

UNIQUENESS AND CONTINUOUS DEPENDENCE FOR A VISCOELASTIC PROBLEM WITH MEMORY IN DOMAINS WITH TIME DEPENDENT CRACKS

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ABSTRACT. We study some hyperbolic partial integro-differential systems in domains with time dependent cracks. In particular, we give conditions on the cracks which imply the uniqueness of the solution with prescribed initial-boundary conditions, and its continuous dependence on the cracks.

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1. INTRODUCTION

The study of models of viscoelastic materials with memory has a long history that goes back to Boltzmann ([1] and [2]) and Volterra ([17] and [18]). Recent results on this subject can be found in [8], [11], [12], and [15]. For particular values of the parameters, the Maxwell model for viscoelastic materials is governed by the following system of partial differential equations in $Q := \Omega \times [0, T]$ with a memory term:

$$\ddot{u}(t) - \operatorname{div}((\mathbb{C} + \mathbb{V})Eu(t)) + \operatorname{div}\left(\int_0^t e^{\tau-t} \mathbb{V}Eu(\tau) d\tau\right) = \ell(t), \quad (1.1)$$

where $\Omega \subset \mathbb{R}^d$ is the reference configuration, $[0, T]$ is the time interval, $u(t)$, $Eu(t)$, and $\ddot{u}(t)$ are the displacement at time t , the symmetric part of its gradient, and its second derivative with respect to time, \mathbb{C} and \mathbb{V} are the elasticity and viscosity tensors, and $\ell(t)$ is the external load at time t .

In this paper we study problem (1.1) with a prescribed time dependent growing crack Γ_t , $t \in [0, T]$, namely

$$\ddot{u}(t) - \operatorname{div}((\mathbb{C} + \mathbb{V})Eu(t)) + \operatorname{div}\left(\int_0^t e^{\tau-t} \mathbb{V}Eu(\tau) d\tau\right) = \ell(t) \quad \text{in } Q_{cr}, \quad (1.2)$$

where $Q_{cr} := \{(x, t) : t \in [0, T], x \in \Omega \setminus \Gamma_t\}$. Problem (1.2) is complemented by initial conditions at $t = 0$ for u and \dot{u} and by boundary conditions on $\partial\Omega$ and Γ_t .

The existence of a solution of (1.2) is proved in [14]. Our first result (Theorem 2.7) is the uniqueness of the solution under strong regularity assumptions on the sets Γ_t and on their dependence on t . More precisely, we assume the same regularity conditions that were used in [6] and [3] to prove the uniqueness of the solution in Q_{cr} of the problem without the memory term, i.e.,

$$\ddot{u}(t) - \operatorname{div}((\mathbb{C} + \mathbb{V})Eu(t)) = \ell(t) \quad \text{in } Q_{cr}. \quad (1.3)$$

To prove our uniqueness result we write problem (1.2) in the equivalent form

$$\ddot{u}(t) - \operatorname{div}((\mathbb{C} + \mathbb{V})Eu(t)) = \ell(t) - \operatorname{div}F_u(t) \quad \text{in } Q_{cr}, \quad (1.4)$$

where

$$F_u(t) := \int_0^t e^{\tau-t} \mathbb{V}Eu(\tau) d\tau.$$

This allows us to estimate u in terms of F_u using the energy inequality for the solution of (1.3). Then we estimate F_u in terms of u using just the definition of F_u . Uniqueness is obtained from the combined estimate.

Our second result (Theorem 4.1) is the continuous dependence of the solutions of (1.2) on the cracks. More precisely, we consider a sequence Γ_t^n of time dependent cracks and the solutions u^n of problem (1.2) with Γ_t replaced by Γ_t^n . Under suitable assumption on the convergence of Γ_t^n to Γ_t we prove that the sequence u^n converges to the solution u of (1.2). Our assumptions of Γ_t^n are similar to those considered in [6] and [3] to prove the corresponding result for (1.3).

To prove the continuous dependence we write our problem in the form (1.4) and we regard u^n as a fixed point for a suitable operator depending on n , which is a contraction if T is small enough. Under this assumption the convergence of u^n is a consequence of a general results on fixed points of contractions (Lemma 4.2). To show that its hypotheses are satisfied, we use the continuous dependence on the cracks of the solutions of problem (1.3) (see [6] and [3]) and we obtain the result if T is small enough. If T is large we divide the interval $[0, T]$ into smaller intervals where we can apply the previous result.

2. FORMULATION OF THE PROBLEM

The reference configuration of our problem is a bounded open set $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with Lipschitz boundary $\partial\Omega$. We assume that $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, where $\partial_D\Omega$ and $\partial_N\Omega$ are disjoint (possibly empty) Borel sets, on which we prescribe Dirichlet and Neumann boundary conditions respectively.

For every $x \in \overline{\Omega}$ the elasticity tensor $\mathbb{C}(x)$ and the viscosity tensor $\mathbb{V}(x)$ are prescribed elements of the space $\mathcal{L}(\mathbb{R}_{sym}^{d \times d}; \mathbb{R}_{sym}^{d \times d})$ of linear maps from $\mathbb{R}_{sym}^{d \times d}$ into $\mathbb{R}_{sym}^{d \times d}$, where $\mathbb{R}_{sym}^{d \times d}$ is the space of real $d \times d$ symmetric matrices. The euclidean scalar product between the matrices A and B is denoted by $A : B$. We assume that the functions $\mathbb{C}, \mathbb{V} : \overline{\Omega} \rightarrow \mathcal{L}(\mathbb{R}_{sym}^{d \times d}; \mathbb{R}_{sym}^{d \times d})$ satisfy the following properties, for suitable constants $\alpha_0 > 0$ and $M_0 > 0$:

- (H1) (regularity) \mathbb{C} is of class C^1 and $\max_{x \in \overline{\Omega}} |\mathbb{C}(x)| \leq M_0$;
- (H2) (symmetry) $\mathbb{C}(x)A : B = A : \mathbb{C}(x)B$ for every $x \in \overline{\Omega}$ and $A, B \in \mathbb{R}_{sym}^{d \times d}$;
- (H3) (coerciveness) $\mathbb{C}(x)A : A \geq \alpha_0 |A|^2$ for every $x \in \overline{\Omega}$ and $A \in \mathbb{R}_{sym}^{d \times d}$;
- (H4) (regularity) \mathbb{V} is of class C^1 and $\max_{x \in \overline{\Omega}} |\mathbb{V}(x)| \leq M_0$;
- (H5) (symmetry) $\mathbb{V}(x)A : B = A : \mathbb{V}(x)B$ for every $x \in \Omega$ and $A, B \in \mathbb{R}_{sym}^{d \times d}$;
- (H6) (coerciveness) $\mathbb{V}(x)A : A \geq \alpha_0 |A|^2$ for every $x \in \overline{\Omega}$ and $A \in \mathbb{R}_{sym}^{d \times d}$.

Throughout the paper we study the problem in the time interval $[0, T]$, with $T > 0$. For $t \in [0, T]$ the crack at time t is given by a subset Γ_t of the intersection between $\overline{\Omega}$ and a suitable $d - 1$ dimensional manifold Γ (regarded as the crack path). We assume that

- (H7) Γ is a complete $(d - 1)$ -dimensional C^2 manifold with boundary;
- (H8) $\Omega \cap \partial\Gamma = \emptyset$ and $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$, where \mathcal{H}^{d-1} denotes the $(d - 1)$ -dimensional Hausdorff measure;
- (H9) for every $x \in \Gamma \cap \partial\Omega$ there exists an open neighborhood U_x of x in \mathbb{R}^d such that $U_x \cap (\Omega \setminus \Gamma)$ is the union of two non empty disjoint open sets U_x^+ and U_x^- with Lipschitz boundary;
- (H10) Γ_t is closed, $\Gamma_t \subset \Gamma \cap \overline{\Omega}$ for every $t \in [0, T]$, and $\Gamma_s \subset \Gamma_t$ for every $s < t$ (irreversibility of the fracture process).

Moreover we assume that there exist $\Phi, \Psi : [0, T] \times \overline{\Omega} \rightarrow \overline{\Omega}$ with the following properties:

- (H11) Φ, Ψ are of class $C^{2,1}$;
- (H12) $\Psi(t, \Phi(t, y)) = y$ and $\Phi(t, \Psi(t, x)) = x$ for every $x, y \in \overline{\Omega}$;

- (H13) $\Phi(t, \Gamma) = \Gamma$, $\Phi(t, \Gamma_0) = \Gamma_t$, and $\Phi(t, y) = y$ for every $t \in [0, T]$ and every y in a neighborhood of $\partial\Omega$;
- (H14) $\Phi(0, y) = y$ for every $y \in \bar{\Omega}$;
- (H15) $|\dot{\Phi}(t, y)|^2 < \frac{m_{det}(\Psi)\alpha_0}{M_{det}(\Psi)K}$ for every $y \in \bar{\Omega}$, where the dot denotes the derivative with respect to t , $m_{det}(\Psi) := \min \det D\Psi$, $M_{det}(\Psi) := \max \det D\Psi$. and K is the constant in Korn's inequality in Lemma 2.2 below.

We shall prove that our hypotheses imply that Korn's inequality holds on $\Omega \setminus \Gamma$. We begin with the following technical lemma.

Lemma 2.1. *Under hypotheses (H7)-(H9), the set $\Omega \setminus \Gamma$ is the union of a finite number of connected open sets with Lipschitz boundary.*

Proof. Since Γ is a C^2 manifold of dimension $d - 1$, for every $x \in \Gamma \cap \Omega$ there exists an open neighborhood U_x of x in \mathbb{R}^d such that $U_x \cap (\Omega \setminus \Gamma)$ is the union of two non empty disjoint open sets U_x^+ and U_x^- with Lipschitz boundary. By our hypothesis on $\Gamma \cap \partial\Omega$ the same property holds, more in general, for every $x \in \Gamma \cap \bar{\Omega}$. Since $\Gamma \cap \bar{\Omega}$ is compact, there exists a finite number of points $x_1, \dots, x_m \in \Gamma \cap \bar{\Omega}$ such that $\Gamma \cap \bar{\Omega} \subset \cup_{i=1}^m U_{x_i}$.

Since Ω has Lipschitz boundary, for every $y \in \partial\Omega \setminus \cup_{i=1}^m U_{x_i} \subset \partial\Omega \setminus \Gamma$ there exists an open neighborhood V_y of y in \mathbb{R}^d such that $V_y \cap (\Omega \setminus \Gamma)$ has Lipschitz boundary. By compactness there exists a finite number of points $y_1, \dots, y_n \in \partial\Omega \setminus \cup_{i=1}^m U_{x_i}$ such that $\partial\Omega \setminus \cup_{i=1}^m U_{x_i} \subset \cup_{j=1}^n V_{y_j}$.

Since $\bar{\Omega} \setminus (\cup_{i=1}^m U_{x_i} \cup \cup_{j=1}^n V_{y_j})$ is compact and is contained in the open set $\Omega \setminus \Gamma$, there exists an open set W with Lipschitz boundary such that $\bar{\Omega} \setminus (\cup_{i=1}^m U_{x_i} \cup \cup_{j=1}^n V_{y_j}) \subset W \subset \Omega \setminus \Gamma$. Therefore

$$\Omega \setminus \Gamma = W \cup \bigcup_{i=1}^m U_{x_i}^+ \cup \bigcup_{i=1}^m U_{x_i}^- \cup \bigcup_{j=1}^n (V_{y_j} \cap (\Omega \setminus \Gamma)).$$

Since every open sets with Lipschitz boundary is the union of a finite number of connected open sets with Lipschitz boundary, the conclusion follows. \square

For every $u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$ Du denotes jacobian matrix in the sense of distributions on $\Omega \setminus \Gamma$ and Eu is its symmetric part, i.e.,

$$Eu := \frac{1}{2}(Du + Du^T).$$

Lemma 2.2. *Under hypotheses (H7)-(H9), there exists a constant K , depending only on Ω and Γ , such that*

$$\|Du\|^2 \leq K(\|u\|^2 + \|Eu\|^2) \quad (2.1)$$

for every $u \in H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$, where $\|\cdot\|$ denotes the L^2 norm.

Proof. The result is a consequence of the second Korn's inequality (see, e.g., [13, Theorem 2.4]), applied to the sets with Lipschitz boundary provided by Lemma 2.1. \square

Remark 2.3. Under hypotheses (H7)-(H9), using a localization argument (see the proof of Lemma 2.1) we can prove that the trace operator is well defined and continuous from $H^1(\Omega \setminus \Gamma; \mathbb{R}^d)$ into $L^2(\partial\Omega; \mathbb{R}^d)$.

We now introduce the function spaces that will be used in the precise formulation of problem (1.2). We set

$$V := H^1(\Omega \setminus \Gamma; \mathbb{R}^d) \quad \text{and} \quad H := L^2(\Omega; \mathbb{R}^d). \quad (2.2)$$

For every finite dimensional Hilbert space Y the symbols (\cdot, \cdot) and $\|\cdot\|$ denote the scalar product and the norm in the $L^2(\Omega; Y)$, according to the context. The space V is endowed with the norm

$$\|u\|_V := (\|u\|^2 + \|Du\|^2)^{1/2}. \quad (2.3)$$

For every $t \in [0, T]$ we define

$$V_t := H^1(\Omega \setminus \Gamma_t; \mathbb{R}^d) \quad \text{and} \quad V_t^D := \{u \in V_t \mid u|_{\partial_D \Omega} = 0\}, \quad (2.4)$$

where $u|_{\partial_D \Omega}$ denotes the trace of u on $\partial_D \Omega$. We note that V_t and V_t^D are closed linear subspaces of V .

We define

$$\mathcal{V} := \{v \in L^2(0, T; V) \cap H^1(0, T; H) \mid v(t) \in V_t \text{ for a.e. } t \in (0, T)\}, \quad (2.5)$$

which is a Hilbert space with the norm

$$\|v\|_{\mathcal{V}} := (\|v\|_{L^2(0, T; V)}^2 + \|\dot{v}\|_{L^2(0, T; H)}^2)^{\frac{1}{2}}, \quad (2.6)$$

where the dot denotes the distributional derivative with respect to t . Moreover we set

$$\mathcal{V}^D := \{v \in \mathcal{V} \mid v(t) \in V_t^D \text{ for a.e. } t \in (0, T)\} \quad (2.7)$$

and note that it is a closed linear subspace of \mathcal{V} . Since $H^1(0, T; H) \hookrightarrow C^0([0, T]; H)$ we have $\mathcal{V} \hookrightarrow C^0([0, T], H)$. In particular $v(0)$ and $v(T)$ are well defined as elements of H , for every $v \in \mathcal{V}$.

We set

$$\tilde{H} := L^2(\Omega; \mathbb{R}_{sym}^{d \times d}). \quad (2.8)$$

On the forcing term $\ell(t)$ of (1.2) we assume that

$$\ell(t) := f(t) - \operatorname{div} F(t), \quad (2.9)$$

where

$$f \in L^2(0, T; H) \quad \text{and} \quad F \in H^1(0, T; \tilde{H}) \quad (2.10)$$

are prescribed function. As usual the divergence of a matrix valued function is the vector valued function whose components are obtained taking the divergence of the rows.

As for the Dirichlet boundary condition on $\partial_D \Omega$, it is obtained by prescribing a function

$$u_D \in H^2(0, T; H) \cap H^1(0, T; V_0). \quad (2.11)$$

We impose that for a.e. $t \in [0, T]$ the trace of the solution $u(t)$ is equal to the trace $u_D(t)$ on $\partial_D \Omega$, i.e., $u(t) - u_D(t) \in V_t^D$.

About the initial data we fix

$$u^0 \in V_0 \quad \text{and} \quad u^1 \in H. \quad (2.12)$$

Moreover, we assume the compatibility condition

$$u^0 - u_D(0) \in V_0^D. \quad (2.13)$$

We are now in a position to give the precise definition of solution of problem (1.2).

Definition 2.4 (Solution for visco-elastodynamics with cracks). We say that u is a weak solution of problem (1.2) of visco-elastodynamics on the cracked domains $\Omega \setminus \Gamma_t$, with external load $\ell = f - \operatorname{div} F$, Dirichlet boundary condition u_D on $\partial_D \Omega$, natural Neumann boundary condition on $\partial_N \Omega \cup \Gamma_t$, and initial conditions u^0 and u^1 , if

$$u \in \mathcal{V} \quad \text{and} \quad u - u_D \in \mathcal{V}^D, \quad (2.14)$$

$$\begin{aligned} & - \int_0^T (\dot{u}(t), \dot{\varphi}(t)) dt + \int_0^T ((C + \mathbb{V})Eu(t), E\varphi(t)) dt - \int_0^T \int_0^t e^{\tau-t} (\mathbb{V}Eu(\tau), E\varphi(t)) d\tau dt \\ & = \int_0^T (f(t), \varphi(t)) dt + \int_0^T (F(t), E\varphi(t)) dt \quad \text{for all } \varphi \in \mathcal{V}^D \text{ with } \varphi(0) = \varphi(T) = 0, \end{aligned} \quad (2.15)$$

$$u(0) = u^0 \quad \text{in } H \quad \text{and} \quad \dot{u}(0) = u^1 \quad \text{in } (V_0^D)^*, \quad (2.16)$$

where $(V_0^D)^*$ denotes the topological dual of V_0^D for $t = 0$.

Remark 2.5. If u satisfy (2.14) and (2.15), it is possible to prove that $\dot{u} \in H^1(0, T; (V_0^D)^*)$ (see [14, Remark 4.6]), which implies $\dot{u} \in C^0([0, T]; (V_0^D)^*)$. In particular $\dot{u}(0)$ is well defined as an element of $(V_0^D)^*$.

Remark 2.6. Under suitable regularity assumptions, u is a solution in the sense of Definition 2.4 if and only if $u(0) = u^0$, $\dot{u}(0) = u^1$, and for every $t \in [0, T]$

$$\begin{aligned} \ddot{u}(t) - \operatorname{div}((\mathbb{C} + \mathbb{V})Eu(t)) + \operatorname{div}\left(\int_0^t e^{\tau-t} \mathbb{V}Eu(\tau) d\tau\right) &= f(t) - \operatorname{div}F(t) && \text{in } \Omega \setminus \Gamma_t, \\ u(t) &= u_D(t) && \text{on } \partial_D\Omega, \\ \left((\mathbb{C} + \mathbb{V})Eu(t) - \int_0^t e^{\tau-t} \mathbb{V}Eu(\tau) d\tau\right)\nu &= F(t)\nu && \text{on } \partial_N\Omega, \\ \left((\mathbb{C} + \mathbb{V})Eu(t) - \int_0^t e^{\tau-t} \mathbb{V}Eu(\tau) d\tau\right)^\pm \nu &= F(t)^\pm \nu && \text{on } \Gamma_t, \end{aligned}$$

where ν is the unit normal and the symbol \pm denotes suitable limits on each side of Γ_t .

The last two conditions represent the natural Neumann boundary conditions on $\partial_N\Omega$ and on the faces of Γ_t .

To describe the boundedness properties of the solutions of problem (2.14)-(2.16), we introduce the space

$$\mathcal{V}^\infty := \{v \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; H) \mid v(t) \in V_t \text{ for a.e. } t \in (0, T)\}, \quad (2.17)$$

which is a Banach space with the norm

$$\|v\|_{\mathcal{V}^\infty} := \|v\|_{L^\infty(0, T; V)} + \|\dot{v}\|_{L^\infty(0, T; H)}. \quad (2.18)$$

As for the continuity properties, it is convenient to introduce the space of weakly continuous functions with values in a Banach space X with topological dual X^* , defined by

$$C_w^0([0, T]; X) := \{v : [0, T] \rightarrow X \mid t \mapsto \langle h, v(t) \rangle \text{ is continuous for every } h \in X^*\}.$$

We are now in position to state one of the main results of the paper.

Theorem 2.7. *Assume (H1)-(H15) and (2.10)-(2.13). Then there exists a unique solution of problem (2.14)-(2.16). Moreover $u \in \mathcal{V}^\infty$, $u \in C_w^0([0, T]; V)$, and $\dot{u} \in C_w^0([0, T]; H)$.*

The existence of a solution is proved in [14] under much weaker assumptions on the cracks Γ_t . The uniqueness will be proved in the next section.

3. UNIQUENESS

In our proof of Theorem 2.7 we shall use some known results about existence and uniqueness for the system of elastodynamics on cracked domains, where the memory terms is not present. We set

$$\mathbb{A} := \mathbb{C} + \mathbb{V} \quad (3.1)$$

and we consider \mathbb{A} as the elasticity tensor of the auxiliary problem defined below.

Definition 3.1 (Solution for elastodynamics with cracks). We say that v is a weak solution of problem (1.3) of elastodynamics on the cracked domains $\Omega \setminus \Gamma_t$, with external load $\ell = f - \operatorname{div}F$, Dirichlet boundary condition u_D on $\partial_D\Omega$, natural Neumann boundary

condition on $\partial_N \Omega \cup \Gamma_t$, and initial conditions u^0 and u^1 , if

$$v \in \mathcal{V} \quad \text{and} \quad v - u_D \in \mathcal{V}^D, \quad (3.2)$$

$$\begin{aligned} & - \int_0^T (\dot{v}(t), \dot{\varphi}(t)) dt + \int_0^T (\mathbb{A}Ev(t), E\varphi(t)) dt = \int_0^T (f(t), \varphi(t)) dt \\ & + \int_0^T (F(t), E\varphi(t)) dt \quad \text{for all } \varphi \in \mathcal{V}^D \text{ with } \varphi(0) = \varphi(T) = 0, \end{aligned} \quad (3.3)$$

$$v(0) = u^0 \quad \text{in } H \quad \text{and} \quad \dot{v}(0) = u^1 \quad \text{in } (V_0^D)^*. \quad (3.4)$$

The following technical lemma will be used in the proof of Theorem 3.3.

Lemma 3.2. *Let v be a weak solution according to Definition 3.1 satisfying $\dot{v}(0) = 0$ in the sense of $(V_0^D)^*$. Then (3.3) holds for every $\varphi \in \mathcal{V}^D$ such that $\varphi(0) \in V_0^D$ and $\varphi(t) = 0$ in a neighborhood of T , even if the condition $\varphi(0) = 0$ is not satisfied.*

Proof. Let φ as in the statement. For every $\varepsilon > 0$, we define

$$\varphi_\varepsilon(t) := \begin{cases} \frac{t}{\varepsilon}\varphi(0) & \text{for } t \in [0, \varepsilon], \\ \varphi(t - \varepsilon) & \text{for } t \in (\varepsilon, T]. \end{cases}$$

Then $\varphi_\varepsilon \in \mathcal{V}^D$ and $\varphi_\varepsilon(0) = \varphi_\varepsilon(T) = 0$, for ε small enough (3.3) holds for φ_ε . We observe that

$$\int_0^T (\dot{v}(t), \dot{\varphi}_\varepsilon(t)) dt = \frac{1}{\varepsilon} \int_0^\varepsilon (\dot{v}(t), \varphi(0)) dt + \int_\varepsilon^T (\dot{v}(t), \dot{\varphi}(t - \varepsilon)) dt \rightarrow \int_0^T (\dot{v}(t), \dot{\varphi}(t)) dt$$

as $\varepsilon \rightarrow 0$, where we have used the initial condition in the first term and the continuity of translations in the second one. In a similar way we can pass to the limit in the other terms of equation (3.3). \square

We are now in a position to state the existence and uniqueness result for the solutions of elastodynamics with cracks.

Theorem 3.3. *Assume (H1)-(H15) and (2.10)-(2.13). Then there exists a unique solution v of problem (3.2)-(3.4). Moreover $v \in \mathcal{V}^\infty$, $v \in C_w^0([0, T]; V)$, and $\dot{v} \in C_w^0([0, T]; H)$.*

Proof. In the case $F = 0$ the existence result, together with an energy bound, is proved in [3] and [16] (a previous result in the scalar case is proved in [5]). When F is present, the same proof can be repeated with obvious modifications (for instance it is enough to repeat the arguments of [14] with $\mathbb{V} = 0$).

As for uniqueness, it can be proved as in [7, Example 4.2 and Theorem 4.3]. Since in that paper the initial conditions are given in a different sense, we have to replace [7, Proposition 2.10] by our Lemma 3.2. The uniqueness result and the existence of a solution with bounded energy imply that the solution satisfies $v \in \mathcal{V}^\infty$. This fact, together with the continuity of v in H and $\dot{v} \in (V_0^D)^*$ (Remark 2.5), implies that $v \in C_w^0([0, T]; V)$ and $\dot{v} \in C_w^0([0, T]; H)$ (see, e.g., [10, Chapitre XVIII, §5, Lemme 6]). \square

For every $v \in C_w^0([0, T]; V)$, with $\dot{v} \in C_w^0([0, T]; H)$, the energy of v is defined for every $t \in [0, T]$ as

$$\mathcal{E}_v(t) := \frac{1}{2} \|\dot{v}(t)\|^2 + \frac{1}{2} (\mathbb{A}Ev(t), Ev(t)). \quad (3.5)$$

Under the same assumption on v , when $u_D = 0$ the work done by the external forces on the displacement v in the time interval $[0, t] \subset [0, T]$ can be written as

$$\mathcal{W}_v(t) := \int_0^t (f(s), \dot{v}(s)) ds - \int_0^t (\dot{F}(s), Ev(s)) ds + (F(t), Ev(t)) - (F(0), Ev(0)), \quad (3.6)$$

see for instance [14, Remarks 5.9 and 5.11].

Theorem 3.4. *Under the assumptions of Theorem 3.3, if $u_D = 0$, then the unique solution v of problem (3.2)-(3.4) satisfies the energy inequality*

$$\mathcal{E}_v(t) \leq \mathcal{E}_v(0) + \mathcal{W}_v(t) \quad \text{for all } t \in [0, T]. \quad (3.7)$$

For a proof we refer to [7, Corollary 3.2] and [14, Remark 5.11].

Proposition 3.5. *Under the assumptions of Theorem 3.3, suppose in addition that $u_D = 0$ and $u^0 = 0$. Then there exists a positive constants A , depending on the constant K in Korn's inequality (2.1) and on the constant α_0 in (H3), but not on T , f , F , and u^1 , such that the solution v of problem (3.2)-(3.4) satisfies*

$$\|v\|_{\mathcal{V}^\infty} \leq A(1+T) \left(\|u^1\| + \|F\|_{L^\infty(0,T;\tilde{H})} + T^{1/2} (\|\dot{F}\|_{L^2(0,T;\tilde{H})} + \|f\|_{L^2(0,T;H)}) \right). \quad (3.8)$$

Proof. Under our assumption we have

$$\mathcal{W}_v(t) := \int_0^t (f(s), \dot{v}(s)) ds - \int_0^t (\dot{F}(s), Ev(s)) ds + (F(t), Ev(t)) \quad \text{and} \quad \mathcal{E}_v(0) = \frac{1}{2} \|u^1\|^2.$$

Recalling (H3), (H6), and (3.7) we have

$$\begin{aligned} \frac{1}{2} \|\dot{v}(t)\|^2 + \frac{\alpha_0}{2} \|Ev(t)\|^2 &\leq T^{1/2} \|\dot{F}\|_{L^2(0,T;\tilde{H})} \|Ev\|_{L^\infty(0,T;\tilde{H})} + \|F\|_{L^\infty(0,T;\tilde{H})} \|Ev\|_{L^\infty(0,T;\tilde{H})} \\ &\quad + T^{1/2} \|f\|_{L^2(0,T;H)} \|\dot{v}\|_{L^\infty(0,T;H)} + \frac{1}{2} \|u^1\|^2. \end{aligned}$$

for all $t \in [0, T]$. We set

$$S := \sup_{t \in [0, T]} (\|\dot{v}(t)\|^2 + \|Ev(t)\|^2)^{1/2}.$$

From the previous inequality we obtain

$$\min\{1/2, \alpha_0/2\} S \leq T^{1/2} \|\dot{F}\|_{L^2(0,T;\tilde{H})} + \|F\|_{L^\infty(0,T;\tilde{H})} + T^{1/2} \|f\|_{L^2(0,T;H)} + \|u^1\|. \quad (3.9)$$

Since $v(t) = \int_0^t \dot{v}(s) ds$ we have $\sup_{t \in [0, T]} \|v(t)\| \leq TS$. Using Korn's inequality (2.1) we obtain $\sup_{t \in [0, T]} \|Dv(t)\| \leq K^{1/2} S$. Therefore

$$\|u\|_{\mathcal{V}^\infty} \leq S + K^{1/2} S + TS,$$

which, together with (3.9), gives (3.8). \square

Let $\mathcal{L} : \mathcal{V}^\infty \rightarrow H^1(0, T; \tilde{H})$ be the linear operator defined by

$$(\mathcal{L}u)(t) := \int_0^t e^{\tau-t} \mathbb{V}Eu(\tau) d\tau \quad (3.10)$$

for every $u \in \mathcal{V}^\infty$ and $t \in [0, T]$. Since

$$(\hat{\mathcal{L}}u)(t) = \mathbb{V}Eu(t) - \int_0^t e^{\tau-t} \mathbb{V}Eu(\tau) d\tau,$$

it is easy to check that \mathcal{L} is bounded. Indeed we have

$$\|\mathcal{L}u\|_{L^\infty(0,T;\tilde{H})} \leq T \|\mathbb{V}\|_\infty \|u\|_{\mathcal{V}^\infty}, \quad (3.11)$$

$$\|\hat{\mathcal{L}}u\|_{L^2(0,T;\tilde{H})} \leq (T^{1/2} + T^{3/2}) \|\mathbb{V}\|_\infty \|u\|_{\mathcal{V}^\infty}. \quad (3.12)$$

Corollary 3.6. *Under the assumptions of Theorem 3.3 there exists a positive constant B , depending on the constant K in Korn's inequality (2.1) and on the constant α_0 in (H3), but not on T and \mathbb{V} , such that, if u satisfies (3.2)-(3.4) with u^0 , u^1 , u_D , and f replaced by zero and F replaced by $\mathcal{L}u$, then*

$$\|u\|_{\mathcal{V}^\infty} \leq B(T + T^3) \|\mathbb{V}\|_\infty \|u\|_{\mathcal{V}^\infty}. \quad (3.13)$$

Proof. By Proposition 3.5, (3.11), and (3.12) we have

$$\|u\|_{\mathcal{V}^\infty} \leq A \left((1+T)T + (T^{1/2} + T^{3/2})^2 \right) \|\mathbb{V}\|_\infty \|u\|_{\mathcal{V}^\infty},$$

which implies (3.13). \square

We are now in a position to prove the uniqueness result.

Proof of Theorem 2.7. The existence result is obtained in [11] under more general hypotheses. To prove uniqueness, we assume by contradiction that there exist two distinct solutions u_1 and u_2 of problem (2.14)-(2.16). Then $u := u_1 - u_2$ is a solution of the same problem with u^0 , u^1 , u_D , f , and F replaced by zero. Therefore u satisfies (3.2)-(3.4) with u^0 , u^1 , u_D , and f replaced by zero and F replaced by $\mathcal{L}u$. By Theorem 3.3 this implies that $u \in C_w([0, T]; V)$ and $\dot{u} \in C_w([0, T]; H)$.

We set

$$t_0 := \inf\{t \in [0, T] \mid u(t) \neq 0\}.$$

Since u is not identically zero, we have $t_0 < T$. We fix $\delta \in (0, T - t_0)$ such that

$$B(\delta + \delta^3) \|\mathbb{V}\|_\infty < 1, \quad (3.14)$$

where B is the constant in (3.13), and we define $t_1 := t_0 + \delta$. In order to study the problem on $[t_0, t_1]$ we define the spaces \mathcal{V}_{t_0, t_1}^D and $\mathcal{V}_{t_0, t_1}^\infty$ as \mathcal{V}^D and \mathcal{V}^∞ (see (2.7) and (2.17)), with 0 and T replaced by t_0 and t_1 .

It is clear that $u \in \mathcal{V}_{t_0, t_1}^D$ and since $Eu(\tau) = 0$ for every $\tau \in [0, t_0]$ we have

$$- \int_{t_0}^{t_1} (\dot{u}(t), \dot{\varphi}(t)) dt + \int_{t_0}^{t_1} (\mathbb{A}Eu(t), E\varphi(t)) dt - \int_{t_0}^{t_1} \int_{t_0}^t e^{\tau-t} (\nabla Eu(\tau), E\varphi(t)) d\tau dt = 0$$

for every $\varphi \in \mathcal{V}_{t_0, t_1}^D$ such that $\varphi(t_0) = \varphi(t_1) = 0$. Moreover, since $u \in C_w([0, T]; V)$, $\dot{u} \in C_w([0, T]; H)$, and u is identically zero on $[0, t_0]$, we have that $u(t_0) = 0$ and $\dot{u}(t_0) = 0$. By (3.13), applied with 0 and T replaced by t_0 and t_1 , we have

$$\|u\|_{\mathcal{V}_{t_0, t_1}^\infty} \leq B(\delta + \delta^3) \|\mathbb{V}\|_\infty \|u\|_{\mathcal{V}_{t_0, t_1}^\infty}.$$

Using (3.14) we obtain $u = 0$ on $[t_0, t_1]$. This contradicts the definition of t_0 and concludes the proof. \square

4. CONTINUOUS DEPENDENCE ON THE DATA

In this section we consider a sequence $\{\Gamma_t^n\}_{t \in [0, T]}$ of time dependent cracks and we want to study the convergence, as $n \rightarrow +\infty$, of the solutions of the corresponding viscoelastic problems. For completeness we assume that also the other data of the problem depend on n .

For every $n \in \mathbb{N}$, let $\mathbb{C}^n, \mathbb{V}^n: \bar{\Omega} \rightarrow \mathcal{L}(\mathbb{R}_{sym}^{d \times d}, \mathbb{R}_{sym}^{d \times d})$, let Γ^n be a $(d-1)$ -dimensional C^2 manifold, let $\{\Gamma_t^n\}_{t \in [0, T]}$ be a family of closed subsets of Γ^n , and let $\Phi^n, \Psi^n: [0, T] \times \bar{\Omega} \rightarrow \bar{\Omega}$. We assume that

(H16) $\mathbb{C}^n, \mathbb{V}^n$ satisfy (H1)-(H6) with constants α_0 and M_0 independent of n ;

(H17) Γ^n and $\{\Gamma_t^n\}_{t \in [0, T]}$ satisfy (H7)-(H10);

(H18) Φ^n, Ψ^n satisfy (H11)-(H15) (with Γ and Γ_t replaced by Γ^n and Γ_t^n), the latter with the constant K that appears in (4.7).

Let $\mathbb{R}^{d \times d}$ be the space of $d \times d$ real matrices. For every pair of normed spaces X and Y let $\mathcal{L}(X; Y)$ be the space of linear and continuous maps between X and Y . For every $x \in \bar{\Omega}$ it is convenient to consider the extensions $\mathbb{C}_e(x), \mathbb{V}_e(x), \mathbb{C}_e^n(x), \mathbb{V}_e^n(x) \in \mathcal{L}(\mathbb{R}^{d \times d}; \mathbb{R}_{sym}^{d \times d})$ of the linear maps $\mathbb{C}(x), \mathbb{V}(x), \mathbb{C}^n(x), \mathbb{V}^n(x)$ defined as

$$\mathbb{C}_e^n(x)[A] := \mathbb{C}^n(x)[A_{sym}] \quad \text{and} \quad \mathbb{V}_e^n(x)[A] := \mathbb{V}^n(x)[A_{sym}] \quad \text{for all } A \in \mathbb{R}^{d \times d}, \quad (4.1)$$

$$\mathbb{C}_e(x)[A] := \mathbb{C}(x)[A_{sym}] \quad \text{and} \quad \mathbb{V}_e(x)[A] := \mathbb{V}(x)[A_{sym}] \quad \text{for all } A \in \mathbb{R}^{d \times d}, \quad (4.2)$$

where A_{sym} is the symmetric part of the matrix A . Moreover we set

$$\mathbb{A}_e^n := \mathbb{C}_e^n + \mathbb{V}_e^n \quad \text{and} \quad \mathbb{A}_e := \mathbb{C}_e + \mathbb{V}_e. \quad (4.3)$$

For technical reasons we use a change of variable which maps Γ_0^n into Γ_0 . This is done by means of diffeomorphisms $\Theta^n, \Xi^n: \bar{\Omega} \rightarrow \bar{\Omega}$ such that

- (H19) Θ^n and Ξ^n are of class $C^{2,1}$;
- (H20) $\Theta^n(\Xi^n(x)) = x$ and $\Xi^n(\Theta^n(x)) = x$ for every $x \in \bar{\Omega}$;
- (H21) $\det D\Theta^n(x) > 0$ for every $x \in \bar{\Omega}$;
- (H22) $\Theta^n(\Gamma \cap \bar{\Omega}) = \Gamma^n \cap \bar{\Omega}$, and $\Theta^n(\Gamma_0) = \Gamma_0^n$;
- (H23) $\Theta^n(\partial_D \Omega) = \partial_D \Omega$ and $\Theta^n(\partial_N \Omega) = \partial_N \Omega$.

We now introduce the function spaces that will be used in the formulation of the n -th viscoelastic problem. For every $n \in \mathbb{N}$ and $t \in [0, T]$ let V^n, V_t^n , and $V_t^{n,D}$ be defined as V, V_t , and V_t^D (see (2.2) and (2.4)) with Γ and Γ_t replaced by Γ^n and Γ_t^n . Let $\mathcal{V}^n, \mathcal{V}^{n,D}$, and $\mathcal{V}^{n,\infty}$ be defined as $\mathcal{V}, \mathcal{V}^D$, and \mathcal{V}^∞ (see (2.5), (2.7), and (2.17)) with V_t and V_t^D replaced by V_t^n and $V_t^{n,D}$.

For every $n \in \mathbb{N}$ we fix

$$u^{0,n} \in V_0^n, \quad u^{1,n} \in H, \quad u_D^n \in H^2(0, T; H) \cap H^1(0, T; V_0^n), \quad (4.4)$$

$$f^n \in L^2(0, T; H), \quad F^n \in H^1(0, T; \tilde{H}), \quad (4.5)$$

and we suppose that $u^{0,n}$ and u_D^n satisfy the compatibility condition

$$u^{0,n} - u_D^n(0) \in V_0^{n,D}. \quad (4.6)$$

Now we give the detailed regularity and convergence hypotheses on the data. First of all we assume that there exists a constant $K > 0$ such that for every $n \in \mathbb{N}$ the following Korn inequality is satisfied:

$$\|Dv\|^2 \leq K(\|v\|^2 + \|Ev\|^2) \quad \text{for every } v \in H^1(\Omega \setminus \Gamma^n; \mathbb{R}^d). \quad (4.7)$$

We set $\underline{H} = L^2(\Omega, \mathbb{R}^{d \times d})$. Concerning the convergence of our data we assume that

$$\|\Phi^n - \Phi\|_{C^2} \rightarrow 0, \quad \|\Psi^n - \Psi\|_{C^2} \rightarrow 0, \quad (4.8)$$

$$\|\mathbb{C}^n - \mathbb{C}\|_{C^1} \rightarrow 0, \quad \|\mathbb{V}^n - \mathbb{V}\|_{C^1} \rightarrow 0 \quad (4.9)$$

$$\|u_D^n - u_D\|_{H^2(0, T; H)} \rightarrow 0, \quad \|Du_D^n - Du_D\|_{H^1(0, T; \underline{H})} \rightarrow 0, \quad (4.10)$$

$$\|f^n - f\|_{L^2(0, T; H)} \rightarrow 0, \quad \|F^n - F\|_{H^1(0, T; \tilde{H})} \rightarrow 0, \quad (4.11)$$

$$\|u^{0,n} - u^0\| \rightarrow 0, \quad \|Du^{0,n} - Du^0\| \rightarrow 0, \quad \|u^{1,n} - u^1\| \rightarrow 0, \quad (4.12)$$

$$\|\Theta^n - Id\|_{C^2} \rightarrow 0, \quad \|\Xi^n - Id\|_{C^2} \rightarrow 0. \quad (4.13)$$

It follows from (H19)-(H21) and (4.13) that

$$m_{det}(\Psi^n) \rightarrow m_{det}(\Psi) \quad \text{and} \quad M_{det}(\Psi^n) \rightarrow M_{det}(\Psi^n) \quad \text{as } n \rightarrow \infty. \quad (4.14)$$

For every $n \in \mathbb{N}$ we consider the solution u^n of the problem

$$u^n \in \mathcal{V}^n \quad \text{and} \quad u^n - u_D^n \in \mathcal{V}^{n,D}, \quad (4.15)$$

$$\begin{aligned} & - \int_0^T (\dot{u}^n(t), \dot{\varphi}(t)) dt + \int_0^T ((\mathbb{C}^n + \mathbb{V}^n)Eu^n(t), E\varphi(t)) dt \\ & - \int_0^T \int_0^t e^{\tau-t} (\mathbb{V}^n Eu^n(\tau), E\varphi(t)) d\tau dt = \int_0^T (f^n(t), \varphi(t)) dt \\ & + \int_0^T (F^n(t), E\varphi(t)) dt \quad \text{for all } \varphi \in \mathcal{V}^{n,D} \text{ with } \varphi(0) = \varphi(T) = 0, \end{aligned} \quad (4.16)$$

$$u^n(0) = u^{0,n} \quad \text{in } H \quad \text{and} \quad \dot{u}^n(0) = u^{1,n} \quad \text{in } (V_0^{D,n})^*. \quad (4.17)$$

We also consider the solution v^n of the problem

$$v^n \in \mathcal{V}^n \quad \text{and} \quad v^n - u_D^n \in \mathcal{V}^{n,D}, \quad (4.18)$$

$$\begin{aligned} & - \int_0^T (\dot{v}^n(t), \dot{\varphi}(t)) dt + \int_0^T (\mathbb{A}^n E v^n(t), E \varphi(t)) dt = \int_0^T (f^n(t), \varphi(t)) dt \\ & + \int_0^T (F^n(t), E \varphi(t)) dt \quad \text{for all } \varphi \in \mathcal{V}^{n,D} \text{ with } \varphi(0) = \varphi(T) = 0, \end{aligned} \quad (4.19)$$

$$v^n(0) = u^{0,n} \quad \text{in } H \quad \text{and} \quad \dot{v}^n(0) = u^{1,n} \quad \text{in } (V_0^{D,n})^*. \quad (4.20)$$

The notion of convergence for u^n as $n \rightarrow \infty$ can't be given directly because they don't belong to the same space. To overcome this problem we need to embed V^n into a common space. This will be done using the standard embedding $V^n \hookrightarrow H \times \underline{H}$ given by $v \mapsto (v, Dv)$, where the distributional gradient Dv on $\Omega \setminus \Gamma^n$ is regarded as a function defined a.e. on Ω , which belongs to \underline{H} .

We are now in a position to state one the main result of this section.

Theorem 4.1. *Assume (H1)-(H23), (2.10)-(2.13), and (4.4)-(4.13). Let u be the solution of (2.14)-(2.16) and let (for every $n \in \mathbb{N}$) u^n be the solution of (4.15)-(4.17). Then*

$$(u^n(t), Du^n(t), \dot{u}^n(t)) \rightarrow (u(t), Du(t), \dot{u}(t)) \quad \text{in } H \times \underline{H} \times H$$

for every $t \in [0, T]$. Moreover there exists a constant $C > 0$ such that

$$\|u^n(t)\| + \|Du^n(t)\| + \|\dot{u}^n(t)\| \leq C$$

for every $n \in \mathbb{N}$ and $t \in [0, T]$.

The proof is based on the following lemma.

Lemma 4.2. *Let X a complete metric space, let $G_n, G: X \rightarrow X$ with $n \in \mathbb{N}$ be maps with same contraction constant $\lambda \in (0, 1)$, and let x_n, x be the corresponding fixed points. Suppose that $G_n(y) \rightarrow G(y)$ for every $y \in X$. Then $x_n \rightarrow x$.*

Proof. We have $d(x_n, x) = d(G_n(x_n), G(x)) \leq d(G_n(x_n), G_n(x)) + d(G_n(x), G(x)) \leq \lambda d(x_n, x) + d(G_n(x), G(x))$, hence $(1-\lambda)d(x_n, x) \leq d(G_n(x), G(x)) \rightarrow 0$, as $n \rightarrow +\infty$. \square

Let $p \in (2, +\infty)$ be fixed. In order to apply the previous lemma we will identify u_n and u with the fixed points of suitable operators defined in the Banach space

$$\mathcal{W}^p := L^p((0, T); H \times \underline{H} \times H), \quad (4.21)$$

where on $H \times \underline{H} \times H$ we consider the Hilbert product norm defined by

$$\|(h_1, h_2, h_3)\|_{H \times \underline{H} \times H} := \left(\|h_1\|^2 + \|h_2\|^2 + \|h_3\|^2 \right)^{1/2} \quad (4.22)$$

for every $(h_1, h_2, h_3) \in H \times \underline{H} \times H$. In order to define the sequence of maps whose fixed points are (u^n, Du^n, \dot{u}^n) and (u, Du, \dot{u}) , we consider the linear operators

$$\mathcal{T}^n : \mathcal{W}^p \longrightarrow H^1(0, T; \tilde{H}) \quad \text{and} \quad \mathcal{T} : \mathcal{W}^p \longrightarrow H^1(0, T; \tilde{H}) \quad (4.23)$$

defined as

$$(\mathcal{T}^n w)(t) := \int_0^t e^{\tau-t} \mathbb{V}_e^n w_2(\tau) d\tau \quad \text{and} \quad (\mathcal{T} w)(t) := \int_0^t e^{\tau-t} \mathbb{V}_e w_2(\tau) d\tau, \quad (4.24)$$

where $w(t) = (w_1(t), w_2(t), w_3(t))$ and $\mathbb{V}_e^n, \mathbb{V}_e$ are as in (4.1) and (4.2). Arguing as in (3.11) and (3.12) we get that

$$\|\mathcal{T} w\|_{L^\infty(0, T; \tilde{H})} \leq T^{-1/p} T \|\mathbb{V}\|_\infty \|w\|_{\mathcal{W}^p}, \quad (4.25)$$

$$\|\widehat{\mathcal{T} w}\|_{L^2(0, T; \tilde{H})} \leq T^{-1/p} (T^{1/2} + T^{3/2}) \|\mathbb{V}\|_\infty \|w\|_{\mathcal{W}^p}, \quad (4.26)$$

and the same estimate holds for $\mathcal{T}^n w$ with \mathbb{V} replaced by \mathbb{V}^n .

Let $\mathcal{G} : \mathcal{W}^p \rightarrow \mathcal{W}^p$ be the operator defined for every $w \in \mathcal{W}^p$ by

$$\mathcal{G}(w) = (z, Dz, \dot{z}), \quad (4.27)$$

where z is the solution of problem (3.2)-(3.4) with F replaced by $F + \mathcal{T}w$. From the definition of \mathcal{G} it follows that (u, Du, \dot{u}) is a fixed point of map \mathcal{G} if and only if u is the solution of the problem considered in Theorem 4.1.

Similarly, let $\mathcal{G}^n : \mathcal{W}^p \rightarrow \mathcal{W}^p$ be the operator defined for every $w \in \mathcal{W}^p$ by

$$\mathcal{G}^n(w) = (z^n, Dz^n, \dot{z}^n), \quad (4.28)$$

where z^n is the solution of problem (4.18)-(4.20) with F replaced by $F^n + \mathcal{T}^n w$. From the definition of \mathcal{G}^n it follows that u^n is the solution of problem (4.15)-(4.17) if and only if (u^n, Du^n, \dot{u}^n) is a fixed point of map \mathcal{G}^n .

The following lemma provides a uniform Lipschitz estimate for the operators \mathcal{G}^n .

Proposition 4.3. *There exist a positive constants B , independent of n and T , such that*

$$\|\mathcal{G}^n(w_1) - \mathcal{G}^n(w_2)\|_{\mathcal{W}^p} \leq B(T + T^3)\|w_1 - w_2\|_{\mathcal{W}^p}, \quad (4.29)$$

for every $w_1, w_2 \in \mathcal{W}^p$.

Proof. Let us fix $w_1, w_2 \in \mathcal{W}^p$ and set $w := w_1 - w_2$. We observe that $\mathcal{G}^n(w_1) - \mathcal{G}^n(w_2) = (v^n, Dv^n, \dot{v}^n)$ where z^n is the solution of problem (4.15)-(4.17) with F^n replaced by $\mathcal{T}^n w$ and $u_D^n, f^n, u^{0,n}, u^{1,n}$ replaced by zero. From Theorem 3.5 and from the uniform bound of the data there exists a positive constants A , independent of n and T , such that

$$\|z^n\|_{\mathcal{V}^\infty} \leq A(1 + T)\|\mathcal{T}^n w\|_{L^\infty(0, T; \tilde{H})} + A(T^{1/2} + T^{3/2})\|\widehat{\mathcal{T}^n w}\|_{L^2(0, T; \tilde{H})}. \quad (4.30)$$

Using (4.25) and (4.26) we get

$$\|(z^n, Dz^n, \dot{z}^n)\|_{\mathcal{W}^p} \leq A\left((1 + T)T + (T^{1/2} + T^{3/2})^2\right)\|\mathbb{V}^n\|_\infty\|w\|_{\mathcal{W}^p} \quad (4.31)$$

which gives (4.29) taking into account (4.9). \square

To apply Lemma 4.2 we have to prove that

$$\mathcal{G}^n(w) \rightarrow \mathcal{G}(w) \quad \text{in } \mathcal{W}^p,$$

for every $w \in \mathcal{W}^p$. In order to prove this we will use the results for the wave equation developed in [4]. Unfortunately these results can not be applied directly because they are obtained under the assumptions:

- (a) $\Gamma_0^n = \Gamma_0$ for all $n \in \mathbb{N}$,
- (b) the forcing terms belong to $L^2(0, T; H)$.

To overcome the difficulties due to (a) we need some preliminary results. The first one is an uniform bound of the solution of problems (4.18)-(4.20).

Proposition 4.4. *Assume (H1)-(H23), (4.4)-(4.10), and (4.12)-(4.13). Let let v^n be the solution of (4.18)-(4.20). Then there exists a positive constant C such that*

$$\|v^n\|_{\mathcal{V}^{n, \infty}} \leq C \quad \text{for every } n \in \mathbb{N}. \quad (4.32)$$

Proof. We note that $v_0^n(t) := v^n(t) - u^{0,n} + u_D^n(0) - u_D^n(t)$ is the solution of (4.18)-(4.20) with u^0 replaced by 0, $u^{1,n}$ replaced by $u^{1,n} - \dot{u}_D^n(0)$, u_D^n replaced by 0, f^n replaced by $f^n - \ddot{u}_D^n$, and F^n replaced by $F^n - \mathbb{A}^n E u_D^n - \mathbb{A}^n E(u^{n,0} - u_D^n(0))$. Then we can apply Proposition 3.5 and (4.8)-(4.13) to obtain that $\|v_0^n\|_{\mathcal{V}^n}$ is equibounded. By (4.10) and (4.12) we get (4.32). \square

The next proposition deals with the case of solution of (4.18)-(4.20) when F^n is replaced by 0.

Proposition 4.5. *Assume (H1)-(H23), (4.4)-(4.10), and (4.12)-(4.13). Given $g \in L^2(0, T; H)$, let v^n be the solution of (4.18)-(4.20) with f^n replaced by g and F^n replaced by 0. Let v be the solution in (3.2)-(3.4) with f replaced by g and F replaced by 0. Then for every $t \in [0, T]$ we have*

$$(v^n(t), Dv^n(t), \dot{v}^n(t)) \rightarrow (v(t), Dv(t), \dot{v}(t)) \quad \text{in } H \times \underline{H} \times H. \quad (4.33)$$

In order to prove this proposition it is convenient to use the following elementary result, whose proof, based on a change of variables, is omitted (for a similar result see [6, Lemma A.7]).

Lemma 4.6. *For every $n \in \mathbb{N}$ let $h^n, h \in H$ and let $\Lambda^n, \Lambda : \bar{\Omega} \rightarrow \bar{\Omega}$ be C^1 diffeomorphisms. Assume that $h^n \rightarrow h$ in H and $\Lambda^n \rightarrow \Lambda$ in C^1 . Assume also that $\det D\Lambda^n(x) > 0$ and $\det D\Lambda(x) > 0$ for every $x \in \bar{\Omega}$ and $n \in \mathbb{N}$. Then $h^n \circ \Lambda^n \rightarrow h \circ \Lambda$ as $n \rightarrow \infty$ in H .*

Proof of Proposition 4.5. To overcome the difficulty due to the fact that we may have $\Gamma_0^n \neq \Gamma_0$, by a change of variables we transform our problem into a problem with new cracks $\hat{\Gamma}_t^n$ satisfying $\hat{\Gamma}_0^n = \Gamma_0$ for every n , to which we can apply the results of [3] and [4].

For every n and t we define $\hat{\Gamma}_t^n := \Xi^n(\Gamma_t^n) \subset \Gamma$ and observe that $\hat{\Gamma}_t^n$ satisfies (H10). The vector spaces \hat{V}_t^n and $\hat{V}_t^{n,D}$ are defined as V_t^n and $V_t^{n,D}$ (see (2.4)) with Γ_t replaced by $\hat{\Gamma}_t^n$, while $\hat{\mathcal{V}}^n$ and $\hat{\mathcal{V}}^{n,D}$ are defined as \mathcal{V}^n and $\mathcal{V}^{n,D}$ (see (2.5) and (2.7)) with V_t and V_t^D replaced by \hat{V}_t^n and $\hat{V}_t^{n,D}$.

For every $t \in [0, T]$ let $\hat{v}^n(t) := v^n(t) \circ \Theta^n$, $\hat{u}_D^n(t) := u_D^n(t) \circ \Theta^n$, $\hat{u}^{0,n} := u^{0,n} \circ \Theta^n$, $\hat{u}^{1,n} := u^{1,n} \circ \Theta^n$, and $\hat{g}^n(t) := g(t) \circ \Theta^n$. It is easy to see that $\hat{v}^n \in \hat{\mathcal{V}}^n$, $\hat{v}^n - \hat{u}_D^n \in \hat{\mathcal{V}}^{n,D}$, $\hat{v}^n(0) = \hat{u}^{0,n}$, $\hat{v}^n(0) = \hat{u}^{1,n}$.

To write the equation satisfied by \hat{v}^n we introduce $\hat{\mathbb{A}}^n : \bar{\Omega} \rightarrow \mathcal{L}(\mathbb{R}^{d \times d}; \mathbb{R}^{d \times d})$ defined as

$$\hat{\mathbb{A}}^n(y)[A] := \mathbb{A}_e^n(\Theta^n(y))[AD\xi^n(\Theta^n(y))](D\xi^n(\Theta^n(y)))^T \quad \text{for all } A \in \mathbb{R}^{d \times d}, \quad (4.34)$$

where \mathbb{A}^n is defined in (4.3). We note that $\hat{\mathbb{A}}^n$ is of class C^1 , with equibounded C^1 norm. Moreover it is symmetric on $\mathcal{L}(\mathbb{R}^{d \times d}, \mathbb{R}^{d \times d})$.

Setting $h^n(x) := \nabla[\det D\xi^n(x)]$, we introduce $\mathbb{L}^n : \bar{\Omega} \rightarrow \mathcal{L}(\mathbb{R}^{d \times d}, \mathbb{R}^d)$ defined as

$$\mathbb{L}^n(y)[A] = \mathbb{A}_e^n(\Theta^n(y))[AD\xi^n(\Theta^n(y))]h^n(\Theta^n(y)) \det D\Theta^n(y) \quad \text{for all } A \in \mathbb{R}^{d \times d}.$$

Let $\varphi \in \hat{\mathcal{V}}^{n,D}$ with $\varphi(0) = \varphi(T) = 0$. Using $(\varphi(t) \circ \Xi^n) \det D\xi^n$ as test function in the equation for $v^n(t)$ we get

$$\begin{aligned} & - \int_0^T (\dot{\hat{v}}^n(t), \dot{\varphi}(t)) dt + \int_0^T (\hat{\mathbb{A}}^n D\hat{v}^n(t), D\varphi(t)) dt + \int_0^T (\mathbb{L}^n D\hat{v}^n(t), \varphi(t)) dt = \\ & \int_0^T (\hat{g}^n(t), \varphi(t)) dt. \end{aligned}$$

By Proposition (4.4) the sequence $\|v^n\|_{\mathcal{V}^n}$ is bounded and in particular $\|Dv^n(t)\|$ is uniformly bounded with respect to n and t . By the definition of \hat{v}^n and (4.13) also $\|D\hat{v}^n(t)\|$ is uniformly bounded with respect to n and t . Since $\det D\xi^n \rightarrow 1$ in $C^1(\bar{\Omega})$, we have $\nabla[\det D\xi^n] \rightarrow 0$ in $C^0(\bar{\Omega}, \mathbb{R}^d)$, which implies that $\mathbb{L}^n \rightarrow 0$ uniformly as $n \rightarrow +\infty$. From this fact and the uniform bound on $\|D\hat{v}^n(t)\|$ we get

$$\|\mathbb{L}^n D\hat{v}^n(t)\| \rightarrow 0 \text{ as } n \rightarrow +\infty, \quad (4.35)$$

uniformly in t . Therefore, setting

$$\hat{f}^n := \hat{g}^n - \mathbb{L}^n D\hat{v}^n, \quad (4.36)$$

we conclude that

$$\hat{v}^n \in \hat{\mathcal{V}}^n \quad \text{and} \quad \hat{v}^n - \hat{u}_D^n \in \hat{\mathcal{V}}^{n,D}, \quad (4.37)$$

$$- \int_0^T (\dot{\hat{v}}^n(t), \dot{\varphi}(t)) dt + \int_0^T (\hat{\mathbb{A}}^n D\hat{v}^n(t), D\varphi(t)) dt = \int_0^T (\hat{f}^n(t), \varphi(t)) dt,$$

$$\text{for all } \varphi \in \hat{\mathcal{V}}^{n,D} \text{ such that } \varphi(0) = \varphi(T) = 0, \quad (4.38)$$

$$\hat{v}^n(0) = \hat{u}^{0,n} \quad \text{in } H \quad \text{and} \quad \dot{\hat{v}}^n(0) = \hat{u}^{1,n} \quad \text{in } (V_0^D)^*. \quad (4.39)$$

In order to apply the results of [4] we define $\hat{\Phi}^n(t, y) := \Xi^n(\Phi^n(t, \Theta^n(y)))$, $\hat{\Psi}^n(t, x) := \Xi^n(\Psi^n(t, \Theta^n(x)))$. We observe that $\hat{\Phi}^n$ and $\hat{\Psi}^n$ satisfy (H11)-(H14) with Γ_t replaced by $\hat{\Gamma}_t^n$. Since in general $\hat{\mathbb{A}}^n[A] \neq \hat{\mathbb{A}}^n[A_{sym}]$ for some $A \in \mathbb{R}^{d \times d}$, we cannot apply the results of [3]. However it is possible to use the results of [4] which hold under more general assumptions involving the tensor

$$\begin{aligned} \hat{\mathbb{B}}^n(t, y)[A] &:= \hat{\mathbb{A}}^n(\hat{\Phi}^n(t, y))[AD\hat{\Psi}^n(t, \hat{\Phi}^n(t, y))]D\hat{\Psi}^n(t, \hat{\Phi}^n(t, y))^T \\ &\quad - A\hat{\Psi}^n(t, \hat{\Phi}^n(t, y)) \otimes \dot{\hat{\Psi}}^n(t, \hat{\Phi}^n(t, y)), \end{aligned}$$

for all $A \in \mathbb{R}^{d \times d}$, $t \in [0, T]$, $y \in \bar{\Omega}$. We claim that there exists two constants $c_0, c_1 > 0$ (independent of n) such that, for n large enough, we have

$$(\hat{\mathbb{B}}^n(t)D\varphi, D\varphi) \geq c_0\|\varphi\|_{V_0}^2 - c_1\|\varphi\|^2 \quad (4.40)$$

for all $\varphi \in V_0$ and $t \in [0, T]$. This is the hypothesis on $\hat{\mathbb{B}}^n$ required in [4].

To prove the claim we use (H3), (H15), and (4.13) (which are satisfied uniformly in n) and by standard computations (see, for instance, [4, Section 1.2]) we obtain

$$\begin{aligned} (\hat{\mathbb{B}}^n(t)D\varphi, D\varphi) &\geq \int_{\Omega} |D\varphi(y)D\Xi^n(\Theta^n(y))D\Psi^n(t, \Phi^n(t, \Theta^n(y)))|^2 \omega^n(t, y) dy \\ &\quad - \alpha_0 \min_{[0, T] \times \bar{\Omega}} \{\det D\Xi^n \det D\Psi^n\} \int_{\Omega} |\varphi(\Xi^n(\Psi^n(t, y)))|^2 dy \end{aligned} \quad (4.41)$$

where

$$\omega^n(t, y) := \frac{\alpha_0 m_{det}(\Psi^n)}{K M_{det}(\Psi^n)} \min_{\bar{\Omega}} \{\det D\Xi^n\} \min_{\bar{\Omega}} \{\det D\Theta^n\} - |\dot{\Phi}^n(t, \Theta^n(y))|^2,$$

while $m_{det}(\Psi^n)$, $M_{det}(\Psi^n)$, α_0 , and K are the constants that appear in (H15), (H16), and (4.7). Since the inverse of the matrices $D\Xi^n(x)D\Psi^n(t, \Phi^n(t, x))$ are bounded uniformly with respect to n , t , and x , there exists a constant $\beta > 0$ such that

$$\int_{\Omega} |D\varphi(y)D\Xi^n(\Theta^n(y))D\Psi^n(t, \Phi^n(t, \Theta^n(y)))|^2 \omega^n(t, y) dy \geq \beta \int_{\Omega} |D\varphi(y)|^2 \omega^n(t, y) dy$$

for all n and t . Moreover by (4.8) and (4.13) there exists a constant $\gamma > 0$ such that

$$\alpha_0 \min_{[0, T] \times \bar{\Omega}} \{\det D\Xi^n \det D\Psi^n\} \int_{\Omega} |\varphi(\Xi^n(\Psi^n(t, y)))|^2 dy \leq \gamma \int_{\Omega} |\varphi(y)|^2 dy$$

for all n and t . Therefore (4.41) gives

$$(\hat{\mathbb{B}}^n(t)D\varphi, D\varphi) \geq \beta \int_{\Omega} |D\varphi(y)|^2 \omega^n(t, y) dy - \gamma \int_{\Omega} |\varphi(y)|^2 dy. \quad (4.42)$$

To conclude the proof of the claim, we define

$$\omega(t, y) := \frac{\alpha_0 m_{det}(\Psi)}{K M_{det}(\Psi)} - |\dot{\Phi}(t, y)|^2.$$

By (4.8), (4.13) and (4.14), we have $\omega^n \rightarrow \omega$ uniformly on $[0, T] \times \bar{\Omega}$. By (H15) and by continuity there exists $\varepsilon > 0$ such that $\omega(t, y) \geq 2\varepsilon$ for all $(t, y) \in [0, T] \times \bar{\Omega}$. By uniform convergence there exists n_ε such that $\omega^n(t, y) \geq \varepsilon$ for all $(t, y) \in [0, T] \times \bar{\Omega}$ and for all

$n > n_\varepsilon$. This inequality together with (4.42) implies (4.40) and concludes the proof of the claim.

By (4.8) and (4.13) we get $\hat{\Phi}^n \rightarrow \Phi$ and $\hat{\Psi}^n \rightarrow \Psi$ in C^2 , while (4.34) and (4.13) give $\hat{\mathbb{A}}^n \rightarrow \mathbb{A}$ in C^1 . Moreover applying Lemma 4.6 to the functions and their derivatives we can prove that $\hat{u}^{0,n} \rightarrow u^0$ in V_0 , $\hat{u}^{1,n} \rightarrow u^1$ in H , $\hat{u}_D^n \rightarrow u_D$ in $H^2(0, T; H) \cap H^1(0, T; V_0)$, and $\hat{g}^n \rightarrow g$ in $L^2(0, T; H)$. Using (4.35) and (4.36) we have that $\hat{f}^n \rightarrow g$ in $L^2(0, T; H)$. We are now in a position to apply [4, Theorem 1.4.1] to problem (4.37)-(4.39) and we obtain

$$(\hat{v}^n(t), D\hat{v}^n(t), \dot{\hat{v}}^n(t)) \rightarrow (v(t), Dv(t), \dot{v}(t)) \quad \text{in } H \times \underline{H} \times H$$

for every $t \in [0, T]$. Since

$$v^n(t, \cdot) = \hat{v}^n(t, \Xi^n(\cdot)), \quad Dv^n(t, \cdot) = D\hat{v}^n(t, \Xi^n(\cdot))D\Xi^n(\cdot), \quad \dot{v}^n(t, \cdot) = \dot{\hat{v}}^n(t, \Xi^n(\cdot)),$$

using Lemma 4.6 we get (4.33) for every $t \in [0, T]$. \square

To use Proposition 4.5 in the proof of the convergence $\mathcal{G}^n(w) \rightarrow \mathcal{G}(w)$ we need the following approximation result.

Lemma 4.7. *Let $G \in H^1((0, T); \tilde{H})$. For every $\varepsilon > 0$ there exists a compact neighborhood K_ε of $\Gamma \cap \bar{\Omega}$ and $G_\varepsilon \in H^1((0, T); \tilde{H})$ such that $G_\varepsilon(t) \in C_c^\infty(\Omega \setminus K_\varepsilon; \mathbb{R}_{sym}^{d \times d})$ for every $t \in [0, T]$ and*

$$\|G_\varepsilon - G\|_{L^\infty(0, T; \tilde{H})} + \|\dot{G}_\varepsilon - \dot{G}\|_{L^2(0, T; \tilde{H})} < \varepsilon.$$

Remark 4.8. By (H22) and (4.13) for every $\varepsilon > 0$ there exists n_ε such that $\Gamma^n \subset K_\varepsilon$, for $n > n_\varepsilon$. From the properties of G_ε follows that

$$(G_\varepsilon(t), Ev) = -(\operatorname{div} G_\varepsilon(t), v) \quad (4.43)$$

for all $t \in [0, T]$ and for all $v \in V_n$, for $n > n_\varepsilon$.

Proof of Lemma 4.7. Given a partition of $[0, T]$, we can consider the piecewise affine interpolation of the values of F at the nodes. It is well known that this interpolation converges in $H^1(0, T; \tilde{H})$ to F as the fineness of the partition tends to zero. To conclude, it is enough to approximate in \tilde{H} the values of F at the nodes by elements of $C_c^\infty(\Omega \setminus \Gamma; \mathbb{R}_{sym}^{d \times d})$ and to consider the corresponding piecewise affine interpolation. \square

Proposition 4.9. *Assume (H1)-(H23) and (4.7)-(4.13). Let v^n be the solution of (4.18)-(4.20) and let v be the solution of (3.2)-(3.4). Then for every $t \in [0, T]$ we have*

$$(v^n(t), Dv^n(t), \dot{v}^n(t)) \rightarrow (v(t), Dv(t), \dot{v}(t)) \quad \text{in } H \times \underline{H} \times H. \quad (4.44)$$

Moreover

$$(v^n, Dv^n, \dot{v}^n) \rightarrow (v, Dv, \dot{v}) \quad \text{in } \mathcal{W}^p = L^p((0, T); H \times \underline{H} \times H). \quad (4.45)$$

Proof. Let $\varepsilon > 0$, let G_ε the function in Lemma 4.7 with $G = F$. Let v_ε^n solution of (4.18)-(4.20) with f^n and F^n replaced by f and G_ε , let v^ε solution of (3.2)-(3.4) with F replaced by G_ε . By (4.11) there exists n_ε such that

$$\|f^n - f\|_{L^2(0, T; H)} + \|F^n - G_\varepsilon\|_{L^\infty(0, T; \tilde{H})} + \|\dot{F}^n - \dot{G}_\varepsilon\|_{L^2(0, T; \tilde{H})} < \varepsilon \quad (4.46)$$

for every $n > n_\varepsilon$. The function $v^n - v_\varepsilon^n$ is the solution of problem (4.18)-(4.20) with f^n and F^n replaced by $f^n - f$ and $F^n - G_\varepsilon$ and u_D^n , f^n , $u^{n,0}$, $u^{1,n}$ replaced by zero. Then by Proposition 3.5 there exists a constant $C(T)$ depending on T (independent of n and ε) such that

$$\|v^n - v_\varepsilon^n\|_{\mathcal{V}^{n,\infty}} \leq C(T)\varepsilon. \quad (4.47)$$

for every $n > n_\varepsilon$. Similarly we can prove

$$\|v - v_\varepsilon\|_{\mathcal{V}^\infty} \leq C(T)\varepsilon. \quad (4.48)$$

Changing the value of n_ε , by (4.43) we have that v_ε^n is the solution of (4.18)-(4.20) with f^n replaced by $g_\varepsilon := f - \operatorname{div} G_\varepsilon$ and F^n replaced by 0, while v_ε is the solution of (3.2)-(3.4)

with f replaced by $g_\varepsilon := f - \operatorname{div}G_\varepsilon$ and F replaced by 0. By Proposition 4.5 for every $t \in [0, T]$ we have

$$(v_\varepsilon^n(t), Dv_\varepsilon^n(t), \dot{v}_\varepsilon^n(t)) \rightarrow (v_\varepsilon(t), Dv_\varepsilon(t), \dot{v}_\varepsilon(t)) \quad \text{in } H \times \underline{H} \times H. \quad (4.49)$$

Since

$$\begin{aligned} & \|(v^n(t), Dv^n(t), \dot{v}^n(t)) - (v(t), Dv(t), \dot{v}(t))\| \leq \|v^n - v_\varepsilon^n\|_{\mathcal{V}^{n,\infty}} \\ & + \|(v_\varepsilon^n(t), Dv_\varepsilon^n(t), \dot{v}_\varepsilon^n(t)) - (v_\varepsilon(t), Dv_\varepsilon(t), \dot{v}_\varepsilon(t))\| + \|v - v_\varepsilon\|_{\mathcal{V}^\infty}, \end{aligned}$$

by (4.47)-(4.49) we get

$$\limsup_{n \rightarrow +\infty} \|(v^n(t), Dv^n(t), \dot{v}^n(t)) - (v(t), Dv(t), \dot{v}(t))\| \leq 2C(T)\varepsilon$$

for every $t \in [0, T]$. By the arbitrariness of ε we obtain (4.44). Finally, using the estimate in Proposition 4.4 and the Dominated Convergence Theorem we obtain (4.45). \square

Corollary 4.10. *Assume (H1)-(H23) and (4.7)-(4.13). Then for every $w \in \mathcal{W}^p$ we have*

$$\mathcal{G}^n(w) \rightarrow \mathcal{G}(w) \quad \mathcal{W}^p.$$

Proof. By (4.9) we get $\mathcal{T}^n w \rightarrow \mathcal{T}w$ in $H^1(0, T; \tilde{H})$ for every $w \in \mathcal{W}^p$. The result follows from Proposition 4.9 with F^n and F replaced by $F^n + \mathcal{T}^n w$ and $F + \mathcal{T}w$. \square

As a consequence of Lemma 4.2, Proposition 4.3, and Corollary 4.10 we obtain the continuous dependence result when T is small enough.

Theorem 4.11. *Assume that $B(T + T^3) < 1$, where B is the constant in Proposition 4.3. Then the conclusion of Theorem 4.1 holds.*

Proof. By Corollary 4.10 $\mathcal{G}^n(w) \rightarrow \mathcal{G}(w)$ in \mathcal{W}^p for every $w \in \mathcal{W}^p$. By Proposition 4.3 the maps \mathcal{G}^n have the same contraction constant $B(T + T^3) < 1$. Then we are in a position to apply Lemma 4.2 and we get

$$w^n := (u^n, Du^n, \dot{u}^n) \rightarrow (u, Du, \dot{u}) =: w \quad \text{in } \mathcal{W}^p = L^p((0, T); H \times \underline{H} \times H). \quad (4.50)$$

From this convergence and (4.9), we obtain $\mathcal{T}^n w^n \rightarrow \mathcal{T}w$ in $H^1(0, T; \tilde{H})$ and we can apply Proposition 4.9, with forcing term F^n and F replaced by $F^n + \mathcal{T}^n w^n$ and $F + \mathcal{T}w$. Since $F^n + \mathcal{T}^n w^n \rightarrow F + \mathcal{T}w$ in $H^1(0, T; \tilde{H})$ we get

$$(u^n(t), Du^n(t), \dot{u}^n(t)) \rightarrow (u(t), Du(t), \dot{u}(t)) \quad \text{in } H \times \underline{H} \times H$$

for every $t \in [0, T]$. We can apply Proposition 4.4 with F^n replaced by $F^n + \mathcal{T}^n w^n$ and we obtain that there exists a constant $C > 0$ such that

$$\|u^n(t)\| + \|Du^n(t)\| + \|\dot{u}^n(t)\| \leq C$$

for every $n \in \mathbb{N}$ and $t \in [0, T]$. \square

We are now in a position prove Theorem 4.1 without additional assumptions on T .

Proof of Theorem 4.1. There exists $k \in \mathbb{N}$ such that $T_0 := T/k$ satisfies $B(T_0 + T_0^3) < 1$. By Theorem 4.11 we have

$$(u^n(t), Du^n(t), \dot{u}^n(t)) \rightarrow (u(t), Du(t), \dot{u}(t)) \quad \text{in } H \times \underline{H} \times H \quad \text{for all } t \in [0, T], \quad (4.51)$$

$$(u^n, Du^n, \dot{u}^n) \rightarrow (u, Du, \dot{u}) \quad \text{in } L^p((0, T_0); H \times \underline{H} \times H). \quad (4.52)$$

If $k = 1$ the proof is finished, otherwise we consider the problem on the interval $[T_0, 2T_0]$.

Note that $u^n(T_0) \in V^n$ and $\dot{u}^n(T_0) \in H$ are well defined, because $u \in C_w^0([0, T_0]; V^n)$ and $\dot{u} \in C_w^0([0, T_0]; H)$. Since $u^n(t) \in V_t^n$ for a.e. $t \in (0, T_0)$, it easy to see that $u^n(T_0) \in V_{T_0}^n$. In order to study the problem on $[T_0, 2T_0]$ we define the spaces $\mathcal{V}_{T_0, 2T_0}$, $\mathcal{V}_{T_0, 2T_0}^D$, $\mathcal{V}_{T_0, 2T_0}^\infty$,

$\mathcal{V}_{T_0, 2T_0}^n$, $\mathcal{V}_{T_0, 2T_0}^{n, D}$, $\mathcal{V}_{T_0, 2T_0}^{n, \infty}$, and $\mathcal{W}_{T_0, 2T_0}^p$ as \mathcal{V} , \mathcal{V}^D , \mathcal{V}^∞ , \mathcal{V}^n , $\mathcal{V}^{n, D}$, $\mathcal{V}^{n, \infty}$, and \mathcal{W}^p with 0 and T replaced by T_0 and $2T_0$. For every $t \in [T_0, 2T_0]$ we set

$$G(t) := F(t) + \int_0^{T_0} e^{\tau-t} \nabla E u(\tau) d\tau \quad \text{and} \quad G^n(t) := F^n(t) + \int_0^{T_0} e^{\tau-t} \nabla^n E u^n(\tau) d\tau.$$

Let v be the solution of the problem

$$\begin{aligned} v &\in \mathcal{V}_{T_0, 2T_0} \quad \text{and} \quad v - u_D \in \mathcal{V}_{T_0, 2T_0}^D, \\ &- \int_{T_0}^{2T_0} (\dot{v}(t), \dot{\varphi}(t)) dt + \int_{T_0}^{2T_0} (\mathbb{A} E v(t), E \varphi(t)) dt - \int_{T_0}^{2T_0} \int_{T_0}^t e^{\tau-t} (\nabla E v(\tau), E \varphi(t)) d\tau dt \\ &= \int_{T_0}^{2T_0} (f(t), \varphi(t)) dt + \int_{T_0}^{2T_0} (G(t), E \varphi(t)) dt \quad \text{for every } \varphi \in \mathcal{V}_{T_0, 2T_0}^D \text{ with } \varphi(T_0) = \varphi(2T_0) = 0, \\ &v(T_0) = u(T_0) \quad \text{in } H \quad \text{and} \quad \dot{v}(T_0) = \dot{u}(T_0) \quad \text{in } (V_{T_0}^D)^*. \end{aligned}$$

For every $n \in \mathbb{N}$ let v^n be the solution of the problem

$$\begin{aligned} v^n &\in \mathcal{V}_{T_0, 2T_0}^n \quad \text{and} \quad v^n - u_D^n \in \mathcal{V}_{T_0, 2T_0}^{n, D}, \\ &- \int_{T_0}^{2T_0} (\dot{v}^n(t), \dot{\varphi}(t)) dt + \int_{T_0}^{2T_0} (\mathbb{A}^n E v^n(t), E \varphi(t)) dt - \int_{T_0}^{2T_0} \int_{T_0}^t e^{\tau-t} (\nabla^n E v^n(\tau), E \varphi(t)) d\tau dt \\ &= \int_{T_0}^{2T_0} (f^n(t), \varphi(t)) dt + \int_{T_0}^{2T_0} (G^n(t), E \varphi(t)) dt \quad \text{for every } \varphi \in \mathcal{V}_{T_0, 2T_0}^{n, D} \text{ with } \varphi(T_0) = \varphi(2T_0) = 0, \\ &v^n(T_0) = u^n(T_0) \quad \text{in } H \quad \text{and} \quad \dot{v}^n(T_0) = \dot{u}^n(T_0) \quad \text{in } (V_{T_0}^{n, D})^*. \end{aligned}$$

We note that, by the definition of G and G^n , the restrictions of u and u^n to $[T_0, 2T_0]$ satisfy the problems for v and v^n . By uniqueness we have that $v = u$ and $v^n = u^n$ on $[T_0, 2T_0]$.

For every $x \in \bar{\Omega}$ and $[T_0, 2T_0]$ we define $\Phi_{T_0}(t, x) := \Phi(t, \Psi(T_0, x))$, $\Psi_{T_0}(t, x) := \Psi(t, \Phi(T_0, x))$, $\Phi_{T_0}^n(t, x) := \Phi^n(t, \Psi^n(T_0, x))$, $\Psi_{T_0}^n(t, x) := \Psi^n(t, \Phi^n(T_0, x))$ which satisfy (H11)-(H15), (4.8) with 0 and T replaced by T_0 and $2T_0$. For every $x \in \bar{\Omega}$ we define $\Theta_{T_0}^n(x) := \Phi^n(T_0, \Theta^n(\Psi(T_0, x)))$, $\Xi_{T_0}^n(x) := \Phi(T_0, \Xi^n(\Psi^n(T_0, x)))$ and we observe that they satisfy (H19)-(H23) and (4.13) with 0 and T replaced by T_0 and $2T_0$.

By (4.51) we have that $(u^n(T_0), Du^n(T_0), \dot{u}^n(T_0)) \rightarrow (u(T_0), Du(T_0), \dot{u}(T_0))$ in $H \times \underline{H} \times H$ while (4.9), (4.11), and (4.52) give $G^n \rightarrow G$ in $H^1(0, T; \bar{H})$. We are now in a position to apply Theorem 4.11 on $[T_0, 2T_0]$ to obtain

$$(u^n(t), Du^n(t), \dot{u}^n(t)) \rightarrow (u(t), Du(t), \dot{u}(t)) \quad \text{in } H \times \underline{H} \times H,$$

for all $t \in [T_0, 2T_0]$. Moreover there exists a constant $C > 0$ such that

$$\|u^n(t)\| + \|Du^n(t)\| + \|\dot{u}^n(t)\| \leq C$$

for every $n \in \mathbb{N}$ and $t \in [T_0, 2T_0]$. The conclusion can be obtained by iterating this process a finite number of times. \square

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