

Superconformal index of low-rank gauge theories via the Bethe Ansatz

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We study the Bethe Ansatz formula for the superconformal index, in the case of 4d $\mathcal{N} = 4$ super-Yang-Mills with gauge group $SU(N)$. We observe that not all solutions to the Bethe Ansatz Equations (BAEs) contribute to the index, and thus formulate “reduced BAEs” such that all and only their solutions contribute. We then propose, sharpening a conjecture of Arabi Ardehali *et al.* [1], that there is a one-to-one correspondence between solutions to the reduced BAEs and vacua of the 4d $\mathcal{N} = 1^*$ theory on S^1 . We test the proposal in the case of $SU(2)$ and $SU(3)$. In the case of $SU(3)$, we confirm that there is a continuous family of solutions, whose contribution to the index is non-vanishing.

1 Introduction and results

The AdS/CFT correspondence [2–4] provides us with a non-perturbative definition of quantum gravity in anti-de-Sitter (AdS) space, in terms of an ordinary quantum field theory (QFT) living on its conformal boundary. In particular, ensembles of states in the boundary theory capture the physics of black holes in the bulk [5]. This means that a counting of quantum microstates in the boundary theory, performed by studying suitable partition functions in the large N limit, can reproduce and explain the Bekenstein-Hawking entropy of black holes in AdS [6]. The last few years have seen a lot of activity in this direction, in the context of supersymmetric black holes in which explicit and precise computations can be performed, see for instance [7–43].

An interesting case is that of type IIB string theory (and its supergravity low-energy limit) in $\text{AdS}_5 \times S^5$, whose physics is captured by the four-dimensional $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory with gauge group $SU(N)$ [2]. The microstates of 1/16 BPS black holes in AdS_5 are captured by the superconformal index [44, 45], a supersymmetric partition function that can be computed exactly. The large N limit can be studied with a variety of methods [19, 20, 34, 41, 45–47]. From it, one can extract the Bekenstein-Hawking entropy, but also find hints of new physics. For instance, one can compute perturbative corrections expected to come from higher-derivative terms [40], as well as non-perturbative corrections coming from Euclidean complex saddles of the gravitational action [48, 49] (thus highlighting the importance of Euclidean complex saddles [18]) and corrections expected from D-branes.

A particularly effective way to compute the large N limit is to use the Bethe Ansatz (BA) approach [20], based on the Bethe Ansatz formulation of the superconformal index [50, 51]. The standard integral formula for the index [44–47] can be recast as a sum over the solution set to certain transcendental equations, dubbed Bethe Ansatz Equations (BAEs) because of their similarity with the ones appearing in the context of integrable systems. One particular solution to the BAEs reproduces the Bekenstein-Hawking entropy of BPS black holes in AdS [20]. On the other hand, other solutions can be associated with complex semiclassical saddles of the gravitational path-integral [48, 49]. A comprehensive picture of all the contributions from BA solutions and the physics they predict is an interesting open problem.

Even for finite N , the complete list of solutions to the BAEs is not known. A subclass of exact solutions was found in [10, 52], and we will refer to them as Hong-Liu (HL) solutions. These are discrete solutions, in one-to-one correspondence with subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$ of order N , and are parametrized by three integers $\{m, n, r\}$ with $N = mn$ and $r \in \mathbb{Z}_n$. On the other hand, a very interesting conjecture was put forward in [1]: that there could be a sort of correspondence between some solutions to the BAEs and the vacua of the 4d $\mathcal{N} = 1^*$

theory — which is $\mathcal{N} = 4$ SYM deformed by $\mathcal{N} = 1$ preserving mass terms — compactified on S^1 . The set of vacua of the $\mathcal{N} = 1^*$ theory [53] (that we will review below) include both discrete massive vacua, which are easily seen to be in one-to-one correspondence with the HL solutions, as well as massless Coulomb vacua, which give rise to continuous families of vacua upon compactification on S^1 . Indeed, the authors of [1] gave evidence for the existence of continuous families of solutions to the BAEs. On the other hand, they also pointed out that there are many more solutions to the BAEs (both discrete and continuous) that do not fit into the pattern.

In this Letter we will make the conjectural correspondence sharper, by studying in detail the low-rank cases of gauge groups $SU(2)$ and $SU(3)$.

Our first result (Section 2) is that, for generic N , most of the solutions to the BAEs lead to contributions that cancel out. This means that we can formulate a more restrictive set of equations, that we call “reduced Bethe Ansatz Equations”, such that only solutions to the latter actually contribute. We then conjecture that there is a one-to-one correspondence between solutions to the reduced BAEs and vacua of the $\mathcal{N} = 1^*$ theory on S^1 .

We test our conjecture in the cases of $SU(2)$ and $SU(3)$ gauge group. In the case of $SU(2)$ (Section 3) we show that the reduced BAEs have only 3 solutions, which indeed are precisely the discrete HL solutions. We then show, by expanding the index order by order in the operator dimension and computing many terms, that those 3 solutions exactly reproduce the superconformal index (as it follows from the integral formula).

In the case of $SU(3)$ (Section 4), we are able to analytically solve the reduced BAEs and find all solutions. They comprise the 4 HL solutions, as well as a continuous family which has complex dimension 1 and is made of a single connected component. This is precisely in correspondence with the vacua of the $\mathcal{N} = 1^*$ theory on S^1 . We also show that, in this case, the HL solutions alone do not reproduce the index, and thus the contribution from the continuous family must be non-trivial. We leave the interesting issue of evaluating such a contribution for future work.

The method we use, in the $SU(3)$ case, to solve the reduced BAEs and to exhibit the structure of discrete and continuous solutions sprouts from the observation that the equations factorize into multiple hyperplanes and curves, at whose intersections lie the solutions. We are hopeful that this method can be generalized to higher N .

Note added: While this work was completed, its results had been announced in [54], and we were in the process of writing up this Letter, we received the paper [55] which has considerable overlap with our work.

2 Bethe Ansatz formula for the superconformal index

We study the superconformal index of 4d $\mathcal{N} = 4$ SYM [44, 45], which counts (with sign) operators in short representations (1/16 BPS) of the superconformal algebra, preserving a complex supercharge \mathcal{Q} . Using $\mathcal{N} = 1$ notation, the field content of $\mathcal{N} = 4$ SYM is given by a vector multiplet and three chiral multiplets X_1, X_2, X_3 , all in the adjoint representation of the gauge group, with superpotential $W = \text{Tr}(X_1[X_2, X_3])$. The R-symmetry is $SU(4)_R$, and we choose the Cartan generators $R_{1,2,3}$ such that each one of them assigns charge 2 to one of the chiral multiplets and zero to the other two. States are further labeled by two angular momenta $J_{1,2}$ with semi-integer eigenvalues, generating rotations of two orthogonal planes in \mathbb{R}^4 , and we define the fermion number as $F = 2J_1$. All fields in the theory have integer charges under $R_{1,2,3}$, moreover

$$F = 2J_{1,2} = R_{1,2,3} \pmod{2} . \quad (2.1)$$

The superconformal index is defined as

$$\mathcal{I}(p, q, y_1, y_2) = \text{Tr} \left[(-1)^F e^{-\beta\{\mathcal{Q}, \mathcal{Q}^\dagger\}} p^{J_1 + \frac{1}{2}R_3} q^{J_2 + \frac{1}{2}R_3} y_1^{\frac{1}{2}(R_1 - R_3)} y_2^{\frac{1}{2}(R_2 - R_3)} \right] . \quad (2.2)$$

Here p, q, y_1, y_2 are fugacities, and it is convenient to introduce chemical potentials $\sigma, \tau, \Delta_1, \Delta_2$ such that

$$p = e^{2\pi i\sigma} , \quad q = e^{2\pi i\tau} , \quad y_j = e^{2\pi i\Delta_j} . \quad (2.3)$$

The index is well-defined for $|p|, |q| < 1$. It is a single-valued function of the fugacities, therefore periodic under integer shifts of the chemical potentials, because all states have integer charges with respect to the exponents in (2.2). Besides, it will be convenient to introduce an auxiliary fugacity y_3 and chemical potential Δ_3 such that¹

$$\Delta_1 + \Delta_2 + \Delta_3 - \sigma - \tau \in \mathbb{Z} . \quad (2.4)$$

By standard arguments [56], the index only counts states annihilated by \mathcal{Q} and \mathcal{Q}^\dagger and therefore is independent of β .

When reading off the (weighted) multiplicities of BPS operators from the index in a “low-temperature” expansion, it may be convenient to use the original parametrization of [45] in terms of fugacities (t, y, v_1, v_2) . They are related to our fugacities by

$$p = t^3 y , \quad q = t^3 / y , \quad y_1 = t^2 v_1 , \quad y_2 = t^2 v_2 , \quad y_3 = t^2 / v_1 v_2 , \quad (2.5)$$

¹This will restore the permutation symmetry acting on the index $a = 1, 2, 3$ in R_a, y_a and Δ_a . Such a symmetry is the Weyl group of the global symmetry $SU(3) \subset SU(4)_R$ that commutes with \mathcal{Q} .

and recall that y_3 is an auxiliary variable defined by $y_1 y_2 y_3 = pq$. In terms of these fugacities, the index reads

$$\mathcal{I} = \text{Tr}_{\text{BPS}}(-1)^F t^{2(\Delta+J_+)} y^{2J_-} v_1^{\mathfrak{q}_1} v_2^{\mathfrak{q}_2} . \quad (2.6)$$

Here $J_{\pm} = (J_1 \pm J_2)/2$ are the Cartan generators of the $SU(2) \times SU(2)$ Lorentz group, Δ is the dimension of the operators, the trace is over BPS states — all and only states satisfying $\Delta = 2J_+ + (R_1 + R_2 + R_3)/2$ — while $\mathfrak{q}_{1,2} = (R_{1,2} - R_3)/2$ are the Cartan generators of the $SU(3) \subset SU(4)_R$ that commutes with \mathcal{Q} .

Integral formulation. The index is independent of the gauge coupling, and thus can be computed exactly [44–47] in terms of a certain contour integral. Specializing to the case of gauge group $SU(N)$, the integral formula reads

$$\mathcal{I}_{SU(N)} = \kappa_N \int_{\mathbb{T}^{N-1}} \mathcal{Z}(u; \Delta, \sigma, \tau) \prod_{j=1}^{N-1} \frac{dz_j}{2\pi i z_j} , \quad (2.7)$$

where $z_j = e^{2\pi i u_j}$, and u_j with $j = 1, \dots, N$ are gauge holonomies along the Cartan generators. We regard u_j with $j = 1, \dots, N-1$ as the independent variables, while the last holonomy, u_N , is fixed by the $SU(N)$ constraint

$$\sum_{j=1}^N u_j = 0 . \quad (2.8)$$

The prefactor is

$$\kappa_N = \frac{1}{N!} \left(\mathcal{I}_{U(1)} \right)^{N-1} , \quad (2.9)$$

written in terms of the index

$$\mathcal{I}_{U(1)}(p, q, y_1, y_2) = (p; p)_{\infty} (q; q)_{\infty} \prod_{a=1}^3 \tilde{\Gamma}(\Delta_a; \sigma, \tau) \quad (2.10)$$

of the free $\mathcal{N} = 4$ vector multiplet. Here $(z; q)_{\infty}$ is the q -Pochhammer symbol while $\tilde{\Gamma}(u; \sigma, \tau)$ is the elliptic gamma function, both defined in Appendix A. The integrand is

$$\mathcal{Z}(u; \Delta, \sigma, \tau) = \prod_{i \neq j}^N \frac{\prod_{a=1}^3 \tilde{\Gamma}(u_{ij} + \Delta_a; \sigma, \tau)}{\tilde{\Gamma}(u_{ij}; \sigma, \tau)} , \quad (2.11)$$

where $u_{ij} = u_i - u_j$. This expression can be slightly simplified using the identity

$$\prod_{i \neq j}^N \frac{1}{\tilde{\Gamma}(u_{ij}; \sigma, \tau)} = \prod_{i < j}^N \theta_0(u_{ij}; \sigma) \theta_0(u_{ji}; \tau) , \quad (2.12)$$

written in terms of the elliptic function θ_0 defined in Appendix A. The contour of integration is along $N-1$ copies of the unit circle, as long as the fugacities satisfy $|pq| < |y_a| < 1$ for all $a = 1, 2, 3$ (otherwise a deformation of the contour is necessary in order to avoid crossing poles).

Bethe Ansatz formulation. In order to compute the large N limit, it is convenient to consider the alternative Bethe Ansatz formula [50, 51]. Such a formula applies whenever the angular chemical potentials can be brought, with suitable integer shifts, to have rational ratio, $\sigma/\tau \in \mathbb{Q}$. This means that $\sigma = a\omega$, $\tau = b\omega$ for positive coprime integers a, b and a chemical potential ω with $\text{Im } \omega > 0$. Our discussion of the Bethe Ansatz equations will be general, however, for simplicity when computing the index we will restrict to the case of two equal angular fugacities:

$$p = q \quad \Leftrightarrow \quad \sigma = \tau . \quad (2.13)$$

In this case the Bethe Ansatz formula simplifies to

$$\mathcal{I}(q, y_1, y_2) = \kappa_N \sum_{\hat{u} \in \text{BAEs}} \mathcal{Z}(\hat{u}; \Delta, \tau) H(\hat{u}; \Delta, \tau)^{-1} . \quad (2.14)$$

The sum is over the solution set $\{\hat{u}\}$ to a system of transcendental equations, dubbed Bethe Ansatz Equations. Defining the $U(N)$ Bethe operators as

$$Q_j(u; \Delta, \tau) = e^{-2\pi i \sum_{k=1}^N u_{jk}} \prod_{a=1}^3 \prod_{k=1}^N \frac{\theta_0(u_{kj} + \Delta_a; \tau)}{\theta_0(u_{jk} + \Delta_a; \tau)} \quad (2.15)$$

for $j = 1, \dots, N$, the $SU(N)$ BA equations are given by

$$1 = \frac{Q_i}{Q_N} \quad \text{for } i = 1, \dots, N-1 . \quad (2.16)$$

The unknowns are the ‘‘complexified $SU(N)$ holonomies’’ u_i living on a torus of modular parameter τ , namely with identifications

$$u_i \sim u_i + 1 \sim u_i + \tau \quad \text{for } i = 1, \dots, N-1 , \quad (2.17)$$

while u_N is fixed by the constraint (2.8). The BAEs (2.16) are invariant under such shifts. In fact, a stronger property holds: Q_i are invariant under shifts of the components of the antisymmetric tensor u_{ij} by 1 or τ , even relaxing the condition that $u_{ij} = u_i - u_j$. This will be used later. It was proven in [51] that only the solutions that are not invariant under any non-trivial element of the Weyl group of $SU(N)$ (namely, only solutions with all u_i different on the torus) actually contribute to the sum in (2.14). In the general case $\sigma/\tau \in \mathbb{Q}$, the BA formula takes a more complicated form, however the BAEs are the same as in (2.16) with the only difference that τ is replaced by ω . Therefore the analysis we will make of the BAEs and their solutions will be valid in the general case.

The prefactor κ_N and the integrand \mathcal{Z} in (2.14) are the same as in the integral formula described before. On the other hand, H is a Jacobian defined as

$$H = \det \left[\frac{1}{2\pi i} \frac{\partial \log(Q_i/Q_N)}{\partial u_j} \right]_{i,j=1,\dots,N-1} . \quad (2.18)$$

Notice that Q_i , κ_N , \mathcal{Z} and H are all invariant under integer shifts of τ , Δ_1 and Δ_2 , in accord with the fact that the superconformal index (2.14) is a single-valued function of the fugacities.

The superconformal index of the $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$ is related to the one for gauge group $SU(N)$ by the simple relation

$$\mathcal{I}_{U(N)} = \mathcal{I}_{U(1)} \mathcal{I}_{SU(N)} . \quad (2.19)$$

Reduced Bethe Ansatz equations. It turns out that not all solutions (faithfully acted upon by the Weyl group) of (2.16) actually contribute to the sum in (2.14). In order to understand this point, notice that $\prod_{i=1}^N Q_i = 1$ identically. It follows that the solutions to the BAEs (2.16) break into N ‘‘sectors’’ parametrized by $\lambda \in \mathbb{Z}_N$ and given by

$$Q_j = e^{2\pi i \lambda / N} \quad \text{for } j = 1, \dots, N , \quad (2.20)$$

each sector corresponding to a different integer value of $\lambda = 0, 1, \dots, N - 1$ and containing a subset of the solutions. Now, the Bethe operators have the property that they implement on \mathcal{Z} the shifts of complexified $SU(N)$ holonomies on the torus [50, 51]:

$$\mathcal{Z}(u - \delta_k \tau) = \frac{Q_k(u)}{Q_N(u)} \mathcal{Z}(u) \quad \text{for } k = 1, \dots, N - 1 , \quad (2.21)$$

where $u - \delta_k \tau = (u_1, \dots, u_k - \tau, \dots, u_{N-1}, u_N + \tau)$ denotes a shift of the k -th and N -th components of u , so as to preserve the $SU(N)$ constraint. On the other hand, if we regard $\mathcal{Z}(u)$ as function of N independent holonomies u_i of $U(N)$, we find

$$\mathcal{Z}(u - \bar{\delta}_k \tau) = (-1)^{N-1} Q_k(u) \mathcal{Z}(u) \quad \text{for } k = 1, \dots, N , \quad (2.22)$$

where $u - \bar{\delta}_k \tau = (u_1, \dots, u_k - \tau, \dots, u_N)$ denotes a shift of the k -th component only.² The BAEs only depend on the differences u_{ij} and, as already noticed, are invariant under shifts of u_{ij} by multiples of 1 and τ . It follows that one solution for u_{ij} on the torus gives rise to multiple solutions for u_i , of the form $u_i = u_i^{(0)} + (\alpha + \beta\tau)/N$ for $i = 1, \dots, N - 1$ and $u_N = u_N^{(0)} + (1 - N)(\alpha + \beta\tau)/N$, where $\alpha, \beta = 0, \dots, N - 1$ (some of these solutions could be equivalent up to the Weyl group action). The function \mathcal{Z} , to be evaluated in (2.14) on those solutions, only depends on u_{ij} , thus the dependence on α, β is the same as if u_N were shifted by $-(\alpha + \beta\tau)$. There is no dependence on α , while the dependence on β is a phase $((-1)^{N-1} Q_N(u))^\beta$ described by (2.22). We conclude that in all but one sector of BA solutions, the sum over β leads to a cancelation because the sum of phases vanishes. The exception is the sector in which $(-1)^{N-1} Q_N(u) = 1$.

²The factor $(-1)^{N-1}$ did not appear in [51] because that paper only dealt with semi-simple gauge groups.

Therefore, in the Bethe Ansatz formula (2.14) we can restrict to BA solutions solely in the sector $\lambda = \frac{N(N-1)}{2} \bmod N$, namely to solutions of

$$Q_i = (-1)^{N-1} \quad \text{for } i = 1, \dots, N, \quad (2.23)$$

which moreover are faithfully acted upon by the Weyl group. We call these the ‘‘reduced Bethe Ansatz Equations’’. Recall that the product of the N equations is identically equal to 1, therefore one of them could be removed from the set.

Hong-Liu (HL) solutions. The full set of solutions to (2.23) is not known, however, a large set was found in [10, 52] and we will refer to them as HL solutions. They are labelled by three positive integers:

$$\{m, n, r\} \quad \text{such that} \quad N = m \cdot n, \quad r \in \mathbb{Z}_n. \quad (2.24)$$

The solutions are

$$u_j \equiv u_{j\hat{k}} = \bar{u} + \frac{\hat{j}}{m} + \frac{\hat{k}}{n} \left(\tau + \frac{r}{m} \right). \quad (2.25)$$

Here we have decomposed the index $j = 0, \dots, N - 1$ into the indices $\hat{j} = 0, \dots, m - 1$ and $\hat{k} = 0, \dots, n - 1$. Moreover, \bar{u} is a constant chosen in such a way to solve the $SU(N)$ constraint. Since what enters in all formulas are the differences $u_{ij} = u_i - u_j$, to each HL solution is associated a multiplicity that we will discuss below. Notice that the HL solutions are in one-to-one correspondence with subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$ of order N , that is with sublattices of index N in generic two-dimensional lattices. It is worth remarking that, surprisingly enough, the HL solutions (2.25) do not depend on the flavor chemical potentials Δ_a , although the BAEs do.

The BAEs (2.23) are invariant under $SL(2, \mathbb{Z})$ modular transformations of the torus, namely under the generators

$$T : \begin{cases} \tau \mapsto \tau + 1 \\ u \mapsto u \end{cases} \quad S : \begin{cases} \tau \mapsto -1/\tau \\ u \mapsto u/\tau \end{cases} \quad C : \begin{cases} \tau \mapsto \tau \\ u \mapsto -u \end{cases}. \quad (2.26)$$

It follows that the HL solutions form orbits under $PSL(2, \mathbb{Z})$, completely classified by the integer $d = \gcd(m, n, r)$. The action of $PSL(2, \mathbb{Z})$ is given by

$$T : \{m, n, r\} \mapsto \{m, n, r + m\}, \quad S : \{m, n, r\} \mapsto \left\{ \gcd(n, r), \frac{mn}{\gcd(n, r)}, \frac{m(n-r)}{\gcd(n, r)} \right\}. \quad (2.27)$$

For given N , the total number of HL solutions is given by the divisor function

$$\sigma_1(N) = \sum_{k|N} k. \quad (2.28)$$

For each integer d such that d^2 divides N , there exist a separate $PSL(2, \mathbb{Z})$ orbit of HL solutions generated by $\{d, \frac{N}{d}, 0\}$, which is isomorphic to the orbit generated by $\{1, N/d^2, 0\}$ in the case of $SU(N/d^2)$. The number of elements in such an orbit is given by $\psi(N/d^2)$, expressed in terms of the Dedekind psi function³

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right) \quad (2.29)$$

where the product is over all prime numbers that divide n .

Multiplicities. From each solution in terms of u_{ij} on the torus, one gets up to N^2 solutions in terms of u_i , related by a shift of the “center of mass” of the first $N - 1$ components. Some of those, however, could be equivalent up to the Weyl group action, and thus the exact multiplicity should be determined by a case-by-case analysis. In the case of the HL solutions, one easily verifies that each solution has multiplicity N , namely it gives rise to N inequivalent solutions in terms of u_i that cannot be identified by the action of the Weyl group.

Besides, since we are only interested in BA solutions that are not fixed by any non-trivial element of the Weyl group, there is an obvious multiplicity by $N!$ related to the action of the Weyl group. Therefore, each HL solution has total multiplicity $N \cdot N!$.

2.1 Correspondence with the $\mathcal{N} = 1^*$ theory

It has been pointed out in [1] that the Hong-Liu ones are not the only solutions to the BAEs. Furthermore, by a combination of analytical and numerical work, evidence was given that for $N \geq 3$ there exist continuous families of solutions. Very interestingly, the authors of [1] put forward the idea of a correspondence between a subset of the solutions to the BAEs and the vacua of the $\mathcal{N} = 1^*$ $SU(N)$ theory on S^1 . This, in particular, allowed them to predict the appearance of continuous branches of BA solutions for various values of N . On the other hand, it was already noticed in that work that, even for $N = 2$, the BAEs admit solutions that do not have a counterpart in the $\mathcal{N} = 1^*$ theory, therefore the problem of understanding in which terms the correspondence could be correct and precise remained open.

In this Letter we present a sharper version of the conjecture: we propose that there is a one-to-one correspondence between solutions to the reduced Bethe Ansatz equations, and vacua of the $\mathcal{N} = 1^*$ $SU(N)$ theory on S^1 . If the conjecture is correct, it implies that the

³Notice that, consistently, $\sum_{d^2|N} \psi\left(\frac{N}{d^2}\right) = \sigma_1(N)$.

HL solutions are the only discrete solutions to the reduced BAEs. It also implies a precise characterization of the number of continuous families and their dimensionality, for each value of N . In Sections 3 and 4 we provide some evidence of the conjecture.

Let us review the $\mathcal{N} = 1^* SU(N)$ theory, which is obtained from $\mathcal{N} = 4$ SYM by an $\mathcal{N} = 1$ preserving mass deformation

$$W_{\text{def}} = \sum_{a=1}^3 \frac{m_a}{2} X_a^2. \quad (2.30)$$

Its vacua have been analyzed in [53, 57]. F- and D-term equations reduce to the classification of isomorphism classes of homomorphisms $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{su}(N)$ modulo $SL(N, \mathbb{C})$, which correspond to partitions of N . Since the chiral superfields X_a all have bare masses, possible massless fields come from the gauge sector. At the classical level, the $SU(N)$ gauge group is broken to the subgroup H that commutes with the image of ρ . Quantum mechanically, all simple $SU(n)$ factors of H confine leading to a number n of massive sectors; on the other hand, if H contains $U(1)$ factors, then one obtains a massless Coulomb phase.

The net result can be summarized as follows. Consider all integer partitions of N . For each partition, let n_j be the number of times the integer j appears in it, and so label the partition by a string

$$(n_1, \dots, n_j) \quad \text{such that} \quad \sum_{j=1}^N j n_j = N. \quad (2.31)$$

Then the subgroup H is given by

$$H = \left[\prod_{j \text{ s.t. } n_j \neq 0}^N U(n_j) \right] / U(1). \quad (2.32)$$

Thus, such a partition contributes to the moduli space with a certain number of branches of a certain dimension, given by:

- dimension = (number of non-zero n_j 's) - 1 (these are isolated gapped vacua if the dimension is zero, or Coulomb branch vacua if it is bigger than zero);
- number of components = product of non-zero n_j 's.

It turns out [53] that the massive vacua are in one-to-one correspondence with subgroups of order N of $\mathbb{Z}_N \times \mathbb{Z}_N$, and thus in correspondence with the HL solutions to the BAEs. On the other hand, upon compactification on S^1 , each Coulomb vacuum gives rise to a continuous family of vacua with complex dimension equal to the number of photons.

3 The case of $SU(2)$

Let us analyze in detail the case of gauge group $SU(2)$. Defining $u \equiv u_{21} = u_2 - u_1$, the refined Bethe Ansatz equation (2.23) can be rewritten as

$$\frac{\theta_1(\Delta_1 + u) \theta_1(\Delta_2 + u) \theta_1(-\Delta_1 - \Delta_2 + u)}{\theta_1(\Delta_1 - u) \theta_1(\Delta_2 - u) \theta_1(-\Delta_1 - \Delta_2 - u)} = -1, \quad (3.1)$$

in terms of the standard Jacobi theta function θ_1 (see Appendix A). For the sake of clarity, here and in the following we leave the dependence of the Jacobi theta functions θ_r on τ implicit. Let us define the function

$$f(u; \Delta, \tau) = \theta_1(u) \theta_1(\Delta_1 + u) \theta_1(\Delta_2 + u) \theta_1(-\Delta_1 - \Delta_2 + u). \quad (3.2)$$

Then, multiplying and dividing (3.1) by $\theta_1(u)$ and using that θ_1 is an odd function of u (while $\theta_{2,3,4}$ are even functions of u), we can recast the reduced BAE as

$$f(u; \Delta, \tau) - f(-u; \Delta, \tau) = 0. \quad (3.3)$$

Notice that we have introduced some spurious solutions, namely the zeros of $\theta_1(u)$, but we will remove them at the end. Using standard identities among the theta functions, see for instance [58], we obtain

$$f(u; \Delta, \tau) = c_1 \theta_1(2u) + c_2 \theta_2(2u) + c_3 \theta_3(2u) + c_4 \theta_4(2u) \quad (3.4)$$

where the coefficients are given by

$$\begin{aligned} c_1 &= -\frac{1}{2} \theta_1(\Delta_1) \theta_1(\Delta_2) \theta_1(\Delta_1 + \Delta_2) \\ c_2 &= -\frac{1}{2} \theta_2(\Delta_1) \theta_2(\Delta_2) \theta_2(\Delta_1 + \Delta_2) \\ c_3 &= +\frac{1}{2} \theta_3(\Delta_1) \theta_3(\Delta_2) \theta_3(\Delta_1 + \Delta_2) \\ c_4 &= -\frac{1}{2} \theta_4(\Delta_1) \theta_4(\Delta_2) \theta_4(\Delta_1 + \Delta_2). \end{aligned} \quad (3.5)$$

From the parity properties of the theta functions, the reduced BAE becomes

$$c_1(\Delta, \tau) \theta_1(2u) = 0. \quad (3.6)$$

For generic values of Δ_a and τ , the coefficient c_1 is non-zero and thus the full set of solutions is given by $u = (m + n\tau)/2$ for integers m, n . On the torus there are 3 non-vanishing solutions, while the solution $u = 0$ is the spurious one that we introduced above and should be removed.⁴ The 3 solutions are

$$u = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau + 1}{2}. \quad (3.7)$$

⁴Notice also that $u = 0$ would lead to configurations (u_1, u_2) that are fixed by the non-trivial element of the Weyl group, and thus would have to be excluded anyway.

These are precisely the 3 Hong-Liu solutions $\{2, 1, 0\}$, $\{1, 2, 0\}$, $\{1, 2, 1\}$, respectively, as it follows from (2.25). Therefore, for $N = 2$, the HL solutions exhaust the full set of solutions to the reduced BAEs, in agreement with the proposed correspondence with vacua of the $\mathcal{N} = 1^*$ theory.

Evaluation of the index. According to the BA formula (in the special case $\sigma = \tau$ for simplicity), we have the identity

$$\mathcal{I}_{SU(2)}(q, y_1, y_2) = \mathcal{I}_{SU(2)}^{\{2,1,0\}} + \mathcal{I}_{SU(2)}^{\{1,2,0\}} + \mathcal{I}_{SU(2)}^{\{1,2,1\}} . \quad (3.8)$$

On the left-hand side of this identity, $\mathcal{I}_{SU(2)}$ is full index computed with the integral formula (2.7). On the right-hand side, instead, each of the three terms $\mathcal{I}_{SU(2)}^{\{m,n,r\}}$ is the contribution to the Bethe Ansatz formula (2.14) from one of the HL solutions:

$$\mathcal{I}_{SU(N)}^{\{m,n,r\}} = N \cdot N! \kappa_N \mathcal{Z}(\hat{u}; \Delta, \tau) H(\hat{u}; \Delta, \tau)^{-1} \Big|_{\hat{u} \in \{m,n,r\}} , \quad (3.9)$$

where \hat{u}_j are as in the HL solution $\{m, n, r\}$ given in (2.25), while the factor $N \cdot N!$ comes from the multiplicity. We have verified (3.8) numerically for many values of the fugacities. This confirms that, for $N = 2$, the contributions from the three HL solutions exactly reproduce the superconformal index.

To gain a more interesting physical understanding, we can perform a “low-temperature” expansion of both sides of (3.8), corresponding to an expansion of the index into BPS operators, order by order in the operator dimension and starting from the identity. In order to do that, it is convenient to use the set of fugacities (t, y, v_1, v_2) in (2.6). The restriction to $p = q$ corresponds to $y = 1$. On the other hand, the dependence on v_1, v_2 organizes into characters of $SU(3)$ that we indicate by χ_{dim} . For instance

$$\begin{aligned} \chi_{\mathbf{1}} &= 1 \\ \chi_{\mathbf{3}} &= v_1 + v_2 + v_1^{-1} v_2^{-1} \\ \chi_{\mathbf{6}} &= v_1^2 + v_2^2 + v_1 v_2 + v_1^{-1} + v_2^{-1} + v_1^{-2} v_2^{-2} , \end{aligned} \quad (3.10)$$

and so on. With computer assistance, we perform a Taylor or Laurent expansion of both sides of (3.8) around $t = 0$.

Expanding the integral expression, left-hand side of (3.8), we obtain the following. Switch-

ing off also the flavor fugacities ($v_1 = v_2 = 1$) we find⁵

$$\mathcal{I}_{SU(2)} = 1 + 6t^4 - 6t^5 - 7t^6 + 18t^7 + 6t^8 - 36t^9 + 6t^{10} + 84t^{11} - 80t^{12} - 132t^{13} + 309t^{14} + \mathcal{O}(t^{15}). \quad (3.12)$$

Notice that the expansion only contains integer powers of t , and the coefficients are well-defined because the number of operators with bounded dimension is finite. Including the flavor fugacities we find

$$\begin{aligned} \mathcal{I}_{SU(2)} = & 1 + \chi_6 t^4 - 2\chi_3 t^5 + (1 - \chi_8) t^6 + 2(\chi_{\bar{3}} + \chi_6) t^7 + (\chi_{15} - 3\chi_3) t^8 \\ & - 2(\chi_{10} + \chi_8) t^9 + (3\chi_6 + 4\chi_{\bar{3}} - \chi_{24}) t^{10} + \mathcal{O}(t^{11}). \end{aligned} \quad (3.13)$$

In terms of Dynkin labels: $\mathbf{10} = [3, 0]$, $\mathbf{15} = [4, 0]$, $\mathbf{15}' = [2, 1]$, $\mathbf{21} = [5, 0]$ and $\mathbf{24} = [3, 1]$.

Similarly, we expand the BA contributions (3.9) for each one of the three HL solutions. Suppressing flavor fugacities we find

$$\begin{aligned} \mathcal{I}_{SU(2)}^{\{2,1,0\}} &= -2 - 12t - 42t^2 - 124t^3 - 348t^4 - 900t^5 - 2142t^6 - 4860t^7 - 10644t^8 + \mathcal{O}(t^9) \\ \mathcal{I}_{SU(2)}^{\{1,2,0\}} &= -\frac{1}{4}t^{-\frac{3}{2}} - \frac{3}{4}t^{-\frac{1}{2}} + \frac{3}{2} - 3t^{\frac{1}{2}} + 6t - \frac{43}{4}t^{\frac{3}{2}} + 21t^2 - \frac{153}{4}t^{\frac{5}{2}} + 62t^3 - 105t^{\frac{7}{2}} \\ &\quad + 177t^4 - \frac{1131}{4}t^{\frac{9}{2}} + 447t^5 - \frac{2775}{4}t^{\frac{11}{2}} + \frac{2135}{2}t^6 - 1635t^{\frac{13}{2}} + 2439t^7 + \mathcal{O}(t^{\frac{15}{2}}) \\ \mathcal{I}_{SU(2)}^{\{1,2,1\}} &= +\frac{1}{4}t^{-\frac{3}{2}} + \frac{3}{4}t^{-\frac{1}{2}} + \frac{3}{2} + 3t^{\frac{1}{2}} + 6t + \frac{43}{4}t^{\frac{3}{2}} + 21t^2 + \frac{153}{4}t^{\frac{5}{2}} + 62t^3 + 105t^{\frac{7}{2}} \\ &\quad + 177t^4 + \frac{1131}{4}t^{\frac{9}{2}} + 447t^5 + \frac{2775}{4}t^{\frac{11}{2}} + \frac{2135}{2}t^6 + 1635t^{\frac{13}{2}} + 2439t^7 + \mathcal{O}(t^{\frac{15}{2}}). \end{aligned} \quad (3.14)$$

Interestingly, the contributions from the two $\{1, 2, r\}$ solutions contain many unwanted features: fractional coefficients, and fractional powers as well as negative powers of t . However the unwanted powers of t cancel out in the sum, and the coefficients sum up to integers. We have verified that the sum of the three contributions above exactly matches the expansion (3.12) of the integral expression, up to order $\mathcal{O}(t^{40})$. Including the flavor fugacities we

⁵A much longer list of coefficients can be found in [37] for gauge group $U(2)$, and then the coefficients for $SU(2)$ can be derived using (2.19). Switching off flavor fugacities, one finds

$$\mathcal{I}_{U(1)} = 1 + 3t^2 - 2t^3 + 3t^4 + 6t^7 - 6t^8 + 12t^{10} - 18t^{11} + 27t^{12} - 12t^{13} - 27t^{14} + \mathcal{O}(t^{15}). \quad (3.11)$$

find

$$\begin{aligned}
\mathcal{I}_{SU(2)}^{\{2,1,0\}} &= -2 - 4\chi_{\bar{3}}t - 2(3\chi_{\mathbf{3}} + 2\chi_{\bar{6}})t^2 - 4(\chi_{\bar{10}} + 2\chi_{\mathbf{8}} + 5)t^3 \\
&\quad - 4(\chi_{\bar{15}} + 2\chi_{\bar{15}'} + 2\chi_{\mathbf{6}} + 10\chi_{\bar{3}})t^4 + \mathcal{O}(t^5) \\
\mathcal{I}_{SU(2)}^{\{1,2,0\}} &= -\frac{1}{4}t^{-\frac{3}{2}} - \frac{1}{4}\chi_{\bar{3}}t^{-\frac{1}{2}} + \frac{3}{2} - \frac{1}{4}(\chi_{\bar{6}} + 2\chi_{\mathbf{3}})t^{\frac{1}{2}} + 2\chi_{\bar{3}}t - \frac{1}{4}(\chi_{\bar{10}} + 2\chi_{\mathbf{8}} + 17)t^{\frac{3}{2}} \\
&\quad + (2\chi_{\bar{6}} + 3\chi_{\mathbf{3}})t^2 - \frac{1}{4}(\chi_{\bar{15}} + 2\chi_{\bar{15}'} + 3\chi_{\mathbf{6}} + 30\chi_{\bar{3}})t^{\frac{5}{2}} + 2(\chi_{\bar{10}} + 2\chi_{\mathbf{8}} + 5)t^3 \\
&\quad - \frac{1}{4}(\chi_{\bar{21}} + 2\chi_{\bar{24}} + 3\chi_{\bar{15}'} + 31\chi_{\bar{6}} + 40\chi_{\mathbf{3}})t^{\frac{7}{2}} + (2\chi_{\bar{15}} + 4\chi_{\bar{15}'} + \frac{9}{2}\chi_{\mathbf{6}} + 20\chi_{\bar{3}})t^4 + \mathcal{O}(t^{\frac{9}{2}}) \\
\mathcal{I}_{SU(2)}^{\{1,2,1\}} &= +\frac{1}{4}t^{-\frac{3}{2}} + \frac{1}{4}\chi_{\bar{3}}t^{-\frac{1}{2}} + \frac{3}{2} + \frac{1}{4}(\chi_{\bar{6}} + 2\chi_{\mathbf{3}})t^{\frac{1}{2}} + 2\chi_{\bar{3}}t + \frac{1}{4}(\chi_{\bar{10}} + 2\chi_{\mathbf{8}} + 17)t^{\frac{3}{2}} \\
&\quad + (2\chi_{\bar{6}} + 3\chi_{\mathbf{3}})t^2 + \frac{1}{4}(\chi_{\bar{15}} + 2\chi_{\bar{15}'} + 3\chi_{\mathbf{6}} + 30\chi_{\bar{3}})t^{\frac{5}{2}} + 2(\chi_{\bar{10}} + 2\chi_{\mathbf{8}} + 5)t^3 \\
&\quad + \frac{1}{4}(\chi_{\bar{21}} + 2\chi_{\bar{24}} + 3\chi_{\bar{15}'} + 31\chi_{\bar{6}} + 40\chi_{\mathbf{3}})t^{\frac{7}{2}} + (2\chi_{\bar{15}} + 4\chi_{\bar{15}'} + \frac{9}{2}\chi_{\mathbf{6}} + 20\chi_{\bar{3}})t^4 + \mathcal{O}(t^{\frac{9}{2}})
\end{aligned} \tag{3.15}$$

We have verified that the sum of these three contributions reproduces the expansion (3.13) of the integral expression, up to order $\mathcal{O}(t^{10})$.

4 The case of $SU(3)$

We consider now the case of gauge group $SU(3)$. We define $u \equiv u_{21}$ and $v \equiv u_{31}$. The Weyl group S_3 of the $SU(3)$ gauge group permutes the triplet (u_1, u_2, u_3) while it acts in the following way on (u, v) :

$$\begin{aligned}
s_{12} : (u, v) &\mapsto (-u, v - u) & s_R : (u, v) &\mapsto (v - u, -u) \\
s_{13} : (u, v) &\mapsto (u - v, -v) & s_L : (u, v) &\mapsto (-v, u - v) \\
s_{23} : (u, v) &\mapsto (v, u) .
\end{aligned} \tag{4.1}$$

The three reduced BAEs (2.23) are mapped one into the others by the action of the Weyl group, therefore let us discuss how to manipulate the first of them. Introducing the function f defined in (3.2), we can recast that equation in the form⁶

$$0 = Q_1(u, v; \Delta, \tau) - 1 = \frac{f(u; \Delta, \tau) f(v; \Delta, \tau) - f(-u; \Delta, \tau) f(-v; \Delta, \tau)}{f(-u; \Delta, \tau) f(-v; \Delta, \tau)} . \tag{4.2}$$

Using the identity (3.4), the numerator becomes

$$\text{num} = \left[c_1 \theta_1(2u) \sum_{k=2,3,4} c_k \theta_k(2v) \right] + \left[c_1 \theta_1(2v) \sum_{k=2,3,4} c_k \theta_k(2u) \right] . \tag{4.3}$$

⁶Notice that the reduced BAEs concern the antisymmetric part of the product of two f 's. This generalizes to any N : the solutions to the reduced BAEs can always be written as the zeros of the antisymmetric part of the product of $N - 1$ functions f .

Collecting terms with the same coefficient c_i and using the identities (A.6), we rewrite it as

$$\begin{aligned} \text{num} = \frac{c_1 \theta_1(u+v)}{\theta_2(0) \theta_3(0) \theta_4(0)} & \left[[3] \theta_3(u+v) \theta_2(u-v) \theta_4(u-v) \right. \\ & \left. - [4] \theta_4(u+v) \theta_2(u-v) \theta_3(u-v) - [2] \theta_2(u+v) \theta_3(u-v) \theta_4(u-v) \right], \end{aligned} \quad (4.4)$$

where we defined $[r] = \theta_r(0) \theta_r(\Delta_1) \theta_r(\Delta_2) \theta_r(\Delta_1 + \Delta_2)$. They obey $[3] = [4] + [2]$ [58], then using (A.6) again we obtain

$$\begin{aligned} \text{num} = \frac{2 c_1 \theta_1(u+v) \theta_1(u) \theta_1(v)}{\theta_2(0)^2 \theta_3(0)^2 \theta_4(0)^2} & \times \\ & \times \left[[2] \theta_4(0) \theta_4(u) \theta_4(v) \theta_4(u-v) - [4] \theta_2(0) \theta_2(u) \theta_2(v) \theta_2(u-v) \right]. \end{aligned} \quad (4.5)$$

This allows us to rewrite the first reduced BAE as

$$0 = Q_1 - 1 = \frac{2 c_1 \theta_1(u+v) h(u, v; \Delta, \tau)}{\theta_2(0) \theta_3(0)^2 \theta_4(0) \prod_{\Delta \in \{\Delta_1, \Delta_2, -\Delta_1 - \Delta_2\}} \theta_1(\Delta - u) \theta_1(\Delta - v)} \quad (4.6)$$

where we defined the function

$$h(u, v; \Delta, \tau) = \theta_2(\Delta_1) \theta_2(\Delta_2) \theta_2(\Delta_1 + \Delta_2) \theta_4(u) \theta_4(v) \theta_4(u-v) - (2 \leftrightarrow 4). \quad (4.7)$$

Crucially, this function is invariant under the gauge Weyl group. The second reduced BAE can be obtained by acting with the gauge Weyl group on the first one. Since the denominator of (4.6) does not have poles, the reduced BAEs can be brought to the factorized form

$$\begin{cases} 0 = \theta_1(u+v) h(u, v; \Delta, \tau) \\ 0 = \theta_1(2u-v) h(u, v; \Delta, \tau) \end{cases} \quad (4.8)$$

From here we clearly see the structure of the solutions.

First, there are discrete solutions that follow from solving

$$0 = \theta_1(u+v) = \theta_1(2u-v). \quad (4.9)$$

These equations represent hyperplanes inside T^2 , and the discrete solutions are at the intersections of two hyperplanes. There are four solutions on T^2 , up to identifying those that are related by the Weyl group action and dropping those that are fixed by some non-trivial element of the Weyl group (*i.e.*, solutions in which either u or v vanishes):

$$(u, v) = \left(\frac{1}{3}, \frac{2}{3} \right), \left(\frac{\tau}{3}, \frac{2\tau}{3} \right), \left(\frac{\tau+1}{3}, \frac{2\tau+2}{3} \right), \left(\frac{\tau+2}{3}, \frac{2\tau+1}{3} \right). \quad (4.10)$$

These are precisely the 4 HL solutions $\{3, 1, 0\}$, $\{1, 3, 0\}$, $\{1, 3, 1\}$, $\{1, 3, 2\}$, respectively, according to (2.25).

Second, there is a continuous family of solutions obtained by solving the single equation

$$0 = h(u, v; \Delta, \tau). \quad (4.11)$$

We will analyze some properties of these solutions below, but we already see that the continuous family has complex dimension 1. The solutions to (4.11), as opposed to the HL solutions, depend on the flavor fugacities Δ_a . There are however six special points on the curve,⁷ all related by the action of the S_3 Weyl group, that do not depend on Δ_a :

$$(u, v) = \left(\frac{1}{2}, \frac{\tau}{2} \right). \quad (4.12)$$

These are smooth points, which do not seem to have any other special property.

Evaluation of the index. The Bethe Ansatz formula (2.14) was derived in [50, 51] assuming that all solutions to the BAEs are isolated. It is then clear that, in the presence of continuous families of solutions as it happens in this case, the formula has to be modified somehow in order to correctly evaluate the contribution of those families. We could hope that, for some reason, continuous families do not contribute. Let us test this possibility, by comparing the results of the integral formula (2.7) with that of the BA formula (2.14), in which the sum is restricted to isolated solutions to the BAEs (here the HL solutions), in the low-temperature limit.

The expansion of the integral formula (2.7), setting the flavor fugacities $v_1 = v_2 = 1$, gives

$$\mathcal{I}_{SU(3)} = 1 + 6t^4 - 6t^5 + 3t^6 + 6t^7 - 16t^9 + 27t^{10} + 18t^{11} - 87t^{12} + 96t^{13} + 54t^{14} + \mathcal{O}(t^{15}), \quad (4.13)$$

while switching on the $SU(3)$ fugacities it gives

$$\begin{aligned} \mathcal{I}_{SU(3)} = & 1 + \chi_{\mathbf{6}}t^4 - 2\chi_{\mathbf{3}}t^5 + (\chi_{\mathbf{10}} - \chi_{\mathbf{8}} + 1)t^6 + 2\chi_{\mathbf{\bar{3}}}t^7 + (\chi_{\mathbf{15}} - \chi_{\mathbf{15}'} + \chi_{\mathbf{\bar{6}}} - 2\chi_{\mathbf{3}})t^8 \\ & - 2\chi_{\mathbf{8}}t^9 + (\chi_{\mathbf{21}} - \chi_{\mathbf{15}'} + \chi_{\mathbf{6}} + 5\chi_{\mathbf{\bar{3}}})t^{10} + (2\chi_{\mathbf{15}'} - 2\chi_{\mathbf{15}} + 4\chi_{\mathbf{\bar{6}}} - 2\chi_{\mathbf{3}})t^{11} + \mathcal{O}(t^{12}). \end{aligned} \quad (4.14)$$

⁷These points have been studied in [57] in the context of the $\mathcal{N} = 1^*$ theory on S^1 . In the context of the Bethe Ansatz equations, they have been noticed in [1].

The expansion of the contributions (3.9) of the HL solutions to the BA formula gives:

$$\begin{aligned}
\mathcal{I}_{SU(3)}^{\{3,1,0\}} &= 3 + 6\chi_{\bar{3}}t + 3(5\chi_{\bar{6}} - 2\chi_{\mathbf{3}})t^2 + 6(3\chi_{\bar{10}} + 2\chi_{\mathbf{8}} - 6)t^3 \\
&\quad + 3(10\chi_{\bar{15}} - \chi_{\bar{15}'} + 24\chi_{\mathbf{6}} - 13\chi_{\bar{3}})t^4 + 6(5\chi_{\bar{21}} + 6\chi_{\bar{24}} + 8\chi_{\mathbf{15}'} - 22\chi_{\bar{6}} + 30\chi_{\mathbf{3}})t^5 \\
&\quad + 3(15\chi_{\bar{28}} - \chi_{\bar{35}} + 16\chi_{\bar{24}} + 71(\chi_{\mathbf{27}} + \chi_{\mathbf{24}}) - 99\chi_{\bar{10}} - 69\chi_{\mathbf{10}} - 85\chi_{\mathbf{8}} + 164)t^6 + \mathcal{O}(t^7) \\
\mathcal{I}_{SU(3)}^{\{1,3,0\}} &= \frac{\chi_{\mathbf{3}} - \chi_{\bar{3}}}{9(\chi_{\mathbf{6}} - 6\chi_{\mathbf{3}} + \chi_{\bar{3}} + 9)} t^{-4} \\
&\quad + \frac{2(\chi_{\mathbf{15}'} - \chi_{\bar{15}'} + 6\chi_{\mathbf{8}} - 7\chi_{\mathbf{6}} + \chi_{\bar{6}} + 11\chi_{\mathbf{3}} - 17\chi_{\bar{3}} + 6)}{9(\chi_{\mathbf{10}} + 2\chi_{\mathbf{8}} - 9\chi_{\mathbf{6}} - 9\chi_{\bar{3}} + 27\chi_{\mathbf{3}} - 26)} t^{-3} + \mathcal{O}(t^{-2}) \\
\mathcal{I}_{SU(3)}^{\{1,3,1\}} &= \frac{\chi_{\bar{3}} + e^{\pi i/3}\chi_{\mathbf{3}}}{9(6\chi_{\mathbf{3}} + e^{-\pi i/3}(\chi_{\mathbf{6}} + \chi_{\bar{3}}) + 9e^{\pi i/3})} t^{-4} \\
&\quad - \frac{2(\chi_{\mathbf{15}'} + \chi_{\bar{6}} + 11\chi_{\mathbf{3}} - e^{\pi i/3}(6\chi_{\mathbf{8}} + 6) + e^{-\pi i/3}(\chi_{\bar{15}'} + 7\chi_{\mathbf{6}} + 17\chi_{\bar{3}}))}{9(e^{\pi i/3}(\chi_{\mathbf{10}} + 2\chi_{\mathbf{8}} - 26) - 9e^{-\pi i/3}(\chi_{\mathbf{6}} + \chi_{\bar{3}}) - 27\chi_{\mathbf{3}})} t^{-3} + \mathcal{O}(t^{-2})
\end{aligned} \tag{4.15}$$

while $\mathcal{I}_{SU(3)}^{\{1,3,2\}}$ is obtained from $\mathcal{I}_{SU(3)}^{\{1,3,1\}}$ by taking the complex conjugate of all coefficients. Notice that $\mathcal{I}_{SU(3)}^{\{1,3,0\}}$ does not have a well-defined limit as $v_{1,2} \rightarrow 1$.

One immediately sees that the sum of the contributions from the 4 HL solutions does not reproduce the index. In particular, there remain negative powers of t , fractional coefficients and characters in the denominators. This implies that the contribution from the continuous family of solutions is necessary in order to correctly reproduce the index. It is an open issue how to compute the contribution from the continuous family, to which we hope to return in future work.

4.1 The continuous family of solutions

Let us study in more detail the curve $h(u, v; \Delta, \tau) = 0$, for generic values of the flavor chemical potentials Δ_a . We rewrite the equation as

$$\frac{\theta_4(u)\theta_4(v)\theta_4(u-v)}{\theta_2(u)\theta_2(v)\theta_2(u-v)} = \frac{\theta_4(\Delta_1)\theta_4(\Delta_2)\theta_4(\Delta_1 + \Delta_2)}{\theta_2(\Delta_1)\theta_2(\Delta_2)\theta_2(\Delta_1 + \Delta_2)}. \tag{4.16}$$

If we regard the left-hand side as a function of u (for fixed v), we see that it is doubly-periodic with periods 1 and τ , with two simple poles at $u = \frac{1}{2}$ and $u = v + \frac{1}{2}$, and two simple zeros at $u = \frac{\tau}{2}$ and $u = v + \frac{\tau}{2}$, *i.e.*, it is an elliptic function of order 2. This means that it gives two values $u_{\pm}(v)$ of u for each value of v , and they satisfy $u_+ + u_- = v \pmod{\mathbb{Z} + \mathbb{Z}\tau}$. In other words we have a branched double cover of the torus.

The equation can be formally solved in terms of Jacobi elliptic functions (see for instance [59]). Define the new variables $z = 2K(m)u$, $\omega = 2K(m)v$ and $z_i = 2K(m)\Delta_i$, where $K(m)$

is the complete elliptic integral of the first kind, and m is the parameter (see Appendix A). Then the curve can be rewritten as

$$\operatorname{cn}(z) \operatorname{cn}(\omega) \operatorname{cn}(z - \omega) = \operatorname{cn}(z_1) \operatorname{cn}(z_2) \operatorname{cn}(z_1 + z_2). \quad (4.17)$$

We use the Jacobi elliptic functions $\operatorname{cn}(z, m)$, $\operatorname{sn}(z, m)$, $\operatorname{dn}(z, m)$ and keep the dependence on the parameter m implicit. Then, employing addition formulas, we can isolate the dependence on z :

$$\operatorname{sn}^2\left(z - \frac{\omega}{2}\right) = \frac{\operatorname{cn}(z_1) \operatorname{cn}(z_2) \operatorname{cn}(z_1 + z_2) - \operatorname{cn}^2(\omega/2) \operatorname{cn}(\omega)}{m \operatorname{sn}^2(\omega/2) \operatorname{cn}(z_1) \operatorname{cn}(z_2) \operatorname{cn}(z_1 + z_2) - \operatorname{dn}^2(\omega/2) \operatorname{cn}(\omega)}. \quad (4.18)$$

The two values for u can finally be written in terms of the inverse function arcsn :

$$u_{\pm}(v) = \frac{v}{2} \pm \frac{1}{2K(m)} \operatorname{arcsn} \left(\sqrt{\frac{\operatorname{cn}(z_1) \operatorname{cn}(z_2) \operatorname{cn}(z_1 + z_2) - \operatorname{cn}^2(\omega/2) \operatorname{cn}(\omega)}{m \operatorname{sn}^2(\omega/2) \operatorname{cn}(z_1) \operatorname{cn}(z_2) \operatorname{cn}(z_1 + z_2) - \operatorname{dn}^2(\omega/2) \operatorname{cn}(\omega)}}} \right). \quad (4.19)$$

The two equations $u = u_{\pm}(v)$ describe the curve as a double cover of the torus. The two sheets of the covering meet at all points where $u_+(v) = u_-(v) = v/2$. We could ask whether these are smooth points where the covering is branched, or rather are self-intersection points of the curve or intersection points of two connected components the curve is composed of. We can see that these must be smooth branch points with a simple argument. The points lie on the hyperplane $\theta_1(2u - v) = 0$ on which $Q_2 = 1$. However, we could have equivalently described the curve as $v = v_{\pm}(u)$. If those points were intersection points, and so if they had an intrinsic geometric characterization, they would also solve $v_+(u) = v_-(u) = u/2$ and so would also lie on the hyperplane $\theta_1(u - 2v) = 0$ on which $Q_3 = 1$. However the intersections of the two hyperplanes (which intersect also the hyperplane $\theta_1(u + v) = 0$) are precisely the HL solutions, and one can easily check that they do not lie on the curve (4.11). This contradiction implies that there are no special intersection points on the curve, only smooth branch points. In particular this implies that the curve is composed of a single connected component.

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A Special functions

We use fugacities and chemical potentials related by

$$z = e^{2\pi i u} , \quad p = e^{2\pi i \sigma} , \quad q = e^{2\pi i \tau} , \quad (\text{A.1})$$

with $|p|, |q| < 1$. The q -Pochhammer symbol is defined as

$$(z; q)_\infty = \prod_{j=0}^{\infty} (1 - zq^j) . \quad (\text{A.2})$$

The elliptic theta function θ_0 is defined as

$$\theta_0(u; \tau) = (z; q)_\infty (q/z; q)_\infty . \quad (\text{A.3})$$

The elliptic gamma function is defined as

$$\tilde{\Gamma}(u; \sigma, \tau) = \prod_{m,n=0}^{\infty} \frac{1 - p^{m+1} q^{n+1} / z}{1 - p^m q^n z} . \quad (\text{A.4})$$

We use the following definitions for the Jacobi theta functions:

$$\begin{aligned} \theta_1(u; \tau) &= 2 e^{\pi i \tau / 4} \sin(\pi u) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n z)(1 - q^n / z) = i e^{\pi i \tau / 4 - \pi i u} (q; q)_\infty \theta_0(u; \tau) \\ \theta_2(u; \tau) &= 2 e^{\pi i \tau / 4} \cos(\pi u) \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n z)(1 + q^n / z) = e^{\pi i \tau / 4 - \pi i u} (q; q)_\infty \theta_0(u + \frac{1}{2}; \tau) \\ \theta_3(u; \tau) &= \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-\frac{1}{2}} z)(1 + q^{n-\frac{1}{2}} / z) = (q; q)_\infty \theta_0(u + \frac{1}{2} + \frac{\tau}{2}; \tau) \\ \theta_4(u; \tau) &= \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-\frac{1}{2}} z)(1 - q^{n-\frac{1}{2}} / z) = (q; q)_\infty \theta_0(u + \frac{\tau}{2}; \tau) . \end{aligned} \quad (\text{A.5})$$

Notice that θ_1 is odd while $\theta_{2,3,4}$ are even under $u \rightarrow -u$. One set of identities we use is

$$\begin{aligned} \theta_1(u+v) \theta_2(u-v) \theta_3(0) \theta_4(0) &= \theta_1(u) \theta_2(u) \theta_3(v) \theta_4(v) + \theta_3(u) \theta_4(u) \theta_1(v) \theta_2(v) \\ \theta_1(u+v) \theta_3(u-v) \theta_2(0) \theta_4(0) &= \theta_1(u) \theta_3(u) \theta_2(v) \theta_4(v) + \theta_2(u) \theta_4(u) \theta_1(v) \theta_3(v) \\ \theta_1(u+v) \theta_4(u-v) \theta_2(0) \theta_3(0) &= \theta_1(u) \theta_4(u) \theta_2(v) \theta_3(v) + \theta_2(u) \theta_3(u) \theta_1(v) \theta_4(v) \\ \theta_2(u+v) \theta_3(u-v) \theta_2(0) \theta_3(0) &= \theta_2(u) \theta_3(u) \theta_2(v) \theta_3(v) - \theta_1(u) \theta_4(u) \theta_1(v) \theta_4(v) \\ \theta_3(u+v) \theta_4(u-v) \theta_3(0) \theta_4(0) &= \theta_3(u) \theta_4(u) \theta_3(v) \theta_4(v) - \theta_1(u) \theta_2(u) \theta_1(v) \theta_2(v) , \end{aligned} \quad (\text{A.6})$$

see for instance [58]. Here $\theta_r(u) \equiv \theta_r(u; \tau)$.

The Jacobi elliptic functions are related to the Jacobi theta functions by

$$\begin{aligned} \operatorname{cn}(2K(m)u, m) &= \left(\frac{1-m}{m}\right)^{1/4} \frac{\theta_2(u; \tau)}{\theta_4(u; \tau)} \\ \operatorname{sn}(2K(m)u, m) &= \left(\frac{1}{m}\right)^{1/4} \frac{\theta_1(u; \tau)}{\theta_4(u; \tau)} \\ \operatorname{dn}(2K(m)u, m) &= (1-m)^{1/4} \frac{\theta_3(u; \tau)}{\theta_4(u; \tau)}. \end{aligned} \tag{A.7}$$

The parameter m is related to the modulus τ by

$$m(\tau) = \frac{\theta_2(0; \tau)^4}{\theta_3(0; \tau)^4}. \tag{A.8}$$

The function $K(m)$ is the complete elliptic integral of the first kind, and is related to τ by

$$\tau = i \frac{K(1-m)}{K(m)} \quad \text{or} \quad 2K(m) = \frac{\theta_3(0; \tau)^2}{\pi}. \tag{A.9}$$

We use the addition formula

$$\operatorname{cn}(x+y, m) \operatorname{cn}(x-y, m) = \frac{\operatorname{cn}^2(y, m) - \operatorname{sn}^2(x, m) \operatorname{dn}^2(y, m)}{1 - m \operatorname{sn}^2(x, m) \operatorname{sn}^2(y, m)}. \tag{A.10}$$

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