x) \quad \|a(t)\|_{L^1(-L,\, L)},\;\; \|b(t)\|_{L^1(-L,\, L)} \le \frac{\mathcal{O}(1)}{\sqrt{t}}. \tag{A.30}$$

Proof of (A.30) In the following, $\alpha(t,\, x-y)$ and $\beta(t,\, x)$ will denote functions that satisfy

$$\|\alpha(t)\|_{L^1(-L,\, L)}, \quad \|\beta(t)\|_{L^1(-L,\, L)} \le \frac{\mathcal{O}(1)}{\sqrt{t}}.$$

By the identities (2.4) and (A.17),

$$\tilde{\Theta}_x^{\lambda_2^*}(t,\, x,\, y) = \Delta_x^{\lambda_2^*}(t,\, x,\, y) = \Bigg(\phi^{\lambda_2^*}(t,\, x,\, y)\sum_{m=-\infty}^{m=+\infty} G(t, x+2mL-y) - G(t, x+2mL+y)\Bigg)_x.$$

One has

$$\begin{aligned} \Bigg|\Bigg(\phi^{\lambda_2^*}(t,\, x,\, y)\sum_{m=-\infty}^{m=+\infty} G(t, x+2mL-y)\Bigg)_x\Bigg| \le &\sum_{m\ge 0}\Big|G_x^{\lambda_2^*}(t,\, x-y+2mL)\Big| + \lambda_2^*\sum_{n>0} G^{\lambda_2^*}(t,\, 2nL-x+y) \\ &+ \sum_{n>0} G_x^{\lambda_2^*}(t,\, 2nL-x+y) \le \alpha(x-y), \end{aligned}$$

where we have set $n := -m$. To complete the proof of (A.30), it is convenient to observe that

$$G_x^{\lambda_2^*}(t,\, x+y) \le G^{\lambda_2^*}(t,\, x) \quad \forall\, x \ge \big(\lambda_2^* t + \sqrt{2t}\big), \quad \forall\, y \ge 0$$

and that

$$\begin{aligned} &|G_x^{\lambda_2^*}(t,\, x+y)| \le G_x^{\lambda_2^*}(t,\, x) + G_x(t,\, \sqrt{2t})\,\chi_{\{\,0\,\le\, x\,\le\, \sqrt{2t}+\lambda_2^* t\}} \le \beta(x) \\ &|G_x^{\lambda_2^*}(t,\, 2L-x-y)| \le G_x^{\lambda_2^*}(t,\, L-x) + G_x(t,\, \sqrt{2t})\,\chi_{\{\,L-\sqrt{2t}-\lambda_2^* t\,\le\, x\,\le\, L\}} \le \beta(x), \quad \forall\, x,\, y \in [0,\, L] \end{aligned}$$

where χ_E denotes the characteristic function of the set E. Hence

$$\begin{aligned}
&\left|\left(\phi^{\lambda_2^*}(t,\,x,\,y)\sum_{m=-\infty}^{m=+\infty} G(t, x+2mL+y)\right)_x\right| \leq \sum_{m>0} G_x^{\lambda_2^*}(t,\,x+y+2mL) + G_x^{\lambda_2^*}(t,\,x+y)\\
&\quad + \lambda_2^* \sum_{n>0} G^{\lambda_2^*}(t,\,2nL-x-y) + \sum_{n>0} G_x^{\lambda_2^*}(t,\,2nL-x-y)\\
&\quad \leq \sum_{m>0} G_x^{\lambda_2^*}(t,\,x+2mL) + \beta(x) + \lambda_2^* \sum_{n>0} G^{\lambda_2^*}(t,\,L-x) + \sum_{n>1} G_x^{\lambda_2^*}(t,\,(2n-1)L-x) + G_x^{\lambda_2^*}(t,\,2L-x-y)\\
&\quad \leq \beta(x),
\end{aligned}$$

which concludes the proof of (A.30). □

• If $t \leq 1$ then

$$\int_0^L |v_{2x}(t,\,x)|dx \leq \frac{\mathcal{O}(1)\delta_1}{\sqrt{t}}. \tag{A.31}$$

Proof. Let $t \leq 1$. From the equality

$$u_{xx} = v_1\Big(\mathrm{D}\tilde{r}_1 u_x + v_{1x}\tilde{r}_{1v} + \sigma_{1x}\tilde{r}_{1\sigma}\Big) + v_{1x}\tilde{r}_1 + p_1\Big(\mathrm{D}\hat{r}_1 u_x + p_{1x}\hat{r}_{1p}\Big) + p_{1x}\hat{r}_1 + v_{2x}r_2 + p_{2x}r_2, \tag{A.32}$$

and from the bounds $\|p_{1x}(t)\|_{L^1} \leq \mathcal{O}(1)\delta_1$ and $\|u_{xx}(t)\| \leq \mathcal{O}(1)\delta_1/\sqrt{t}$, it follows that

$$\|v_{1x}(t)\| = \|\langle \ell_1,\, u_{xx}(t)\rangle - p_{1x}(t)\|_{L^1} \leq \frac{\mathcal{O}(1)\delta_1}{\sqrt{t}},$$

where $\ell_1 = (1,\,0)$. Hence

$$\|w_1(t)\|_{L^1} \leq \mathcal{O}(1)\|v_1(t)\|_{L^1} + \|v_{1x}(t)\|_{L^1} + \mathcal{O}(1)\|p_1(t)\|_{L^1} + \|p_{1x}(t)\|_{L^1} \leq \frac{\mathcal{O}(1)\delta_1}{\sqrt{t}}.$$

From the estimates

$$\begin{aligned}
&\|w_1'(x=0)\|_{L^1(0,\,+\infty)} = \|\langle \ell_1,\, u_{b\,0}''\rangle\|_{L^1(0,\,+\infty)} \leq \delta_1\\
&\|w_1'(x=L)\|_{L^1(0,\,+\infty)} = \|\langle \ell_1,\, u_{b\,L}''\rangle\|_{L^1(0,\,+\infty)} \leq \delta_1\\
&\|w_1(t=0)\|_{L^1(0,\,L)} = \|\langle \ell_1,\, u_0'' - A(u_0)u_0'\rangle\|_{L^1(0,\,L)} \leq \mathcal{O}(1)\delta_1,
\end{aligned}$$

and from the representation formula

$$\begin{aligned}
w_{1x}(t,\,x) = &\int_0^L \Delta_x^{\lambda_1^*}(t,\,x,\,y)w_1(0,\,y)dy + J_x^{\lambda_1^*\,0}(t,\,x)w_1(0,\,0) + J_x^{\lambda_1^*\,L}(t,\,x)w_1(0,\,L)\\
&+ \int_0^t J_x^{\lambda_1^*\,0}(t-s,\,x)w_1'(s,\,0)ds + \int_0^t J_x^{\lambda_1^*\,L}(t-s,\,x)w_1'(s,\,L)ds\\
&+ \int_0^t\int_0^L \Delta_x^{\lambda_1^*}(t-s,\,x,\,y)\Big((\lambda_1^*-\lambda_1)w_{1y} - \lambda_{1y}w_1\Big)(s,\,y)dsdy,
\end{aligned} \tag{A.33}$$

it follows that

$$\|w_{1x}(t)\|_{L^1} \leq \frac{\mathcal{O}(1)\delta_1}{\sqrt{t}}.$$

Hence

$$\|\sigma_{1x}(t)v_1(t)\|_{L^1} = \left\|\theta'\left(w_{1x}(t) - \frac{w_1}{v_1}v_{1x}(t)\right)\right\|_{L^1} \leq \frac{\mathcal{O}(1)\delta_1}{\sqrt{t}}.$$

and therefore from (A.32) one gets (A.31). □

• If $t \geq 1$ then

$$\int_0^L |v_{2x}(t,\,x))|dx \leq \mathcal{O}(1)\delta_1 \tag{A.34}$$

Proof. One can repeat the same computations performed to prove (A.31), using, instead of (A.33), the following representation formula (which holds if $t \geq 1$):

$$
\begin{aligned}
w_{1x}(t, x) = &\int_0^L \Delta_x^{\lambda_1^*}(1, x, y) w_1(t-1, y)dy + J_x^{\lambda_1^*\,0}(1, x) w_1(t-1, 0) + J_x^{\lambda_1^*\,L}(1, x) w_1(t-1, L) \\
&+ \int_0^1 J_x^{\lambda_1^*\,0}(1-s, x) w_1'(t-1+s, 0)ds + \int_0^1 J_x^{\lambda_1^*\,L}(1-s, x) w_1'(t-1+s, L)ds \\
&+ \int_0^1 \int_0^L \Delta_x^{\lambda_1^*}(1-s, x, y)\Big((\lambda_1^* - \lambda_1) w_{1y} - \lambda_{1y} w_1\Big)(t-1+s, y)dsdy.
\end{aligned}
$$

□

Let

$$
\mathcal{I}(T) := \sup_{\substack{\tau \in (-T, T) \\ x \in (-L, L)}} \int_{\max\{0, \tau\}}^{\min\{T, T+\tau\}} \int_{\max\{0, \xi\}}^{\min\{L, L+\xi\}} |v_1(t, x)|\, |v_{2x}(t-\tau, x-\xi)| dtdx.
$$

It holds:

$$
\int_0^T \int_0^L |v_1(t, x)|\, |v_{2x}(t, x)| dxdt \leq \mathcal{I}(T).
$$

Moreover, thanks to the estimates (A.31) and (A.34),

$$
\int_{\max\{0, \tau\}}^{\max\{2, 2+\tau\}} \int_{\max\{0, \xi\}}^{\min\{L, L+\xi\}} |v_1(t, x)|\, |v_{2x}(t-\tau, x-\xi)| \leq \mathcal{O}(1) \|v_1\|_{L^\infty} \delta_1 \int_0^2 \left\{1 + \frac{1}{\sqrt{t}}\right\} dt \leq \mathcal{O}(1)\delta_1^2.
$$

Hence we are left to estimate the term

$$
\int_{\max\{2, 2+\tau\}}^{\min\{T, T+\tau\}} \int_{\max\{0, \xi\}}^{\min\{L, L+\xi\}} |v_1(t, x)|\, |v_{2x}(t-\tau, x-\xi)| dxdt
$$

in the case $T \geq 2$: to do this, we will exploit the representation formula (A.25) and the estimate (A.30).

One has

$$
\begin{aligned}
&\int_{\max\{2, 2+\tau\}}^{\min\{T, T+\tau\}} \int_{\max\{0, \xi\}}^{\min\{L, L+\xi\}} v_1(t, x) \int_0^L \tilde{\Theta}_x^{\lambda_2^*}(1, x-\xi, y) v_2(t-1-\tau, y) \\
&\leq \int_{\max\{2, 2+\tau\}}^{\min\{T, T+\tau\}} \int_{\max\{0, \xi\}}^{\min\{L, L+\xi\}} v_1(t, x) \int_0^L a(1, x-\xi-y) v_2(t-1-\tau, y) \\
&\quad + \int_{\max\{2, 2+\tau\}}^{\min\{T, T+\tau\}} \int_{\max\{0, \xi\}}^{\min\{L, L+\xi\}} v_1(t, x) \int_0^L b(1, x-\xi) v_2(t-1-\tau, y) \\
&\leq \int_{-L}^L a(1, z) \int_{\max\{2, 2+\tau\}}^{\min\{T, T+\tau\}} \int_{\max\{0, z+\xi\}}^{\min\{L, L+\xi+z\}} v_1(t, x) v_2(t-1-\tau, x-z-\xi) d\xi \\
&\quad + \int_{\max\{0, \xi\}}^{\min\{L, L+\xi\}} b(1, x-\xi) \left(\int_{\max\{2, 2+\tau\}}^{\min\{T, T+\tau\}} v_1(t, x) \left(\int_0^L v_2(t-1-\tau, y) dy \right) dt \right) dx \leq \mathcal{O}(1)\delta_1^2,
\end{aligned}
$$

and

$$\int_{\max\{2,\,2+\tau\}}^{\min\{T,\,T+\tau\}}\int_{\max\{0,\,\xi\}}^{\min\{L,\,L+\xi\}} v_1(t,\,x)\int_0^1\int_0^L \tilde{\Theta}_x^{\lambda_2^*}(1-s,\,x-\xi,\,y)\Big((\lambda_2^*-\lambda_2)v_{2y}\Big)(t-\tau-1+s,\,y)dydsdxdt$$
$$\leq \delta_1 \int_{\max\{2,\,2+\tau\}}^{\min\{T,\,T+\tau\}}\int_{\max\{0,\,\xi\}}^{\min\{L,\,L+\xi\}} v_1(t,\,x)\int_0^1\int_0^L a(1-s,\,x-\xi-y)v_{2y}(t-\tau-1+s,\,y)dydsdxdt$$
$$+\,\delta_1 \int_{\max\{2,\,2+\tau\}}^{\min\{T,\,T+\tau\}}\int_{\max\{0,\,\xi\}}^{\min\{L,\,L+\xi\}} v_1(t,\,x)\int_0^1\int_0^L b(1-s,\,x-\xi)v_{2y}(t-\tau-1+s,\,y)dydsdxdt$$
$$\leq \delta_1 \int_0^1\int_{-L}^L a(1-s,\,z)\bigg(\int_{\max\{0,\,\xi+z\}}^{\min\{L,\,L+z+\xi\}}\int_{\max\{2,\,2+\tau\}}^{\min\{T,\,T+\tau\}} v_1(t,\,x)v_{2x}(t-\tau-1+s,\,x-\xi-z)dxdt\bigg)dzds$$
$$+\,\delta_1 \int_0^1\int_{\max\{0,\,\xi\}}^{\min\{L,\,L+\xi\}} b(1-s,\,x-\xi)\bigg(\int_{\max\{2,\,2+\tau\}}^{\min\{T,\,T+\tau\}} v_1(t,\,x)\bigg(\int_0^1 v_{2y}(t-\tau-1+s,\,y)dy\bigg)dt\bigg)dxds$$
$$\leq \mathcal{O}(1)\delta_1\mathcal{I}(T)+\mathcal{O}(1)\delta_1^3.$$

With analogous computations one can estimate the other terms that comes from the representation formula (A.25) and hence prove that $\mathcal{I}(T) \leq \mathcal{O}(1)\delta_1^2$.

A.3.4 Proof of the estimate (4.17)

Since in the following we will often refer to equations (3.10) and (3.11), we recall them:

$$\sigma_1 = \lambda_1^* - \theta\bigg(\frac{w_1}{v_1}+\lambda_1^*\bigg),$$

where the cut-off θ is given by

$$\theta(s)=\begin{cases} s & \text{if } |s|\leq\hat{\delta} \\ 0 & \text{if } |s|\geq 3\hat{\delta} \\ \text{smooth connection if} & \hat{\delta}\leq s\leq 3\hat{\delta}\end{cases} \qquad \delta_1 << \hat{\delta} \leq \frac{1}{3}.$$

It follows that $|w_1+\sigma_1 v_1|\neq 0$ only when the function θ is not the identity, i.e. when $|w_1+\lambda_1^* v_1| > \hat{\delta}|v_1|$. Since

$$w_1 = v_{1x} - \lambda_1 v_1 + p_{1x} - \lambda_1 p_1,$$

the condition $|w_1+\sigma_1 v_1|\neq 0$ implies

$$|v_{1x} + (\lambda_1^*-\lambda_1)v_1 + p_{1x} - \lambda_1 p_1| > \hat{\delta}|v_1|.$$

There are therefore two possible cases:

1. $$|v_{1x}+(\lambda_1^*-\lambda_1)v_1| \geq \frac{1}{2}\hat{\delta}|v_1|,$$

   and therefore, since $|\lambda_1^*-\lambda_1|\leq\mathcal{O}(1)\delta_1$ and $\delta_1 << \hat{\delta}$,

   $$|v_{1x}|\geq\frac{\hat{\delta}}{3}|v_1|.$$

2. $$|v_{1x}|<\frac{\hat{\delta}}{3}|v_1| \quad\Longrightarrow\quad |p_{1x}-\lambda_1 p_1| > \frac{\hat{\delta}}{2}|v_1|.$$

If case 1 holds, then

$$|w_1 + \sigma_1 v_1| = |v_{1x} + (\sigma_1 - \lambda_1)v_1 + p_{1x} - \lambda_1 p_1|$$
$$\leq |v_{1x}| + \delta_1|v_1| + |p_{1x} - \lambda_1 p_1| \leq \mathcal{O}(1)|v_{1x}| + |p_{1x} - \lambda_1 p_1|$$

and therefore

$$\Big(|v_1| + |w_1| + |v_{1x}| + |w_{1x}|\Big)\Big(|w_1 + \sigma_1 v_1|\Big) \leq \mathcal{O}(1)\Big(|v_{1x}| + |p_1| + |p_{1x}| + |w_{1x}|\Big)\Big(\mathcal{O}(1)|v_{1x}| + |p_{1x} - \lambda_1 p_1|\Big)$$
$$\leq \mathcal{O}(1)\Big(|v_{1x}| + |p_1| + |p_{1x}| + |w_{1x}|\Big)|p_{1x} - \lambda_1 p_1| + \mathcal{O}(1)|v_{1x}|^2 + \mathcal{O}(1)|v_{1x}|\Big(|p_1| + |p_{1x}|\Big) + \mathcal{O}(1)|w_{1x}|^2.$$

Since

$$|p_1|,\, |p_{1x}| \leq \mathcal{O}(1)\delta_1 \exp(-cx/2),$$

it follows that, if case 1 holds, then one is left to prove

$$\int_0^T \int_0^L \chi_{\{|(w_1/v_1)+\lambda_1^*| \geq \hat{\delta}\}} \Big(|v_{1x}|^2 + |w_{1x}|^2\Big)(t,x)dxdt \leq \mathcal{O}(1)\delta_1^2. \tag{A.35}$$

On the other hand, if case 2 holds then

$$|v_{1x} + (\sigma_1 - \lambda_1)v_1 + p_{1x} - \lambda_1 p_1| \leq \frac{4}{3}\hat{\delta}|v_1| + |p_{1x} - \lambda_1 p_1| \leq \mathcal{O}(1)|p_{1x} - \lambda_1 p_1|,$$

and therefore

$$\int_0^T \int_0^L \Big(|v_1| + |w_1| + |v_{1x}| + |w_{1x}|\Big)\Big(|w_1 + \sigma_1 v_1|\Big)(s,\, x)dsdx$$
$$\leq \mathcal{O}(1)\int_0^T \int_0^L \Big(|v_1| + |w_1| + |v_{1x}| + |w_{1x}|\Big)|p_{1x} - \lambda_1 p_1|(s,\, x)dsdx \leq \mathcal{O}(1)\delta_1^2,$$

thanks to the exponential decay of $|p_1|$ and $|p_{1x}|$.

To prove (A.35) it is convenient to introduce a new cutoff function:

$$\psi(s) = \begin{cases} 0 & \text{if } |s| \leq 3/5\,\hat{\delta} \\ 1 & \text{if } |s| \geq 4/5\,\hat{\delta} \\ \text{smooth connection if} & 3/5\,\hat{\delta} \leq |s| \leq 4/5\,\hat{\delta}. \end{cases}$$

Moreover, in the following we will only prove that

$$\int_0^T \int_0^L \chi_{\{|(w_1/v_1)+\lambda_1^*| \geq \hat{\delta}\}} |v_{1x}|^2(t,x)dxdt \leq \mathcal{O}(1)\delta_1^2, \tag{A.36}$$

because the estimate

$$\int_0^T \int_0^L \chi_{\{|(w_1/v_1)+\lambda_1^*| \geq \hat{\delta}\}} |w_{1x}|^2(t,x)dxdt \leq \mathcal{O}(1)\delta_1^2.$$

can be obtained with similar techniques.

As we have already observed, it is sufficient to show

$$\int_0^T \int_0^L |v_{1x}|^2 \psi\Big(\frac{w_1}{v_1} + \lambda_1^*\Big)(t,x)dxdt \leq \mathcal{O}(1)\delta_1^2.$$

Multiplying the equation

$$v_{1t} + (\lambda_1 v_1)_x - v_{1xx} = 0$$

by ψv_1, we get

$$0 = \int_0^L \int_0^T \left(\frac{d}{dt}(\frac{v_1^2}{2}\psi) - \frac{v_1^2}{2}(\psi_t + \lambda_1\psi_x - \psi_{xx}) + \psi|v_{1x}|^2 + \frac{v_1^2}{2}\lambda_{1x}\psi - v_1^2\psi_{xx} \right) dxdt$$
$$+ \int_0^T \Big[\psi v_1(\lambda_1 v_1 - v_{1x})\Big]_{x=0}^{x=L} dt + \int_0^T \Big[\frac{v_1^2}{2}(\psi_x - \lambda_1\psi)\Big]_{x=0}^{x=L} dt. \tag{A.37}$$

Indeed,

$$\frac{d}{dt}(\frac{v_1^2}{2}\psi) = v_1 v_{1t}\psi + \frac{v_1^2}{2}\psi_t$$

and

$$\int_0^L \int_0^T \big(\lambda_1 v_1 - v_{1x}\big)_x \psi v_1 dxdt = \int_0^L \int_0^T (v_{1x} - \lambda_1 v_1)(\psi v_1)_x dxdt + \int_0^T \Big[\psi v_1(\lambda_1 v_1 - v_{1x})\Big]_{x=0}^{x=L} dt$$
$$= \int_0^L \int_0^T \psi_x \left(\frac{v_1^2}{2}\right)_x + \psi v_{1x}^2 - \lambda_1 \psi_x v_1^2 - \lambda_1 \psi \left(\frac{v_1^2}{2}\right)_x dxdt + \int_0^T \Big[\psi v_1(\lambda_1 v_1 - v_{1x})\Big]_{x=0}^{x=L} dt$$
$$= \int_0^L \int_0^T \psi v_{1x}^2 + \left(\frac{v_1^2}{2}\right)(\lambda_{1x}\psi - \lambda_1\psi_x + \psi_{xx} - 2\psi_{xx}) dxdt + \int_0^T \Big[\psi v_1(\lambda_1 v_1 - v_{1x})\Big]_{x=0}^{x=L} dt$$
$$+ \int_0^T \Big[\frac{v_1^2}{2}(\psi_x - \lambda_1\psi)\Big]_{x=0}^{x=L} dt.$$

One can develop the term $\psi_t + \lambda_1\psi_x - \psi_{xx}$ and, since

$$\psi_t = \psi'\left(\frac{w_{1t}v_1 - w_1 v_{1t}}{v_1^2}\right), \qquad \psi_x = \psi'\left(\frac{w_{1x}v_1 - w_1 v_{1x}}{v_1^2}\right),$$
$$\psi_{xx} = \psi''\left(\frac{w_1}{v_1}\right)_x^2 + \psi'\left(\frac{w_{1xx}v_1 - v_{1xx}w_1}{v_1^2} - 2\frac{v_{1x}(w_{1x}v_1 - w_1 v_{1x})}{v_1^3}\right), \tag{A.38}$$

one obtains

$$v_1^2(\psi_t + \lambda_1\psi_x - \psi_{xx}) = \psi' v_1(w_{1t} + (\lambda_1 w_1)_x - w_{1xx}) - \psi' w_1(v_{1t} + (\lambda_1 v_1)_x - v_{1xx})$$
$$- \psi'' v_1^2 \left(\frac{w_1}{v_1}\right)_x^2 + 2\psi' v_{1x} v_1 \left(\frac{w_1}{v_1}\right)_x.$$

Thus, inserting the last formula into (A.37), we obtain

$$\int_0^L \int_0^T \psi|v_{1x}|^2 = -\frac{1}{2}\int_0^L \Big[v_1^2 dx\Big]_{t=0}^{t=T} + \int_0^T \Big[\psi v_1(v_{1x} - \lambda_1 v_1)\Big]_{x=0}^{x=L} dt + \int_0^T \Big[\frac{v_1^2}{2}(\psi_x - \lambda_1\psi)\Big]_{x=0}^{x=L} dt$$
$$- \frac{1}{2}\int_0^L \int_0^T \psi'' v_1^2 \left(\frac{w_1}{v_1}\right)_x^2 + \psi' v_{1x} v_1 \left(\frac{w_1}{v_1}\right)_x + v_1^2\psi_{xx} - \frac{v_1^2}{2}\lambda_{1x}\psi.$$

The boundary terms are bounded by $\mathcal{O}(1)\delta_1^2$ since $\|v_1\|_{L^\infty} \leq \mathcal{O}(1)\delta_1$ and thanks to the estimates of Proposition 4.2. Since by (4.16)

$$\int_0^T \int_0^L \chi_{\{|\lambda_1^* + w_1/v_1| \leq 3\hat{\delta}\}} v_1^2 \left(\frac{w_1}{v_1}\right)_x^2 dxds \leq \mathcal{O}(1)\delta_1^2,$$

we are left to estimate the following terms:

- $$\int_0^T\int_0^L \left|\psi' v_{1x} v_1 \left(\frac{w_1}{v_1}\right)_x\right| dsdx \le \int_0^T\int_0^L \left|\psi' v_{1x}\Big(w_{1x} - \frac{w_1}{v_1} v_{1x}\Big)\right| dsdx$$
$$\le \mathcal{O}(1)\int_0^T\int_0^L \left|\psi'\Big(|v_1| + |p_{1x} - \lambda_1 p_1|\Big)\Big(w_{1x} - \frac{w_1}{v_1} v_{1x}\Big)\right| dsdx$$
$$\le \mathcal{O}(1)\int_0^T\int_0^L |v_1 w_{1x} - v_{1x} w_1| dsdx + \mathcal{O}(1)\int_0^T\int_0^L |p_{1x} - \lambda_1 p_1|\Big(|w_{1x}| + \mathcal{O}(1)|v_{1x}|\Big)$$

Indeed, if $\psi' \neq 0$ then $|\lambda_1^* - w_1/v_1| \le \hat{\delta}$ and hence

$$|v_{1x}| \le \mathcal{O}(1)|v_1| + |p_{1x} - \lambda_1 p_1|.$$

- $$\int_0^T\int_0^L \psi'\Big(w_{1xx}v_1 - w_1 v_{1xx}\Big) dsdx = \mathcal{O}(1)\int_0^T\int_0^L \Big(w_{1x}v_1 - w_1 v_{1x}\Big)_x dsdx$$
$$\le \mathcal{O}(1)\int_0^T \Big[w_{1x}v_1 - w_1 v_{1x}\Big]_{x=0}^{x=L} \le \mathcal{O}(1)\delta_1^2$$

- $$\Big|\int_0^L\int_0^T \frac{v_1^2}{2}\lambda_{1x}\psi\Big| = \Big|\int_0^L\int_0^T \frac{v_1^2}{2}(\lambda_1 - \lambda_1^*)_x \psi\Big|$$
$$\le \Big|\int_0^L\int_0^T (\lambda_1 - \lambda_1^*)\Big(\frac{v_1^2}{2}\psi\Big)_x\Big| + \Big|\int_0^T \Big[(\lambda_1 - \lambda_1^*)\frac{v_1^2}{2}\psi\Big]_{x=0}^{x=L}$$
$$\le \mathcal{O}(1)\delta_1\Big|\int_0^T \Big[\frac{v_1^2}{2}\psi\Big]_{x=0}^{x=L} + \mathcal{O}(1)\delta_1^2 \le \mathcal{O}(1)\delta_1^2.$$

Acknowledgments. The author wishes to thank Stefano Bianchini for having proposed the problem and for many helpful suggestions.

References

[1] D. Amadori. Initial-boundary value problems for nonlinear systems of conservation laws. *NoDEA*, 4:1–42, 1997.

[2] D. Amadori and R. M. Colombo. Viscosity Solutions and Standard Riemann Semigroup for Conservation Laws with Boundary. *Rend. Sem. Mat. Univ. Padova*, 99:219–245, 1998.

[3] F. Ancona and S. Bianchini. Vanishing viscosity solutions for general hyperbolic systems with boundary. *Preprint IAC-CNR 28*, 2003.

[4] S. Bianchini and A. Bressan. BV estimates for a class of viscous hyperbolic systems. *Indiana Univ. Math. J.*, 49:1673–1713, 2000.

[5] S. Bianchini and A. Bressan. A case study in vanishing viscosity. *Discrete Contin. Dynam. Systems*, 7:449–476, 2001.

[6] S. Bianchini and A. Bressan. A center manifold technique for tracing viscous waves. *Comm. Pure Applied Anal.*, 1:161–190, 2002.

[7] S. Bianchini and A. Bressan. Vanishing viscosity solutions of non linear hyperbolic systems. *Ann. of Math.*, 161:223–342, 2005.

[8] A. Bressan. Contractive metrics for nonlinear hyperbolic systems. *Indiana Univ. Math. J.*, 37:409–421, 1988.

[9] A. Bressan. Global solution to systems of conservation laws by wave-front-tracking. *J. Math. Anal. Appl.*, 170:414–432, 1992.

[10] A. Bressan. The unique limit of the Glimm scheme. *Arch. Rational Mech. Anal.*, 130:205–230, 1995.

[11] A. Bressan. *Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy Problem.* Oxford University Press, Oxford, 2000.

[12] A. Bressan and R. M. Colombo. The semigroup generated by 2×2 conservation laws. *Arch. Rational Mech. Anal.*, 133:1–75, 1995.

[13] A. Bressan, G. Crasta, and B. Piccoli. Well-posedness of the Cauchy problem for $n \times n$ conservation laws. *Mem. Amer. Math. Soc.*, 694, 2000.

[14] A. Bressan and P. Goatin. Olenik type estimates and uniqueness for $n \times n$ conservation laws. *J. Differential Equations*, 156:26–49, 1999.

[15] A. Bressan and P. LeFloch. Uniqueness of weak solutions to systems of conservation laws. *Arch. Rational Mech. Anal.*, 140:301–317, 1997.

[16] A. Bressan and M. Lewicka. A uniqueness condition for hyperbolic systems of conservation laws. *Discrete Contin. Dynam. Systems*, 6:673–682, 2000.

[17] A. Bressan, T. P. Liu, and T. Yang. L^1 stability estimates for $n \times n$ conservation laws. *Arch. Rational Mech. Anal.*, 149:1–22, 1999.

[18] F. H. Clarke. *Optimization and Nonsmooth Analysis.* Wiley-Interscience, 1983.

[19] C. Donadello and A. Marson. Stability of front tracking solutions to the initial and boundary value problem for systems of conservation laws. *Preprint*, 2005.

[20] J. Glimm. Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.*, 18:697–715, 1965.

[21] J. Goodman. *Initial Boundary Value Problems for Hyperbolic Systems of Conservation Laws.* PhD Thesis, California University, 1982.

[22] P. Lax. Hyperbolic systems of conservation laws II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[23] T. P. Liu. The Riemann Problem for General Systems of Conservation Laws. *J. Differential Equations*, 18:218–234, 1975.