SISSA 27/2012/EP
ZTF-12-02

Stationary rotating black holes in theories with gravitational Chern-Simons Lagrangian term

L. Bonora[a], M. Cvitan[b], P. Dominis Prester[c], S. Pallua[b], I. Smolić[b]

[a] *International School for Advanced Studies (SISSA/ISAS), Via Bonomea 265, 34136 Trieste, Italy*

[b] *Physics Department, Faculty of Science, University of Zagreb, p.p. 331, HR-10002 Zagreb, Croatia*

[c] *Department of Physics, University of Rijeka, Ul. Radmile Matejčić 2, HR 51000 Rijeka, Croatia*

Email: bonora@sissa.it, mcvitan@phy.hr, pprester@phy.uniri.hr, pallua@phy.hr, ismolic@phy.hr

Abstract

We study the effects of introducing purely gravitational Chern-Simons Lagrangian terms in ordinary Einstein gravity on stationary rotating black hole solutions and on the associated thermodynamical properties, in a generic number of dimensions which support these terms (i.e. in $D = 4k - 1$). We analyze the conditions, namely the number of vanishing angular momenta, under which the contributions of the Chern-Simons term to the equations of motion and the black hole entropy vanish. The particular case of a 7-dimensional theory in which a purely gravitational Chern-Simons term is added to the Einstein-Hilbert Lagrangian in $D = 7$ dimensions is investigated in some detail. As we have not been able to find exact analytic solutions in nontrivial cases, we turn to perturbation theory and calculate the first-order perturbative correction to the Myers-Perry metric in the case where all angular momenta are equal. The expansion parameter is a dimensionless combination linear in the Chern-Simons coupling constant and the angular momentum. Corrections to horizon and ergosurface properties, as well as black hole entropy and temperature, are presented.

Keywords: Rotating black holes, gravitational Chern-Simons terms

1 Introduction

Black holes are probably the most spectacular prediction of General Relativity. From a theoretical perspective, a crucial moment which lent credibility to the assumption of their existence in reality was Kerr's analytic construction of a stationary rotating black hole solution in Einstein gravity in four spacetime dimensions [1]. With the development of string theory and other extra dimensions and/or higher derivative theories, it has become important to extend the Kerr solution to higher number of dimensions D and/or to more general diffeomorphism covariant theories of gravity. The generalization to $D > 4$, in Einstein gravity, was done by Myers and Perry in [2]. Since then, a number of corresponding black hole solutions in different supergravity theories were constructed (for reviews see, e.g., [3, 4]). However, despite a lot of effort, there is still not a single explicit analytic black hole solution in any generalized theory of gravity with higher curvature terms in the action. A related problem, important also on phenomenological grounds, is that one would like to have dynamical solutions, e.g., with in-falling matter, in which a Kerr black hole is created; however so far none has been found.

It is not hard to locate the roots for this failure of extending the Kerr solution in the abovementioned directions. The Kerr solution (and its Myers-Perry generalization) belongs to a special class of spacetimes for which the metric can be written in Kerr-Schild form with flat seed metric. This dramatically reduces the number of unknown functions from the start. The failure of attempts that used the Kerr-Schild ansatz in some higher-curvature theories of gravity shows that the ansatz has limited use for black hole constructions, and that the Einstein action is somewhat special in this respect. Without some alternative simplifying property of the metric, the task of finding analytic stationary rotating black hole solutions in any $D > 3$ theory seems to be hopeless. A possible strategy is to turn to different types of perturbative calculations, with the hope of extracting some information which could be useful for nonperturbative constructions.

In this paper we study asymptotically flat stationary rotating black hole solutions in theories with purely gravitational Chern-Simons terms [5] in the action in $D > 3$ spacetime dimensions. One can name several reasons why these terms are interesting by themselves, including their special properties. Though they give diffeomorphism covariant contribution to the equations of motion [6, 7], they are not manifestly diff-covariant. This leads to interesting consequences, e.g., for the black hole entropy [8, 9] and anomalies for the boundary theories (as in AdS constructions) [6]. Topological considerations [10] become relevant due to these terms, which moreover break parity in the purely gravitational sector. Gravitational Chern-Simons terms are present in some superstring/M theory low energy effective actions (depending on type and compactification), and though they appear more frequently in the form of mixed gauge-gravitational Chern-Simons Lagrangian terms,[1] some compactifications to 7-dimensional spacetime may lead to purely gravitational Chern-Simons Lagrangian terms. It should be recalled that, despite the mentioned recent developments, there is much less understanding of the consequences of gravitational Chern-Simons terms in $D > 3$, then in the simplest case of $D = 3$ [18, 19] which has been thoroughly studied in the literature (for the reviews see [20, 21, 12]). One of the aims of this paper is to try and fill some of these gaps.

The contribution to the equations of motion due to gravitational Chern-Simons Lagrangian terms is, at least apparently, terribly involved in $D > 3$. Such terms exist only in $D = 4k - 1$, $k \in \mathbb{N}$, which implies that stationary rotating black holes are characterized by $2k - 1$ angular momenta. However, due to their special properties, connected to parity violation, it is possible to obtain some exact results. For example, we show that if the solution for the metric has "enough" isometries (which, in the case of interest here, typically occurs when two or more angular momenta vanish) then adding a gravitational Chern-Simons term in the action does not change the black hole solutions. So, to find situations where a gravitational Chern-Simons contribution is nontrivial, one has to consider rotating black holes with at least $2k - 2$ nonvanishing angular momenta. This is very complicated already in $D = 7$. For this reason

[1] The role of mixed gauge-gravitational Chern-Simons terms for black hole constructions in superstring effective theories is reviewed in [11, 12, 13, 14, 15]. In some cases it was shown that all higher-derivative α'-corrections to near-horizon properties of extremal black holes are originated solely by such Chern-Simons terms, though low energy effective actions contain infinite number of higher-derivative terms [16, 17].

we have turned to perturbative calculations in a special case, that of a $D = 7$ solution in which all angular momenta are equal. We have constructed the lowest order corrections to the Myers-Perry metric in an expansion in the Chern-Simons coupling constant and angular momentum, and we have showed that the gravitational Chern-Simons term affects all the black hole characteristics we have calculated − horizon, ergoregion and black hole entropy (at least in this perturbative sense). Our perturbative solution does not allow expressing the metric in Kerr-Schild form with a flat seed metric. This implies that to find exact analytic solutions, if they exist, in such more general theories with gravitational Chern-Simons Lagrangian terms, one needs a new ansatz.

The outline of the paper is as follows. Section 2 is devoted to establishing some general results. We show that a gravitational Chern-Simons Lagrangian term does not change stationary rotating black hole solutions and the corresponding black hole entropy if two or more angular momenta are zero. This is a consequence of the more general theorem derived in [22]. In Section 3 we specialize to the particular theory in $D = 7$ obtained by adding a gravitational Chern-Simons Lagrangian term to Einstein-Hilbert action. In Section 4 we turn to the perturbative calculation in the Chern-Simons coupling constant, in the special case when all three angular momenta are equal. A few Appendices are devoted to details of calculations.

2 A few general considerations

We are interested in gravity theories in $D = 2n - 1$ dimensions ($n \in 2\mathbf{N}$) with Lagrangians of the form

$$\mathbf{L} = \mathbf{L}_0 + \lambda\, \mathbf{L}_{\mathrm{gCS}} \tag{1}$$

where $\mathbf{L}_0$ is some general manifestly diffeomorphism-invariant Lagrangian density and $\mathbf{L}_{\mathrm{gCS}}$ is the purely gravitational Chern-Simons (gCS) Lagrangian density given by

$$\mathbf{L}_{\mathrm{gCS}} = n \int_0^1 dt\, \mathrm{str}(\mathbf{\Gamma}\, \mathbf{R}_t^{n-1}) \tag{2}$$

Here $\mathbf{R}_t = t d\mathbf{\Gamma} + t^2 \mathbf{\Gamma}\mathbf{\Gamma}$, $\mathbf{\Gamma}$ is the Levi–Civita connection and str denotes a symmetrized trace, which is an example of an invariant symmetric polynomial of the Lie algebra of the $SO(1, D-1)$ group. In (1) λ denotes the gCS coupling constant, which is dimensionless and may be quantized [10, 20, 24]. Since the $n = 2$ ($D = 3$) case is studied in detail in the literature, we shall focus on $n \geq 4$ cases.

Adding gravitational CS terms to the Lagrangian brings about additional terms in the equations of motion. It was shown in [6] that the equation for the metric tensor $g_{\alpha\beta}$ acquires an additional term $C^{\alpha\beta}$ which, for the gCS term (2), is of the form

$$C^{\alpha\beta} = -\frac{n}{2^{n-1}}\, \epsilon^{\nu_1 \cdots \nu_{2n-2}(\alpha}\, \nabla_\rho \left(R^{\beta)}{}_{\sigma_1\nu_1\nu_2}\, R^{\sigma_1}{}_{\sigma_2\nu_3\nu_4} \cdots R^{\sigma_{n-3}}{}_{\sigma_{n-2}\nu_{2n-5}\nu_{2n-4}} R^{\sigma_{n-2}\rho}{}_{\nu_{2n-3}\nu_{2n-2}} \right) \tag{3}$$

The tensor $C^{\alpha\beta}$ is symmetric, traceless and covariantly conserved

$$C^{\alpha\beta} = C^{\beta\alpha} \;, \qquad C^\alpha_\alpha = 0 \;, \qquad \nabla_\alpha C^{\alpha\beta} = 0 \tag{4}$$

In $D = 3$ $C^{\alpha\beta}$ is known as Cotton tensor, and in higher dimensions it can be regarded as some sort of generalization thereof [6].

The peculiar properties of gCS terms make them rather special. They have a topological character (leading to quantization of their coupling constant), they are not manifestly diffeomorphism covariant but their contribution to equations of motion (3) is diff-covariant, they are parity-odd, and conformally covariant [5, 6]. We are interested in investigating how they affect black hole solutions found in theories where they are absent, once they are added to the theory. However, as we elaborated in [22, 7], it appears that it is not easy to find physically interesting configurations for which the gCS contribution to the equations of motion (3) is nonvanishing and are at the same time simple enough to be analytically tractable.[2] In [22] we proved a theorem for any metric in D dimensions of the form

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = g_{ab}(y)\, dy^a dy^b + f(y)\, h_{ij}(z)\, dz^i dz^j \tag{5}$$

where local coordinates are split as $x^\mu = (y^a, z^i)$, $\mu = 1, \ldots, D$, $a = 1, \ldots, d$, and $i = 1, \ldots, p$ $(d+p = D)$, and $g_{ab}(y)$ and $h_{ij}(z)$ are arbitrary tensors depending only on the $\{y^a\}$ and $\{z^i\}$ coordinates, respectively. It turns out that if $d > 1$ and $p > 1$ the gCS contribution to the equations of motion vanishes, i.e.,

$$C^{\mu\nu}[g] = 0 \ . \tag{6}$$

Due to the conformal covariance of the $C^{\mu\nu}$ tensor, the theorem extends to any metric which is conformally equivalent to (5).

As discussed in [22], this theorem covers many classes of metrics usually discussed in the literature. In particular, it also applies to all spacetimes with local $SO(k)$ isometry, with $k \geq 3$. It appears that if we want to study gCS Lagrangian terms with nontrivial influence, stationary rotating asymptotically flat black hole solutions are the next simplest objects.

We shall be interested also in thermodynamics of black holes. It was shown in [8, 9] that a gCS Lagrangian term (2) brings in an additional term in the black hole entropy formula. For a theory with Lagrangian (1) the latter is given by

$$S = S_0 + \lambda S_{\text{gCS}} \ . \tag{7}$$

S_0 is Wald black hole entropy [25] due to the Lagrangian $\mathbf{L}_0$. In coordinate systems of the type standardly used in the literature (like the generalized Boyer-Lindquist type of coordinates we use in this paper) S_{gCS} can be calculated from

$$S_{\text{gCS}}[g] = 4\pi n \int_{\mathcal{B}} \mathbf{\Gamma}_N \mathbf{R}_N^{n-2} \ , \tag{8}$$

where $\mathcal{B}$ is the $(D-2)$-dimensional bifurcation surface of the black hole horizon [9]. In a forthcoming paper, [22], by using conformal invariance of (8), we shall prove a theorem according to which, for black hole metrics of the form (5), with $p \geq 1$ and coordinates z tangential to the bifurcation surface of the horizon, the gCS entropy term (8) vanishes.

Using the just mentioned theorems, we can already state one general result. If for stationary rotating black hole p of angular momenta J_i are zero, then the spacetime usually has $SO(2p)$ isometry. Let us restrict to the cases in which this is valid.[3] Then, if $p \geq 2$ such spacetime falls under the class of the above theorems guaranteeing $C^{\mu\nu} = 0$ and $S_{\text{gCS}} = 0$. This leads us to the following clearcut statement:

If in the theory with some arbitrary Lagrangian $\mathbf{L}_0$, a solution has two or more vanishing angular momenta J_i, then introducing a Lagrangian gCS term (as in (1)) does not change the solution nor the corresponding black hole entropy.

If the black hole solution with only one vanishing angular momentum is also of the form (5), then by the second theorem the gCS entropy term (8) again vanishes. However, though this indeed applies to *all known* stationary rotating black hole solutions (e.g., the Myers-Perry black holes we discuss in the next section), for the general Lagrangian (1) there is no guarantee that solutions with only one angular momentum vanishing are of the form (5). Indeed, we shall show in the next section on an explicit example that, when only one angular momentum is vanishing, a gCS term, due to its parity-odd structure, forces the solution to depart from the form (5).

---

[2]Notable exceptions are nontrivial analytically tractable solutions obtained in [24] by "squashing" maximally symmetric spaces. Such solutions may play a role in AdS/CFT constructions.

[3]We restrict ourselves here to "standard" black holes with horizon topology given by a sphere S^{D-2}. In this case the above symmetry statement is valid when there is no matter outside the horizon. However, it can be violated if there is matter with symmetry breaking energy-momentum tensor (e.g., rigid matter which does not rotate in corresponding directions but with the shape which breaks the $SO(2p)$ isometry). Such systems are excluded in our analysis.

In conclusion, we see that if we want to study the problem in which gCS Lagrangian terms have non-trivial influence on stationary rotating black hole solutions, we cannot take more then one angular momentum to be zero, because in those cases both solution and entropy are unchanged when we "switch on" coupling constant λ in (1). If only one angular momentum is zero, the solution is generally affected, but the first order correction in gCS coupling λ of the gCS entropy term vanishes. So, to find a *completely* non-trivial problem, in which all interesting ingredients are non-vanishing, we need to analyze black holes with all angular momenta nonvanishing. If we add to this that in $D = 3$ dimensions it is known that a gCS term does not change rotating black hole solutions such as BTZ black hole (though it contributes to horizon and asymptotic charges such as entropy, mass and angular momentum), it follows that we have to go to $D \geq 7$ dimensions.

3 Stationary rotating black holes in $D = 7$

Following the conclusion of the previous section, from now on we specialize to the simplest non-trivial case with action

$$\mathbf{L} = \mathbf{L}_0 + \lambda\,\mathbf{L}_{\mathrm{gCS}} = \frac{1}{16\pi G_N}\boldsymbol{\epsilon}R + \lambda\,\mathbf{L}_{\mathrm{gCS}}. \tag{9}$$

Such theory in $D = 3$ is known as topologically massive gravity and was first considered in [18, 19]. We are interested in finding stationary rotating asymptotically flat black hole solutions in $D = 7$.

3.1 Myers-Perry black holes

For $\lambda = 0$ we have ordinary general relativity with Einstein-Hilbert Lagrangian for which stationary rotating asymptotically flat black holes, with the horizon topology of the 5-sphere S^5, are described by Myers-Perry solutions (MP BH) [2, 26]. Here we review the basic properties of Myers-Perry solutions we shall need in our calculations.

In generalized Boyer-Lindquist coordinates the MP metric in $D = 7$ is given by

$$ds^2_{\mathrm{MP}} = -dt^2 + \frac{\mu\, r^2}{\Pi\, F}\left(dt - \sum_{i=1}^{3} a_i \mu_i^2\, d\phi_i\right)^2 + \frac{\Pi\, F}{\Pi - \mu\, r^2}dr^2 + \sum_{i=1}^{3}(r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) \tag{10}$$

where

$$F = F(r, \vec{\mu}) = 1 - \sum_{i=1}^{3} \frac{a_i^2\, \mu_i^2}{r^2 + a_i^2}\ , \qquad\qquad \Pi = \Pi(r) = \prod_{i=1}^{3}(r^2 + a_i^2) \tag{11}$$

and the coordinates μ_i are not all independent but satisfy

$$\sum_{i=1}^{3} \mu_i^2 = 1\ \ . \tag{12}$$

From the asymptotic behavior of the metric (10) it can be shown [2] that four free parameters μ and a_i ($i = 1, 2, 3$) determine the mass M and angular momenta J_i with

$$M = \frac{5\,\pi^2}{16\,G_N}\mu\ \ , \tag{13}$$

$$J_i = \frac{\pi^2}{8\,G_N}\mu\, a_i = \frac{2}{5}M a_i\ \ . \tag{14}$$

We shall assume $\mu > 0$ from now on. The event horizon of the MP BH is located at $r = r_H$ where the horizon radius r_H is the largest solution of the polynomial equation

$$\Pi(r_H) - \mu\, r_H^2 = 0 \quad . \tag{15}$$

Eq. (15) is a cubic equation in r^2, with three solutions which we denote $r^2_{\rm min}$, r^2_-, and $r^2_{\rm max} \equiv r^2_H$. The exact expressions for roots is rather awkward (see [27]) and we shall not use it. For later purposes we note the obvious relation (obtained from one of Vieta's formulae)

$$r^2_{\rm min}\, r^2_-\, r^2_H = -(a_1 a_2 a_3)^2 \quad . \tag{16}$$

To keep our analysis simple we restrict to the case in which the largest solution satisfies $r^2_{\rm max} = r^2_H > 0$.[4] A necessary, but not sufficient, condition for this is $\mu > \sum_i \prod_{j\neq i} a_j^2$. In this case all the roots are real, and satisfy $r^2_{\rm min} < 0 \leq r_- \leq r^2_H$. The surface defined by $r = r_-$ is the inner horizon, which is hidden from the outside observer by event horizon $r = r_H$.

Using (15) and (10) one obtains that the horizon area is given by

$$A_H = \pi^3 \mu\, r_H \quad . \tag{17}$$

The ergosurface is an infinite redshifted surface, located outside the event horizon, defined by the condition $g_{tt} = 0$, which for MP BH metric (10) leads to an equation

$$\Pi(r)\, F(r, \vec{\mu}) = \mu\, r^2 \quad . \tag{18}$$

As we are interested in black hole thermodynamics, let us quote that the entropy S, temperature T, and angular velocities Ω_i of the MP BH are given by

$$S = \frac{A_H}{4\,G_N} = \frac{\pi^3}{4\,G_N}\mu\, r_H \quad , \tag{19}$$

$$T = \frac{\kappa}{2\pi} = \frac{\Pi'(r_H) - 2\mu\, r_H}{4\pi\mu\, r_H^2} \quad , \tag{20}$$

$$\Omega_i = \frac{a_i}{r_H^2 + a_i^2} \quad . \tag{21}$$

MP black holes with coincident inner and outer horizon radii, $r_- = r_H$, obviously have $T = 0$, which means that they are extremal black holes.

A general MP BH in $D = 2m + 1$ with generic choice of parameters μ and $\vec{a}$ is quite complicated to analyze. One reason is that for generic choice of the parameters μ and $\vec{a}$ one has a rather "modest" isometry group $\mathbf{R} \times U(1)^m$. There are two mechanisms by which one can straightforwardly enlarge the isometry group in a simple way and/or simplify calculations:

(a) Taking k of the angular momenta J_i vanishing, which for a MP black hole means taking the corresponding a_i to vanish. This enlarges the factor $U(1)^k$ to $SO(2k)$ in the isometry group.

(b) Taking k of the angular momenta J_i to be equal, which for a MP black hole means taking the corresponding a_i to be equal. This enhances the factor $U(1)^k$ to $U(k)$. If all J_i are equal, then we obtain cohomogeneity-1 metrics in which all "angular" dependence is determined, and the only freedom left is in a number of functions of the radial coordinate r.

In case (a), already if just one $a_j = 0$, a direct consequence is that the radius of the inner horizon is $r_- = 0$, and the polynomial in (15) is of one order smaller, which simplifies solving for the event horizon radius r_H. In the case of our main interest, $D = 7$, by taking $a_3 = 0$ we obtain

$$r_H = \frac{1}{\sqrt{2}} \left(-(a_1^2 + a_2^2) + \sqrt{4\mu + (a_1^2 - a_2^2)^2} \right)^{1/2} \tag{22}$$

[4]For a discussion of the subtleties of extending spacetime to the $r^2 < 0$ region see [2] and a review [26].

where a (necessary and sufficient) condition to have $r_H^2 > 0$ is $\mu > a_1^2 a_2^2$. We can now make further simplifications either by applying (a) again, or (b). By taking also $a_2 = 0$ the isometry group is enlarged from $\mathbf{R} \times U(1)^3$ to $\mathbf{R} \times U(1) \times SO(4)$. If, on the other hand, we restrict to $a_1 = a_2 \equiv a$, then the symmetry is enlarged to $\mathbf{R} \times U(1) \times U(2)$ and we obtain a simple expression for r_H

$$r_H = \left(\sqrt{\mu} - a^2\right)^{1/2} \tag{23}$$

Another variant of the possibility (c) in $D = 7$ is to have all three parameters a_i equal, $a_1 = a_2 = a_3 \equiv a$, with isometry group $\mathbf{R} \times U(3)$. From (15) and $\mu > 0$ then it follows that $r_H^2 > 0$ requires $\mu > 27\, a^4/4$. From (11) and (12) it follows that F is a function of r only

$$F = F(r) = 1 - \frac{a^2}{r^2 + a^2} = \frac{r^2}{r^2 + a^2} \tag{24}$$

which, together with (18), yields an especially simple expression for the location of the ergosurface: $r = r_e$, where

$$r_e = \left(\sqrt{\mu} - a^2\right)^{1/2} \tag{25}$$

3.2 Adding gCS Lagrangian terms: exact results

We now turn our attention to the full Lagrangian (9) with $\lambda \neq 0$, for which we would like to find solutions describing stationary rotating black holes which we denote $\bar{g}_{\mu\nu}$. Equations of motion now read

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R - 16\pi\, G_N \lambda\, C^{\alpha\beta} = 0 \tag{26}$$

where $C_{\mu\nu}$ is the contribution of the gCS term which in $D = 7$ is obtained by putting $n = 4$ in (3)

$$C^{\alpha\beta} = -\frac{1}{2}\, \epsilon^{\nu_1 \cdots \nu_6 (\alpha}\, \nabla_\rho \left(R^{\beta)}{}_{\sigma_1 \nu_1 \nu_2}\, R^{\sigma_1}{}_{\sigma_2 \nu_3 \nu_4}\, R^{\sigma_2 \rho}{}_{\nu_5 \nu_6} \right) \tag{27}$$

Contracting (26) with $g_{\alpha\beta}$ and using the fact that $C^{\alpha\beta}$ is traceless, (4), it follows that $R = 0$. Inserting this back in (26) we obtain the equations of motion in simpler form

$$R^{\alpha\beta} - 16\pi\, G_N \lambda\, C^{\alpha\beta} = 0 \ . \tag{28}$$

The entropy is given by

$$S[\bar{g}] = S_0[\bar{g}] + \lambda S_{\mathrm{gCS}}[\bar{g}] = \frac{A_H[\bar{g}]}{4\, G_N} + 16\pi\lambda \int_{\mathcal{B}} \mathbf{\Gamma}_N[\bar{g}] \mathbf{R}_N[\bar{g}]^2 \ , \tag{29}$$

where $A_H[\bar{g}]$ is the horizon area calculated from the metric $\bar{g}_{\mu\nu}$ which is a solution to the full equations of motion (28). It is convenient for later discussions to write solutions of (28) in the following form

$$\bar{g}_{\alpha\beta} = g_{(0)\alpha\beta} + \delta g_{\alpha\beta} \ , \qquad g_{(0)\alpha\beta} = (g_{\mathrm{MP}})_{\alpha\beta} \tag{30}$$

where g_{MP} is Myers-Perry black hole, which is a solution for $\lambda = 0$. For a generic MP black hole metric we obtain (see Appendix A.3)

$$S_{\mathrm{gCS}}[g_{\mathrm{MP}}] = 128\, \pi^4 \frac{\mu}{r_H} a_1 a_2 a_3 \left(\sum_{i=1}^{3} \frac{1}{r_H^2 + a_i^2} \right)^3 \tag{31}$$

Observe that (31) automatically vanishes when one or more angular momentum parameters a_i vanish. The result (31) is especially interesting when $\delta g_{\alpha\beta} = 0$, in which case it gives the full gCS contribution to the black hole entropy. In generic cases, when $\delta g_{\alpha\beta} \neq 0$, it gives a part of the first-order correction to the black hole entropy in the perturbative expansion in λ (the second part comes from $S_0[\bar{g}]$ term in (29).)

There is little hope to find exact solutions with generic angular momenta of such highly involved field equation as (28)-(27). We now turn to analysis of special cases with enhanced isometry group, and thereafter we turn to perturbative calculations.

3.2.1 $J_2 = J_3 = 0$, $J_1 \neq 0$

Let us us start with the most symmetric case involving rotating black holes in $D = 7$. As noted in Sec. 3.1, when two angular momenta are zero, e.g., $J_2 = J_3 = 0$, the symmetry enhances to $\mathbf{R} \times U(1) \times SO(4)$. From the general discussion in Sec. 2 we then know that both solution and the black hole entropy remain the same as in the $\lambda = 0$ case. This means

$$\bar{g}_{\alpha\beta} = g_{(0)\alpha\beta} = (g_{\rm MP})_{\alpha\beta} \tag{32}$$

and

$$S[\bar{g}] = S_0[g_0] = \frac{A_H[g_0]}{4\,G_N} \quad . \tag{33}$$

where S_0 is Bekenstein-Hawking entropy and g_0 is the MP black hole with $a_2 = a_3 = 0$. As a check, we see that result (33) also follows directly from (31). If we want to see nontrivial effects of the gCS Lagrangian term we have to go to less symmetric cases.

3.2.2 $J_3 = 0$, $J_1 \neq 0$, $J_2 \neq 0$

Now we take just one vanishing angular momentum, e.g., J_3 (so $J_3 = 0$, $J_1 \neq 0$, $J_2 \neq 0$). In this case in general there is no important enhancement of the symmetry group of isometries. For the corresponding MP black hole by explicit calculation we have established that

$$C^{\alpha\beta}[g_{\rm MP}] \neq 0 \;\; , \qquad \text{when} \;\; J_3 = 0 \; , \; J_1 \neq 0 \; , \; J_2 \neq 0 \tag{34}$$

so a gCS contribution to the equations of motion are in this case nontrivial and MP black holes are no longer solutions, i.e.,

$$\bar{g}_{\alpha\beta} \neq (g_{\rm MP})_{\alpha\beta} \;\; , \qquad \text{when} \;\; J_3 = 0 \; , \; J_1 \neq 0 \; , \; J_2 \neq 0 \tag{35}$$

The equations of motion still look too complicated to offer much hope for finding exact solutions. However, we can get some information from a perturbative analysis. Direct calculation shows that nonvanishing components in (34) are $C^{t\phi_3}[g_{\rm MP}]$, $C^{\phi_1\phi_3}[g_{\rm MP}]$ and $C^{\phi_2\phi_3}[g_{\rm MP}]$, which shows that a perturbative solution (around $\lambda = 0$) is not of the form (5) when $\lambda \neq 0$.

Let us turn our attention to the black hole entropy. If we plug the MP metric with $J_3 = 0$ (which means $a_3 = 0$) into the gCS entropy term, from (31) we obtain

$$S_{\rm gCS}[g_{\rm MP}] = 0 \;\; , \qquad \text{(for a MP BH with } a_3 = 0\text{)} \tag{36}$$

It follows that up to first-order in a perturbative expansion in λ, the black hole entropy is given by Bekenstein-Hawking area formula. However, as a perturbed solution is not of the form (5), it is possible that a gCS entropy term gives nonvanishing contribution starting from second order in λ.

3.2.3 $J_i = J \neq 0$ for all $i = 1, 2, 3$

The case in which all angular momenta are equal and nonvanishing deserves a special place. On the one hand, it keeps all the non-trivial consequences of the most generic case. This means that all quantities (except charges defined at asymptotic infinity), both geometric and thermodynamic, are affected by the presence of the gCS Lagrangian term.[5] On the other hand the symmetry group of isometries enhances to $\mathbf{R} \times U(3)$ which induces significant constraints on the metric. This combination makes this case an ideal laboratory for calculations, and we shall explore it in detail perturbatively in Sec. 4.

We have already shown how results in this case simplify for $\lambda = 0$, which is for MP black holes with $a_i = a \neq 0$, $i = 1, 2, 3$. Let us just note that the result (31) also simplifies and becomes

$$S_{\rm gCS}[g_{\rm MP}] = 3456\,\pi^4 \left(\frac{a}{r_H} \right)^3 \tag{37}$$

where r_H is horizon radius of MP black hole.

[5] We shall show this explicitly in Sec. 4.

3.2.4 gCS terms and interior of black holes

Here we pause for the moment to address an interesting issue raised in [23] on the basis of 3-dimensional analysis, which can be put as a question "Does gravitational Chern-Simons terms see interior of black holes?". We shall argue here that in $D > 3$ the answer is negative, and that the apparently positive answer in $D = 3$ is probably a coincidence.

Let us first state the issue. It is known that in $D = 3$ the Hilbert-Einstein action supplemented with a negative cosmological constant term leads to the BTZ solutions [28] describing stationary rotating black holes. The difference with our problem, aside from the number of dimensions, is the presence of the negative cosmological constant term $\Lambda = -1/\ell^2$ (which is necessary in $D = 3$ if we want to have black hole solutions at all) implying that BTZ solutions are asymptotically AdS. Including a gCS Lagrangian term in $D = 3$ does not affect stationary rotating black hole solutions (they are still BTZ) but does change the entropy, which can be written in the form [23] (in Appendix B one can find a short review of $D = 3$ case)

$$S = \frac{A_H}{4\,G_N} - \mathrm{sign}(j)\frac{\beta}{\ell}\frac{A_-}{4\,G_N} \quad , \qquad \beta \equiv 32\pi G_N \lambda \tag{38}$$

where ℓ is the standard parametrization of the cosmological constant, A_- is the area of the inner horizon, and j is angular momentum parameter We see that the gCS contribution to the entropy is divided into two parts, one coming from Einstein-Hilbert action and depending only on a geometrical property (area) of the (outer) event horizon, and one coming from gCS Lagrangian term and depending only on a geometrical property (again area) of the inner horizon. In [23] it was speculated that this may not be coincidental but indicates that a gCS term "seems to see interior of the black hole".

We investigate here the same assertion in $D > 3$. As a warm-up, let us see what happens when one or more angular momenta vanish. Applying the general analysis of Sec. 2, we know that when two or more angular momenta vanish, (i) the solution is unchanged, so it is given by the corresponding Myers-Perry metric (10) (with two or more parameters a_i vanishing), (ii) $S_{\mathrm{gCS}} = 0$. From (i) and (16) it follows that radius and area of inner horizon vanish. This, combined with (ii), shows that (38) is true in this case. However, we shall now show that this is not true for generic angular momenta when $S_{\mathrm{gCS}} \neq 0$. We note that in $D > 3$ it is impossible to have $\delta g_{\alpha\beta} = 0$ while at the same time $S_{\mathrm{gCS}} \neq 0$, a situation present for $D = 3$ BTZ black holes. It is this fact that makes the analysis more complicated, and in fact we are forced to turn to a perturbative expansion.

We treat the gCS coupling λ as a perturbation parameter. In this way a solution for the metric to (28)-(27) can be expanded as

$$\bar{g}_{\alpha\beta} = \sum_{k=0}^{\infty} \lambda^k \, g^{(k)}_{\alpha\beta} \quad , \qquad\qquad g^{(0)}_{\alpha\beta} = (g_{\mathrm{MP}})_{\alpha\beta} \tag{39}$$

where $(g_{\mathrm{MP}})_{\alpha\beta}$ is the (generic) MP metric. Using this in (29) one can obtain a similar expansion in λ for the black hole entropy. However, we prefer here to write this expansion in the following way

$$S[\bar{g}] = \frac{A_H[\bar{g}]}{4\,G_N} + \lambda\, S_{\mathrm{gCS}}[\bar{g}] = \frac{A_H[\bar{g}]}{4\,G_N} + \lambda\, S_{\mathrm{gCS}}[g^{(0)}] + O(\lambda^2) \tag{40}$$

Comparing with (38) we see that since $S_{\mathrm{gCS}}[\bar{g}]$ should be some function of intrinsic geometric quantities connected to the inner horizon of the solution $\bar{g}_{\alpha\beta}$ (like, e.g., $A_-[\bar{g}]$), then $S_{\mathrm{gCS}}[g^{(0)}]$ should give the same for the MP metric $g^{(0)}_{\alpha\beta}$.

We have already calculated this in $D = 7$ and the result is presented in Eq. (31). We have not found any interpretation of this result in terms of geometric quantities linked to the inner horizon, or more generally, in terms of some other simple geometrical properties interior to event horizon r_H. This conclusion does not change if we generalize to (A)dS black holes (by introducing a cosmological constant Λ in Lagrangian $\mathbf{L}_0$), at least not for generic values of Λ.[6]

[6] This follows simply from the fact that the limit $\Lambda \to 0$ is well-defined and smooth in $D > 3$, so it leads to our asymptotically flat results and corresponding conclusions.

Why and how the area of the inner horizon appears in $D = 3$ in (38)? For our argument it is enough to restrict our attention to the more symmetric case in which all angular momenta are equal, which for MP black holes in $D = 2m+1$ dimensions (m is odd integer) requires $a_i = a$, $i = 1, \ldots, m$. Let us assume that the formula (37) generalizes to

$$S_{\rm gCS}[g^{(0)}] = c_m \left(\frac{a}{r_H} \right)^m \tag{41}$$

where c_m are some constants.[7]

In $D = 3$ ($m = 1$) $g^{(0)}$ is BTZ black hole metric, for which one has

$$a = (r_H r_-)/\ell \ , \tag{42}$$

where r_- is again radius of the inner horizon. Using (42) one obtains

$$S_{\rm gCS}[g_{\rm BTZ}] = c_1 \frac{a}{r_H} = c_1 \frac{r_-}{\ell} = \frac{c_1}{2\pi} \frac{A_-}{\ell} \tag{43}$$

which is in fact the way how Eq. (38) is originally obtained. However, the above mechanism is not possible in $D > 3$, because generally

$$\prod_{i=1}^{m} |a_i| = \prod_{i=1}^{m} |r_i^2|^{1/2} \tag{44}$$

in the asymptotically flat case ($\Lambda = 0$), and

$$\prod_{i=1}^{m} |a_i| = \frac{1}{\ell} \prod_{i=1}^{m+1} |r_i^2|^{1/2} \tag{45}$$

in the asymptotically (A)dS case ($\Lambda = \pm 1/\ell^2$), where r_i^2 are complete set of roots of the horizon-defining polynomial equation (Eq. (15) in $\Lambda = 0$ case). Only in $D = 3$ ($m = 1$) (where there are only AdS black holes) one has $a^m = r_H r_-$, so that after dividing by r_H one is left with r_- alone in (41).[8] Other roots, aside r_H and r_-, are not defining other inner horizons and are, as far as we know, deplete of any direct geometrical meaning. We now see that the fact that in $D = 3$ one has $S_{\rm gCS} \propto A_-$ is probably just a coincidental consequence of the more fundamental relation (41).

4 Perturbative calculations in $D = 7$: case $a_i = a$

4.1 Is perturbative expansion in λ viable?

Searching for exact solutions to the equations of motion (28)

$$R_{\nu\sigma}[\bar{g}] = 16\pi G_N \lambda \, C_{\nu\sigma}[\bar{g}] \ , \tag{46}$$

where $G_{\nu\sigma}$ is the Einstein tensor and $C_{\nu\sigma}$ the contribution of gCS Lagrangian term (3), is probably futile. So we would like to turn to a perturbative analysis. But, of course, we have to be sure that a perturbative expansion in the gCS coupling λ makes sense at all. Due to topological reasons it was argued

[7] However simple (41) may look, for $D > 3$ ($m > 1$) we again did not managed to find any interpretation for it purely in terms of intrinsic geometric properties of the inner horizon $r = r_-$.

[8] In general $D = 2m + 1$, m odd, dimensional analysis tells us there could be terms in $S_{\rm gCS}$ of the form $a^m/(\ell^{m-1} r_H)$, in which r_H cancels. However, only in $D = 3$ one is left just with r_-.

in the literature [20, 24, 10] that only for special discrete ("quantized") values of λ, defined through some "quantization condition" of the form

$$\lambda_n = n\,\lambda_1 \ , \qquad\qquad n \in \mathrm{Z} \ , \tag{47}$$

can one give unambiguous meaning to a gCS term in the action.[9] The value of the constant λ_1 depends on what is exactly the space of allowed configurations. Taken at face value, this quantization may invalidate perturbation theory in λ.

We would like to argue that even if (47) is correct[10], perturbation theory in λ can be made meaningful. One can achieve this by scaling additional parameters of the theory, which for the stationary black holes are G_N, μ and a_i. As in this case there are two independent dimensionless parameters, there are several ways one can do this. We present two possibilities:[11]

(a) We take as two independent dimensionless parameters $c_{\lambda N} \equiv \lambda G_N/\mu^{5/4}$ and $(a/\mu^{1/4})$, and take $c_{\lambda N} \ll 1$ by making the scaling parameter $G_N/\mu^{5/4}$ sufficiently small while keeping a and μ fixed. It is obvious that an expansion in λ can be trivially written as an expansion in $c_{\lambda N}$. This is the well-known scenario when one takes Planck length $l_{\mathrm{Pl}} = G_N^{1/5}$ to be much smaller then physical scales in the problem.

(b) We define a dimensionless parameter $c_{\lambda a} \equiv \lambda\, G_N a/\mu^{3/2}$, and take $c_{\lambda a} \ll 1$ by making the scaling parameter $(a/\mu^{1/4})$ sufficiently small while keeping $G_N/\mu^{5/4}$ fixed. This is meaningful because, as we show below, one can write the expansion in λ as an expansion in $c_{\lambda a}$ with good convergence properties for small $a/\mu^{1/4}$.

We are interested here in the case (b). Let us first discuss two subtleties. In both cases, (a) and (b), we can formally treat the expansion in λ independently of the expansions of other quantities which are small in the relevant scaling parameters ($G_N/\mu^{5/4}$ and $a/\mu^{1/4}$, respectively). This is because one can make the effective coupling $c_{\lambda_n} \ll 1$ for arbitrarily high n in quantization law (47), by making the relevant scaling parameter sufficiently small. However, for specifically chosen λ_n, at the end of calculation one should group all the terms with the same powers of the small scaling parameters ($G_N/\mu^{5/4}$ and $a/\mu^{1/4}$, respectively).

We would like to argue that the claim in (b) is sound. We start from the equations of motion (46) and consider a perturbative solution in λ around Myers-Perry metric (10). It is obvious that a perturbative expansion for the metric can be written in the form

$$\bar{g}_{\nu\sigma} = \sum_{k=0}^{\infty} c_{\lambda N}^k \, g_{\nu\sigma}^{(k)} \tag{48}$$

where $c_{\lambda N} = \lambda\, G_N/\mu^{5/4}$, $g_{\nu\sigma}^{(0)}$ is MP black hole solution with all parameters a_i equal, $a_i = a$, and $g_{\nu\sigma}^{(k)}$ depend on μ and a (but not on λ and G_N). We assume that $c_{\lambda N}$ is small enough so that expansion (48) is convergent. If λ is quantized, and so assumes finite value from the set (47), one can make $c_{\lambda N}$ small as we like by appropriately tuning Newton's constant G_N.

Now we want to show that in the perturbative expansion every power of λ is accompanied by a factor of a. Following formally a standard procedure we insert (48) in (46) and collect terms with the same power of λ. It is important to note that $g_{\nu\sigma}^{(0)}$ is analytic in a around $a = 0$, as are all operators obtained

[9] For $D = 3$ it was argued in [20], for $D = 7$ in [24], and for general case in [10]. The argument is based on a standard application of path-integral quantization to gravity.

[10] One way to counter (47) is by noting that the argument used in obtaining (47) is quantum mechanical, and assumes that "naive" path integral formulation of gravity in which one integrates over metrics (or connections and vielbeins) is meaningful in nonperturbative regime. This is normally a standard quantization prescription, but gravity is hardly "normal" theory, especially in $D > 3$ where general relativity cannot be put in the form of the gauge theory. Indeed, we know basically nothing for sure about quantum gravity, so a skeptical view on the correctness of the quantization of gCS coupling constant is not unmotivated.

[11] For sake of clarity, we restrict ourself here to the case where all parameters a_i are equal, $a_i = a$.

by expanding both sides in (46). This allows us to make Taylor expansions in a. In the first order one gets (we show this explicitly in Sec. 4.2)

$$G'[g^{(0)}] \cdot g^{(1)} = C[g^{(0)}] \tag{49}$$

where, for the sake of simplicity, we use an abstract notation (the symbol $G'[g^{(0)}]$ is in fact a linear differential operator acting on $g^{(1)}_{\nu\sigma}$). The key point is that right hand side (i.e.,the gCS term) generates an extra factor of a^2 (in view of the a-dependence), in such a way that every component in $g^{(1)}_{\nu\sigma}$ has an extra factor of a compared with $g^{(0)}_{\nu\sigma}$. This means that if we make the redefinition $g^{(1)}_{\rho\sigma} \equiv a^2 h^{(1)}_{\nu\sigma}$ the expansion in (48) becomes[12]

$$\bar{g}_{\nu\sigma} = g^{(0)}_{\nu\sigma} + (c_{\lambda N} a^2) h^{(1)}_{\nu\sigma} + \sum_{k=2}^{\infty} c^k_{\lambda N}\, g^{(k)}_{\nu\sigma} \tag{50}$$

At the second order we obtain a (differential) equation

$$G'[g^{(0)}] \cdot g^{(2)} = a^2\, C'[g^{(0)}] \cdot h^{(1)} - a^4\, G''[g^{(0)}] \cdot g^{(1)} \cdot g^{(1)} \tag{51}$$

It can be shown that $C'[g^{(0)}] \cdot h^{(1)} \propto a^2$. It then follows from (51) that $g^{(2)}_{\nu\sigma}$ has (at least) an extra multiplicative factor a^4 compared with $g^{(0)}_{\nu\sigma}$. Defining $g^{(2)}_{\nu\sigma} \equiv a^4 h^{(2)}_{\nu\sigma}$ and using this in (50) we get

$$\bar{g}_{\nu\sigma} = g^{(0)}_{\nu\sigma} + (c_{\lambda N} a^2) h^{(1)}_{\nu\sigma} + (c_{\lambda N} a^2)^2 h^{(2)}_{\nu\sigma} + \sum_{k=3}^{\infty} c^k_{\lambda N}\, g^{(k)}_{\nu\sigma} \tag{52}$$

Repeating this procedure we finally get

$$\bar{g}_{\nu\sigma} = \sum_{k=0}^{\infty} (c_{\lambda N} a^2)^k h^{(k)}_{\nu\sigma} \tag{53}$$

where $h^{(0)}_{\nu\sigma} \equiv g^{(0)}_{\nu\sigma}$ and all $h^{(k)}_{\nu\sigma}$ are analytic in a around $a = 0$. We now see that our perturbative expansion is an effective expansion in (λa^2).

We can write (53) in the following form

$$\bar{g}_{\nu\sigma} = \sum_{k=0}^{\infty} (c_{\lambda a})^k\, \tilde{h}^{(k)}_{\nu\sigma} \quad , \qquad \tilde{h}^{(k)}_{\nu\sigma} = (\mu^{1/4} a)^k h^{(k)}_{\nu\sigma} \tag{54}$$

where $c_{\lambda a} = \lambda\, G_N a/\mu^{3/2}$ is a dimensionless parameter. What is interesting in this new parametrization is that $\tilde{h}^{(k)}_{\nu\sigma}$, beside being analytic in a, also satisfies

$$\lim_{a \to 0} \tilde{h}^{(k)}_{\nu\sigma} = 0 \tag{55}$$

We now see that (54) is expansion in (λa) with the coefficients which become very small when $a/\mu^{1/4}$ is small, improving the convergence of the expansion in that regime. Comparing (54) with the expansion (48), we conclude that (54) can be made sensible even for λ and $G_N/\mu^{5/4}$ finite, if we take a small enough. This is exactly our claim in (b).

[12] In fact, as we show in Sec. 4.2, $g^{(1)}_{t\phi_i}$ contains a multiplicative factor of a^2, while all other components of $g^{(1)}_{\nu\sigma}$ have a multiplicative factor a^3.

4.2 Perturbative expansion in λ: Equations of motion

Our aim is to find perturbative stationary rotating asymptotically flat black hole solutions in $D = 7$ in a theory with Lagrangian (9) to first-order in gCS coupling λ. For simplicity we specialize to the case when all angular momenta J_i, $i = 1, 2, 3$, are equal. We perturb around MP black holes which are parametrized by two numbers (μ, a), because in this case $a_i = a$, $i = 1, 2, 3$. As we discussed in Sec. 3, this case is rich enough to expect all relevant quantities to be perturbed by the gCS terms.

As in (48) we search for the perturbative solution

$$\bar{g}_{\nu\sigma} = g^{(0)}_{\nu\sigma} + \alpha\, g^{(1)}_{\nu\sigma} + O(\alpha^2)\ , \tag{56}$$

where for convenience we defined

$$\alpha \equiv 16\pi\, G_N \lambda\ \ . \tag{57}$$

Putting (56) in EOM (46) and using gauge condition

$$g^{(0)\nu\rho} g^{(1)}_{\nu\rho} = 0\ \ , \qquad\qquad \nabla^\nu g^{(1)}_{\nu\rho} = 0, \tag{58}$$

one obtains

$$-\frac{1}{2}\nabla^\beta\nabla_\beta\, g^{(1)}_{\nu\sigma} + R^\beta{}_{\nu\sigma}{}^\rho g^{(1)}_{\beta\rho} = C_{\nu\sigma}[g^{(0)}] \tag{59}$$

In (58) and (59) covariant derivative ∇_ν, Riemann tensor $R_{\beta\nu\sigma\rho}$ and $C_{\nu\sigma}$ are constructed from the unperturbed metric $g^{(0)}_{\nu\sigma}$, which is also used for raising and lowering indices. By solving (59) one obtains the first-order correction to metric $g^{(1)}_{\nu\sigma}$.

In our case $g^{(0)}_{\nu\sigma}$ is the MP black hole metric with all angular momenta equal, i.e.,

$$a_i = a\ \ , \qquad\qquad i = 1, 2, 3\ \ . \tag{60}$$

From (10), (11) and (12) one gets

$$\begin{aligned} ds^2_{(0)} &\equiv g^{(0)}_{\nu\sigma} dx^\nu dx^\sigma \\ &= -dt^2 + \frac{\mu\, r^2}{\Pi\, F}\left(dt - a\sum_{i=1}^{3}\mu_i^2\, d\phi_i\right)^2 + \frac{\Pi\, F}{\Pi - \mu\, r^2} dr^2 + (r^2 + a^2)\sum_{i=1}^{3}(d\mu_i^2 + \mu_i^2 d\phi_i^2) \end{aligned} \tag{61}$$

where now

$$F = F(r) = 1 - \frac{a^2}{r^2 + a^2} = \frac{r^2}{r^2 + a^2}\ , \qquad\qquad \Pi = \Pi(r) = (r^2 + a^2)^3 \tag{62}$$

Condition (60) substantially simplifies the MP metric. In fact, it can be shown that the dependence on the coordinates $\vec{\mu}$ is completely fixed by the enhanced symmetries induced by (60). We use this to write $g^{(1)}_{\nu\sigma}$ in the following form

$$\begin{aligned} ds^2_{(1)} &\equiv g^{(1)}_{\nu\sigma} dx^\nu dx^\sigma \\ &= f_t(r)\left(\mu - (a^2 + r^2)^2\right) dt^2 + f_r(r)\frac{r^2(a^2 + r^2)^2}{\Pi - \mu r^2} dr^2 + h(r)(a^2 + r^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) \\ &\quad - f_{t\phi}(r)\frac{a\,\mu}{(a^2 + r^2)^2}\mu_i^2 dt\, d\phi_i + f_\phi(r)\frac{a^2\mu}{(a^2 + r^2)^2}\mu_i^2\mu_j^2 d\phi_i d\phi_j \end{aligned} \tag{63}$$

where f_t, f_r, h, $f_{t\phi}$, and f_ϕ are five unknown functions of the coordinate r alone, to be found by solving the equations of motion. We see that in the special case (60), due to the enhancement of symmetry, the problem generally (i.e., not only in perturbation theory) boils down to solving a system of *ordinary* differential equations, which is of immense help.

Writing (61) and (63) in the gauge conditions (58) imposes two constraints on unknown functions, which we use to express $f_t(r)$ and $f_{t\phi}(r)$ in terms of the remaining three functions. The result is

$$\begin{aligned}
g^{(1)}_{tt} &= \frac{1}{3r(r^2+a^2)^3}\Big\{2r\left[\mu\, a^2 f_\phi(r) + \left(4(r^2+a^2)^3 - \mu(2r^2+a^2)\right) f_r(r)\right.\\
&\quad \left. + \left(5(r^2+a^2)^3 - 2a^2\mu\right) h(r)\right] + (r^2+a^2)\left((r^2+a^2)^3 - r^2\mu\right) f_r'(r)\Big\}\\
g^{(1)}_{t\phi_i} &= \frac{\mu_i^2}{6a\mu r(r^2+a^2)^3}\Big\{\mu a^2 r\left(\mu(3r^2+5a^2) - (r^2+a^2)^3\right) f_\phi(r)\\
&\quad + r\left(5(r^2+a^2)^6 - \mu(r^2-6a^2)(r^2+a^2)^3 - 2\mu^2 a^2(2r^2+a^2)\right) f_r(r)\\
&\quad - r\left(5(r^2+a^2)^6 - 3\mu(5r^2+3a^2)(r^2+a^2)^3 + 4\mu^2 a^4\right) h(r)\\
&\quad + (r^2+a^2)\left((r^2+a^2)^6 - \mu(r^2-a^2)(r^2+a^2)^3 - r^2a^2\mu^2\right) f_r'(r)\Big\}\\
g^{(1)}_{rr} &= \frac{r^2(r^2+a^2)^2}{(r^2+a^2)^3 - r^2\mu} f_r(r)\\
g^{(1)}_{\mu_1\mu_1} &= \frac{(r^2+a^2)(1-\mu_2^2)}{1-\mu_1^2-\mu_2^2} h(r)\\
g^{(1)}_{\mu_1\mu_2} &= \frac{(r^2+a^2)\mu_1\mu_2}{1-\mu_1^2-\mu_2^2} h(r)\\
g^{(1)}_{\mu_2\mu_2} &= \frac{(r^2+a^2)(1-\mu_1^2)}{1-\mu_1^2-\mu_2^2} h(r)\\
g^{(1)}_{\phi_i\phi_j} &= \frac{a^2\mu\,\mu_i^2\mu_j^2}{(r^2+a^2)^2} f_\phi(r) + \delta_{ij}(r^2+a^2)\mu_i^2\, h(r)
\end{aligned} \tag{64}$$

Inserting (61) and (64) in the EOM (59) we obtain the following system of differential equations for the remaining unknown functions $f_r(r)$, $h(r)$, and $f_\phi(r)$

$$\begin{aligned}
f_r''(r) &= \frac{\mu\, r^2(7r^2+3a^2) - (15r^2-a^2)(r^2+a^2)^3}{r(r^2+a^2)\left[(r^2+a^2)^3 - \mu\, r^2\right]} f_r'(r) - \frac{8r^2\left(5(r^2+a^2)^3 - a^2\mu\right)}{(r^2+a^2)^2\left[(r^2+a^2)^3-\mu\, r^2\right]} f_r(r)\\
&\quad + \frac{8a^2\mu\, r^2}{(r^2+a^2)^2\left[(r^2+a^2)^3-\mu\, r^2\right]} f_\phi(r) + \frac{8r^2\left(5(r^2+a^2)^3 - 2a^2\mu\right)}{(r^2+a^2)^2\left[(r^2+a^2)^3-\mu\, r^2\right]} h(r)\\
&\quad + \frac{3456\, a^3\mu^3 r^2(7r^2-a^2)}{(r^2+a^2)^{10}\left[(r^2+a^2)^3-\mu\, r^2\right]}
\end{aligned} \tag{65}$$

$$\begin{aligned}
h''(r) &= \frac{2r}{r^2+a^2} f_r'(r) + \frac{4r^2\left(2(r^2+a^2)^3 - a^2\mu\right)}{(r^2+a^2)^2\left[(r^2+a^2)^3-\mu\, r^2\right]} f_r(r)\\
&\quad - \frac{4a^2\mu\, r^2}{(r^2+a^2)^2\left[(r^2+a^2)^3-\mu\, r^2\right]} f_\phi(r) - \frac{(5r^2-a^2)(r^2+a^2)^2 - \mu\, r^2}{r\left[(r^2+a^2)^3-\mu\, r^2\right]} h'(r)\\
&\quad - \frac{8r^2\left((r^2+a^2)^3 - a^2\mu\right)}{(r^2+a^2)^2\left[(r^2+a^2)^3-\mu\, r^2\right]} h(r) + \frac{1728\, a^3\mu^3 r^2}{(r^2+a^2)^9\left[(r^2+a^2)^3-\mu\, r^2\right]}
\end{aligned} \tag{66}$$

$$
\begin{aligned}
f_\phi''(r) \;=\; & -\frac{2r\left(5(r^2+a^2)^3+4a^2\mu\right)}{a^2\mu(r^2+a^2)}f_r'(r)-\frac{4r^2\left(10(r^2+a^2)^6+3a^2\mu(r^2+a^2)^3-4a^4\mu^2\right)}{a^2\mu(r^2+a^2)^2\left[(r^2+a^2)^3-\mu\, r^2\right]}f_r(r) \\
& +\frac{5r^2+a^2}{r(r^2+a^2)}f_\phi'(r)+\frac{4r^2\left(5(r^2+a^2)^3+4a^2\mu\right)}{(r^2+a^2)^2\left[(r^2+a^2)^3-\mu\, r^2\right]}f_\phi(r) \\
& -\frac{2r\left(5(r^2+a^2)^6-3\mu(5r^2+a^2)(r^2+a^2)^3+4a^4\mu^2\right)}{a^2\mu(r^2+a^2)\left[(r^2+a^2)^3-\mu\, r^2\right]}h'(r) \\
& +\frac{8r^2\left(5(r^2+a^2)^6-a^2\mu(r^2+a^2)^3-4a^4\mu^2\right)}{a^2\mu(r^2+a^2)^2\left[(r^2+a^2)^3-\mu\, r^2\right]}h(r) \\
& -\frac{1728\, a\,\mu^2 r^2\left((215r^2-57a^2)(r^2+a^2)^3-2\mu(105r^4-33a^2r^2-2a^4)\right)}{(r^2+a^2)^{10}\left[(r^2+a^2)^3-\mu\, r^2\right]} \qquad (67)
\end{aligned}
$$

4.3 Solving at lowest order in a

Equations (65)-(67) still appear nasty enough to be solved exactly, so we turn to slowly rotating black holes, i.e., $a/\mu^{1/4} \ll 1$.[13] In this regime, solutions of (65)-(66), with proper asymptotic behavior to describe asymptotically flat black holes, are given by

$$
\begin{aligned}
f_r(r) &= \frac{432}{5}\frac{a^3\mu^3}{r^{16}\,(r^4-\mu)}+O(a^5) \\
f_\phi(r) &= -1296\,\frac{a\,\mu^2}{r^{14}}-\frac{5\,r^6}{a^2\mu}h(r)+O(a^3) \\
h(r) &= \frac{2592}{5}\frac{a^3}{\mu^2}\,\tilde{h}(r^4/\mu)+O(a^5)\ , \qquad (68)
\end{aligned}
$$

where the function $\tilde{h}(u)$ is given by

$$
\begin{aligned}
\tilde{h}(u) \;=\; & -Q_{1/2}(2u-1)\int_1^u\frac{dx}{x^5}P_{1/2}(2x-1) \\
& +P_{1/2}(2u-1)\left(\int_\infty^u\frac{dx}{x^5}Q_{1/2}(2x-1)-i\frac{\pi}{2}\int_1^\infty\frac{dx}{x^5}P_{1/2}(2x-1)\right) \qquad (69)
\end{aligned}
$$

and P_ν and Q_ν are standard Legendre functions. Details of the calculation are presented in Appendix A. Using (68) and (69) in (64) we obtain $g^{(1)}_{\mu\nu}$ at the lowest order in a

$$g^{(1)}_{tt}=-\frac{6048}{5}\frac{a^3\mu^3}{r^{20}}+O(a^5) \qquad (70a)$$

$$g^{(1)}_{t\phi_i}=-\frac{72}{5}\frac{a^2\mu^3(43\,r^4-45\,\mu)}{r^{18}(r^4-\mu)}\mu_i^2+O(a^4) \qquad (70b)$$

$$g^{(1)}_{rr}=\frac{432}{5}\frac{a^3\mu^3}{r^{12}\,(r^4-\mu)^2}+O(a^5) \qquad (70c)$$

$$g^{(1)}_{\mu_1\mu_1}=\frac{2592}{5}\frac{a^3}{\mu^2}r^2\tilde{h}(r^4/\mu)\frac{1-\mu_2^2}{1-\mu_1^2-\mu_2^2}+O(a^5) \qquad (70d)$$

$$g^{(1)}_{\mu_1\mu_2}=\frac{2592}{5}\frac{a^3}{\mu^2}r^2\tilde{h}(r^4/\mu)\frac{\mu_1\mu_2}{1-\mu_1^2-\mu_2^2}+O(a^5) \qquad (70e)$$

[13] A similar double perturbative perturbative expansion was performed in [30] for the case of perturbation of Einstein gravity with massless scalar field in $D=4$ by a mixed Chern-Simons Lagrangian term. In contrast to our case, in this theory the lowest-order correction does not capture changes in the horizon properties like area and temperature.

$$g^{(1)}_{\mu_2\mu_2} = \frac{2592}{5}\frac{a^3}{\mu^2}r^2\tilde{h}(r^4/\mu)\frac{1-\mu_1^2}{1-\mu_1^2-\mu_2^2} + O(a^5) \tag{70f}$$

$$g^{(1)}_{\phi_i\phi_j} = -1296\frac{a^3\mu^3}{r^{18}}\left[1+2\frac{r^{20}}{\mu^5}\tilde{h}(r^4/\mu)\right]\mu_i^2\mu_j^2 + \delta_{ij}\frac{2592}{5}\frac{a^3}{\mu^2}r^2\,\tilde{h}(r^4/\mu)\,\mu_i^2 + O(a^5) \tag{70g}$$

where $i,j = 1,2,3$ and $\mu_3^2 = 1-\mu_1^2-\mu_2^2$. Note that the "ugly" part containing $\tilde{h}$ cancels in $g^{(1)}_{tt}$ and $g^{(1)}_{t\phi_i}$ in the lowest order in a.

Let us check that our perturbed solution still describes an asymptotically flat black hole. We will do this by checking the behavior of the perturbed metric in two limits - asymptotic infinity and near-horizon. For this we need the corresponding behavior of the function $\tilde{h}(u)$ which we defined in (69).

The asymptotic behaviour of the function $\tilde{h}(u)$ in the $u\to\infty$ limit is of the form

$$\tilde{h}(u) = Cu^{-3/2} + O(u^{-5/2}) \ . \tag{71}$$

where the constant C is

$$C = -\frac{\pi}{16}\int_1^\infty \frac{dx}{x^5}P_{1/2}(2x-1) \approx -0.0593\ldots \tag{72}$$

This means that $\tilde{h}(r^4/\mu) \propto 1/r^6$, so the asymptotic behavior of (70) at the limit $r\to\infty$ in the lowest order in a is

$$g^{(1)}_{tt} \sim O(r^{-20})\ , \quad g^{(1)}_{t\phi_i} \sim O(r^{-18})\ , \quad g^{(1)}_{rr} \sim O(r^{-20})\ , \quad g^{(1)}_{\mu_i\mu_j} \sim O(r^{-4})\ , \quad g^{(1)}_{\phi_i\phi_j} \sim O(r^{-4})\ . \tag{73}$$

We see explicitly that the perturbed solution is still asymptotically flat and that the fall-off conditions (73) guarantee that the metric perturbation (70) does not change the relations between asymptotic quantities (energy and angular momentum) and black hole parameters (μ and a). However, we should ask what happens in higher orders in the perturbation parameter $a/\mu^{1/4}$. To answer this we have performed a detailed analysis by perturbatively solving eqs.(65)-(66) in the regime $r \gg \mu^{1/4}$, using (70) as starting point, to all relevant orders in $u = r^4/\mu$ and $a/\mu^{1/4}$. We have found that $g^{(1)}_{\mu\nu}$ has the following asymptotic behavior at $r\to\infty$

$$g^{(1)}_{tt} \sim \frac{a^5}{r^{16}}\ , \quad g^{(1)}_{t\phi_i} \sim \frac{a^4}{r^{10}}\ , \quad g^{(1)}_{rr} \sim \frac{a^5}{r^{12}}\ , \quad g^{(1)}_{\mu_i\mu_j} \sim \frac{a^3}{r^4}\ , \quad g^{(1)}_{\phi_i\phi_j} \sim \frac{a^3}{r^4}\ . \tag{74}$$

We see that after including *all* orders of $a/\mu^{1/4}$ in $g^{(1)}_{\mu\nu}$, asymptotics have changed, but not in a significant way - the conclusion is that Myers-Perry relations (13) and (14) are still valid.

In the limit $r\to\mu^{1/4}$ (i.e., $u\to 1$), the function $\tilde{h}$ has the following expansion

$$\tilde{h}(u) = \tilde{h}(1) + \frac{1}{4}(77+23\,\tilde{h}(1))(u-1) + \frac{5}{64}(847+173\,\tilde{h}(1))(u-1)^2 + O(u^3) \tag{75}$$

where

$$\tilde{h}(1) = -\int_1^\infty \frac{dx}{x^5}\left(Q_{1/2}(2x-1) + i\frac{\pi}{2}P_{1/2}(2x-1)\right) \approx -0.15336\ldots \tag{76}$$

This implies that the metric perturbation (70) has the expected behavior for the black hole in the vicinity of the horizon, which, at the zeroth-order in a, is located at $r = \mu^{1/4}$. We shall see that part of the expansion (75) proportional to the "ugly" constant $\tilde{h}(1)$ does not contribute to near-horizon quantities (event horizon and ergosurface[14] properties), as it cancels in the calculations.

Now we are ready to calculate corrections to various black hole parameters. Below we present the main results while technical details of the calculations can be found in Appendix A.1.

[14]Generically, the ergosurface is not in the near-horizon region. However, as we are doing a perturbative calculation in a, for $|a|/\mu^{1/4} \ll 1$ the ergosurface is perturbatively close to the horizon.

4.3.1 Event horizon

We can find the location of the event horizon in standard fashion from

$$\bar{g}^{rr}(\bar{r}_H) = 0 \tag{77}$$

From (56), (61) and (70) follows

$$\bar{g}^{rr}(r) = \frac{(r^2+a^2)^3 - r^2\mu}{r^2(r^2+a^2)^2} - \alpha\left(\frac{432}{5}\frac{a^3\mu^3}{r^{20}} + O(a^5)\right) + \sum_{k=2}^{\infty}\alpha^k\, O(a^{2k}) \tag{78}$$

which, plugged in (77), gives

$$\bar{r}_H = r_{H0} + \alpha\left(\frac{108}{5}\frac{a^3}{\mu^{7/4}} + O(a^5)\right) + \sum_{k=2}^{\infty}\alpha^k\, O(a^{2k}) \tag{79}$$

where r_{H0} is the horizon radius for a MP black hole with $a_i = a$. Taking into account the possibility that the gCS coupling constant λ (which by (57) implies the same for α) is quantized, we must eventually view formally the double expansion in (79) (over α and a) as a single expansion (over a). [15] The final result is

$$\bar{r}_H = \mu^{1/4} - \frac{3}{4}\frac{a^2}{\mu^{1/4}} + \frac{108}{5}\frac{\alpha\, a^3}{\mu^{7/4}} + O(a^4) \tag{80}$$

We note that the same result for the event horizon is obtained from an analysis of circular orbits, which leads to the horizon condition

$$\left(\sum_i g_{t\phi_i}\right)^2 - g_{tt}\sum_{i,j} g_{\phi_i\phi_j} = 0 \tag{81}$$

The details can be found in Appendix A.1.

The location of the horizon is a coordinate dependent result, so by itself the result (79)-(80) does not say much.[16] We have to calculate proper, coordinate independent, quantities connected with event horizon. One such obvious is the proper area of the horizon, which we also need to find the black hole entropy. In Appendix A.2 we show that it is given by

$$\bar{A}_H = A_H^{(0)} - \alpha\left(540\pi^3\frac{a^3}{\mu^{3/4}} + O(a^5)\right) + \sum_{k=2}^{\infty}\alpha^k\, O(a^{2k}) \tag{82}$$

where the first term on the right side is the horizon area of the Myers-Perry black hole

$$A_H^{(0)} = \pi^3\mu\, r_{H0} \tag{83}$$

By expanding r_{H0} (horizon radius of the MP black hole) in a we obtain

$$\bar{A}_H = \pi^3\left(\mu^{5/4} - \frac{3}{4}a^2\mu^{3/4} - 540\,\alpha\frac{a^3}{\mu^{3/4}}\right) + O(a^4) \tag{84}$$

Now we see that the gCS Lagrangian term induces a real change on geometry of black hole solutions.

[15] We explained this in detail in Sec. 4.1.

[16] Note that (79)-(80) naively suggest that for $a > 0$ the gCS term tends to"enlarge" the horizon (at lowest order of perturbation around $a = 0$), but calculating the horizon area (82) shows that it actually tends to "shrink" it.

4.3.2 Ergosurface

The location of the ergosurface is obtained from the infinite red-shift condition

$$\bar{g}_{tt}(\bar{r}_e) = 0 \ . \tag{85}$$

From (56), (61) and (70)

$$\bar{g}_{tt}(r) = -1 + \frac{\mu}{(r^2 + a^2)^2} - \alpha \left(\frac{6048}{5} \frac{a^3 \mu^3}{r^{20}} + O(a^5) \right) + \sum_{k=2}^{\infty} \alpha^k \, O(a^{2k}) \tag{86}$$

follows. By inserting this in (85) we obtain that the ergosurface is defined by the condition $r = \bar{r}_e$, where

$$\bar{r}_e = \sqrt{\mu^{1/2} - a^2} - \alpha \left(\frac{1512}{5} \frac{a^3}{\mu^{7/4}} + O(a^5) \right) + \sum_{k=2}^{\infty} \alpha^k \, O(a^{2k}) \tag{87}$$

By expanding the first term (which is the ergosurface radius of the MP black hole) and collecting powers of a we obtain

$$\bar{r}_e = \mu^{1/4} - \frac{1}{2} \frac{a^2}{\mu^{1/4}} - \frac{1512}{5} \frac{\alpha \, a^3}{\mu^{7/4}} + O(a^4) \tag{88}$$

4.3.3 Angular velocity

If we write the horizon generating null Killing vector $\bar{\chi}$ as

$$\bar{\chi} = \frac{\partial}{\partial t} + \bar{\Omega}_H \sum_i \frac{\partial}{\partial \phi_i} \tag{89}$$

then $\bar{\Omega}_H$ is the angular velocity of the horizon. We can obtain it from the null-condition on the horizon

$$\bar{\chi}^2(\bar{r}_H) \equiv \bar{\chi}^\mu \bar{\chi}^\nu \bar{g}_{\mu\nu}\big|_{r=\bar{r}_H} = 0 \tag{90}$$

From (89) and the form of the metric it follows

$$\bar{\Omega}_H = - \left. \frac{\sum_i \bar{g}_{t\phi_i}}{\sum_{i,j} \bar{g}_{\phi_i \phi_j}} \right|_{r=\bar{r}_H} \tag{91}$$

Putting (56), (61), (70), and (79) in (91) we obtain

$$\bar{\Omega}_H = \frac{a}{r_{H0}^2 + a^2} - \alpha \left(648 \frac{a^2}{\mu^2} + O(a^4) \right) + \sum_{k=2}^{\infty} \alpha^k \, O(a^{2k}) \tag{92}$$

By expanding the first term (which is Ω_H of the MP black hole) and collecting powers of a we obtain

$$\bar{\Omega}_H = \frac{a}{\sqrt{\mu}} - 648 \, \alpha \frac{a^2}{\mu^2} + \frac{1}{2} \frac{a^3}{\mu} + O(a^4) \tag{93}$$

4.3.4 Surface gravity and black hole temperature

The surface gravity $\bar{\kappa}$ is defined by

$$\bar{\chi}^\mu \bar{\nabla}_\mu \bar{\chi}^\nu = \bar{\kappa} \bar{\chi}^\nu \qquad \text{on the horizon } r = \bar{r}_H \tag{94}$$

Using (89), (92), (56), (61), (70), and (79) we obtain

$$\bar{\kappa} = \frac{3 r_{H0}}{r_{H0}^2 + a^2} - \frac{1}{r_{H0}} + \alpha \left(1944 \frac{a^3}{\mu^{9/4}} + O(a^5) \right) + \sum_{k=2}^{\infty} \alpha^k \, O(a^{2k}) \tag{95}$$

By expanding the first term (which is κ of the MP black hole) and collecting powers of a we obtain

$$\bar{\kappa} = \frac{2}{\mu^{1/4}} - \frac{3}{2}\frac{a^2}{\mu^{3/4}} + 1944\,\alpha\frac{a^3}{\mu^{9/4}} + O(a^4) \tag{96}$$

The black hole temperature $\bar{T}_H$ is obtained from surface gravity via

$$\bar{T}_H = \frac{\bar{\kappa}}{2\pi} \tag{97}$$

4.3.5 Black hole entropy

As discussed in Sec. 2 a black hole entropy in our case is given by

$$\bar{S}_{\rm bh} = \bar{S}_{\rm BH} + \lambda\bar{S}_{\rm gCS} \tag{98}$$

where $S_{\rm BH}$ is the Bekenstein-Hawking entropy proportional to the proper horizon area A_H

$$\bar{S}_{\rm BH} = \frac{\bar{A}_H}{4\,G_N} \tag{99}$$

and $S_{\rm gCS}$ is the contribution induced by Lagrangian gCS term given by [8, 9]

$$S_{\rm gCS} = 16\pi\int_{\mathcal{B}} \mathbf{\Gamma}_N \mathbf{R}_N^2 \tag{100}$$

By using (56), (61), (70), (79) and (57) we obtain

$$\bar{S}_{\rm BH} = \frac{A_H^{(0)}}{4\,G_N} - \lambda\left(2160\pi^4\frac{a^3}{\mu^{3/4}} + O(a^5)\right) + \sum_{k=2}^{\infty}\alpha^k\,O(a^{2k}) \tag{101}$$

where the first term is the entropy of the Myers-Perry black hole. In the same way we obtain

$$\bar{S}_{\rm gCS} = 3456\,\pi^4\left(\frac{a}{r_{H0}}\right)^3 + \sum_{k=2}^{\infty}\alpha^k\,O(a^{2k}) \tag{102}$$

Interestingly the simple result (102) is a-exact in lowest order in λ. Plugging (101) and (102) into (98) gives us the black hole entropy

$$\bar{S}_{\rm bh} = \frac{A_H^{(0)}}{4\,G_N} + \lambda\left((6\pi)^4\frac{a^3}{\mu^{3/4}} + O(a^5)\right) + \sum_{k=2}^{\infty}\alpha^k\,O(a^{2k}) \tag{103}$$

or, written purely as expansion in a,

$$\bar{S}_{\rm bh} = \frac{\pi^3}{4\,G_N}\left(\mu^{5/4} - \frac{3}{4}a^2\mu^{3/4}\right) + (6\pi)^4\lambda\frac{a^3}{\mu^{3/4}} + O(a^4) \tag{104}$$

5 Conclusion

We have investigated in some detail the influence of adding a purely gravitational Chern-Simons Lagrangian term in the action of some diffeomorphism covariant theory of gravity, on asymptotically flat stationary rotating black hole solutions and corresponding black hole entropy. We have shown that the structure of the Chern-Simons term, characterized by its parity violating properties, does not have any effect when two or more angular momenta vanish. Perturbative arguments indicate that, instead, in cases when at most one angular momentum is zero, the influence of gravitational Chern-Simons terms is nontrivial, both on the solutions and the entropy.

In an attempt to find black hole solutions we have specialized to what seems to be the simplest nontrivial case, i.e. Einstein gravity supplemented with a gravitational Chern-Simons Lagrangian term in $D = 7$, and black holes with all angular momenta equal. We have calculated the first-order correction of the gravitational Chern-Simons entropy term and argued that it does not correspond to any geometric property of the interior of black hole (like the inner horizon surface area), contrary to the conjecture made in [23] which was based on the analysis of rotating AdS black holes in $D = 3$. Due to the complexity of the equations of motion, we have not been able to find exact analytic solutions. We have turned to a double perturbative expansion, in Chern-Simons coupling constant and angular momentum, and constructed the first-order correction to Myers-Perry solution. We have explicitly calculated corrections to horizon area, ergoregion and black hole entropy, all of which are nonvanishing. A perturbative analysis shows that the influence of the gravitational Chern-Simons Lagrangian term is completely nontrivial: it changes the type of metric - the perturbed metric does not seem to fall into the Kerr-Schild class with a flat seed metric. This is unfortunate because the Kerr-Schild ansatz was the crucial tool used for constructing Kerr and Myers-Perry solutions. It remains to be seen whether our perturbative results can suggest some new ansatz which could be used in analytic constructions.

An obvious extension would be to include a cosmological constant and consider asymptotically (A)dS solutions. In Einstein gravity exact analytic solutions of this type were obtained in [31], so one could naively expect that extension of our treatment to this case should be straightforward. However this is not the case - introduction of cosmological constants seriously complicates the calculations. This interesting problem is currently under investigation.

Acknowledgments

One of us (L.B.) would like to thank the Theoretical Physics Department, University of Zagreb, for hospitality and financial support during his visits there. M.C., P.D.P., S.P. and I.S. would like to thank SISSA for hospitality and financial support during visits there and would also like to acknowledge support by the Croatian Ministry of Science, Education and Sport under the contract no. 119-0982930-1016. The work of L.B. was supported in part by the MIUR-PRIN contract 2009-KHZKRX. Visits have been financially supported in part by the University of Zagreb Development Fund.

Appendix

A Some technical details

A.1 Perturbed horizon, ergosphere, angular velocity, surface gravity

We are interested in finding the position of the horizon (see e.g. [32] page 63, [33] page 373) by looking at the massless test particle in the circular motion around the black hole[17]

$$u = \frac{\partial}{\partial t} + \omega_i \frac{\partial}{\partial \phi_i}$$

[17]Our conventions are essentially the same as in [34, 25], except for the definition of the binormal to the horizon $\epsilon_{\mu\nu}$ which has the opposite sign.

Because of the symmetry induced by $a_i = a$ it suffices to restrict ourselves to the case $\omega_i = \Omega$ where Ω is chosen such that $u^2 = 0$. Now, we use the fact that the horizon is the surface where for $u^2 = 0$. There must be only one solution Ω to this equation, i.e. for

$$g_{tt} + 2\Omega \sum_i g_{t\phi_i} + \Omega^2 \sum_{i,j} g_{\phi_i\phi_j} = 0 \tag{105}$$

only one solution to

$$\Omega = \frac{-\sum_i g_{t\phi_i} \pm \sqrt{(\sum_i g_{t\phi_i})^2 - g_{tt}\sum_{i,j} g_{\phi_i\phi_j}}}{\sum_{i,j} g_{\phi_i\phi_j}} \tag{106}$$

This will be so when

$$\left(\sum_i g_{t\phi_i}\right)^2 - g_{tt}\sum_{i,j} g_{\phi_i\phi_j} = 0 \tag{107}$$

In that case

$$\Omega = -\left.\frac{\sum_i g_{t\phi_i}}{\sum_{i,j} g_{\phi_i\phi_j}}\right|_{r=r_H} = \frac{a}{a^2 + r_H^2} \tag{108}$$

In summary from (107) we get the horizon position r_H. Plugging this into (106) we get the angular velocity of the horizon (108). This Ω can then be used to define the horizon generating Killing vector $\chi = \frac{\partial}{\partial t} + \Omega \sum_i \frac{\partial}{\partial \phi_i}$. The surface gravity κ is defined as

$$\chi^\sigma \nabla_\sigma \chi^\nu = \kappa \chi^\nu \quad \text{at the horizon } r = r_H \tag{109}$$

where χ^ν is the Killing vector that is null on the horizon. However, for technical reasons, to calculate κ we use a different formula. This is because the coordinates we use (Boyer-Lindquist) are not regular at the horizon where (109) is to be evaluated. A formula which is more suitable, as it can be evaluated in the vicinity of the horizon, is (see Eq. (12.5.16) in [34])

$$\kappa^2 = -\lim_{\text{horizon}} \frac{(\chi^\sigma \nabla_\sigma \chi^\nu)(\chi^\sigma \nabla_\sigma \chi_\nu)}{\chi^\sigma \chi_\sigma} \tag{110}$$

Note the minus sign. The well known result for the unperturbed case (Myers-Perry with $a_i = a$) is

$$\kappa = 3\frac{r_H^{1/3}}{\mu^{1/3}} - \frac{1}{r_H} = \frac{3r_H}{a^2 + r_H^2} - \frac{1}{r_H} \tag{111}$$

The ergosurface can be found from the equation $g_{tt} = 0$, which gives

$$r_e = \sqrt{\sqrt{\mu} - a^2} \tag{112}$$

We now want to find the position of the horizon $\bar{r}$ for the perturbed metric $\bar{g}$. We apply the prescription (107) to the perturbed metric $g + \delta g$. To first order we have:

$$\begin{aligned} &\delta r \frac{\partial}{\partial r}\left\{\left(\sum_i g_{t\phi_i}\right)^2 - g_{tt}\sum_{i,j} g_{\phi_i\phi_j}\right\} + \\ &+ 2\left(\sum_i g_{t\phi_i}\right)\left(\sum_i \delta g_{t\phi_i}\right) - g_{tt}\sum_{i,j}\delta g_{\phi_i\phi_j} - \delta g_{tt}\sum_{i,j}\delta g_{\phi_i\phi_j} = 0 \end{aligned} \tag{113}$$

Plugging in the explicit expressions we obtain:

$$\delta r = \left.\frac{(a^2+r^2)(\Pi - r^2\mu)(f_r(r)+4h(r))}{2r((\Pi+(r^2-a^2)\mu))}\right|_{r=r_H} \tag{114}$$

We note that the new horizon is located on $r = \text{const}$, i.e. δr is not a function of μ_i. We assume that $h(r)$ and $(\Pi - r^2\mu)f_r(r)$ are regular at the horizon $r = r_H$ (the explicit solution (68) justifies this). For convenience we define $f_{r2}(r)$ to be

$$f_{r2}(r) = (\Pi(r) - r^2\mu)f_r(r) \tag{115}$$

which is regular and nonvanishing at the horizon. We obtain

$$\delta r = \left.\frac{(a^2+r^2)f_{r2}(r)}{2\mu r(2r^2-a^2)}\right|_{r=r_H} \tag{116}$$

To check the previous result, since the horizon is at $r = \text{const}$, we can use $\bar{g}^{rr} = 0$ (see e.g. [35] page 190-191) to determine its location. We get

$$g^{rr} + \delta g^{rr} = 0 \tag{117}$$

or

$$g^{rr} - g^{rr}g^{rr}\delta g_{rr} = 0 \tag{118}$$

expanding $\bar{r}_H = r_H + \delta r$, and using $g^{rr}\delta g_{rr} = f_r(r)$

$$\delta r \frac{\partial}{\partial r} g^{rr} - g^{rr} f_r(r) = 0 \tag{119}$$

or

$$\delta r = \left.\frac{g^{rr} f_r(r)}{\partial_r g^{rr}}\right|_{r=r_H} = \left.\frac{(a^2+r^2)f_{r2}(r)}{2\mu r(2r^2-a^2)}\right|_{r=r_H} \tag{120}$$

For the perturbed case we define

$$\bar{\chi}^\sigma \bar{\nabla}_\sigma \bar{\chi}^\nu = \bar{\kappa}\bar{\chi}^\nu \quad \text{at the horizon } r_{\bar{H}} = r_H + \delta r \tag{121}$$

where

$$\bar{\chi} = \frac{\partial}{\partial t} + \bar{\Omega}\sum_i \frac{\partial}{\partial \phi_i} \tag{122}$$

The perturbed angular velocity $\bar{\Omega}$ can be found from the condition

$$\bar{\chi}^\sigma \bar{\chi}^\nu \bar{g}_{\sigma\nu}|_{r=r_{\bar{H}}=r_H+\delta r} = 0, \tag{123}$$

which gives

$$\bar{\Omega} = -\left.\frac{\sum_i \bar{g}_{t\phi_i}}{\sum_{i,j}\bar{g}_{\phi_i\phi_j}}\right|_{r=r_{\bar{H}}=r_H+\delta r} \tag{124}$$

The result is

$$\begin{aligned}\bar{\Omega} = \frac{a}{a^2+r_H^2} \quad - \quad &\Bigg(-\frac{\left(2a^4+13a^2r_H^2+2r_H^4\right)f_{r2}(r_H)}{6a\left(a^2-2r_H^2\right)\left(a^2+r_H^2\right)^2\mu} - \frac{f'_{r2}(r_H)}{6ar_H\mu} \\ &+ \frac{a(a^2-2r_H^2)(a^2+r_H^2)f_\phi(r_H)}{6r_H^2\mu} \\ &+ \frac{\left(a^2-2r_H^2\right)\left(a^2+r_H^2\right)\left(4a^2+5r_H^2\right)h(r_H)}{6ar_H^2\mu}\Bigg) + O(\alpha)^2\end{aligned}$$

The surface gravity for the perturbed black hole is

$$\bar{\kappa} = \frac{3r_H}{a^2+r_H^2} - \frac{1}{r_H} \quad + \quad \left(\frac{\left(-2a^4 - 13a^2 r_H^2 - 2r_H^4\right) f_{r2}(r_H)}{2r_H\left(a^2-2r_H^2\right)\left(a^2+r_H^2\right)^2 \mu} - \frac{f_{r2}'(r_H)}{2r_H^2\mu} + \right. \\ + \frac{\left(a^4 - 2a^2 r_H^2\right) f_\phi(r_H)}{2r_H\left(a^2+r_H^2\right)^2} + \\ \left. + \frac{\left(a^2-2r_H^2\right)\left(4a^2+5r_H^2\right) h(r_H)}{2r_H\left(a^2+r_H^2\right)^2} \right) + O(\alpha)^2$$

The condition for the ergosurface $\bar{r}_e = r_e + \delta r_e$ is $\bar{g}_{tt} = 0$ in the perturbed case. Therefore

$$\delta r_e \partial_r g_{tt} + \delta g_{tt} = 0 \tag{125}$$

or

$$\bar{r}_e = \sqrt{\sqrt{\mu} - a^2} + \frac{(a^2+r^2) f_{r2}'(r)}{12\mu r^2} + \frac{a^2 f_\phi(r)}{6r} + \left(\frac{5r}{6} - \frac{a^2}{3r}\right) h(r) + \frac{f_{r2}(r)}{6\mu r} \tag{126}$$

Plugging in the explicit solutions (68-69) one gets

$$\bar{\Omega} = \frac{a}{\sqrt{\mu}} - \frac{648\alpha a^2}{\mu^2} + O(a)^3 + O(\alpha)^2 \tag{127}$$

$$\bar{\kappa} = \frac{2}{\mu^{1/4}} - \frac{3a^2}{2\mu^{3/4}} + \frac{1944\alpha a^3}{\mu^{9/4}} + O(a)^4 + O(\alpha)^2 \tag{128}$$

$$\bar{r}_e = \mu^{1/4} - \frac{a^2}{2\mu^{1/4}} - \frac{1512 a^3 \alpha}{5\mu^{7/4}} + O(a)^4 + O(\alpha)^2 \tag{129}$$

$$\bar{r}_H = \mu^{1/4} - \frac{3a^2}{4\mu^{1/4}} + \frac{108\alpha a^3}{5\mu^{7/4}} + O(a)^4 + O(\alpha)^2 \tag{130}$$

A.2 Area of the perturbed horizon: $A_{\bar{H}}[\bar{g}]$

The area can be calculated as an integral of the square root of the determinant of the metric $q(\bar{H}, \bar{g})$ induced from the perturbed metric $\bar{g}$ on the perturbed horizon $\bar{H}$. Before writing down the formula for the area, let us first introduce a notation fit for our case with two horizons: unperturbed H and perturbed $\bar{H}$, and two metric tensors: unperturbed g and perturbed $\bar{g}$.

For any metric g and any $(D-2)$-dimensional spacelike surface H we can find the basis n^ν, l^ν, t_i^ν such that $D-2$ vectors t_i^ν are tangential to H and n^μ and l^ν are normal to H, so that we have

$$n^2 = 0, \quad l^2 = 0, \quad n^\nu l_\nu = -1, \quad t_i^\nu n_\nu = 0, \quad t_i^\nu l_\nu = 0 \tag{131}$$

Then we define

$$h_{\sigma\nu} = \frac{n_\sigma l_\nu + l_\sigma n_\nu}{n^\rho l_\rho}$$

Note that these basis vectors depend on H as well as on g. Therefore $h_{\sigma\nu}$ does too, and we denote it as $h(H,g)_{\sigma\nu}$. We define the metric $q(H,g)_{\sigma\nu}$ induced on H from g as

$$q(H,g)_{\sigma\nu} = g_{\sigma\nu} - h(H,g)_{\sigma\nu} \tag{132}$$

Now the formula for the area can be written as:

$$A_{\bar{H}}[\bar{g}] = \int_{\bar{H}} \sqrt{q(\bar{H},\bar{g})} d^5x = (2\pi)^3 \int_0^1 d\mu_1 \int_0^{\sqrt{1-\mu_1^2}} d\mu_2 \sqrt{q(\bar{H},\bar{g})} \tag{133}$$

We wish to express this as an integration that involves unperturbed quantities. We denote $q_0 = q(H,g)$. We use the formula $\delta\sqrt{q} = \frac{1}{2}\sqrt{q}q^{\sigma\nu}\delta q_{\sigma\nu}$ to relate the square roots of the determinants of the following metrics induced on the perturbed horizon (to first order):

$$\begin{aligned}
\sqrt{q(\bar{H},\bar{g})} - \sqrt{q(\bar{H},g)} &= \frac{1}{2}\sqrt{q(\bar{H},g)}q(\bar{H},g)^{\sigma\nu}(q(\bar{H},\bar{g})_{\sigma\nu} - q(\bar{H},g)_{\sigma\nu}) \\
&= \frac{1}{2}\sqrt{q(H,g)}q(H,g)^{\sigma\nu}(q(H,\bar{g})_{\sigma\nu} - q(H,g)_{\sigma\nu}) \\
&= \frac{1}{2}\sqrt{q_0}{q_0}^{\sigma\nu}(q(H,\bar{g})_{\sigma\nu} - q(H,g)_{\sigma\nu}) \\
&= \frac{1}{2}\sqrt{q_0}{q_0}^{\sigma\nu}(\delta g_{\sigma\nu} + n_\sigma\delta l_\nu + \delta n_\sigma l_\nu + l_\sigma \delta n_\nu + \delta l_\sigma n_\nu) \\
&= \frac{1}{2}\sqrt{q_0}{q_0}^{ab}\delta g_{ab} \\
&= \frac{1}{2}\sqrt{q_0}(g^{\sigma\nu} - h(H,g)^{\sigma\nu})\delta g_{\sigma\nu} \\
&= -\frac{1}{2}\sqrt{q_0}h(H,g)^{\sigma\nu}\delta g_{\sigma\nu} \\
&= \sqrt{q_0}n^\sigma l^\nu \delta g_{\sigma\nu}
\end{aligned} \tag{134}$$

where we used the gauge condition (58). In the last line, n^ν and l^ν denote a pair of null vectors that satisfy (131) for the unperturbed metric g on the unperturbed horizon H. We can express the difference $\sqrt{q(\bar{H},g)} - \sqrt{q(H,g)}$ to first order using δr:

$$\sqrt{q(\bar{H},g)} - \sqrt{q(H,g)} = \delta r \partial_r \sqrt{q_0} \tag{135}$$

So

$$\sqrt{q(\bar{H},\bar{g})} - \sqrt{q_0} = \sqrt{q_0}n^\sigma l^\nu \delta g_{\sigma\nu} + \delta r \partial_r \sqrt{q_0} \tag{136}$$

Denoting by $A_{H(r)}[g]$ the area of the surface $H(r)$ (defined by r = const, t = const) measured by the unperturbed metric $g_{\sigma\nu}$, i.e. $A_{H(r)}[g] = \int_{H(r)} \sqrt{q(H(r),g)}d^5x$. For the Myers-Perry 7D geometry we obtain

$$A_{H(r)}[g] = \pi^3 \left(a^2 + r^2\right) \sqrt{\left(a^2+r^2\right)^3 + a^2\mu} \tag{137}$$

When $r = r_H$, this reduces to $A_H[g] = \pi^3\mu r_H$, which is the area of the (unperturbed) MP black hole horizon. The correction (135) is

$$\delta r \partial_r A_{H(r)}[g]\Big|_{r=r_H} = \delta r A_H[g] \left(\frac{3r_H^{5/3}}{\mu^{2/3}} + \frac{2r_H^{1/3}}{\mu^{1/3}}\right) \tag{138}$$

Hence

$$\begin{aligned}
A_{\bar{H}}[\bar{g}] &= A_H[g] + \delta r \partial_r A_{H(r)}[g] + \int_H \sqrt{q_0}n^\sigma l^\nu \delta g_{\sigma\nu} d^5x + O(\alpha^2) \\
&= A_H[g] + \delta r \partial_r A_{H(r)}[g] + A_H[g] n^\sigma l^\nu \delta g_{\sigma\nu} + O(\alpha^2)
\end{aligned} \tag{139}$$

To find $n^\sigma l^\nu \delta g_{\sigma\nu}$ we need to choose n^ν and l^ν. The choice

$$n_\nu = \frac{1}{\sqrt{2}}\left(\frac{dt_\nu}{\sqrt{-g^{tt}}} + \frac{dr_\nu}{\sqrt{g^{rr}}}\right) \;,\quad l_\nu = \frac{1}{\sqrt{2}}\left(\frac{dt_\nu}{\sqrt{-g^{tt}}} - \frac{dr_\nu}{\sqrt{g^{rr}}}\right)$$

satisfies (131). Using the ansatz for δg (64) we have

$$n^\sigma l^\nu \delta g_{\sigma\nu}\Big|_{r=r_H} = \frac{1}{2}\left(\left(1-\frac{r_H^{4/3}}{\mu^{1/3}}\right) f_\phi(r_H) + \left(4+\frac{r_H^{4/3}}{\mu^{1/3}}\right) h(r_H)\right) \tag{140}$$

Then

$$\begin{aligned} A_{\bar{H}}[\bar{g}] &= A_H[g]\left(1+\frac{f_{r2}(r_H)}{2r_H^2-a^2}\left(\frac{3r_H^{4/3}}{2\mu^{4/3}}+\frac{1}{\mu}\right)\right. \\ &\quad \left.+\frac{1}{2}\left(\left(1-\frac{r_H^{4/3}}{\mu^{1/3}}\right) f_\phi(r_H)+\left(4+\frac{r_H^{4/3}}{\mu^{1/3}}\right) h(r_H)\right)\right) + O(\alpha^2) \end{aligned} \tag{141}$$

where $f_{r2}(r)$ is defined in (115). Finally plugging in the explicit solutions (68)-(69), to the lowest order in a we find

$$\begin{aligned} A_{\bar{H}}[\bar{g}] &= A_H[g]\left(1-\frac{540\alpha a^3}{\mu^2}+O(a^5)\right)+O(\alpha^2) \\ &= \pi^3\mu^{5/4}\left(1-\frac{3a^2}{4\mu^{1/2}}-\frac{540\alpha a^3}{\mu^2}+O(a^4)\right)+O(\alpha^2) \end{aligned} \tag{142}$$

A.3 Entropy: $S_{\text{gCS}}[g_{\text{MP}}]$

Now we give the details of the calculation of (31). Here $g_{\rho\sigma}$ denotes g_{MP} i.e. the unperturbed 7-dimensional Myers-Perry solution (10), and $\Gamma^c{}_{\rho\sigma}$ and R_{abcd} are constructed from $g_{\rho\sigma}$. In $D=7$, Eq. (8) becomes

$$\begin{aligned} S_{\text{gCS}}[g] &= 16\pi\int_H \mathbf{\Gamma}_N \mathbf{R}_N^2 \\ &= 16\pi\int_H \mathbf{\Gamma}_N\wedge\mathbf{R}_N\wedge\mathbf{R}_N \\ &= 16\pi\frac{5!}{4}\int_H (\Gamma_N)_{[\sigma_1}(R_N)_{\sigma_2\sigma_3}(R_N)_{\sigma_4\sigma_5]} \\ &= \frac{16\pi}{4}\int_H \varepsilon^{\sigma_1\ldots\sigma_5}(\Gamma_N)_{\sigma_1}(R_N)_{\sigma_2\sigma_3}(R_N)_{\sigma_4\sigma_5} d^5x \end{aligned} \tag{143}$$

Here $\mathbf{\Gamma}_N = \frac{1}{2}\text{tr}\boldsymbol{\epsilon}\mathbf{\Gamma}$, $\mathbf{R}_N = \frac{1}{2}\text{tr}\boldsymbol{\epsilon}\mathbf{R}$ and $\boldsymbol{\epsilon}$ is the binormal to the horizon, normalized as $\epsilon_{\rho\sigma}\epsilon^{\rho\sigma}=-2$ with $\epsilon_{tr}<0$.[18] Also, the totally antisymmetric tensor density ε satisfies $\varepsilon^{3\ldots D}=\varepsilon_{3\ldots D}=1$. The binormal has only one nonzero component which is $\epsilon_{tr}=-\epsilon_{rt}$, and since $\epsilon_{\rho\sigma}\epsilon^{\rho\sigma}=-2$, we have $(\epsilon_{tr})^2 g^{tt}g^{rr}=-1$, so that $\epsilon_{tr}=-\frac{1}{\sqrt{-g^{tt}g^{rr}}}$. Then

$$\Gamma_{N\beta} = \frac{1}{2}\epsilon_{\sigma\rho}\Gamma^{\rho\sigma}{}_\beta = \frac{1}{2}\frac{1}{\sqrt{-g^{tt}g^{rr}}}(\Gamma^{tr}{}_\beta - \Gamma^{rt}{}_\beta) \tag{144}$$

$$R_{N\beta\gamma} = \frac{1}{2}\epsilon_{\sigma\rho}R^{\rho\sigma}{}_{\beta\gamma} = \frac{1}{\sqrt{-g^{tt}g^{rr}}}R^{tr}{}_{\beta\gamma} \tag{145}$$

We note that for Myers-Perry 7D solution it happens that $\Gamma^{tr}{}_a=-\Gamma^{rt}{}_a$, $\Gamma^{tr}{}_2=0$, $\Gamma^{tr}{}_3=0$, $\Gamma^{tr}{}_4=0$, $R^{tr}{}_{34}=0$, $R^{tr}{}_{56}=0$, $R^{tr}{}_{57}=0$, $R^{tr}{}_{67}=0$. (The coordinates are: $t=x^1$, $r=x^2$, $\mu^1=x^3$, $\mu^2=x^4$,

[18]This normalization is due to the definition of $\epsilon_{\rho\sigma}$ in Eq. (4.7) of [9] as $\nabla_\sigma\xi^\rho\big|_{\mathcal{B}}=\kappa\,\epsilon^\rho{}_\sigma$.

$\phi^1 = x^5$, $\phi^2 = x^6$, $\phi^3 = x^7$.) We have $-g^{tt}g^{rr} = \frac{r^2\mu+(\Pi-r^2\mu)F}{F^2\Pi}$. So at the horizon $r = r_H$, defined as the largest real solution to eq.(15), we have

$$\sqrt{-g^{tt}g^{rr}} = \frac{1}{F(r_H, \mu_1, \mu_2)} \tag{146}$$

Therefore,

$$\begin{aligned} S_{\mathrm{gCS}}[g] = \frac{16\cdot 8\pi}{4}(2\pi)^3 \int_H F^3(& \Gamma^{tr}{}_5(-R^{tr}{}_{36}R^{tr}{}_{47} + R^{tr}{}_{37}R^{tr}{}_{46}) + \\ & \Gamma^{tr}{}_6(+R^{tr}{}_{35}R^{tr}{}_{47} - R^{tr}{}_{37}R^{tr}{}_{45}) + \\ & \Gamma^{tr}{}_7(-R^{tr}{}_{35}R^{tr}{}_{46} + R^{tr}{}_{36}R^{tr}{}_{45}))d\mu^1 d\mu^2 \end{aligned} \tag{147}$$

The integrations over ϕ^1, ϕ^2, ϕ^3 produced $(2\pi)^3$. The remaining integrations over μ^1, μ^2 run each from 0 to 1 with the constraint $\mu_1^2 + \mu_2^2 \leq 1$. Plugging in Γ and R we get

$$\begin{aligned} S_{\mathrm{gCS}}[g] &= \frac{16\pi}{4}(2\pi)^3 \frac{32 a_1 a_2 a_3 r_H}{\mu^3} Q \times \\ &\times \int_H \frac{\mu_1\mu_2}{F^5}\left(9r_H^4 + \sum_{i=1}^3 \left(2a_i^4\nu_i^4 + 9r_H^2 a_i^2\nu_i^2\right) - \sum_{i<j} a_i^2 a_j^2 \left(\mu_i^2\mu_j^2 - 5\nu_i^2\nu_j^2\right)\right) d\mu_1 d\mu_2 \end{aligned} \tag{148}$$

where $\nu_i^2 = 1 - \mu_i^2$, $\mu_3^2 = 1 - \mu_1^2 - \mu_2^2$ and

$$\begin{aligned} Q &= \frac{1}{2}\sum_{i\neq j}(r_H^2 + a_i^2)(r_H^2 + a_j^2) = \sum_{i=1}^3 \frac{\mu r_H^2}{r_H^2 + a_i^2} \\ &= a_1^2a_2^2 + a_2^2a_3^2 + a_1^2a_3^2 + 2r_H^2(a_1^2 + a_2^2 + a_3^2) + 3r_H^4 \end{aligned} \tag{149}$$

Since

$$\begin{aligned} F &= 1 - \sum_{i=1}^3 \frac{a_i^2\mu_i^2}{a_i^2 + r^2} \\ &= \frac{r^2}{\Pi}(r^4 + a_1^2a_2^2\mu_3^2 + a_2^2a_3^2\mu_1^2 + a_1^2a_3^2\mu_2^2 + r^2\sum_{i=1}^3 a_i^2\nu_i^2) \end{aligned} \tag{150}$$

which, at the horizon, becomes

$$F(r_H) = \frac{1}{\mu}(r_H^4 + a_1^2a_2^2\mu_3^2 + a_2^2a_3^2\mu_1^2 + a_1^2a_3^2\mu_2^2 + r_H^2\sum_{i=1}^3 a_i^2\nu_i^2), \tag{151}$$

we obtain

$$S_{\mathrm{gCS}}[g] = -\frac{16\pi}{4}\ (2\pi)^3\ 32\ a_1a_2a_3\ \mu^2 r_H\ Q\ I, \tag{152}$$

where

$$I = \frac{1}{4}\int_0^1 d\mu_1^2 \int_0^{1-\mu_1^2} d\mu_2^2 \frac{9r_H^4 + \sum_{i=1}^3(2a_i^4\nu_i^4 + 9r_H^2a_i^2\nu_i^2) - \sum_{i<j} a_i^2a_j^2(\mu_i^2\mu_j^2 - 5\nu_i^2\nu_j^2)}{(r_H^4 + a_1^2a_2^2\mu_3^2 + a_2^2a_3^2\mu_1^2 + a_1^2a_3^2\mu_2^2 + r_H^2\sum_{i=1}^3 a_i^2\nu_i^2)^5}$$

This can be integrated, the result is $I = \frac{Q^2}{8\mu^4 r_H^8}$. The final result for $S_{\mathrm{gCS}}[g]$ is then

$$S_{\mathrm{gCS}}[g] = 128\pi^4 \frac{a_1a_2a_3}{\mu^2 r_H^7} Q^3 \tag{153}$$

or

$$S_{\rm gCS}[g] = 128\,\pi^4 \frac{\mu}{r_H} a_1 a_2 a_3 \left(\sum_{i=1}^{3} \frac{1}{r_H^2 + a_i^2} \right)^3 \tag{154}$$

The last part of (154) can be written by using the thermodynamical quantities of the Myers-Perry black hole:

$$a_1 a_2 a_3 \left(\sum_{i=1}^{3} \frac{1}{r_H^2 + a_i^2} \right)^3 = J_1 J_2 J_3 \left(\sum_{i=1}^{3} \frac{\omega_i}{J_i} \right)^3 \tag{155}$$

where J_i are angular momenta, and ω_i are (conjugated) angular velocities. In the simpler case with all angular momenta equal, $a_i = a$, (154) becomes

$$S_{\rm gCS}[g] = 3456\,\pi^4 \left(\frac{a}{r_H} \right)^3 \tag{156}$$

B BTZ black hole in $D = 3$

For the sake of completeness and comparison with the $D = 7$ case, we briefly review here the $D = 3$ case (for more details one can consult Refs. [18, 19, 28, 29, 23, 21]). In the process we emphasize the important simplifications that occur in $D = 3$ compared to $D > 3$. The Lagrangian is

$$\mathbf{L} = \frac{1}{16\pi G_N} \boldsymbol{\epsilon}\,(R - 2\Lambda) + \lambda\,\mathbf{L}_{gCS} \tag{157}$$

where $\boldsymbol{\epsilon}$ is the volume 3-form, Λ is the cosmological constant and gCS Lagrangian term is obtained by taking $n = 2$ in (2)

$$\mathbf{L}_{gCS} = 2 \int_0^1 dt\ \mathrm{str}(\boldsymbol{\Gamma}\,\mathbf{R}_t) \tag{158}$$

The equations of motion following from Lagrangian (157) are

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} - 16\pi G_N \lambda\, C^{\mu\nu} = 0 \tag{159}$$

where from (3) it follows that $C^{\mu\nu}$, known as Cotton tensor, is

$$C^{\mu\nu} = -\epsilon^{\alpha\beta(\mu}\,\nabla_\rho\, R^{\nu)\rho}{}_{\alpha\beta} = 2\,\epsilon^{\alpha\beta(\mu}\,\nabla_\alpha\, R^{\nu)}{}_\beta \tag{160}$$

The second equality follows from the fact that in $D = 3$ the Riemann tensor is not independent but is completely determined by the Ricci tensor

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} - \frac{1}{2}(g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}) R \tag{161}$$

By contracting (159) with $g_{\mu\nu}$ and using the fact that $C^{\mu\nu}$ is traceless, we obtain that all solutions satisfy a relation

$$R = 6\Lambda, \tag{162}$$

so that (159) can be equivalently written as

$$R^{\mu\nu} - 2\Lambda\, g^{\mu\nu} - 16\pi\lambda\, C^{\mu\nu} = 0. \tag{163}$$

Let us first focus on the theory with $\lambda = 0$, that is General Relativity with cosmological constant. From (163) and (161) it follows that every solution necessarily satisfies

$$R_{\lambda\mu\nu\kappa} = \Lambda(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}) \tag{164}$$

This means that every solution is a maximally symmetric metric, locally equivalent to AdS space when $\Lambda < 0$, to dS space when $\Lambda > 0$, and to flat Minkowski space when $\Lambda = 0$. This is a special property valid only in $D = 3$, and is connected to the fact that General Relativity in $D = 3$ does not possess local dynamical degrees of freedom in the bulk (there are no gravitational waves). Because of this it was for long thought that this theory could not host black hole solutions. However, stationary rotating black hole solutions, known as BTZ black holes, were found for $\Lambda = -1/\ell^2 < 0$. The underlying reason is that one can make black hole spacetimes by identifying points in AdS space by means of discrete subgroups of isometry group $SO(2,2)$ ("quotienting" of AdS space). This is not possible in dS ($\Lambda > 0$) or Minkowski spaces ($\Lambda = 0$), so in General Relativity in three dimensions there are only AdS black holes. In Reissner-Nordstrom type coordinates the BTZ black hole metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\phi + N(r)dt)^2 \tag{165}$$

where

$$f(r) = \frac{r^2}{\ell^2} - \frac{j^2}{r^2} - m \;, \qquad N(r) = -\frac{j}{r^2} \tag{166}$$

and m and j parametrize the black hole solutions. For $m < 0$ has BTZ metric has naked singularity (with the exception of $m = -1$, $j = 0$ for which (165) is the ordinary AdS space), so we take $m \geq 0$. By introducing $r_\pm$ defined by

$$m = \frac{r_+^2 - r_-^2}{\ell^2} \;, \qquad |j| = \frac{r_+ r_-}{\ell} \tag{167}$$

we can write the function $f(r)$ as

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} \tag{168}$$

We see that r_+ and r_- are locations of event horizons, and from $m \geq 0$ it follows that $r_+ \geq r_-$, so r_+ is the outer horizon (a "true" black hole event horizon), and r_- is the inner horizon. BTZ black holes also possess ergoregions bounded by $r_+ < r < r_e$, where

$$r_e = \ell\sqrt{m} \tag{169}$$

The mass and angular momentum of BTZ black hole in General Relativity are

$$M_0 = \frac{m}{8G_N} \;, \qquad J_0 = \frac{j}{4G_N} \tag{170}$$

while the black hole entropy (given by the Bekenstein-Hawking expression) and temperature are

$$S_0 = \frac{A_H}{4G_N} = \frac{\pi r_+}{2G_N} \;, \qquad T_0 = \frac{r_+^2 - r_-^2}{2\pi r_+} \tag{171}$$

Let us now turn on the gCS coupling and take $\lambda \neq 0$. One important effect of the presence of the gCS Lagrangian term is that now the theory possesses massive propagating waves.[19] Interestingly, the BTZ metric (165) continues to satisfy the equations of motion (163) with $\lambda \neq 0$, because

$$C_{\mu\nu}[g_{\mathrm{BTZ}}] = 0 \;. \tag{172}$$

[19]However, to avoid ghost modes one has to take the opposite sign for Einstein-Hilbert Lagrangian term, which means taking $G_N < 0$ in (157). This theory with such a choice is usually referred as Topologically Massive Gravity. Here we note a crucial difference with corresponding theories in $D > 3$, where the standard Einstein-Hilbert action has massless propagating modes, and adding a purely gravitational CS Lagrangian term in backgrounds like (A)dS and flat Minkowski does not introduce new propagating modes [22].

There are at least two ways to obtain this. One is to use the fact that the BTZ metric is locally AdS_3, and an AdS metric is locally conformally equivalent to flat Minkowski space for which $C_{\mu\nu} = 0$. Another one is by using the fact that the covariant derivative of the Riemann tensor in maximally symmetric spaces is zero, and again because the BTZ metric is locally equivalent to an AdS_3 space; using this in (160) we obtain that $C_{\mu\nu}$ vanishes. The consequence is that all results obtained from geometric properties, which include $r_\pm$, r_e, A_H and T_{bh} remain the same as for $\lambda = 0$. But, quantities like "asymptotic charges" which are calculated from the Lagrangian are generically expected to change. Indeed one gets that energy and angular momentum get shifted[20]

$$M = M_0 - \frac{32\pi G_N \lambda}{\ell^2} J_0 \ , \qquad J = J_0 - 32\pi G_N \lambda M_0 \tag{173}$$

The black hole entropy formula receives an additional term

$$S = S_0 + \lambda S_{\mathrm{gCS}} \tag{174}$$

where S_0 is the same as in (171), and S_{gCS} is obtained from (8) by taking $n = 2$,

$$S_{\mathrm{gCS}} = 8\pi \int_{\mathcal{B}} \mathbf{\Gamma}_N = 4\pi \int_{\mathcal{B}} \epsilon^\mu{}_\nu \, \Gamma^\nu{}_{\mu\phi}. \tag{175}$$

The second equality in (175) follows from the fact that the bifurcation surface $\mathcal{B}$ on the event horizon (which is a closed line in $D = 3$) for BTZ metric (165) is given by $t = \mathrm{const}$, $r = \mathrm{const} \to r_+$, so

$$dt \,\big|_{\mathcal{B}} = dr \,\big|_{\mathcal{B}} = 0 \tag{176}$$

The binormal $\epsilon^\mu{}_\nu$ can be calculated by constructing a pair of vectors h^μ and k^μ which are orthonormal

$$h^2 = -1 \ , \qquad k^2 = 1 \ , \qquad h \cdot k \equiv h^\mu k_\mu = 0 \tag{177}$$

and also normal to $\mathcal{B}$. By (176) this means that they have to satisfy

$$h_\phi \,\big|_{\mathcal{B}} = k_\phi \,\big|_{\mathcal{B}} = 0 \tag{178}$$

The binormal is then

$$\epsilon_{\mu\nu} = h_\mu k_\nu - k_\mu h_\nu \tag{179}$$

From (165) and (178) it follows that one possible choice is

$$h^\mu = \frac{1}{\sqrt{f}} (\partial_t - N \partial_\phi) \ , \qquad k^\mu = \sqrt{f}\, \partial_r \ , \tag{180}$$

where f and N are functions defined in (166). Expressed as a dual one-forms, h and k are

$$h_\mu = -\sqrt{f}\, dt \ , \qquad k_\mu = \frac{1}{\sqrt{f}}\, dr \tag{181}$$

Using this in (179) we obtain that the nonvanishing components of the binormal (179) for the BTZ metric are

$$\epsilon_{tr} = -\epsilon_{rt} = -1 \tag{182}$$

The relevant components of Levi-Civita connection, evaluated on $\mathcal{B}$, are

$$\Gamma^{tr}{}_\phi \,\big|_{\mathcal{B}} = -\Gamma^{rt}{}_\phi \,\big|_{\mathcal{B}} = -\frac{j}{r_+} \tag{183}$$

[20] In asymptotically AdS spacetimes one cannot simply read mass and angular momentum from specific coefficients in the expansion of the metric around asymptotic infinity, as is possible in the asymptotically flat case.

Now we have all ingredients to calculate $S_{\rm gCS}$

$$\begin{aligned} S_{\rm gCS}[g_{\rm BTZ}] &= 8\pi \int \mathbf{\Gamma}_N = 4\pi \int_0^{2\pi} d\phi\, \epsilon_{\mu\nu} \Gamma^{\nu\mu}{}_\phi \Big|_{r=r_+} = -8\pi \int_0^{2\pi} d\phi\, \frac{j}{r_+} \\ &= -16\pi^2 \frac{j}{r_+} = -\mathrm{sign}(j)\, 16\pi^2 \, \frac{r_-}{\ell} \end{aligned} \tag{184}$$

Finally, the complete expression for the entropy is

$$S = \frac{\pi\, r_+}{2\, G_N} - \mathrm{sign}(j)\, 16\pi^2 \lambda \frac{r_-}{\ell} \tag{185}$$

which is equal to the expression (38), because the outer horizon area is $A_H \equiv A_+ = 2\pi r_+$ and the inner horizon area is $A_- = 2\pi r_-$.

C Perturbative calculation in the asymptotic infinity region

In Section 4.3 we indicated that our perturbative solution (70) (lowest order in a)[21] does not necessarily give a leading order contribution at asymptotic infinity $r \to \infty$, in an expansion in $1/r$. This means that we still have to convince ourself that (70) indeed corresponds to some exact solution which is asymptotically flat Minkowski with finite mass and angular momenta. The idea is to perturbatively solve the equations of motion as expansion in $1/r$ such that in the lowest order in a this new perturbative solution is consistent with (70). After some laborious work one gets the following perturbative solution

$$\begin{aligned} f_r(r) &= \frac{432\, a^3 \mu^3 r^2}{5\left((r^2+a^2)^3 - \mu r^2\right)(r^2+a^2)^8} \Bigg\{ 1 - \frac{1}{1001}\frac{a^2}{r^2}\left[-91 + 18\frac{\mu}{r^4} + \frac{32}{3}\left(\frac{\mu}{r^4}\right)^2 + \frac{1280}{187}\left(\frac{\mu}{r^4}\right)^3 \right. \\ &\left. + \frac{215040}{46189}\left(\frac{\mu}{r^4}\right)^4 \right] + \frac{1}{3003}\frac{a^4}{r^4}\left[\frac{389}{3}\frac{\mu}{r^4} + \frac{5281}{32}\left(\frac{\mu}{r^4}\right)^2 + \frac{70540423}{430848}\left(\frac{\mu}{r^4}\right)^3 \right. \\ &\left. + \frac{17673550205}{116852736}\left(\frac{\mu}{r^4}\right)^4 \right] + \frac{a^6}{r^6}\left[-\frac{227}{3003}\frac{\mu}{r^4} - \frac{841055}{4900896}\left(\frac{\mu}{r^4}\right)^2 - \frac{6068098579}{24582894336}\left(\frac{\mu}{r^4}\right)^3 \right. \\ &\left. - \frac{5365625105215}{17896347076608}\left(\frac{\mu}{r^4}\right)^4 \right] + \frac{a^8}{r^8}\left[\frac{346}{3003}\frac{\mu}{r^4} + \frac{683615}{1633632}\left(\frac{\mu}{r^4}\right)^2 + \frac{5119195267}{6145723584}\left(\frac{\mu}{r^4}\right)^3 \right. \\ &\left. + \frac{23201035787035}{17896347076608}\left(\frac{\mu}{r^4}\right)^4 \right] \Bigg\} - \frac{3\, a^3\, C}{10\, \mu^{1/2} r^6} \Bigg\{ \frac{a^2}{r^2}\left[2\frac{\mu}{r^4} + \frac{19}{8}\left(\frac{\mu}{r^4}\right)^2 + \frac{161}{64}\left(\frac{\mu}{r^4}\right)^3 \right. \\ &\left. + \frac{1323}{512}\left(\frac{\mu}{r^4}\right)^4 \right] + \frac{a^4}{r^4}\left[D + (D-12)\frac{\mu}{r^4} + (D-21)\left(\frac{\mu}{r^4}\right)^2 + \frac{1}{352}(352\, D - 10419)\left(\frac{\mu}{r^4}\right)^3 \right. \\ &\left. + \left(\frac{128128\, D - 4874739}{128128}\right)\left(\frac{\mu}{r^4}\right)^4 \right] + \frac{a^6}{r^6}\left[-5D + \frac{21}{8}(16-3D)\frac{\mu}{r^4} + \frac{9}{64}(736 - 77\, D)\left(\frac{\mu}{r^4}\right)^2 \right. \\ &\left. + \frac{1}{1536}(290664 - 21205\, D)\left(\frac{\mu}{r^4}\right)^3 + \frac{1}{172032}(51525792 - 2888851\, D)\left(\frac{\mu}{r^4}\right)^4 \right] \Bigg\} \end{aligned}$$

[21] It is implicitly assumed that we are still restricted to the first-order correction in the gCS coupling λ.

$$
\begin{aligned}
h(r) &= \frac{5184}{385}\frac{a^3\mu^3}{r^{20}}\left[1+\frac{100}{117}\frac{\mu}{r^4}+\frac{320}{429}\left(\frac{\mu}{r^4}\right)^2+\frac{62720}{94809}\left(\frac{\mu}{r^4}\right)^3+\frac{3211264}{5404113}\left(\frac{\mu}{r^4}\right)^4\right]\left\{1+\frac{a^2}{r^2}\left[-\frac{755}{72}\right.\right.\\
&\left. -\frac{82705}{33696}\frac{\mu}{r^4}-\frac{557979647}{252315648}\left(\frac{\mu}{r^4}\right)^2-\frac{314914279261105}{154571593752576}\left(\frac{\mu}{r^4}\right)^3-\frac{5988320920522710149}{3141601398052356096}\left(\frac{\mu}{r^4}\right)^4\right]\\
&+\frac{a^4}{r^4}\left[\frac{4345}{72}+\frac{1029245}{33696}\frac{\mu}{r^4}+\frac{1593594078715}{47183026176}\left(\frac{\mu}{r^4}\right)^2+\frac{9821329285794745}{266987298299904}\left(\frac{\mu}{r^4}\right)^3\right.\\
&\left.+\frac{16293952469907467641638 7}{411235623005053412966 4}\left(\frac{\mu}{r^4}\right)^4\right]+\frac{a^6}{r^6}\left[-\frac{3025}{12}-\frac{2318825}{11232}\frac{\mu}{r^4}-\frac{542871773465}{1965959424}\left(\frac{\mu}{r^4}\right)^2\right.\\
&\left.-\frac{345032173308603785}{978953427099648}\left(\frac{\mu}{r^4}\right)^3-\frac{297982639365346239860165}{685392705008422354944}\left(\frac{\mu}{r^4}\right)^4\right]+\frac{a^8}{r^8}\left[\frac{30745}{36}\right.\\
&+\frac{2601475}{2592}\frac{\mu}{r^4}+\frac{42798224525}{26687232}\left(\frac{\mu}{r^4}\right)^2+\frac{282898493882717 89}{11890122596352}\left(\frac{\mu}{r^4}\right)^3\\
&\left.\left.+\frac{1216168521971070006591639 5}{363785358812162634547 2}\left(\frac{\mu}{r^4}\right)^4\right]\right\}+\frac{2592\,a^3\,C}{5\,\sqrt{\mu}\,r^6}\left[1+\frac{3}{4}\frac{\mu}{r^4}+\frac{75}{128}\left(\frac{\mu}{r^4}\right)^2+\frac{245}{512}\left(\frac{\mu}{r^4}\right)^3\right.\\
&\left.+\frac{6615}{16384}\left(\frac{\mu}{r^4}\right)^4\right]\left\{1-3\frac{a^2}{r^2}\left[1+\frac{1}{2}\frac{\mu}{r^4}+\frac{17}{32}\left(\frac{\mu}{r^4}\right)^2+\frac{2437}{4928}\left(\frac{\mu}{r^4}\right)^3+\frac{816743}{1757184}\left(\frac{\mu}{r^4}\right)^4\right]\right.\\
&+\frac{a^4}{r^4}\left[6+\frac{D}{4}+\left(\frac{27}{4}+\frac{D}{128}\right)\frac{\mu}{r^4}+\left(\frac{693}{64}+\frac{11\,D}{1536}\right)\left(\frac{\mu}{r^4}\right)^2+\left(\frac{135111}{9856}+\frac{41\,D}{8192}\right)\left(\frac{\mu}{r^4}\right)^3\right.\\
&\left.+\left(\frac{133194135}{8200192}+\frac{121\,D}{32768}\right)\left(\frac{\mu}{r^4}\right)^4\right]-\frac{a^6}{r^6}\left[10+\frac{5}{4}D+\left(\frac{75}{4}+\frac{71\,D}{128}\right)\frac{\mu}{r^4}+\left(\frac{2739}{64}\right.\right.\\
&\left.\left.\left.+\frac{20343}{39424}D\right)\left(\frac{\mu}{r^4}\right)^2+\left(\frac{64213}{896}+\frac{1686253}{3514368}D\right)\left(\frac{\mu}{r^4}\right)^3+\left(\frac{66706965}{630784}+\frac{23324683}{51544064}D\right)\left(\frac{\mu}{r^4}\right)^4\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
f_\phi(r) &= -\frac{2592\,a\,C}{5\,\mu^{3/2}}\left\{\frac{a^2}{r^2}\left[4\frac{\mu}{r^4}+\frac{27}{8}\left(\frac{\mu}{r^4}\right)^2+\frac{45}{16}\left(\frac{\mu}{r^4}\right)^3+\frac{1225}{512}\left(\frac{\mu}{r^4}\right)^4\right]-\frac{a^4}{r^4}\left[12\frac{\mu}{r^4}+\frac{135}{8}\left(\frac{\mu}{r^4}\right)^2\right.\right.\\
&\left.+\frac{3789}{176}\left(\frac{\mu}{r^4}\right)^3+\frac{1165707}{46592}\left(\frac{\mu}{r^4}\right)^4\right]+\frac{a^6}{r^6}\left[\left(24+\frac{9}{8}D\right)\frac{\mu}{r^4}+\left(\frac{405}{8}+\frac{15}{16}D\right)\left(\frac{\mu}{r^4}\right)^2\right.\\
&\left.\left.+\left(\frac{4113}{44}+\frac{1225}{1536}D\right)\left(\frac{\mu}{r^4}\right)^3+\left(\frac{73721565}{512512}+\frac{2835}{4096}D\right)\left(\frac{\mu}{r^4}\right)^4\right]\right\}-\frac{1296\,a\,\mu^2r^2}{\left(r^2+a^2\right)^8}\left\{1+\frac{a^2}{r^2}\left[-\frac{1}{33}\right.\right.\\
&\left.+\frac{24}{455}\frac{\mu}{r^4}+\frac{32}{693}\left(\frac{\mu}{r^4}\right)^2+\frac{7680}{187187}\left(\frac{\mu}{r^4}\right)^3+\frac{14336}{388531}\left(\frac{\mu}{r^4}\right)^4\right]-\frac{a^4}{r^4}\left[\frac{94}{715}\frac{\mu}{r^4}+\frac{71497}{288288}\left(\frac{\mu}{r^4}\right)^2\right.\\
&\left.+\frac{824128219}{2425943520}\left(\frac{\mu}{r^4}\right)^3+\frac{274143678797}{662827669504}\left(\frac{\mu}{r^4}\right)^4\right]+\frac{a^6}{r^6}\left[\frac{1182}{5005}\frac{\mu}{r^4}+\frac{3914045}{4900896}\left(\frac{\mu}{r^4}\right)^2\right.\\
&\left.+\frac{10444744351}{6584703840}\left(\frac{\mu}{r^4}\right)^3+\frac{136132959681349}{53689041229824}\left(\frac{\mu}{r^4}\right)^4\right]-\frac{a^8}{r^8}\left[\frac{1836}{5005}\frac{\mu}{r^4}+\frac{126995}{63648}\left(\frac{\mu}{r^4}\right)^2\right.\\
&\left.\left.+\frac{63286760347}{11523231720}\left(\frac{\mu}{r^4}\right)^3+\frac{86254981787935}{7669863032832}\left(\frac{\mu}{r^4}\right)^4\right]\right\}-\frac{\left(5\left(a^2+r^2\right)^3-2a^2\mu\right)}{a^2\mu}h(r) \qquad (186)
\end{aligned}
$$

The asymptotic perturbative solution (186) satisfies the system of equations (65)-(67) for all values of the integration constants C and D. However, to be consistent with full-space solution (in lowest order in a) (68), C has to be fixed as in (72). This leaves us with one free integration constant, which is D. Plugging (186) into (64) we obtain the following asymptotic expansion for the first order correction in λ

of the metric tensor:

$$g^{(1)}_{tt} = 1728\left\{\frac{1}{5}CD\frac{a^7}{\sqrt{\mu}}\frac{1}{r^{10}} - CD\frac{a^9}{\sqrt{\mu}}\frac{1}{r^{12}} - \frac{1}{4}C(a^4D+3\mu)\sqrt{\mu}\,a^5\frac{1}{r^{16}} + O(r^{-18})\right\} \quad (187)$$

$$g^{(1)}_{t\phi_i} = 864\,\mu_i^2\left\{\frac{CD\,a^6}{2\,\mu^{3/2}}\frac{1}{r^4} - \frac{CD\,a^8}{\mu^{3/2}}\frac{1}{r^6} - \frac{CD\,a^6}{5\,\sqrt{\mu}}\frac{1}{r^8} - C\left(\frac{8}{5}\sqrt{\mu}\,a^4 - \frac{11\,D\,a^8}{5\,\sqrt{\mu}}\right)\frac{1}{r^{10}} + O(r^{-12})\right\} \quad (188)$$

$$g^{(1)}_{rr} = -\frac{5184}{5}\left\{\frac{CD\,a^7}{2\,\sqrt{\mu}}\frac{1}{r^{10}} + C\sqrt{\mu}\,a^5\frac{1}{r^{12}} + O(r^{-14})\right\} \quad (189)$$

$$g^{(1)}_{\mu_i\mu_j} = \chi_{ij}(\vec{\mu})\left\{\frac{2592\,C\,a^3}{5\,\sqrt{\mu}}\frac{1}{r^4} + O(r^{-6})\right\} \quad (190)$$

$$g^{(1)}_{\phi_i\phi_j} = \delta_{ij}\mu_i^2\left(\frac{2592\,C\,a^3}{5\,\sqrt{\mu}}\frac{1}{r^4} + O(r^{-6})\right) + \mu_i^2\mu_j^2\left(-2592\,C\frac{a^3}{\sqrt{\mu}}\frac{1}{r^4} + O(r^{-6})\right) \quad (191)$$

From (187)-(191) it is obvious that the perturbative solution is asymptotically Minkowski, with finite mass and angular momenta. Eq. (187) is telling us that $M^{(1)}$ (first-order correction in λ of mass) vanishes to all orders in a. However, from (188) we see that due to the existence of a term $1/r^4$ angular momenta receive a correction, such that $J^{(1)} \propto Da^6$. But, if we take for integration constant $D = 0$, we obtain an asymptotic perturbative solution with vanishing correction of the angular momenta $J^{(1)}$ to all orders in a. Putting $D = 0$ one gets the asymptotic behavior (74).

References

[1] R. P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, Phys. Rev. Lett. **11** (1963) 237.

[2] R. C. Myers and M. J. Perry, *Black Holes in Higher Dimensional Space-Times*, Annals Phys. **172** (1986) 304.

[3] D. Youm, *Black holes and solitons in string theory*, Phys. Rept. **316** (1999) 1 [hep-th/9710046].

[4] K. -i. Maeda and M. Nozawa, *Black hole solutions in string theory*, Prog. Theor. Phys. Suppl. **189** (2011) 310 [arXiv:1104.1849 [hep-th]].

[5] S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. Math. **99** (1974) 48.

[6] S. N. Solodukhin, *Holographic description of gravitational anomalies*, JHEP **0607** (2006) 003 [hep-th/0512216].

[7] L. Bonora, M. Cvitan, P. D. Prester, S. Pallua and I. Smolić, *Gravitational Chern-Simons Lagrangian terms and spherically symmetric spacetimes*, Class. Quant. Grav. **28** (2011) 195009 [arXiv:1105.4792 [hep-th]].

[8] Y. Tachikawa, *Black hole entropy in the presence of Chern-Simons terms*, Class. Quant. Grav. **24** (2007) 737 [hep-th/0611141].

[9] L. Bonora, M. Cvitan, P. Dominis Prester, S. Pallua and I. Smolić, *Gravitational Chern-Simons Lagrangians and black hole entropy*, JHEP **1107** (2011) 085 [arXiv:1104.2523 [hep-th]].

[10] L. Bonora, M. Cvitan, P. Dominis Prester, S. Pallua and I. Smolić, *Gravitational Chern-Simons terms and black hole entropy. Global aspects*, arXiv:1207.6969 [hep-th].

[11] T. Mohaupt, *Supersymmetric black holes in string theory*, Fortsch. Phys. **55** (2007) 519 [hep-th/0703035 [HEP-TH]].

[12] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, Gen. Rel. Grav. **40** (2008) 2249 [arXiv:0708.1270 [hep-th]].

[13] A. Castro, J. L. Davis, P. Kraus and F. Larsen, *String Theory Effects on Five-Dimensional Black Hole Physics*, Int. J. Mod. Phys. A **23** (2008) 613 [arXiv:0801.1863 [hep-th]].

[14] B. de Wit and S. Katmadas, *Near-Horizon Analysis of D=5 BPS Black Holes and Rings*, JHEP **1002** (2010) 056 [arXiv:0910.4907 [hep-th]].

[15] P. Dominis Prester, *alpha'-Corrections and Heterotic Black Holes*, PoS BHGRS (2008) 033 [arXiv:1001.1452 [hep-th]].

[16] P. Dominis Prester and T. Terzić, *α'-exact entropies for BPS and non-BPS extremal dyonic black holes in heterotic string theory from ten-dimensional supersymmetry*, JHEP **0812** (2008) 088 [arXiv:0809.4954 [hep-th]].

[17] M. Cvitan, P. Dominis Prester and A. Ficnar, *α'^2-corrections to extremal dyonic black holes in heterotic string theory*, JHEP **0805** (2008) 063 [arXiv:0710.3886 [hep-th]].

[18] S. Deser, R. Jackiw and S. Templeton, *Three-dimensional massive gauge theories,* Phys. Rev. Lett. **48** (1982) 975.

[19] S. Deser, R. Jackiw and S. Templeton, *Topologically massive gauge theories,* Ann. Phys., NY **140** (1982) 372.

[20] E. Witten, *Three-Dimensional Gravity Revisited*, arXiv:0706.3359 [hep-th].

[21] P. Kraus, *Lectures on black holes and the AdS(3)/CFT(2) correspondence*, Lect. Notes Phys. **755** (2008) 193 [hep-th/0609074].

[22] L. Bonora, M. Cvitan, P. Dominis Prester, S. Pallua and I. Smolić, *Symmetries and gravitational Chern-Simons Lagrangian terms*, in preparation.

[23] S. N. Solodukhin, *Holography with gravitational Chern-Simons,* Phys. Rev. **D74** (2006) 024015 [hep-th/0509148].

[24] H. Lu and Y. Pang, "Seven-Dimensional Gravity with Topological Terms," Phys. Rev. D **81** (2010) 085016 [arXiv:1001.0042 [hep-th]].

[25] V. Iyer and R. M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, Phys. Rev. D **50** (1994) 846 [gr-qc/9403028].

[26] R. C. Myers, *Myers-Perry black holes*, arXiv:1111.1903 [gr-qc].

[27] J. Doukas, *Exact constraints on D$\leq$ 10 Myers Perry black holes and the Wald Problem*, Phys. Rev. D **84** (2011) 064046 [arXiv:1009.6118 [hep-th]].

[28] M. Banados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, Phys. Rev. Lett. **69** (1992) 1849 [hep-th/9204099].

[29] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, Phys. Rev. D **48** (1993) 1506 [gr-qc/9302012].

[30] N. Yunes and F. Pretorius, *Dynamical Chern-Simons Modified Gravity. I. Spinning Black Holes in the Slow-Rotation Approximation*, Phys. Rev. D **79** (2009) 084043 [arXiv:0902.4669 [gr-qc]].

[31] G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, *Rotating black holes in higher dimensions with a cosmological constant*, Phys. Rev. Lett. **93** (2004) 171102 [hep-th/0409155].

[32] V. P. Frolov, and I. D. Novikov, "Black hole physics: Basic concepts and new developments," Kluwer Academic Publishers (1998) 770p

[33] T. Padmanabhan, "Gravitation: Foundations and frontiers," Cambridge, UK: Cambridge Univ. Pr. (2010) 700p

[34] R. M. Wald, "General Relativity," Chicago, Usa: Univ. Pr. (1984) 491p

[35] E. Poisson, "A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics," Cambridge, UK: Cambridge Univ. Pr. (2004) 233p