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GROMOV-WITTEN THEORY OF TAME DELIGNE-MUMFORD STACKS IN MIXED CHARACTERISTIC

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Abstract. We define Gromov-Witten classes and invariants of smooth proper tame Deligne-Mumford stacks of finite presentation over a Dedekind domain. We prove that they are deformation invariants and verify the fundamental axioms. For a smooth proper tame Deligne-Mumford stack over a Dedekind domain, we prove that the invariants of fibers in different characteristics are the same. We show that genus zero Gromov-Witten invariants define a potential which satisfies the WDVV equation and we deduce from this a reconstruction theorem for genus zero Gromov-Witten invariants in arbitrary characteristic.

Contents

1. Introduction

Gromov-Witten theory of orbifolds was introduced in the symplectic setting in [7], followed by an adaptation to the algebraic setting in [1] and [2], where Abramovich, Graber and Vistoli developed the Gromov-Witten theory of Deligne-Mumford stacks in characteristic zero, using the moduli stack of twisted stable maps into $\mathcal{X}$, denoted by $\mathcal{K}_{g,n}(\mathcal{X},\beta)$. This stack was constructed in [4] and it is the necessary analogue of Kontsevich's moduli stack of stable maps for smooth projective varieties when replacing the variety with a Deligne-Mumford stack. The stack $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ is defined for any tame proper Deligne-Mumford stack $\mathcal{X}$ of finite presentation over a noetherian base scheme S and is a proper algebraic stack over S ([4]). When the base is a field k of characteristic zero, $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ is Deligne-Mumford and admits a perfect obstruction theory ([2]). This leads to a virtual fundamental class $[\mathcal{K}_{g,n}(\mathcal{X},\beta)]^{\mathrm{virt}} \in A_*(\mathcal{K}_{g,n}(\mathcal{X},\beta))$ and the Gromov-Witten invariants of $\mathcal{X}$ are obtained by integrating cohomology classes on $\mathcal{X}$ against $[\mathcal{K}_{g,n}(\mathcal{X},\beta)]^{\mathrm{virt}}$.

In this paper we define Gromov-Witten classes and invariants associated to smooth proper tame Deligne-Mumford stacks of finite presentation over a Dedekind domain. The main motivation for us is to compare the invariants in different characteristics for stacks defined in mixed characteristic. We hope that this approach could give a useful insight into the Gromov-Witten theory in characteristic zero, providing a new technique for computing Gromov-Witten invariants.

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We consider a modified version, which we denote by $\mathcal{K}_{g,n}({}^{\mathcal{X}}/_{S}, \beta_\eta)$, of Abramovich, Graber and Vistoli' stack of twisted stable maps. The stack $\mathcal{K}_{g,n}({}^{\mathcal{X}}/_{S}, \beta_\eta)$ parametrizes twisted stable maps to $\mathcal{X}$, but we take β_η to be a cycle class over the generic fiber $\mathcal{X}_\eta$ of $\mathcal{X}$ rather than over $\mathcal{X}$ itself (section 2). This stack turns out to be more convenient when we want to compare the Gromov-Witten invariants in mixed characteristic.

The fundamental ingredient for the construction of Gromov-Witten invariants is the virtual fundamental class $[\mathcal{K}_{g,n}(\mathcal{X}, \beta)]^{\text{virt}} \in A_*(\mathcal{K}_{g,n}(\mathcal{X}, \beta))$. In the language of [6], a virtual fundamental class $[\mathcal{M}]^{\text{virt}} \in A_*(\mathcal{M})$ is defined in the Chow group with rational coefficients, for a Deligne-Mumford stack $\mathcal{M}$ endowed with a perfect obstruction theory. The main problem in developing Gromov-Witten theory in positive or mixed characteristic is that in general the stack $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ is not Deligne-Mumford. For istance this happens for $\mathcal{K}_{0,0}(\mathbb{P}^1_k, p)$, when k is a field of characteristic $p > 0$, because the map $f \colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ such that $f(x_0, x_1) = (x_0^p, x_1^p)$ is stable but has stabilizer

$$\mu_p = \operatorname{Spec} {}^{k[x]}/_{(x^p - 1)} = \operatorname{Spec} {}^{k[x]}/_{(x-1)^p},$$

which is not reduced. When the base is a field of characteristic $p > 0$, then $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ is still Deligne-Mumford for certain values of the fixed discrete parameters g, n, β which are big with respect to p ([1]). However, this is not satisfactory from the point of view of Gromov-Witten theory, because most of the properties of Gromov-Witten invariants (e.g. WDVV equation, Getzler relations) involve all the invariants at the same time.

The definition of virtual fundamental class for Artin stacks was not feasible at the beginning because of the lack for Artin stacks of two useful technical devices: Chow groups and the cotangent complex. We now have these devices at our disposal. Chow groups and intersection theory for Artin stacks over a field are defined in [14] and we verified that Kresch's theory naturally extends to stacks over a Dedekind domain (see Appendix B). A working theory for the cotangent complex of a morphism of Artin stacks is provided by [15], [23], [16]. Nonetheless the presence of these tools is not enough to overcome all the difficulties in the absolute case: the existing construction relies on the correspondence between Picard stacks and 2-term complexes of abelian sheaves, whereas the cotangent complex of an Artin stack has three terms, so that one cannot exploit directly the above correspondence. A first step in this direction was done by Francesco Noseda in his PhD thesis ([20]), even though his construction was not completely proven to be *intrinsic* and therefore may depend on the chosen resolution of the perfect obstruction theory (this point is crucial to prove the functoriality of the virtual fundamental class). To our knowledge, no intrinsic construction of the virtual fundamental class of an Artin stack has been done so far.

However, for the purpose of this work, it is enough to define a relative version of the virtual fundamental class of an Artin stack, generalizing the construction in [6] 7. In particular, we describe a way of constructing virtual fundamental classes of Artin stacks which admits a Deligne-Mumford type morphism into a smooth Artin stack over a scheme S and a relative perfect obstruction theory (section 3). In section 4 we apply this to the natural forgetful functor $\theta \colon \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \mathfrak{M}^{\text{tw}}_{g,n}$ into the stack of twisted curves $\mathfrak{M}^{\text{tw}}_{g,n}$ constructed in [4], after we exhibited a perfect relative obstruction theory for θ, and we construct a virtual fundamental class $[\mathcal{K}_{g,n}(\mathcal{X}, \beta)]^{\text{virt}} \in A_*(\mathcal{K}_{g,n}(\mathcal{X}, \beta))$.

A Dedekind domain D can be thought of as a space whose points corresponds to fields of different characteristics; a Deligne-Mumford stack $\mathcal{Y}$ over D is a family of Deligne-Mumford stacks - the fibers - each of which is defined over a point of D. We prove the following result, provinding a comparison between invariants in different characteristics (section 5).

1. **Theorem.** *Let $\mathcal{Y}$ be a smooth proper tame Deligne-Mumford stack of finite presentation over a Dedekind domain D. Then the Gromov-Witten theories of the geometric fibers of $\mathcal{Y}$ are equivalent (i.e., the Gromov-Witten invariants of the fibers are the same).*

As an application, if one starts with a smooth proper Deligne-Mumford stack $\mathcal{X}$ over $\mathbb{C}$ and can put it in a family over a Dedekind domain, which has a fiber with known Gromov-Witten invariants, then one gets the invariants of $\mathcal{X}$.

When the base is an algebraically closed field k, we consider the l-adic étale cohomology, for a prime l different from the characteristic of k,

$$H^r(\overline{\mathcal{I}}_\mu(\mathcal{X}), \mathbb{Z}_l) = \varprojlim_m H^r_{\text{ét}}(\overline{\mathcal{I}}_\mu(\mathcal{X}), {}^{\mathbb{Z}}/{}_{l^m\mathbb{Z}})$$

of the rigidified ciclotomic inertia stack $\overline{\mathcal{I}}_\mu(\mathcal{X})$ and we set

$$H^*_{\text{st}}(\mathcal{X}) = \sum_r H^r(\overline{\mathcal{I}}_\mu(\mathcal{X}), \mathbb{Q}_l(\overline{r})),$$

where $\overline{r}$ is the integral part of $r/2$. Then, in section 6, we prove that Gromov-Witten invariants define an associative and supercommutative product on the quantum cohomology ring $H^*_{\text{st}}(\mathcal{X})$.

Future work. A natural generalization would be to develop a Gromov-Witten theory for tame Artin stacks, using the moduli stack of twisted stable maps constructed in [3]. The main problem is that the natural forgetful functor $\theta\colon \mathcal{K}_{g,n}(\mathcal{X}, \beta) \to \mathfrak{M}^{\text{tw}}_{g,n}$ is not of Deligne-Mumford type in general, and therefore the relative cotangent complex of θ has three terms, so that one cannot use the construction described in 4.

In another direction, it would be interesting to prove a degeneration formula in the mixed characteristic setting. This would give a useful tool to compute Gromov-Witten invariants of Deligne-Mumford stacks in characteristic zero out of *simpler* invariants of tame Deligne-Mumford stacks in positive characteristic. For istance, this would apply to the fake projective plane constructed by Mumford in [19] using p-adic uniformization. We imagine this is far from easy, but we hope to return to these points in a future paper.

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Notations. We write $({}^{\mathrm{Sch}}/S)$ for the category of schemes over a base scheme S. For a scheme $X \in ({}^{\mathrm{Sch}}/S)$, we denote by $A_*({}^X/S)$ the group of numerical equivalence classes of cycles. All stacks are Artin stacks in the sense of [5], [15] and are of finite type over a base scheme. Unless otherwise specified, the words "stack of twisted stable maps" refer to $\mathcal{K}_{g,n}({}^{\mathcal{X}}/S, \beta_\eta)$ in Definition 2.4. We recall that a Deligne-Mumford stack $\mathcal{X}$ is tame if for every algebraically closed field k and morphism $\overline{x}\colon \operatorname{Spec} k \to \mathcal{X}$ the stabilizer group of x in $\mathcal{X}$ has order invertible in k.

2. The stack of twisted stable maps

Let D be a Dedekind domain and set $S = \operatorname{Spec} D$. Let $\mathcal{X}$ be a proper tame Deligne-Mumford stack of finite presentation over S, admitting a projective coarse moduli scheme X. We fix an ample invertible sheaf $\mathscr{O}(1)$ on X. We fix integers $g \geq 0$, $n \geq 0$. Let η be the generic point of S and set $X_\eta = X \times_S \eta$. Fix $\beta_\eta \in A_1({}^{X_\eta}/\eta)$.

2.1. **Twisted curves and twisted stable maps.** For any closed point $s \in S$, we denote by X_s the fiber over s. Let $\mathfrak{m}_s \subset D$ be the maximal ideal corresponding to s and consider the localization $R = D_{\mathfrak{m}_s}$ of D at $\mathfrak{m}_s$. Let us set $\widetilde{X}_s = X \times_S \operatorname{Spec} R$ and let $X_s \xrightarrow{i} X$ and $X_\eta \xrightarrow{j} X$ be the natural inclusions. Notice that R is a dicrete valuation ring and, by [12] 20.3, there exists a specialization homomorphism

$$\sigma_s \colon A_*({}^{X_\eta}/{}_\eta) \to A_*({}^{X_s}/{}_s),$$

sending a cycle α to $i^!\widetilde{\alpha}$, for some $\widetilde{\alpha} \in A_*({}^{\tilde{X}_s}/{}_R)$ such that $j^*\widetilde{\alpha} = \alpha$. By [12] 20.3.5, there exists an induced specialization homomorphism

$$\overline{\sigma_s} \colon A_*({}^{X_{\overline{\eta}}}/{}_{\overline{\eta}}) \to A_*({}^{X_{\overline{s}}}/{}_{\overline{s}}),$$

where $\overline{\eta}$ and $\overline{s}$ are geometric points over η and s. We denote by $\overline{\beta_\eta} \in A_1({}^{X_{\overline{\eta}}}/{}_{\overline{\eta}})$ the cycle class induced by β_η and we notice that $\overline{\sigma_s}\left(\overline{\beta_\eta}\right) = \overline{\sigma_s(\beta_\eta)}$.

2.1. **Definition.** Let T be a scheme over S. A *stable n-pointed map of genus g and class β_η into X* is the data $(C \xrightarrow{\pi} T, t_i, f)$, where

(1) the morphism π is a projective flat family of curves;
(2) the geometric fibers of π are reduced with at most nodes as singularities;
(3) the sheaf $\pi_*\omega_{C/T}$ is locally free of rank g (where $\omega_{C/T}$ is the relative dualizing sheaf);
(4) the morphisms $t_1, \ldots, t_n$ are sections of π which are disjoint and land in the smooth locus of π;
(5) $f \colon C \to X$ is a morphism of S-schemes;
(6) the group scheme $\operatorname{Aut}(C, f, \pi, t_i)$ of automorphisms of C, which commute with f, π and t_i, is finite over T;
(7) for every geometric point $\overline{t} \to T$, we consider the following induced morphisms

$$C_{\overline{t}} = C \times_T \overline{t} \xrightarrow{f_{\overline{t}}} X_{\overline{t}} = X \times_S \overline{t} \xrightarrow{\tau} X_{\overline{s}} = X \times_S \overline{s} \to X_s = X \times_S s \xrightarrow{i} X,$$

where $s = \operatorname{Spec} k \in S$ is the image of $\overline{t}$ and $\overline{s} = \operatorname{Spec} \overline{k}$, with $\overline{k}$ a separable closure of k, then we have $f_{\overline{t}*}[C_{\overline{t}}] = \tau^*\overline{\sigma_s}\left(\overline{\beta_\eta}\right)$.

2.2. Remark. Notice that a stable map of class β_η is a stable map of class β (in the sense of [4] 4.3.1) for some $\beta \in A_1({}^X/{}_S)$ such that $j^*\beta = \beta_\eta$.

2.3. **Definition.** Let T be a scheme over S. A *twisted stable n-pointed map of genus g and class β_η into* $\mathcal{X}$ over T is the data $(\mathcal{C} \to T, \{\Sigma_i^{\mathcal{C}}\}_{i=1}^n, f \colon \mathcal{C} \to \mathcal{X})$ where

(1) the following natural diagram is commutative

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & X \end{array}$$

(2) $(\mathcal{C} \to T, \{\Sigma_i^{\mathcal{C}}\}_{i=1}^n)$ is a twisted nodal n-pointed curve of genus g over T;
(3) the morphism $\mathcal{C} \to \mathcal{X}$ is representable;
(4) let Σ_i^C be the image of $\Sigma_i^{\mathcal{C}}$ in C, then $(C \to T, \{\Sigma_i^C\}_{i=1}^n, f \colon C \to X)$ is a stable n-pointed map of class β_η.

2.4. **Definition.** We denote by $\mathcal{K}_{g,n}({}^{\mathcal{X}}/{}_S, \beta_\eta)$ the category fibered in groupoids over $({}^{\mathrm{Sch}}/{}_S)$ of twisted stable n-pointed maps of genus g and class β_η into $\mathcal{X}$.

2.5. **Theorem.** *The category $\mathcal{K}_{g,n}({}^{\mathcal{X}}/{}_S, \beta_\eta)$ is a proper Artin stack over S, admitting a projective coarse moduli scheme $K_{g,n}({}^{\mathcal{X}}/{}_S, \beta_\eta) \to S$.*

Proof. Let $d = \deg \beta_\eta$. It is enough to show that $\mathcal{K}_{g,n}({}^{\mathcal{X}}/{}_S, \beta_\eta)$ is an open and closed substack of $\mathcal{K}_{g,n}({}^{\mathcal{X}}/{}_S, d)$ and then apply [4] 1.4.1. Notice that $\mathcal{K}_{g,n}({}^{\mathcal{X}}/{}_S, \beta_\eta) = \bigsqcup \mathcal{K}_{g,n}({}^{\mathcal{X}}/{}_S, \beta)$, where the union is over $\beta \in A_1({}^X/{}_S)$ such that $j^*\beta = \beta_\eta$. By [4] 1.4.1, $\mathcal{K}_{g,n}({}^{\mathcal{X}}/{}_S, \beta_\eta)$ is an open substack of $\mathcal{K}_{g,n}(\mathcal{X}/S, d)$, because it is a union of open substacks. On the other hand $\mathcal{K}_{g,n}(\mathcal{X}/S, d) \setminus \mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta) = \bigsqcup \mathcal{K}_{g,n}(\mathcal{X}/S, \beta)$ is open, where the union is over $\beta \in A_1(X/S)$ such that $\deg \beta = d$, $j^*\beta \neq \beta_\eta$. It follows that $\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)$ is a closed substack of $\mathcal{K}_{g,n}(\mathcal{X}/S, d)$. □

2.6. We denote by $\mathfrak{M}^{\mathrm{tw}}_{g,\,n/S}$ the stack of twisted n-pointed curves of genus g as defined in [4] 4.1.2. Recall that $\mathfrak{M}^{\mathrm{tw}}_{g,\,n/S}$ is a smooth Artin stack, locally of finite type over S. Moreover, the stack $\mathfrak{M}^{\mathrm{tw},\leq N,\Gamma}_{g,\,n/S}$, classifying twisted curves $(\mathcal{C}, \{\Sigma^{\mathcal{C}}_i\})$ such that the order of the stabilizer group at every point is at most N and the coarse space C of $\mathcal{C}$ has dual graph Γ, is a smooth Artin stack of finite type over S ([22] 1.9–1.12).

2.7. **Definition.** Let $\mathcal{C} \to \mathfrak{M}^{\mathrm{tw}}_{g,\,n/S}$ be the universal twisted nodal curve. We define the algebraic stack $\underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}, \mathcal{X})$ over $\mathfrak{M}^{\mathrm{tw}}_{g,\,n/S}$ as follows

(1) for every S-scheme T, an object of $\underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}, \mathcal{X})(T)$ is a twisted pointed curve $(\mathcal{C}_T \to T, \{\Sigma_i\}_{i=1}^n)$ over T together with a representable morphism of S-stacks $f\colon \mathcal{C}_T \to \mathcal{X}$;
(2) a morphism from $(\mathcal{C}_T \to T, \{\Sigma^{\mathcal{C}}_i\}, f)$ to $(\mathcal{C}_{T'} \to T', \{\Sigma'_i\}, f')$ consists of data (F, α), where $F\colon \mathcal{C}_T \to \mathcal{C}_{T'}$ is a morphism of twisted curves and $\alpha\colon f \to f' \circ F$ is an isomorphism.

2.8. Remark. There is a canonical functor $\overline{\theta}\colon \underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}, \mathcal{X}) \to \mathfrak{M}^{\mathrm{tw}}_{g,\,n/S}$ which forgets the map into $\mathcal{X}$. Moreover, since stability is an open condition, the stack $\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)$ is an open substack of $\underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}, \mathcal{X})$.

2.9. **Proposition.** *The natural forgetful functor*

$$\theta\colon \mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta) \to \mathfrak{M}^{tw}_{g,\,n/S}$$

which forgets the morphism into $\mathcal{X}$ is of Deligne-Mumford type.

Proof. Let $U \to \mathfrak{M}^{\mathrm{tw}}_{g,\,n/S}$ be a morphism from a scheme U over S and let us denote $\mathcal{C}_U = \mathcal{C} \times_{\mathfrak{M}^{\mathrm{tw}}_{g,n}} U$ the corresponding twisted pointed curve over U. Form the fiber diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}, \mathcal{X}) \arrow[d, "\overline{\theta}"] \\ U \arrow[r] & \mathfrak{M}^{\mathrm{tw}}_{g,\,n/S} \end{tikzcd}$$

and notice that $V = \underline{\mathrm{Hom}}_U(\mathcal{C}_U, \mathcal{X})$. Since $\mathcal{C}_U$ and $\mathcal{X}$ are Deligne-Mumford stacks it follows, by [21] 1.1, that V is a Deligne-Mumford stack and hence $\overline{\theta}$ is of Deligne-Mumford type. The statement follows from the fact that $\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)$ is an open substack of $\underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}, \mathcal{X})$. □

2.10. Remark. For every S-scheme T, a morphism $T \to \mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)$ corresponds to a stable map $(C_T \xrightarrow{\pi_T} T, t_i, f_T)$ over T, then, by descent theory, the identity of $\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)$ corresponds to a universal stable map $(\mathscr{C} \xrightarrow{\pi} \mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta), \sigma_i, \psi)$.

2.2. **Evaluation maps.** Let $\overline{\mathcal{I}}_\mu(\mathcal{X})$ be stack of cyclotomic gerbes of $\mathcal{X}$ as described in [2] 3.3. Recall that $\overline{\mathcal{I}}_\mu(\mathcal{X})$ is proper, since $\mathcal{X}$ is proper; moreover, if $\mathcal{X}$ is smooth then $\overline{\mathcal{I}}_\mu(\mathcal{X})$ is smooth ([2] 3.4).

2.11. Remark ([2] 3.5). There is an involution $\iota\colon \overline{\mathcal{I}}_\mu(\mathcal{X}) \to \overline{\mathcal{I}}_\mu(\mathcal{X})$ defined over each $\overline{\mathcal{I}}_{\mu_r}(\mathcal{X})$ as follows. Consider the inversion automorphism $\tau\colon \mu_r \to \mu_r$ sending ξ to ξ^{-1}. For every object $(\mathcal{G}, \psi)$ of $\overline{\mathcal{I}}_{\mu_r}(\mathcal{X})$, we can change the banding of the gerbe $\mathcal{G} \to T$ through τ and get another object ${}^\tau\mathcal{G} \to \mathcal{X}$ of $\overline{\mathcal{I}}_{\mu_r}(\mathcal{X})$.

2.12. Notation. We denote $\Delta\colon \overline{\mathcal{I}}_\mu(\mathcal{X}) \to \overline{\mathcal{I}}_\mu(\mathcal{X})^2$ the morphism, which we will call *diagonal*, induced by the identity and the involution ι.

2.13. **Definition** ([2] 4.4.1)**.** The *i-th evaluation map* $e_i\colon \mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta) \to \overline{\mathcal{I}}_\mu(\mathcal{X})$ is the morphism that associates to every twisted stable map $(\mathcal{C}\to T, \{\Sigma_i^{\mathcal{C}}\}_i, f\colon \mathcal{C}\to\mathcal{X})$ the diagram

$$\begin{tikzcd} \Sigma_i^{\mathcal{C}} \arrow[r, "f"] \arrow[d] & \mathcal{X} \\ T & \end{tikzcd}$$

The *i-th twisted evaluation map* $\check{e}_i\colon \mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta) \to \overline{\mathcal{I}}_\mu(\mathcal{X})$ is the composition $\iota\circ e_i$, where ι is the involution described in Remark 2.11.

2.14. Remark. Let us notice that the evaluation map e_i is the composition

$$\mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta) \xrightarrow{\Gamma_{e_i}} \mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta)\times_S \overline{\mathcal{I}}_\mu(\mathcal{X}) \xrightarrow{\pi} \overline{\mathcal{I}}_\mu(\mathcal{X}),$$

where Γ_{e_i} is the graph of e_i and π is the projection. By the following cartesian diagram

$$\begin{tikzcd} \mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta) \arrow[r, "e_i"] \arrow[d, "\Gamma_{e_i}"'] & \overline{\mathcal{I}}_\mu(\mathcal{X}) \arrow[d, "\Delta"] \\ \mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta)\times_S \overline{\mathcal{I}}_\mu(\mathcal{X}) \arrow[r, "e_i\times\mathrm{id}"] & \overline{\mathcal{I}}_\mu(\mathcal{X})\times_S \overline{\mathcal{I}}_\mu(\mathcal{X}) \end{tikzcd}$$

it follows that Γ_{e_i} is a regular local immersion, hence, by [14] 6.1, there exists a Gysin map $\Gamma_{e_i}^!$. Moreover $\underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C},\mathcal{X})$ is flat over $\mathfrak{M}^{\mathrm{tw}}_{g,n}$ therefore, since $\mathfrak{M}^{\mathrm{tw}}_{g,n}$ is smooth over S and $\mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta)$ is an open substack of $\underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C},\mathcal{X})$, we get that π is flat. Then we can define the pull-back $e_i^* = \Gamma_{e_i}^!\circ\pi^*$.

2.15. Notation. We write $e^*(\underline{\gamma}) = \prod_{i=1}^n e_i^*(\gamma_i)$ for every $\underline{\gamma} = \gamma_1\otimes\cdots\otimes\gamma_n$.

3. Relative intrinsic normal cone

In this section we extend the construction in [6] to the case of a morphism of Deligne-Mumford type of Artin stacks over a scheme S. In particular, we describe a way of constructing virtual fundamental classes for Artin stacks which admits a Deligne-Mumford type morphism into a smooth Artin stack, with the additional condition that $\mathcal{M}$ admits a stratification by global quotients in the sense of [14] 3.5.3. Moreover, we give a criterion to verify whether a complex is an obstruction theory.

3.1. Cones and cone stacks.

Let S be a scheme and let $\mathcal{M}$ be an Artin S-stack. We consider the lisse-étale topos $\mathcal{M}_{\text{lis-ét}}$ of $\mathcal{M}$. Let $\mathscr{S}^\bullet$ be a quasi-coherent sheaf of graded $\mathscr{O}_{\mathcal{M}}$-algebras in the topos $\mathcal{M}_{\text{lis-ét}}$ such that

(1) the canonical morphism $\mathscr{O}_{\mathcal{M}}\to\mathscr{S}^0$ is an isomorphism,
(2) $\mathscr{S}^1$ is coherent,
(3) $\mathscr{S}^\bullet$ is locally generated by $\mathscr{S}^1$.

3.1. **Definition** ([26] 1.18)**.** The *cone* associated to $\mathscr{S}^\bullet$ is the S-stack $C(\mathscr{S}^\bullet)$ associated to the groupoid $\operatorname{Spec}\mathscr{S}^\bullet_R \rightrightarrows \operatorname{Spec}\mathscr{S}^\bullet_U$, where $R\rightrightarrows U$ is a presentation of $\mathcal{M}$ and $\mathscr{S}^\bullet_U$ (respectively $\mathscr{S}^\bullet_R$) is the restriction of $\mathscr{S}^\bullet$ to U (respectively R). A morphism of cones over $\mathcal{M}$ is induced by a graded morphism of sheaves of graded $\mathscr{O}_{\mathcal{M}}$-algebras.

3.2. Remark. The natural morphism $\mathscr{S}^\bullet\to\mathscr{S}^0$ induces a morphism of S-stacks $0\colon\mathcal{M}\to C(\mathscr{S}^\bullet)$ called the *vertex* of $C(\mathscr{S}^\bullet)$. Moreover the morphism $\mathscr{S}^\bullet\to\mathscr{S}^\bullet[x]$ induces an action $\gamma\colon\mathbb{A}^1_S\times_S C(\mathscr{S}^\bullet)\to C(\mathscr{S}^\bullet)$.

3.3. **Definition.** If $\mathscr{F}$ is a coherent sheaf of $\mathscr{O}_{\mathcal{M}}$-modules over $\mathcal{M}$, the cone $C(\mathscr{F})$ associated to $\mathrm{Sym}(\mathscr{F})$ is called an *abelian cone*. An abelian cone $C(\mathscr{F})$ is a *vector bundle* over $\mathcal{M}$ if $\mathscr{F}$ is a locally free coherent sheaf over $\mathcal{M}$.

3.4. Remark. The natural morphism $\mathrm{Sym}(\mathscr{S}^1) \to \mathscr{S}^\bullet$ is surjective, because $\mathscr{S}^\bullet$ is locally generated by $\mathscr{S}^1$, hence the induced morphism of cones $C(\mathscr{S}^\bullet) \to C(\mathscr{S}^1)$ is a closed immersion. The abelian cone $C(\mathscr{S}^1)$ is called the *abelianization* of $C = C(\mathscr{S}^\bullet)$ and it is denoted by $A(C)$. Moreover a morphism of cones $C \to C'$ induces a morphism $A(C) \to A(C')$. In particular the abelianization defines a functor A from the category of cones over $\mathcal{M}$ to the category of abelian cones over $\mathcal{M}$.

3.5. **Definition** ([6] 1.2)**.** A sequence of morphisms of cones

$$0 \to E \xrightarrow{i} C \to C' \to 0$$

is *exact* if E is a vector bundle and locally over $\mathcal{M}$ there is a morphism of cones $C \to E$ splitting i and inducing an isomorphism $C \cong E \times C'$.

3.6. Remark. A sequence of coherent sheaves on $\mathcal{M}$

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{E} \to 0,$$

with $\mathscr{E}$ locally free, is exact if and only if

$$0 \to C(\mathscr{E}) \to C(\mathscr{F}) \to C(\mathscr{F}') \to 0$$

is exact ([12] Example 4.1.7).

3.7. **Definition** ([6] 1.8–1.9)**.** A *cone stack* over $\mathcal{M}$ is an algebraic $\mathcal{M}$-stack $\mathfrak{C}$ together with a section and an $\mathbb{A}^1_S$-action such that, smooth locally on $\mathcal{M}$, there exist a cone C, a vector bundle E over $\mathcal{M}$ and a morphism of abelian cones $E \to A(C)$ such that C is invariant under the induced action of E on $A(C)$, and there exists an $\mathbb{A}^1_S$-equivariant morphism $[C/E] \to \mathfrak{C}$ which is an isomorphism (for the definitions of $\mathbb{A}^1_S$-action and $\mathbb{A}^1_S$-equivariant morphism and 2-isomorphism we refer to [6] 1.5). A morphism of cone stacks is an $\mathbb{A}^1_S$-equivariant morphism of $\mathcal{M}$-stacks. A 2-isomorphism of cone stacks is an $\mathbb{A}^1_S$-equivariant 2-isomorphism. An *abelian cone stack* over $\mathcal{M}$ is a cone stack $\mathfrak{C}$ such that smooth locally $\mathfrak{C} \cong [C/E]$, where C is an abelian cone. A *vector bundle stack* over $\mathcal{M}$ is a cone stack $\mathfrak{C}$ such that smooth locally $\mathfrak{C} \cong [C/E]$, where C is a vector bundle.

3.8. Remark. Abelian cone stacks over $\mathcal{M}$ form a 2-category denoted by $(ACS/\mathcal{M})$. We consider the associated homotopy category $\mathrm{Ho}(ACS/\mathcal{M})$.

3.2. **Abelian cone stacks and complexes of sheaves.** Let $C^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$ be the category of complexes $(E^\bullet, d_E)$ of coherent sheaves in the topos $\mathcal{M}_{\text{lis-ét}}$ such that $h^i(E^\bullet, d_E) = 0$, for $i \neq 0, -1$; consider the subcategory $\hat{C}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$ of complexes $(E^\bullet, d_E)$ with $\ker d_E^0$ locally free.

3.9. **Definition** ([6] 2)**.** Let $\psi, \varphi\colon (E^\bullet, d_E) \to (F^\bullet, d_F)$ be morphisms in the category $\hat{C}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$. A *homotopy* $\varkappa\colon \psi \to \varphi$ is a morphism $\varkappa\colon E^\bullet \to F^\bullet[1]$ in $\hat{C}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$ such that

$$\begin{cases} \varkappa^{i+1} d_E^i = \varphi^i - \psi^i \\ d_F^i \varkappa^{i+1} = \varphi^{i+1} - \psi^{i+1}. \end{cases}$$

3.10**.** We can view $\hat{C}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$ as a 2-category, where the 2-morphisms are homotopies. We define a morphism of 2-categories

$$\hat{h}\colon \hat{C}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))^{\mathrm{opp}} \to (ACS/\mathcal{M})$$

such that $\hat{h}(E^\bullet) = [C(E^{-1})/C(E^0)]$ if $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$, and $\hat{h}(E^\bullet) = \hat{h}(\tau_{[-1,0]}E^\bullet)$ in general. In the following we can assume, for every complex $E^\bullet$, that $E^i = 0$ for $i \neq -1, 0$. If $\psi\colon E^\bullet \to F^\bullet$

is a morphism of complexes, then it induces a commutative diagram of abelian cones

$$\begin{tikzcd}
C(F^0) \arrow[r] \arrow[d] & C(F^{-1}) \arrow[d] \\
C(E^0) \arrow[r] & C(E^{-1})
\end{tikzcd}$$

which gives a morphism of cones $\hat{h}(\psi)\colon \hat{h}(F^\bullet) \to \hat{h}(E^\bullet)$. Finally, $\varkappa\colon E^0 \to F^{-1}$ is a homotopy of morphisms ψ, φ of complexes from $E^\bullet$ to $F^\bullet$, then $\varkappa \circ d_E = \varphi^{-1} - \psi^{-1}$ and $d_F \circ \varkappa = \varphi^0 - \psi^0$. The 2-morphism $\hat{h}(\varkappa)\colon \hat{h}(\psi) \to \hat{h}(\varphi)$ is defined in the following way. For every $\mathcal{M}$-scheme U and every $(P, f) \in \hat{h}(F^\bullet)(U)$, let $\{U_i\}$ be an open cover of U such that $U_i \times_U P \cong U_i \times_{\mathcal{M}} C(F^0)$, then

$$\hat{h}(\varkappa)(U)(P, f)\colon \hat{h}(\psi)(U)(P, f) \to \hat{h}(\varphi)(U)(P, f)$$

is obtained by gluing the isomorphisms

$$U_i \times_{\mathcal{M}} C(E^0) \xrightarrow{(\mathrm{id}_{U_i}, C(\varkappa)\circ f_i|_{U_i \times_{\mathcal{M}} \{0_F\}} \circ p_1 + p_2)} U_i \times_{\mathcal{M}} C(E^0),$$

where $C(\varkappa)$ is the morphism of cones induced by $\varkappa$. In particular $\hat{h}(\varkappa)$ is a 2-isomorphism.

3.11. **Lemma.** *Let $\psi\colon E^\bullet \to F^\bullet$ be a morphism in $\hat{C}^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}ét}))$ such that the diagram of cones*

$$\begin{tikzcd}
C(F^0) \arrow[r, "C(d_F)"] \arrow[d, "C(\psi^0)"'] & C(F^{-1}) \arrow[d, "C(\psi^{-1})"] \\
C(E^0) \arrow[r, "C(d_E)"] & C(E^{-1})
\end{tikzcd}$$

is cartesian and the morphism

$$C(d_E) + C(\psi^{-1})\colon C(E^{-1}) \times_{\mathcal{M}} C(F^{-1}) \to C(E^{-1})$$

is surjective, then $\hat{h}(\psi)$ is an isomorphism of cone stacks.

Proof. Let us notice that $\hat{h}(F^\bullet) \cong \hat{h}(F^\bullet \oplus E^{-1})$, because the following diagram

$$\begin{tikzcd}
C(F^{-1}) \times_{\mathcal{M}} C(E^{-1}) \times_{\mathcal{M}} C(F^0) \times_{\mathcal{M}} C(E^{-1}) \arrow[rr] \arrow[d] & & C(F^{-1}) \times_{\mathcal{M}} C(E^{-1}) \arrow[d] \\
C(F^{-1}) \times_{\mathcal{M}} C(E^{-1}) \times_{\mathcal{M}} C(F^0) \arrow[r] \arrow[d] & C(F^{-1}) \times_{\mathcal{M}} C(F^0) \arrow[r] \arrow[d] & C(F^{-1}) \arrow[d] \\
C(F^{-1}) \times_{\mathcal{M}} C(E^{-1}) \arrow[r] & C(F^{-1}) \arrow[r] & \hat{h}(F^\bullet)
\end{tikzcd}$$

is cartesian (the projection $C(F^{-1}) \times_{\mathcal{M}} C(E^{-1}) \to C(F^{-1})$ is surjective). Therefore we can assume that $C(\psi^{-1})$ is surjective. Moreover the following diagram is cartesian

$$\begin{tikzcd}
C(F^{-1}) \times_{\mathcal{M}} C(F^0) \arrow[rr] \arrow[d] & & C(F^{-1}) \arrow[d] \\
C(F^{-1}) \times_{\mathcal{M}} C(E^0) \arrow[r] \arrow[d] & C(E^{-1}) \times_{\mathcal{M}} C(E^0) \arrow[r] \arrow[d] & C(E^{-1}) \arrow[d] \\
C(F^{-1}) \arrow[r] & C(E^{-1}) \arrow[r] & \hat{h}(E^\bullet)
\end{tikzcd}$$

(the upper square is cartesian because $C(F^0) = C(F^{-1}) \times_{C(E^{-1})} C(E^0)$). It follows that $\hat{h}(\psi)$ is an isomorphism. □

3.12. **Proposition.** *Let $\hat{D}^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}\acute{e}t}))$ be the derived category of complexes $(E^\bullet, d_E)$ of coherent sheaves in the topos $\mathcal{M}_{lis\text{-}\acute{e}t}$ such that* $\ker d_E^0$ *is locally free and* $h^i(E^\bullet, d_E) = 0$ *for $i \neq -1, 0$. Let $Ho({}^{ACS}/_{\mathcal{M}})$ be the homotopy category associated to $({}^{ACS}/_{\mathcal{M}})$. The functor $\hat{h}$ induces a functor of categories*

$$\hat{D}^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}\acute{e}t}))^{opp} \to Ho({}^{ACS}/_{\mathcal{M}}).$$

Proof. We need to prove the following two facts: (1) if $\psi\colon E^\bullet \to F^\bullet$ is a quasi-isomorphism then $\hat{h}(\psi)$ is an isomorphism; (2) if $\psi, \varphi\colon E^\bullet \to F^\bullet$ are morphisms of complexes and $\xi\colon \hat{h}(\psi) \to \hat{h}(\varphi)$ is an $\mathbb{A}^1_S$-equivariant 2-isomorphism then there exists a unique homotopy $\varkappa\colon \psi \to \varphi$ such that $\xi = \hat{h}(\varkappa)$. We can assume $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$ and $F^\bullet = [F^{-1} \xrightarrow{d_F} F^0]$ with E^0 and F^0 locally free.

For the second statement, we define a morphism $C(\varkappa)\colon C(F^{-1}) \to C(E^0)$ as follows. Let T be an $\mathcal{M}$-scheme and let $f\colon T \to C(F^{-1})$ be a morphism. Then f defines an element $(P = T \times_{\mathcal{M}} C(F^0), f_P)$ in $\hat{h}(F^\bullet)(T)$. The images $\hat{h}(\psi)(T)(P, f_P)$ and $\hat{h}(\varphi)(T)(P, f_P)$ are trivial, hence $\xi(T)(P, f_P)$ corresponds to a morphism $g\colon T \to C(E^0)$. We define $C(\varkappa)(T)(f) = g$. Then $C(\varkappa)$ induces a homomorphism $\varkappa\colon E^0 \to F^{-1}$. Moreover $\xi = \hat{h}(\varkappa)$ and $\varkappa$ is unique by construction.

Let us now prove the first statement. We have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & h^{-1}(E^\bullet) \arrow[r] \arrow[d, "\wr"'] & E^{-1} \arrow[r, "d_E"] \arrow[d, "\psi^{-1}"'] & E^0 \arrow[r] \arrow[d, "\psi^0"] & h^0(E^\bullet) \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & h^{-1}(F^\bullet) \arrow[r] & F^{-1} \arrow[r, "d_F"] & F^0 \arrow[r] & h^0(F^\bullet) \arrow[r] & 0
\end{tikzcd}$$

It follows that $E^{-1} = E^0 \times_{F^0} F^{-1}$. Moreover we have an exact sequence

$$0 \to E^{-1} \xrightarrow{(d_E, \psi^{-1})} E^0 \oplus F^{-1} \xrightarrow{\psi^0 - d_F} F^0 \to 0,$$

which induces an exact sequence of cones

$$0 \to C(F^0) \xrightarrow{(C(\psi^0), -C(d_F))} C(E^0) \times_{\mathcal{M}} C(F^{-1}) \xrightarrow{C(d_E) + C(\psi^{-1})} C(E^{-1}) \to 0.$$

Hence the induced diagram of cones

$$\begin{tikzcd}
C(F^0) \arrow[r, "C(d_F)"] \arrow[d, "C(\psi^0)"'] & C(F^{-1}) \arrow[d, "C(\psi^{-1})"] \\
C(E^0) \arrow[r, "C(d_E)"] & C(E^{-1})
\end{tikzcd}$$

is cartesian and the statement follows from Lemma 3.11. □

3.13. **Lemma.** *Let $D^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}\acute{e}t}))$ be the derived category of complexes of coherent sheaves in the topos $\mathcal{M}_{lis\text{-}\acute{e}t}$ with cohomology sheaves concentrated in degree -1 and 0. If $\mathcal{M}$ admits a stratification by global quotients then the natural functor*

$$\hat{D}^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}\acute{e}t})) \to D^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}\acute{e}t}))$$

is an equivalence of categories.

Proof. Since $\mathcal{M}$ can be stratified by global quotient stacks (in the sense of [14] 3.5.4), each of which has the resolution property by [25] 1.1, it is enough to prove the statement when $\mathcal{M}$ has the resolution property. Notice that the functor is fully faithfull. We want to show that every complex $E^\bullet$ in $D^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$ is in the essential image. We can assume $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$ because $\tau_{\leq 0}E^\bullet$ is quasi-isomorphic to $E^\bullet$ and $\tau_{[-1,0]}E^\bullet$. Since $\mathcal{M}$ has the resolution property, there exists a locally free sheaf F^0 and a surjective morphism $\varphi^0\colon F^0 \to E^0$. We form the cartesian diagram

$$\begin{array}{ccc} F^{-1} & \xrightarrow{d_F} & F^0 \\ {\scriptstyle\varphi^{-1}}\downarrow & & \downarrow{\scriptstyle\varphi^0} \\ E^{-1} & \xrightarrow{d_E} & E^0 \end{array}$$

then $F^\bullet = [F^{-1} \xrightarrow{d_F} F^0] \in \hat{D}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$. We claim that $\varphi\colon F^\bullet \to E^\bullet$ is a quasi-isomorphism. Since φ^0 is surjective, we have immediately that $h^0(\varphi)$ is surjective and the following sequence

$$0 \to F^{-1} \xrightarrow{(d_F,\varphi^{-1})} F^0 \oplus E^{-1} \xrightarrow{\varphi^0 - d_E} E^0 \to 0$$

is exact. Using this we get that $F^\bullet$ is quasi-isomorphic to $F^0 \oplus E^\bullet$, which is quasi isomorphic to $E^\bullet$. $\square$

3.14. **Lemma.** *Let $D^{[-1,0]}_{coh}(\mathcal{M}_{lis\text{-}\acute{e}t})$ be the derived category of complexes of sheaves of $\mathscr{O}_\mathcal{M}$-modules in the topos $\mathcal{M}_{lis\text{-}\acute{e}t}$ with coherent cohomology sheaves concentrated in degree -1 and 0. If $\mathcal{M}$ admits a stratification by global quotients then the natural functor*

$$D^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}\acute{e}t})) \to D^{[-1,0]}_{coh}(\mathcal{M}_{lis\text{-}\acute{e}t})$$

is an equivalence of categories.

Proof. Since $\mathcal{M}$ can be stratified by global quotient stacks (in the sense of [14] 3.5.4), each of which has the resolution property by [25] 1.1, it is enough to prove the statement when $\mathcal{M}$ has the resolution property. First we show that the functor is fully faithfull. Let $E^\bullet, F^\bullet \in D^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$, we want to show that the canonical map

$$\mathrm{Hom}_{D^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))}(E^\bullet, F^\bullet) \to \mathrm{Hom}_{D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M}_{\text{lis-ét}})}(E^\bullet, F^\bullet)$$

is a bijection. We can assume $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$ and $F^\bullet = [F^{-1} \xrightarrow{d_F} F^0]$. Recall that $\mathrm{Hom}(\bullet, F^\bullet)$ is a cohomological functor. Using the following distinguished triangle

$$E^{-1} \xrightarrow{d_E} E^0 \to E^\bullet \xrightarrow{+1} E^{-1}[1],$$

we can reduce to the case where $E^\bullet$ is a coherent sheaf E, similarly $F^\bullet = F$. By resolution property, there exists a locally free sheaf P^0 and a surjective morphism $\psi\colon P^0 \to E$. Set $P^{-1} = \ker\psi$, then $P^\bullet = [P^{-1} \to P^0]$ is a complex of locally free sheaves quasi-isomorphic to E, hence $E = P^\bullet$ in $D^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$. Using the distinguished triangle

$$P^{-1} \to P^0 \to P^\bullet \xrightarrow{+1} P^{-1}[1],$$

we can reduce to the case where $E^\bullet$ is a locally free sheaf E. Let $E' = {}^{E}/_{\mathscr{O}_\mathcal{M}}$, then $\mathrm{rk}\, E' < \mathrm{rk}\, E$, hence we can reeduce to $E = \mathscr{O}_\mathcal{M}$. That is, we have reduce to showing that

$$\mathrm{Hom}_{D^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))}(\mathscr{O}_\mathcal{M}, F[n]) \to \mathrm{Hom}_{D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M}_{\text{lis-ét}})}(\mathscr{O}_\mathcal{M}, F[n])$$

is a bijection for every coherent sheaf F and $n = -1, 0$. If $n = -1$, both groups are zero. If $n = 0$ then both sides are $\Gamma(\mathcal{M}, F)$.

It remains to show that every complex $E^\bullet \in D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M}_{\text{lis-ét}})$ is in the essential image. We can assume $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$. We have the following exact sequence of complexes of sheaves

$$0 \to h^{-1}(E^\bullet)[1] \to E^\bullet \to [\operatorname{im} d_E \to E^0] \to 0,$$

which induces a distinguished triangle

$$h^{-1}(E^\bullet)[1] \to E^\bullet \to [\operatorname{im} d_E \to E^0] \xrightarrow{+1} h^{-1}(E^\bullet)[2].$$

Notice that $[\operatorname{im} d_E \to E^0] = h^0(E^\bullet)$ in $D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M}_{\text{lis-ét}})$. Then we have a distinguished triangle

$$h^{-1}(E^\bullet)[1] \to E^\bullet \to h^0(E^\bullet) \xrightarrow{+1} h^{-1}(E^\bullet)[2].$$

Since $h^0(E^\bullet)$ and $h^{-1}(E^\bullet)$ are coherent, the morphism $h^0(E^\bullet)[-1] \xrightarrow{+1} h^{-1}(E)[1]$ corresponds to a morphism $\psi\colon h^0(E^\bullet)[-1] \to h^{-1}(E)[1]$ in $D^{\leq 0}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$. Completing ψ to a distinguished triangle in $D^{\leq 0}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$ and mapping it to $D^{\leq 0}_{\mathrm{coh}}(\mathcal{M}_{\text{lis-ét}})$, we deduce that $E^\bullet$ is quasi-isomorphic to the mapping cone of ψ, hence it is in the essential image. ☐

3.15. If $\mathcal{M}$ admits a stratification by global quotients then the functor $\hat{h}$ induces a functor

$$h^1/h^0\colon D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M}_{\text{lis-ét}})^{\mathrm{opp}} \to \mathrm{Ho}(ACS/\mathcal{M}).$$

3.16. **Proposition.** *Let $\psi\colon E^\bullet \to F^\bullet$ be a morphism in $D^{[-1,0]}_{coh}(\mathcal{M}_{lis\text{-}\acute{e}t})$. If $\mathcal{M}$ admits a stratification by global quotients then $h^1/h^0(\psi)$ is an isomorphism if and only if $h^0(\psi)$ and $h^{-1}(\psi)$ are isomorphisms.*

Proof. By Lemma 3.13 and Lemma 3.14, we can assume

$$E^\bullet = [E^{-1} \to E^0],\ F^\bullet = [F^{-1} \to F^0]$$

with E^0, F^0 locally free and E^{-1}, F^{-1} coherent. Let $G = E^0 \times_{F^0} F^{-1}$, then

$$0 \to G \to E^0 \oplus F^{-1} \to F^0$$

is exact. Notice that $E^0 \oplus F^{-1} \to F^0$ is surjective if and only if $h^0(\psi)$ is surjective. Let us assume that $h^0(\psi)$ is surjective, then we get an exact sequence of cones

$$0 \to C(F^0) \to C(E^0) \times C(F^{-1}) \to C(G) \to 0.$$

Applying Lemma 3.11 we obtain $[C(F^{-1})/C(F^0)] \cong [C(G)/C(E^0)]$, hence the following diagram

$$\begin{tikzcd}
C(G) \arrow[r] \arrow[d] & C(E^{-1}) \arrow[d] \\
h^1/h^0(F^\bullet) \arrow[r, "h^1/h^0(\psi)"] & h^1/h^0(E^\bullet)
\end{tikzcd}$$

is cartesian and in particular $h^1/h^0(\psi)$ is representable. If moreover $h^0(\psi)$ is an isomorphism and $h^{-1}(\psi)$ is surjective, then the morphism $E^{-1} \to G$ is surjective, hence $C(G) \to C(E^{-1})$ is a closed immersion, which implies that $h^1/h^0(\psi)$ is a closed immersion. If $h^{-1}(\psi)$ is also an isomorphism then $E^{-1} \to G$ is an isomorphism and so $C(G) \cong C(E^{-1})$; it follows that $h^1/h^0(\psi)$ is an isomorphism.

Viceversa, if $h^1/h^0(\psi)$ is representable then the induced morphism on automorphisms of objects is injective. Hence we have that the morphism

$$C(h^0(\psi))\colon C(h^0(F^\bullet)) \to C(h^0(E^\bullet))$$

is a closed immersion, which implies that $h^0(\psi)$ is surjective. If moreover $h^1/h^0(\psi)$ is a closed immersion then $C(G) \to C(E^{-1})$ is a closed immersion, hence $E^{-1} \to G$ is surjective. It follows that $h^0(\psi)$ is injective and $h^{-1}(\psi)$ is surjective. Finally, if $h^1/h^0(\psi)$ is an isomorphism then $C(G) \cong C(E^{-1})$, hence $E^{-1} \cong G$, from which we get that $h^{-1}(\psi)$ is injective. ☐

3.17. **Theorem.** *If $\mathcal{M}$ admits a stratification by global quotients then the functor*

$$h^1/h^0 \colon D^{[-1,0]}_{coh}(\mathcal{M}_{lis\text{-}\acute{e}t})^{opp} \to Ho({}^{ACS}/\mathcal{M})$$

is an equivalence of categories.

Proof. By Proposition 3.16, it follows that h^1/h^0 is fully faithfull. It remains to show that every abelian cone stack $\mathfrak{C}$ over $\mathcal{M}$ is in the essential image of h^1/h^0. By definition, for every smooth $\mathcal{M}$-scheme U, there exist a coherent sheaf E_U^{-1} and a locally free sheaf E_U^0 over U such that $\mathfrak{C} \times_{\mathcal{M}} U \cong [C(E_U^{-1})/C(E_U^0)]$. The collection $\{E_U^{-1} \to E_U^0\}_U$ defines a complex $[E^{-1} \to E^0] \in D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M}_{\text{lis-ét}})$. □

3.3. Relative intrinsic normal cone.

3.18. **Theorem** ([15] 17.3, [23], [16] 2.2.5)**.** *Let S be a scheme and let $\mathcal{M}$, $\mathfrak{M}$ be Artin S-stacks. Let $f \colon \mathcal{M} \to \mathfrak{M}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Then there exists $L_f^\bullet \in D^{\leq 1}_{qcoh}(\mathcal{M}_{lis\text{-}\acute{e}t})$ such that*

(1) f is of Deligne-Mumford type if and only if $L_f^\bullet \in D^{\leq 0}_{qcoh}(\mathcal{M}_{lis\text{-}\acute{e}t})$;
(2) for every cartesian diagram

$$\begin{array}{ccc} \mathcal{M}' & \xrightarrow{f'} & \mathfrak{M}' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{M} & \xrightarrow{f} & \mathfrak{M} \end{array}$$

there exists a morphism $Lg^* L_f^\bullet \to L_{f'}^\bullet$; if h is flat, this is an isomorphism;
(3) given two morphisms of S-stacks $\mathcal{M} \xrightarrow{f} \mathfrak{M} \xrightarrow{g} Z$ with $h = g \circ f$, there exists a natural distinguished triangle

$$Lf^* L_g^\bullet \to L_h^\bullet \to L_f^\bullet \to Lf^* L_g^\bullet[1].$$

If moreover f is of Deligne-Mumford type, then

(1) f is smooth if and only if $L_f^\bullet$ is locally free in degree 0;
(2) f is étale if and only if $L_f^\bullet = 0$;
(3) if f factors as $\mathcal{M} \xrightarrow{i} M \xrightarrow{p} \mathfrak{M}$ with i representable and a closed embedding with ideal sheaf $\mathscr{I}$ and p of Deligne-Mumford type and smooth, then

$$\tau_{\geq -1} L_f^\bullet \cong [\mathscr{I}/\mathscr{I}^2 \to \Omega_p|_{\mathcal{M}}].$$

3.19. Remark. If f is of Deligne-Mumford type then $\tau_{\geq -1} L_f^\bullet \in D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M}_{\text{lis-ét}})$.

3.20. **Definition.** Let $f \colon \mathcal{M} \to \mathfrak{M}$ be a morphism of Artin S-stacks. If f is of Deligne-Mumford type, we define the *relative intrinsic normal sheaf* of f as the abelian cone stack $\mathfrak{N}_f = h^1/h^0(\tau_{\geq -1} L_f^\bullet)$.

3.21. Remark. Notice that, smooth locally on $\mathcal{M}$ and $\mathfrak{M}$, the morphism f factors as $\mathcal{M} \xrightarrow{i} M \xrightarrow{p} \mathfrak{M}$, with i a closed embedding and p representable and smooth. More explicitly, let V be a smooth atlas for $\mathfrak{M}$ and let U be an affine scheme which is an ètale atlas for $\mathcal{M} \times_{\mathfrak{M}} V$. In particular there exists a closed embedding $j \colon U \hookrightarrow \mathbb{A}^n_S$. Let us set $M = \mathbb{A}^n_S \times_S V$ and let $f_U \colon U \to V$ be the morphism induced by f, then f_U factors as $U \xrightarrow{i} M \xrightarrow{p} V$, where i is a closed embedding and p is smooth. Moreover, by Theorem 3.18, we have $\mathfrak{N}_{f_U} \cong [A(C_i)/T_p|_U]$, where $C_i = C(\mathscr{I}/\mathscr{I}^2)$ and $\mathscr{I}$ is the ideal sheaf corresponding to i.

3.22. **Proposition.** *There exists a unique closed subcone stack $\mathfrak{C}_f \subseteq \mathfrak{N}_f$ such that*

(1) if f factors as $p \circ i$, with i representable closed embedding and p representable smooth, then $\mathfrak{C}_f = [C_i/T_p|_{\mathcal{M}}]$;

(2) for every smooth morphism $V \to \mathfrak{M}$, let $g\colon U = V \times_{\mathfrak{M}} \mathcal{M} \to V$ be the induced morphism, then $\mathfrak{C}_g \cong \mathfrak{C}_f \times_{\mathcal{M}} U$.

If moreover $\mathfrak{M}$ is purely dimensional of pure dimension n, then $\mathfrak{C}_f$ is purely dimensional of pure dimension n.

Proof. By Remark 3.21, smooth locally on $\mathcal{M}$, there exists a commutative diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d] & M \arrow[d, "p"] \\ \mathcal{M} \arrow[r, "f"] & \mathfrak{M} \end{tikzcd}$$

where U is a scheme, i is a closed embedding and the vertical arrows are smooth. Let us set $f_U = p \circ i$, then $\mathfrak{N}_{f_U} \cong [A(C_i)/T_p|_U]$. By [6] 3.2, the action of $T_p|_U$ on $A(C_i)$ leaves C_i invariant, hence we can define the quotient stack $[C_i/T_p|_U]$ which is a subcone stack of $\mathfrak{N}_{f_U}$. First of all, we show that $[C_i/T_p|_U]$ does not depend on the factorization chosen. Let $U \xrightarrow{i'} M' \xrightarrow{p'} \mathfrak{M}$ be an other factorization of f_U and consider the following commutative diagram

$$\begin{tikzcd} & M' \arrow[dr, "p'"] & \\ U \arrow[ur, "i'"] \arrow[r, "j"] \arrow[dr, "i"'] & M \times_{\mathfrak{M}} M' \arrow[u] \arrow[r, "q"] \arrow[d, "\pi"] & \mathfrak{M} \\ & M \arrow[ur, "p"'] & \end{tikzcd}$$

In particular j is a closed embedding and both q and π are smooth. Hence it is enough to check

$$[C_i/T_p|_U] = [C_j/T_q|_U]$$

as closed substacks of $\mathfrak{N}_{f_U}$. By [12], Example 4.2.6, we have the following cartesian diagram

$$\begin{tikzcd} C_j \arrow[r] \arrow[d] & C_i \arrow[d] \\ A(C_j) \arrow[r, "\varphi"] & A(C_i) \end{tikzcd}$$

Moreover, by Remark 3.21, both $A(C_i)$ and $A(C_j)$ are smooth atlases for $\mathfrak{N}_{f_U}$ and the following diagram

$$\begin{tikzcd} A(C_j) \arrow[r, "\alpha_j"] \arrow[d, "\varphi"'] & \mathfrak{N}_{f_U} \\ A(C_i) \arrow[ur, "\alpha_i"'] & \end{tikzcd}$$

is commutative; therefore it is enough to show that the inverse image of $[C_i/T_p|_U]$ and $[C_j/T_q|_U]$ in $A(C_j)$ are the same. We have

$$\varphi^{-1}(\alpha_i^{-1}([C_i/T_p|_U])) = \varphi^{-1}(C_i) = C_j = \alpha_j^{-1}([C_j/T_q|_U]).$$

It follows that $[C_i/T_p|_U]$ depends only on f_U, hence $\mathfrak{C}_{f_U} = [C_i/T_p|_U]$ is well-defined.

Now we want to prove that, given $f_U = p \circ i$ and $f_{U'} = p' \circ i'$ as above, the cone-stacks $\mathfrak{C}_{f_U}$ and $\mathfrak{C}_{f_{U'}}$ agree on $U \times_{\mathcal{M}} U'$. Consider the following commutative diagram

$$\begin{tikzcd}
V = U \times_{\mathcal{M}} U' \arrow[r] \arrow[d] & U \arrow[d] \arrow[dr, "i"] & \\
U' \arrow[r] \arrow[dr, "i'"'] & \mathcal{M} \arrow[dr, "f"] & M \arrow[d, "p"] \\
& M' \arrow[r, "p'"'] & \mathfrak{M}
\end{tikzcd}$$

where i and i' are closed embeddings, the maps p, p' and the vertacal morphisms are smooth. Let $N = M \times_{\mathfrak{M}} M'$, then we have the following commutative diagram

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "v"'] & N \arrow[r, "q"] \arrow[d, "\pi"] & \mathfrak{M} \arrow[d, equal] \\
U \arrow[r, "i"] & M \arrow[r, "p"] & \mathfrak{M}
\end{tikzcd}$$

with v, q and π smooth and j a closed embedding. Let $f_V = q \circ j$. It is enough to show that $\mathfrak{C}_{f_V} \cong v^*\mathfrak{C}_{f_U}$. Recall that $\mathfrak{C}_{f_U} = [C_i/T_p|_U]$, hence

$$v^*\mathfrak{C}_{f_U} = v^*[C_i/i^*T_p] = [v^*C_i/j^*\pi^*T_p].$$

By Example 4.2.6 in [12], there is a short exact sequence of cones over V

$$0 \to j^*T_\pi \to C_j \to v^*C_i \to 0.$$

Since π, p and q are smooth, by Theorem 3.18, we have a short exact sequence

$$0 \to T_\pi \to T_q \to \pi^*T_p \to 0,$$

hence, pulling back via j and noticing that $j^*\pi^*T_p = v^*i^*T_p$, we get the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & j^*T_\pi \arrow[r] \arrow[d, equal] & j^*T_q \arrow[r] \arrow[d] & v^*i^*T_p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & j^*T_\pi \arrow[r] & C_j \arrow[r] & v^*C_i \arrow[r] & 0
\end{tikzcd}$$

from which we obtain

$$\mathfrak{C}_{f_V} = [C_j/j^*T_q] \cong [v^*C_i/v^*i^*T_p] = v^*[C_i/i^*T_p] = v^*\mathfrak{C}_{f_U}.$$

Finally, let us assume that $\mathfrak{M}$ is purely dimensional of pure dimension n. Let $U \xrightarrow{u} \mathcal{M}$ be a smooth atlas such that $f_U = f \circ u$ factors as $U \xrightarrow{i} M \xrightarrow{p} \mathfrak{M}$, with i a closed embedding and p smooth (such an atlas exists by Remark 3.21). Then $\mathfrak{C}_{f_U} = [C_i/i^*T_p]$. We can assume that M is purely dimensional of pure dimension m and p is smooth of relative dimension $m - n$. By [12], B.6.6, we have that C_i is purely dimensional of pure dimension m. Moreover, we have the following cartesian diagram

$$\begin{tikzcd}
i^*T_p \times C_i \arrow[r] \arrow[d] & C_i \arrow[d] \\
C_i \arrow[r] & {[C_i/i^*T_p]}
\end{tikzcd}$$

from which we get that $C_i \to [C_i/i^*T_p]$ is surjective and smooth of relative dimension $m - n$. It follows that $[C_i/i^*T_p]$ is purely dimensional of pure dimension n. $\square$

3.23. **Definition.** The unique closed subcone stack $\mathfrak{C}_f$ of $\mathfrak{N}_f$ is called the *relative intrinsic normal cone* of f.

3.24. **Proposition.** *Consider the following commutative diagram of algebraic Artin stacks over a scheme S,*

$$
(1)\qquad
\begin{array}{ccc}
\mathcal{M}' & \xrightarrow{f'} & \mathfrak{M}' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\
\mathcal{M} & \xrightarrow{f} & \mathfrak{M}
\end{array}
$$

where the morphisms f and f' are of Deligne-Mumford type. Then there exists a natural morphism $\alpha\colon \mathfrak{C}_{f'} \to g^*\mathfrak{C}_f$ such that

(1) if (1) is cartesian then α is a closed immersion;
(2) if moreover the morphism h is flat then α is an isomorphism.

Proof. Let $U \xrightarrow{u} \mathcal{M}$ and $U' \xrightarrow{u'} \mathcal{M}'$ be smooth affine atlases with a morphism $g_U : U \to U'$ such that $u' \circ g_u = g \circ u$. There exist closed embeddings $U \hookrightarrow \mathbb{A}^n_S \hookleftarrow U'$, for some n. Let $V \to \mathfrak{M}$ be a smooth atlas and let $M = V \times_S \mathbb{A}^n_S$. Then $f_U = f \circ u$ factors as $U \xrightarrow{i} M \xrightarrow{p} \mathfrak{M}$, with i a closed embedding and p smooth. Moreover $f'_{U'} = f' \circ u'$ factors as $U' \xrightarrow{i'} M' \xrightarrow{p'} \mathfrak{M}'$, where $M' = M \times_{\mathfrak{M}} \mathfrak{M}'$, the morphism i' is a closed embedding and p' is smooth. We have the following commutative diagram

$$
\begin{array}{ccccc}
U' & \xrightarrow{i'} & M' & \xrightarrow{p'} & \mathfrak{M}' \\
\downarrow{\scriptstyle \widetilde{g}} & & \downarrow{\scriptstyle \widetilde{h}} & & \downarrow{\scriptstyle h} \\
U & \xrightarrow{i} & M & \xrightarrow{p} & \mathfrak{M}
\end{array}
$$

By Theorem 3.18, there is a morphism $T_{p'} \to \widetilde{h}^*T_p$, which is an isomorphism if h is flat. Moreover there exists a morphism $\widetilde{\alpha}\colon C_{i'} \to \widetilde{g}^*C_i$, induced by the natural map $\mathscr{I} \otimes_{\mathscr{O}_U} \mathscr{O}_{U'} \to \mathscr{I}'$, where $\mathscr{I}$ is the ideal sheaf of U in M and $\mathscr{I}'$ is the ideal sheaf of U' in M' ([12], Appendix B.6). Then we get a commutative diagram

$$
\begin{array}{ccc}
T_{p'}|_{U'} & \longrightarrow & C_{i'} \\
\downarrow & & \downarrow{\scriptstyle \widetilde{\alpha}} \\
\widetilde{g}^*(T_p|_U) & \longrightarrow & \widetilde{g}^*C_i
\end{array}
$$

from which we obtain a morphism of stacks

$$\alpha\colon \mathfrak{C}_{f'_{U'}} = [{}^{C_{i'}}/_{T_{p'}|_{U'}}] \to [{}^{\widetilde{g}^*C_i}/_{\widetilde{g}^*(T_p|_U)}] = \widetilde{g}^*\mathfrak{C}_{f_U}.$$

If (1) is cartesian then $U' = U \times_M M'$ and the morphism $\hat{\alpha}$ is a closed embedding, since the morphism $\mathscr{I} \otimes_{\mathscr{O}_U} \mathscr{O}_{U'} \to \mathscr{I}'$ is surjective ([12], Appendix B.6). Moreover, by Theorem 3.18, we have $T_{p'} \cong \widetilde{h}^*T_p$. It follows that α is a closed embedding. If moreover h is flat then also $\widetilde{h}$ is flat and hence $\widetilde{\alpha}$ is an isomorphism, because $\mathscr{I} \otimes_{\mathscr{O}_U} \mathscr{O}_{U'} \cong \mathscr{I}'$ ([12], Appendix B.6). Hence α is an isomorphism. □

3.4. **Perfect obstruction theories.** Let $f\colon \mathcal{M} \to \mathfrak{M}$ be a morphism of Artin stacks over S. Assume that f is of Deligne-Mumford type.

3.25. **Definition** ([6] 4.4)**.** Let $E^\bullet \in D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M})$. A morphism $\varphi\colon E^\bullet \to \tau_{\geq -1}L^\bullet_f$ in $D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M})$ is called a *relative obstruction theory* for f if $h^0(\varphi)$ is an isomorphism and $h^{-1}(\varphi)$ is surjective.

3.26. Remark. If $(E^\bullet, \varphi)$ is a relative obstruction theory for f, then, by Proposition 3.16, the morphism ${}^{h^1}/_{h^0}(\varphi)\colon \mathfrak{N}_f \to {}^{h^1}/_{h^0}(E^\bullet)$ is a closed embedding.

3.27. **Theorem.** *A pair $(E^\bullet, \varphi)$ is a relative obstruction theory for f if and only if, for any geometric point $\overline{s}$ of S, for any small extension $A' \to A = {}^{A'}\!/_{I}$ in $({}^{Art}\!/_{\hat{\mathscr{O}}_{S,\overline{s}}})$ and any commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "g"] \arrow[d, "i"'] & \mathcal{M} \arrow[d, "f"] \\ \operatorname{Spec} A' \arrow[r, "h'"] & \mathfrak{M} \end{tikzcd}$$

the obstruction $h^1(\varphi^\vee)(ob_f(g,h')) \in h^1((g^*E^\bullet)^\vee) \otimes I$ vanishes if and only if there exists a morphism $g' \colon \operatorname{Spec} A' \to \mathcal{M}$ such that $g' \circ i = g$, $f \circ g' = h'$, and moreover if $h^1(\varphi^\vee)(ob_f(g,h')) = 0$ then the set of isomorphism classes of such morphisms g' is a torsor under $h^0((g^*E^\bullet)^\vee) \otimes I$.

Proof. If $(E^\bullet, \varphi)$ is a relative obstruction theory for f, the statement follows immediately from Proposition A.11. Viceversa, let assume that the second part of the statement holds and let show that $h^0(\varphi)$ is an isomorphism and $h^{-1}(\varphi)$ is surjective. Since the statement is local, we can assume that $\mathcal{M}$ is an affine scheme $\operatorname{Spec} R$. Then, by assumptions, for every R-algebra B and B-module N, there is a bijection $\hom(h^0(L_f^\bullet) \otimes B, N) \to \hom(h^0(E^\bullet) \otimes B, N)$, which implies that $h^0(\varphi)$ is an isomorphism. We can assume that f factors as $\mathcal{M} \xrightarrow{i} M \xrightarrow{p} \mathfrak{M}$ with i a closed embedding with ideal sheaf $\mathscr{I}$ into an affine scheme M and p smooth. We can further assume that E^0 is locally free, E^{-1} is a coherent sheaf, $E^i = 0$ for $i \neq 0, -1$ and φ^{-1} surjective. Then we easily see that the complex $G \to i^*\Omega_p$, where G is the cokernel of $\ker \varphi^0 \times_{E^0} E^{-1} \to E^{-1}$, is quasi-isomorphic to $E^\bullet$. Therefore we can assume $E^0 = i^*\Omega_p$ and we have to prove that $E^{-1} \to {}^{\mathscr{I}}\!/_{\mathscr{I}^2}$ is surjective; let F be its image. Let $\mathcal{M} = \operatorname{Spec} A$, $F' \subset \mathscr{I}$ the inverse image of F, and $\operatorname{Spec} A' \subset M$ the subscheme defined by F'; let $g \colon \operatorname{Spec} A \to \mathcal{M}$ be the identity. We can extend g to the inclusion $g' \colon \operatorname{Spec} A' \to M$. Let $\pi \colon {}^{\mathscr{I}}\!/_{\mathscr{I}^2} \to {}^{\mathscr{I}}\!/_{F'}$ be the natural projection. By assumption π factors via E^0 if and only if g extends to a map $\operatorname{Spec} A' \to \mathcal{M}$, if and only if $\pi \circ \varphi^{-1} \colon E^{-1} \to {}^{\mathscr{I}}\!/_{F'}$ factors via E^0. As $\pi \circ \varphi^{-1}$ is the zero map, it certainly factors. Therefore π also factors. Moreover, the fact that π factors via E^0 together with $\pi \circ \varphi^{-1} = 0$ implies $\pi = 0$, hence $\varphi^{-1} \colon E^{-1} \to {}^{\mathscr{I}}\!/_{\mathscr{I}^2}$ is surjective. $\square$

3.28. **Definition** ([6] 5.1)**.** Let $(E^\bullet, \varphi)$ be a relative obstruction theory for f. We say that $(E^\bullet, \varphi)$ is *perfect* (of perfect amplitude contained in $[-1, 0]$) if, smooth locally over $\mathcal{M}$, it is isomorphic to $[E^{-1} \to E^0]$ with E^{-1}, E^0 locally free sheaves over $\mathcal{M}$.

3.29. Remark. A relative obstruction theory $(E^\bullet, \varphi)$ is perfect if and only if ${}^{h^1}\!/_{h^0}(E^\bullet)$ is a vector bundle stack over $\mathcal{M}$.

3.5. **Virtual fundamental class.** Let D be a Dedekind domain and set $S = \operatorname{Spec} D$. Let $f \colon \mathcal{M} \to \mathfrak{M}$ be a morphism of Deligne-Mumford type of Artin stacks over S. Assume that $\mathfrak{M}$ is purely dimensional of pure dimension m and that $\mathcal{M}$ admits a stratification by global quotients. Let $(E^\bullet, \varphi)$ be a perfect relative obstruction theory for f, we denote by

$$\mu \colon \mathfrak{E}_f = {}^{h^1}\!/_{h^0}(E^\bullet) \to \mathcal{M}$$

the associated vector bundle stack of rank r. By Remark 3.26, the relative intrinsic normal cone $\mathfrak{C}_f$ is a closed substack of $\mathfrak{E}_f$. Moreover, by Theorem B.2 and [14] Proposition 3.5.10, the flat pullback

$$\mu^* \colon A_*({}^{\mathcal{M}}\!/_{k}) \to A_{*+r}({}^{\mathfrak{E}_f}\!/_{k})$$

is an isomorphism and we denote the inverse by $0^!$.

3.30. **Definition.** The *virtual fundamental class* of $\mathcal{M}$ relative to $(E^\bullet, \varphi)$ is the cycle class

$$[\mathcal{M}, E^\bullet]^{\mathrm{virt}} = 0^![\mathfrak{C}_f] \in A_*({}^{\mathcal{M}}\!/_{S}).$$

3.31. Remark. The intrinsic cone $\mathfrak{C}_f$ is purely dimensional of pure dimension m, therefore $[\mathcal{M}, E^\bullet]^{\mathrm{virt}} \in A_{m-r}({}^{\mathcal{M}}\!/_{S})$ and $m - r$ is called the *virtual dimension* of $\mathcal{M}$.

3.32. **Proposition.** *Consider the following cartesian diagram of Artin stacks over S,*

$$\begin{tikzcd} \mathcal{M}' \arrow[r, "f'"] \arrow[d, "g"'] & \mathfrak{M}' \arrow[d, "h"] \\ \mathcal{M} \arrow[r, "f"] & \mathfrak{M} \end{tikzcd}$$

where f and f' are of Deligne-Mumford type, $\mathfrak{M}$ and $\mathfrak{M}'$ are smooth and purely dimensional of pure dimension m, $\mathcal{M}$ and $\mathcal{M}'$ admit stratifications by global quotients. Let $(E^\bullet, \varphi)$ be a perfect relative obstruction theory for f. If h is flat or a regular local immersion (of constant dimension) then

$$h^![\mathcal{M}, E^\bullet]^{virt} = [\mathcal{M}', Lg^*E^\bullet]^{virt}.$$

Proof. Let us notice that $Lg^*E^\bullet$ is a perfect relative obstruction theory for f'. The statement follows by Theorem B.2 in the same way as in [6] 7.2. Let $\mathfrak{C}_{f'} = {}^{h^1}/{}_{h^0}(Lg^*E^\bullet)$ and let $0'^!$ the inverse of μ'^*, where $\mu' \colon \mathfrak{C}_{f'} \to \mathcal{M}'$. If h is flat then, by Proposition 3.24, we have $g^*\mathfrak{C}_f \cong \mathfrak{C}_{f'}$, hence $h^![\mathfrak{C}_f] = [\mathfrak{C}_{f'}]$. Therefore we get

$$h^![\mathcal{M}, E^\bullet]^{\mathrm{virt}} = h^!0^![\mathfrak{C}_f] = 0'^!h^![\mathfrak{C}_f] = 0'^![\mathfrak{C}_{f'}] = [\mathcal{M}', Lg^*E^\bullet]^{\mathrm{virt}}.$$

If h is a regular local immersion, let consider $\rho \colon \mathfrak{N}_h \to \mathfrak{M}'$ and let $\widetilde{0} \colon g^*\mathfrak{C}_f \to \mathfrak{N}_h \times_{\mathfrak{M}} \mathfrak{C}_f$ be the zero section. Then $\widetilde{0}^![\mathfrak{C}_{g^*\mathfrak{C}_f/\mathfrak{C}_f}] = h^![\mathfrak{C}_f]$, by definition of $h^!$, and

$$\widetilde{0}^![\rho^*\mathfrak{C}_{f'}] = \widetilde{0}^!\rho^*[\mathfrak{C}_{f'}] = [\mathfrak{C}_{f'}].$$

Moreover, by Theorem B.2 and [6] 3.3–3.5, we have that $[\mathfrak{C}_{g^*\mathfrak{C}_f/\mathfrak{C}_f}] = [\rho^*\mathfrak{C}_{f'}]$. Hence $h^![\mathfrak{C}_f] = [\mathfrak{C}_{f'}]$ and one concludes as before. $\square$

3.33. Let us consider a cartesian diagram

$$\begin{tikzcd} \mathcal{M}' \arrow[r, "g"] \arrow[d, "f'"'] & \mathcal{M} \arrow[d, "f"] \\ \mathfrak{M}' \arrow[r, "h"] & \mathfrak{M} \end{tikzcd}$$

where vertical arrow are morphisms of Deligne-Mumford type and h is a local complete intersection morphism of S-stacks with finite unramified diagonal over S. Let $E^\bullet$ and $E'^\bullet$ be perfect obstruction theories for f and f' respectively. Then $E^\bullet$ and $E'^\bullet$ are *compatible* over h if there exists a homomorphism of distinguished triangles

$$\begin{tikzcd} g^*E^\bullet \arrow[r] \arrow[d] & E'^\bullet \arrow[r] \arrow[d] & f'^*L^\bullet_h \arrow[r] \arrow[d] & g^*E^\bullet[1] \arrow[d] \\ g^*L^\bullet_{\mathcal{M}} \arrow[r] & L^\bullet_{\mathcal{M}'} \arrow[r] & L^\bullet_g \arrow[r] & g^*L^\bullet_{\mathcal{M}}[1] \end{tikzcd}$$

in $D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M}')$.

3.34. **Proposition.** *Let $E^\bullet$ and $E'^\bullet$ be compatible perfect obstruction theories as above. If either h is smooth or $\mathfrak{M}$ and $\mathfrak{M}'$ are smooth over S, then*

$$h^![\mathcal{M}, E^\bullet]^{virt} = [\mathcal{M}', E'^\bullet]^{virt}.$$

Proof. By [6] 2.7, the distinguished triangle

$$g^*E^\bullet \to E'^\bullet \to f'^*L^\bullet_h \to g^*E^\bullet[1]$$

induces a short exact sequence of vector bundle stacks

$$g^*\mathfrak{N}_h \to \mathfrak{C}_{f'} \xrightarrow{\psi} g^*\mathfrak{C}_f$$

over $\mathcal{M}'$. If h is smooth, by [6] 3.14, the following diagram is cartesian

$$\begin{tikzcd} \mathfrak{C}_{f'} \arrow[r] \arrow[d] & g^*\mathfrak{C}_f \arrow[d] \\ \mathfrak{E}_{f'} \arrow[r] & g^*\mathfrak{E}_f \end{tikzcd}$$

hence $0^!_{g^*\mathfrak{E}_f}[g^*\mathfrak{C}_f] = 0^!_{\mathfrak{E}_{f'}}[\mathfrak{C}_{f'}]$ and we have

$$h^![\mathcal{M}, E^\bullet]^{\mathrm{virt}} = h^! 0^!_{\mathfrak{E}_f}[\mathfrak{C}_f] = 0^!_{g^*\mathfrak{E}_f}[g^*\mathfrak{C}_f] = [\mathcal{M}', E'^\bullet]^{\mathrm{virt}}.$$

If $\mathfrak{M}$ and $\mathfrak{M}'$ are smooth then h factors as $\mathfrak{M}' \xrightarrow{\Gamma_h} \mathfrak{M}' \times_S \mathfrak{M} \xrightarrow{p} \mathfrak{M}$, where Γ_h is the graph of h, which is a regular local immersion, and p is smooth. We have the following cartesian diagram

$$\begin{tikzcd} \mathcal{M}' \arrow[r] \arrow[d, "f'"'] & \mathfrak{M}' \times_S \mathcal{M} \arrow[r] \arrow[d, "\widetilde{f}"] & \mathcal{M} \arrow[d, "f"] \\ \mathfrak{M}' \arrow[r, "\Gamma_h"] & \mathfrak{M}' \times_S \mathfrak{M} \arrow[r, "p"] & \mathfrak{M} \end{tikzcd}$$

and we consider the obstruction theory $\Omega_{\mathfrak{M}'} \oplus E^\bullet$ for $\widetilde{f}$. Notice that $\Omega_{\mathfrak{M}'} \oplus E^\bullet$ is compatible with $E^\bullet$ over p and with $E'^\bullet$ over Γ_h. Then, by the first part, we can assume that h is a regular local immersion. By [6] 5.9, $[f'^*\mathfrak{N}_h \times_{\mathcal{M}'} \mathfrak{C}_{f'}] = \psi^*[\mathfrak{C}_{g^*\mathfrak{C}_f/\mathfrak{C}_f}]$ and hence

$$\begin{aligned}[\mathcal{M}', E'^\bullet]^{\mathrm{virt}} = 0^!_{\mathfrak{E}_{f'}}[\mathfrak{C}_{f'}] &= 0^!_{f'^*\mathfrak{N}_h \times_{\mathcal{M}'} \mathfrak{E}_{f'}}[f'^*\mathfrak{N}_h \times_{\mathcal{M}'} \mathfrak{C}_{f'}] = 0^!_{f'^*\mathfrak{N}_h \times_S g^*\mathfrak{E}_f}[\mathfrak{C}_{g^*\mathfrak{C}_f/\mathfrak{C}_f}] \\ &= 0^!_{g^*\mathfrak{E}_f} h^![\mathfrak{C}_f] = h^! 0^!_{\mathfrak{E}_f}[\mathfrak{C}_f] = h^![\mathcal{M}, E^\bullet]^{\mathrm{virt}}.\end{aligned}$$ □

4. A virtual fundamental class

Let D be a Dedekind domain and set $S = \operatorname{Spec} D$. Let $\mathcal{X}$ be a smooth proper tame Deligne-Mumford stack of finite presentation over S, admitting a projective coarse moduli scheme X. We want to define a virtual fundamental class for $\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)$ relative to the forgetful morphism

$$\theta \colon \mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta) \to \mathfrak{M}^{\mathrm{tw}}_{g,\,n/S},$$

following the construction of 3.5. For this, we need a perfect relative obstruction theory for θ.

4.1. **The stack of morphisms.** With notations as in Definition 2.7, notice that $\overline{\mathscr{C}} = \mathcal{C} \times_{\mathfrak{M}^{\mathrm{tw}}_{g,n}} \underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}, \mathcal{X})$ is a universal family for $\underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}, \mathcal{X})$ and we have the following commutative diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d, "\pi"] \arrow[rr, bend left, "\psi"] & \overline{\mathscr{C}} \arrow[r, "\overline{\psi}"'] \arrow[d, "\overline{\pi}"] & \mathcal{X} \\ \mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta) \arrow[r] & \underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}, \mathcal{X}) & \end{tikzcd}$$

4.1. **Lemma.** *We have* $F^\bullet = R\overline{\pi}_*(\overline{\psi}^*\Omega_{\mathcal{X}/S} \otimes \omega_{\overline{\pi}})[-1] \in D^{(-1,0)}_{coh}\left(\underline{Hom}_{\mathfrak{M}^{tw}_{g,n}}(\mathcal{C}, \mathcal{X})\right)$ *and* $h^1/h^0(F^\bullet)$ *is a vector bundle stack.*

Proof. Since $\mathcal{X}$ is smooth over S, the sheaf $\Omega_{\mathcal{X}/S}$ is a vector bundle over $\mathcal{X}$. The dualizing sheaf $\omega_{\overline{\pi}}$ is a line bundle over $\overline{\mathscr{C}}$, because $\overline{\mathscr{C}}$ is a family of curves with at most nodal singularities (which are Gorenstein). Hence $\overline{\psi}^*\Omega_{\mathcal{X}/S} \otimes \omega_{\overline{\pi}}$ is a vector bundle on $\overline{\mathscr{C}}$. Recall that the cohomology of the total pushforward is the higher pushforward sheaf. Moreover, $\overline{\pi}$ is a flat projective morphism of relative dimension 1, so the i-pushforward vanishes for $i > 1$ by the cohomology and base-change theorem ([11] Corollary 8.3.4), therefore

$$R\overline{\pi}_*(\overline{\psi}^*\Omega_{\mathcal{X}/S} \otimes \omega_{\overline{\pi}}) \in D^{(0,1)}_{\mathrm{coh}}\left(\underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C},\mathcal{X})\right).$$

Set $\mathscr{F} = \overline{\psi}^*\Omega_{\mathcal{X}/S} \otimes \omega_{\overline{\pi}}$. Let $\mathscr{L}$ be a $\overline{\pi}$-ample line bundle then, for n big enough, $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections and $R^0\overline{\pi}_*(\mathscr{F} \otimes \mathscr{L}^{-n}) = 0$. Thus we have a surjection

$$\mathscr{G} = \overline{\pi}^*\overline{\pi}_*(\mathscr{F} \otimes \mathscr{L}^n) \otimes \mathscr{L}^{-n} \to \mathscr{F},$$

and we denote by $\mathscr{K}$ the kernel. Notice that $\mathscr{K}$ is a vector bundle because it is the kernel of a surjection of vector bundles. Hence we get the following exact sequence

$$0 \to R^0\overline{\pi}_*\mathscr{K} \to R^0\overline{\pi}_*\mathscr{G} \to R^0\overline{\pi}_*\mathscr{F} \to R^1\overline{\pi}_*\mathscr{K} \to R^1\overline{\pi}_*\mathscr{G} \to R^1\overline{\pi}_*\mathscr{F} \to 0.$$

Since $R^0\overline{\pi}_*(\mathscr{F} \otimes \mathscr{L}^{-n}) = 0$, we have that $R^0\overline{\pi}_*\mathscr{G} = 0$ and thus $R^0\overline{\pi}_*\mathscr{K} = 0$. As a consequence, $R^1\overline{\pi}_*\mathscr{K}$ and $R^1\overline{\pi}_*\mathscr{G}$ are vector bundles and $F^\bullet$ is quasi-isomorphic to $[R^1\overline{\pi}_*\mathscr{K} \to R^1\overline{\pi}_*\mathscr{G}]$. $\square$

4.2. We define a morphism $\overline{\varphi}\colon F^\bullet \to \tau_{\geq -1}L^\bullet_{\overline{\theta}}$ in $D^{(-1,0)}_{\mathrm{coh}}\left(\underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C},\mathcal{X})\right)$ as follows. By adjuction, this is equivalent to define a morphism

$$(\overline{\psi}^*\Omega_{\mathcal{X}/S} \otimes \omega_{\overline{\pi}})[-1] \to L\overline{\pi}^!(L^\bullet_{\overline{\theta}}).$$

Recall that if $\overline{\pi}$ is a flat proper Gorenstein morphism of relative dimension N, then $L\overline{\pi}^! = \overline{\pi}^* \otimes \omega_{\overline{\pi}}[-N]$. This applies in our case with $N = 1$ and we get $L\overline{\pi}^! = \overline{\pi}^* \otimes \omega_{\overline{\pi}}[-1]$. Hence to give the morphism $\overline{\varphi}$ is equivalent to giving a morphism $\overline{\psi}^*\Omega_{\mathcal{X}/S} \to \overline{\pi}^*L^\bullet_{\overline{\theta}}$. Notice that $\Omega_{\mathcal{X}/S} = L^\bullet_{\mathcal{X}/S}$, since $\mathcal{X}$ is smooth over S (Theorem 3.18). We define the morphism $\overline{\psi}^*L^\bullet_{\mathcal{X}/S} \to \overline{\pi}^*L^\bullet_{\overline{\theta}}$ as the composition

$$\overline{\psi}^*L^\bullet_{\mathcal{X}/S} \to L^\bullet_{\overline{\mathscr{C}}/S} \to L^\bullet_{\overline{\mathscr{C}}/\mathcal{C}} \cong \overline{\pi}^*L^\bullet_{\overline{\theta}},$$

where $\mathcal{C}$ is the universal curve of $\mathfrak{M}^{\mathrm{tw}}_{g,\,n/S}$, the isomorphism $L^\bullet_{\overline{\mathscr{C}}/\mathcal{C}} \cong \overline{\pi}^*L^\bullet_{\overline{\theta}}$ follows from the fact that $\overline{\mathscr{C}} = \mathcal{C} \times_{\mathfrak{M}^{\mathrm{tw}}_{g,n}} \underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C},\mathcal{X})$ and the morphism $\mathcal{C} \to \mathfrak{M}^{\mathrm{tw}}_{g,\,n/S}$ is flat (Theorem 3.18).

4.3. **Proposition.** *The pair $(F^\bullet, \overline{\varphi})$ defined above is a perfect relative obstruction theory for $\overline{\theta}$.*

Proof. Let $\mathrm{Spec}\,\overline{k} \xrightarrow{\overline{x}} \underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C},\mathcal{X})$ be a geometric point. Then $\overline{x}$ corresponds to a twisted pointed curve $\mathcal{C}_{\overline{x}}$ over $\overline{k}$ together with a representable morphism $\overline{\psi}_{\overline{x}}\colon \mathcal{C}_{\overline{x}} \to \mathcal{X}$, obtained by pulling back $(\overline{\mathscr{C}}, \overline{\psi})$ along $\overline{x}$. Using Serre duality and cohomology and base chage theorem ([11] Corollary 8.3.4), we have

$$H^i(\mathcal{C}_{\overline{x}}, \overline{\psi}^*_{\overline{x}}T_{\mathcal{X}/S}) = H^{1-i}(\mathcal{C}_{\overline{x}}, \overline{x}^*(\overline{\psi}^*\Omega_{\mathcal{X}/S} \otimes \omega_{\overline{\pi}}))^\vee = h^{i-1}((F^\bullet[-1])^\vee) = h^i((L\overline{x}^*F^\bullet)^\vee).$$

Now, let $A' \to A = {}^{A'}\!/_I$ be a small extension in $({}^{\mathrm{Art}}\!/_{\hat{\mathscr{O}}_{S,\overline{s}}})$ and consider a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}\,A \arrow[r, "g"] \arrow[d, "i"'] & \underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C},\mathcal{X}) \arrow[d, "\overline{\theta}"] \\
\mathrm{Spec}\,A' \arrow[r, "h'"] & \mathfrak{M}^{\mathrm{tw}}_{g,\,n/S}
\end{tikzcd}$$

In particular h' corresponds to a family of twisted curves $\mathcal{C}_{A'}$ over A', obtained by pulling back $\mathcal{C} \to \mathfrak{M}^{\mathrm{tw}}_{g,\,n/S}$ along h', and g corresponds to a family of twisted curves $\mathcal{C}_A$ over A together with a representable morphism $\overline{\psi}_A\colon \mathcal{C}_A \to \mathcal{X}$, obtained by pulling back $(\overline{\mathscr{C}}, \overline{\psi})$ along g. Thus g extends to $g'\colon\ \mathrm{Spec}\,A' \to \underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C},\mathcal{X})$ if and only if $\overline{\psi}_A$ extends to $\overline{\psi}_{A'}\colon \mathcal{C}_{A'} \to \mathcal{X}$ if and only if, by Proposition A.11 and Proposition A.13, $h^1(\overline{\varphi}^\vee)(\mathrm{ob}_{\overline{\theta}}(g,h'))$ is zero in $H^1(\mathcal{C}_{\overline{x}}, \overline{\psi}^*_{\overline{x}}T_{\mathcal{X}/S}) \otimes I$. Moreover the extensions, if they exist, form a torsor under $H^0(\mathcal{C}_{\overline{x}}, \overline{\psi}^*_{\overline{x}}T_{\mathcal{X}/S}) \otimes I$. By Theorem 3.27, $(F^\bullet, \overline{\varphi})$ is a relative obstruction theory for $\overline{\theta}$ and, by Lemma 4.1, $F^\bullet$ is perfect. $\square$

4.2. **A perfect obstruction theory for $\mathcal{K}_{g,n}({\mathcal{X}}/{S}, \beta_\eta)$.**

4.4. **Corollary.** *Let $E^\bullet = R\pi_*(\psi^*\Omega_{\mathcal{X}/S} \otimes \omega_\pi)[-1]$ and let $\varphi\colon E^\bullet \to \tau_{\geq -1} L^\bullet_\theta$ be the morphism induced by $\overline{\varphi}$. Then $(E^\bullet, \varphi)$ is a perfect relative obstruction theory for θ.*

Proof. Since the natural inclusion $j\colon \mathcal{K}_{g,n}({\mathcal{X}}/{S}, \beta_\eta) \hookrightarrow \underline{\mathrm{Hom}}_{\mathfrak{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}, \mathcal{X})$ is an open immersion, it follows that $Lj^*L^\bullet_{\overline{\theta}} = L^\bullet_\theta$, $Lj^*F^\bullet = E^\bullet$, and $\varphi = j^*\overline{\varphi}$. Hence, by Lemma 4.1, we have $E^\bullet \in D^{(-1,0)}_{\mathrm{coh}}(\mathcal{K}_{g,n}({\mathcal{X}}/{S}, \beta_\eta))$. By Proposition 4.3, we know that $(F^\bullet, \overline{\varphi})$ is a perfect obstruction theory for $\overline{\theta}$, hence $h^0(\overline{\varphi})$ is an isomorphism and $h^{-1}(\overline{\varphi})$ is surjective. Since the pullback j^* is an exact functor, we have that $h^0(\varphi)$ is an isomorphism and $h^{-1}(\varphi)$ is surjective, which implies the statement. □

4.5. **Definition.** We define the *virtual fundamental class* of $\mathcal{K}_{g,n}({\mathcal{X}}/{S}, \beta_\eta)$ to be

$$[\mathcal{K}_{g,n}({\mathcal{X}}/{S}, \beta_\eta)]^{\mathrm{virt}} = [\mathcal{K}_{g,n}({\mathcal{X}}/{S}, \beta_\eta), E^\bullet]^{\mathrm{virt}} \in A_*({\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)}/{S}).$$

4.6. Remark. Consider the vector bundle stack $\mu\colon \mathfrak{E}_\theta = {h^1}/{h^0}(E^\bullet) \to \mathcal{K}_{g,n}({\mathcal{X}}/{S}, \beta_\eta)$. Then, for a geometric point $\overline{x}$ of a component $\mathcal{K}$ of $\mathcal{K}_{g,n}({\mathcal{X}}/{S}, \beta_\eta)$, by Riemann-Roch theorem ([2] 7.2.1),

$$\begin{aligned}
\operatorname{rk} \overline{x}^*\mathfrak{E}_\theta &= \dim h^{-1}(L\overline{x}^*E^\bullet) - \dim h^0(L\overline{x}^*E^\bullet)\\
&= \dim H^1(\mathscr{C}_{\overline{x}}, \psi^*_{\overline{x}}T_{\mathcal{X}/S}) - \dim H^0(\mathscr{C}_{\overline{x}}, \psi^*_{\overline{x}}T_{\mathcal{X}/S})\\
&= (g-1)\operatorname{rk}(\psi^*_{\overline{x}}T_{\mathcal{X}/S}) - c_1(\psi^*_{\overline{x}}T_{\mathcal{X}/S}) \cdot [\mathscr{C}_{\overline{x}}] + \sum_{i=1}^n \operatorname{age}(\Sigma_i)\\
&= (g-1)\dim_S \mathcal{X} - c_1(T_{\mathcal{X}/S}) \cdot \psi_{\overline{x}*}[\mathscr{C}_{\overline{x}}] + \sum_{i=1}^n \operatorname{age}(\Sigma_i),
\end{aligned}$$

where $\operatorname{age}(\Sigma_i) = \operatorname{age}(\psi^*_{\overline{x}}T_{\mathcal{X}/S}|_{\Sigma_i})$ denotes the age of a locally free sheaf as defined in [2] 7.1 (recall that the age is constant on connected components of $\overline{\mathcal{I}}_\mu(\mathcal{X})$). Thus the dimension of $[\mathcal{K}]^{\mathrm{virt}}$ is

$$\dim_S \mathfrak{M}^{\mathrm{tw}}_{g,\,{n}/{S}} - \operatorname{rk} \overline{x}^*\mathfrak{E}_\theta = (\dim_S \mathcal{X} - 3)(1-g) + c_1(T_{\mathcal{X}_\eta/\eta}) \cdot \psi_{\overline{x}*}[\mathscr{C}_{\overline{x}}] - \sum_{i=1}^n \operatorname{age}(\Sigma_i) + n.$$

4.3. **Properties.**

4.7 ([2] 5.1)**.** Let $\mathfrak{D}^{\mathrm{tw}}(g_1, A|g_2, B)$ be the category fibered in groupoids over $({\mathrm{Sch}}/{S})$ which parametrizes nodal twisted curves with a distinguished node separating the curve in two components, one of genus g_1 containing the markings in a subset $A \subset \{1, \dots, n\}$, the other of genus g_2 containing the markings in the complementary set B. The category $\mathfrak{D}^{\mathrm{tw}}(g_1, A|g_2, B)$ is a smooth algebraic stack, locally of finite presentation over S. Let $g = g_1 + g_2$, there is a natural representable morphism

$$\mathrm{gl}\colon \mathfrak{D}^{\mathrm{tw}}(g_1, A|g_2, B) \to \mathfrak{M}^{\mathrm{tw}}_{g,n}$$

induced by gluing the two families of curves into a family of reducible curves with a distinguished node.

4.8. **Proposition.** *(1) Consider the evaluation morphisms $\check{e}_{\check{\bullet}}\colon \mathcal{K}_{g_1, A\sqcup\check{\bullet}}(\mathcal{X}, \beta_1) \to \overline{\mathcal{I}}_\mu(\mathcal{X})$ and $e_\bullet\colon \mathcal{K}_{g_2, B\sqcup\bullet}(\mathcal{X}, \beta_2) \to \overline{\mathcal{I}}_\mu(\mathcal{X})$. There exists a natural representable morphism*

$$\mathcal{K}_{g_1, A\sqcup\check{\bullet}}(\mathcal{X}, \beta_1) \times_{\overline{\mathcal{I}}_\mu(\mathcal{X})} \mathcal{K}_{g_2, B\sqcup\bullet}(\mathcal{X}, \beta_2) \to \mathcal{K}_{g_1+g_2, A\sqcup B}(\mathcal{X}, \beta_1 + \beta_2).$$

(2) Consider the evaluation morphisms $\check{e}_{\check{\bullet}} \times e_\bullet\colon \mathcal{K}_{g-1, A\sqcup\{\check{\bullet}, \bullet\}}(\mathcal{X}, \beta_\eta) \to \overline{\mathcal{I}}_\mu(\mathcal{X})^2$ and the diagonal $\Delta\colon \overline{\mathcal{I}}_\mu(\mathcal{X}) \to \overline{\mathcal{I}}_\mu(\mathcal{X})^2$ (2.12). There exists a natural representable morphism

$$\mathcal{K}_{g-1, A\sqcup\{\check{\bullet}, \bullet\}}(\mathcal{X}, \beta_\eta) \times_{\overline{\mathcal{I}}_\mu(\mathcal{X})^2} \overline{\mathcal{I}}_\mu(\mathcal{X}) \to \mathcal{K}_{g,A}(\mathcal{X}, \beta_\eta).$$

(3) We have a cartesian diagram

$$\begin{tikzcd} \coprod_{\beta_1+\beta_2=\beta_\eta} \mathcal{K}_{g_1,A\sqcup\check{\bullet}}(\mathcal{X},\beta_1)\times_{\overline{\mathcal{I}}_\mu(\mathcal{X})}\mathcal{K}_{g_2,B\sqcup\bullet}(\mathcal{X},\beta_2) \arrow[r] \arrow[d] & \mathcal{K}_{g_1+g_2,A\sqcup B}(\mathcal{X},\beta_\eta) \arrow[d] \\ \mathfrak{D}^{tw}(g_1,A|g_2,B) \arrow[r, "gl"] & \mathfrak{M}^{tw}_{g_1+g_2,A\sqcup B} \end{tikzcd}$$

Proof. Follows in the same way as in [2] 5.2. □

4.9. By [2] 6.2.4, the morphism gl is finite and unramified, therefore, by [14] 4.1, it induces a pull-back homomorphism on Chow groups

$$\mathrm{gl}^!\colon A_*(\mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta))\to\bigoplus_{\beta_1+\beta_2=\beta_\eta}A_*(\mathcal{K}_{g_1,A\sqcup\check{\bullet}}(\mathcal{X},\beta_1)\times_{\overline{\mathcal{I}}_\mu(\mathcal{X})}\mathcal{K}_{g_2,B\sqcup\bullet}(\mathcal{X},\beta_2)).$$

4.10. **Proposition.** *Consider the diagonal* $\Delta\colon\overline{\mathcal{I}}_\mu(\mathcal{X})\to\overline{\mathcal{I}}_\mu(\mathcal{X})^2$ *(2.12). We have*

(1) $gl^![\mathcal{K}_{g,A\sqcup B}(\mathcal{X},\beta_\eta)]^{virt}=\sum_{\beta_1+\beta_2=\beta_\eta}\Delta^!([\mathcal{K}_{g_1,A\sqcup\check{\bullet}}(\mathcal{X},\beta_1)]^{virt}\times[\mathcal{K}_{g_2,B\sqcup\bullet}(\mathcal{X},\beta_2)]^{virt})$;

(2) $gl^![\mathcal{K}_{g,A}(\mathcal{X},\beta_\eta)]^{virt}=\Delta^![\mathcal{K}_{g-1,A\sqcup\{\check{\bullet},\bullet\}}(\mathcal{X},\beta_\eta)]^{virt}$.

Proof. For the first part, by Proposition 4.8 and Proposition 3.32,

$$\mathrm{gl}^![\mathcal{K}_{g,A\sqcup B}(\mathcal{X},\beta_\eta)]^{\mathrm{virt}}=\sum_{\beta_1+\beta_2=\beta_\eta}[\mathcal{K}_{g_1,A\sqcup\check{\bullet}}(\mathcal{X},\beta_1)\times_{\overline{\mathcal{I}}_\mu(\mathcal{X})}\mathcal{K}_{g_2,B\sqcup\bullet}(\mathcal{X},\beta_2)]^{\mathrm{virt}}.$$

Let us denote for simplicity $\mathcal{K}^{(1)}=\mathcal{K}_{g_1,A\sqcup\check{\bullet}}(\mathcal{X},\beta_1)$ and $\mathcal{K}^{(2)}=\mathcal{K}_{g_2,B\sqcup\bullet}(\mathcal{X},\beta_2)$. Let $E_j^\bullet$ be the perfect obstruction theory of $\mathcal{K}^{(j)}$ as constructed in section 4.2, then $E_1^\bullet\oplus E_2^\bullet$ is the perfect obstruction theory of $\mathcal{K}^{(1)}\times_k\mathcal{K}^{(2)}$. Let $E_{1,2}^\bullet$ be the perfect obstruction theory of $\mathcal{K}^{(1)}\times_{\overline{\mathcal{I}}_\mu(\mathcal{X})}\mathcal{K}^{(2)}$. By considering the normalization sequence for a family of nodal curves with a distinguished node Σ over $\mathcal{K}^{(1)}\times_{\overline{\mathcal{I}}_\mu(\mathcal{X})}\mathcal{K}^{(2)}$, we get the following distinguished triangle, as in [2] 5.3.1,

$$E_{1,2}^\bullet\to E_1^\bullet\oplus E_2^\bullet\to E_\Sigma^\bullet,$$

where $E_\Sigma^\bullet$ can be identified with the cotangent complex of the map Δ in the same way as in [2] 3.6.1. Then, by Proposition 3.34, we get

$$\Delta^!([\mathcal{K}_{g_1,A\sqcup\check{\bullet}}(\mathcal{X},\beta_1)]^{\mathrm{virt}}\times[\mathcal{K}_{g_2,B\sqcup\bullet}(\mathcal{X},\beta_2)]^{\mathrm{virt}})=[\mathcal{K}_{g_1,A\sqcup\check{\bullet}}(\mathcal{X},\beta_1)\times_{\overline{\mathcal{I}}_\mu(\mathcal{X})}\mathcal{K}_{g_2,B\sqcup\bullet}(\mathcal{X},\beta_2)]^{\mathrm{virt}}.$$

For the second part of the statement, we observe that, since Δ is a regular embedding,

$$\Delta^![\mathcal{K}_{g-1,A\sqcup\{\check{\bullet},\bullet\}}(\mathcal{X},\beta_\eta)]^{\mathrm{virt}}=[\mathcal{K}_{g-1,A\sqcup\{\check{\bullet},\bullet\}}(\mathcal{X},\beta_\eta)\times_{\overline{\mathcal{I}}_\mu(\mathcal{X})^2}\overline{\mathcal{I}}_\mu(\mathcal{X})]^{\mathrm{virt}},$$

and, by Proposition 3.32, the right-hand side is equal to $\mathrm{gl}^![\mathcal{K}_{g,A}(\mathcal{X},\beta_\eta)]^{\mathrm{virt}}$. □

5. Gromov-Witten classes and invariants

5.1. **Gromov-Witten classes.** Let D be a Dedekind domain, set $S=\operatorname{Spec}D$ and denote by η the generic point of S. Let $\mathcal{X}$ be a smooth proper tame Deligne-Mumford stack of finite presentation over S, admitting a projective coarse moduli scheme X. Set $X_\eta=X\times_S\eta$. Fix $\beta_\eta\in A_1(X_\eta/\eta)$ and $g,n\geq 0$ with $2g+n\geq 3$.

5.1. Remark. If $S=\operatorname{Spec}k$ with k an algebraically closed field and if l is a prime different from the characteristic of k, we can define the l-adic étale cohomology as

$$H^r(\overline{\mathcal{I}}_\mu(\mathcal{X}),\mathbb{Z}_l)=\varprojlim_m H^r_{\text{ét}}(\overline{\mathcal{I}}_\mu(\mathcal{X}),\mathbb{Z}/l^m\mathbb{Z}).$$

Moreover $H^r(\overline{\mathcal{I}}_\mu(\mathcal{X}),\mathbb{Q}_l)=H^r(\overline{\mathcal{I}}_\mu(\mathcal{X}),\mathbb{Z}_l)\otimes_{\mathbb{Z}_l}\mathbb{Q}_l$ and we have the cycle map

$$\mathrm{cl}\colon A^r(\overline{\mathcal{I}}_\mu(\mathcal{X})/k)_{\mathbb{Q}}\to H^{2r}(\overline{\mathcal{I}}_\mu(\mathcal{X}),\mathbb{Q}_l(r))$$

as described in [18] VI.9. We set $H^*(\overline{\mathcal{I}}_\mu(\mathcal{X})) = \sum_r H^r(\overline{\mathcal{I}}_\mu(\mathcal{X}), \mathbb{Q}_l(\overline{r}))$, where $\overline{r}$ is the integral part of $r/2$.

5.2. **Definition** (Gromov-Witten classes)**.** We define the linear operator

$$\mathrm{I}^{\mathcal{X}}_{g,n,\beta_\eta} \colon A^*(\overline{\mathcal{I}}_\mu(\mathcal{X})/S)^{\otimes n}_{\mathbb{Q}} \to A^*(\overline{\mathcal{M}}_{g,\,n/S}/S)_{\mathbb{Q}}$$

such that, given $\underline{\gamma} \in A^*(\overline{\mathcal{I}}_\mu(\mathcal{X})/S)^{\otimes n}_{\mathbb{Q}}$,

$$\mathrm{I}^{\mathcal{X}}_{g,n,\beta_\eta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = q_*\left(e^*(\underline{\gamma}) \cap [\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\mathrm{virt}}\right),$$

where $q \colon \mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta) \to \overline{\mathcal{M}}_{g,\,n/S}$ forgets the map to $\mathcal{X}$, passes to the coarse curve and stabilizes. If moreover $S = \operatorname{Spec} k$ with k an algebraically closed field, we can define

$$\mathrm{I}^{\mathcal{X}}_{g,n,\beta_\eta} \colon H^*(\overline{\mathcal{I}}_\mu(\mathcal{X}))^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,\,n/S})$$

as above, where, abusing the notation, we write $[\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\mathrm{virt}}$ instead of the corresponding homology class $\mathrm{cl}\left([\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\mathrm{virt}}\right)$.

5.3. **Definition** (Gromov-Witten invariants)**.** We define

$$\langle \mathrm{I}^{\mathcal{X}}_{g,n,\beta_\eta}\rangle(\underline{\gamma}) = \int_{\mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta)} \left(e^*(\underline{\gamma}) \cap [\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\mathrm{virt}}\right),$$

for $\underline{\gamma} = \gamma_1 \otimes \cdots \otimes \gamma_n \in A^*(\overline{\mathcal{I}}_\mu(\mathcal{X})/S)^{\otimes n}_{\mathbb{Q}}$. If $S = \operatorname{Spec} k$ with k an algebraically closed field then $\langle \mathrm{I}^{\mathcal{X}}_{g,n,\beta_\eta}\rangle(\underline{\gamma})$ is defined for every $\underline{\gamma} \in H^*(\overline{\mathcal{I}}_\mu(\mathcal{X}))^{\otimes n}$.

5.4. Notation. When $S = \operatorname{Spec} k$, we have $X_\eta = X$ and hence we will simply write β instead of β_η.

5.5. Remark. We have that

$$\begin{aligned}
\int_{\overline{\mathcal{M}}_{g,\,n/S}} \mathrm{I}^{\mathcal{X}}_{g,n,\beta_\eta}(\underline{\gamma}) &= \int_{\overline{\mathcal{M}}_{g,\,n/S}} q_*\left(e^*(\underline{\gamma}) \cap [\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\mathrm{virt}}\right) \\
&= \int_{\mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta)} \left(e^*(\underline{\gamma}) \cap [\mathcal{K}_{g,n}(\mathcal{X}/S, \beta_\eta)]^{\mathrm{virt}}\right) \\
&= \langle \mathrm{I}^{\mathcal{X}}_{g,n,\beta_\eta}\rangle(\underline{\gamma}).
\end{aligned}$$

5.6. **Definition** ([2] 6.1.1)**.** We define a locally constant function $\mathfrak{r} \colon \overline{\mathcal{I}}_\mu(\mathcal{X}) \to \mathbb{Z}$ by evaluating on geometric points, $\mathfrak{r}(\overline{x}, \mathcal{G}) = r$, where $\mathcal{G}$ is a gerbe banded by μ_r. We can view $\mathfrak{r}$ as an element of $A^0(\overline{\mathcal{I}}_\mu(\mathcal{X}))$.

5.7 (Alternative definition)**.** Following the formalism of [2], we could define $\mathrm{I}^{\mathcal{X}}_{g,n,\beta_\eta}$ as a linear operator $A^*(\overline{\mathcal{I}}_\mu(\mathcal{X})/S)^{\otimes n}_{\mathbb{Q}} \to A^*(\overline{\mathcal{I}}_\mu(\mathcal{X})/S)_{\mathbb{Q}}$ such that

$$\mathrm{I}^{\mathcal{X}}_{g,n,\beta_\eta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = \mathfrak{r} \cdot \check{e}_{n+1*}\left(\left(\prod_{i=1}^n e_i^*(\gamma_i)\right) \cap [\mathcal{K}_{g,n+1}(\mathcal{X}/S, \beta_\eta)]^{\mathrm{virt}}\right).$$

With this definition,

$$\begin{aligned}
&\int_{\overline{\mathcal{I}}_\mu(\mathcal{X})} \frac{1}{\mathfrak{r}} \mathrm{I}^{\mathcal{X}}_{g,n-1,\beta_\eta}(\gamma_1\otimes\cdots\otimes\gamma_{n-1})\cap\iota^*(\gamma_n)\\
&\qquad = \int_{\overline{\mathcal{I}}_\mu(\mathcal{X})} \check{e}_{n*}\left(\left(\prod_{i=1}^{n-1} e_i^*(\gamma_i)\right)\cap[\mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta)]^{\mathrm{virt}}\right)\cap\iota^*(\gamma_n)\\
&\qquad = \int_{\overline{\mathcal{I}}_\mu(\mathcal{X})} \check{e}_{n*}\left(\left(\prod_{i=1}^{n-1} e_i^*(\gamma_i)\right)\cap[\mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta)]^{\mathrm{virt}}\cap\check{e}^*_{n+1}\iota^*(\gamma_n)\right)\\
&\qquad = \int_{\mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta)} \left(\left(\prod_{i=1}^{n-1} e_i^*(\gamma_i)\right)\cap[\mathcal{K}_{g,n}(\mathcal{X}/S,\beta_\eta)]^{\mathrm{virt}}\cap e^*_{n}(\gamma_n)\right)\\
&\qquad = \langle \mathrm{I}^{\mathcal{X}}_{g,n,\beta_\eta}\rangle(\underline{\gamma}).
\end{aligned}$$

5.8. Remark. Let $\mathcal{M}$ be a proper Artin stack over a field k. Let L be a finite algebraic extension of k, then $\mathcal{M}_L=\mathcal{M}\times_k L\xrightarrow{\rho_L}\mathcal{M}$ is smooth and finite of degree $[L:k]$. By [12] 1.7.4, $\rho_{L*}\rho_L^*=[L:k]$, therefore ρ_L^* gives an isomorphism $A_*(\mathcal{M}/k)_{\mathbb{Q}}\cong A_*(\mathcal{M}_L/L)_{\mathbb{Q}}$. Let $\overline{k}$ be an algebraic closure of k and set $\overline{\mathcal{M}}=\mathcal{M}\times_k\overline{k}$, then $A_*(\overline{\mathcal{M}}/\overline{k})=\varinjlim_L A_*(\mathcal{M}_L/L)$, where the limit is over all finite algebraic extensions L of k such that $L\subset\overline{k}$. There is an induced homomorphism $\rho\colon A_*(\mathcal{M}/k)\to A_*(\overline{\mathcal{M}}/\overline{k})$ which gives an isomorphism $A_*(\mathcal{M}/k)_{\mathbb{Q}}\cong A_*(\overline{\mathcal{M}}/\overline{k})_{\mathbb{Q}}$; for all $\beta\in A_*(\mathcal{M}/k)$ we set $\overline{\beta}=\rho(\beta)$. The same holds for bivariant Chow groups $A^*(\bullet)_{\mathbb{Q}}$.

5.9. **Proposition.** *Let $\mathcal{X}$ be a smooth proper tame Deligne-Mumford stack of finite presentation over a field k, admitting a projective coarse moduli scheme X, and set $\overline{\mathcal{X}}=\mathcal{X}\times_k\overline{k}$. Then, for all $\underline{\gamma}\in A^*(\mathcal{I}_\mu(\mathcal{X})/k)_{\mathbb{Q}}^{\otimes n}$,*

$$I^{\mathcal{X}}_{g,n,\beta}(\underline{\gamma})=I^{\overline{\mathcal{X}}}_{g,n,\overline{\beta}}(\underline{\gamma}).$$

Proof. Let L be a finite algebraic extension of k and set $\mathcal{X}_L=\mathcal{X}\times_k L$. Let $\beta_L=\rho_L^*\beta$. Notice that $\mathcal{K}_{g,n}(\mathcal{X}_L/L,\beta_L)\cong\mathcal{K}_{g,n}(\mathcal{X}/k,\beta)\times_k L$ and thus, by Proposition 3.32,

$$[\mathcal{K}_{g,n}(\mathcal{X}/k,\beta)]^{\mathrm{virt}}=[\mathcal{K}_{g,n}(\mathcal{X}_L/L,\beta_L)]^{\mathrm{virt}}\in A_*(\mathcal{K}_{g,n}(\mathcal{X}/k,\beta)/k)_{\mathbb{Q}}\cong A_*(\mathcal{K}_{g,n}(\mathcal{X}_L/L,\beta_L)/L)_{\mathbb{Q}}.$$

Then for all $\underline{\gamma}\in A^*(\mathcal{I}_\mu(\mathcal{X})/k)_{\mathbb{Q}}^{\otimes n}$, we have $\mathrm{I}^{\mathcal{X}_L}_{g,n,\beta_L}(\underline{\gamma})=\mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\underline{\gamma})$ and therefore, passing to the limit, we get $\mathrm{I}^{\overline{\mathcal{X}}}_{g,n,\overline{\beta}}(\underline{\gamma})=\mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\underline{\gamma})$. □

5.2. **Comparison of invariants in mixed characteristic.** Let D be a Dedekind domain, set $B=\operatorname{Spec}D$. We denote by $\eta=\operatorname{Spec}K$ the generic point of B and let $b_0,b_1\in B$ be closed points of B. Let $\pi\colon\mathcal{Y}\to B$ be a smooth proper tame Deligne-Mumford stack of finite presentation over B, admitting a projective coarse moduli scheme Y and set $\mathcal{Y}_\eta=\mathcal{Y}\times_B\eta$, $\mathcal{Y}_h=\mathcal{Y}\times_B b_h$ for $h=0,1$. By [12] 20.3, there are specialization morphisms $\sigma_h\colon A_*(\mathcal{Y}_\eta/\eta)\to A_*(\mathcal{Y}_h/b_h)$ for $h=0,1$. Let $b_h=\operatorname{Spec}k_h$ and let $\overline{k}_h$ be an algebraic closure of k_h for $h=0,1$. We set $\overline{b}_h=\operatorname{Spec}\overline{k}_h$. Recall that the cospecialization map gives an isomorphism $H^*(\overline{\mathcal{I}}_\mu(\overline{\mathcal{Y}}_0))\cong H^*(\overline{\mathcal{I}}_\mu(\overline{\mathcal{Y}}_1))$, where $\overline{\mathcal{Y}}_h=\mathcal{Y}_h\times_{k_h}\overline{k}_h$ for $h=0,1$ ([18] VI.4.1).

5.10. **Theorem.** *Let $\beta\in A_1(Y_\eta/\eta)$ and set $\beta_h=\sigma_h(\beta)$ for $h=0,1$. Then*

$$I^{\overline{\mathcal{Y}}_0}_{g,n,\overline{\beta}_0}(\underline{\gamma})=I^{\overline{\mathcal{Y}}_1}_{g,n,\overline{\beta}_1}(\underline{\gamma}),$$

for every $\underline{\gamma}\in H^*(\overline{\mathcal{I}}_\mu(\overline{\mathcal{Y}}_0))^{\otimes n}\cong H^*(\overline{\mathcal{I}}_\mu(\overline{\mathcal{Y}}_1))^{\otimes n}$.

Proof. Let R_h be the localization of D at b_h for $h=0,1$, then R_h is a discrete valuation ring with generic point η and closed point b_h. Let $\hat{R}_h$ be the completion of R_h, then $\hat{R}_h$ is a complete discrete valuation ring with closed point b_h and generic point $\eta\times_{R_h}\hat{R}_h$. Moreover $R_0\otimes_D R_1=K$ and hence $\eta\times_{R_0}\hat{R}_0=\eta\times_{R_1}\hat{R}_1$. We denote by $\hat{\eta}=\operatorname{Spec}\hat{K}$ the generic point of $\hat{R}_h$. Set $\hat{\mathcal{Y}}_h = \mathcal{Y} \times_D \hat{R}_h$ and $\hat{\mathcal{Y}}_\eta = \mathcal{Y} \times_D \hat{\eta}$. Let $i_h\colon \mathcal{Y}_h \to \hat{\mathcal{Y}}_h$ and $j_h\colon \hat{\mathcal{Y}}_\eta \to \hat{\mathcal{Y}}_h$ be the natural inclusions. Let $\hat{\beta} \in A_1(\hat{Y}_\eta/\hat{\eta})$ be the pullback of β. We have the following cartesian diagram

$$\begin{tikzcd}
\mathcal{K}_{g,n}(\hat{\mathcal{Y}}_\eta/\hat{\eta}, \hat{\beta}) \arrow[r, "\hat{\jmath}"] \arrow[d] & \mathcal{K}_{g,n}(\hat{\mathcal{Y}}_h/\hat{R}_h, \hat{\beta}) \arrow[d] & \mathcal{K}_{g,n}(\mathcal{Y}_h/b_h, \beta_h) \arrow[l, "\hat{\imath}"'] \arrow[d] \\
\mathfrak{M}_{g,\,n/\hat{\eta}} \arrow[r, "\tilde{\jmath}"] & \mathfrak{M}_{g,\,n/\hat{R}_h} & \mathfrak{M}_{g,\,n/b_h} \arrow[l, "\tilde{\imath}"']
\end{tikzcd}$$

Let $\overline{K}$ be an algebraic closure of $\hat{K}$. We set $\overline{\beta} = \rho(\hat{\beta}) \in A_1(\overline{Y}_\eta/\overline{\eta})$, where $\overline{\eta} = \operatorname{Spec}\overline{K}$ and $\overline{\mathcal{Y}}_\eta = \mathcal{Y} \times_D \overline{\eta}$. By [12] 20.3.5 and Theorem B.2, there exists a specialization homomorphism

$$\hat{\sigma}_h\colon A_*(\mathcal{K}_{g,n}(\overline{\mathcal{Y}}_\eta/\overline{\eta}, \overline{\beta})/\overline{\eta})_{\mathbb{Q}} \to A_*(\mathcal{K}_{g,n}(\overline{\mathcal{Y}}_h/\overline{b}_h, \overline{\beta}_h)/\overline{b}_h)_{\mathbb{Q}},$$

and, by the functoriality of the virtual fundamental class (Proposition 3.32),

$$\hat{\sigma}_h([\mathcal{K}_{g,n}(\overline{\mathcal{Y}}_\eta/\overline{\eta}, \overline{\beta})]^{\mathrm{virt}}) = [\mathcal{K}_{g,n}(\overline{\mathcal{Y}}_h/\overline{b}_h, \overline{\beta}_h)]^{\mathrm{virt}}.$$

By [18] VI.4.1, there are isomorphisms $H^*(\overline{\mathcal{I}}_\mu(\overline{\mathcal{Y}}_\eta)) \cong H^*(\overline{\mathcal{I}}_\mu(\overline{\mathcal{Y}}_h))$ for $h = 0, 1$, compatible with evaluation maps. It follows that, for $h = 0, 1$, $\mathrm{I}^{\overline{\mathcal{Y}}_h}_{g,n,\overline{\beta}_h}(\underline{\gamma}) = \mathrm{I}^{\overline{\mathcal{Y}}_\eta}_{g,n,\overline{\beta}}(\underline{\gamma})$ for $\underline{\gamma} \in H^*(\overline{\mathcal{I}}_\mu(\overline{\mathcal{Y}}_h))^{\otimes n}$. □

5.11. **Corollary.** *Let $\mathcal{X}$ be a smooth proper tame Deligne-Mumford stack of finite presentation over a field k, admitting a projective coarse moduli scheme X. Then the Gromov-Witten invariants $\langle I^{\mathcal{X}}_{g,n,\beta}\rangle$ are invariant under deformations of $\mathcal{X}$.*

5.3. **Axioms.** Let $\mathcal{X}$ be a smooth proper tame Deligne-Mumford stack of finite presentation over an algebraically closed field k, admitting a projective coarse moduli scheme X.

5.3.1. *Effectivity.* Let $A_1(X/k)_+$ be the set of $\beta \in A_1(X/k)$ such that $\beta \cdot c_1(\mathscr{L}) \geq 0$ for every ample line bundle $\mathscr{L}$. Then $\mathrm{I}^{\mathcal{X}}_{g,n,\beta} = 0$, for all $\beta \notin A_1(X/k)_+$.

Proof. If $\mathcal{K}_{g,n}(\mathcal{X}/k, \beta) \neq \emptyset$ then $\beta = f_*[C]$ for some stable map (C, x_i, f), hence $\beta \in A_1(X/k)_+$. It follows that $\mathcal{K}_{g,n}(\mathcal{X}/k, \beta) = \emptyset$ for every $\beta \notin A_1(X/k)_+$, and thus $[\mathcal{K}_{g,n}(\mathcal{X}/k, \beta)]^{\mathrm{virt}} = 0$. □

5.3.2. *S_n-covariance.* For all $\gamma_j \in H^{m_j}(\overline{\mathcal{I}}_\mu(\mathcal{X}))$, we have

$$\mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_i \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_n) = (-1)^{m_i m_{i+1}} \mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{i+1} \otimes \gamma_i \otimes \cdots \otimes \gamma_n).$$

Proof. The statement follows from the following ([18] VI.8)

$$\gamma_1 \otimes \cdots \otimes \gamma_i \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_n = (-1)^{m_i m_{i+1}} \gamma_1 \otimes \cdots \otimes \gamma_{i+1} \otimes \gamma_i \otimes \cdots \otimes \gamma_n \in H^*(\overline{\mathcal{I}}_\mu(\mathcal{X})^n).$$ □

5.3.3. *Grading.* Let us set $H^*_{\mathrm{st}}(\mathcal{X}) = H^*(\overline{\mathcal{I}}_\mu(\mathcal{X}))$. We consider $H^*_{\mathrm{st}}(\mathcal{X})$ as a graded group with the following grading $H^m_{\mathrm{st}}(\mathcal{X}) = \bigoplus_\Omega H^{m-2\mathrm{age}(\Omega)}(\Omega)$, where the sum is taken over all connected components Ω of $\overline{\mathcal{I}}_\mu(\mathcal{X})$. We have

$$\mathrm{I}^{\mathcal{X}}_{g,n,\beta}\colon \bigotimes_{i=1}^n H^{m_i}_{\mathrm{st}}(\mathcal{X}) \to H^{\sum m_i + 2((g-1)\dim_k \mathcal{X} - c_1(T_{\mathcal{X}/k})\cdot\beta)}(\overline{\mathcal{M}}_{g,\,n/k}).$$

Proof. Let $\overline{x} = (\mathcal{C} \to \mathcal{X}, \Sigma_1, \ldots, \Sigma_n)$ be a geometric point of a component $\mathcal{K}$ of $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ then, for $i = 1, \ldots, n$, we have evaluation maps $e_i\colon \mathcal{K} \to \Omega_i$ for connected components Ω_j of $\overline{\mathcal{I}}_\mu(\mathcal{X})$. Since the age only depends on the connected component, we have $\mathrm{age}(\Sigma_i) = \mathrm{age}(\Omega_i)$. The virtual fundamental class $[\mathcal{K}]^{\mathrm{virt}}$ is a cycle class of dimension

$$(\dim_S \mathcal{X} - 3)(1 - g) + c_1(T_{\mathcal{X}/k}) \cdot \beta - \sum_{i=1}^n \mathrm{age}(\Sigma_i) + n.$$

Notice that $\gamma_i \in H^{m_i}_{\mathrm{st}}(\mathcal{X}) = H^{m_i - 2\mathrm{age}(\Omega_i)}(\Omega_i)$. It follows that $\mathrm{I}^{\mathcal{X}}_{g,n,\beta,\Omega_{n+1}}(\underline{\gamma})$ has degree

$$2(3g-3+n)-2\left((\dim_k \mathcal{X}-3)(1-g)+n+c_1(T_{\mathcal{X}/k})\cdot\beta-\sum_{i=1}^{n}\mathrm{age}(\Sigma_i)\right)+$$
$$+\sum_{i=1}^{n}(m_i-2\mathrm{age}(\Omega_i)) = \sum_{i=1}^{n} m_i + 2((g-1)\dim_k \mathcal{X} - c_1(T_{\mathcal{X}/k})\cdot\beta). \quad \square$$

5.3.4. *Fundamental class.* Let $\varphi_n\colon \overline{\mathcal{M}}_{g,\, n\,+\,1/k} \to \overline{\mathcal{M}}_{g,\, n/k}$ be the natural functor that forgets the last marked point and stabilizes. We have

$$\mathrm{I}^{\mathcal{X}}_{g,n+1,\beta}(\bullet \otimes \mathrm{id}) = \varphi_n^* \mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\bullet),$$
$$\mathrm{I}^{\mathcal{X}}_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}) = \begin{cases} \int_{\overline{\mathcal{I}}_\mu(\mathcal{X})} \frac{1}{\mathfrak{r}}\gamma_1 \cup \iota^*\gamma_2 & \text{if } \beta = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us form the cartesian diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "j"] \arrow[d, "\hat\theta"] & \mathcal{N} \arrow[r, "\widetilde\varphi"] \arrow[d, "\widetilde\theta"] & \mathcal{K}_{g,n}(\mathcal{X}/k, \beta) \arrow[d, "\theta_n"] \\
\mathfrak{M}^{\mathrm{tw}}_{g,\, n\,+\,1/k} \arrow[r] & \mathfrak{N} \arrow[r] \arrow[d] & \mathfrak{M}^{\mathrm{tw}}_{g,\, n/k} \arrow[d] \\
 & \overline{\mathcal{M}}_{g,\, n\,+\,1/k} \arrow[r, "\varphi_n"] & \overline{\mathcal{M}}_{g,\, n/k}
\end{tikzcd}$$

and notice that $\mathcal{M}$ is the algebraic stack of twisted stable maps of genus g and class β with $n+1$ gerbes which remain stable if we forget the last gerbe. In particular there is a regular embedding $i\colon \mathcal{M} \to \mathcal{K}_{g,n+1}(\mathcal{X}/k, \beta)$ which commute with θ_{n+1} and $\hat\theta$. If we define a virtual fundamental class $[\mathcal{M}]^{\mathrm{virt}}$ relative to $\hat\theta$ as described in section 4.2 then

$$i^![\mathcal{K}_{g,n+1}(\mathcal{X}/k, \beta)]^{\mathrm{virt}} = [\mathcal{M}]^{\mathrm{virt}}.$$

If we define a virtual fundamental class $[\mathcal{N}]^{\mathrm{virt}}$ relative to $\widetilde\theta$ then, by Proposition 3.32,

$$j^*\widetilde\varphi^*[\mathcal{K}_{g,n}(\mathcal{X}/k, \beta)]^{\mathrm{virt}} = j^*[\mathcal{N}]^{\mathrm{virt}} = [\mathcal{M}]^{\mathrm{virt}}.$$

Let $\widetilde q\colon \mathcal{N} \to \overline{\mathcal{M}}_{g,\, n\,+\,1/k}$ and let $\pi\colon \overline{\mathcal{I}}_\mu(\mathcal{X})^{n+1} \to \overline{\mathcal{I}}_\mu(\mathcal{X})^n$ be the projection on the first n components. Moreover we denote $\widetilde e = e_{(n)} \circ \widetilde\varphi$, $\hat e = e_{(n+1)} \circ i$ and observe that $q_{n+1} \circ i = \widetilde q \circ j$. We have that

$$\begin{aligned}
\mathrm{I}^{X}_{g,n+1,\beta}(\underline{\gamma} \otimes \mathrm{id}) &= q_{n+1*}\left(e^*_{(n+1)}(\underline{\gamma} \otimes \mathrm{id}) \cap [\mathcal{K}_{g,n+1}(\mathcal{X}/k, \beta)]^{\mathrm{virt}}\right) \\
&= q_{n+1*} i_*\left(\hat e^*(\underline{\gamma} \otimes \mathrm{id}) \cap i^![\mathcal{K}_{g,n+1}(\mathcal{X}/k, \beta)]^{\mathrm{virt}}\right) \\
&= \widetilde q_* j_*\left(\hat e^*(\underline{\gamma} \otimes \mathrm{id}) \cap j^*\widetilde\varphi^*[\mathcal{K}_{g,n}(\mathcal{X}/k, \beta)]^{\mathrm{virt}}\right) \\
&= \widetilde q_*\left(j_*\hat e^*(\underline{\gamma} \otimes \mathrm{id}) \cap \widetilde\varphi^*[\mathcal{K}_{g,n}(\mathcal{X}/k, \beta)]^{\mathrm{virt}}\right) \\
&= \widetilde q_*\widetilde\varphi^*\left(e^*_{(n)}(\underline{\gamma}) \cap [\mathcal{K}_{g,n}(\mathcal{X}/k, \beta)]^{\mathrm{virt}}\right) \\
&= \varphi_n^* q_{n*}\left(e^*_{(n)}(\underline{\gamma}) \cap [\mathcal{K}_{g,n}(\mathcal{X}/k, \beta)]^{\mathrm{virt}}\right) \\
&= \varphi_n^* \mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\underline{\gamma}).
\end{aligned}$$

The remaining part of the proof follows from the same arguments of [2] 8.2.1. $\square$

5.3.5. *Divisor.* We have, for all $\gamma \in H^2(X)$,

$$\varphi_* \mathrm{I}^{\mathcal{X}}_{g,n+1,\beta}(\bullet \otimes \gamma) = (\beta \cdot \gamma)\mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\bullet).$$

Proof. Consider the functor

$$\overline{\varphi}\colon \mathcal{K}_{g,n+1}({}^{X}\!/k, \beta) \to \mathcal{K}_{g,n}({}^{\mathcal{X}}\!/k, \beta)$$

which forgets the last gerbe and stabilizes, and let

$$\widetilde{\varphi} = \varphi \times e_{n+1} \colon \mathcal{K}_{g,n+1}({}^{X}\!/k, \beta) \to \mathcal{K}_{g,n}({}^{\mathcal{X}}\!/k, \beta) \times_k \overline{\mathcal{I}}_\mu(\mathcal{X})$$

. By the Künneth formula ([18] VI.8), we can write

$$\widetilde{\varphi}_*[\mathcal{K}_{g,n+1}({}^{X}\!/k, \beta)]^{\mathrm{virt}} = [\mathcal{K}_{g,n}({}^{\mathcal{X}}\!/k, \beta)]^{\mathrm{virt}} \otimes \beta' + \alpha,$$

where $\beta' \in H^*(\overline{\mathcal{I}}_\mu(\mathcal{X}))$ and $\alpha \in H^m(\mathcal{K}_{g,n}({}^{\mathcal{X}}\!/k, \beta)) \otimes H^l(\overline{\mathcal{I}}_\mu(\mathcal{X}))$, with m less than the degree of $[\mathcal{K}_{g,n}({}^{\mathcal{X}}\!/k, \beta)]^{\mathrm{virt}}$. The class β' can be calculated by restricting to what happens over a generic point of $\mathcal{K}_{g,n}({}^{\mathcal{X}}\!/k, \beta)$. Representing such a point by $\xi = (\mathcal{C}, \Sigma_1, \ldots, \Sigma_n, f)$, we have the cartesian diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d] & \xi \times_k \overline{\mathcal{I}}_\mu(\mathcal{X}) \arrow[r] \arrow[d] & \xi \arrow[d, "i"] \\
\mathcal{K}_{g,n+1}({}^{X}\!/k, \beta) \arrow[r, "\widetilde{\varphi}"] & \mathcal{K}_{g,n}({}^{X}\!/k, \beta) \times_k \overline{\mathcal{I}}_\mu(\mathcal{X}) \arrow[r, "\pi"] & \mathcal{K}_{g,n}({}^{X}\!/k, \beta)
\end{tikzcd}$$

where, for ξ generic, the map i is a regular embedding, hence

$$i^! \widetilde{\varphi}_*[\mathcal{K}_{g,n+1}({}^{X}\!/k, \beta)]^{\mathrm{virt}} = f_* i^! [\mathcal{K}_{g,n+1}({}^{X}\!/k, \beta)]^{\mathrm{virt}} = f_*[\mathcal{C}] = \beta,$$

on the other hand

$$i^! \widetilde{\varphi}_*[\mathcal{K}_{g,n+1}({}^{X}\!/k, \beta)]^{\mathrm{virt}} = i^! \left([\mathcal{K}_{g,n}({}^{X}\!/k, \beta)]^{\mathrm{virt}} \otimes \beta' + \alpha \right) = \beta'.$$

It follows that $\beta' = \beta$. Then

$$\begin{aligned}
\varphi_* \mathrm{I}^{\mathcal{X}}_{g,n+1,\beta}(\underline{\gamma} \otimes \gamma) &= \varphi_* q_{n+1*} \left(e^*_{(n+1)}(\underline{\gamma} \otimes \gamma) \cap [\mathcal{K}_{g,n+1}({}^{\mathcal{X}}\!/k, \beta)]^{\mathrm{virt}} \right) \\
&= q_{n*} \pi_* \widetilde{\varphi}_* \left(\widetilde{\varphi}^* (e_{(n)} \times \mathrm{id})^* (\underline{\gamma} \otimes \gamma) \cap [\mathcal{K}_{g,n+1}({}^{\mathcal{X}}\!/k, \beta)]^{\mathrm{virt}} \right) \\
&= q_{n*} \pi_* \left((e_{(n)} \times \mathrm{id})^* (\underline{\gamma} \otimes \gamma) \cap \widetilde{\varphi}_* [\mathcal{K}_{g,n+1}({}^{\mathcal{X}}\!/k, \beta)]^{\mathrm{virt}} \right) \\
&= q_{n*} \pi_* \left((e_{(n)} \times \mathrm{id})^* (\underline{\gamma} \otimes \gamma) \cap \left([\mathcal{K}_{g,n}({}^{\mathcal{X}}\!/k, \beta)]^{\mathrm{virt}} \times \beta + \alpha \right) \right) \\
&= q_{n*} \left(e^*_{(n)}(\underline{\gamma}) \cap [\mathcal{K}_{g,n}({}^{\mathcal{X}}\!/k, \beta)]^{\mathrm{virt}} \right) (\beta \cdot \gamma) \\
&= (\beta \cdot \gamma) \mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\underline{\gamma}).
\end{aligned}$$

□

5.3.6. *Splitting.* Let $g_1, g_2, n_1, n_2 \geq 0$ be integers with $2g_j + n_j + 1 \geq 3$, and set $g = g_1 + g_2$, $n = n_1 + n_2$. Let

$$\mathrm{gl}\colon \overline{\mathcal{M}}_{g_1,\, n_1\,+\,1/k} \times_k \overline{\mathcal{M}}_{g_2,\, n_2\,+\,1/k} \to \overline{\mathcal{M}}_{g,\, n/k},$$

be the natural functor that identifies the last marked points. Let $\underline{\gamma} = \gamma_1 \otimes \cdots \otimes \gamma_n$, then

$$\mathrm{gl}^! \mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\underline{\gamma}) = \sum_{\beta_1+\beta_2=\beta} \mathrm{I}^{\mathcal{X}}_{g_1,n_1+1,\beta_1} \otimes \mathrm{I}^{\mathcal{X}}_{g_2,n_2+1,\beta_2}(\underline{\gamma} \otimes [\Delta]),$$

where Δ is the diagonal in $\overline{\mathcal{I}}_\mu(\mathcal{X})^2$ (2.12).

Proof. Let us notice that $A_1({}^{X}\!/\!_k)_+$ is a commutative semigroup then, by effectivity, the sum is finite. Denote for simplicity $\mathcal{K}^{(\beta_j)} = \mathcal{K}_{g_j,n_j+2}({}^{\mathcal{X}}\!/\!_k, \beta_j)$ for $j = 1, 2$. Let us consider the following commutative diagram

$$\begin{tikzcd} \mathcal{K}^{(\beta_1)} \times_k \mathcal{K}^{(\beta_2)} \arrow[drr, "q_{1,2}"'] & & \mathcal{K}^{(\beta_1)} \times_{\overline{\mathcal{I}}_\mu(\mathcal{X})} \mathcal{K}^{(\beta_2)} \arrow[ll, "\widetilde{\Delta}"'] \arrow[r] \arrow[d, "\widetilde{q}"] & \mathcal{K}_{g,n}({}^{\mathcal{X}}\!/\!_k, \beta) \arrow[d, "q"] \\ & & \overline{\mathcal{M}}_{g_1,\, n_1+1/k} \times_k \overline{\mathcal{M}}_{g_2,\, n_2+1/k} \arrow[r, "\mathrm{gl}"] & \overline{\mathcal{M}}_{g,\, n/k} \end{tikzcd}$$

where the square is cartesian by Proposition 4.8. Moreover, we have the following cartesian diagram

$$\begin{tikzcd} \mathcal{K}^{(\beta_1)} \times_{\overline{\mathcal{I}}_\mu(\mathcal{X})} \mathcal{K}^{(\beta_2)} \arrow[r, "\widetilde{\Delta}"] \arrow[d, "\widetilde{e}"'] & \mathcal{K}^{(\beta_1)} \times_k \mathcal{K}^{(\beta_2)} \arrow[d, "e_{1,2}"] \\ \overline{\mathcal{I}}_\mu(\mathcal{X})^{n+1} \arrow[r, "\mathrm{id}\times\Delta"] & \overline{\mathcal{I}}_\mu(\mathcal{X})^{n+2} \end{tikzcd}$$

By Proposition 4.10,

$$\mathrm{gl}^![\mathcal{K}_{g,n}({}^{\mathcal{X}}\!/\!_k, \beta)]^{\mathrm{virt}} = \sum_{\beta_1+\beta_2=\beta} \Delta^!([\mathcal{K}_{g_1,n_1+1}(\mathcal{X}, \beta_1)]^{\mathrm{virt}} \times [\mathcal{K}_{g_2,n_2+1}(\mathcal{X}, \beta_2)]^{\mathrm{virt}}).$$

Then we have

$$\begin{aligned} \mathrm{gl}^!\mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\underline{\gamma}) &= \mathrm{gl}^!q_*\left(e^*(\underline{\gamma}) \cap [\mathcal{K}_{g,n}({}^{\mathcal{X}}\!/\!_k, \beta)]^{\mathrm{virt}}\right) \\ &= \widetilde{q}_*\mathrm{gl}^!\left(e^*(\underline{\gamma}) \cap [\mathcal{K}_{g,n}({}^{\mathcal{X}}\!/\!_k, \beta)]^{\mathrm{virt}}\right) \\ &= \widetilde{q}_*\left(\widetilde{e}^*\pi^*(\underline{\gamma}) \cap \mathrm{gl}^![\mathcal{K}_{g,n}({}^{\mathcal{X}}\!/\!_k, \beta)]^{\mathrm{virt}}\right) \\ &= \sum_{\beta_1+\beta_2=\beta} q_{1,2*}\widetilde{\Delta}_*\left(\widetilde{e}^*(\underline{\gamma} \otimes \mathrm{id}) \cap \Delta^!([\mathcal{K}_{g_1,n_1+1}(\mathcal{X}, \beta_1)]^{\mathrm{virt}} \times [\mathcal{K}_{g_2,n_2+1}(\mathcal{X}, \beta_2)]^{\mathrm{virt}})\right) \\ &= \sum_{\beta_1+\beta_2=\beta} q_{1,2*}\left(\widetilde{\Delta}_*\widetilde{e}^*(\underline{\gamma} \otimes \mathrm{id}) \cap ([\mathcal{K}_{g_1,n_1+1}(\mathcal{X}, \beta_1)]^{\mathrm{virt}} \times [\mathcal{K}_{g_2,n_2+1}(\mathcal{X}, \beta_2)]^{\mathrm{virt}})\right) \\ &= \sum_{\beta_1+\beta_2=\beta} q_{1,2*}\left(e_{1,2}^*(\underline{\gamma} \otimes [\Delta]) \cap ([\mathcal{K}_{g_1,n_1+1}(\mathcal{X}, \beta_1)]^{\mathrm{virt}} \times [\mathcal{K}_{g_2,n_2+1}(\mathcal{X}, \beta_2)]^{\mathrm{virt}})\right) \\ &= \sum_{\beta_1+\beta_2=\beta} \mathrm{I}^{\mathcal{X}}_{g_1,n_1+1,\beta_1} \otimes \mathrm{I}^{\mathcal{X}}_{g_2,n_2+1,\beta_2}(\underline{\gamma} \otimes [\Delta]). \end{aligned}$$

□

5.3.7. *Genus reduction.* Let $\mathrm{gl}\colon \overline{\mathcal{M}}_{g-1,\, n+2/k} \to \overline{\mathcal{M}}_{g,\, n/k}$ be the natural functor that identifies the last gerbes. We have

$$\mathrm{gl}^!\mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\bullet) = \mathrm{I}^{\mathcal{X}}_{g-1,n+2,\beta}(\bullet \otimes [\Delta]),$$

where Δ is the diagonal in $\overline{\mathcal{I}}_\mu(\mathcal{X})^2$ (2.12).

Proof. Let us consider the following commutative diagram

$$\begin{tikzcd} \mathcal{K}_{g-1,n+2}({}^{\mathcal{X}}\!/\!_k, \beta) \arrow[drr, "q_{n+2}"'] & & \mathcal{K}_{g-1,n+2}({}^{\mathcal{X}}\!/\!_k, \beta) \times_{\overline{\mathcal{I}}_\mu(\mathcal{X})^2} \overline{\mathcal{I}}_\mu(\mathcal{X}) \arrow[ll, "\widetilde{\Delta}"'] \arrow[r] \arrow[d, "\widetilde{q}"] & \mathcal{K}_{g,n}({}^{\mathcal{X}}\!/\!_k, \beta) \arrow[d, "q_n"] \\ & & \overline{\mathcal{M}}_{g-1,\, n+2/k} \arrow[r, "\mathrm{gl}"] & \overline{\mathcal{M}}_{g,\, n/k} \end{tikzcd}$$

where the square is cartesian. Moreover, we have the following cartesian diagram

$$\begin{tikzcd}
\mathcal{K}_{g-1,n+2}(\mathcal{X}/k,\beta)\times_{\overline{\mathcal{I}}_\mu(\mathcal{X})^2}\overline{\mathcal{I}}_\mu(\mathcal{X}) \arrow[r, "\widetilde{\Delta}"] \arrow[d, "\widetilde{e}"'] & \mathcal{K}_{g-1,n+2}(\mathcal{X}/k,\beta) \arrow[d, "e_{(n+2)}"] \\
\overline{\mathcal{I}}_\mu(\mathcal{X})^{n+1} \arrow[r, "\mathrm{id}\times\Delta"] & \overline{\mathcal{I}}_\mu(\mathcal{X})^{n+2}
\end{tikzcd}$$

By Proposition 4.10,

$$\mathrm{gl}^![\mathcal{K}_{g,n}(\mathcal{X}/k,\beta)]^{\mathrm{virt}} = \Delta^!([\mathcal{K}_{g-1,n+2}(\mathcal{X},\beta)]^{\mathrm{virt}}).$$

Then we have

$$\begin{aligned}
\mathrm{gl}^!\mathrm{I}^{\mathcal{X}}_{g,n,\beta}(\underline{\gamma}) &= \mathrm{gl}^!q_{n*}\left(e^*_{(n)}(\underline{\gamma})\cap[\mathcal{K}_{g,n}(\mathcal{X}/k,\beta)]^{\mathrm{virt}}\right)\\
&= \widetilde{q}_*\mathrm{gl}^!\left(e^*_{(n)}(\underline{\gamma})\cap[\mathcal{K}_{g,n}(\mathcal{X}/k,\beta)]^{\mathrm{virt}}\right)\\
&= \widetilde{q}_*\left(\widetilde{e}^*\pi^*(\underline{\gamma})\cap\mathrm{gl}^![\mathcal{K}_{g,n}(\mathcal{X}/k,\beta)]^{\mathrm{virt}}\right)\\
&= q_{n+2*}\widetilde{\Delta}_*\left(\widetilde{e}^*(\underline{\gamma}\otimes\mathrm{id})\cap\Delta^![\mathcal{K}_{g-1,n+2}(\mathcal{X},\beta)]^{\mathrm{virt}}\right)\\
&= q_{n+2*}\left(\widetilde{\Delta}_*\widetilde{e}^*(\underline{\gamma}\otimes\mathrm{id})\cap[\mathcal{K}_{g-1,n+2}(\mathcal{X},\beta)]^{\mathrm{virt}}\right)\\
&= q_{n+2*}\left(e^*_{(n+2)}(\underline{\gamma}\otimes[\Delta])\cap[\mathcal{K}_{g-1,n+2}(\mathcal{X},\beta)]^{\mathrm{virt}}\right)\\
&= \mathrm{I}^{\mathcal{X}}_{g-1,n+2,\beta}(\underline{\gamma}\otimes[\Delta]).
\end{aligned}$$

□

6. Genus zero invariants in positive characteristic

6.1. **Gromov-Witten potential.** Let $\mathcal{X}$ be a smooth proper tame Deligne-Mumford stack of finite presentation over an algebraically closed field k (of arbitrary characteristic), admitting a projective coarse moduli scheme X. Fix $\beta\in A_1(X/k)$ and $n\geq 0$. Let l be a prime different from the characteristic of k.

6.1. Remark. Recall that we defined on the group $H^*_{\mathrm{st}}(\mathcal{X})=H^*(\overline{\mathcal{I}}_\mu(\mathcal{X}))$ the following grading $H^m_{\mathrm{st}}(\mathcal{X})=\bigoplus_\Omega H^{m-2\mathrm{age}(\Omega)}(\Omega)$, where the sum is taken over all connected components Ω of $\overline{\mathcal{I}}_\mu(\mathcal{X})$. By [18] V.1.11, $H^*_{\mathrm{st}}(\mathcal{X})=\sum_r H^r(\overline{\mathcal{I}}_\mu(\mathcal{X}),\mathbb{Q}_l(\overline{r}))$ is finitely generated over $\mathbb{Q}_l$. Let $T_0=1,T_1,\dots,T_m$ be generators for $H^*_{\mathrm{st}}(\mathcal{X})$. For each $i=1,\dots,m$, we introduce a variable t_i of the same degree of T_i, such that the t_i supercommute, which means

$$t_it_j=(-1)^{\deg t_i\deg t_j}t_jt_i,$$

and $t_i^2=0$ if t_i has odd degree.

6.2. Remark. If $\gamma_i\in H^{m_i}_{\mathrm{st}}(\mathcal{X})$ then $\langle\mathrm{I}^{\mathcal{X}}_{0,n,\beta}\rangle(\gamma_1\otimes\cdots\otimes\gamma_n)\in\mathbb{Q}_l$ is zero unless

$$\sum_{i+1}^n m_i = 2(\dim_k\mathcal{X}+c_1(T_{\mathcal{X}/k})\cdot\beta).$$

6.3. Notation. We denote the vector $(a_0,\dots,a_m)$ as $\underline{a}$; we set $|\underline{a}|=a_0+\cdots+a_m$ and $\underline{a}!=a_0!\cdots a_m!$. Moreover we set $\langle\mathrm{I}^{\mathcal{X}}_{0,n,\beta}\rangle=0$ for $n<3$.

6.4. **Definition.** Let $\gamma=\sum_{i=0}^m t_iT_i$ (regarding T_i and t_i as supercommuting variables). We define the *genus* 0 *Gromov-Witten potential* as the formal series

$$\Phi(\gamma)=\sum_{n\geq 0}\sum_{\beta\in A_1(X/k)}\frac{1}{n!}\langle\mathrm{I}^{\mathcal{X}}_{0,n,\beta}\rangle(\gamma^n)q^\beta,$$

where q^β is a free variable of degree $\beta \cdot c_1(T_{\mathcal{X}/k})$ and

$$\frac{1}{n!}\langle \mathrm{I}^{\mathcal{X}}_{0,n,\beta}\rangle(\gamma^n) = \sum_{|\underline{a}|=n} \epsilon(\underline{a})\langle \mathrm{I}^{\mathcal{X}}_{0,n,\beta}\rangle(T^{\underline{a}})\frac{t^{\underline{a}}}{\underline{a}!},$$

with $\epsilon(\underline{a}) = \pm 1$ determined by

$$(t_0T_0)^{a_0}\cdots(t_mT_m)^{a_m} = \epsilon(\underline{a})T_0^{a_0}\cdots T_m^{a_m}t_0^{a_0}\cdots t_m^{a_m}.$$

6.5. Remark. By effectivity axiom, the Gromov-Witten potential is a formal series in $\mathcal{R} = R[\![t_0,\dots,t_m]\!]$, with $R = \mathbb{Q}_l[\![q^\beta;\ \beta \in A_1({}^{X}\!/k)_+]\!]$.

6.2. **Quantum product.** By [18] VI.8, $H^*(\overline{\mathcal{I}}_\mu(\mathcal{X}) \times_k \overline{\mathcal{I}}_\mu(\mathcal{X})) = H^*(\overline{\mathcal{I}}_\mu(\mathcal{X})) \otimes H^*(\overline{\mathcal{I}}_\mu(\mathcal{X}))$. Let Δ be the diagonal in $\overline{\mathcal{I}}_\mu(\mathcal{X})^2$ (2.12), then

$$[\Delta] = \sum_{e,f} g^{ef}T_e \otimes T_f.$$

6.6. **Definition.** We define

$$T_i * T_j = \sum_{e,f}\frac{\partial^3\Phi}{\partial t_i\partial t_j\partial t_e}g^{ef}T_f.$$

Extending this linearly gives the *(big) quantum product* on $H^*_{\mathrm{st}}(\mathcal{X},\mathcal{R})$.

6.7. Remark. Notice that the Gromov-Witten invariants with $n < 3$ do not affect the quantum product.

6.8. **Lemma.** *For all i, j, h, we have*

$$\frac{\partial^3\Phi(\gamma)}{\partial t_i\partial t_j\partial t_h} = \sum_{n\geq 0}\sum_{\beta\in A_1({}^{X}\!/k)}\frac{1}{n!}\langle I^{\mathcal{X}}_{0,n+3,\beta}\rangle(T_i\otimes T_j\otimes T_h\otimes\gamma^n)q^\beta.$$

Proof. For simplicity, we will assume that $H^*_{\mathrm{st}}(\mathcal{X},\mathcal{R})$ has only even cohomology so that we don't have to worry about signs. We have

$$\frac{\partial^3\Phi(\gamma)}{\partial t_i\partial t_j\partial t_h} = \frac{\partial^3}{\partial t_i\partial t_j\partial t_h}\sum_{n,\ \beta}\sum_{|\underline{a}|=n}\langle \mathrm{I}^{\mathcal{X}}_{0,n,\beta}\rangle(T^{\underline{a}})\frac{t^{\underline{a}}}{\underline{a}!}q^\beta = \sum_{n,\ \beta}\sum_{|\underline{a}|=n}\langle \mathrm{I}^{\mathcal{X}}_{0,n,\beta}\rangle(T^{\underline{a}'})\frac{t^{\underline{a}'}}{\underline{a}'!}q^\beta,$$

where $\underline{a}' = \underline{a} - e_i - e_j - e_h$ and $e_i = (0,\dots,0,1,0,\dots,0)$ with 1 in the i-th position. Moreover

$$\begin{aligned}\sum_{n,\ \beta}\frac{1}{n!}\langle \mathrm{I}^{\mathcal{X}}_{0,n+3,\beta}\rangle(T_i\otimes T_j\otimes T_h\otimes\gamma^n)q^\beta &= \sum_{n,\ \beta}\sum_{|\underline{a}|=n}\langle \mathrm{I}^{\mathcal{X}}_{0,n+3,\beta}\rangle(T_i\otimes T_j\otimes T_h\otimes T^{\underline{a}})\frac{t^{\underline{a}}}{\underline{a}!}q^\beta\\ &= \sum_{n,\ \beta}\sum_{|\underline{a}|=n+3}\langle \mathrm{I}^{\mathcal{X}}_{0,n+3,\beta}\rangle(T^{\underline{a}+e_i+e_j+e_h})\frac{t^{\underline{a}}}{\underline{a}!}q^\beta.\end{aligned}$$ □

6.9. **Theorem** (WDVV equation)**.** *The Gromov-Witten potential satisfies the equation*

$$\sum_{e,f}\frac{\partial^3\Phi}{\partial t_i\partial t_j\partial t_e}g^{ef}\frac{\partial^3\Phi}{\partial t_f\partial t_h\partial t_l} = (-1)^{\deg t_i(\deg t_j+\deg t_h)}\sum_{e,f}\frac{\partial^3\Phi}{\partial t_j\partial t_h\partial t_e}g^{ef}\frac{\partial^3\Phi}{\partial t_f\partial t_i\partial t_l},$$

for all i, j, h, l.

Proof. For simplicity, we will assume that $H^*_{\mathrm{st}}(\mathcal{X},\mathcal{R})$ has only even cohomology so that we don't have to worry about signs. If we set

$$F(ij|hl) = \frac{\partial^3\Phi}{\partial t_i\partial t_j\partial t_e}g^{ef}\frac{\partial^3\Phi}{\partial t_f\partial t_h\partial t_l},$$

then we want to show that $F(ij|hl) = F(jh|il)$. Consider the following cartesian diagram

$$\begin{tikzcd}
 & D(ij|hl) \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{0,\,n\,+\,4}/k \arrow[d, "\rho"] \\
\operatorname{Spec} k = \overline{\mathcal{M}}_{0,\,\{i,\,j\}\,\cup\,\bullet}/k \arrow[r, phantom, "\times_k"] & \overline{\mathcal{M}}_{0,\,\{h,\,l\}\,\cup\,\bullet}/k \arrow[r, "\mathrm{gl}"] & \overline{\mathcal{M}}_{0,\,4}/k
\end{tikzcd}$$

where the image of gl is a boundary point of $\overline{\mathcal{M}}_{0,\,4}/k \cong \mathbb{P}^1_k$. Since the boundary points are linearly equivalent, the same is true for the fibers of ρ over these points, hence $D(ij|hl)$ and $D(jh|il)$ are linearly equivalent divisors in $\overline{\mathcal{M}}_{0,\,n\,+\,4}/k$. Let $A \cup B$ be a partition of $\{1,\ldots,n+4\}$ such that $i,j \in A$ and $h,l \in B$. Let us set $\overline{\mathcal{M}}_{A,B} = \overline{\mathcal{M}}_{0,\,A\,\cup\,\bullet}/k \times_k \overline{\mathcal{M}}_{0,\,B\,\cup\,\bullet}/k$ and form the fiber square

$$\begin{tikzcd}
D(A|B) \arrow[r] \arrow[d] & D(ij|hl) \arrow[d] \\
\overline{\mathcal{M}}_{A,B} \arrow[r, "\mathrm{gl}"] & \overline{\mathcal{M}}_{0,\,n\,+\,4}/k
\end{tikzcd}$$

then $D(ij|hl) = \bigsqcup_{\substack{A\cup B=\{1,\ldots,n+4\}\\ i,j\in A,\ h,l\in B}} D(A|B)$. We set

$$\overline{\mathcal{K}}^{(\beta_1,\beta_2)} = \mathcal{K}_{0,A\cup\bullet}(\mathcal{X}/k,\beta_1) \times_k \mathcal{K}_{0,B\cup\bullet}(\mathcal{X}/k,\beta_2).$$

Let us set for simplicity $\gamma_{n_1} = T_i \otimes T_j \otimes \gamma^{n_1}$ and $\gamma_{n_2} = T_h \otimes T_l \otimes \gamma^{n_2}$. Then, by Lemma 6.8 and splitting axiom,

$$\begin{aligned}
F(ij|hl) &= \sum_{\beta_1,\beta_2,n_1,n_2,e,f} \frac{1}{n_1!n_2!}\langle \mathrm{I}^{\mathcal{X}}_{0,n_1+3,\beta_1}\rangle(T_e\otimes\gamma_{n_1})g^{ef}\langle \mathrm{I}^{\mathcal{X}}_{0,n_2+3,\beta_2}\rangle(T_f\otimes\gamma_{n_2})q^{\beta_1+\beta_2}\\
&= \sum_{\beta,n}\sum_{\substack{\beta_1+\beta_2=\beta\\ n_1+n_2=n}}\sum_{e,f}\frac{1}{n_1!n_2!}\int_{[\mathcal{K}^{(\beta_1,\beta_2)}]^{\mathrm{virt}}} g^{e,f}e^*_{(1,2)}(T_e\otimes\gamma_{n_1}\otimes T_f\otimes\gamma_{n_2})q^{\beta}\\
&= \sum_{\beta,n}\sum_{\substack{\beta_1+\beta_2=\beta\\ n_1+n_2=n}}\frac{1}{n_1!n_2!}\int_{[\mathcal{K}^{(\beta_1,\beta_2)}]^{\mathrm{virt}}} e^*_{(1,2)}(\gamma_{n_1}\otimes[\Delta]\otimes\gamma_{n_2})q^{\beta}\\
&= \sum_{\beta,n}\sum_{\substack{A\cup B=\{1,\ldots,n+4\}\\ i,j\in A,\ h,l\in B}}\frac{1}{n!}\int_{\overline{\mathcal{M}}_{A,B}} \mathrm{gl}^!\mathrm{I}^{\mathcal{X}}_{0,n+4,\beta}(T_i\otimes T_j\otimes T_h\otimes T_l\otimes\gamma^n)q^{\beta}\\
&= \sum_{\beta,n}\sum_{\substack{A\cup B=\{1,\ldots,n+4\}\\ i,j\in A,\ h,l\in B}}\frac{1}{n!}\int_{D(A|B)} \mathrm{I}^{\mathcal{X}}_{0,n+4,\beta}(T_i\otimes T_j\otimes T_h\otimes T_l\otimes\gamma^n)q^{\beta}\\
&= \sum_{\beta,n}\frac{1}{n!}\int_{D(ij|hl)} \mathrm{I}^{\mathcal{X}}_{0,n+4,\beta}(T_i\otimes T_j\otimes T_h\otimes T_l\otimes\gamma^n)q^{\beta}.
\end{aligned}$$

Since $D(ij|hl)$ and $D(jh|il)$ are linerly equivalent, it follows that $F(ij|hl) = F(jh|il)$. □

6.10. **Proposition.** *The quantum product is supercommutative with identity T_0 and associative.*

Proof. By Lemma 6.8 and S_n-covariance axiom,

$$\begin{aligned}
T_i * T_j &= \sum_{\beta,n,e,f} \frac{1}{n!} \langle \mathrm{I}^{\mathcal{X}}_{0,n+3,\beta} \rangle (T_i \otimes T_j \otimes T_e \otimes \gamma^n) g^{ef} T_f q^\beta \\
&= \sum_{\beta,n,e,f} \frac{1}{n!} (-1)^{\deg T_i \deg T_j} \langle \mathrm{I}^{\mathcal{X}}_{0,n+3,\beta} \rangle (T_j \otimes T_i \otimes T_e \otimes \gamma^n) g^{ef} T_f q^\beta \\
&= (-1)^{\deg T_i \deg T_j} T_j * T_i.
\end{aligned}$$

Let $\Delta \colon \overline{\mathcal{I}}_\mu(\mathcal{X}) \to \overline{\mathcal{I}}_\mu(\mathcal{X})^2$ be the diagonal (2.12) and let $p_i \colon \overline{\mathcal{I}}_\mu(\mathcal{X})^2 \to \overline{\mathcal{I}}_\mu(\mathcal{X})$ be the natural projections for $i = 1, 2$. By the fundamental class axiom,

$$\begin{aligned}
T_i &= p_{2*}\Delta_*(\Delta^! p_1^* \iota^*(T_i)) \\
&= p_{2*}(p_1^* \iota^*(T_i) \cup [\Delta]) \\
&= \sum_{e,f} g^{ef} p_{2*}((\iota^*(T_i) \otimes T_0) \cup (T_e \otimes T_f)) \\
&= \sum_{e,f} g^{ef} p_{2*}((\iota^*(T_i) \cup T_e) \otimes T_f) \\
&= \sum_{e,f} \langle \mathrm{I}^{\mathcal{X}}_{0,3,0} \rangle (T_0 \otimes T_i \otimes T_e) g^{ef} T_f.
\end{aligned}$$

Moreover, we have $\langle \mathrm{I}^{\mathcal{X}}_{0,n+3,\beta} \rangle (\bullet \otimes T_0) = 0$ unless $\beta = 0$ and $n = 3$. Therefore

$$\begin{aligned}
T_0 * T_i &= \sum_{\beta,n,e,f} \frac{1}{n!} \langle \mathrm{I}^{X}_{0,n+3,\beta} \rangle (T_0 \otimes T_i \otimes T_e \otimes \gamma^n) g^{ef} T_f q^\beta \\
&= \sum_{e,f} \langle \mathrm{I}^{X}_{0,3,0} \rangle (T_0 \otimes T_i \otimes T_e) g^{ef} T_f = T_i.
\end{aligned}$$

Finally, we prove that the quantum product is associative. For simplicity, we will assume that $H^*_{\mathrm{st}}(\mathcal{X}, \mathcal{R})$ has only even cohomology so that we don't have to worry about signs. We have

$$(T_i * T_j) * T_h = \sum_{e,f} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^{ef} T_e * T_h = \sum_{c,d,e,f} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^{ef} \frac{\partial^3 \Phi}{\partial t_f \partial t_h \partial t_c} g^{cd} T_d$$

and

$$T_i * (T_j * T_h) = (-1)^{\deg T_i (\deg T_j + \deg T_h)} (T_j * T_h) * T_i,$$

since the quantum product is supercommutative. Therefore, associativity follows from Theorem 6.9. □

6.3. Reconstruction for genus zero Gromov-Witten invariants.

6.11. **Theorem.** *If $H^*_{st}(\mathcal{X})$ is generated by $H^2_{st}(\mathcal{X})$ then every genus zero Gromov-Witten invariant can be uniquely reconstructed starting with the following system of Gromov-Witten invariants*

$$\left\{ I^{\mathcal{X}}_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes \gamma_3) \,\middle|\, \beta \cdot c_1(T_{\mathcal{X}/k}) \leq \dim_k \mathcal{X} + 1, \ \deg \gamma_3 = 2 \right\}.$$

Proof. Apply the WDVV equation (Theorem 6.9) to $\gamma_1 \otimes \cdots \otimes \gamma_{n+1}$ with indeces $\{i, j, h, l\} = \{1, 2, n, n+1\}$. Let us define a partial order on pairs (β, n), with $n \geq 3$ and $\beta \in A_1({}^X/_k)_+$, by setting $(\beta, n) > (\beta', n')$ if and only if either $\beta = \beta' + \beta''$ or $\beta = \beta'$ and $n > n'$. Then there are four terms of higher order in the WDVV equation each of the form

$$\mathrm{I}_{a,b} = \sum_{e,f} \langle \mathrm{I}^{\mathcal{X}}_{0,3,0} \rangle (\gamma_a \otimes \gamma_b \otimes T_e) g^{ef} \langle \mathrm{I}^{\mathcal{X}}_{0,n-1,\beta} \rangle (T_f \otimes (\otimes_{s \neq a,b} \gamma_s)),$$

with $(a, b) \in \{(1, 2), (n, n+1), (2, n), (1, n+1)\}$. As shown in the proof of Proposition 6.10, we have

$$\gamma_a \cup \gamma_b = \sum_{e,f} \langle \mathrm{I}^{\mathcal{X}}_{0,3,0} \rangle (\gamma_a \otimes \gamma_b \otimes T_e) g^{ef} T_f,$$

hence $\mathrm{I}_{a,b} = \langle \mathrm{I}^{\mathcal{X}}_{0,n-1,\beta}\rangle(\gamma_a \cup \gamma_b \otimes (\otimes_{s\neq a,b}\gamma_s))$. Let us consider $\langle \mathrm{I}^{\mathcal{X}}_{0,n,\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_n)$. If $\deg \gamma_n = 2$, then we can apply divisor axiom to reduce n. Otherwise, since $H^*_{\mathrm{st}}(\mathcal{X})$ is generated by $H^2_{\mathrm{st}}(\mathcal{X})$, we can write $\gamma_n = \sum_i \delta'_i \cup \delta_i$, with $\deg \delta_i = 2$. By linearity, we can assume $\gamma_n = \delta' \cup \delta$, with $\deg \delta = 2$. Apply the construction above with $\gamma_n = \delta'$ and $\gamma_{n+1} = \delta$. Then, by WDVV equation, we get

$$\begin{aligned}
&\pm\langle \mathrm{I}^{\mathcal{X}}_{0,n-1,\beta}\rangle(\gamma_1 \cup \gamma_2 \otimes \gamma_3 \otimes \cdots \gamma_{n-1} \otimes \delta' \otimes \delta) \pm \langle \mathrm{I}^{\mathcal{X}}_{0,n-1,\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta' \cup \delta)\\
&\pm\langle \mathrm{I}^{\mathcal{X}}_{0,n-1,\beta}\rangle(\gamma_1 \cup \delta \otimes \gamma_2 \otimes \cdots \gamma_{n-1} \otimes \delta') \pm \langle \mathrm{I}^{\mathcal{X}}_{0,n-1,\beta}\rangle(\gamma_1 \otimes \gamma_2 \cup \delta' \otimes \gamma_\otimes \cdots \gamma_{n-1} \otimes \delta) =\\
&= \text{a combination of higher order terms.}
\end{aligned}$$

By divisor axiom, the first and the fourth summands are lifted from $\overline{\mathcal{M}}_{0,\, n\, -\, 1/k}$. Moreover in the third summand we have $\deg \delta' < \deg \gamma_n$. If $\deg \delta' = 2$ then, by divisor axiom, we can reduce n, otherwise we repeat this trick and in a finite number of iterations we will reduce n. Finally, we can apply the procedure described above to $\langle \mathrm{I}^{\mathcal{X}}_{0,3,\beta}\rangle(\gamma_1 \otimes \gamma_2 \otimes \gamma_3)$ and decrease $\deg \gamma_3 \geq 2$. ☐

Appendix A. Deformation theory

We review some results of deformation theory (for further details see [24], [13]).

Let S be a scheme. We consider Artin stacks of finite type over S. Fix a geometric point $\operatorname{Spec} \overline{k} \xrightarrow{\overline{s}} S$ of S. Let $\Lambda = \hat{\mathscr{O}}_{S,\overline{s}}$ and consider the category (Art/Λ) of local artinian Λ-algebras with residue field $\overline{k}$.

A.1. Let $f\colon X \to Y$ be a morphism of finite type of Artin stacks. Recall that f is smooth if and only if, for every geometric point $\overline{x}$ of X, for every Artinian local $\hat{\mathscr{O}}_{Y,f(\overline{x})}$-algebra A and for every ideal $I \subset A$ such that $I^2 = 0$, the canonical map

$$\mathrm{Hom}_Y(\operatorname{Spec} A, X) \to \mathrm{Hom}_Y(\operatorname{Spec} A/I, X)$$

is surjective.

A.2. Let $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{\mathrm{opp}}$ be a category fibered in groupoids. Let us consider two morphisms $\pi'\colon A' \to A$ and $\pi''\colon A'' \to A$ in (Art/Λ), with π'' surjective. We form the cartesian diagram

$$\begin{tikzcd}
A' \times_A A'' \arrow[r, "q''"] \arrow[d, "q'"'] & A'' \arrow[d, "\pi''"] \\
A' \arrow[r, "\pi'"] & A
\end{tikzcd}$$

then the functors

$$\mathcal{F}(\pi') \circ \mathcal{F}(q'), \mathcal{F}(\pi'') \circ \mathcal{F}(q'')\colon \mathcal{F}(A' \times_A A'') \to \mathcal{F}(A)$$

are isomorphic and we get an induced functor

$$\Psi\colon \mathcal{F}(A' \times_A A'') \to \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'').$$

A.3. **Definition.** A *deformation category* over Λ is a category fibered in groupoids $\mathcal{F} \to (\mathrm{Art}/\Lambda)^{\mathrm{opp}}$ such that, given morphisms $\pi'\colon A' \to A$ and $\pi''\colon A'' \to A$ in (Art/Λ), with π'' surjective, the functor Ψ is an equivalence of categories.

A.4. **Definition.** A *small extension* is an exact sequence

$$0 \to I \to A' \to A \to 0$$

in (Art/Λ) such that $I\mathfrak{m}_{A'} = 0$, where $\mathfrak{m}_{A'}$ is the maximal ideal of A'.

A.5. **Definition.** Let $\mathcal{F}$, $\mathcal{G}$ be deformation categories and $\nu\colon \mathcal{F} \to \mathcal{G}$ a functor of categories fibered in groupoids. Let $T^1\nu$ and $T^2\nu$ be $\overline{k}$-vector spaces. We say that ν has *tangent space* $T^1\nu$ and *obstruction space* $T^2\nu$ if, for every small extension

$$0 \to I \to A' \to A \to 0,$$

there is a functorial exact sequence

$$T^1\nu \otimes_{\overline{k}} I \to \mathcal{F}(A') \to \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(A') \xrightarrow{\mathrm{ob}_\nu} T^2\nu \otimes_{\overline{k}} I.$$

A.6. **Definition.** Let $\mathcal{F}$, $\mathcal{G}$ be deformation categories and $\nu\colon \mathcal{F} \to \mathcal{G}$ a functor of categories fibered in groupoids. Let $A' \to A = {A'}/{I}$ be a small extension and fix objects $\sigma' \in \mathcal{F}(A')$, $\sigma \in \mathcal{F}(A)$, $\tau \in \mathcal{G}(A)$, $\tau' \in \mathcal{G}(A')$ such that $\mathcal{F}(i)(\sigma') = \sigma$, $\nu(A')(\sigma') = \tau'$ and $\mathcal{G}(i)(\tau') = \tau = \nu(A)(\sigma)$. Let $\mathrm{Aut}_{A'}(\sigma')$ be the group of authomorphisms of σ' in $\mathcal{F}(A')$. There is a natural homomorphism

$$\mathrm{Aut}_{A'}(\sigma') \to \mathrm{Aut}_A(\sigma) \times_{\mathrm{Aut}_A(\tau)} \mathrm{Aut}_{A'}(\tau').$$

An *infinitesimal automorphism* of σ' is an element of the kernel of this homomorphism.

A.7. Notation. Let $\sigma \in \mathcal{F}(A)$, $\tau' \in \mathcal{G}(A')$ be objects such that $\mathcal{G}(i)(\tau') = \nu(A)(\sigma)$. If $\mathrm{ob}_\nu(\sigma, \tau') = 0$, we denote by $\mathcal{S}_\nu$ the set of isomorphism classes of $\sigma' \in \mathcal{F}(A')$ such that$\mathcal{F}(i)(\sigma') = \sigma$, $\nu(A')(\sigma') = \tau'$.

A.8. Let $F\colon X \to Y$ be a Deligne-Mumford type morphism of algebraic Artin stacks over S. Let $\operatorname{Spec} \overline{k} \xrightarrow{\overline{x}} X$ be a geometric point of X. Let $\Lambda = \hat{\mathscr{O}}_{S,\overline{x}}$. Consider the deformation category $h_{X,\overline{x}}$ such that, for all $A \in ({\mathrm{Art}}/{\Lambda})$, the objects of $h_{X,\overline{x}}(A)$ are morphisms $f_X\colon \operatorname{Spec} A \to X$ such that $f_X|_{\operatorname{Spec}\overline{k}} = \overline{x}$. There is a natural functor $\nu_F\colon h_{X,\overline{x}} \to h_{Y,\overline{x}}$ given by the composition with F.

A.9. **Lemma.** *Let $L^\bullet_F$ be the relative cotangent complex of F. If F is representable then, for every geometric point $\overline{x}$ of X and for every small extension $A' \to A = {A'}/{I}$ in $({\mathrm{Art}}/{\hat{\mathscr{O}}_{S,\overline{x}}})$,*

(1) there is a functorial surjective set-theoretical map

$$ob_F\colon h_{X,\overline{x}}(A) \times_{h_{Y,\overline{x}}(A)} h_{Y,\overline{x}}(A') \to h^1((L\overline{x}^*L^\bullet_F)^\vee) \otimes I$$

such that $ob_F(f_X, f'_Y) = 0$ if and only if there exists $f'_X \in h_{X,\overline{x}}(A')$ such that $f'_X \circ i = f_X$ and $F \circ f'_X = f'_Y$;

(2) if $ob_F(f_X, f'_Y) = 0$ then the set of isomorphism classes of $f'_X \in h_{X,\overline{x}}(A')$, such that $f'_X \circ i = f_X$ and $F \circ f'_X = f'_Y$, is a torsor under $h^0((L\overline{x}^*L^\bullet_F)^\vee) \otimes I$;

(3) if $ob_F(f_X, f'_Y) = 0$ and $f'_X \in h_{X,\overline{x}}(A')$ is such that $F \circ f'_X = f'_Y$, $f'_X \circ i = f_X$, then the group of infinitesimal authomorphisms of f'_X with respect to (f_X, f'_Y) contains only the identity.

Proof. Let $v\colon V \to Y$ be a smooth surjective morphism from a scheme V and form the fibre diagram

$$\begin{tikzcd} U \arrow[r, "G"] \arrow[d, "u"'] & V \arrow[d, "v"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

By Theorem 3.18, $Lu^*L^\bullet_F \cong L^\bullet_G$. By A.1, $\overline{x} \to X$ factors through u and we know, by deformation theory of schemes, that there exists a functorial exact sequence

$$0 \to h^0((L\overline{x}^*L^\bullet_G)^\vee) \otimes I \to h_{U,\overline{x}}(A') \to h_{U,\overline{x}}(A) \times_{h_{V,\overline{x}}(A)} h_{V,\overline{x}}(A') \xrightarrow{\mathrm{ob}_G} h^1((L\overline{x}^*L^\bullet_G)^\vee) \otimes I \to 0.$$

Let us consider a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "f_X"] \arrow[d, "i"'] & X \arrow[d, "F"] \\ \operatorname{Spec} A' \arrow[r, "f'_Y"] & Y \end{tikzcd}$$

By A.1, there exists $f'_{V,1}\colon \operatorname{Spec} A' \to V$ such that $v \circ f'_{V,1} = f'_Y$. Then there exists a unique morphism $f_{U,1}\colon \operatorname{Spec} A \to U$ such that $u \circ f_{U,1} = f_X$, $G \circ f_{U,1} = f'_{V,1} \circ i$. If $f'_{V,2}\colon \operatorname{Spec} A' \to V$ is another morphism such that $v \circ f'_{V,2} = f'_Y$ then there exists a unique morphism $f'_{V \times_Y V}\colon \operatorname{Spec} A' \to$

$V \times_Y V$ such that $v_j \circ f'_{V\times_Y V} = f'_{V,j}$ for $j = 1, 2$, where $v_j \colon V \times_Y V \to V$ are the projections. Let us form the fibre diagram

$$\begin{tikzcd} U \times_Y V \arrow[r, "H"] \arrow[d, "u_j"'] & V \times_Y V \arrow[d, "v_j"] \\ U \arrow[r, "G"] \arrow[d, "u"'] & V \arrow[d, "v"] \\ X \arrow[r, "F"] & Y \end{tikzcd}$$

There exists a unique morphism $f_{U\times_Y V} \colon \operatorname{Spec} A \to U \times_Y V$ such that $u_j \circ f_{U\times_Y V} = f_{U,j}$ for $j = 1, 2$, and $H \circ f_{U\times_Y V} = f'_{V\times_Y V} \circ i$. Therefore

$$\mathrm{ob}_G(f_{U,j}, f'_{V,j}) = \mathrm{ob}_G(u_j \circ f_{U\times_Y V}, v_j \circ f'_{V\times_Y V}) = \mathrm{ob}_H(f_{U\times_Y V}, f'_{V\times_Y V}).$$

Hence, if we set $\mathrm{ob}_F(f_X, f'_Y) = \mathrm{ob}_G(f_{U,1}, f'_{V,1})$, this gives a well-defined surjective map

$$\mathrm{ob}_F \colon h_{X,\overline{x}}(A) \times_{h_{Y,\overline{x}}(A)} h_{Y,\overline{x}}(A') \to h^1((L\overline{x}^* L^\bullet_G)^\vee) \otimes I \cong h^1((L\overline{x}^* L^\bullet_F)^\vee) \otimes I.$$

Moreover, if $\mathrm{ob}_F(f_X, f'_Y) = 0$ then there exist $f_U \in h_{U,\overline{x}}(A)$, $f'_V \in h_{V,\overline{x}}(A')$ such that $u \circ f_U = f_X$, $v \circ f'_V = f'_Y$ and $\mathrm{ob}_G(f_U, f'_V) = 0$. It follows that there exists a morphism $f'_U \in h_{U,\overline{x}}(A')$ such that $f'_U \circ i = f_U$, $G \circ f'_U = f'_V$, and $f'_X = u \circ f'_U \in h_{X,\overline{x}}(A')$ is such that $f'_X \circ i = f_X$, $F \circ f'_X = f'_Y$. Therefore $T^2_{\nu_F} = h^1((L\overline{x}^* L^\bullet_F)^\vee)$ is an obstruction space for ν_F.

Let us now fix $\overline{f}_X \in h_{X,\overline{x}}(A)$, $\overline{f}'_Y \in h_{Y,\overline{x}}(A')$, $\overline{f}'_V \in h_{V,\overline{x}}(A')$ such that $F \circ \overline{f}_X = \overline{f}'_Y \circ i$, $v \circ \overline{f}'_V = \overline{f}'_Y$, and let $\overline{f}_U \in h_{U,\overline{x}}(A)$ be the unique morphism such that $u \circ \overline{f}_U = \overline{f}_X$, $G \circ \overline{f}_U = \overline{f}'_V \circ i$. Assume that $\mathrm{ob}_F(\overline{f}_X, \overline{f}'_Y) = 0$. There is a natural map $\rho \colon \mathcal{S}_G \to \mathcal{S}_F$ such that $\rho(f'_U) = u \circ f'_U$. We know that $\mathcal{S}_G$ is a torsor under $h^0((L\overline{x}^* L^\bullet_G)^\vee) \otimes I$. We claim that ρ is an isomorphism and thus $\mathcal{S}_F$ is a torsor under

$$h^0((L\overline{x}^* L^\bullet_G)^\vee) \otimes I \cong h^0((L\overline{x}^* L^\bullet_F)^\vee) \otimes I.$$

Let $f'_X \in \mathcal{S}_F$, then $F \circ f'_X = v \circ \overline{f}'_V$ and therefore there exists a unique morphism $f'_U \in h_{U,\overline{x}}(A')$ such that $G \circ f'_U = \overline{f}'_V$, $u \circ f'_U = f'_X$. It follows that $f'_U \in \mathcal{S}_G$ and $f'_X = \rho(f'_U)$.

Finally, let $f_X \in h_{X,\overline{x}}(A)$, $f_Y \in h_{Y,\overline{x}}(A)$, $f'_Y \in h_{Y,\overline{x}}(A')$ such that $F \circ f_X = f_Y = f'_Y \circ i$. Assume that $\mathrm{ob}_F(f_X, f'_Y) = 0$ and fix $f'_X \in h_{X,\overline{x}}(A')$ such that $F \circ f'_X = f'_Y$, $f'_X \circ i = f_X$. We claim that the natural homomorphism

$$\mathrm{Aut}_X(f'_X) \to \mathrm{Aut}_X(f_X) \times_{\mathrm{Aut}_Y(f_Y)} \mathrm{Aut}_Y(f'_Y),$$

where $\mathrm{Aut}_X(f'_X)$ is the authomorphism group of f'_X in X, is injective. Since F is representable, if an authomorphism α of f'_X induces the identity of $F \circ f'_X$ in Y then $\alpha = \mathrm{id}$. □

A.10. **Proposition.** *Let $L^\bullet_X$ be the cotangent complex of X. Then, for every geometric point $\overline{x}$ of X and for every small extension $A' \to A = {}^{A'}/_{I}$ in $({}^{Art}/_{\hat{\mathcal{O}}_{S,\overline{x}}})$,*

(1) there is a functorial set-theoretical map

$$ob_X \colon h_{X,\overline{x}}(A) \to h^1((L\overline{x}^* L^\bullet_X)^\vee) \otimes I$$

such that $ob_X(f_X) = 0$ if and only if there exists $f'_X \in h_{X,\overline{x}}(A')$ such that $f'_X \circ i = f_X$;

(2) if $ob_X(f_X) = 0$ then the set of isomorphism classes of $f'_X \in h_{X,\overline{x}}(A')$ such that $f'_X \circ i = f_X$ is a torsor under $h^0((L\overline{x}^* L^\bullet_X)^\vee) \otimes I$;

(3) if $ob_X(f_X) = 0$ and $f'_X \in h_{X,\overline{x}}(A')$ is such that $f'_X \circ i = f_X$, then the group of infinitesimal authomorphisms of f'_X with respect to f_X is isomorphic to $h^{-1}((L\overline{x}^* L^\bullet_X)^\vee) \otimes I$.

Proof. Let us consider a representable smooth and surjective morphism $u \colon U \to X$ from a scheme U, then the map $\overline{x} \to X$ factors through u. Recall that, by Theorem 3.18, $h^1((L\overline{x}^* L^\bullet_X)^\vee) \cong h^1((L\overline{x}^* L^\bullet_U)^\vee)$. We have the following exact sequence

$$0 \to h^{-1}((L\overline{x}^* L^\bullet_X)^\vee) \to h^0((L\overline{x}^* L^\bullet_u)^\vee) \to h^0((L\overline{x}^* L^\bullet_U)^\vee) \to h^0((L\overline{x}^* L^\bullet_X)^\vee) \to 0.$$

Moreover, by deformation theory of schemes, we know that there is a functorial exact sequence

$$0 \to h^0((L\overline{x}^* L_U^\bullet)^\vee) \otimes I \to h_{U,\overline{x}}(A') \to h_{U,\overline{x}}(A) \xrightarrow{\mathrm{ob}_U} h^1((L\overline{x}^* L_U^\bullet)^\vee) \otimes I \to 0.$$

By A.1, for every $f_X \in h_{X,\overline{x}}(A)$, there exists $f_{U,1} \in h_{U,\overline{x}}(A)$ such that $u \circ f_{U,1} = f_X$. If $f_{U,2} \in h_{U,\overline{x}}(A)$ is another morphism such that $u \circ f_{U,2} = f_X$ then there exists a unique morphism $f_{U \times_X U} \in h_{U \times_X U,\overline{x}}(A)$ such that $u_j \circ f_{U \times_X U} = f_{U,j}$ for $j = 1, 2$, where $u_j \colon U \times_X U \to U$ are the projections. By Theorem 3.18, u_1 and u_2 induces isomorphisms $h^1((L\overline{x}^* L_{U \times_X U}^\bullet)^\vee) \cong h^1((L\overline{x}^* L_U^\bullet)^\vee)$. Therefore

$$\mathrm{ob}_U(f_{U,j}) = \mathrm{ob}_U(u_j \circ f_{U \times_X U}) = \mathrm{ob}_{U \times_X U}(f_{U \times_X U}).$$

Hence, if we set $\mathrm{ob}_X(f_X) = \mathrm{ob}_U(f_{U,1})$, this gives a well-defined surjective map

$$\mathrm{ob}_X \colon h_{X,\overline{x}}(A) \to h^1((L\overline{x}^* L_U^\bullet)^\vee) \otimes I \cong h^1((L\overline{x}^* L_X^\bullet)^\vee) \otimes I.$$

Moreover, if $\mathrm{ob}_F(f_X) = 0$ then there exist $f_U \in h_{U,\overline{x}}(A)$ such that $u \circ f_U = f_X$ and $\mathrm{ob}_U(f_U) = 0$. Thus there exists a morphism $f'_U \in h_{U,\overline{x}}(A')$ such that $f'_U \circ i = f_U$ and $f'_X = u \circ f'_U \in h_{X,\overline{x}}(A')$ is such that $f'_X \circ i = f_X$. Therefore $h^1((L\overline{x}^* L_X^\bullet)^\vee)$ is an obstruction space for $h_{X,\overline{x}}$.

By Theorem 3.18, we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \to h^{-1}((L\overline{x}^* L_X^\bullet)^\vee) \arrow[r] \arrow[d, equal] & h^0((L\overline{x}^* L_{\widetilde{u}}^\bullet)^\vee) \arrow[r, "\rho_{\widetilde{u}}"] \arrow[d, "\rho_{u_j}"] & h^0((L\overline{x}^* L_{U \times_X U}^\bullet)^\vee) \arrow[r, "\rho_{U \times_X U}"] \arrow[d, "\rho_{u_j}"] & h^0((L\overline{x}^* L_X^\bullet)^\vee) \arrow[r] \arrow[d, equal] & 0 \\
0 \to h^{-1}((L\overline{x}^* L_X^\bullet)^\vee) \arrow[r] & h^0((L\overline{x}^* L_u^\bullet)^\vee) \arrow[r, "\rho_u"] & h^0((L\overline{x}^* L_U^\bullet)^\vee) \arrow[r, "\rho_U"] & h^0((L\overline{x}^* L_X^\bullet)^\vee) \arrow[r] & 0
\end{tikzcd}$$

where we set $\widetilde{u} = u \circ u_j$. Let us fix $\overline{f}_X \in h_{X,\overline{x}}(A)$, $\overline{f}_U \in h_{U,\overline{x}}(A)$, $\overline{f}_{U \times_X U} \in h_{U \times_X U,\overline{x}}(A)$ such that $u \circ \overline{f}_U = \overline{f}_X$, $u_j \circ \overline{f}_{U \times_X U} = \overline{f}_U$. Let us assume that $\mathrm{ob}_X(\overline{f}_X) = 0$ and fix $\overline{f}'_X \in \mathcal{S}_X$, $\overline{f}'_U \in \mathcal{S}_U$, $\overline{f}'_{U \times_X U} \in \mathcal{S}_{U \times_X U}$ such that $u \circ \overline{f}'_U = \overline{f}'_X$, $u_j \circ \overline{f}'_{U \times_X U} = \overline{f}'_U$. There is a natural surjective map $\mathcal{S}_U \to \mathcal{S}_X$ given by composition with u. We know that $\mathcal{S}_U$ and $\mathcal{S}_U$ are torsors under $h^0((L\overline{x}^* L_U^\bullet)^\vee) \otimes I$ and $h^0((L\overline{x}^* L_u^\bullet)^\vee) \otimes I$, respectively. Let $\alpha_U \in h^0((L\overline{x}^* L_U^\bullet)^\vee) \otimes I$ be such that $\rho_U(\alpha_U)$ is the identity $e_X \in h^0((L\overline{x}^* L_X^\bullet)^\vee) \otimes I$, then $\alpha_U = \rho_u(\alpha_u)$ for some $\alpha_u \in h^0((L\overline{x}^* L_u^\bullet)^\vee) \otimes I$. It follows that $\alpha_U \cdot \overline{f}'_U = \overline{f}_U$ and therefore $\alpha_U = e_U$. As a consequence, for every $\alpha_X \in h^0((L\overline{x}^* L_X^\bullet)^\vee) \otimes I$, we can define $\alpha_X \cdot \overline{f}'_X = u \circ (\alpha_U \cdot \overline{f}'_U) \in \mathcal{S}_X$ for some $\alpha_U \in h^0((L\overline{x}^* L_U^\bullet)^\vee) \otimes I$ such that $\rho_U(\alpha_U) = \alpha_X$. We claim that this defines an action of $h^0((L\overline{x}^* L_X^\bullet)^\vee) \otimes I$ over $\mathcal{S}_X$ which is transitive and free. Let $f'_X \in \mathcal{S}_X$, then there exists $f'_U \in \mathcal{S}_U$ such that $u \circ f'_U = f'_X$. Moreover $f'_U = \alpha_U \cdot \overline{f}'_U$ for some $\alpha_U \in h^0((L\overline{x}^* L_U^\bullet)^\vee) \otimes I$, therefore $f'_X = \rho_U(\alpha_U) \cdot \overline{f}'_X$. Now if $\alpha_X \cdot \overline{f}'_X = \overline{f}'_X$ then $\alpha_X = \rho_U(\alpha_U)$ and $u \circ (\alpha_U \cdot \overline{f}'_U) = u \circ \overline{f}'_U$. As a cosequence there exists $\alpha'_{U \times_X U} \in h^0((L\overline{x}^* L_{U \times_X U}^\bullet)^\vee) \otimes I$ such that $\rho_{u_1}(\alpha'_{U \times_X U}) \cdot \overline{f}'_U = \alpha_U \cdot \overline{f}'_U$ and $\rho_{u_2}(\alpha'_{U \times_X U}) \cdot \overline{f}'_U = \overline{f}'_U$. Hence we have that $\rho_{u_2}(\alpha_{U \times_X U}) = e_U$, $\rho_{u_1}(\alpha_{U \times_X U}) = \alpha_U$, and $\alpha_{U \times_X U} = \rho_{\widetilde{u}}(\alpha_{\widetilde{u}})$. It follows that

$$\alpha_X = \rho_U(\alpha_U) = \rho_U(\rho_{u_1}(\alpha_{U \times_X U})) = \rho_{U \times_X U}(\alpha_{U \times_X U}) = e_X,$$

and this proves that $\mathcal{S}_X$ is a torsor under $h^0((L\overline{x}^* L_X^\bullet)^\vee) \otimes I$.

Finally, let us consider an object $f_X \in h_{X,\overline{x}}(A)$ such that $\mathrm{ob}_X(f_X) = 0$ and let us fix $f'_X \in h_{X,\overline{x}}(A')$ such that $f'_X \circ i = f_X$. We claim that the kernel of the natural homomorphism $\mathrm{Aut}_X(f'_X) \xrightarrow{i_X^*} \mathrm{Aut}_X(f_X)$ is isomorphic to $h^{-1}((L\overline{x}^* L_X^\bullet)^\vee) \otimes I$. Let consider $f_U \in h_{U,\overline{x}}(A)$, $f'_U \in h_{U,\overline{x}}(A')$ such that $f'_U \circ i = f_U$, $u \circ f'_U = f'_X$. We know that we have the following commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Aut}_U(f'_U) \arrow[r] \arrow[d, equal] & \mathrm{Aut}_U(f_U) \times_{\mathrm{Aut}_X(f_X)} \mathrm{Aut}_X(f'_X) \arrow[r, "\omega"] \arrow[d] & h^0((L\overline{x}^* L^\bullet_u)^\vee) \otimes I \arrow[d, "\rho_u"] \\
\mathrm{Aut}_U(f'_U) \arrow[r] & \mathrm{Aut}_U(f_U) \arrow[r] & h^0((L\overline{x}^* L^\bullet_U)^\vee) \otimes I
\end{tikzcd}$$

where $\mathrm{Aut}_U(f'_U) = \mathrm{Aut}_U(f_U) = \{\mathrm{id}\}$ since U is a scheme. Therefore

$$\ker(i^*_X) = \mathrm{Aut}_U(f_U) \times_{\mathrm{Aut}_X(f_X)} \mathrm{Aut}_X(f'_X) \subset h^0((L\overline{x}^* L^\bullet_u)^\vee) \otimes I.$$

Moreover, an element of $h^0((L\overline{x}^* L^\bullet_u)^\vee) \otimes I$ acts trivially on $\mathcal{S}_u$ if and only if it is in the image of ω. Therefore $\mathrm{Aut}_U(f_U) \times_{\mathrm{Aut}_X(f_X)} \mathrm{Aut}_X(f'_X) = h^{-1}((L\overline{x}^* L^\bullet_X)^\vee) \otimes I$. □

A.11. **Proposition.** *Let $L^\bullet_F$ be the relative cotangent complex of F. Then, for every geometric point $\overline{x}$ of X and for every small extension $A' \to A = {}^{A'}/{}_{I}$ in $({}^{Art}/{}_{\hat{\mathcal{O}}_{S,\overline{x}}})$,*

(1) there is a functorial surjective set-theoretical map

$$ob_F \colon h_{X,\overline{x}}(A) \times_{h_{Y,\overline{x}}(A)} h_{Y,\overline{x}}(A') \to h^1((L\overline{x}^* L^\bullet_F)^\vee) \otimes I$$

such that $ob_F(f_X, f'_Y) = 0$ if and only if there exists $f'_X \in h_{X,\overline{x}}(A')$ such that $f'_X \circ i = f_X$ and $F \circ f'_X = f'_Y$;

(2) if $ob_F(f_X, f'_Y) = 0$ then the set of isomorphism classes of $f'_X \in h_{X,\overline{x}}(A')$, such that $f'_X \circ i = f_X$ and $F \circ f'_X = f'_Y$, is a torsor under $h^0((L\overline{x}^* L^\bullet_F)^\vee) \otimes I$;

(3) if $ob_F(f_X, f'_Y) = 0$ and $f'_X \in h_{X,\overline{x}}(A')$ is such that $F \circ f'_X = f'_Y$, $f'_X \circ i = f_X$, then the group of infinitesimal authomorphisms of f'_X with respect to (f_X, f'_Y) contains only the identity.

Proof. Let $v \colon V \to Y$ be a smooth surjective morphism from a scheme V, then U is a Deligne-Mumford stack. Let $w \colon W \to U$ be an étale surjective morphism from a scheme W then, by Theorem 3.18, $Lu^* L^\bullet_F \cong L^\bullet_G$ and $Lw^* L^\bullet_G \cong L^\bullet_{G \circ w}$. By A.1, $\overline{x} \to X$ factors through u and $u \circ w$. Moreover $h_{W,\overline{x}} \cong h_{U,\overline{x}}$, because w is étale ([18] I.3.22), and, by deformation theory of schemes, we get a functorial exact sequence

$$0 \to h^0((L\overline{x}^* L^\bullet_G)^\vee) \otimes I \to h_{U,\overline{x}}(A') \to h_{U,\overline{x}}(A) \times_{h_{V,\overline{x}}(A)} h_{V,\overline{x}}(A') \xrightarrow{\mathrm{ob}_G} h^1((L\overline{x}^* L^\bullet_G)^\vee) \otimes I \to 0.$$

Therefore the first and the second part of the statement follows by the proof of Lemma A.9.

By Theorem 3.18, we have the following exact sequence

$$0 \to h^{-1}((L\overline{x}^* L^\bullet_X)^\vee) \to h^{-1}((L\overline{x}^* L^\bullet_Y)^\vee) \to$$
$$\to h^0((L\overline{x}^* L^\bullet_F)^\vee) \to h^0((L\overline{x}^* L^\bullet_X)^\vee) \to h^0((L\overline{x}^* L^\bullet_Y)^\vee).$$

Let $f_X \in h_{X,\overline{x}}(A)$, $f_Y \in h_{Y,\overline{x}}(A)$, $f'_Y \in h_{Y,\overline{x}}(A')$ such that $F \circ f_X = f_Y = f'_Y \circ i$. Assume that $\mathrm{ob}_F(f_X, f'_Y) = 0$ and fix $f'_X \in h_{X,\overline{x}}(A')$ such that $F \circ f'_X = f'_Y$, $f'_X \circ i = f_X$. By Proposition A.10, we have the following commutative diagram with exatc rows

$$\begin{tikzcd}
 & & \mathrm{Aut}_X(f'_X) \arrow[r] \arrow[d, equal] & \mathrm{Aut}_X(f_X) \times_{\mathrm{Aut}_Y(f_Y)} \mathrm{Aut}_Y(f'_Y) \arrow[d] \\
0 \arrow[r] & h^{-1}((L\overline{x}^* L^\bullet_X)^\vee) \otimes I \arrow[r] \arrow[d, hook] & \mathrm{Aut}_X(f'_X) \arrow[r] \arrow[d] & \mathrm{Aut}_X(f_X) \arrow[d] \\
0 \arrow[r] & h^{-1}((L\overline{x}^* L^\bullet_Y)^\vee) \otimes I \arrow[r] & \mathrm{Aut}_Y(f'_Y) \arrow[r] & \mathrm{Aut}_Y(f_Y)
\end{tikzcd}$$

from which we deduce that the only infinitesimal automorphism is the identity. □

A.12**.** Let X be a smooth S-scheme and let $\operatorname{Spec}\overline{k} \xrightarrow{\overline{x}} X$ be a geometric point of X. Let $C \to \operatorname{Spec}\overline{x}$ be a flat morphism of schemes, with C separated. Let $\Lambda = \hat{\mathscr{O}}_{S,\overline{x}}$. We define the deformation category Def_C such that, for all $A \in ({}^{\mathrm{Art}}/\Lambda)$, the objects of $\operatorname{Def}_C(A)$ are flat morphisms $C_A \xrightarrow{\pi_A} \operatorname{Spec} A$ such that the following diagram is cartesian

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d] & C_A \arrow[d, "\pi_A"] \\ \operatorname{Spec}\overline{x} \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

If $\pi\colon A' \to A$ is a morphism in $({}^{\mathrm{Art}}/\Lambda)$, then

$$\operatorname{Def}_C(\pi)\colon \operatorname{Def}_C(A') \to \operatorname{Def}_C(A)$$

sends $\pi_{A'}\colon C_{A'} \to \operatorname{Spec} A'$ to $C_{A'} \times_{\operatorname{Spec} A'} \operatorname{Spec} A \to \operatorname{Spec} A$. Given a morphism of schemes $f\colon C \to X$, we define the deformation category $\operatorname{Def}_{C,f}$ such that, for all $A \in ({}^{\mathrm{Art}}/\Lambda)$, the objects of $\operatorname{Def}_{C,f}(A)$ are pairs of morphisms $(C_A \xrightarrow{\pi_A} \operatorname{Spec} A, f_A)$ where π_A is an object of Def_C and $f_A\colon C_A \to X$ is such that $f_A \circ g = f$. There is a natural functor $\nu_f\colon \operatorname{Def}_{C,f} \to \operatorname{Def}_C$ which forgets the morphism to X.

A.13. **Proposition.** *The functor ν_f has tangent and obstruction spaces, for $i = 1, 2$,*

$$T^i\nu_f = H^{i-1}(C, f^*T_{X/S}).$$

Proof. Given a small extension

$$0 \to I \to A' \to A \to 0, \tag{2}$$

we want to study the functor

$$\operatorname{Def}_{C,f}(A') \to \operatorname{Def}_{C,f}(A) \times_{\operatorname{Def}_C(A)} \operatorname{Def}_C(A').$$

An object of $\operatorname{Def}_{C,f}(A) \times_{\operatorname{Def}_C(A)} \operatorname{Def}_C(A')$ is a cartesian diagram

$$\begin{tikzcd} C_A \arrow[r] \arrow[d, "\pi_A"'] & C_{A'} \arrow[d, "\pi_{A'}"] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \end{tikzcd}$$

togheter with a morphism $f_A\colon C_A \to X$. We want to know whether there exists a morphism $f_{A'}\colon C_{A'} \to X$ such that $f_{A'}|_{C_A} = f_A$ and how unique is such a morphism.

Assume $X = \operatorname{Spec} T$ and $C = \operatorname{Spec} R$, then $C_{A'} = \operatorname{Spec} R_{A'}$ and $C_A = \operatorname{Spec} R_A$ with $R_{A'} = R \otimes_\Lambda A'$ and $R_A = R \otimes_\Lambda A$. By assumption $\pi_{A'}$ is flat, then the tensor product $R_{A'} \otimes_{A'} \bullet$ is exact and from the sequence (2) we get an exact sequence

$$0 \to R \otimes_\Lambda I \to R_{A'} \to R_A \to 0.$$

Notice that $(R_{A'} \otimes_{A'} I)(R_{A'} \otimes_{A'} \mathfrak{m}_{A'}) = 0$, because $I\mathfrak{m}_{A'} = 0$. The morphism f_A induces a homomorphism $f_A^\sharp\colon T \to R_A$. We have that $T = {}^{T'}/{}_J$, with $T' = B[x_1, \dots, x_n]$. There exists always a lifting $g_{A'}^\sharp\colon T' \to R_{A'}$ of $f_A^\sharp$ and the set of such liftings is a principal homogeneous space under $\operatorname{Hom}_{T'}(\Omega_{T'/B}, R \otimes_\Lambda I)$. Moreover there exists a homomorphism

$$\alpha\colon \operatorname{Hom}_{T'}(\Omega_{T'/B}, R \otimes_\Lambda I) \to \operatorname{Hom}_{T'}({}^J/{}_{J^2}, R \otimes_\Lambda I),$$

induced by restriction, such that $\ker\alpha = T^1\nu_f \otimes I$ and $\operatorname{coker}\alpha = T^2\nu_f \otimes I$. Since

$$R \otimes_\Lambda I \cong (R \otimes_\Lambda \overline{k}) \otimes_{\overline{k}} I \cong R \otimes_{\overline{k}} I,$$

we have

$$\operatorname{Hom}_{T'}(\Omega_{T'/B}, R \otimes_\Lambda I) \cong \operatorname{Hom}_{T'}(\Omega_{T'/B}, R) \otimes_{\overline{k}} I \cong H^0(C, f^*T_{\mathbb{A}^n_{S/S}}|_X)$$
$$\operatorname{Hom}_{T'}({}^J/{}_{J^2}, R \otimes_\Lambda I) \cong \operatorname{Hom}_{T'}({}^J/{}_{J^2}, R) \otimes_{\overline{k}} I \cong H^0(C, f^*({}^J/{}_{J^2})^\vee).$$

We have an exact sequence

$$0 \to {}^{J}/_{J^2} \to \Omega_{T'/B} \otimes_{T'} T \to \Omega_{T/B} \to 0,$$

from which we deduce the following exact sequence

$$0 \to H^0(C, f^*T_{X/S}) \to H^0(C, f^*T_{\mathbb{A}^n_S/S}|_X) \to H^0(C, f^*({}^{J}/_{J^2})^\vee) \to H^1(C, f^*T_{X/S}),$$

where $H^1(C, f^*T_{X/S}) = 0$, because C is affine and $T_{X/S}$ is locally free. Hence $T^2\nu_f = 0$ and $T^1\nu_f = H^0(C, f^*T_{X/S})$.

In general, we consider coverings $\{X_i = \operatorname{Spec} T_i\}$, $\{C_i = \operatorname{Spec} R_i\}$, $\{C_{A,i} = \operatorname{Spec} R_{A,i}\}$ and $\{C_{A',i} = \operatorname{Spec} R_{A',i}\}$ by open affine schemes of X, C, C_A and $C_{A'}$, respectively, where $R_{A,i} = R_i \otimes_\Lambda A$ and $R_{A',i} = R_i \otimes_\Lambda A'$. We can chooce these covers in such a way that $f_A(C_{A,i}) \subset X_i$. To define $f_{A'}$ is the same as to define $f_{A',i} \colon C_{A',i} \to X_i$ such that $f_{A',i}|_{C_{A',ij}} = f_{A',j}|_{C_{A',ij}}$, where $C_{A',ij} = C_{A',i} \times_{C_{A'}} C_{A',j}$. By the affine case, there exist always a collection of liftings $\{f_{A',i}\}$. We can assume that $C_{A',ij}$ is affine (otherwise we consider an affine cover), then there exists a unique element $\eta_{ij} \in H^0(C_{ij}, f^*T_{X/S}) \otimes I$ such that $\eta_{ij}(f_{A',i}) = f_{A',j}$. Over C_{hij} we have

$$\eta_{hj}(f_{A',h}) = \eta_{ij}(\eta_{hi}(f_{A',h})),$$

hence $\eta_{hj} = \eta_{ij} + \eta_{hi}$ and so $\{\eta_{ij}\} \in C^1(\{C_i\}, f^*T_{X/S}) \otimes I$ is a cocycle. For each i, the set of liftings $\{f_{A',i}\}$ is a principal homogeneous space under $H^0(C_i, f^*T_{X/S}) \otimes I$. Therefore, given another collection of liftings $\{\tilde{f}_{A',i}\}$, there exists a unique collection $\{\eta_i \in H^0(C_i, f^*T_{X/S}) \otimes I\}$ such that $\eta_i(\{\tilde{f}_{A',i}\}) = \{f_{A',i}\}$. Let $\tilde{\eta}_{ij} \in H^0(C_{ij}, f^*T_{X/S}) \otimes I$ be such that $\tilde{\eta}_{ij}(\tilde{f}_{A',i}) = \tilde{f}_{A',j}$. Over C_{ij} we have

$$\eta_j(\tilde{\eta}_{ij}(\tilde{f}_{A',i})) = f_{A',j} = \eta_{ij}(f_{A',i}) = \eta_{ij}(\eta_i(\tilde{f}_{A',i})),$$

hence $\tilde{\eta}_{ij} = \eta_{ij} + \eta_i - \eta_j$ and the cocycle $\{\eta_{ij}\}$ is unique up to a coboundary. Now there exists a collection of morphisms $f_{A',i}$ which coincide on $C_{A',ij}$ if and only if the class of $\{\eta_{ij}\}$ in $H^1(\{C_i\}, f^*T_{X/S}) \otimes I$ is zero. In this case the set of such collections is equal to the set of $\{\eta_i \in H^0(C_i, f^*T_{X/S}) \otimes I\}$ and the gluing condition is equivalent to $\eta_i = \eta_j$ on C_{ij}. It follows that $\{\eta_i\} \in H^0(\{C_i\}, f^*T_{X/S}) \otimes I$. Finally

$$H^r(\{C_i\}, f^*T_{X/S}) \cong H^r(C, f^*T_{X/S}),$$

because C is separated. □

Appendix B. Intersection theory on Artin stacks over Dedekind domains

Intersection theory for schemes of finite type over a field was developed by Fulton and MacPherson ([12]) and was extended by Vistoli to a $\mathbb{Q}$-valued intersection theory on Deligne-Mumford stacks ([26]). In [9], Edidin and Graham define equivariant Chow groups, which provide integer-valued Chow groups for global quotient stacks. In [14], Kresch takes the idea of Edidin, Graham and Totaro further and develops an intersection theory on Artin stacks over a field together with an integer-valued intersection product on smooth Artin stacks which admit stratifications by global quotient stacks. Using an appropriate definition of relative dimension, one can define Chow groups for schemes over a Dedekind domain and show that they satisfy the properties expected from intersection theory ([12] 20). It follows that the theories in [26] and [9] are valid for stacks over a Dedekind domain ([26], [9] 6.2).

Although not mentioned in [14], one can verify that the theory can be extended to Artin stacks over a Dedekind domain. As a consequence we get that Manolache's construction of the virtual pullback in [17] is valid for Deligne-Mumford type morphisms of Artin stacks over a Dedekind domain. As a consequence we are able to extend Manolache's proof of Costello's pushforward formula to proper morphisms of Artin stacks with quasi-finite diagonal.

B.1. **Chow groups of Artin stacks with quasi-finite diagonal.** Let D be a Dedekind domain and let $\mathcal{M}$ be an Artin stack over $S = \operatorname{Spec} D$. For an integral closed substack $Z \subset \mathcal{M}$, we define the relative dimension ([12] 20.1)

$$\dim_S Z = \operatorname{trdeg}_{k(T)} k(Z) - \operatorname{codim}_S T,$$

where T is the closure of the image of $\mathcal{Z}$ in S, and $k(Z)$, $k(T)$ are function fields ([26] 1.14).

B.1. **Definition.** We denote by $Z_*(\mathcal{M}/S)$ the free abelian group on the set of integral closed substacks of $\mathcal{M}$, graded by relative dimension. Let $W_j(\mathcal{M}/S) = \bigoplus_{\mathcal{Z}} k(Z)^*$, where the sum is taken over all integral substacks Z of $\mathcal{M}$ of relative dimension $j+1$. There is a homomorphism $\partial\colon W_j(\mathcal{M}/S) \to Z_j(\mathcal{M}/S)$ which locally for the smooth topology sends a rational function to the corresponding Weil divisor. The *Chow groups* of $\mathcal{M}$ are defined to be $A_j(\mathcal{M}/S) = Z_j(\mathcal{M}/S)/\partial W_j(\mathcal{M}/S)$.

B.2. **Theorem** ([14] 3.5.7, 5.3.1)**.** *Let $\mathcal{M}$ be an Artin stack with quasi-finite diagonal over a Dedekind domain D. Then $A_*(\mathcal{M}/S) \cong A_*^{Kresch}(\mathcal{M}/S)$, where $A_*^{Kresch}(\mathcal{M}/S)$ are Kresch's Chow groups (*[14] *2.1.11).*

B.3. **Theorem** ([10] 2.7)**.** *Let $\mathcal{M}$ be an Artin stack with quasi-finite diagonal over a Dedekind domain D. Then there exists a finite surjective morphism $U \to \mathcal{M}$ from a scheme U.*

B.4. Remark ([10] 2.4). The morphism $U \to \mathcal{M}$ is strongly representable.

B.2. **Proper pushforward.** Let $\pi\colon \mathcal{N} \to \mathcal{M}$ be a proper morphism of Artin S-stacks. If $\mathcal{M}$ and $\mathcal{N}$ have quasi-finite diagonal then it is possible to define a nonrepresentable proper pushforward π_* as follows ([10] 2.8).

B.5. **Definition.** Let $u\colon U \to \mathcal{M}$ be a finite and surjective morphism from a scheme U. We define the *proper pushforward*

$$u_*\colon A_*(U/S) \to A_*(\mathcal{M}/S)$$

by $u_*[Z] = \deg(Z/u(Z))[u(Z)]$, where $\deg(Z/u(Z)) = \deg(V \times_{\mathcal{M}} U/V)$ for a smooth atlas $V \to u(Z)$.

B.6. Remark. Notice that $V \times_{\mathcal{M}} U \cong V \times_{u(Z)} Z$ is a scheme and the degree $\deg(Z/u(Z))$ is independent of the chosen atlas. Let $V' \to u(Z)$ be another smooth atlas and set $W = V \times_{u(Z)} V'$. Then

$$\deg(V \times_{\mathcal{M}} U/V) = \deg(W \times_{\mathcal{M}} U/W) = \deg(V' \times_{\mathcal{M}} U/V').$$

B.7. Remark. The proper pushforward commutes with projective pushforward and flat pullback (this follows easily from the properties of the relative degree).

B.8. Notation. We set $A_*(\mathcal{M}/S)_{\mathbb{Q}} = A_*(\mathcal{M}/S) \otimes_{\mathbb{Z}} \mathbb{Q}$.

B.9. **Lemma.** *Let $u\colon U \to \mathcal{M}$ be a finite and surjective morphism from a scheme U and let u_1, $u_2\colon U \times_{\mathcal{M}} U \to U$ be the projections. Then we have the following exact sequence*

$$A_j(U \times_{\mathcal{M}} U/S)_{\mathbb{Q}} \xrightarrow{u_{1*}-u_{2*}} A_j(U/S)_{\mathbb{Q}} \xrightarrow{u_*} A_j(\mathcal{M}/S)_{\mathbb{Q}} \to 0.$$

Proof. For surjectivity of u_*, let $[\mathcal{Z}] \in A_j(\mathcal{M}/S)_{\mathbb{Q}}$. Let Z be a j-dimensional component of $\mathcal{Z} \times_{\mathcal{M}} U$, then $[Z] \in A_j(U/S)_{\mathbb{Q}}$ and

$$u_*\left(\frac{1}{\deg(Z/\mathcal{Z})}[Z]\right) = [\mathcal{Z}] \in A_j(\mathcal{M}/S)_{\mathbb{Q}}.$$

Notice that also $u_*\colon W_*(U/S) \to W_*(\mathcal{M}/S)$ is surjective. Moreover, for $[Z] \in A_j(U \times_{\mathcal{M}} U/S)_{\mathbb{Q}}$,

$$\begin{aligned} u_*(u_{1*} - u_{2*})[Z] &= u_*(\deg(Z/u_1(Z))[u_1(Z)] - \deg(Z/u_2(Z))[u_2(Z)]) \\ &= \deg(Z/u(u_1(Z)))[u(u_1(Z))] - \deg(Z/u(u_2(Z))))[u(u_2(Z))] = 0. \end{aligned}$$

So it is enough to show that every $\alpha = \sum_{i=1}^{s} n_i[Z_i] \in Z_j({}^{U}/_{S})_{\mathbb{Q}}$ such that $u_*(\alpha) = 0$ in $Z_j({}^{\mathcal{M}}/_{S})_{\mathbb{Q}}$ lies in the image of $u_{1*} - u_{2*}$. Since

$$u_*(\alpha) = \sum_{i=1}^{s} n_i \deg({}^{Z_i}/_{u(Z_i)})[u(Z_i)] = 0,$$

we may assume that $u(Z_i) = \mathcal{Z}$ for $i = 1, \ldots, s$. Therefore we get $\sum_{i=1}^{s} n_i d_i = 0$, where we set $d_i = \deg({}^{Z_i}/_{\mathcal{Z}})$. For $i = 2, \ldots, s$, let V_i be a j-dimensional component of $Z_1 \times_{\mathcal{Z}} Z_i$, then

$$u_{2*}[V_i] = \deg({}^{V_i}/_{Z_i})[Z_i] = e_i d_1 [Z_i],$$

where we set $e_i = \deg({}^{V_i}/_{Z_i \times_{\mathcal{Z}} Z_1})$. By properties of relative degree,

$$u_{1*}[V_i] = \deg({}^{V_i}/_{Z_1})[Z_1] = e_i d_i [Z_1],$$

and it follows that

$$(u_{1*} - u_{2*}) \sum_{i=2}^{s} \frac{n_i}{e_i d_1} [V_i] = \alpha. \qquad \square$$

B.10. Remark. *If $p\colon T \to U$ is a finite surjective morphism from a scheme T and we set $t = u \circ p$, then we have the following commutative diagram with exact rows*

$$\begin{tikzcd}
A_*({}^{T \times_{\mathcal{M}} T}/_{S})_{\mathbb{Q}} \arrow[r, "t_{1*} - t_{2*}"] \arrow[d] & A_*({}^{T}/_{S})_{\mathbb{Q}} \arrow[r, "t_*"] \arrow[d, "p_*"] & A_*({}^{\mathcal{M}}/_{S})_{\mathbb{Q}} \arrow[r] \arrow[d, equal] & 0 \\
A_*({}^{U \times_{\mathcal{M}} U}/_{S})_{\mathbb{Q}} \arrow[r, "u_{1*} - u_{2*}"] & A_*({}^{U}/_{S})_{\mathbb{Q}} \arrow[r, "u_*"] & A_*({}^{\mathcal{M}}/_{S})_{\mathbb{Q}} \arrow[r] & 0
\end{tikzcd}$$

B.11**.** Let $u\colon U \to \mathcal{M}$ be a finite surjective morphism from a scheme U and form the fibre diagram

$$\begin{tikzcd}
V \arrow[r, "\pi'"] \arrow[d, "v"'] & U \arrow[d, "u"] \\
\mathcal{N} \arrow[r, "\pi"] & \mathcal{M}
\end{tikzcd}$$

Then V is a scheme and v is finite surjective. Moreover, by Lemma B.9, we get the following commutative diagram with exact rows

$$\begin{tikzcd}
A_*({}^{V \times_{\mathcal{N}} V}/_{S})_{\mathbb{Q}} \arrow[r, "v_{1*} - v_{2*}"] \arrow[d] & A_*({}^{V}/_{S})_{\mathbb{Q}} \arrow[r, "v_*"] \arrow[d, "\pi'_*"] & A_*({}^{\mathcal{N}}/_{S})_{\mathbb{Q}} \arrow[r] & 0 \\
A_*({}^{U \times_{\mathcal{M}} U}/_{S})_{\mathbb{Q}} \arrow[r, "u_{1*} - u_{2*}"] & A_*({}^{U}/_{S})_{\mathbb{Q}} \arrow[r, "u_*"] & A_*({}^{\mathcal{M}}/_{S})_{\mathbb{Q}} \arrow[r] & 0
\end{tikzcd}$$

which induces a map $\pi_*\colon A_*({}^{\mathcal{N}}/_{S})_{\mathbb{Q}} \to A_*({}^{\mathcal{M}}/_{S})_{\mathbb{Q}}$.

B.12. **Lemma.** *The map π_* does not depend on the choice of the finite surjective morphism $u\colon U \to \mathcal{M}$.*

Proof. Let $u'\colon U' \to \mathcal{M}$ be a finite surjective morphism from a scheme U' and consider $T = U \times_{\mathcal{M}} U'$ with the natural morphism $p\colon T \to U$, which is finite surjective. Let us form the fibre diagram

$$\begin{tikzcd}
W \arrow[r, "\pi''"] \arrow[d, "q"'] & T \arrow[d, "p"] \\
V \arrow[r, "\pi'"] \arrow[d, "v"'] & U \arrow[d, "u"] \\
\mathcal{N} \arrow[r, "\pi"] & \mathcal{M}
\end{tikzcd}$$

and set $t = u \circ p$, $w = v \circ q$. Let us denote by $\widetilde{\pi}_*$ the pullback defined via $t\colon T \to \mathcal{M}$. Let $\alpha \in A_*(\mathcal{N}/S)_{\mathbb{Q}}$ and let $\alpha'' \in A_*(W/S)_{\mathbb{Q}}$ such that $w_*\alpha'' = \alpha$, then

$$\widetilde{\pi}_*\alpha = t_*\pi''_*\alpha'' = u_*p_*\pi''_*\alpha'' = u_*\pi'_*(q_*\alpha'') = \pi_*\alpha,$$

where the last equality follows from the fact that $v_*(q_*\alpha'') = t_*\alpha'' = \alpha$. □

B.13. **Definition.** We call $\pi_*\colon A_*(\mathcal{N}/S)_{\mathbb{Q}} \to A_*(\mathcal{M}/S)_{\mathbb{Q}}$ the *proper pushforward* for π.

B.3. **Costello's pushforward formula.** In [17] Manolache uses the virtual pullback to give a short proof of Costello's pushforward formula ([8] 5.0.1). Here we apply Manolache's construction to prove the pushforward formula in a more general setting.

B.14. **Proposition.** *Let D be a Dedekind domain. Let us consider a cartesian diagram*

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow{f} & \mathcal{M}_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathfrak{M}_1 & \xrightarrow{g} & \mathfrak{M}_2 \end{array}$$

where

(1) $\mathfrak{M}_1, \mathfrak{M}_2$ are Artin stacks over D of the same pure dimension,
(2) $\mathcal{M}_1, \mathcal{M}_2$ are Artin stacks over D with quasi-finite diagonal,
(3) g is a Deligne-Mumford type morphism of degree d,
(4) f is proper,
(5) for $i = 1, 2$, p_i admits perfect obstruction theory $E_i^\bullet$ such that $f^*E_2^\bullet \cong E_1^\bullet$.

Then

$$f_*[\mathcal{M}_1, E_1^\bullet]^{virt} = d[\mathcal{M}_2, E_2^\bullet]^{virt}$$

in each of the following cases

(a) g is projective,
(b) $\mathfrak{M}_1, \mathfrak{M}_2$ are Deligne-Mumford stacks and g is proper,
(c) $\mathfrak{M}_1, \mathfrak{M}_2$ have quasi-finite diagonal and g is proper.

Proof. Since in each of the cases listed above we are able to pushforward along g, the statement follows by the same argument of [17] 5.29, after noticing that non-representable proper pushforward commutes with virtual pullback (this can be shown in the same way as in [17] 4.1). □

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