On the area of the graph of a piecewise smooth map from the plane to the plane with a curve discontinuity*

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Abstract

In this paper we provide an estimate from above for the value of the relaxed area functional $\overline{\mathcal{A}}(\mathbf{u},\Omega)$ for an $\mathbb{R}^2$-valued map $\mathbf{u}$ defined on a bounded domain Ω of the plane and discontinuous on a $\mathcal{C}^2$ simple curve $\overline{J}_{\mathbf{u}} \subset \Omega$, with two endpoints. We show that, under certain assumptions on $\mathbf{u}$, $\overline{\mathcal{A}}(\mathbf{u},\Omega)$ does not exceed the area of the regular part of $\mathbf{u}$, with the addition of a singular term measuring the area of a *disk-type* solution $\Sigma_{\min}$ of the Plateau's problem spanning the two traces of $\mathbf{u}$ on $\overline{J}_{\mathbf{u}}$. The result is valid also when $\Sigma_{\min}$ has self-intersections. A key element in our argument is to show the existence of what we call a *semicartesian parametrization* of $\Sigma_{\min}$, namely a conformal parametrization of $\Sigma_{\min}$ defined on a suitable parameter space, which is the identity in the first component. To prove our result, various tools of parametric minimal surface theory are used, as well as some results from Morse theory.

1 Introduction

Given a bounded open set $\Omega \subset \mathbb{R}^2 = \mathbb{R}^2_{(x,y)}$ and a map $\mathbf{v} = (v_1, v_2) : \Omega \to \mathbb{R}^2 = \mathbb{R}^2_{(\xi,\eta)}$ of class $\mathcal{C}^1$, the area $\mathcal{A}(\mathbf{v},\Omega)$ of the graph of $\mathbf{v}$ in Ω is given by

$$\mathcal{A}(\mathbf{v},\Omega) = \int_\Omega |\mathcal{M}(\nabla\mathbf{v})| \, dx \, dy,$$

where $|\cdot|$ denotes the euclidean norm, $\nabla\mathbf{v}$ is the Jacobian matrix of $\mathbf{v}$ and $\mathcal{M}(\nabla\mathbf{v})$ is the vector whose components are the determinants of all minors$^{(1)}$ of $\nabla\mathbf{v}$, hence

$$|\mathcal{M}(\nabla\mathbf{v})| = \sqrt{1 + |\nabla v_1|^2 + |\nabla v_2|^2 + (\partial_x v_1 \partial_y v_2 - \partial_y v_1 \partial_x v_2)^2}.$$

The polyconvex [4] functional $\mathcal{A}(\mathbf{v},\Omega)$ has linear growth and measures the area of the graph of $\mathbf{v}$, a smooth two-codimensional surface in $\mathbb{R}^4 = \mathbb{R}^2_{(x,y)} \times \mathbb{R}^2_{(\xi,\eta)}$. When considering the

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$^{(1)}$Including the determinant of order zero, which by definition is equal to one.

perspective of the direct method of the Calculus of Variations, it is important to assign a reasonable notion of area also to the graph of a *nonsmooth* map, namely to extend the functional $\mathcal{A}(\cdot,\Omega)$ out of $\mathcal{C}^1(\Omega;\mathbb{R}^2)$ in a natural way. We agree in defining this extended area as the $L^1(\Omega;\mathbb{R}^2)$-lower semicontinuous envelope $\overline{\mathcal{A}}(\cdot,\Omega)$ (or relaxed functional for short) of $\mathcal{A}(\cdot,\Omega)$, i.e.,

$$\overline{\mathcal{A}}(\mathbf{v},\Omega) := \inf\left\{\liminf_{\varepsilon\to 0^+}\mathcal{A}(\mathbf{v}_\varepsilon,\Omega)\right\} \tag{1.1}$$

where the infimum is taken over all sequences[(2)] $(\mathbf{v}_\varepsilon)\subset\mathcal{C}^1(\Omega;\mathbb{R}^2)$ converging to $\mathbf{v}$ in $L^1(\Omega;\mathbb{R}^2)$. The interest of definition (1.1) is clearly seen in the scalar case[(3)], where this notion of extended area is useful for solving non-parametric minimal surface problems, under various type of boundary conditions (see for instance [9], [13], [8]). We recall that in the scalar case $\mathcal{A}(\cdot,\Omega)$ happens to be convex, and $\overline{\mathcal{A}}(\cdot,\Omega)$ is completely characterized: its domain is the space $\mathrm{BV}(\Omega)$ of functions with bounded variation in Ω, and its expression is suitably given in integral form.

The analysis of the properties of $\overline{\mathcal{A}}(\mathbf{v},\Omega)$ for maps $\mathbf{v}$ from a subset of the plane to the plane is much more difficult [8]; geometrically, the problem is to understand which could be the most "economic" way, in terms of two-dimensional area in $\mathbb{R}^4$, of approximating a *nonsmooth two-codimensional graph* of a map $\mathbf{v}$ of bounded variation, with graphs of smooth maps, where the approximation takes place in $L^1(\Omega;\mathbb{R}^2)$. It is the aim of the present paper to address this problem for discontinuous maps $\mathbf{v}$ of class $\mathrm{BV}(\Omega;\mathbb{R}^2)$, having a $\mathcal{C}^2$-curve of discontinuity and satisfying suitable properties.

In [1] Acerbi and Dal Maso studied the relaxation of polyconvex functionals with linear growth in arbitrary dimension and codimension. In particular, they proved that $\overline{\mathcal{A}}(\cdot,\Omega)=\mathcal{A}(\cdot,\Omega)$ on $\mathcal{C}^1(\Omega;\mathbb{R}^2)$, and that for $p\in[2,+\infty]$,

$$\overline{\mathcal{A}}(\mathbf{v},\Omega)=\int_\Omega|\mathcal{M}(\nabla\mathbf{v})|\,dx\,dy,\qquad \mathbf{v}\in W^{1,p}(\Omega;\mathbb{R}^2),$$

and the exponent p is optimal. Concerning the representation of $\overline{\mathcal{A}}(\cdot,\Omega)$ in $\mathrm{BV}(\Omega;\mathbb{R}^2)$, they proved [1, Theorem 2.7] that the domain of $\overline{\mathcal{A}}(\cdot,\Omega)$ is contained in $\mathrm{BV}(\Omega;\mathbb{R}^2)$, and

$$\overline{\mathcal{A}}(\mathbf{v},\Omega)\geq\int_\Omega|\mathcal{M}(\nabla\mathbf{v})|\,dx\,dy+|D^s\mathbf{v}|(\Omega),\qquad \mathbf{v}\in\mathrm{BV}(\Omega;\mathbb{R}^2), \tag{1.2}$$

where $\nabla\mathbf{v}$ and $D^s\mathbf{v}$ denote the absolutely continuous and the singular part of the distributional gradient $D\mathbf{v}$ of $\mathbf{v}$, respectively. In addition, if $\mathbf{v}\in\mathrm{BV}(\Omega;\{\alpha_1,\dots,\alpha_m\})$ where $\alpha_1,..,\alpha_m$ are vectors of $\mathbb{R}^2$, and denoting by $\mathcal{L}^2$ and $\mathcal{H}^1$ the Lebesgue measure and the one-dimensional Hausdorff measure in $\mathbb{R}^2$ respectively,

$$\overline{\mathcal{A}}(\mathbf{v},\Omega)=\mathcal{L}^2(\Omega)+\sum_{\substack{k,\,l\,\in\,\{1,..,m\}\\ k\,<\,l}}|\alpha_k-\alpha_l|\,\mathcal{H}^1(J_{kl}), \tag{1.3}$$

provided $\partial\Omega$ and the jump curves J_{kl} forming the jump set $J_\mathbf{v}$ of $\mathbf{v}$ are smooth enough and that $\mathbf{v}$ takes locally only two vectors around J_{kl}, see [1, Theorem 2.14] for the details. Finally, and

[(2)]In this paper we consider families of functions (or functionals, or points) indicized by a continuous parameter; with a small abuse of language, these families are still called sequences.

[(3)]Namely, for functions $v:\Omega\to\mathbb{R}$.

maybe more interestingly, it is proven in [1, Section 3] that the relaxed area is not subadditive with respect to Ω, thus in particular it does not admit an integral representation, hence it is *non-local.* The non-subadditivity of $\overline{\mathcal{A}}(\mathbf{v}, \cdot)$, conjectured by De Giorgi in [5], concerns the triple junction map $\mathbf{u}_{\rm tr}$, which is a map defined on the unit disk of the source plane, and assumes as values three non-collinear vectors on three circular congruent sectors. The proof given in [1] does not supply the precise value of $\overline{\mathcal{A}}(\mathbf{u}_{\rm tr}, \Omega)$, however it provides a nontrivial lower bound and an upper bound. The upper bound was refined in [3], where the authors exhibited an approximating sequence (conjectured to be optimal[(4)], at least under symmetry assumptions) constructed by solving three (similar) Plateau-type problems coupled at the triple point[(5)]. The singular contribution concentrated over to the triple point arising in this construction, consists of a term penalizing the length of the Steiner-graph connecting the three values in the target space $\mathbb{R}^2$, with weight two. If the construction of [3] were optimal, it would shed some light on the nonlocality phenomenon addressed in [5] and [1].
The question arises as to whether the nonlocality is due to the special form of the triple junction map $\mathbf{u}_{\rm tr}$, or whether it can be obtained for other qualitatively different maps $\mathbf{v}$. We are not still able to answer this question, which nevertheless can be considered as the main motivation of the present paper. In this direction, our idea is to study the properties of $\overline{\mathcal{A}}(\cdot, \Omega)$, for maps generalizing those in (1.3), with no triple or multiple junctions. Namely, we are interested in $\overline{\mathcal{A}}(\mathbf{u}, \Omega)$, where $\mathbf{u}$ is regular enough in $\Omega \setminus \overline{J}_{\mathbf{u}}$, and the jump set $J_{\mathbf{u}}$ is a $\mathcal{C}^2$ simple curve compactly contained[(6)] in Ω. It is worth anticipating that we are concerned here only with an estimate from above of the value of the relaxed area, and we shall not face the problem of the estimate from below. Nevertheless, we believe our construction of the recovery sequence to be optimal, at least for a reasonably large class of maps.
Referring to the next sections for the details, we now briefly sketch the main results and the ideas of the present paper. Suppose that $\mathbf{u} \in \mathrm{BV}(\Omega; \mathbb{R}^2)$ is a vector valued map regular enough in $\Omega \setminus \overline{J}_{\mathbf{u}}$, and let us parametrize $\overline{J}_{\mathbf{u}}$ with a map $\alpha : t \in [a, b] \to \alpha(t) \in \overline{J}_{\mathbf{u}}$. Denote by $\mathbf{u}^{\pm}$ the two traces of $\mathbf{u}$ on $\overline{J}_{\mathbf{u}}$, and let $\gamma^{\pm}$, defined in $[a, b]$, be the composition of $\mathbf{u}^{\pm}$ with the parametrization α. Let us define Γ as the union of the graphs of γ^+ and γ^-. Our regularity assumptions ensure that Γ is a rectifiable, simple and closed space curve, with a special structure, due to the fact that it is union of graphs of two vector maps defined in the same interval $[a, b]$ (Definition 2.1). Finally, let us denote by $\Sigma_{\min}$ an area minimizer solution of the Plateau's problem for Γ, in the class of surfaces spanning Γ and having the *topology of the disk* [6]. Suppose that $\Sigma_{\min}$ admits what we call a *semicartesian parametrization* (Definition 2.2), namely a global parametrization whose first component coincides with the parameter $t \in [a, b]$. Our first result reads as follows.

Theorem 1.1. *Under the above assumptions, there exists a sequence* $(\mathbf{u}_\varepsilon)$ *of sufficiently*

[(4)] In the sense that equality should hold in (1.1) along the above mentioned sequence.

[(5)] The construction of [3] is intrinsically four-dimensional and cannot be reduced to a three-dimensional construction.

[(6)] As one can deduce from our proofs, the case when $\overline{J}_{\mathbf{u}} \cap \partial\Omega \neq \emptyset$ requires a separate study, leading to a Plateau-type problem with partial free boundary, and will be investigated elsewhere. Also, the case when $\overline{J}_{\mathbf{u}} \subset \Omega$ is a closed simple curve is out of the scope of the present paper, since it leads to the study of minimal immersions of an annulus in $\mathbb{R}^3$.

regular[7] *maps converging to* $\mathbf{u}$ *in* $L^1(\Omega;\mathbb{R}^2)$ *such that*

$$\lim_{\varepsilon\to 0^+} \overline{\mathcal{A}}(\mathbf{u}_\varepsilon,\Omega) = \int_{\Omega\setminus \overline{J}_{\mathbf{u}}} |\mathcal{M}(\nabla \mathbf{u})|\, dx\, dy + \mathcal{H}^2(\Sigma_{\min}). \tag{1.4}$$

In particular

$$\overline{\mathcal{A}}(\mathbf{u},\Omega) \le \int_{\Omega\setminus \overline{J}_{\mathbf{u}}} |\mathcal{M}(\nabla \mathbf{u})|\, dx\, dy + \mathcal{H}^2(\Sigma_{\min}).$$

Under the hypothesis that there exists a semicartesian parametrization

$$X(t,s) = (t, X_2(t,s), X_3(t,s))$$

of $\Sigma_{\min}$ defined on a plane domain $D \subset \mathbb{R}^2_{(t,s)}$, the key point of the construction stands in the definition of $\mathbf{u}_\varepsilon$ in a suitable neighborhood of the jump $J_{\mathbf{u}}$. For (x,y) in this neighbourhood we define the pair of functions $(t(x,y), s(x,y)) \in D$ corresponding to the parametrization of the nearest point on $\overline{J}_{\mathbf{u}}$ to (x,y), and to the signed distance from $J_{\mathbf{u}}$, respectively. Next, we define

$$\mathbf{u}_\varepsilon(x,y) := \left(X_2\left(t(x,y), \frac{s(x,y)}{\varepsilon}\right), X_3\left(t(x,y), \frac{s(x,y)}{\varepsilon}\right)\right) \tag{1.5}$$

for (x,y) such that $\left(t(x,y), \frac{s(x,y)}{\varepsilon}\right) \in D$. Note carefully that, in this way, the definition of $\mathbf{u}_\varepsilon$ cannot be reduced to a one-dimensional profile, being intrinsically two-dimensional. The explicit computation (`step 9` of the proof of Theorem 3.1) of the area of the graph of $\mathbf{u}_\varepsilon$ localized in this region is the source of the term

$$\mathcal{H}^2(\Sigma_{\min})$$

appearing in (1.4).
It is interesting to comment on the role of the term

$$(\partial_x \mathbf{u}_{\varepsilon 1} \partial_y \mathbf{u}_{\varepsilon 2} - \partial_y \mathbf{u}_{\varepsilon 1} \partial_x \mathbf{u}_{\varepsilon 2})^2 \tag{1.6}$$

in the details of the computation. If X is semicartesian, the area of $\Sigma_{\min}$ is given by

$$\int_D \sqrt{|\partial_s X_2|^2 + |\partial_s X_3|^2 + (\partial_t X_2 \partial_s X_3 - \partial_s X_2 \partial_t X_3)^2}\, dt\, ds.$$

The first two addenda under the square root are obtained, in the limit, from $|\nabla \mathbf{u}_{\varepsilon 1}|^2 + |\nabla \mathbf{u}_{\varepsilon 2}|^2$, while the last addendum is originated in the limit exactly by (1.6).
Various technical difficulties are present in the estimate of $\mathcal{A}(\mathbf{u}_\varepsilon,\cdot)$ outside of the above mentioned neighbourhood of $J_{\mathbf{u}}$. Far from $J_{\mathbf{u}}$ we set $\mathbf{u}_\varepsilon := \mathbf{u}$, while in a (small) intermediate neighbourhood the map $\mathbf{u}_\varepsilon$ is suitably defined in such a way that the corresponding contribution of the area is negligible. The technical point behind this construction is to guarantee that $\mathbf{u}_\varepsilon$ is sufficiently smooth. In Theorem 3.1 we study the case in which $\Sigma_{\min}$ is the graph of a map defined on a two-dimensional convex domain, the so-called non-parametric case; here an approximating argument leads to the Lipschitz regularity of $\mathbf{u}_\varepsilon$ in Ω. In Theorem 4.1, instead,

[7] $(\mathbf{u}_\varepsilon) \subset \mathrm{Lip}(\Omega;\mathbb{R}^2)$ in Theorem 3.1, and $(\mathbf{u}_\varepsilon) \subset W^{1,2}(\Omega;\mathbb{R}^2)$ in Theorem 4.1.

we study a more general situation, managing in building a sequence $(\mathbf{u}_\varepsilon)$ in $W^{1,2}(\Omega;\mathbb{R}^2)$. In this case we need to modify the domain of the semicartesian parametrization, in order to gain the L^1 integrability of the gradients of $\mathbf{u}_\varepsilon$ and to make a further regularization near the *crack tips*, that is the end points of $J_{\mathbf{u}}$, (see `steps 1` and `2` of Theorem 4.1).

Several other comments are in order concerning Theorem 1.1. First of all, and as already mentioned, our result provides only an estimate from above of the value of $\overline{\mathcal{A}}(\mathbf{u},\Omega)$. Only if Γ is contained in a plane, we are able to prove that inequality (1.4) is actually an equality$^{(8)}$, so that $(\mathbf{u}_\varepsilon)$ becomes a recovery sequence. This case is a slight generalization of the piecewise constant case (1.3) considered in [1], and seems not enough for answering the nonlocality question on $\overline{\mathcal{A}}$.

After this remark, we come back to the important issue of the *semicartesian parametrization.* First of all, a semicartesian parametrization represents an intermediate situation between the non-parametric case, and the general case in which $\Sigma_{\min}$ is just an area-minimizing surface spanning Γ and having the topology of the disk. We stress that the assumptions on Γ that ensure the existence of a semicartesian parametrization of $\Sigma_{\min}$ are not so restrictive$^{(9)}$; for example the analytic curves displayed in Figures 1(a) and (b) satisfy the hypotheses of Theorem 1.2 below, and thus the corresponding $\Sigma_{\min}$ admit a semicartesian parametrization and Theorem 4.1 applies. Observe that the surface $\Sigma_{\min}$ in Figure 1(a) (area-minimizing and with the topology of the disk) has self-intersections$^{(10)}$. In this case the map $\mathbf{u}_\varepsilon$ defined in (1.5) is not injective; of course, the source of this phenomenon is due to the higher codimension of graph$(\mathbf{u})$, and it does not arise in the scalar case.

Let us now inspect the delicate problem of the existence of a domain $D \subset \mathbb{R}^2_{(t,s)}$ and a semicartesian parametrization $X : D \to \mathbb{R}^3$. Besides the non-parametric case, in this paper we exhibit other sufficient conditions for the existence of a semicartesian parametrization, and we refer to Theorem 5.1 for all details.

Theorem 1.2. *Suppose that* Γ *admits a parametrization which is analytic, and nondegenerate in the sense of* (5.1) *at the junctions between* γ^- *and* γ^+*. Then* $\Sigma_{\min}$ *admits a semicartesian parametrization.*

Before commenting on the proof, which represent maybe the most technical part of the present paper, we want to briefly discuss Figure 1(a), since it is a sort of prototypical example in our work. The boundary of the represented surface satisfies all hypotheses of Theorem 1.2. It is built as the union of two graphs of two analytic maps $\gamma^\pm : [a,b] \to \mathbb{R}^2_{(\xi,\eta)}$. We take the graph of γ^- as the (planar) half-circle starting from the south pole S and ending at the north pole N. The graph of γ^+ is the remaining part of the boundary. Clearly γ^- and γ^+ *join in an*

$^{(8)}$We believe the sequence $(\mathbf{u}_\varepsilon)$ to be a recovery sequence much more generally, at least when $\Sigma_{\min}$ can be identified with the support of the "vertical component" of a cartesian current [8] obtained by minimizing the mass among all cartesian currents coinciding with the graph of $\mathbf{u}$ out of the jump. In this respect, we observe that the precise knowledge of several qualitative properties of $\Sigma_{\min}$ is required in order to prove Theorems 1.1 and 1.2. For this reason generalizing the proof using an area-mininizing cartesian current seems not to be easy.

$^{(9)}$Roughly speaking, we can say (as we shall prove) that the special structure of Γ as union of two graphs, "propagates" into $\Sigma_{\min}$, ensuring the existence of a semicartesian parametrization.

$^{(10)}$It is possible to find embedded surfaces spanning the same boundary with non zero-genus and lower area, see for example [12, Figure 8.1.1 and Figure 8.1.2]. Nevertheless our argument seems to be hardly generalizable to surfaces not of disk-type.

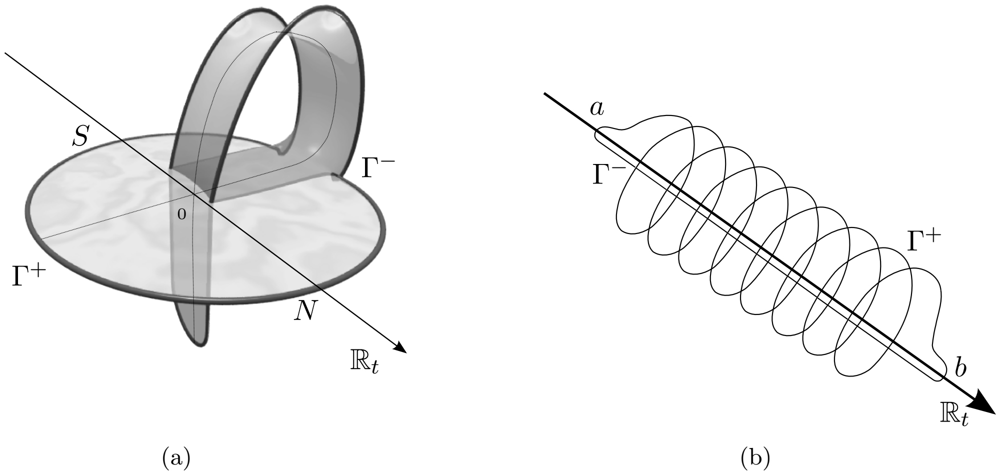

Figure 1: (a): an example of $\Sigma_{\min}$ with self-intersections admitting a semicartesian parametrization. We also plot the intersection of $\Sigma_{\min}$ with the plane $\{\bar{t} = 0\}$: this is a non-simple curve connecting $(\bar{t}, \gamma^-(\bar{t}))$ and $(\bar{t}, \gamma^+(\bar{t}))$. (b): an other analytic curve Γ leading to a $\Sigma_{\min}$ admitting a semicartesian parametrization. In this case γ^- is approximatively constant in $[a+\delta, b-\delta]$ for some small $\delta > 0$, so that its graph Γ^- is almost a segment (we cannot require constancy due to analiticity). The graph Γ^+ of γ^+ is, instead, an helix around Γ^-. It is clear that this situation is very far from the non-parametric case. The qualitative properties of Γ in correspondence to the points a and b are not arbitrary, and will be discussed in detail in the next sections (see also the assumptions in Theorem 1.2).

analytic way. We stress that for $\bar{t} \in (a, b)$ the intersection of the plane $\{t = \bar{t}\}$ with Γ is just the set of points $\{(\bar{t}, \gamma^-(\bar{t})), (\bar{t}, \gamma^+(\bar{t}))\}$, while the intersection with the surface $\Sigma_{\min}$ is a connected, possibly non-simple, curve[(11)]. Moreover, near the two poles, Γ is essentially a circumference, and this implies, as we shall see later (`step 4` in the proof of Theorem 6.1) that the nondegeneracy assumption mentioned in the statement of Theorem 1.2 is satisfied.
The analiticity of Γ in Theorem 1.2 is a strong assumption: indeed it forces $\mathbf{u}$ to have a rather rigid structure, in particular near the crack tips, and it also implies that the traces $\mathbf{u}^-$ and $\mathbf{u}^+$ cannot be independent. As we shall clarify below, the reason for which we require analyticity is that we need to exclude of branch points and boundary branch points on $\Sigma_{\min}$. Finding sufficient conditions on Γ ensuring the existence of a semicartesion parametrization of $\Sigma_{\min}$, without assuming analyticity, requires further investigation.
Roughly speaking, the proof of Theorem 1.2 runs as follows. First we need to guarantee that no plane orthogonal to the $t-$axis is tangent to $\Sigma_{\min}$ since, under this transversality condition, a classical result provides a *local* semicartesian parametrization (Theorems 6.1 and 7.5). Let us consider a conformal parametrization Y of $\Sigma_{\min}$ defined on the unit disk B; thanks to the analyticity of Γ, it is possible to extend $\Sigma_{\min}$ to a minimal surface Σ^{ext}, parametrized on B^{ext}, an open set containing B, by an analytic map $Y^{\text{ext}} = (Y_1^{\text{ext}}, Y_2^{\text{ext}}, Y_3^{\text{ext}})$ coinciding with Y on B. Now we define a *height* function h, defined on B^{ext} and returning for each point

[(11)]The surface in [12, Figure 8.1.2] mentioned in footnote (10) does not satisfy this property.

(u, v) the $t-$coordinate of its image through $Y^{\rm ext}$, that is

$$h : B^{\rm ext} \to \mathbb{R}_t,$$
$$h(u, v) := Y_1^{\rm ext}(u, v).$$

We now observe that the tangent plane to $\Sigma^{\rm ext}$ at $Y^{\rm ext}(u, v)$ is orthogonal to the $t-$axis if and only if (u, v) is a critical point for h. Thus in order to get the desired transversality property, we need to exclude the presence of critical points of h on $\overline{B}$, except for a minimum and a maximum on ∂B, which exist since h is continuous. Internal maxima and minima are excluded by a geometric argument, and saddle points are excluded by using a Morse relation for closed domains (see Appendix 8). In this step, proven in Theorem 6.1, the analiticity of Γ is once more crucial, because it prevents $\Sigma_{\rm min}$ to have boundary or internal branch points; this regularity and the nondegeneracy hypotheses on the parametrization of Γ imply that h is a Morse function satisfying the requirements of Theorem 8.1.
In this way we have obtained the existence of a *local* semicartesian parametrization. Using the simple connectedness of $\Sigma_{\rm min}$, it is finally possible to globalize the argument, and provide a semicartesian parametrization (Section 6.2). We notice here that several properties of the (a priori unknown) parameter domain D can be proven, as shown in Section 6.3: in particular, it turns out that ∂D is union of the graphs of two functions $\sigma^\pm$, which are locally Lipschitz (but not Lipschitz) with a local Lipschitz constant controlled by the Lipschitz constant of $\gamma^\pm$. We refer to Section 6 for the details of the proofs, but it is clear that the analyticity assumption is fundamental in most of the arguments.

The plan of the paper is the following. In Section 2 we fix some notation and we introduce the space $\mathrm{D}(\Omega; \mathbb{R}^2)$ (some properties of which are given in Section 9). We also give the definition of semicartesian parametrization. In Section 3 we prove Theorem 1.1 for maps whose associated Plateau's problem admits a non parametric solution. In Section 4 we provide a generalization of this result for possibly self-intersecting area-minimizing surfaces, underlying that what is really important is that the solution of the Plateau's problem admits a semicartesian parametrization. In Section 5 we give some sufficient conditions on $\mathbf{u}$ for the existence of a semicartesian parametrization of $\Sigma_{\rm min}$, see Theorem 5.1, the proof of which is given in Section 6 and is the most technical part of the paper. In Sections 7 and 8 we collect some classical results of minimal surfaces and Morse Theory needed in our proofs.

2 Notation

If $n \geq 2$, we denote by $\cdot, |\cdot|$ the euclidean scalar product and norm in $\mathbb{R}^n$, respectively, and by $\overline{E}$ and $\mathrm{int}(E)$ the closure and the interior part of a set $E \subseteq \mathbb{R}^n$. $\mathcal{H}^2$ is the Hausdorff measure in $\mathbb{R}^n$ and $\mathcal{L}^2$ is the Lebesgue measure in $\mathbb{R}^2$. $B \subset \mathbb{R}^2 = \mathbb{R}^2_{(u,v)}$ is the open unit disk and ∂B is its boundary. We choose an arc-length parametrization

$$\mathbf{b} : \theta \in [0, 2\pi) \to \mathbf{b}(\theta) \in \partial B, \tag{2.1}$$

and take $\theta_{\rm s}, \theta_{\rm n} \in [0, 2\pi)$, with $\theta_{\rm s} < \theta_{\rm n}$, so that

$$\mathbf{b}(\theta_{\rm s}) = (0, -1), \qquad \mathbf{b}(\theta_{\rm n}) = (0, 1).$$

For a differentiable map $Y : B \to \mathbb{R}^3$, the components are denoted by $Y = (Y_1, Y_2, Y_3)$, and the partial derivatives by $Y_u = \partial_u Y = (\partial_u Y_1, \partial_u Y_2, \partial_u Y_3)$ and $Y_v = \partial_v Y = (\partial_v Y_1, \partial_v Y_2, \partial_v Y_3)$. Ω is a bounded open subset of the source space $\mathbb{R}^2_{(x,y)}$, while the target space is denoted by $\mathbb{R}^2_{(\xi,\eta)}$. When no confusion is possible, we often write $\mathbb{R}^2$ in place of the source or of the target space.
As in the introduction, if $\mathbf{v} \in \mathrm{BV}(\Omega; \mathbb{R}^2)$ we denote by $\nabla \mathbf{v}$ and $D^s \mathbf{v}$ the absolutely continuous and the singular part of the distributional gradient of $\mathbf{v}$, respectively.
With $\mathrm{D}(\Omega; \mathbb{R}^2)$ we denote the subset of $\mathrm{BV}(\Omega; \mathbb{R}^2)$ on which the relaxed area functional admits the following integral representation:

$$\overline{\mathcal{A}}(\mathbf{v}, \Omega) = \int_\Omega |\mathcal{M}\,(\nabla \mathbf{v})\,|\, dx\, dy < +\infty. \tag{2.2}$$

As we have already noticed in the introduction, $W^{1,p}(\Omega; \mathbb{R}^2)$ is contained in $\mathrm{D}(\Omega; \mathbb{R}^2)$ for every $p \in [2, +\infty]$. In Appendix 9 we report the characterization of $\mathrm{D}(\Omega; \mathbb{R}^2)$ given in [1] and we prove that the functional $\overline{\mathcal{A}}$ can be obtained also by relaxing from $\mathrm{D}(\Omega; \mathbb{R}^2)$.
We now give the useful definition of *semicartesian parametrization*.

Definition 2.1 (Union of two graphs). A closed simple rectifiable curve $\Gamma \subset \mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)}$, is said to be *union of two graphs* if there exists an interval $[a, b] \subset \mathbb{R}_t$ such that Γ is the union of the graphs of two continuous maps $\gamma^\pm \in \mathcal{C}([a, b]; \mathbb{R}^2) \cap \mathrm{Lip}_{\mathrm{loc}}((a, b); \mathbb{R}^2)$. That is $\Gamma = \Gamma^+ \cup \Gamma^-$ where

$$\Gamma^\pm = \{(t, \xi, \eta) : t \in [a, b], (\xi, \eta) = \gamma^\pm(t)\}.$$

When necessary, we shall say that Γ is union of the graphs of $\gamma^\pm$.

Definition 2.2 (Semicartesian parametrization). A disk-type surface Σ in $\mathbb{R}^3$ (possibly with self intersections) is said to admit a *semicartesian parametrization* if $\Sigma = X(D)$, where

- $D \subset \mathbb{R}^2_{(t,s)}$ is given by

$$D = \{(t, s) : t \in [a, b],\ \sigma^-(t) \le s \le \sigma^+(t)\}, \tag{2.3}$$

  with $\sigma^\pm \in \mathrm{Lip}_{\mathrm{loc}}((a, b))$ satisfying

$$\begin{aligned} \sigma^-(a) &= 0 = \sigma^+(a), \\ \sigma^-(b) &= \sigma^+(b), \\ \sigma^- &< \sigma^+ \text{ in } (a, b); \end{aligned} \tag{2.4}$$

- $X \in W^{1,2}(D; \mathbb{R}^3)$ has the following form:

$$X(t, s) = (t, X_2(t, s), X_3(t, s)) \quad \text{a.e. } (t, s) \in D. \tag{2.5}$$

Sometimes we refer to a semicartesian parametrization as to a global semicartesian parametrization; on the other hand, a local semicartesian parametrization is a $W^{1,2}$ map of the form (2.5), defined in a neighourhood of a point.

3 Non-parametric case: graph over a convex domain

As explained in the introduction, our aim is to estimate from above the area of the graph of a discontinuous map with a curve discontinuity compactly contained in Ω. In this section we study a case which leads to consider a non-parametric Plateau's problem over a convex domain.

3.1 Hypotheses on u and statement for the non-parametric case

Let $\Omega \subset \mathbb{R}^2 = \mathbb{R}^2_{(x,y)}$ be a bounded open set and assume that

$$\mathbf{u} = (\mathrm{u}_1, \mathrm{u}_2) : \Omega \to \mathbb{R}^2 = \mathbb{R}^2_{(\xi,\eta)}$$

satisfies the following properties (u1) – (u4):

(u1) $\mathbf{u} \in \mathrm{BV}(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$ and $\overline{J}_{\mathbf{u}}$ is a non-empty *simple* curve of class $\mathcal{C}^2$ (not reduced to a point) contained in Ω. We shall write

$$\overline{J}_{\mathbf{u}} = \alpha([a,b]),$$

where a and b are two real numbers with $a < b$, and

$$\alpha : t \in [a,b] \subset \mathbb{R} = \mathbb{R}_t \to \alpha(t) \in \overline{J}_{\mathbf{u}}$$

is an *arc-length* parametrization of $\overline{J}_{\mathbf{u}}$ of class $\mathcal{C}^2$. Note that we are assuming that if $t_1, t_2 \in [a,b]$, $t_1 \neq t_2$ then $\alpha(t_1) \neq \alpha(t_2)$, and moreover

$$\overline{J}_{\mathbf{u}} \cap \partial\Omega = \emptyset.$$

In particular, the two distinct crack tips are $\overline{J}_{\mathbf{u}} \setminus J_{\mathbf{u}} = \{\alpha(a), \alpha(b)\} \subset \Omega$ (see Figure 2(a)).

(u2) $\mathbf{u} \in W^{1,\infty}\left(\Omega \setminus \overline{J}_{\mathbf{u}};\mathbb{R}^2\right)$; by the Sobolev embeddings (see for example [2, Theorem 4.12]) we have $\mathbf{u} \in \mathcal{C}\left(\Omega \setminus \overline{J}_{\mathbf{u}};\mathbb{R}^2\right)$.

As a consequence of (u1) and (u2), there exists the trace of $\mathbf{u}$ on $\overline{J}_{\mathbf{u}}$ on each side of the jump:

$$\begin{aligned} \gamma^-(t) = \gamma^-[\mathbf{u}](t) = (\gamma_1^-(t), \gamma_2^-(t)) := \mathbf{u}^-(\alpha(t)) \in \mathbb{R}^2, \\ \gamma^+(t) = \gamma^+[\mathbf{u}](t) = (\gamma_1^+(t), \gamma_2^+(t)) := \mathbf{u}^+(\alpha(t)) \in \mathbb{R}^2, \end{aligned} \qquad t \in [a,b],$$

and the functions

$$t \in [a,b] \longrightarrow \gamma^\pm(t)$$

are Hölder continuous(12).
Notice that

$$\gamma^-(a) = \gamma^+(a), \qquad \gamma^-(b) = \gamma^+(b). \tag{3.1}$$

(u3) $\gamma^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$ and there exists a finite set of points $t_0 := a < t_1 < \cdots < t_m < t_{m+1} = b$ of $[a,b]$ such that $\gamma^\pm \in \mathcal{C}^2((t_i,t_{i+1})) \cap \mathcal{C}^1([t_i,t_{i+1}])$ for any $i = 0,\ldots,m$. Moreover we require

$$\gamma^-(t) \neq \gamma^+(t), \qquad t \in (a,b). \tag{3.2}$$

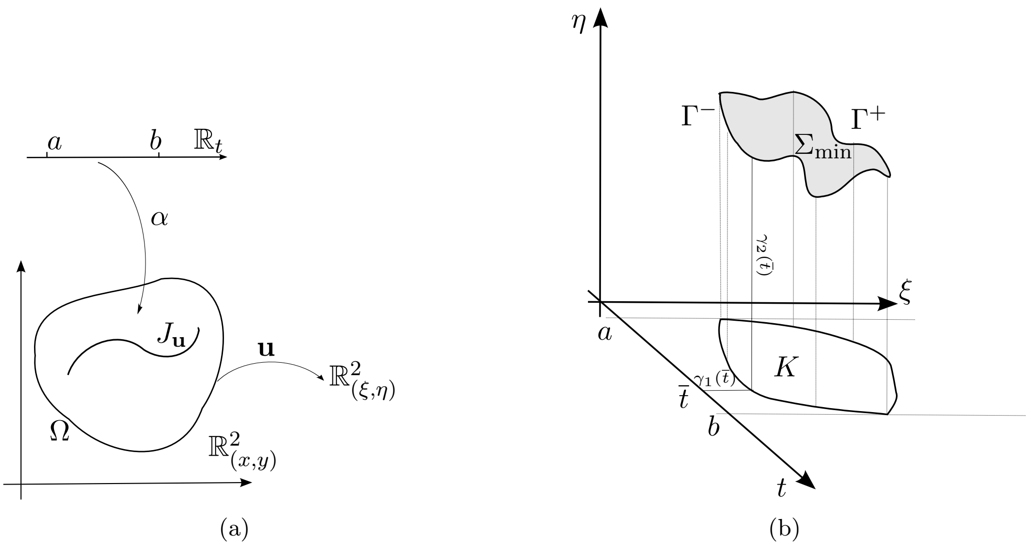

Figure 2: (a): the domain Ω, the arc-length parametrization of the jump of the map $\mathbf{u}$. Notice that the closure of the jump is contained in Ω. (b): the Lipschitz curve Γ, union of the graphs on $[a,b]$ of the vector valued functions γ^- and γ^+. K is a closed convex set in $\mathbb{R}^2_{(t,\xi)}$, having non empty interior, and $\Sigma_{\min}$ is the area-minimizing surface spanning Γ. We observe that ∂K is not differentiable at $(a,\gamma_1^+(a))$ and $(b,\gamma_1^+(b))$, and Γ is not differentiable at $(a,\gamma^+(a))$, $(b,\gamma^+(b))$.

In order to state our last assumption (u4), we denote by $\Gamma^\pm = \Gamma^\pm[\mathbf{u}]$ the graphs of the maps $\gamma^\pm$,

$$\Gamma^- = \Gamma^-[\mathbf{u}] := \{(t,\xi,\eta) \in [a,b]\times\mathbb{R}^2 : (\xi,\eta) = \gamma^-(t)\},$$
$$\Gamma^+ = \Gamma^+[\mathbf{u}] := \{(t,\xi,\eta) \in [a,b]\times\mathbb{R}^2 : (\xi,\eta) = \gamma^+(t)\},$$

and we set

$$\Gamma = \Gamma[\mathbf{u}] := \Gamma^- \cup \Gamma^+. \tag{3.3}$$

In view of assumptions (u2) and (u3), $\Gamma \subset \mathbb{R}^3$ is a closed, simple, Lipschitz and piecewise $\mathcal{C}^2$ curve obtained as union of two curves; moreover $(a,\gamma^+(a))$ and $(b,\gamma^+(b))$ (coinciding with $(a,\gamma^-(a))$ and $(b,\gamma^-(b))$ respectively) are *nondifferentiability points* of Γ. The next assumption requires introducing the projection on a plane spanned by t and one of the two coordinates, say ξ, in the target space $\mathbb{R}^2_{(\xi,\eta)}$. We suppose that:

(u4) the orthogonal projection of Γ on the plane $\mathbb{R}^2_{(t,\xi)}$ is the boundary of a closed convex set K with non-empty interior. In particular, without loss of generality,

$$\gamma_1^-(t) < \gamma_1^+(t)\,, \quad t \in (a,b),$$

and we assume that γ_1^- is convex and γ_1^+ is concave. Moreover thanks to hypothesis (u3),

$$\gamma_1^\pm \in \mathrm{Lip}([a,b])$$

and therefore $(a,\gamma_1^-(a))$ and $(b,\gamma_1^-(b))$ are nondifferentiability points of ∂K.

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(12) Indeed $W^{1,\infty}(U) \subset \mathcal{C}^{0,\lambda}(\overline{U})$ for every $\lambda \in (0,1)$ if $U \subset \mathbb{R}^2$ is a smooth enough open set (see again [2, Theorem 4.12]), and we can consider, for each trace, a sufficiently smooth open set $U \subset \Omega$ such that $\overline{J}_{\mathbf{u}} \subset \partial U$.

Summarizing, $\partial K = \mathrm{graph}(\gamma_1^-) \cup \mathrm{graph}(\gamma_1^+)$ is of class $\mathcal{C}^1$ up to a finite set of points containing $(a, \gamma_1^-(a))$ and $(b, \gamma_1^-(b))$. In particular, ∂K is *not* of class $\mathcal{C}^2$.

Remark 3.1. The hypothesis that Γ has corners in $(a, \gamma^-(a))$ and $(b, \gamma^-(b))$ is related to the regularity assumptions made on $\mathbf{u}$ in (u2): requiring that Γ is differentiable at $(a, \gamma^-(a))$ and $(b, \gamma^-(b))$ would *prevent* $\mathbf{u}$ to belong to $W^{1,\infty}\left(\Omega \setminus \overline{J}_{\mathbf{u}}; \mathbb{R}^2\right)$. On the other hand, it is useful to require $\mathbf{u} \in W^{1,\infty}\left(\Omega \setminus \overline{J}_{\mathbf{u}}; \mathbb{R}^2\right)$: indeed, in this case, we can infer (see the proof of Theorem 3.1, for example `step 8`) that the approximating maps $\mathbf{u}_\varepsilon$ are Lipschitz and thus in particular that they can be used to estimate $\overline{\mathcal{A}}(\mathbf{u}, \Omega)$. In Section 4 we manage in weakening this requirement (compare condition ($\tilde{\mathrm{u}}$2)).

Before stating our first result, we need the following definition (for further details, see Section 7).

Definition 3.1. We denote by $\Sigma_{\min} \subset \mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)}$ an area-minimizing surface of disk-type spanning Γ, that is the image of the unit disk through a solution of the Plateau's problem (7.1) for Γ.

Now we are in a position to state our first theorem.

Theorem 3.1. *Suppose that* $\mathbf{u}$ *satisfies assumptions* (u1)-(u4)*. Then there exists a sequence*

$$(\mathbf{u}_\varepsilon)_\varepsilon \subset \mathrm{Lip}(\Omega; \mathbb{R}^2) \tag{3.4}$$

converging to $\mathbf{u}$ *in* $L^1(\Omega; \mathbb{R}^2)$ *as* $\varepsilon \to 0^+$ *such that*

$$\lim_{\varepsilon \to 0^+} \overline{\mathcal{A}}(\mathbf{u}_\varepsilon, \Omega) = \overline{\mathcal{A}}(\mathbf{u}, \Omega \setminus \overline{J}_{\mathbf{u}}) + \mathcal{H}^2(\Sigma_{\min}) = \int_\Omega |\mathcal{M}(\nabla \mathbf{u})| \; dx\, dy + \mathcal{H}^2(\Sigma_{\min}). \tag{3.5}$$

In particular

$$\overline{\mathcal{A}}(\mathbf{u}, \Omega) \leq \int_\Omega |\mathcal{M}(\nabla \mathbf{u})| \; dx\, dy + \mathcal{H}^2(\Sigma_{\min}). \tag{3.6}$$

3.2 Proof of Theorem 3.1

The proof of Theorem 3.1 is rather long, and we split it into several steps.

`Step 1`. Definition of the function z and representation of the surface $\Sigma_{\min}$.

Since Γ in (3.3) admits a convex one-to-one parallel projection, we can apply Theorem 7.7. In particular, there exists a scalar function $z \in \mathcal{C}(K) \cap \mathcal{C}^\omega(\mathrm{int}(K))$ such that

$$\Sigma_{\min} = \left\{(t, \xi, \eta) \in \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)} : \; (t, \xi) \in K, \, \eta = z(t, \xi)\right\} = \mathrm{graph}(z),$$

where z solves

$$\begin{cases} \mathrm{div}\left(\dfrac{\nabla z}{\sqrt{1 + |\nabla z|^2}}\right) = 0 & \text{in } \mathrm{int}(K), \\ z = \phi & \text{on } \partial K, \end{cases} \tag{3.7}$$

and

$$\phi = \gamma_2^\pm \quad \text{on } \mathrm{graph}(\gamma_1^\pm).$$

Remark 3.2. It is worthwhile to stress the different role played in (3.7) by the two components of the traces $\gamma^{\pm}$: the *first components* $\gamma_1^{\pm}$ determine the boundary of the domain K where solving the non-parametric Plateau's problem, the Dirichlet condition of which is given by the *second components* $\gamma_2^{\pm}$ (see Figure 2(b)).

Remark 3.3. $\Sigma_{\min}$ is the unique area-minimizing surface among all graph-like surfaces on $\mathrm{int}(K)$ satisfying the Dirichlet condition in (3.7)

Due to the presence of corners in ∂K, we cannot directly infer from Theorem 7.8 that $z \in \mathrm{Lip}(\overline{K})$. Since the Lipschitz regularity of z is strictly related to the Lipschitz regularity of $\mathbf{u}_\varepsilon$, in order to ensure inclusion (3.4) a smoothing argument is required (see Figure 3).

`Step 2`. Smoothing of ∂K and $\gamma_2^{\pm}$: definition of the function z_μ and of the surface $\Sigma_{\min}^{\mu}$.
Since ∂K has only a finite number of nondifferentiability points, we smoothen the corners of K obtaining, for a suitable $\overline{\mu} > 0$ small enough, a sequence $(K_\mu)_{\mu \in (0,\overline{\mu})}$ of sets with the following properties:

- each K_μ is convex, closed, with non-empty interior and is contained in K. Moreover, K_μ coincides with K out of the disks of radius μ centered at the nondifferentiability points of ∂K;
- $\partial K_\mu \in \mathcal{C}^2$;
- $\mu_1 < \mu_2$ implies $K_{\mu_1} \supset K_{\mu_2}$,

see Figure 3(a).
In order to apply Theorem 7.8, we need not only to smoothen the set K, but also the Dirichlet condition $\gamma_2^{\pm}$ at the same time. Firstly we observe that since both K and K_μ are convex sets and $K_\mu \subset K$, there exist a point $O \in K_\mu$ and a projection π_μ acting as follows:

$$\begin{aligned} \pi_\mu : \partial K_\mu &\to \partial K \\ p &\to \pi_\mu(p), \end{aligned}$$

where $\pi_\mu(p)$ is the unique point of ∂K lying on the half-line rising from O and passing through p.
Now using this projection and again the fact that $\gamma_2^{\pm}$ are Lipschitz and piecewise $\mathcal{C}^2$, for every $\mu \in (0, \overline{\mu})$ we can define a function ϕ_μ with the following properties:

- $\phi_\mu : \partial K_\mu \to \mathbb{R}$ is of class $\mathcal{C}^2$;
- ϕ_μ coincides with $\gamma_2^{\pm}$ on $\partial K_\mu \cap \partial K$ out of the balls of radius μ centered at the nondifferentiability points of $\gamma_2^{\pm}$;
- the Hausdorff distance between the graph of ϕ_μ and Γ is less than μ;
- there holds
$$\frac{\left|\phi_\mu(p) - \gamma_2^{\pm}(\pi_\mu(p))\right|}{|p - \pi_\mu(p)|} \leq C, \qquad p \in \partial K_\mu, \tag{3.8}$$
  where C is a positive constant independent of μ.

For any $\mu \in (0, \overline{\mu})$ let us denote by z_μ the solution to

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla z_\mu}{\sqrt{1+|\nabla z_\mu|^2}}\right) = 0 & \text{in } \operatorname{int}(K_\mu), \\ z_\mu = \phi_\mu & \text{on } \partial K_\mu. \end{cases}$$

Theorem 7.8 yields

$$z_\mu \in \operatorname{Lip}(K_\mu) \cap \mathcal{C}^\omega(\operatorname{int}(K_\mu)).$$

We denote by $\Sigma^\mu_{\min}$ the graph of z_μ. Applying [15, §305] it follows[(13)]

$$\lim_{\mu \to 0^+} \mathcal{H}^2(\Sigma^\mu_{\min}) = \mathcal{H}^2(\Sigma_{\min}). \tag{3.9}$$

In order to assert that the maps $\mathbf{u}_\varepsilon$ in `step 6` are Lipschitz continuous, in particular close to the crack tips of $J_{\mathbf{u}}$, we need to extend z_μ to K.

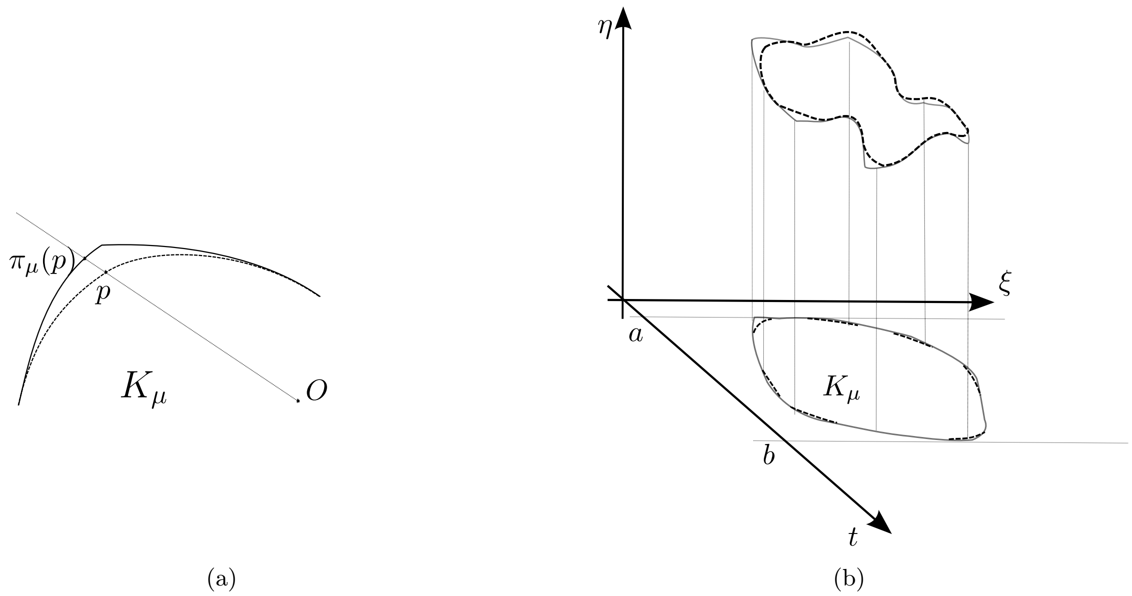


Figure 3: (a): the approximation process near a nondifferentiability point of ∂K and the action of the projection map π_μ. (b): the graph of the boundary value ϕ_μ, approximating the space curve Γ.

`Step 3`. Extension of z_μ on K: definition of the extended surface $\widehat{\Sigma}_\mu$.
We consider again the projection π_μ defined in the previous step and we observe that for every point $(t, \xi) \in K \setminus K_\mu$ there exist a unique $p \in \partial K_\mu$ and $\rho \in (0, 1]$ such that

$$(t, \xi) = \rho p + (1 - \rho)\pi_\mu(p).$$

Thus we extend z_μ to K defining

$$\widehat{z}_\mu(t, \xi) := \begin{cases} \rho\phi_\mu(p) + (1-\rho)\phi(\pi_\mu(p)), & (t, \xi) \in K \setminus K_\mu, \\ z_\mu(t, \xi), & (t, \xi) \in K_\mu. \end{cases}$$

[(13)] An argument leading to an equality of the type (3.9) in a nonsmooth situation was proved in [3].

Notice that

$$\widehat{z}_\mu = z \qquad \text{on } \partial K. \tag{3.10}$$

We denote by $\widehat{\Sigma}_\mu$ the graph of $\widehat{z}_\mu$ on K. Property (3.8) gives a uniform control of the gradient of $\widehat{z}_\mu$ on $K \setminus K_\mu$, which implies that

$$\lim_{\mu \to 0^+} \mathcal{H}^2(\widehat{z}_\mu(K \setminus K_\mu)) = 0.$$

Thus from (3.9)

$$\lim_{\mu \to 0^+} \mathcal{H}^2(\widehat{\Sigma}_\mu) = \mathcal{H}^2(\Sigma_{\min}). \tag{3.11}$$

Remark 3.4. By construction, we have that $\widehat{z}_\mu$ is Lipschitz continuous.

`Step 4`. Definition of the parameter space D.
For our goals, it is convenient to choose a parameter space D different from K, for parametrizing $\Sigma_{\min}$ and $\widehat{\Sigma}_\mu$. Set

$$\sigma(t) := \frac{\gamma_1^+(t) - \gamma_1^-(t)}{2}, \qquad t \in [a, b].$$

Let $D \subset \mathbb{R}^2_{(t,s)}$ be defined as follows:

$$D := \{(t, s) : \ t \in [a, b], |s| \leq \sigma(t)\},$$

which has the same qualitative properties of K, in particular $\partial D = \text{graph}(\sigma) \cup \text{graph}(-\sigma)$, and D has two angles in correspondence of $t = a$ and $t = b$ (same angles as the corresponding ones of K). We notice that the segment $(a, b) \times \{0\}$ is contained in $\text{int}(D)$, see Figure 4.

`Step 5`. Definition of the maps X and X_μ.
The construction of the function $\mathbf{u}_\varepsilon$ in the statement of the theorem is mainly based on the maps

$$X : D \to \mathbb{R}^3, \qquad X_\mu : D \to \mathbb{R}^3,$$

defined as follows: for any $(t, s) \in D$

$$\begin{aligned} X(t, s) :=& \left(t, s + \frac{\gamma_1^+(t) + \gamma_1^-(t)}{2}, z\left(t, s + \frac{\gamma_1^+(t) + \gamma_1^-(t)}{2}\right)\right) \\ =& (t, X_2(t, s), X_3(t, s)), \\ X_\mu(t, s) :=& \left(t, s + \frac{\gamma_1^+(t) + \gamma_1^-(t)}{2}, \widehat{z}_\mu\left(t, s + \frac{\gamma_1^+(t) + \gamma_1^-(t)}{2}\right)\right) \\ =& (t, X_{\mu 2}(t, s), X_{\mu 3}(t, s)). \end{aligned} \tag{3.12}$$

Remark 3.5. We stress that the maps X and X_μ are semicartesian. In particular, where they are differentiable, their gradient never vanishes on D. Observe also that, from Remark 3.4, it follows

$$X_\mu \in \text{Lip}(D; \mathbb{R}^3). \tag{3.13}$$

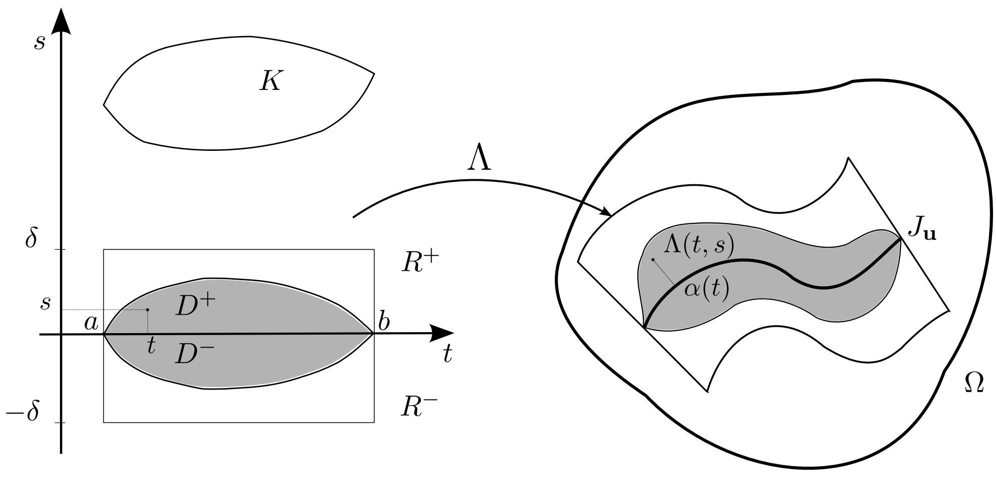


Figure 4: We display the domain $D = \overline{D^+ \cup D^-}$ obtained by symmetrizing K. It is contained in the rectangle $R = [a,b] \times [-\delta, \delta]$ on which it is defined the diffeomorphism Λ; $\Lambda([a,b] \times \{0\})$ is exactly the closure $\overline{J}_{\mathbf{u}}$ of the discontinuity curve.

`Step 6`. Definition of the map $\mathbf{u}_\varepsilon$.
For the definition of $\mathbf{u}_\varepsilon$ we need some preparation. Denote by $^\perp$ the counterclockwise rotation of $\pi/2$ in $\mathbb{R}^2_{(x,y)}$. Hypothesis (u1) implies that there exists $\delta > 0$ and a closed set contained in Ω and containing $\overline{J}_{\mathbf{u}}$ of the form $\Lambda(R)$, where $R := [a,b] \times [-\delta, \delta]$ and

$$\Lambda(t,s) := \alpha(t) + s\dot{\alpha}(t)^\perp, \qquad (t,s) \in R,$$

is a diffeomorphism of class $\mathcal{C}^1(R; \Lambda(R))$, see Figure 4. If $\Lambda^{-1} : \Lambda(R) \to R$ is the inverse of Λ, we have

$$\Lambda^{-1}(x,y) = (t(x,y), s(x,y)),$$

where

- $s(x,y) = d(x,y)$ is the distance of (x,y) from $J_{\mathbf{u}}$ on the side of $J_{\mathbf{u}}$ corresponding to the trace $\mathbf{u}^+$, and minus the distance of (x,y) from $J_{\mathbf{u}}$ on the other side,
- $t(x,y)$ is so that $\alpha(t(x,y)) = (x,y) - d(x,y)\nabla d(x,y)$ is the unique point on $\overline{J}_{\mathbf{u}}$ nearest to (x,y).

Since $\overline{J}_{\mathbf{u}}$ is of class $\mathcal{C}^2$, we have that d is of class $\mathcal{C}^2$ on $\Lambda(R)$[14] and t is of class $\mathcal{C}^1$ on $\Lambda(R)$. We can always suppose

$$D \setminus ((a,0) \cup (b,0)) \subset \mathrm{int}(R), \tag{3.14}$$

since, if not, we choose $c \in (0,1)$ so that $D_c := \{(t,s) \in \mathbb{R}^2 : (t, s/c) \in D\} \subset R$, and we prove the result with D_c in place of D and $X_c(t,s) := X(t, s/c)$ in place of $X(t,s)$.
Set $R^+ := [a,b] \times (0, \delta]$, $R^- := [a,b] \times [-\delta, 0)$, and

$$D^+ := D \cap R^+, \qquad D^- := D \cap R^-.$$

[14] It is sufficient to slightly extend $\overline{J}_{\mathbf{u}}$ and consider d on a small enough tubolar neighborhood of the extension.

For any $\varepsilon \in (0,1)$ let

$$D_\varepsilon := \{(t,s) \in \mathbb{R}^2 : \ (t, s/\varepsilon) \in D\},$$

and

$$D_\varepsilon^\pm := \{(t,s) \in \mathbb{R}^2 : \ (t, s/\varepsilon) \in D^\pm\},$$

so that

$$\mathrm{int}(D_\varepsilon) \supset (a,b) \times \{0\}.$$

We set $R_\varepsilon := [a,b] \times (-\varepsilon\delta, \varepsilon\delta)$ and $R_\varepsilon^+ := [a,b] \times (0, \varepsilon\delta]$, $R_\varepsilon^- := [a,b] \times [-\varepsilon\delta, 0)$. From (3.14), we have $D_\varepsilon \subset R_\varepsilon$.

We are now in a position to define the sequence $(\mathbf{u}_\varepsilon) \subset \mathrm{Lip}(\Omega;\mathbb{R}^2)$. We do this in three steps as follows:

- **outer region.** If $(x,y) \in \Omega \setminus \Lambda(R_\varepsilon)$

$$\mathbf{u}_\varepsilon(x,y) := \mathbf{u}(x,y); \tag{3.15}$$

- **opening the fracture: intermediate region.** If $(x,y) \in \Lambda(R_\varepsilon^\pm \setminus D_\varepsilon^\pm)$

$$\mathbf{u}_\varepsilon(x,y) := \mathbf{u}(T_\varepsilon^\pm(x,y)), \tag{3.16}$$

  where $T_\varepsilon^\pm := \Lambda \circ \Phi_\varepsilon^\pm \circ \Lambda^{-1}$ with

$$\Phi_\varepsilon^+ : R_\varepsilon^+ \setminus D_\varepsilon^+ \to R_\varepsilon^+, \qquad \Phi_\varepsilon^+(t,s) := \left(t, \frac{s - \varepsilon\sigma(t)}{\delta - \sigma(t)}\delta\right),$$

$$\Phi_\varepsilon^- : R_\varepsilon^- \setminus D_\varepsilon^- \to R_\varepsilon^-, \qquad \Phi_\varepsilon^-(t,s) := \left(t, \frac{s + \varepsilon\sigma(t)}{\delta - \sigma(t)}\delta\right).$$

  Notice that $T_\varepsilon^\pm$ is the identity on $\partial R_\varepsilon^\pm \setminus ([a,b] \times \{0\})$, see Figure 5.

- **opening the fracture: inner region.** If $(x,y) \in \Lambda(D_\varepsilon)$

$$\mathbf{u}_\varepsilon(x,y) := \left(X_{\mu_\varepsilon 2}\left(t(x,y), \frac{d(x,y)}{\varepsilon}\right), X_{\mu_\varepsilon 3}\left(t(x,y), \frac{d(x,y)}{\varepsilon}\right)\right), \tag{3.17}$$

  for a suitable choice of the sequence $(\mu_\varepsilon)_\varepsilon$ converging to 0 as $\varepsilon \to 0^+$, that will be selected later[15].

Remark 3.6. We have

$$\mathbf{u}_\varepsilon \in \mathrm{Lip}(\Omega;\mathbb{R}^2). \tag{3.18}$$

Indeed

- by assumption (u2) it follows $\mathbf{u} \in W^{1,\infty}(\Omega \setminus \Lambda(R_\varepsilon);\mathbb{R}^2)$, hence $\mathbf{u}_\varepsilon \in W^{1,\infty}(\Omega \setminus \Lambda(R_\varepsilon);\mathbb{R}^2)$;
- in $\Lambda(D_\varepsilon)$ the regularity of $\mathbf{u}_\varepsilon$ is the same as the Lipschitz regularity of X_{μ_ε}, see (3.13);

[15] See the conclusion of `step 9`.

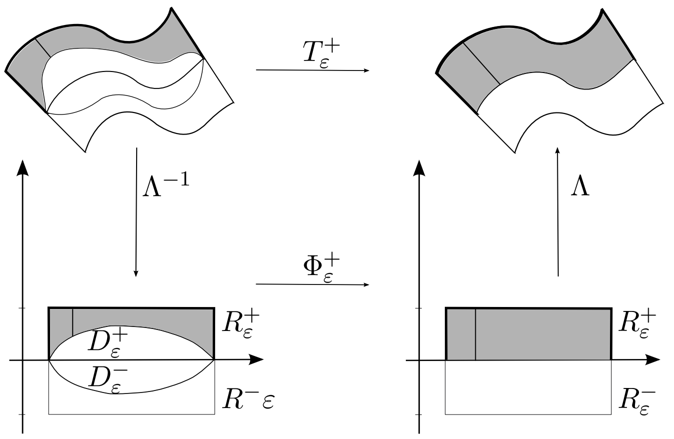


Figure 5: The action of the map T_ε^+. Any oblique small segment on the top left is mapped in the parallel longer segment reaching the fracture, on the top right.

- in $\Lambda(R_\varepsilon \setminus D_\varepsilon)$, $\mathbf{u}_\varepsilon$ is defined as the composition of $\mathbf{u} \in W^{1,\infty}\left(\Omega \setminus \overline{J}_{\mathbf{u}}; \mathbb{R}^2\right)$ and a Lipschitz deformation.

Since by construction $\mathbf{u}_\varepsilon$ is continuous (remember (3.10)), inclusion (3.18) follows.

Remark 3.7. We have

$$\sup_{\varepsilon \in (0,1]} \|\mathbf{u}_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^2)} < +\infty,$$

since $\mathbf{u} \in L^\infty(\Omega; \mathbb{R}^2)$ by assumption (u1) and, for some $\overline{\mu} > 0$, $\sup_{\mu \in (0,\overline{\mu})} \|X_\mu\|_{L^\infty(D)} < +\infty$. Therefore $\mathbf{u}_\varepsilon \to \mathbf{u}$ in $L^1(\Omega; \mathbb{R}^2)$. Indeed

$$\int_\Omega |\mathbf{u}_\varepsilon - \mathbf{u}| \, dx \, dy = \int_{\Lambda(R_\varepsilon)} |\mathbf{u}_\varepsilon - \mathbf{u}| \, dx \, dy \to 0$$

as $\varepsilon \to 0^+$, because the Lebesgue measure of $\Lambda(R_\varepsilon)$ tends to 0.

`Step 7`. We have

$$\lim_{\varepsilon \to 0^+} \overline{\mathcal{A}}(\mathbf{u}_\varepsilon, \Omega \setminus \Lambda(R_\varepsilon)) = \overline{\mathcal{A}}(\mathbf{u}, \Omega \setminus J_{\mathbf{u}}).$$

Indeed by (3.15),

$$\overline{\mathcal{A}}(\mathbf{u}_\varepsilon, \Omega \setminus \Lambda(R_\varepsilon)) = \overline{\mathcal{A}}(\mathbf{u}, \Omega \setminus \Lambda(R_\varepsilon)).$$

Let us show that the contribution to the area in the intermediate region $\Lambda(R_\varepsilon \setminus D_\varepsilon)$ (definition (3.16)) is negligible as $\varepsilon \to 0^+$.

`Step 8`. We have

$$\lim_{\varepsilon \to 0^+} \overline{\mathcal{A}}(\mathbf{u}_\varepsilon, \Lambda(R_\varepsilon^\pm \setminus D_\varepsilon^\pm)) = 0.$$

We make the computation in $\Lambda(R_\varepsilon^+ \setminus D_\varepsilon^+)$, the case in $\Lambda(R_\varepsilon^- \setminus D_\varepsilon^-)$ being similar. To simplify notation, we write T_ε instead of T_ε^+, and set $T_\varepsilon = (T_{\varepsilon 1}, T_{\varepsilon 2})$.

Take a constant $C > 0$ so that

$$\begin{aligned}\mathcal{A}(\mathbf{u}_\varepsilon, \Lambda(R_\varepsilon^+ \setminus D_\varepsilon^+)) &= \int_{\Lambda(R_\varepsilon^+ \setminus D_\varepsilon^+)} |\mathcal{M}(\nabla \mathbf{u}_\varepsilon)|\, dx\, dy \\ &\le C \int_{\Lambda(R_\varepsilon^+ \setminus D_\varepsilon^+)} \left[1 + |\partial_x \mathrm{u}_{\varepsilon 1}| + |\partial_x \mathrm{u}_{\varepsilon 2}| + |\partial_y \mathrm{u}_{\varepsilon 1}| + |\partial_y \mathrm{u}_{\varepsilon 2}| + |\partial_x \mathrm{u}_{\varepsilon 1} \partial_y \mathrm{u}_{\varepsilon 2} - \partial_y \mathrm{u}_{\varepsilon 1} \partial_x \mathrm{u}_{\varepsilon 2}|\right] dx\, dy\end{aligned} \tag{3.19}$$

where for $i = 1, 2$

$$\partial_x \mathrm{u}_{\varepsilon i} = \partial_x \mathrm{u}_i \partial_x T_{\varepsilon 1} + \partial_y \mathrm{u}_i \partial_x T_{\varepsilon 2}, \qquad \partial_y \mathrm{u}_{\varepsilon i} = \partial_x \mathrm{u}_i \partial_y T_{\varepsilon 1} + \partial_y \mathrm{u}_i \partial_y T_{\varepsilon 2}.$$

From the definition of T_ε

$$\nabla T_\varepsilon(x, y) = \nabla \Lambda \left(\Phi_\varepsilon(t(x, y), s(x, y))\right)^T \nabla \Phi_\varepsilon^+(t(x, y), s(x, y)) \cdot \nabla \Lambda^{-1}(x, y).$$

Λ is a $\mathcal{C}^1$ diffeomorphism, thus all components of its Jacobian are bounded; on the other hand the Jacobian of the transformation Φ_ε^+ is

$$\nabla \Phi_\varepsilon^+(t, s) = \begin{bmatrix} 1 & 0 \\ \dfrac{-\delta \dot{\sigma}(t)[\delta \varepsilon - s]}{[\delta - \sigma(t)]^2} & \dfrac{\delta}{\delta - \sigma(t)} \end{bmatrix}.$$

The denominator $(\delta - \sigma(t))$ is strictly positive; moreover $\sigma \in \mathrm{Lip}([a, b])$ and thus all terms of $\nabla \Phi_\varepsilon^+$ are uniformly bounded with respect to ε.
Then, since both ∇T_ε and $\nabla \mathbf{u}$ are bounded, we obtain that also the integrand on the right hand side of (3.19) can be controlled by a constant independent of ε and

$$\lim_{\varepsilon \to 0^+} \mathcal{A}(\mathbf{u}_\varepsilon, \Lambda(R_\varepsilon^+ \setminus D_\varepsilon^+)) = 0.$$

The main point is to show that the definition given in (3.17) in the relevant region $\Lambda(D_\varepsilon)$ is such that the corresponding area gives origin to the term $\mathcal{H}^2(\Sigma_{\min})$ in the limit $\varepsilon \to 0^+$, and it is done in the next step.

`Step 9`. We have

$$\lim_{\varepsilon \to 0^+} \overline{\mathcal{A}}(\mathbf{u}_\varepsilon, \Lambda(D_\varepsilon)) = \mathcal{H}^2(\Sigma_{\min}). \tag{3.20}$$

Let us fix $\mu > 0$; we denote by $\mathbf{u}_\varepsilon^\mu$ the function defined on $\Lambda(D_\varepsilon)$ as

$$\mathbf{u}_\varepsilon^\mu(x, y) := (X_{\mu 2}(t(x, y), d(x, y)/\varepsilon), X_{\mu 3}(t(x, y), d(x, y)/\varepsilon)) = (\mathrm{u}_{\varepsilon 1}^\mu(x, y), \mathrm{u}_{\varepsilon 2}^\mu(x, y)).$$

In $\Lambda(\mathrm{int}(D_\varepsilon))$ we have

$$\nabla \mathrm{u}_{\varepsilon 1}^\mu = \begin{pmatrix} \partial_t X_{\mu 2}\, \partial_x t + \dfrac{1}{\varepsilon}\, \partial_s X_{\mu 2}\, \partial_x d \\ \\ \partial_t X_{\mu 2}\, \partial_y t + \dfrac{1}{\varepsilon}\, \partial_s X_{\mu 2}\, \partial_y d \end{pmatrix}, \qquad \nabla \mathrm{u}_{\varepsilon 2}^\mu = \begin{pmatrix} \partial_t X_{\mu 3}\, \partial_x t + \dfrac{1}{\varepsilon}\, \partial_s X_{\mu 3}\, \partial_x d \\ \\ \partial_t X_{\mu 3}\, \partial_y t + \dfrac{1}{\varepsilon}\, \partial_s X_{\mu 3}\, \partial_y d \end{pmatrix},$$

where the left hand sides and t and d are evaluated at (x, y), while $X_{\mu 2}$ and $X_{\mu 3}$ are evaluated at $(t(x, y), d(x, y)/\varepsilon)$.

Therefore

$$\left|\nabla \mathrm{u}^{\mu}_{\varepsilon 1}\right|^2 + \left|\nabla \mathrm{u}^{\mu}_{\varepsilon 2}\right|^2 = \frac{1}{\varepsilon^2} G_1 + \frac{2}{\varepsilon} G_2 + G_3 \qquad \text{in } \Lambda(\mathrm{int}(D_\varepsilon)), \tag{3.21}$$

with

$$\begin{cases} G_1 := \Big((\partial_s X_{\mu 2})^2 + (\partial_s X_{\mu 3})^2 \Big) \; |\nabla d|^2 = (\partial_s X_{\mu 2})^2 + (\partial_s X_{\mu 3})^2, \\ G_2 := \Big(\partial_t X_{\mu 2} \partial_s X_{\mu 2} + \partial_t X_{\mu 3} \partial_s X_{\mu 3} \Big) \; \nabla t \cdot \nabla d, \\ G_3 := \Big((\partial_t X_{\mu 2})^2 + (\partial_t X_{\mu 3})^2 \Big) \; |\nabla t|^2, \end{cases}$$

where we have used the eikonal equation for the signed distance function

$$|\nabla d|^2 = 1 \qquad \text{in } \mathrm{int}(\Lambda(R)).$$

Notice that $|\nabla t|^2$ is uniformly bounded with respect to ε on D_ε, by the assumption that $\overline{J}_{\mathbf{u}}$ is of class $\mathcal{C}^2$.
A direct computation shows that

$$\partial_x \mathrm{u}^{\mu}_{\varepsilon 1} \partial_y \mathrm{u}^{\mu}_{\varepsilon 2} - \partial_x \mathrm{u}^{\mu}_{\varepsilon 2} \partial_y \mathrm{u}^{\mu}_{\varepsilon 1} = \frac{1}{\varepsilon^2} E_1 + \frac{1}{\varepsilon} \widetilde{E}_2 + E_3, \tag{3.22}$$

with

$$\begin{cases} E_1 := \partial_s X_{\mu 2} \partial_x d \partial_s X_{\mu 3} \partial_y d - \partial_s X_{\mu 2} \partial_y d \partial_s X_{\mu 3} \partial_x d = 0, \\ \widetilde{E}_2 := \partial_t X_{\mu 2} \partial_s X_{\mu 3} \Big(\partial_x t \partial_y d - \partial_y t \partial_x d \Big) + \partial_t X_{\mu 3} \partial_s X_{\mu 2} \Big(\partial_x d \partial_y t - \partial_y d \partial_x t \Big) \\ \qquad = \Big(\partial_t X_{\mu 2} \partial_s X_{\mu 3} - \partial_t X_{\mu 3} \partial_s X_{\mu 2} \Big) \nabla t \cdot \nabla d^{\perp}, \\ E_3 := \partial_t X_{\mu 2} \partial_x t \partial_t X_{\mu 3} \partial_y t - \partial_t X_{\mu 2} \partial_y t \partial_t X_{\mu 3} \partial_x t = 0, \end{cases} \tag{3.23}$$

Set

$$E_2 := \partial_t X_{\mu 2} \partial_s X_{\mu 3} - \partial_t X_{\mu 3} \partial_s X_{\mu 2}. \tag{3.24}$$

From (3.22), (3.23) and (3.24) we have

$$\big(\partial_x \mathrm{u}^{\mu}_{\varepsilon 1} \partial_y \mathrm{u}^{\mu}_{\varepsilon 2} - \partial_x \mathrm{u}^{\mu}_{\varepsilon 2} \partial_y \mathrm{u}^{\mu}_{\varepsilon 1} \big)^2 = \frac{1}{\varepsilon^2} (E_2)^2 \; |\nabla t \cdot \nabla d^{\perp}|^2, \tag{3.25}$$

where again $X_{\mu 2}$ and $X_{\mu 3}$ are evaluated at $(t(x,y), d(x,y)/\varepsilon)$.
Notice that if $(x,y) \in \Lambda(D_\varepsilon)$ then the vector $\nabla d^{\perp}(x,y) = \nabla d^{\perp}(\pi(x,y))$ is tangent to $J_{\mathbf{u}}$ at $\pi(x,y)$, and has unit length. In addition, t is constant along the normal direction to $J_{\mathbf{u}}$, so that if $(x,y) \in \Lambda(D_\varepsilon)$ then $\nabla t(x,y) = \nabla t(\pi(x,y)) + \mathcal{O}(\varepsilon)$, and $\nabla t(\pi(x,y))$ is also tangent to $J_{\mathbf{u}}$, where

$$|\mathcal{O}(\varepsilon)| \leq c \|\kappa\|_{L^\infty(J_{\mathbf{u}})} \max_{t \in [a,b]} (\sigma^+(t) - \sigma^-(t)),$$

κ being the curvature of $J_{\mathbf{u}}$, for a positive constant c independent of ε.
Since α is an arc-length parametrization of $J_{\mathbf{u}}$, it follows that $|\nabla t| = 1$ on $J_{\mathbf{u}}$. Therefore

$$|\nabla t \cdot \nabla d^{\perp}| = 1 + \mathcal{O}(\varepsilon) \qquad \text{on } \Lambda(D_\varepsilon), \tag{3.26}$$

and hence from (3.25)

$$\left(\partial_x \mathrm{u}^\mu_{\varepsilon 1}\partial_y \mathrm{u}^\mu_{\varepsilon 2} - \partial_x \mathrm{u}^\mu_{\varepsilon 2}\partial_y \mathrm{u}^\mu_{\varepsilon 1}\right)^2 = \frac{1}{\varepsilon^2}(E_2)^2(1+\mathcal{O}(\varepsilon)).$$

Whence, from (3.21) and (3.25),

$$\begin{aligned}
&\overline{\mathcal{A}}(\mathbf{u}^\mu_\varepsilon, \Lambda(D_\varepsilon)) \\
&= \int_{\Lambda(D_\varepsilon)} \sqrt{1 + |\nabla \mathrm{u}^\mu_{\varepsilon 1}|^2 + |\nabla \mathrm{u}^\mu_{\varepsilon 2}|^2 + (\partial_x \mathrm{u}^\mu_{\varepsilon 1}\partial_y \mathrm{u}^\mu_{\varepsilon 2} - \partial_x \mathrm{u}^\mu_{\varepsilon 2}\partial_y \mathrm{u}^\mu_{\varepsilon 1})^2}\ dxdy \\
&= \int_{\Lambda(D_\varepsilon)} \sqrt{1 + G_3 + \frac{2}{\varepsilon}\ G_2 + \frac{1}{\varepsilon^2}\Big[G_1 + (E_2)^2(1+\mathcal{O}(\varepsilon))\Big]}\ dxdy.
\end{aligned}$$

The area formula implies that

$$\overline{\mathcal{A}}(\mathbf{u}^\mu_\varepsilon, \Lambda(D_\varepsilon)) = \int_{D_\varepsilon} \sqrt{1 + \widehat{G}_3 + \frac{2}{\varepsilon}\widehat{G}_2 + \frac{1}{\varepsilon^2}\left[\widehat{G}_1 + (\widehat{E}_2)^2(1+\mathcal{O}(\varepsilon))\right]}\ |\det(\Lambda)|\ dtds.$$

Here, for $i = 1, 2, 3$, $\widehat{G}_i$ (respectively $\widehat{E}_2$) equals G_i (respectively E_2) with (x, y) replaced by $\Lambda^{-1}(x, y) = (t, s)$, where we have $\alpha(t) = \pi(x, y)$ and $s = d(x, y)$; in particular $X_{\mu 2}$ and $X_{\mu 3}$ are evaluated at $(t, s/\varepsilon)$. Remember also that $|\det(\Lambda)| = |1 - \kappa s|$, κ being the curvature of $J_{\mathbf{u}}$ at $\alpha(t)$. Making the change of variables $s/\varepsilon \to s$ we get

$$\overline{\mathcal{A}}(\mathbf{u}^\mu_\varepsilon, \Lambda(D_\varepsilon)) = \int_{D} \sqrt{\varepsilon^2 + \varepsilon^2\widehat{G}_3 + 2\varepsilon\widehat{G}_2 + \left[\widehat{G}_1 + (\widehat{E}_2)^2(1+\mathcal{O}(\varepsilon))\right]}\ |1 - \varepsilon\kappa s|\ dtds,$$

where now $X_{\mu 2}$ and $X_{\mu 3}$ are evaluated at (t, s), and we notice that the term $\mathcal{O}(\varepsilon)$ is unaffected by the variable change.
Hence, by our regularity assumption on $\overline{J}_{\mathbf{u}}$ and (3.26), we deduce

$$\lim_{\varepsilon\to 0^+} \overline{\mathcal{A}}(\mathbf{u}^\mu_\varepsilon, \Lambda(D_\varepsilon)) = \int_D \sqrt{\widehat{G}_1 + (\widehat{E}_2)^2}\ \ dtds. \tag{3.27}$$

From (3.12) it follows

$$DX = \begin{bmatrix} 1 & 0 \\ \partial_t X_{\mu 2} & \partial_s X_{\mu 2} \\ \partial_t X_{\mu 3} & \partial_s X_{\mu 3} \end{bmatrix},$$

so that using the area formula

$$\begin{aligned}
\mathcal{H}^2(\widehat{\Sigma}^\mu) &= \int_D \sqrt{\det(DX_\mu^T DX_\mu)}\ dt\, ds \\
&= \int_D \sqrt{(\partial_s X_{\mu 2})^2 + (\partial_s X_{\mu 3})^2 + (\partial_t X_{\mu 2}\partial_s X_{\mu 3} - \partial_t X_{\mu 3}\partial_s X_{\mu 2})^2}\ dt\, ds,
\end{aligned}$$

which coincides with the right hand side of (3.27). This shows that for any $\mu \in (0, \overline{\mu})$

$$\lim_{\varepsilon\to 0^+} \overline{\mathcal{A}}(\mathbf{u}^\mu_\varepsilon, \Lambda(D_\varepsilon)) = \mathcal{H}^2(\widehat{\Sigma}^\mu).$$

Recalling (3.11), by a diagonalization process we can choose $(\mu_\varepsilon)_\varepsilon$ such that, defining

$$\mathbf{u}_\varepsilon(x,y) := \mathbf{u}_\varepsilon^{\mu_\varepsilon}(x,y), \qquad (x,y) \in \Lambda(D_\varepsilon),$$

we get

$$\lim_{\varepsilon \to 0^+} \overline{\mathcal{A}}(\mathbf{u}_\varepsilon, \Lambda(D_\varepsilon)) = \mathcal{H}^2(\Sigma_{\min}).$$

This concludes the proof of (3.20) and hence of (3.5). Inequality (3.6) follows by observing that $\mathrm{Lip}(\Omega;\mathbb{R}^2) \subset \mathrm{D}(\Omega;\mathbb{R}^2)$ (see (2.2)) and applying Lemma 9.1. □

4 Parametric case

In this section we relax the hypotheses of Theorem 3.1, in order to allow area-minimizing surfaces not of graph-type and possibly self-intersecting.

4.1 Hypotheses on u and statement for the parametric case

We consider a map $\mathbf{u} = (\mathrm{u}_1, \mathrm{u}_2)$ satisfying condition (u1) and:

(ũ2)
- $\mathbf{u} \in \mathcal{C}^1(\Omega \setminus \overline{J}_{\mathbf{u}};\mathbb{R}^2)$;
- $\mathcal{M}(\nabla \mathbf{u}) \in L^1(\Omega \setminus \overline{J}_{\mathbf{u}};\mathbb{R}^6)$;
- for $\zeta > 0$ small enough, denoting by B^a_ζ (respectively B^b_ζ) the open disk centered at $\alpha(a)$ (respectively $\alpha(b)$) with radius ζ, we have $\mathrm{u}_1 \in W^{1,2}\left(\Omega \setminus \overline{J_{\mathbf{u}} \cup B^a_\zeta \cup B^b_\zeta}\right)$;
- $\mathrm{u}_2 \in W^{1,2}\left(\Omega \setminus \overline{J}_{\mathbf{u}}\right)$.

(ũ3) The two traces $\gamma^\pm$ belong to $\mathcal{C}\left([a,b];\mathbb{R}^2\right) \cap BV\left([a,b];\mathbb{R}^2\right)$ and (3.1) and (3.2) hold.

(ũ4) $\Gamma = \Gamma[\mathbf{u}]$, defined as in (3.3), is such that the image $\Sigma_{\min}$ of an area-minimizing disk-type solution of the Plateau's problem admits a semicartesian parametrization with domain D (Definition 2.2) satisfying the two following conditions:

- $(a,b) \times \{0\} \subset \mathrm{int}(D)$,
- if $\sigma^+ \notin \mathrm{Lip}([a,b))$, near the point $(a,0)$ the graph of σ^+ is of the form $\{(\tau(s), s)\}$, for $|s|$ small enough, where

$$\tau(s) = a + \alpha_2 s^2 + o(s^2) \tag{4.1}$$

with $\alpha_2 > 0$. The analogue holds near the point $(b,0)$, and similar conditions also for σ^-.

Conditions on Γ ensuring that $\Sigma_{\min}$ admits a semicartesian parametrization are given in Section 5.

Remark 4.1. Theorem 4.1 remains valid if we exchange the hypotheses on the two components of $\mathbf{u}$, that is if we ask $\mathrm{u}_1 \in W^{1,2}\left(\Omega \setminus \overline{J}_{\mathbf{u}}\right)$ and $\mathrm{u}_2 \in W^{1,2}\left(\Omega \setminus \overline{J_{\mathbf{u}} \cup B^a_\zeta \cup B^b_\zeta}\right)$.

An example of map satisfying (u1), (ũ2)-(ũ4) is given in Example 5.1 below.

Remark 4.2. The first two items of hypothesis (ũ2) guarantee that $\mathbf{u} \in \mathrm{D}\left(\Omega \setminus \overline{J}_{\mathbf{u}}; \mathbb{R}^2\right)$. We observe that any $\mathbf{v} \in W^{1,2}(\Omega \setminus J_{\mathbf{v}}; \mathbb{R}^2)$ satisfies (ũ2); the converse conclusion is false, as shown by the map described in Example 5.1. We also observe that assuming in (ũ2) the weaker condition $u_1 \in W^{1,2}_{\mathrm{loc}}\left(\Omega \setminus \overline{J}_{\mathbf{u}}\right)$ is not enough for our proof to work, since we need $\mathbf{u}^{\mu}_{\varepsilon}$ to be, in the intermediate region, of class $W^{1,2}\left(\Lambda(R_\varepsilon \setminus D^{\mu}_{\varepsilon}); \mathbb{R}^2\right)$, see the expression in step 2 below.

Theorem 4.1. *Suppose that* $\mathbf{u}$ *satisfies assumptions* (u1), (ũ2)-(ũ4). *Then there exists a sequence*

$$(\mathbf{u}_\varepsilon)_\varepsilon \subset W^{1,2}(\Omega; \mathbb{R}^2)$$

converging to $\mathbf{u}$ *in* $L^1(\Omega; \mathbb{R}^2)$ *as* $\varepsilon \to 0^+$ *satisfying* (3.5). *Moreover* (3.6) *holds.*

4.2 Proof of Theorem 4.1

As we have already remarked, hypothesis (ũ2) guarantees that $\mathbf{u} \in \mathrm{D}(\Omega \setminus \overline{J}_{\mathbf{u}}; \mathbb{R}^2)$ and hence the expression $\overline{\mathcal{A}}(\mathbf{u}, \Omega \setminus \overline{J}_{\mathbf{u}})$ in (3.5) is meaningful.

To prove the theorem, we follow the line of reasoning of the proof of Theorem 3.1. However we now have to overcome two different problems. More precisely,

- the derivative of $\sigma^\pm$, whose graphs form the boundary of the domain D, could be unbounded at $t = a$ and $t = b$; this implies that also $|\nabla T_\varepsilon|$ could be unbounded;
- near the crack tips the map $\mathbf{u}$ is not regular enough to guarantee straightforwardly that $\mathbf{u}_\varepsilon$ is sufficiently regular.

Also in this case, we split the proof into various steps. In the first step we construct a family of surfaces Σ_μ approximating $\Sigma_{\min}$ and parametrized on suitable domains $D^\mu \subseteq D$ bounded by the graphs of $\sigma^\pm_\mu \in \mathrm{Lip}([a, b])$.

`step 1`. If $\sigma^\pm \in \mathrm{Lip}([a, b])$, we do not need to approximate Σ, thus we can pass directly to the next step with $X_\mu = X$ and $D^\mu = D$. Hence we can assume $\sigma^\pm \in \mathrm{Lip}_{\mathrm{loc}}([a, b]) \setminus \mathrm{Lip}([a, b])$. Let us suppose for example that the graph of σ^+ near the point $(a, 0)$ is in the form (4.1) (while the derivative of σ^+ near b is bounded and $\sigma^- \in \mathrm{Lip}([a, b])$) [(16)], see Figure 6. Then we modify the domain D and the map X near the point $(a, 0)$ as follows.
For a small positive constant c we adopt the following notation:

- ℓ_c is the portion of the line over $[a, b]$ passing through $(a, 0)$ with angular coefficient c^{-1}, that is:

$$\ell_c(t) = \frac{t - a}{c}, \quad t \in [a, b];$$

- P_c is the first intersection point of ℓ_c with ∂D and its coordinates are denoted by $(t_c, \sigma^+(t_c))$ ($t_c > a$ thanks to our assumption about the graph of σ^+).

[(16)] If also the other derivatives blow up, the construction of X_μ and D^μ is similar.

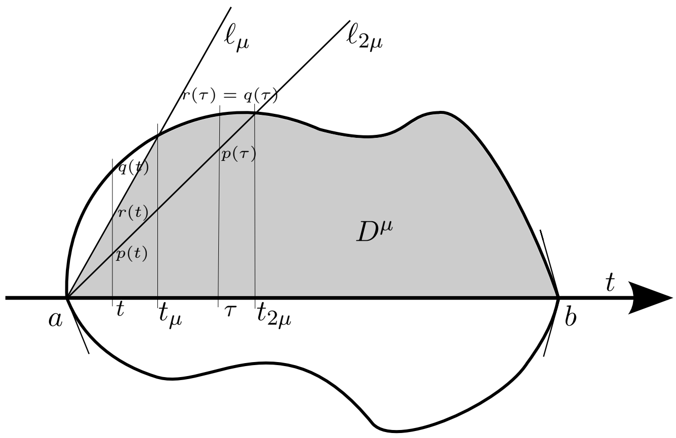


Figure 6: Modification of the domain D when the gradient of σ^+ blows up (in this case only near $t = a$).

For every $\mu > 0$ small enough, we define

$$\sigma_\mu^+ := \begin{cases} \ell_\mu & \text{in } [a, t_\mu), \\ \sigma^+ & \text{in } [t_\mu, b], \end{cases}$$

and

$$D^\mu := \{(t, s) : t \in [a, b], \sigma^-(t) \leq s \leq \sigma_\mu^+(t)\}.$$

In order to define the map X_μ on D^μ, we need to consider also the line $\ell_{2\mu}$ and the corrisponding intersection point $P_{2\mu}$. For any $t \in [a, t_{2\mu}]$, we denote by $p(t)$, $r(t)$ and $q(t)$ the points with first coordinate t on the segment bounded by $(a, 0)$ and $P_{2\mu}$, on ∂D^μ and on ∂D respectively (of course, $r(t) = q(t)$ for $t \in [t_\mu, t_{2\mu}]$), see Figure 6.
Thus we define

$$X_\mu : D^\mu \to \mathbb{R}^3$$

as follows:

$$X_\mu(t, s) := \begin{cases} \phi_\mu(t, s) & \text{if } t \in [a, t_{2\mu}),\ s \geq \ell_{2\mu}(t), \\ X(t, s) & \text{otherwise,} \end{cases}$$

where ϕ_μ is linear on the vertical lines, $\phi_\mu(p(t)) = X(p(t))$ and $\phi_\mu(r(t)) = X(q(t))$. We observe that X_μ is still a semicartesian parametrization.
Denoting by Σ_μ the image of D^μ through X_μ, we have $\mathcal{H}^2(\Sigma_\mu) \to \mathcal{H}^2(\Sigma)$ as $\mu \to 0^+$ ([15, §305]).

`step 2.`
For every $\mu > 0$ small enough, let us define the sequence $(\mathbf{u}_\varepsilon^\mu)$ as follows:

$$u_{\varepsilon 1}^\mu := \begin{cases} \mathrm{u}_1 & \text{in } \Omega \setminus (\Lambda(R_\varepsilon) \cup B^a_{\delta\varepsilon/2} \cup B^b_{\delta\varepsilon/2}), \\ \mathbf{u}(T_\varepsilon^\mu)_1 & \text{in } \Lambda(R_\varepsilon \setminus D_\varepsilon^\mu) \setminus (B^a_{\delta\varepsilon/2} \cup B^b_{\delta\varepsilon/2}), \\ X_{\mu 2}(t, s/\varepsilon) & \text{in } \Lambda(D_\varepsilon^\mu) \setminus (B^a_{\delta\varepsilon/2} \cup B^b_{\delta\varepsilon/2}), \\ \psi_\varepsilon^\mu & \text{in } B^a_{\delta\varepsilon/2} \cup B^b_{\delta\varepsilon/2}, \end{cases}$$

and

$$u^\mu_{\varepsilon 2} := \begin{cases} \mathrm{u}_2 & \text{in } \Omega \setminus \Lambda(R_\varepsilon), \\ \mathbf{u}(T^\mu_\varepsilon)_2 & \text{in } \Lambda(R_\varepsilon \setminus D^\mu_\varepsilon), \\ X_{\mu 3}(t, s/\varepsilon) & \text{in } \Lambda(D^\mu_\varepsilon), \end{cases}$$

where:

- Λ, R, $t = t(x, y)$, $s = s(x, y)$ are defined as in `step 6` of the proof of Theorem 3.1;
- T^μ_ε is defined as at (3.16), with D^μ in place of D and $\sigma^\pm_\mu$ instead of $\pm\sigma$;
- $B^a_{\delta\varepsilon/2}$ (respectively $B^b_{\delta\varepsilon/2}$) is the disk centered at $\alpha(a)$ (respectively $\alpha(b)$) with radius $\delta\varepsilon/2$;
- the function ψ^μ_ε is linear along the radii and is equal to u_1 in $\alpha(a)$ and to $u^\mu_{\varepsilon 1}$ on $\partial B^a_{\delta\varepsilon/2}$ (similarly in the other end point of the jump).

By construction and thanks to hypotheses (u1), (ũ2)-(ũ4), the sequence $(\mathbf{u}^\mu_\varepsilon)$ is in $W^{1,2}(\Omega; \mathbb{R}^2)$. We have

- $\overline{\mathcal{A}}\left(\mathbf{u}^\mu_\varepsilon, \Omega \setminus \left(\Lambda(R_\varepsilon) \cup B^a_{\delta\varepsilon/2 \cup B^a_{\delta\varepsilon/2}}\right)\right) = \overline{\mathcal{A}}\left(\mathbf{u}, \Omega \setminus \left(\Lambda(R_\varepsilon) \cup B^a_{\delta\varepsilon/2} \cup B^a_{\delta\varepsilon/2}\right)\right)$ by definition;
- $\overline{\mathcal{A}}\left(\mathbf{u}^\mu_\varepsilon, \Lambda(R_\varepsilon \setminus D^\mu_\varepsilon) \setminus (B^a_{\delta\varepsilon/2} \cup B^a_{\delta\varepsilon/2})\right) \to 0$ as $\varepsilon \to 0^+$ because of the estimates done in `step 8` of the proof of Theorem 3.1 since $|\nabla T^\mu_\varepsilon|$ is bounded;
- $\overline{\mathcal{A}}\left(\mathbf{u}^\mu_\varepsilon, \Lambda(D^\mu_\varepsilon) \setminus (B^a_{\delta\varepsilon/2} \cup B^a_{\delta\varepsilon/2})\right) \to \mathcal{H}^2(\Sigma_\mu)$ as $\varepsilon \to 0^+$: indeed the computations done in `step 9` of Theorem 3.1 work, since they depend only on the fact that the parametrization is in semicartesian form.

Thus

$$\lim_{\varepsilon \to 0^+} \overline{\mathcal{A}}\left(\mathbf{u}^\mu_\varepsilon, \Omega \setminus (J_\mathbf{u} \cup ((B^a_{\delta\varepsilon/2} \cup B^b_{\delta\varepsilon/2}) \setminus \Lambda(R_\varepsilon))\right) = \overline{\mathcal{A}}(\mathbf{u}, \Omega \setminus \overline{J}_\mathbf{u}) + \mathcal{H}^2(\Sigma_\mu).$$

In order to compute $\overline{\mathcal{A}}(\mathbf{u}^\mu_\varepsilon, B^a_{\delta\varepsilon/2})$ we observe that

$$|\nabla \psi^\mu_\varepsilon| \le \frac{C}{\varepsilon} \tag{4.2}$$

for some constant $C > 0$ independent of ε and μ. Thus for a possibly different value of the constant C (still independent of ε and μ),

$$\begin{aligned}
&\overline{\mathcal{A}}(\mathbf{u}^\mu_\varepsilon, B^a_\varepsilon \setminus \overline{J}_\mathbf{u}) \\
&\le C \int_{B^a_{\delta\varepsilon/2} \setminus \overline{J}_\mathbf{u}} [1 + |\nabla \psi^\mu_\varepsilon| + |\nabla u^\mu_{\varepsilon 2}| + |\partial_x \psi^\mu_\varepsilon \ \partial_y u^\mu_{\varepsilon 2} - \partial_y \psi^\mu_\varepsilon \ \partial_x u^\mu_{\varepsilon 2}|] \ dx\, dy \\
&\le C \left[\mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon)\right] + (1 + C) \int_{B^a_{\delta\varepsilon/2} \setminus \overline{J}_\mathbf{u}} |\nabla u^\mu_{\varepsilon 2}| \ dx\, dy,
\end{aligned}$$

where we have used (4.2). Recalling that on $\Lambda(D_\varepsilon^\mu)$ we have $u_{\varepsilon 2}(x,y) = X_{\mu 3}(t, s/\varepsilon)$, the term

$$\int_{\left(B^a_{\delta\varepsilon/2}\cap\Lambda(D^\mu_\varepsilon)\right)\setminus\overline{J}_{\mathbf{u}}} |\nabla u^\mu_{\varepsilon 2}|\; dx\, dy$$

is negligible as $\varepsilon \to 0^+$. On the other hand on $B^a_{\delta\varepsilon/2} \setminus \Lambda(R_\varepsilon)$ we have $u^\mu_{\varepsilon 2} = \mathrm{u}_2$, and thus

$$\int_{B^a_{\delta\varepsilon/2}\setminus\Lambda(R_\varepsilon)} |\nabla u^\mu_{\varepsilon 2}| dx\, dy = \mathcal{O}(\varepsilon^2).$$

Finally we get an analogous result also on $B^a_{\delta\varepsilon/2} \cap \Lambda(R_\varepsilon \setminus D^\mu_\varepsilon)$ since here $u^\mu_{\varepsilon 2}$ is defined as $(\mathbf{u}(T^\mu_\varepsilon))_2$ and T^μ_ε has bounded gradient and tends to the identity.
Thus the area contribute on $B^a_{\delta\varepsilon/2}$ is asymptotically negligible (and similarly on $B^b_{\delta\varepsilon/2}$).
Finally, since $\mathcal{H}^2(\Sigma_\mu)$ tends to $\mathcal{H}^2(\Sigma_{\min})$ as $\mu \to 0^+$, we can choose $\mathbf{u}_\varepsilon$ as $\mathbf{u}^{\mu_\varepsilon}_\varepsilon$ for a suitable sequence (μ_ε) converging to zero, so that we get (3.5). Recalling that $W^{1,2}(\Omega;\mathbb{R}^3) \subset \mathrm{D}(\Omega;\mathbb{R}^2)$ and applying Lemma 9.1 we obtain (3.6).

Remark 4.3. If $\mathbf{u}$ satisfies (u1), (ũ2), (ũ3) and $\Gamma[\mathbf{u}]$, defined as in (3.3), is contained in a plane Π, then

$$\overline{\mathcal{A}}(\mathbf{u}, \Omega) = \overline{\mathcal{A}}(\mathbf{u}, \Omega \setminus \overline{J}_{\mathbf{u}}) + \mathcal{H}^2(\Sigma_{\min}).$$

Indeed $\Sigma_{\min}$ is of course the portion of Π bounded by Γ; moreover, thanks to the definition of Γ, the plane Π cannot be orthogonal to the versor $(1,0,0)$. Thus either the projection of $\Sigma_{\min}$ on the plane $\mathbb{R}^2_{(t,\xi)}$ or its projection on $\mathbb{R}^2_{(t,\eta)}$ is a domain with non-empty interior. On the symmetrization of this domain we can define a semicartesian parametrization of $\Sigma_{\min}$ and, applying Theorem 4.1, we find

$$\overline{\mathcal{A}}(\mathbf{u}, \Omega) \leq \overline{\mathcal{A}}(\mathbf{u}, \Omega \setminus \overline{J}_{\mathbf{u}}) + \mathcal{H}^2(\Sigma_{\min}).$$

On the other hand, in this case $\mathcal{H}^2(\Sigma_{\min}) = |D^s\mathbf{u}|(\Omega)$ thus, using relation (1.2), we have also

$$\overline{\mathcal{A}}(\mathbf{u}, \Omega) \geq \overline{\mathcal{A}}(\mathbf{u}, \Omega \setminus \overline{J}_{\mathbf{u}}) + \mathcal{H}^2(\Sigma_{\min}).$$

5 On the existence of semicartesian parametrizations

Now our goal is to state some conditions on Γ which allow to construct a semicartesian parametrization for the corresponding area-minimizing surface, in order to apply Theorem 4.1 for suitable maps $\mathbf{u}$. Theorem 5.1 provides some sufficient conditions: roughly, we shall assume that Γ is the union of the graphs of two analytic curves, joining in an analytic way and satisfying a further assumption of non degeneracy. We stress that the analiticity forces the gradient of $\mathbf{u}$ to blow up near the crack tips.
The proof of Theorem 5.1 is quite involved and it is postponed to section 6.

We start with the following definition.

Definition 5.1 (Condition (A)). We say that a curve Γ union of two graphs satisfies condition (A) if there exists an injective *analytic* map

$$g = (g_1, g_2, g_3) : \partial B \to \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)}$$

such that

$$\Gamma = g(\partial B)$$

where, still denoting for simplicity by g the composition $g \circ \mathbf{b}$ (see (2.1)), and using the prime for differentiation with respect to θ, the following properties are satisfied:

$$\begin{aligned}
&|g'(\theta)| \neq 0, \qquad \theta \in [0, 2\pi), \\
&g_1' < 0 \text{ in } (\theta_{\rm n}, \theta_{\rm s}), \\
&g_1' > 0 \text{ in } (\theta_{\rm s}, \theta_{\rm n}), \\
&g_1''(\theta_{\rm s}) > 0, \qquad g_1''(\theta_{\rm n}) < 0.
\end{aligned} \tag{5.1}$$

Note carefully that the last three conditions involve the first component of g only.
Our result is the following.

Theorem 5.1 (**Existence of semicartesian parametrizations**). *Let $\Gamma \subset \mathbb{R}^3$ be a curve union of the two graphs of $\gamma^\pm$ and satisfying condition* (A). *Then there exist an analytic, connected, simply connected, and bounded set D and a disk-type area-minimizing solution $X \in \mathcal{C}^\omega(\overline{D}; \mathbb{R}^3)$ of the Plateau's problem for the contour Γ, satisfying Definition 2.2, with X free of interior branch points and of boundary branch points. Moreover,*

(i) near the point $(a, 0)$, the curve ∂D is of the form $\{(\tau(s), s)\}$, for $|s|$ small enough, with τ as in 4.1 and $\alpha_2 > 0$, and similarly near the point $(b, 0)$;

(ii) the Lipschitz constant of $\sigma^\pm$ on a relatively compact subinterval of (a, b) is bounded by the Lipschitz constant of the restriction of $\gamma^\pm$ on the same subinterval.

Remark 5.1. The semicartesian parametrization provided by Theore 5.1 could not satisfy the condition

$$(a, b) \times \{0\} \subset \operatorname{int}(D).$$

We can obtain a semicartesian parametrization fulfilling condition (ũ4) of Theorem 4.1 by symmetrizing the domain, as in `step 4` of the proof of Theorem 3.1

From Remark 5.1 and Theorems 4.1 and 5.1 we get the following result.

Corollary 5.1. *Suppose that $\mathbf{u}$ satisfies* (u1), *(*ũ2*), (*ũ3*) and that $\Gamma[\mathbf{u}]$ satisfies condition* (A). *Then there exists a sequence*

$$(\mathbf{u}_\varepsilon)_\varepsilon \subset W^{1,2}(\Omega; \mathbb{R}^2)$$

converging to $\mathbf{u}$ in $L^1(\Omega; \mathbb{R}^2)$ as $\varepsilon \to 0^+$ satisfying (3.5)*. Hence* (3.6) *holds.*

Remark 5.2. Before proving Theorem 5.1, the following comments are in order.

- Note the special structure of the curve $\Gamma[\mathbf{u}]$ in Corollary 5.1: it is not the graph of an $\mathbb{R}^2$-valued function over $\overline{J}_{\mathbf{u}}$, but it is instead the union of two analytic graphs, *joining together in an analytic way*, of the two $\mathbb{R}^2$-valued functions $\mathbf{u}^\pm$. Since globally the map g is required to be analytic, it results that $\mathbf{u}^\pm$ are not independent.

- The nondegeneracy requirement (5.1) of g at the south and north poles are necesssary in order the proof of Theorem 6.1 to work. In particular, it is needed to ensure that the restriction of the height function h to ∂B is a Morse function (`step 4` of the proof of Theorem 6.1).
- As we shall see, the analiticity requirement in hypothesis (ũ4) is needeed in order to prevent the existence of boundary branch points in a disk-type solution of the Plateau's problem with boundary Γ.

Example 5.1 (**Maps satisfying the hypotheses of Theorem 5.1**). In this example we present a map $\mathbf{u}$ satisfying (u1), (ũ2) and (ũ3) and whose $\Gamma = \Gamma[\mathbf{u}]$ satisfies condition (A) and hence, from Theorem 5.1, also condition (ũ4). The map $\mathbf{u}$ is defined so that Γ is a perturbation of the circle: indeed Γ is exactly the boundary of the unit disk contained in the plane $\mathbb{R}^2_{(t,\xi)}$ if u_2 is identically zero. Consequently, the nondegeneracy conditions expressed in (5.1) hold. It is clear that, starting from $\mathbf{u}$, several other maps satisfying the same conditions can be constructed.

Let Ω be an open connected subset of $\mathbb{R}^2_{(t,s)}$ containing the square $[-1,1]^2$ and let us consider the map $\mathbf{u} = (\mathrm{u}_1, \mathrm{u}_2) : \Omega \to \mathbb{R}^2$ defined by

$$\mathrm{u}_1(t,s) := \begin{cases} \sqrt{1-t^2+s^2}, & \text{if } |t|<1,\ s>0 \\ -\sqrt{1-t^2+s^2}, & \text{if } |t|<1,\ s<0 \\ s, & \text{otherwise} \end{cases}$$

and $\mathrm{u}_2 \in \mathcal{C}^1(\Omega) \cap W^{1,\infty}(\Omega)$ such that condition (A) holds (the simpler example is of course $\mathrm{u}_2 \equiv 0$). We notice that $\mathrm{u}_1 \in \mathrm{Lip}_{\mathrm{loc}}\left(\Omega \setminus \overline{J}_{\mathbf{u}}\right) \cap \mathcal{C}^1\left(\Omega \setminus \overline{J}_{\mathbf{u}}\right)$ (with $\overline{J}_{\mathbf{u}} = [-1,1] \times \{0\}$); it is in $W^{1,1}(\Omega \setminus \overline{J}_{\mathbf{u}})$ but it fails to be in $W^{1,2}(\Omega \setminus \overline{J}_{\mathbf{u}})$, as it can be checked directly (see Remark 4.2). Consequently, since the gradient of u_2 is supposed to be bounded, we get also that $\mathcal{M}(\nabla \mathbf{u})$ is in $L^1(\Omega \setminus \overline{J}_{\mathbf{u}}; \mathbb{R}^6)$. In this particular case Γ is a close simple analytic curve lying on the cylinder with base the unit disk, thus the existence of a semicartesian parametrization is obvious since the area-minimizing surface spanning Γ can be described as a graph on the disk (Theorem 7.7). We stress that Γ satisfies (A) and thus we could apply the argument in the proof of Theorem 5.1. It is easy to modify this example keeping the same behaviour near the poles but losing the convexity of the projection of Γ.

Example 5.2. Two other interesting examples of curves Γ satisfying condition (A) have already been discussed in the introduction and plotted in Figure 1.

6 Proof of Theorem 5.1

In this section we prove Theorem 5.1. The proof is involved, and we split it into various points.

Let $\Sigma_{\min}$ be an area-minimizing surface spanning Γ and having the topology of the disk. Let

$$Y : (u,v) \in \overline{B} \subset \mathbb{R}^2_{(u,v)} \to (Y_1(u,v), Y_2(u,v), Y_3(u,v)) \in \mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)} \tag{6.1}$$

be a conformal parametrization of $\Sigma_{\min}$ (see Theorem 7.1).

Since we can assume the three points condition (Remark 7.1), we suppose that

$$\begin{aligned} Y(0,-1) &= (a, \gamma^+(a)) = (a, \gamma^-(a)) =: S \\ Y(0,1) &= (b, \gamma^+(b)) = (b, \gamma^-(b)) =: N, \end{aligned} \tag{6.2}$$

and we fix a third condition as we wish (respecting the monotonicity on the boundary parametrization), for definitiveness

$$Y(1,0) = ((a+b)/2, \gamma^+((a+b)/2)).$$

In Section 6.1 we show a transversality property. We will make use of Morse relations for manifolds with boundary, in order to exclude, for a suitable Morse function, the presence of critical points of index one. The absence of boundary branch points for Y will be used in the proof.

In Section 6.2 we explain how this transversality property ensures the existence of a local semicartesian parametrization and, using some compactness argument and the simply connectedness of $\Sigma_{\min}$, also of a global semicartesian parametrization.

Finally in Section 6.3 we provide the regularity and the shape of the domain of this semicartesian parametrization.

6.1 A transversality result

Let $\mathcal{P}$ be the family of parallel planes orthogonal to the unit vector $e_t = (1,0,0)$, that is the planes in the form

$$\Big\{(t,\xi,\eta) \in \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)} : t = \text{const}\Big\}.$$

Notice that each point of $\Sigma_{\min}$ is met by some $\Pi \in \mathcal{P}$.

The next result is one of the most delicate parts of the proof of Theorem 5.1.

Theorem 6.1 (Transversality). *In the same hypotheses of Theorem 5.1, none of the planes of $\mathcal{P}$ is tangent to $\Sigma_{\min}$.*

Proof. We have to show that the normal direction to $\Sigma_{\min}$ at a point of $\Sigma_{\min}$ is never parallel to $(1,0,0)$; at self-intersection points of $\Sigma_{\min}$, the statement refers to all normal directions.

Our strategy is to introduce a height function having the planes of the family $\mathcal{P}$ as level sets, namely the function given by the first coordinate t in $\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)}$, restricted to an extension of $\Sigma_{\min}$. The proof consists then in proving that the only critical points of the height function are the minimum and the maximum corresponding to points S and N (see (6.2)).

Since $\partial\Sigma_{\min} = \Gamma$ is non-empty, in order to deal with boundary critical points first of all it is convenient to extend $\Sigma_{\min}$ across Γ.

By condition (A) the curve Γ is analytic; therefore (Theorem 7.4) we can extend $\Sigma_{\min}$ to an analytic minimal surface Σ^{ext} across Γ; Σ^{ext} can be parametrized on a bounded smooth simply connected open set $B^{\text{ext}} \supset B$ through an analytic map $Y^{\text{ext}} = (Y_1^{\text{ext}}, Y_2^{\text{ext}}, Y_3^{\text{ext}})$ which coincides with Y on $\overline{B}$, is harmonic, i.e. $\Delta Y^{\text{ext}} = 0$ in B^{ext}, and satisfies the conformality relations $|Y_u^{\text{ext}}|^2 = |Y_v^{\text{ext}}|^2$, $Y_u^{\text{ext}} \cdot Y_v^{\text{ext}} = 0$ in B^{ext}. In addition, from Theorems 7.2 and 7.3,

Y^{ext} has no interior (i.e., in B) and no boundary (i.e., on ∂B) branch points. Hence, possibly reducing B^{ext}, we can suppose that Y^{ext} has no branch points in B^{ext}.
Therefore, the Gauss map

$$\mathcal{N} : (u,v) \in B^{\text{ext}} \to \mathcal{N}(u,v) := \frac{Y_u^{\text{ext}}(u,v) \wedge Y_v^{\text{ext}}(u,v)}{|Y_u^{\text{ext}}(u,v) \wedge Y_v^{\text{ext}}(u,v)|} \tag{6.3}$$

is well-defined in B^{ext}[(17)].
Let us define

$$h : (u,v) \in B^{\text{ext}} \to h(u,v) := Y_1^{\text{ext}}(u,v) \in \mathbb{R}_t.$$

Observe that $(u_0, v_0) \in B^{\text{ext}}$ is a critical point for h if and only if the plane $\{(t,\xi,\eta) \in \mathbb{R}^3 : t = Y_1^{\text{ext}}(u_0,v_0)\}$ is tangent to Σ^{ext} at $Y^{\text{ext}}(u_0,v_0)$. Indeed, criticality implies $\partial_u Y_1^{\text{ext}}(u_0,v_0) = \partial_v Y_1^{\text{ext}}(u_0,v_0) = 0$, and one checks from (6.3) that

$$\mathcal{N}(u_0,v_0) = (1,\,0,\,0). \tag{6.4}$$

On the other hand, if $\mathcal{N}(u_0,v_0) = (1,0,0)$ the image of any vector through the differential of Y at (u_0,v_0) is orthogonal to $(1,0,0)$. If in particular we consider the image of $e_u = (1,0)$ and $e_v = (0,1)$ we obtain $Y_{1u}(u_0,v_0) = 0 = Y_{1v}(u_0,v_0)$.
From the above observation, it follows that the thesis of the theorem reduces to show that the function h has no critical points in $\overline{B}$, except for $(0,\pm 1)$, for which we shall prove separately that $\mathcal{N}(0,\pm 1) \neq (1,0,0)$.

At first, we shall show that the thesis of the theorem holds true up to a small rotation of Σ^{ext} around a line in the orthogonal space to $(1,0,0)$ that takes a direction in a suitable set to become $(1,0,0)$; moreover this set of directions is dense in a small neighborhood of $(1,0,0)$. In the last step we will show that the statements holds true *without applying this rotation.*

`step 1`. Up to a suitable rotation in $\mathbb{R}^3$, the function h has no degenerate critical points.

We notice that any degenerate critical point of h is a critical point also for the Gauss map. Indeed let $(u_0,v_0) \in B^{\text{ext}}$ be critical: using (6.4) we have for the coefficients of the second fundamental form

$$Y_{uu}^{\text{ext}} \cdot \mathcal{N} = Y_{1uu}^{\text{ext}} = h_{uu}, \qquad Y_{uv}^{\text{ext}} \cdot \mathcal{N} = Y_{1uv}^{\text{ext}} = h_{uv}, \qquad Y_{vv}^{\text{ext}} \cdot \mathcal{N} = Y_{1vv}^{\text{ext}} = h_{vv}.$$

If in addition (u_0,v_0) is degenerate, then the determinant of the Hessian of h at (u_0,v_0) vanishes, and this implies that also the determinant of the second fundamental form is zero. That is (u_0,v_0) is a critical point for the Gauss map.
From Sard's lemma, it follows that we can find a rotation around a line in the orthogonal space to $(1,0,0)$, as close as we want to the identity, so that the t-direction does not belong to the set of critical values of the Gauss map. Moreover such a rotation can be freely chosen in a set that is dense in a neighborhood of the identity. We also remark that for a sufficiently small rotation condition (A) remains valid although the values θ_{n} and θ_{s} of the parameter leading to maximal and minimal value of the t-component are perturbed of a small amount.

(17) $\mathcal{N}$ is also harmonic and satisfies the conformality relations, see [6, Chapter 1.2].

Therefore, from now on we assume that

$$\text{all critical points of } h \text{ in } B^{\text{ext}} \text{ are nondegenerate.}$$

`step 2`. The height function h has no critical points on ∂B.

Suppose by contradiction that there exists $(u,v) \in \partial B \setminus \{(0,\pm 1)\}$ such that $\nabla h(u,v) = 0$, namely (u,v) is a critical point of h different from $(0,\pm 1)$. We claim that if $\tau_{\partial B} \in \mathbb{R}^2$, $|\tau_{\partial B}| = 1$, $\tau_{\partial B}$ tangent to ∂B at (u,v) , then for some $\lambda \neq 0$

$$Y_{\tau_{\partial B}}(u,v) = \lambda \tau_\Gamma(u,v),$$

where $\tau_\Gamma(u,v)$ is a tangent unit vector to Γ at $Y(u,v)$ and $Y_{\tau_{\partial B}}$ is the derivative of Y along $\tau_{\partial B}$. Indeed, since Y is smooth up to ∂B, it follows that $Y_{\tau_{\partial B}}(u,v)$ is tangent to Γ at $Y(u,v)$. Now write $\tau_{\partial B} = \alpha e_u + \beta e_v$, $\alpha^2 + \beta^2 = 1$ and $e_u = (1,0)$, $e_v = (0,1)$. Since

$$Y_{\tau_{\partial B}}(u,v) = \alpha Y_u(u,v) + \beta Y_v(u,v),$$

the conformality relations imply

$$|Y_{\tau_{\partial B}}(u,v)|^2 = (\alpha^2 + \beta^2)|Y_u(u,v)|^2.$$

Then the absence of boundary branch points guarantees that $|Y_{\tau_{\partial B}}(u,v)|^2 \neq 0$. Hence $Y_{\tau_{\partial B}}(u,v)$ is a non-zero vector parallel to $\tau_\Gamma(u,v)$ and the claim follows. Observe now that, by assumption, $\tau_\Gamma(u,v)$ has non-zero t-component, so that

$$\alpha Y^1_u(u,v) + \beta Y^1_v(u,v) \neq 0, \tag{6.5}$$

which contradicts the criticality of (u,v) for h. Thus (6.5) shows that h has no critical points on $\partial B \setminus \{(0,\pm 1)\}$. In order to exclude that S (and similarly N) is a critical point for h, we observe that condition (A) implies that the convex hull of Γ, and hence the convex hull of $\Sigma_{\min}$[(18)], is contained in a wedge having the tangent to Γ at its lowest point as ridge and the two slopes are strictly increasing starting from the ridge. Thus the normal vector to Σ^{ext} in S cannot be parallel to $(1,0,0)$.

As a consequence of `step 2` we can suppose that all critical points of h are contained in B.

`Step 3`. The function h has neither local maxima nor local minima in B.

Indeed, assume by contradiction that $p = Y(u_0,v_0) \in \Sigma_{\min}$, where $(u_0,v_0) \in B$ is a local minimum point for h. Then locally the surface $\Sigma_{\min}$ is contained in a half-space delimited by the tangent plane $\{(t,\xi,\eta) : t = Y_1(u_0,v_0)\}$, the intersection with this tangent plane being locally only the point $Y(p)$. We now construct a competitor surface Σ' as follows: we remove from $\Sigma_{\min}$ a small portion locally around p, obtained by cutting $\Sigma_{\min}$ locally with a plane at a level slightly higher than the minimal value. We fill the removed portion with a portion of plane, and this givs Σ'[(19)]. Then the area of Σ' is strictly smaller than the area of $\Sigma_{\min}$, a contradiction.

[(18)] Any connected minimal surface X with a parameter domain D is contained in the convex hull of $X_{|\partial D}$. See [7, Theorem 1, chapter 4.1].

[(19)] If the cut level is close enough to the critical level, Σ' is the image of a map in $\mathcal{C}(\Gamma)$ (see Appendix 7).

A similar argument holds for a local maximum point and therefore the proof of `step 3` is concluded.

Employing the notation of Section 8 , we have therefore

$$m_0(h, B) = m_2(h, B) = 0.$$

The next step is a consequence of the monotonicity and nondegeneracy assumptions expressed in (5.1), and of the conformality and analiticity of $\Sigma_{\min}$.

`Step 4`. The restriction $h_{|\partial B}$ of h to ∂B is a Morse function; moreover $m_0^-(h_{|\partial_h^- B}) = 1$ and $m_1^-(h_{|\partial_h^- B}) = 0$ (Section 8).

We observe that condition (A) implies that there exists a parametrization of Γ on ∂B whose first components is a Morse function. We have to show that also the parametrization induced by the area-minimizing minimal surface Y has the same property.
As already done for the function g, we denote by $Y^{\text{ext}}_{|\partial B}$ and by $h_{|\partial B}$ the composition $Y^{\text{ext}} \circ \mathbf{b}$ and $h \circ \mathbf{b}$ respectively (see (2.1)) and we use the prime for differentiation with respect to θ. At first, we observe that out of branch points, all the directional derivatives of Y^{ext} are non zero. Thus in particular, from the absence of boundary branch points on ∂B, we deduce that

$$|(Y^{\text{ext}}_{|\partial B} S)'(\theta)| \neq 0, \quad \theta \in [0, 2\pi).$$

On the other hand since g is analytic with differentiable inverse, there exists a $\mathcal{C}^1$ function ψ from $[0, 2\pi]$ in itself such that $\psi(2\pi) = \psi(0) + 2\pi$ and

$$Y^{\text{ext}}_{|\partial B}(\theta) = g(\psi(\theta)), \quad \theta \in [0, 2\pi).$$

Differentiating the last expression and remembering from (5.1) that $|g'| \neq 0$, we get that also ψ' never vanishes, indeed:

$$0 \neq |(Y^{\text{ext}}_{|\partial B})'(\theta)| = |g'(\psi(\theta))||\psi'(\theta)|.$$

From the semicartesian form of Γ, $h_{|\partial B}$ has just a minimum and a maximum in corrispondence of $N = (0, 1)$ and $S = (0, -1)$. From the properties of g we infer that $\psi(\theta_{\rm s})$ is the value of θ corresponding to S and similarly for N. Since

$$h''_{|\partial B}(\theta) = g_1''(\psi(\theta))(\psi(\theta))^2 + g_1'(\psi(\theta))\psi''(\theta),$$

computing for the values corresponding to S and N we get that the first addend is non-zero while the second vanishes. We have thus proven that $h_{|\partial B}$ is a Morse function, with a maximum in $(0, 1)$ and a minimum in $(0, -1)$.

Following once more Section 8 (see (8.1)), we now set

$$\partial_h^- B := \{(u, v) \in \partial B : \nabla h(u, v) \cdot \nu_B(u, v) < 0\},$$

where $\nu_B(u, v)$ denotes the outward unit normal to ∂B at $(u, v) \in \partial B$.
We prove that

$$(0, -1) \in \partial_h^- B \qquad \text{and} \qquad (0, 1) \notin \partial_h^- B.$$

Indeed if $\nabla h(0,-1)\cdot \nu_B(0,-1)\geq 0$, we get a contradiction from the same argument used in `step 2` to prove that $(0,-1)$ is not critical for h. Similarly $(0,1)\notin \partial_h^- B$.

We have thus obtained that

$$m_0^-(h_{|\partial_h^- B}) = 1, \qquad m_1^-(h_{|\partial_h^- B}) = 0.$$

`Step 5`. The function h has no saddle points in B.

The Morse function h (`step 1`) has no points of index zero (minima) in B and no points of index two (maxima) in B by `step 3`: again following the notation of Section 8 (see (8.2)), we have

$$M_0(h, B\cup \partial B) = 1, \qquad M_2(h, B\cup\partial B) = 0.$$

In addition, using `steps 2` and `4` we can apply Theorem 8.1, and obtain, being $\chi(B) = 1$,

$$M_1(h) = M_0(h, B\cup\partial B) + M_2(h, B\cup \partial B) - \chi(B) = 0.$$

`step 6.` It is not necessary to apply any rotation.

It is sufficient to show that the direction given by $(1,0,0)$ is actually not critical for the Gauss map. At first we can assume that Γ is not contained in a plane. Indeed if it were planar, necessarily

$$\mathcal{N}(u,v) = \nu_0, \quad (u,v)\in B$$

for some constant unit vector $\nu_0 \neq (1,0,0)$, since Γ is union of two graphs.
Assuming that Γ is non planar, we reason by contradiction and suppose that there is a degenerate critical point $p = Y(u_0,v_0)$ for the height function h in the inside of $\Sigma_{\rm min}$. This means that (u_0,v_0) is a critical point for the Gauss map, that is the product $\kappa_1\kappa_2$ of the two principal curvatures is 0; because of the minimality of $\Sigma_{\rm min}$ we get that p is an umbilical point, with $\kappa_1 = 0 = \kappa_2$. Recalling that in a non-planar minimal surface the umbilical points are isolated (see for example [6, Remark 2, chapter 5.2]), we can find a direction in a small neighborhood of $(1,0,0)$ that is normal to $\Sigma_{\rm min}$ in a neighborhood of the degenerate critical point p and is not a critical value for the Gauss map. If we rotate $\Sigma_{\rm min}$ taking this direction to become vertical, we have a nondegenerate critical point for the height function which is a contradiction in view of the previous steps. □

6.2 The semicartesian parametrization

We can apply Theorem 7.5 to $\Sigma^{\rm ext}$ with the family of planes of Theorem 6.1, obtaining a *local* semicartesian parametrization. More precisely, for any point $p\in\Sigma^{\rm ext}$ there exists an open domain $D_p \subset \mathbb{R}^2_{(t,s)}$ and an analytic[(20)], conformal semicartesian map X_p parametrizing an open neighbourhood of p on $\Sigma^{\rm ext}$:

$$\begin{aligned} X_p: \quad & D_p \to \Sigma^{\rm ext}, \\ & (t,s_p) \to (t, X_{p2}(t,s_p), X_{p3}(t,s_p)). \end{aligned} \tag{6.6}$$

[(20)]From the proof of Theorem 7.5 one infers that the regularity of the local semicartesian map is the same as the surface.

Proposition 6.1 (Global semicartesian parametrization). *In the hypotheses of Theorem 5.1, $\Sigma_{\min} = X(D)$ admits an analytic parametrization of the form* (2.5).

Proof. The local parametrization in (6.6) is unique up to an additive constant: $s_p \mapsto s_p + \rho$. Indeed, if t_p is the t-coordinate of p, the direction of $\partial_{s_p} X_p$ is given by the intersection of the tangent plane to Σ^{ext} and the plane $\{t = t_p\}$, since its t-component is zero. The vector $\partial_t X_p$ is then uniquely determined by being in the tangent plane to Σ^{ext}, orthogonal to $\partial_{s_p} X_p$ and having 1 as t-component. This in turn determines the norm of $\partial_{s_p} X_p$ and hence $\partial_{s_p} X_p$ itself (up to a choice of the orientation of Σ^{ext})(21). Functions $X_{p2}(t, s_p)$ and $X_{p3}(t, s_p)$ can now be obtained by integrating the vector field $\partial_{s_p} X_p$ along the curve $\{t = t_p\} \cap \Sigma^{\text{ext}}$ and transported as constant along the curves $\{s = \text{const}\}$. Now we can cover $\Sigma_{\min} \cup \Gamma$ with a finite number of such neighbourhoods (local charts) having connected pairwise intersection, and we can choose the constant in such a way that on the intersection of two neighborhoods the different parametrizations coincide. In this way we can "transport" the parametrization from a fixed chart along a chain of pairwise intersecting charts. This definition is wellposed if we can prove that the transported parametrization is independent of the actual chain, or equivalently that transporting the parametrization along a closed chain of charts produces the original parametrization. This is a consequence of the simple connectedness of the surface(22), indeed we can take a closed curve that traverses the original chain of charts and let it shrink until it is contained in a single chart.
Thus we can construct a *global* semicartesian parametrization X defined on a open domain $D^{\text{ext}} \subset \mathbb{R}^2$ as required in hypothesis (ũ4). Eventually

$$D := X^{-1}(\Sigma_{\min} \cup \Gamma) \tag{6.7}$$

is a closed bounded (connected and simply connected) set such that the intersection with the line $\{t = k\}$, for $k \in (a, b)$, is an interval (not reduced to a point); indeed if the intersection were composed by two (or more) connected components, there would be at least 4 points on the intersection of Γ with the plane $\{t = k\}$, and this is impossible since Γ is union of two graphs on t.

□

Before proving that the domain D satisfies the local Lipschitz conditions required by Definition 2.2, we need the following regularity result.

Lemma 6.1. *The domain D defined in* (6.7) *has analytic boundary.*

Proof. The boundary of D is the image of an analytic map defined on ∂B. This latter fact follows directly from the analiticity of the map $Y : B \to \Sigma_{\min}$ (see (6.1)) and of the map $X : D \to \Sigma_{\min}$. The fact that $\Sigma_{\min}$ can have self-intersections is not a problem here because the preimages of points (in either B or D) in a self-intersection are well separated, so that we can restrict to small patches of the surface and reason locally. □

We are now in a position to specify a further property of ∂D(23).

(21)Incidentally we note here that $|\partial_{s_p} X_p| = |\partial_t X_p| \geq 1$ (which excludes branch points).
(22)This is one of the points where it is important to consider disk-type area-minimizing surfaces.
(23)The analiticity of ∂D in particular implies that we cannot have a global Lipschitz constant for $\sigma^{\pm}$, so that the result in Proposition 6.2 is optimal.

Proposition 6.2. *In the hypotheses and with the notation of Proposition 6.1, D has the form in* (2.3)*, where the two functions* $\sigma^{\pm} : [a, b] \to \mathbb{R}$ *satisfy* (2.4) *and condition (ii) of Theorem 5.1.*

Proof. Since, as noticed in Lemma 6.1, $D \cap \{t = k\}$ is a interval, not reduced to a point, for any $k \in (a, b)$, D is in the form (2.3) with $\sigma^- < \sigma^+$ in (a, b); up to traslation we can suppose also $\sigma^+(a) = 0 = \sigma^-(a)$.
Let $(t, s) \in \partial D$ and let $p = X(t, s) \in \Gamma$. Let us suppose that $s = \sigma^-(t)$ (the case $s = \sigma^+(t)$ being similar) and let us write σ in place of σ^- for simplicity. We have to show that

$$|\sigma'(t)| \le |\gamma'(t)|. \tag{6.8}$$

Let $\vartheta(t, s) \in [-\pi/2, \pi/2]$ be the angle between the tangent line to Γ at p (spanned by $\frac{\Gamma'(t)}{|\Gamma'(t)|}$) and the direction of $X_t(t, s)$. Note that if $\vartheta(t, s) \in (-\pi/2, \pi/2)$ we have

$$\mathrm{tg}(\vartheta(t, s)) = \sigma'(t). \tag{6.9}$$

Indeed, take a vector ℓ generating the tangent line to ∂D at (t, s), for instance $\ell = (\sigma'(t), 1)$. Using also the conformality of X, the derivative X_ℓ of X along the direction of ℓ is given by $X_\ell(t, s) = \sigma'(t)X_s(t, s) + X_t(t, s)$, and is a vector generating the tangent line to Γ at p, and (6.9) follows.
Let now $\Theta(t, s) \in [0, \pi/2]$ be the angle between the tangent line to Γ at p and the line generated by $e_t = (1, 0, 0)$. If $\Theta(t, s) \in [0, \pi/2)$ we have, writing γ in place of γ^-,

$$\mathrm{tg}(\Theta(t, s)) = |\gamma'(t)|.$$

Hence, to show (6.8), it is sufficient to show that $\vartheta(t, s) \le \Theta(t, s)$, or equivalently

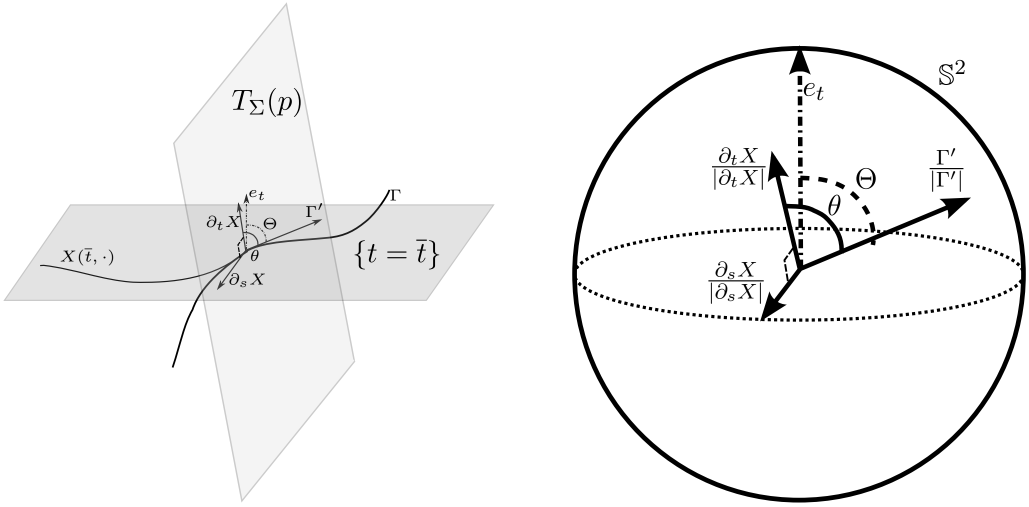
$$\frac{\pi}{2} - \vartheta(t, s) \ge \frac{\pi}{2} - \Theta(t, s). \tag{6.10}$$

Consider $\frac{\Gamma'(t)}{|\Gamma'(t)|}$ as a point on $\mathbb{S}^2 \subset \mathbb{R}^3$ and think of e_t as the vertical direction (Figure 7(b)). We have that $\frac{\pi}{2} - \Theta(t, s)$ is the latitude of $\frac{\Gamma'(t)}{|\Gamma'(t)|}$. On the other hand, remembering that $X_s(t, s)$ is orthogonal to e_t, we have that $\frac{\pi}{2} - \vartheta(t, s)$ (the angle between $\frac{\Gamma'(t)}{|\Gamma'(t)|}$ and $X_s(t, s)$ by conformality) is the geodesic distance (on $\mathbb{S}^2$) between $\frac{\Gamma'(t)}{|\Gamma'(t)|}$ and the point obtained as the intersection between $T_p(\Sigma_{\min})$ and the equatorial plane. Hence inequality (6.10) holds true. □

6.3 Shape of the parameter domain

In order to conclude the proof of Theorem 5.1, we need to study the behaviour of ∂D near $(a, 0)$ and $(b, 0)$.

Proposition 6.3. *Assertion (i) of Theorem 5.1 holds.*



(a) (b)

Figure 7: (a): The dotted vector e_t is perpendicular to the plane $\{t = \bar t\}$ on which we have represented a part of the curve $\{X(\bar t, s) : s \in [\sigma^-(t), \sigma^+(t)]\}$. Γ is also drawn, and passes through the plane $\{t = \bar t\}$ transversally. The other plane is the tangent plane to $\Sigma_{\min}$ at $p = X(\bar t, \sigma^-(\bar t))$ and the three vectors are the conformal basis of the tangent plane $\mathrm{span}\{\partial_t X, \partial_s X\}$ and the vector $\Gamma'(\bar t)$. The angles θ and Θ are also displayed. (b): the same vectors normalized and represented on the sphere $\mathbb{S}^2$.

Proof. Let us consider the point $(a, 0)$. From the analiticity of ∂D (Lemma 6.1) and the fact that $(a, 0)$ minimizes the t-component in ∂D, we can express it locally in a neighborhood of $(a, 0)$ as the graph $(\tau(s), s)$ of a function $\tau : (s^-, s^+) \to \mathbb{R}$ defined in a neighborhood (s^-, s^+) of the origin that can be Taylor expanded as

$$\tau(s) = a + \alpha_2 s^2 + \alpha_3 s^3 + \alpha_4 s^4 + o(s^4), \qquad s \in (s^-, s^+),$$

with $\alpha_2 \geq 0$.
Assume by contradiction that

$$\alpha_2 = 0.$$

Since D is contained in the half-plane $\{t \geq a\}$ it follows that

$$\alpha_3 = 0 \quad \text{and} \quad \alpha_4 \geq 0.$$

We shall now compute the area $A(\varepsilon)$ of

$$\Sigma^\varepsilon_{\min} := \Sigma_{\min} \cap \{t < a + \varepsilon\} = X(D \cap S_\varepsilon)$$

for small positive values of ε, where $S_\varepsilon := \{(t, s) : a \leq t < a + \varepsilon\}$. Using the conformal map X we need to integrate the area element over the set $D \cap S_\varepsilon$. However the integrand is the modulus of the external product of the two derivatives of X with respect to t and to s, which is always greater than or equal to 1, so that, integrating, we get

$$A(\varepsilon) \geq \mathcal{L}^2(D \cap S_\varepsilon) \geq c\varepsilon^{1+1/4} \tag{6.11}$$

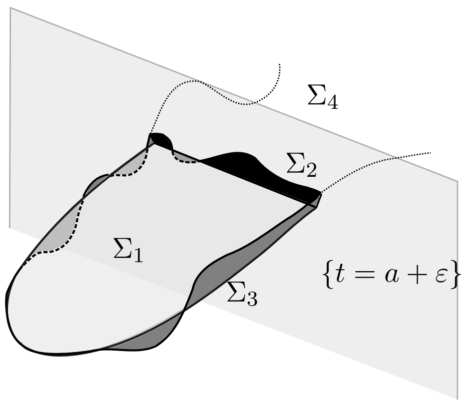


Figure 8: The competitor surface Σ. Σ_1, Σ_2 and Σ_3 are the light gray, black and dark gray surface respectively.

for some positive constant c independent of ε.
We now want to show that the minimality of $\Sigma_{\min}$ entails that $\mathcal{H}^2(\Sigma^\varepsilon_{\min}) \leq c\varepsilon^{1+1/2}$, which is in contradiction with (6.11). Indeed we can compare the area of $\Sigma_{\min}$ with the competitor surface

$$\Sigma := \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4,$$

where (see Figure 8):

- Σ_1 is the parabolic sector delimited by the osculating parabola to Γ in the minimum point and by the plane $\{t = a + \varepsilon\}$;
- Σ_2 is the portion of the plane $\{t = a+\varepsilon\}$ between the curve $\Sigma_{\min} \cap \{t = a+\varepsilon\}$ and the boundary of Σ_1;
- Σ_3 is obtained connecting linearly each point of the osculating parabola with the point of Γ having the same t-coordinate;
- $\Sigma_4 := \Sigma_{\min} \cap \{a + \varepsilon \leq t \leq b\}$.

Notice that Σ is a Lipschitz surface and $\partial\Sigma = \Gamma$. Moreover $\Sigma_{\min} = \Sigma^\varepsilon_{\min} \cup \Sigma_4$ with $\Sigma^\varepsilon_{\min} \cap \Sigma_4 = \emptyset$. Thus, using also the minimality of $\Sigma_{\min}$, we get

$$\mathcal{H}^2(\Sigma_{\min}) = A(\varepsilon) + \mathcal{H}^2(\Sigma_4) \leq \mathcal{H}^2(\Sigma) \leq \sum_{i=1}^{4} \mathcal{H}^2(\Sigma_i),$$

which implies $A(\varepsilon) \leq \mathcal{H}^2(\Sigma_1) + \mathcal{H}^2(\Sigma_2) + \mathcal{H}^2(\Sigma_3)$. Now, we notice that, for a constant c independent of ε:

- $\mathcal{H}^2(\Sigma_1) \leq c\varepsilon^{1+1/2}$, since it is a parabolic sector,
- $\mathcal{H}^2(\Sigma_2) \leq c\varepsilon^{1+1/2}$ because $\Sigma_{\min}$ is bounded by the two planes of the wedge,
- $\mathcal{H}^2(\Sigma_3) = o(\varepsilon^{1+1/2})$ because $\Sigma^\varepsilon_{\min}$ is contained in the inside of a cylindrical shape obtained by translation of Γ in the direction orthogonal to both the tangent vector to Γ in its minimum point and the vector $(1, 0, 0)$.

Thus we get the contradicting relation:

$$c_1\varepsilon^{1+1/2} \geq A(\varepsilon) \geq c_2\varepsilon^{1+1/4},$$

where c_1 and c_2 are two positive constants independent of ε. □

7 Appendix 1: some useful results on the Plateau's problem

In this appendix we briefly collect all definitions and results on the Plateau's problem, with the related references, needed in the proofs of Theorems 3.1, 4.1, 5.1 and 6.1.

7.1 Parametric approach

Let $B \subset \mathbb{R}^2_{(u,v)}$ be the unit open disk and Γ be an oriented(24) rectifiable closed simple curve in $\mathbb{R}^3$. We are interested in minimizing the area functional

$$\int_B |Y_u \wedge Y_v| \, du\, dv$$

in the class(25)

$$\mathcal{C}(\Gamma) = \left\{ Y \in H^{1,2}(B;\mathbb{R}^3) \cap \mathcal{C}(\partial B;\mathbb{R}^3),\ Y_{|\partial B}(\partial B) = \Gamma,\, Y_{|\partial B} \text{ weakly monotonic} \right\}.$$

The set $Y(B)$ for $Y \in \mathcal{C}(\Gamma)$ is called a *disk-type surface* spanning Γ.

Definition 7.1 (**Disk-type area-minimizing solution**). We refer to a solution of the minimum problem

$$\inf_{Y \in \mathcal{C}(\Gamma)} \int_B |Y_u \wedge Y_v| \, du\, dv \tag{7.1}$$

as *disk-type area-minimizing solution of Plateau's problem for the contour* Γ. Its image in $Y(B) \subset \mathbb{R}^3$ is called *area-minimizing surface* spanning Γ, but sometimes, with a small abuse of language, also *area-minimizing solution*, identifying the image and the parametrization. We usually denote such a $Y(B)$ by $\Sigma_{\min}$.

For further details about the formulation of Plateau's problem we refer to [6, chapter 4, p. 270].

Concerning the existence of a solution of (7.1) the following holds.

Theorem 7.1 (**Existence of minimizers and interior regularity**). *Problem* (7.1) *admits a solution* $Y \in \mathcal{C}^2(B) \cap \mathcal{C}(\overline{B})$, *such that*

$$\Delta Y = 0 \qquad \text{in } B \tag{7.2}$$

and the conformality relations hold:

$$|Y_u|^2 = |Y_v|^2 \qquad \text{and} \qquad Y_u \cdot Y_v = 0 \qquad \text{in } B. \tag{7.3}$$

Moreover the restriction $Y_{|\partial B}$ *is a continuous, strictly monotonic map onto* Γ.

(24)The orientation is provided by fixing a homeomorphism from ∂B onto Γ.
(25)Since Γ is rectifiable, we have $\mathcal{C}(\Gamma) \neq \emptyset$.

Proof. See for instance [6, Main Theorem 1, chapter 4, p. 270]. □

Remark 7.1 (Three points condition). One can impose on a minimizer Y the so-called three points condition: this means that we can fix three points ω_1, ω_2 and ω_3 on ∂B and three points P_1, P_2 and P_3 on Γ (in such a way that the orientation of Γ is respected) and find a solution Y of (7.1) such that $Y(\omega_j) = P_j$ for any $j = 1, 2, 3$.

Definition 7.2 (Minimal surface). A map $Y \in \mathcal{C}^2(B) \cap \mathcal{C}(\overline{B})$ satisfying (7.2) and (7.3) mapping ∂B onto Γ in a weakly monotonic way is called a *minimal surface spanning* Γ.

Concerning the regularity of a map $Y : B \to \mathbb{R}^3$ parametrizing a minimal surface, we cannot a priori avoid *branch points*.

Definition 7.3 (Branch point). A point $\omega_0 \in B$ is called an *interior branch point* for $Y \in \mathcal{C}^2(B) \cap \mathcal{C}(\overline{B})$ if

$$|Y_u(\omega_0) \wedge Y_v(\omega_0)| = 0. \tag{7.4}$$

If Y is differentiable on ∂B, and $\omega_0 \in \partial B$ is such that (7.4) holds, then ω_0 is called a *boundary branch point.*

Observe that if ω_0 is a branch point and (7.3) holds, then $Y_u(\omega_0) = Y_v(\omega_0) = 0$.
It is known that interior branch points for a solution of (7.1) can be excluded.

Theorem 7.2 (Absence of interior branch points). *Let Y be as in Theorem 7.1. Then Y has no interior branch points.*

Proof. See [16, Main Theorem]. □

Under the stronger assumption that Γ is analytic the classical Lewy's regularity theorem [11] guarantees that the solution of (7.1) is analytic on $\overline{B}$.

Theorem 7.3 (Absence of boundary branch points). *Let Γ be analytic and Y be a solution of (7.1). Then Y is analytic up to Γ and has no boundary branch points.*

Proof. See [10]. □

Theorem 7.4 (Analytic extension). *Let Γ be analytic and Y be a minimal surface spanning Γ. Then Y can be extended as a minimal surface across Γ, that is there exist an open set $B^{\text{ext}} \supset \overline{B}$ and an analytic map $Y^{\text{ext}} : B^{\text{ext}} \to \mathbb{R}^3$ such that $Y^{\text{ext}} = Y$ in $\overline{B}$ and Y^{ext} satisfies (7.2) and (7.3) in B^{ext}.*

Proof. From [7, Theorem 1, chapter 2.3] one can extend a minimal surface across an analytic subarc of Γ. We apply this result twice to two overlapping subarcs covering Γ. Where the two extensions overlap, they have to coincide due to analiticity. □

The following classical result can be found in [6, p. 66].

Theorem 7.5 (Local semicartesian parametrization). *If a minimal surface Y is intersected by a family of parallel planes $\mathcal{P}$ none of which is tangent to the given surface and if each point of the surface belongs to some plane $\Pi \in \mathcal{P}$, then the intersection lines of these planes with the minimal surface form a family of curves which locally belong to a net of conformal parameters on the surface.*

7.2 Non-parametric approach

Concerning the so-called non-parametric problem and the minimal surface equation, we give the following definition and we refer to [9] for more.

Definition 7.4 (Non-parametric solution). Let $U \subset \mathbb{R}^2$ be a connected, bounded, open set and let $\phi \in \mathcal{C}(\partial U; \mathbb{R}^2)$. A *solution of the minimal surface equation for the boundary datum* ϕ is a solution $z \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$ of

$$\begin{cases} \operatorname{div}\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}}\right) = 0 & \text{in } U \\ z = \phi & \text{on } \partial U. \end{cases} \tag{7.5}$$

The existence of a solution of (7.5) is given by the following result.

Theorem 7.6 (Existence of non-parametric solutions). *Let $U \subset \mathbb{R}^2$ be bounded and open and suppose that ∂U is $\mathcal{C}^2$ and has non negative curvature. Then* (7.5) *admits a solution.*

Proof. See [9, Theorem 13.6]. □

If Γ can be described as the graph of a continuous function defined on the boundary of a bounded convex open set, then the following representation result holds.

Theorem 7.7. *If Γ admits a one-to-one parallel projection onto a plane Jordan curve bounding a convex domain U, then* (7.1) *has a unique solution X, up to conformal $\mathcal{C}^1$ diffeomorphisms of B. Moreover $X(B)$ can be represented as the graph of a solution $z : U \to \mathbb{R}$ of* (7.5) *with boundary datum a function ϕ whose graph is Γ.*

Proof. See [6, Theorem 1, chapter 4.9]. □

We conclude this appendix with a regularity result for a solution of (7.5).

Theorem 7.8. *Let $U \subset \mathbb{R}^2$ be bounded open convex set with ∂U of class $\mathcal{C}^2$ and let z be a solution of* (7.5) *with boundary datum $\phi \in \mathcal{C}^{1,\lambda}(\partial U)$ for some $\lambda \in (0, 1]$. Then $z \in \mathcal{C}^{0,1}(\overline{U})$.*

Proof. See [9, Theorem 13.7]. □

8 Appendix 2: a result from Morse theory

In this short section we report a result from [14, Theorem 10] on critical points of Morse functions. The result holds in any dimension, but we need and state it only for $n = 2$.
Let U be a bounded open subset of $\mathbb{R}^2$ and let B be an open subset of U of class $\mathcal{C}^3$ with $\overline{\mathrm{B}} \subset U$. Suppose that

- $f : U \to \mathbb{R}$ is a Morse function;
- B contains all critical points of f;
- all critical points of the restriction $f_{|\partial \mathrm{B}}$ of f to $\partial\mathrm{B}$ are non degenerate (i.e., $f_{|\partial \mathrm{B}}$ is a Morse function).

Define

$$\partial_f^- \mathrm{B} := \{b \in \partial \mathrm{B} : \ \nabla f(b) \cdot \nu_{\mathrm{B}}(b) < 0\}, \tag{8.1}$$

where $\nu_{\mathrm{B}}(b)$ denotes the outward unit normal to $\partial\mathrm{B}$ at $b \in \partial\mathrm{B}$.
For $i = 0, 1, 2$, denote by $m_i(f, \mathrm{B})$ the number of critical points of index i of f in B and by $m_i(f_{|\partial_f^- \mathrm{B}})$ the number of critical points of index i of $f_{|\partial\mathrm{B}}$ on $\partial_f^-\mathrm{B}$, with $m_2(f_{|\partial_f^- \mathrm{B}}) := 0$.
Define

$$M_i(f, \mathrm{B} \cup \partial\mathrm{B}) := m_i(f, \mathrm{B}) + m_i(f_{|\partial_f^- \mathrm{B}}), \qquad i = 0,\ 1\ ,2. \tag{8.2}$$

The following result holds.

Theorem 8.1. *We have*

$$M_0(f, \mathrm{B} \cup \partial\mathrm{B}) - M_1(f, \mathrm{B} \cup \partial\mathrm{B}) + M_2(f, \mathrm{B} \cup \partial\mathrm{B}) = \chi(\mathrm{B}),$$

where $\chi(\mathrm{B})$ *is the Euler characteristic of* B.

9 Appendix 3: the space $\mathrm{D}(\Omega; \mathbb{R}^2)$

In this section we discuss a property of the space $\mathrm{D}(\Omega; \mathbb{R}^2)$ introduced at the beginning of Section 2.
In [1] the following result is proven.

Theorem 9.1. *Let* $\mathbf{v} \in \mathrm{BV}(\Omega; \mathbb{R}^2)$*. The following conditions are equivalent:*

- $\overline{\mathcal{A}}(\mathbf{v}, \Omega) = \displaystyle\int_\Omega |\mathcal{M}\left(\nabla \mathbf{v}(x)\right)| \, dx\, dy < +\infty$;
- $\mathbf{v} \in W^{1,1}(\Omega; \mathbb{R}^2)$, $\mathcal{M}(\nabla \mathbf{v}) \in L^1(\Omega; \mathbb{R}^6)$ *and there exists a sequence* $(\mathbf{v}^\mu) \subset \mathcal{C}^1(\Omega; \mathbb{R}^2)$ *converging to* $\mathbf{v}$ *in* $L^1(\Omega; \mathbb{R}^2)$ *such that the sequence* $(\mathcal{M}(\nabla \mathbf{v}^\mu))$ *converges to* $\mathcal{M}(\nabla \mathbf{v})$ *in* $L^1(\Omega; \mathbb{R}^6)$.

Hence $\mathrm{D}(\Omega; \mathbb{R}^2)$ is the subset of $\mathrm{BV}(\Omega; \mathbb{R}^2)$ satisfying one of the two equivalent conditions of Theorem 9.1. The following lemma shows that $\overline{\mathcal{A}}$ can be obtained also by relaxing $\mathcal{A}$ in $\mathrm{D}(\Omega; \mathbb{R}^2)$.

Lemma 9.1. *Let* $\mathbf{u} \in \mathrm{BV}(\Omega; \mathbb{R}^2)$*. Then*

$$\overline{\mathcal{A}}(\mathbf{u}, \Omega) = \inf \left\{ \liminf_{\varepsilon \to 0^+} \overline{\mathcal{A}}(\mathbf{u}_\varepsilon, \Omega),\ (\mathbf{u}_\varepsilon)_\varepsilon \subset \mathrm{D}(\Omega; \mathbb{R}^2),\ \mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^1(\Omega; \mathbb{R}^2) \right\}. \tag{9.1}$$

Proof. Trivially $\overline{\mathcal{A}}(u, \Omega)$ is larger than or equal to the right hand side of (9.1), since $\mathcal{C}^1(\Omega; \mathbb{R}^2) \subset \mathrm{D}(\Omega; \mathbb{R}^2)$ and $\overline{\mathcal{A}} = \mathcal{A}$ on $\mathcal{C}^1(\Omega; \mathbb{R}^2)$.
In order to prove the opposite inequality, let $(\mathbf{v}_\varepsilon)$ be a sequence in $\mathrm{D}(\Omega; \mathbb{R}^2)$ such that

$$\lim_{\varepsilon \to 0^+} \overline{\mathcal{A}}(\mathbf{v}_\varepsilon, \Omega) = \inf \left\{ \liminf_{\varepsilon \to 0^+} \overline{\mathcal{A}}(\mathbf{u}_\varepsilon, \Omega),\ (\mathbf{u}_\varepsilon) \subset \mathrm{D}(\Omega; \mathbb{R}^2),\ \mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^1(\Omega; \mathbb{R}^2) \right\}.$$

Thanks to Theorem 9.1, for each $\varepsilon > 0$ we can find a sequence $(\mathbf{v}_\varepsilon^\mu)_\mu$ in $\mathcal{C}^1(\Omega;\mathbb{R}^2)$ converging to $\mathbf{v}_\varepsilon$ in $L^1(\Omega;\mathbb{R}^2)$ as $\mu \to 0^+$ such that

$$\mathcal{A}(\mathbf{v}_\varepsilon^\mu,\Omega) = \int_\Omega |\mathcal{M}\left(\nabla \mathbf{v}_\varepsilon^\mu(x)\right)| \, dx \overset{\mu\to 0^+}{\longrightarrow} \int_\Omega |\mathcal{M}\left(\nabla \mathbf{v}_\varepsilon(x)\right)| \, dx = \overline{\mathcal{A}}(\mathbf{v}_\varepsilon,\Omega).$$

Thus by a diagonal process we obtain a sequence $(\mathbf{v}_\varepsilon^{\mu(\varepsilon)}) \subset \mathcal{C}^1(\Omega;\mathbb{R}^2)$ converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ as $\varepsilon \to 0^+$ such that the right hand side of (9.1) equals

$$\lim_{\varepsilon\to 0^+} \mathcal{A}(\mathbf{u}_\varepsilon^{\mu(\varepsilon)},\Omega) = \lim_{\varepsilon\to 0^+} \overline{\mathcal{A}}(\mathbf{v}_\varepsilon,\Omega),$$

and this concludes the proof.

□

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