Recurrent procedure for the determination of the Free Energy ϵ^2-expansion in the Topological String Theory.

B.A.Dubrovin , A.Ya.Maltsev*

SISSA-ISAS, Via Beirut 2-4 - 34014 Trieste - Italy,
dubrovin@sissa.it , maltsev@sissa.it

Abstract

We present here the iteration procedure for the determination of free energy ϵ^2-expansion using the theory of KdV - type equations. In our approach we use the conservation laws for KdV - type equations depending explicitly on times $t_1, t_2, \ldots$ to find the ϵ^2-expansion of $u(x, t_1, t_2, \ldots)$ after the infinite number of shifts of $u(x, 0, 0, \ldots) \equiv x$ along $t_1, t_2, \ldots$ in recurrent form. The formulas for the free energy expansion are just obtained then as a result of quite simple integration procedure applied to $u_n(x)$.

This work is devoted mainly to the calculation of low-dispersion expansions of the solutions of KdV type equations and their using for calculation of Weil-Petersson volumes of moduli spaces. More precisely we obtain a recurrent procedure for the formulas presented in [2] for such expansions and will refer here to the papers [2] and [3] where the more detailed information and references can be found. Our procedure is based on the quasi-classical expansion for Schrödinger operator and uses also the times-dependent integrals for KdV type equations which, as far as we know, were not mentioned in the previous papers.

*also L.D.Landau Institute for Theoretical Physics, Russian Acad. Sci., maltsev@itp.ac.ru

We shall consider the Free Energy function of Topological String Theory $F(x, t_1, t_2, \ldots)$ such that its second derivative with respect to x $u(x, t_1, t_2, \ldots) = \frac{d^2}{dx^2} F$ satisfies at any $t_1, t_2, \ldots$ the KdV hierarchy with respect to all $t_1, t_2, \ldots$.

The first KdV-equation is written here in the form:

$$\frac{\partial u}{\partial t_2} = u u_x + \frac{\epsilon^2}{12} u_{xxx} \tag{1}$$

where ϵ is small parameter, so we can consider the ϵ^2-expansion of the solution $u(x, t_1, t_2, \ldots)$ of KdV hierarchy and after that try to get $F(x, t_1, t_2, \ldots)$ by the integration with the corresponding normalizing conditions.

We shall describe now the construction which permits to obtain the ϵ^2-expansion of $u(x, t_1, t_2, \ldots)$ using times-dependent conservation laws $J(t_1, t_2, \ldots)$ for KdV hierarchy.

It is well known that (1) can be written in the form:

$$\frac{\partial L}{\partial t_2} = [A_2, L] \tag{2}$$

$$L = -\frac{\epsilon^2}{12} \frac{d^2}{dx^2} - \frac{u(x)}{6} \tag{3}$$

$$A_2 = \frac{\epsilon^2}{3} \frac{d^3}{dx^3} + \frac{1}{2} \left(u \frac{d}{dx} + \frac{d}{dx} u \right) \tag{4}$$

Equation $L\psi = \frac{k^2}{12}$ after the substitution:

$$i\chi(x, k, \epsilon) = \frac{1}{\epsilon} \frac{d \ln \psi}{dx} \tag{5}$$

takes the form:

$$-i\epsilon \frac{d}{dx} \chi + \chi^2 = k^2 + 2u(x) \tag{6}$$

and using this form we can obtain the densities of conservation laws for (1) from the formal expansion:

$$\chi(x, k, \epsilon) \sim k + \sum_{n=1}^{\infty} \frac{\chi_n(x, \epsilon)}{(2k)^n} \tag{7}$$

when $k \to \infty$. Here $\chi_{2n}(x,\epsilon)$ are full derivatives $\chi_{2n}(x,\epsilon) = \partial_x Q_{2n}(x,\epsilon)$, and $\chi_{2n+1}(x,\epsilon)$ - are densities of conservation laws, which are polynomial expressions of $u, \epsilon u_x, \epsilon^2 u_{xx}, \ldots$

$$I_n = \int \chi_{2n+1}(x,\epsilon)dx$$

We can write the higher KdV-type equations, consisting with (1), in the Hamiltonian form:

$$\frac{\partial u}{\partial t_k} = \partial_x \frac{\delta}{\delta u(x)} \int \chi_{2n+1}(x,\epsilon)dx = \partial_x \frac{\delta}{\delta u(x)} \frac{1}{\pi i} \oint (2k)^{2n} \left[\int \chi(x,k,\epsilon)dx \right] dk \tag{8}$$

Theorem 1.

Let us consider equations (8) on the functional space of rapidly decreasing functions: $u(x) \to 0, |x| \to \infty$. Then the functional:

$$J_n(t_n) = \int xu(x)dx - 4(2n-1)t_n \int \chi_{2n-1}(x,\epsilon)dx \tag{9}$$

is the conservation law for n-th KdV-equation (8), depending on the time t_n.[1]

Proof.

Since $\int \chi_{2n-1}(x,\epsilon)dx$ is the conservation law for any of equations (8), we have:

$$\frac{d}{dt_n} J_n = \int x \partial_x \frac{\delta}{\delta u(x)} \int \chi_{2n+1}(y,\epsilon)dydx - 4(2n-1) \int \chi_{2n-1}(x,\epsilon)dx =$$

$$= - \int dx \frac{\delta}{\delta u(x)} I_n - 4(2n-1) I_{n-1}$$

But as can be extracted from (6) and (7): [2]

$$\int \frac{\delta}{\delta u(x)} I_n dx \equiv \int \frac{\delta}{\delta u(x)} \frac{1}{\pi i} (2k)^{2n} \left[\int \chi(y,k,\epsilon)dy \right] dkdx =$$

[1]Firstly the functionals of this type were considered in [1] , where they were restricted on the slowly modulated m-phase algebro-geometric solutions of KdV. Here we derive the analogous formulas for any solution of KdV-type equations.

[2]For the identities of this type see also [4].

$$= \frac{1}{\pi i} \oint (2k)^{2n} \left[2 \frac{d}{d(k^2)} \int \chi(y,k,\epsilon) dy \right] dk = -4(2n-1) I_{n-1} \qquad (10)$$

so we have $\frac{d}{dt_n} J_n = 0$.

Theorem is proved.

Let us now consider the variational derivative of J_n with respect to $u(x)$ corresponding to rapidly decreasing variations $\delta u(x)$, that is the variational derivative of Euler-Lagrange type:

$$\Omega_n(x,\epsilon) = \frac{\delta J_n}{\delta u(x)} \equiv \frac{\partial P_n}{\partial u}(x) - \frac{\partial}{\partial x} \frac{\partial P_n}{\partial u_x}(x) + \frac{\partial^2}{\partial x^2} \frac{\partial P_n}{\partial u_{xx}}(x) - \dots \qquad (11)$$

where $J_n = \int P_n(u, \epsilon u_x, \epsilon^2 u_{xx}, \dots) dx$.

Theorem 2.

If we consider the n-th equation of (8) then the evolution of $\Omega_n(x,\epsilon)$ satisfies the linear equation:

$$\frac{d}{dt_n} \Omega_n(x,\epsilon) = D_n^1(\epsilon, u, u_x, \dots) \frac{d}{dx} \Omega_n + \dots + D_n^{2n-1}(\epsilon, u, u_x, \dots) \frac{d^{2n-1}}{dx^{2n-1}} \Omega_n =$$

$$= \sum_{s=1}^{2n-1} D_n^s(\epsilon, u, u_x, \dots, u_{(2n-s-1)x}) \frac{d^s}{dx^s} \Omega_n \qquad (12)$$

where $D_n^s(\epsilon, u, u_x, \dots)$ are some polynomials of $u, u_x, u_{xx}, \dots$, $\frac{d}{dt_n}$ is full derivative with respect to t_n, $u_{nx} \equiv \frac{d^n}{dx^n} u(x)$.

Proof.

Let us consider any of equations (8) as the flux on the space $(u(x))$ corresponding to vector field:

$$\xi(x) = \partial_x \frac{\delta}{\delta u(x)} \int \chi_{2n+1}(x,\epsilon) dx$$

Since J_n is invariant function, then $\Omega(x,\epsilon)$ is the invariant 1-form on the space $(u(x))$. So its full Lie-derivative with respect to $\xi(x)$ plus partial derivative with respect to t_n must be zero, that is:

$$\frac{\partial}{\partial t_n} \Omega_n(x,\epsilon) + \left(L_\xi \Omega_n \right)(x) = 0$$

where

$$\left(L_{\xi}\Omega_n\right)(x) = \int \xi(y)\frac{\delta}{\delta u(y)}\Omega_n(x)dy + \int \Omega_n(y)\frac{\delta}{\delta u(x)}\xi(y)dy$$

and

$$\frac{\partial}{\partial t_n}\Omega_n(x,\epsilon) + \int \xi(y)\frac{\delta}{\delta u(y)}\Omega_n(x)dy \equiv \frac{d\Omega_n(x)}{dt_n}$$

The expression $\int \Omega_n(y)\frac{\delta}{\delta u(x)}\xi(y)dy$ is the action of the linear differential operator of type (12) on $\Omega_n(x)$.

Theorem is proved.

Corollary.

If $\Omega_n(x) = 0$ at $t_n = 0$ (and u(x) is rapidly decreasing), then $\Omega_n(x) \equiv 0$ at any t_n.

There can be easily formulated the generalizations of Theorems 1 and 2 if we consider the common solution $u(\epsilon, x, t_1, t_2, \ldots, t_n, C_1, C_2, \ldots, C_n)$ of the system of equations:

$$\frac{\partial u}{\partial t_n} = C_n\partial_x\frac{\delta}{\delta u(x)}\int \chi_{2n+1}(x,\epsilon)dx, \ \ n = 1,\ldots,N. \tag{13}$$

Then:

Theorem 1′.

The functional:

$$J(t_1,\ldots,t_N,C_1,\ldots,C_N) = \int xu(x)dx - \sum_{s=1}^{N} 4(2s-1)C_s t_s \int \chi_{2s-1}(x,\epsilon)dx \tag{14}$$

- is the conservation law for all fluxes (13) for $1 \le n \le N$, that is:

$$\frac{d}{dt_n}J(t_1,\ldots,t_n,C_1,\ldots,C_N) \equiv 0, \ \ 1 \le n \le N.$$

The proof is evident since any of $t_s \int \chi_{2s-1}(x,\epsilon)dx, \ s \neq n$ at fixed t_s is the conservation law for n-th KdV-equation and all KdV-equations commute with each other.

Theorem 2′.

The coefficients $\Omega(x)$ of 1-form Ω: $\Omega(x) = \frac{\delta J}{\delta u(x)}$ (Euler-Lagrange derivative) satisfy the system of equations:

$$\frac{d}{dt_n}\Omega(x, t_1, \ldots, t_n) = \sum_{s=1}^{2n-1} C_n D_n^s(\epsilon, u, u_x, \ldots, u_{(2n-s-1)x}) \frac{d^s}{dx^s}\Omega(x)$$

$$n = 1, \ldots, N, \tag{15}$$

and if $\Omega(x)$ is zero at $t_1 = t_2 = \ldots = t_N = 0$ then it is identically zero at any $t_1, t_2, \ldots, t_N$.

It is also evident that we can add to $\Omega(x)$ any invariant form of type

$$\Omega'(x) = \sum_{s=0}^{M} d_s \frac{\delta}{\delta u(x)} \int \chi_{2s+1}(x, \epsilon) dx$$

(where the coefficients d_s do not depend on $t_1, t_2, \ldots$) and Theorems 2, 2′ will remain valid.

We shall need later the invariant forms of type

$$\Omega(x) = x - u(x) + \sum_{s=1}^{\infty} \beta_s t_s \frac{\delta}{\delta u(x)} \int \chi_{2s-1}(x, \epsilon) dx \tag{16}$$

(where $u(x)$ is the variational derivative of the momentum integral $P = \frac{1}{2}\int u^2(x)dx$) for the investigation of asymptotic expansion of $u(x)$ in terms of ϵ^2 after the infinite number of shifts of the initial function $u(x, 0, 0, \ldots) = x$ along times $t_1, t_2, \ldots$ according to KdV equations (8).

All the considerations above were for rapidly decreasing functions $u(x)$. But as can be easily seen, the relations (12), (15) are local expressions of $u, u_x, u_{xx}, \ldots, u_{t_s}, u_{xt_s}, u_{xxt_s}, \ldots$ where we consider $D_n^s, \Omega(x)$ (and $\frac{\delta}{\delta u(x)} \int \chi_{2s+1}(x, \epsilon)dx$) just as local polynomials of $u, u_x, \ldots$ (for the last we use just formal Euler-Lagrange expression for variational derivative in this case) and Theorems 2, 2′ will be valid for 1-forms (11) and (16) for any global in x solution $u(x)$ up to the time $t_1, t_2, \ldots, t_N$ where this global solution exists (so if $\Omega(x)$ is identically zero at $t_1 = t_2 = \ldots = 0$ it will be identically zero in all region where we have a global solution $u(x)$.)

Now we shall consider the following construction:

It is well known that all KdV-type equations (8) in our case can be written in the form:

$$\frac{\partial u}{\partial t_n} = \frac{1}{\alpha_n} u^{n-1} u_x + \epsilon^2 (\ldots) + \epsilon^4 (\ldots) + \ldots \tag{17}$$

Let denote $K_n^m(u, u_x, u_{xx}, \ldots)$ - the corresponding term in n-th KdV equation which has the multiplier ϵ^{2m}, so we have

$$\frac{\partial u}{\partial t_n} = \frac{1}{\alpha_n} u^{n-1} u_x + \sum_{m=1}^{n-1} \epsilon^{2m} K_n^m(u, u_x, u_{xx}, \ldots) \tag{18}$$

Let we are given the function $\Phi(u)$ which is a convergent everywhere (in u-plane) series:

$$\Phi(u) = \sum_{n=0}^{\infty} \gamma_n u^n \tag{19}$$

We can consider the common solution of all KdV-type equations (17) given by the relations

$$\frac{\partial u}{\partial t_n} = \alpha_n \gamma_{n-1} \partial_x \frac{\delta}{\delta u(x)} \int \chi_{2n+1}(y, \epsilon) dy =$$

$$= \alpha_n \gamma_{n-1} \partial_x \frac{\delta}{\delta u(x)} \frac{1}{\pi i} \oint (2k)^{2n} \left[\int \chi(y, k, \epsilon) dy \right] dk \tag{20}$$

up to the times $t_1, t_2, \ldots$, where this global solution $u(x, t_1, t_2, \ldots)$ exists and we put $u(x, 0, 0, \ldots) = x$.

Theorem 3.

If $\Phi(u) = \sum_{n=0}^{\infty} \gamma_n u^n$ is a convergent everywhere series, then the equation

$$\frac{\partial u}{\partial \tau} = \sum_{n=1}^{\infty} \alpha_n \gamma_{n-1} \partial_x \frac{\delta}{\delta u(x)} \int \chi_{2n+1}(x, \epsilon) dx \tag{21}$$

can be represented as:

$$\frac{\partial u}{\partial \tau} = \Phi(u) u_x + \sum_{n=1}^{\infty} \epsilon^{2m} K_m(u, u_x, \ldots, u_{(2m+1)x}) \tag{22}$$

where $K_m(u, u_x, \ldots, u_{(2m+1)x})$ are the polynomials of $u_x, u_{xx}, \ldots$ (containing in each term (2m+1) derivatives with respect to x) with the coefficients depending on u and being expressed in terms of $\Phi(u), \Phi^{(1)}(u), \Phi^{(2)}(u), \ldots,$ $\Phi^{(q)}(u) \equiv \frac{d^q}{du^q}\Phi(u)$.

Proof.

Let us remind that the KdV-hierarchy can be extracted from the Riccati equation (6) for $\chi(x,\epsilon,k)$:

$$-i\epsilon\frac{d}{dx}\chi + \chi^2 = k^2 + 2u(x)$$

and we use the function:

$$\chi_R(x,k) \equiv k + \sum_{n=0}^{\infty} \frac{\chi_{2n+1}(x,\epsilon)}{(2k)^{2n+1}} \tag{23}$$

for the generation of Hamiltonian KdV-fluxes.

As was shown by B.A.Dubrovin (see [5], [6], [7]) the following relation holds:

$$\frac{\delta}{\delta u(x)}\int \chi_R(x,\epsilon,k)dx \ \ (\equiv \frac{\delta}{\delta u(x)}\int \chi(x,\epsilon,k)dx) = \frac{\lambda}{\chi_R(x,\epsilon,k)} \tag{24}$$

in the class of rapidly decreasing or periodic functions $u(x)$. So it is valid as local relation for $u, u_x, \ldots$ in any order of formal expansion of $\chi_R(x,\epsilon,k)$ and $\frac{1}{\chi_R(x,\epsilon,k)}$ in terms of $1/(2k)^{2s+1}$, where we use just their local expressions obtained from the formal equation (6) and $\frac{\delta}{\delta u(x)}$ is just formal Euler-Lagrange expression for derivatives of $\int \chi_{R,(2n+1)}(x,\epsilon)dx$ with respect to $u(x)$:

$$\frac{\delta}{\delta u(x)}\int \chi_{R,(2n+1)}(x,\epsilon)dx = \frac{\partial \chi_{R,(2n+1)}(x,\epsilon)}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \chi_{R,(2n+1)}(x,\epsilon)}{\partial u_x} + \ldots$$

and we have

$$\frac{\delta}{\delta u(x)}\int \chi_{R,(2n+1)}(x,\epsilon)dx = \lambda\left[\frac{1}{\chi_R(x,\epsilon,k)}\right]_{2n+1}$$

($[\ldots]_{2n+1}$ means here the corresponding term in the expansion).

We shall not need the value of constant λ in our case, but what is important that λ does not depend upon ϵ. This fact can be easily obtained from the fact that, as follows from the formal equation (6), the formulas for $\chi_n(x,\epsilon)$ in (7) as the expressions of $u, u_x, u_{xx}, \ldots$ differ from the analogous formulas at $\epsilon = 1$ just by the multiplier ϵ in any differentiation with respect to x (as well known, ϵ can be removed from the initial KdV-equation (1) by scaling transformation). Since the relation (24) in any order of k^{-1} is just the identical equality between two polynomials of $u, u_x, u_{xx}, \ldots$ it will remain true if we replace any differentiation with respect to x by $\epsilon\frac{d}{dx}$. It can be also checked by the direct calculations similar to [6], [7] that here $\lambda = 1$.

So we can represent the equation (21) as:

$$\frac{\partial u}{\partial \tau} = \lambda \sum_{n=1}^{\infty} \alpha_n \gamma_{n-1} \partial_x \frac{1}{\pi i} \oint (2k)^{2n} \frac{dk}{\chi_R(x,\epsilon,k)}$$

and write the closed equation for $A(x,\epsilon,k) \equiv \frac{1}{\chi_R(x,\epsilon,k)}$

$$\frac{1}{2}\epsilon^2 A \frac{d^2}{dx^2} A - \frac{1}{4}\epsilon^2 \left(\frac{d}{dx} A\right)^2 = 1 - (k^2 + 2u(x))A^2 \tag{25}$$

obtained from (6) by substitution $\chi = \chi_R + i\chi_{Im}$, where χ_R is real if $k^2 + 2u(x) \geq 0$ and coincides with the introduced above, and $A = 1/\chi_R(x,\epsilon,k)$, $B = \chi_{Im}(x,\epsilon,k)/\chi_R(x,\epsilon,k)$.

After that we obtain the following system of equations:

$$\frac{\partial u}{\partial \tau} = \lambda \sum_{n=1}^{\infty} \alpha_n \gamma_{n-1} \partial_x \frac{1}{\pi i} \oint (2k)^{2n} A(x,\epsilon,k) dk$$

$$\frac{1}{2}\epsilon^2 A \frac{d^2}{dx^2} A - \frac{1}{4}\epsilon^2 \left(\frac{d}{dx} A\right)^2 = 1 - (k^2 + 2u(x))A^2 \tag{26}$$

where $\frac{1}{\pi i} \int (2k)^{2n} A(x,\epsilon,k) dk$ is just the formal expression meaning that we must take the n-th term in the formal expansion:

$$A(x,\epsilon,k) \sim \sum_{n=0}^{\infty} \frac{A^*_{2n+1}(x,\epsilon)}{(2k)^{2n+1}}, \quad A^*_1(x,\epsilon) \equiv 2,$$

$k \to \infty$.

It is also possible to obtain ϵ^2-expansion of $A(x,\epsilon,k)$ from the expansion (26), which corresponds to quasi-classical limit for Schrödinger operator:

$$A(x,\epsilon,k)=\sum_{n=0}^{\infty}\epsilon^{2n}\hat{A}_n(x,k) \tag{27}$$

where $\hat{A}_0(x,k)=\frac{1}{\sqrt{k^2+2u(x)}}$, and

$$\hat{A}_1(x,k)=\frac{1}{2\sqrt{k^2+2u(x)}}\left[\frac{1}{4}(\hat{A}_{0x})^2-\frac{1}{2}\hat{A}_0\hat{A}_{0xx}\right],$$

$$\hat{A}_n(x,k)=$$

$$=\frac{1}{2\sqrt{k^2+2u(x)}}\left[\frac{1}{4}\sum_{s=0}^{n-1}\left(\frac{d}{dx}\hat{A}_s\right)\left(\frac{d}{dx}\hat{A}_{n-s-1}\right)-\frac{1}{2}\sum_{s=0}^{n-1}\hat{A}_s\frac{d^2}{dx^2}\hat{A}_{n-s-1}\right]-$$

$$-\frac{1}{2}\sqrt{k^2+2u(x)}\left[\sum_{s=1}^{n-1}\hat{A}_s\hat{A}_{n-s}\right],\quad n\geq 2. \tag{28}$$

As can be easily seen, any $\hat{A}_n(x,k)$ is the expression containing only polynomials of $u,u_x,\ldots$ divided by some odd degrees of $\sqrt{k^2+2u(x)}$:

$$\hat{A}_n(x,k)=\sum_{q=1}^{3n}\frac{\hat{D}_n^q(u,u_x,\ldots,u_{2nx})}{(\sqrt{k^2+2u(x)})^{2q+1}}\quad,\quad n\geq 1.$$

Let us now consider the first formal equation of (26) in the form of the formal expansion:

$$\frac{\partial u}{\partial\tau}=\lambda\sum_{n=1}^{\infty}\alpha_n\gamma_{n-1}\partial_x\frac{1}{\pi i}\oint(2k)^{2n}\left[\sum_{s=0}^{\infty}\epsilon^{2s}\sum_{q=0}^{3s}\frac{\hat{D}_s^q(u,u_x,\ldots,u_{2sx})}{(\sqrt{k^2+2u(x)})^{2q+1}}\right]dk$$

where $\hat{D}_s^q(u,u_x,\ldots,u_{2sx})$ do not depend on k and formal integration

$$\frac{1}{\pi i}\oint(2k)^{2n}\frac{dk}{(\sqrt{k^2+2u(x)})^{2q+1}}$$

coincides here with the value of this integral.

The value

$$\lambda \sum_{n=1}^{\infty} \alpha_n \gamma_{n-1} \frac{1}{\pi i} \oint (2k)^{2n} \frac{dk}{\sqrt{k^2+2u(x)}}$$

coincides by the definition with $\int \Phi(u)du$ because $\hat{A}_0(x,k) = 1/\sqrt{k^2+2u(x)}$ and the limit of every KdV-equation at $\epsilon \to 0$ is:

$$\frac{\partial u}{\partial \tau} = \frac{u^{n-1}u_x}{\alpha_n}$$

so we must have $\frac{\partial u}{\partial \tau} = \Phi(u)u_x, \ \ \epsilon = 0$, and any of the values

$$\lambda \sum_{n=1}^{\infty} \alpha_n \gamma_{n-1} \frac{1}{\pi i} \oint (2k)^{2n} \frac{dk}{(\sqrt{k^2+2u(x)})^{2q+1}} \tag{29}$$

is equal by such a way to

$$\frac{(-1)^q}{(2q-1)!!} \frac{d^q}{du^q} \int \Phi(u)du = \frac{(-1)^q}{(2q-1)!!} \Phi^{(q-1)}(u(x)), \ \ q \geq 1.$$

Using these two equalities in the first equation of (26) we obtain the equation (22) in the required form in any order of ϵ^2 after the finite number of steps (28).

Theorem 3 is proved.

Corollary.

If $\Phi(u)$ is such that the equation

$$x - u_0(x,\tau) + \tau \Phi(u_0(x,\tau)) \equiv 0 \tag{30}$$

has the unique solution for any x and $0 \leq \tau \leq 1$ then $u(x,\tau,\epsilon)$ can be represented as the formal expansion in powers of ϵ^2:

$$u(x,\tau,\epsilon) = u_0(x,\tau) + \sum_{n=1}^{\infty} \epsilon^{2n} u_n(x,\tau), \ \ -\infty < x < \infty, \ \ 0 \leq \tau \leq 1, \tag{31}$$

where $u_0(x,\tau)$ satisfies (30).

The proof is evident since we have a linear non-homogeneous evolution equation on every $u_n(x,\tau)$ with the initial data $u_n(x,0) \equiv 0$ which always has a unique solution.

Recurrent formulas for $u_n(x,1)$ in the ϵ^2 - expansion of $u(x,1)$.

Theorem 4.

Let $\Phi(u)$ be a convergent everywhere series: $\Phi(u) = \sum_{n=0}^{\infty} \gamma_n u^n$ such that the equation (30) has a unique solution for any x and $0 \le \tau \le 1$, then the solution $u(x) \equiv u(x,1)$ of (21)

$$\frac{\partial u}{\partial \tau} = \sum_{n=1}^{\infty} \alpha_n \gamma_{n-1} \partial_x \frac{\delta}{\delta u(x)} \int \chi_{2n+1}(x) dx$$

(formal Euler-Lagrange derivative) with the initial data: $u(x,0) \equiv x$ can be represented as the formal expansion in terms of ϵ^2

$$u(x) = u_0(x) + \sum_{n=1}^{\infty} \epsilon^{2n} u_n(x) \tag{32}$$

where $u_0(x)$ satisfies

$$x - u_0(x) + \Phi(u_0(x)) = 0 \tag{33}$$

and the coefficients $u_n(x)$ can be found from the recurrent formulas:

$$A_0(x,k) = \frac{1}{\sqrt{k^2 + 2u_0(x)}}$$

$$u_1(x) = u_{0x} \hat{L} < \frac{1}{2\sqrt{k^2 + 2u_0(x)}} \left(\frac{1}{4}(A_{0x})^2 - \frac{1}{2} A_0 A_{0xx} \right) >$$

$$A_1(x,k) = \frac{1}{2\sqrt{k^2 + 2u_0(x)}} \left(\frac{1}{4}(A_{0x})^2 - \frac{1}{2} A_0 A_{0xx} \right)$$

$$u_n(x) = u_{0x} \hat{L} < \frac{1}{2\sqrt{k^2 + 2u_0(x)}} \{ \frac{1}{4} \sum_{s=0}^{n-1} \left(\frac{d}{dx} A_s \right) \left(\frac{d}{dx} A_{n-s-1} \right) -$$

$$-\frac{1}{2} \sum_{s=0}^{n-1} A_s \frac{d^2}{dx^2} A_{n-s-1} - (k^2 + 2u_0(x)) \sum_{s=1}^{n-1} A_s A_{n-s} -$$

$$- 2 \sum_{z=1}^{n-1} u_z(x) \left(\sum_{s=0}^{n-z} A_s A_{n-z-s} \right) \} > \quad , n \ge 2 \tag{34}$$

$$A_n(x,k) = \frac{1}{2\sqrt{k^2+2u_0(x)}}\{\frac{1}{4}\sum_{s=0}^{n-1}\left(\frac{d}{dx}A_s\right)\left(\frac{d}{dx}A_{n-s-1}\right) -$$

$$-\frac{1}{2}\sum_{s=0}^{n-1}A_s\frac{d^2}{dx^2}A_{n-s-1} - (k^2+2u_0(x))\sum_{s=1}^{n-1}A_sA_{n-s} -$$

$$-2\sum_{z=1}^{n-1}u_z(x)\left(\sum_{s=0}^{n-z}A_sA_{n-z-s}\right)\} - \frac{u_n(x)}{(\sqrt{k^2+2u_0(x)})^3} \quad ,n\geq 2 \tag{35}$$

where all $A_n(x,k)$ have the form

$$A_n(x,k) = \sum_{q=1}^{3n}\frac{D_n^q(u_0,u_{0x},\ldots,u_{0(2n)x})}{(\sqrt{k^2+2u_0(x)})^{2q+1}} \tag{36}$$

and $\hat{L}$ is the linear operator acting on the functions of k so that:

$$\hat{L} < \frac{1}{\sqrt{k^2+2u_0(x)}} >= \Phi(u_0(x)) \tag{37}$$

$$\hat{L} < \frac{1}{(\sqrt{k^2+2u_0(x)})^{2q+1}} >= \frac{(-1)^q}{(2q-1)!!}\Phi^{(q)}(u_0(x)) \quad ,q\geq 1. \tag{38}$$

(Let us note here that $A_n(x,k)$ can be obtained from the introduced previously $A_s^*(x,k)$ if we substitute the function $u(x)$ in the form (31) in all the expressions for $A_s^*(x,k)$.)

Proof.

Let us change the first equation of (26) by the equation $\Omega(x)\equiv 0$ at $\tau=1$, where $\Omega(x)$ is of the form (16) which is identically zero at $t_1=t_2=\ldots=0$ $(u(x,0,0,\ldots)=x)$

$$\Omega(x) = x - u(x) + \sum_{s=1}^{\infty}\beta_s\frac{\delta}{\delta u(x)}\int\chi_{2s-1}(x,\epsilon)dx. \tag{39}$$

It is not very difficult to check using (14) that for $\tau=1$ the coefficients β_n, corresponding to the flux (21) can be expressed in terms of introduced in (17) and (19) α_n and γ_n by formula

$$\beta_n = -4(2n-1)\alpha_n\gamma_{n-1},$$

and, as follows from (10) $\alpha_n = (-1)^n (n+1)!/4^n n(2n-1)!!$

According to the formula (24) we can write this equation in the form:

$$x - u(x,\epsilon) + \sum_{n=0}^{\infty} \lambda\beta_n \frac{1}{\pi i} \oint (2k)^{2n-2} A(x,k,\epsilon) dk = 0 \tag{40}$$

where β_n are such that:

$$\sum_{n=1}^{\infty} \lambda\beta_n \frac{1}{\pi i} \oint (2k)^{2n-2} \frac{dk}{\sqrt{k^2 + 2u_0(x)}} \equiv \Phi(u_0(x)) \tag{41}$$

for any $u_0(x)$ so that at $\epsilon = 0$ we obtain formula (33).

As can be easily shown using the formal representation (40) (just like as in Theorem 3) $\Omega(x)$ can be represented as a formal series on ϵ^2, which at any power of ϵ^2 is just a local expression of $u, u_x, u_{xx}, \ldots$, having the form:

$$\Omega(x) = \sum_{s=0}^{\infty} \epsilon^{2s} N(u, u_x, \ldots, u_{(2s)x}) \tag{42}$$

where $N(u, u_x, \ldots, u_{(2s)x})$ are polynomials of $u_x, u_{xx}, \ldots$ containing $2s$ derivatives with respect to x, with the coefficients depending on $\Phi(u), \Phi^{(1)}(u), \ldots$. This means that the formal series (40) in any order of ϵ^2 is convergent everywhere (as the sum of differentiations of convergent everywhere series (41)) and can be expressed in the appropriate form (42).

It is evident from this fact that at any finite order of ϵ^2 $\Omega(x)$ satisfies to linear differential equation of finite order like (12) according to n-th KdV equation, since it is so for any finite sum (39), and from Theorem 3 we can conclude that it is also so for the evolution with respect to τ.

So that, we can use the local equality $\Omega(x) \equiv 0$, where $\Omega(x)$ is local expression of $u, u_x, \ldots$, in any order of ϵ^2 for $u(x)$ in the form (32) at $\tau = 1$ if $u(x,\tau)$ is a formal global asymptotic solution of (21) for $0 \le \tau \le 1$.

Now let us introduce linear operator $\hat{L}$ acting on functions of k by the formula:

$$\hat{L} < G(k) >= \sum_{n=1}^{\infty} \lambda\beta_n \frac{1}{\pi i} \oint (2k)^{2n-2} G(k) dk$$

By the definition:

$$\hat{L} < \frac{1}{\sqrt{k^2 + 2u_0(x)}} > \equiv \Phi(u_0(x))$$

and it can be easily seen that

$$\hat{L} < \frac{1}{(\sqrt{k^2 + 2u_0(x)})^{2q+1}} > = \frac{(-1)^q}{(2q-1)!!} \Phi^{(q)}(u_0(x))$$

where $\Phi^{(q)}(u) \equiv \frac{d^q}{du^q}\Phi(u)$, for $q \geq 1$.

By the substitution of expansions

$$A(x, k, \epsilon) = \frac{1}{\sqrt{k^2 + 2u_0(x)}} + \sum_{n=1}^{\infty} \epsilon^{2n} A_n(x, k)$$

and

$$u(x, \epsilon) = u_0(x) + \sum_{n=1}^{\infty} \epsilon^{2n} u_n(x)$$

in the system

$$\frac{1}{2}\epsilon^2 A \frac{d^2}{dx^2} A - \frac{1}{4}\epsilon^2 \left(\frac{d}{dx} A\right)^2 = 1 - (k^2 + 2u(x))A^2,$$

$$x - u(x, \epsilon) + \hat{L} < A(x, k, \epsilon) > = 0$$

(that is $u_n(x) = \hat{L} < A_n(x, k) >$), it is easy to obtain (34) and (35) for u_n and A_n, where we used the fact that

$$\hat{L} < \frac{1}{(\sqrt{k^2 + 2u_0(x)})^3} > = -\Phi^{(1)}(u_0(x))$$

and that view (33) $\Phi^{(1)}(u_0(x)) = 1 - \frac{1}{u_{0x}}$.

Formulas (36) for A_n are evident from the recurrent formulas (35).

Theorem 4 is proved.

As can be easily seen from (33), all $\Phi^{(q)}(u_0(x))$ can be expressed in terms of $u_0(x)$ and its derivatives using formula $\Phi^{(q+1)}(u_0(x)) = \frac{1}{u_{0x}} \frac{d}{dx} \Phi^{(q)}(u_0(x))$ and so it can be easily seen that we can represent $u_n(x)$ in the form:

$$u_n(x) = \sum_{s=1}^{N(n)} \frac{U_n^s(u_{0xx}, u_{0xxx}, \ldots)}{(u_{0x})^s} \tag{43}$$

where $U_n^s(u_{0xx}, u_{0xxx}, \ldots)$ are polynomials containing $(2n+s)$ differentiations with respect to x in each term.

All the functions $u_n(x), n \geq 1$ (see [2]) are full double derivatives with respect to x of the functions $F_n(x)$, where $F_1(x) = (1/24)\ln(u_{0x})$ and all $F_n(x), n \geq 2$ have the same form as (43):

$$F_n(x) = \sum_{s=1}^{N(n)-2} \frac{f_n^s(u_{0xx}, u_{0xxx}, \ldots)}{(u_{0x})^s} \tag{44}$$

The condition (44) fixes here uniquely the integration constants in the determination of $F_n(x)$, such that

$$u_n(x) = \frac{d^2}{dx^2} F_n(x)$$

The values $F_n(x)$ in the form (44) are necessary for the calculations of Weil-Petersson volumes of moduli spaces and it is quite easy to propose an algorithm for determination of $F_n(x)$ using $u_n(x)$ in the form (43). Namely we define the algorithm of integration of the expression (43) provided that it is the full derivative with respect to x in the following way:

since $D^{-1}u_n$ must have the same form

$$D^{-1}u_n = \sum_{s=1}^{N(n)-1} \frac{v_n^s(u_{0xx}, u_{0xxx}, \ldots)}{(u_{0x})^s} \tag{45}$$

where v_n^s are polynomials (containing $(2n+s-1)$ differentiations with respect to x in each term), it is easy to define the highest term $v_n^{N(n)-1}(u_{0xx}, u_{0xxx}, \ldots)$ from (43) as

$$v_n^{N(n)-1} = \frac{U_n^{N(n)}(u_{0xx}, u_{0xxx}, \ldots)}{(N(n)-1)u_{0xx}}$$

which must be polynomial of $u_{0xx}, u_{0xxx}, \ldots$ if $u_n(x)$ is the full derivative of (45).

The rest of (45) without the highest term

$$\sum_{s=1}^{N(n)-2} \frac{v_n^s(u_{0xx}, u_{0xxx}, \ldots)}{(u_{0x})^s}$$

is the integral of the value

$$u_n(x) - \frac{U_n^{N(n)}}{(u_{0x})^{N(n)}} - \frac{(v_n^{N(n)-1})_x}{(u_{0x})^{N(n)-1}} = \sum_{s=1}^{N(n)-1} \frac{U_n^s(u_{0xx}, u_{0xxx}, \ldots)}{(u_{0x})^s} - \frac{(v_n^{N(n)-1})_x}{(u_{0x})^{N(n)-1}}$$

and we can repeat the same procedure for $v_n^{N(n)-2}(u_{0xx}, u_{0xxx}, \ldots)$ and so on. The last step must give identically zero for the corresponding rest of sum (43) for $u_n(x)$ - full derivative of the expression (45) and so we shall obtain quite simple formulas for v_n^s in terms of U_n^m and their derivative. Applying this procedure twice to any $u_n(x)$ in the form (43) we shall obtain the required form (44) for the functions $F_n(x)$.

The second author (A.Ya.M.) thanks INTAS (grant INTAS 96-0770) and RFBR (grant 97-01-00281) for partial financial support.

References

[1] B. Dubrovin. *Functionals of the Peierls-Fröhlich Type and the Variational Principle for the Whitham Equations.* // Amer. Math. Soc. Transl. (2) Vol. **179**, 1997.

[2] T. Eguchi , Y. Yamada , S.-K. Yang. *On the Genus Expansion in the Topological String Theory.* // hep-th/9405106, Rev. Math. Phys. 7 (1995) 279.

[3] P. Zograf. *Weil-Petersson volumes of moduli spaces of curves and the genus expansion in two dimensional gravity.* // math.AG/9811026 .

[4] I.M.Gelfand , L.A.Dikii. *Asymptotic behaviour of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg - de Vries equation.* Russian Math. Surveys **30** : 5 (1975) , 77-113 . (Tr. from Uspekhi Mat. Nauk **30** : 5 (1975), 67-100).

[5] B.A.Dubrovin , V.B.Matveev , S.P.Novikov . *Non-linear equations of Korteweg - de Vries type, finite-zone linear operators, and abelian varieties.* Russian Math. Surveys bf 31 : 1 (1976), 59-146. (Tr. from Uspekhi Mat. Nauk **31** : 1 (1976), 55-136).

[6] B.A.Dubrovin. *Inverse problem for periodic finite-zoned potentials in the theory of scattering.* Functional Analysis and its Applications. **9** : 1 (1975), 61-62. (Tr. from Funktsional. Anal. i Prilozhen. **9** : 1 (1975), 65-66).

[7] B.A.Dubrovin. *Periodic Problems for the Korteweg - de Vries Equation in the Class of Finite Band Potentials.* Functional Analysis and its Applications. **9** : 3 (1975), 215-223. (Tr. from Funktsional. Anal. i Prilozhen. **9** : 3 (1975)).