Poincaré-Korn and Korn inequalities for functions with small jump set

(Some results on Dal Maso's *GSBD* functions).

Antonin Chambolle

CMAP, Ecole Polytechnique, CNRS

with S. Conti (Bonn), G. Francfort (Paris-Nord), V. Crismale (CMAP), F. Cagnetti (Sussex), L. Scardia (HW)

Calculus of Variations and Applications: una conferenza per i 65 anni di Gianni Dal Maso (27 Jan.-1 Feb 2020)

Continuity of Neumann linear elliptic problems on varying two-dimensional bounded open sets

Antonin Chambolle and Francesco Doveri

SISSA, Trieste, Italy

January 27, 1996

Continuity of Neumann linear elliptic problems on varying two-dimensional bounded open sets

Antonin Chambolle and Francesco Doveri

SISSA, Trieste, Italy

January 27, 1996

Continuity of Neumann linear elliptic problems on varying two-dimensional bounded open sets

Antonin Chambolle and Francesco Doveri

SISSA, Trieste, Italy

January 27, 1996

Outline

- Displacements with fractures and Gianni's *GSBD* space
- Poincaré and Korn inequalities:
- Known results
- A new result
- Ideas of proof

Displacements with discontinuities

Let $\Omega \subset \mathbb{R}^d$, connected, Lipschitz, $d \geq 2$. Let $K \subset \Omega$ a closed set with with $\mathcal{H}^{d-1}(K) < +\infty$, and $u : \Omega \to \mathbb{R}^d$ measurable such that

$$e(u) := \frac{Du + (Du)^T}{2} \in L^p(\Omega \setminus K)$$

for some $p \in]1, +\infty[$.

- Such functions (for $p = 2$) are in the "energy space" of the "Griffith Energy"

$$\mathcal{E}(u, K) = \int_{\Omega \setminus K} \mathbb{C}e(u) : e(u)dx + \gamma \mathcal{H}^{d-1}(K)$$

  introduced by Francfort and Marigo (1998) in a variational model for fracture growth in the context of linearized elasticity.
- K is the fracture, $e(u)$ the *infinitesimal strain*, $\mathbb{C}$ = "Hooke's law" which expresses the stress in term of the strain, $\gamma > 0$ a parameter called "toughness".

- Natural question: what control does one have on u? on ∇u?

Korn, Poincaré-Korn

If $\mathcal{H}^{d-1}(K) = 0$ and Ω is Lipschitz, one has the well known Korn inequality: $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ and ($p > 1$)

$$\|\nabla u\|_p \leq c(\|e(u)\|_p + \|u\|_p), \qquad (K)$$

as well as (if Ω connected)

$$\|\nabla u - A\|_p \leq c\|e(u)\|_p \qquad (K')$$

for some skew-symmetric A.
(As a consequence,) one also has (if $p < d$)

$$\|u - a\|_{p^*} \leq c\|e(u)\|_p \qquad (PK)$$

for a an "infinitesimal rigid motion", that is, affine with $a(x) = Ax + b$, $A + A^T = 0$, and $p^* = pd/(d-p)$.

Korn, Poincaré-Korn

When $\mathcal{H}^{d-1}(K) > 0$ one has therefore $u \in W^{1,p}_{loc}(\Omega \setminus K)$, but what control can we hope? In particular if $\mathcal{H}^{d-1}(K) << 1$?

More general situation: (Dal Maso, 2011) $u \in GSBD^p(\Omega)$:

- J_u, the intrinsic jump set, is just a countably $(d-1)$-rectifiable set with $\mathcal{H}^{d-1}(J_u) < +\infty$,
- and $e(u) \in L^p$ an "approximate symmetrized gradient".

This space was introduce by Gianni in 2011 as the right **energy space** for Griffith's Energy, extending "$SBD^p(\Omega)$" towards functions with possibly unbounded jumps $\to$ *existence.*

Defined by requiring some control on 1D slices.

In such a space it is not even clear that ∇u exists, so what would "Korn's inequality" mean?...

Known results for $BD/SBD/G(S)BD$

Older results: study of BD, SBD (Suquet 78, Matthies et al 79):

- Kohn's PhD thesis (79) (jumps and singularities)
- Bellettini-Coscia (93) (slicing)
- Bellettini-Coscia-Dal Maso (98) (compactness in SBD)
- Ambrosio-Coscia-Dal Maso (97), Hajłasz (96) (fine properties)
  - Weak L^1 estimate on ∇u

Recent results on Korn / Poincaré-Korn by

- C.-Conti-Francfort (2014/16)
- Friedrich (2015, 16-18, several results)
- Conti-Focardi-Iurlano (2015)

Known results

[A.C., S. Conti, G. Francfort (IUMJ 2016)]: there exists $\omega \subset \Omega$ with $|\omega| \leq c\mathcal{H}^{d-1}(J_u)^{d/(d-1)}$ and a infinitesimal rigid motion with

$$\int_{\Omega\setminus\omega} |u-a|^{pd/(d-1)} dx \leq c \int_\Omega |e(u)|^p dx$$

- No estimate on $\partial\omega$;
- No estimate on ∇u;
- Exponent $< p^*$.

Known results

Series of results by M. Friedrich (2015–18):

- Case $p = 2$, $d = 2$: control of the perimeter $\mathcal{H}^1(\partial^*\omega) \le c\mathcal{H}^1(J_u)$, and of $\nabla u - A$ in $\Omega \setminus \omega$, at the expense of losing a bit in the exponents (< 2 and $< 2^* = \infty$) (preprint 2015);
- $p = d = 2$, "Piecewise Korn" with a control of $\nabla u - \sum_i A_i \chi_{P_i}$ (preprint 2016-2018);
- $d \ge 2$, $p = 2$: control of the perimeter with $\sqrt{\mathcal{H}^{d-1}(J_u)}$, control of $\|\nabla u\|_{L^1}$ out of ω (same preprint);
- $SBD^2 \cap L^\infty \subset SBV$: control of $\|\nabla u\|_1$ if $e(u) \in L^2$, $u \in L^\infty$; $GSBD^2 \subset GBV$ (same).

Applications: with F. Solombrino, existence of quasistatic fracture evolutions in 2D.

Known results

Conti-Focardi-Iurlano (2015), show, for any $p \in (1, \infty)$ and in dimension $d = 2$, given $u \in GSBD^p(\Omega)$,

- that $u = v \in W^{1,p}(\Omega; \mathbb{R}^d)$ except on an exceptional set ω;
- with $Per(\omega) \leq c\mathcal{H}^1(J_u)$ and $\|e(v)\|_p \leq c\|e(u)\|_p$;
- hence Korn (K') and Poincaré-Korn ((PK), with p^*) hold in $\Omega \setminus \omega$.

Application: integral representation of some energies (2015); density estimates for weak minimizers (hence strong) of Griffith's energy (2016).

Extension to higher dimension

With F. Cagnetti (Sussex, Brighton), L. Scardia (HW, Edinburgh)

Theorem. Let $u \in GSBD^p(\Omega)$: there exists ω (small) with $Per(\omega) \leq c\mathcal{H}^{d-1}(J_u)$ and $v \in W^{1,p}(\Omega;\mathbb{R}^d)$ with $u = v$ in $\Omega \setminus \omega$ and $\|e(v)\|_p \leq c\|e(u)\|_p$.
In particular (as (K') and (PK) hold for v):

$$\|\nabla u - A\|_{L^p(\Omega\setminus\omega)} \leq c\|e(u)\|_{L^p(\Omega)}$$
$$\|u - a\|_{L^{p^*}(\Omega\setminus\omega)} \leq c\|e(u)\|_{L^p(\Omega)}.$$

Here "∇u" is the approximate gradient of u which coincides with ∇v a.e. out of ω. (The result is for $p < d$, if $p > d$ we get that u coincides with a Hölder function out of ω.)

Applications?

- Up to now mostly a few remarks:
  - An approximation result for $GSBD^p$ functions (a variant of a recent result with V. Crismale, where now the jump is mostly untouched and $u_n = u$ in most of the domain);
  - The observation that ∇u (the approximate gradient) exists a.e. (as for BD functions).

Idea of proof

- Relies on [CCF 16], a "cleaning lemma" in [CCI 17], and the construction in [Conti, Focardi, Iurlano 15] who have first shown this in $2D$.

A technical detail of [CCF 16]

Theorem [A.C., S. Conti, G. Francfort (IUMJ 2016)]: Let $\delta > 0$ $\theta > 0$, $Q = (-\delta, \delta)^d$, $Q' = (1+\theta)Q$, $Q'' = (1+2\theta)Q$, $p \in (1, \infty)$, $u \in GSBD_p(Q'')$. There exists $c(\theta, p, d) > 0$ such that
1. There exists $\omega \subset Q'$ and an affine function $a : \mathbb{R}^d \to \mathbb{R}^d$ with $e(a) = 0$ ($a = Ax + b$, $A + A^T = 0$) such that:

$$|\omega| \le c\delta \mathcal{H}^{d-1}(J_u)$$

$$\|u - a\|_{L^{dp/(d-1)}(Q' \setminus \omega)} \le c\delta^{1-1/d} \|e(u)\|_{L^p(Q'')}.$$

A technical detail of [CCF 16]

Theorem [A.C., S. Conti, G. Francfort (IUMJ 2016)]: Let $\delta > 0$ $\theta > 0$, $Q = (-\delta, \delta)^d$, $Q' = (1+\theta)Q$, $Q'' = (1+2\theta)Q$, $p \in (1, \infty)$, $u \in GSBD_p(Q'')$. There exists $c(\theta, p, d) > 0$ such that

1. There exists $\omega \subset Q'$ and an affine function $a : \mathbb{R}^d \to \mathbb{R}^d$ with $e(a) = 0$ ($a = Ax + b$, $A + A^T = 0$) such that:

$$|\omega| \leq c\delta \mathcal{H}^{d-1}(J_u)$$

$$\|u - a\|_{L^{dp/(d-1)}(Q' \setminus \omega)} \leq c\delta^{1-1/d} \|e(u)\|_{L^p(Q'')}.$$

2. Letting $v = u\chi_{Q' \setminus \omega} + a\chi_\omega$, and for ϕ a smooth symmetric mollifier with support in $B(0, \theta/2)$,

$$\int_Q |e(v * \phi_\delta) - e(u) * \phi_\delta|^p dx \leq c \left(\frac{\mathcal{H}^{d-1}(J_u)}{\delta^{d-1}} \right)^s \int_{Q''} |e(u)|^p dx$$

for some exponent $s = s(p, d) > 0$.

Detail of [CCF16]

- The proof relies heavily on slicing;
- For $GSBD^p$ functions we use that for a.e. $x, y \in \Omega$, if $[x, y] \cap J_u = \emptyset$, then *(if u is smooth out of J_u)*

$$(u(y) - u(x)) = \int_0^1 \nabla u(x + s(y - x))(y - x) \qquad ds$$

Detail of [CCF16]

- The proof relies heavily on slicing;
- For $GSBD^p$ functions we use that for a.e. $x, y \in \Omega$, if $[x, y] \cap J_u = \emptyset$, then *(if u is smooth out of J_u)*

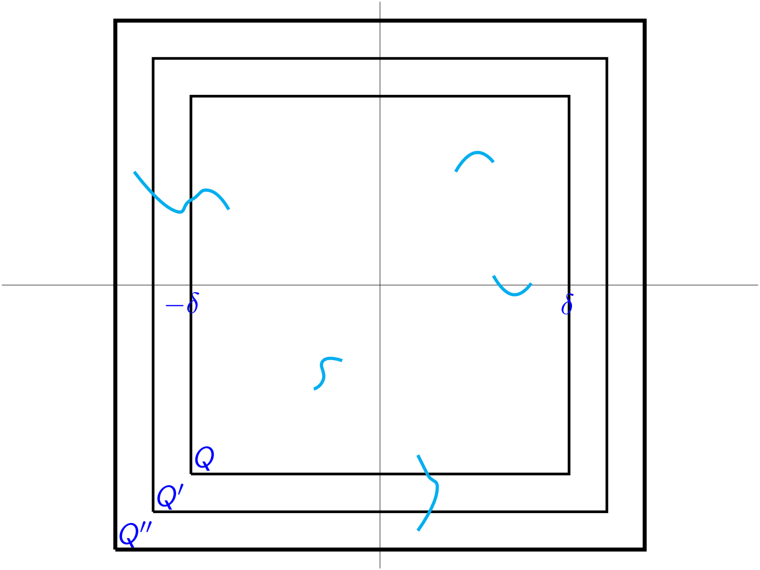
$$
\begin{aligned}
(u(y) - u(x))\cdot(y - x) \\
&= \int_0^1 \nabla u(x + s(y - x))(y - x)\cdot(y - x) ds
\end{aligned}
$$

Detail of [CCF16]

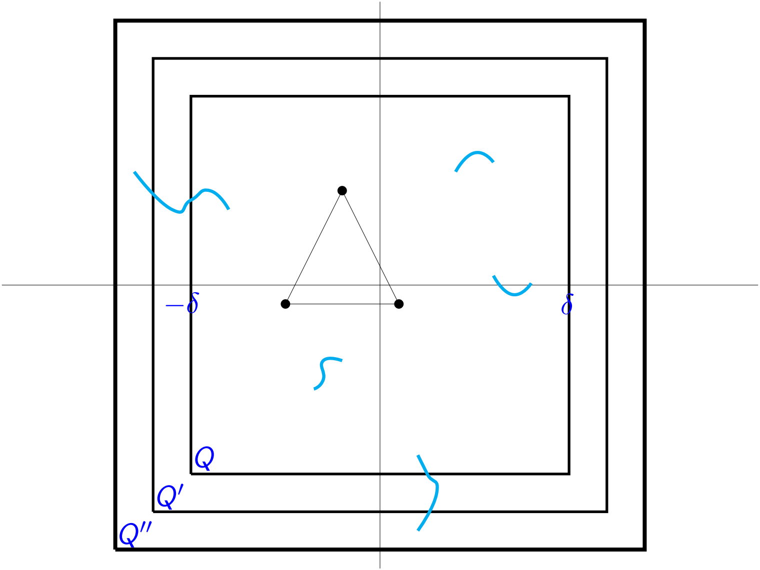
- The proof relies heavily on slicing;
- For $GSBD^p$ functions we use that for a.e. $x, y \in \Omega$, if $[x, y] \cap J_u = \emptyset$, then

$$
\begin{aligned}
(u(y) - u(x)) \cdot (y - x) \\
&= \int_0^1 e(u)(x + s(y - x))(y - x) \cdot (y - x) ds
\end{aligned}
$$

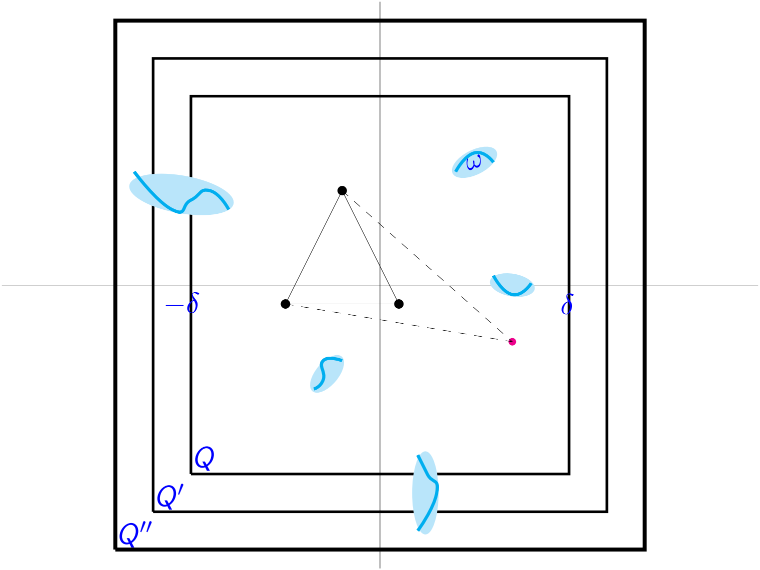
Detail of [CCF 16]



Detail of [CCF 16]



Detail of [CCF 16]



Detail of [CCF16]

- Many applications, such as:
  - A Γ-convergence result for a phase-field approximation of Griffith's energy with a constraint of non-interpenetration in 2D (C-Conti-Francfort)
  - weak minimizers of Griffith are strong in any dimension (C-Conti-Iurlano);
  - compactness and lower semicontinuity in *GSBD* (C-Crismale);
  - existence of strong minimizers for Griffith's Dirichlet problem (C-Crismale)

A first consequence: cleaning lemma

The following is derived from the previous Theorem (cf [C-Conti-Iurlano, 17])

Lemma There exists $\bar{\delta} > 0$ (d, p) such that For every $u \in GSBD^p(B_1)$ with $\delta := \mathcal{H}^{d-1}(J_u)^{1/d} \leq \bar{\delta}$, there is $1 - \sqrt{\delta} < R < 1$ and $\tilde{u} \in GSBD^p(B_1)$ with

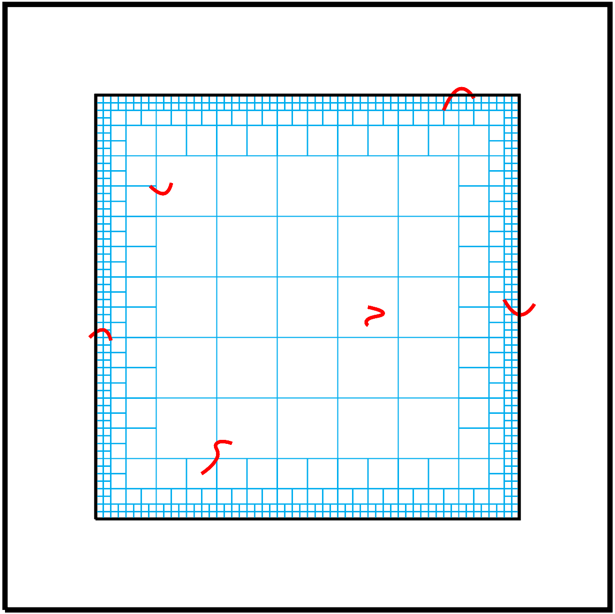
- $\tilde{u} \in C^\infty(B_{1-\sqrt{\delta}})$, $\tilde{u} = u$ in $B_1 \setminus B_R$;
- $\mathcal{H}^{d-1}(J_{\tilde{u}} \setminus J_u) \leq c\sqrt{\delta}\mathcal{H}^{d-1}(J_u \cap B_1 \setminus B_{1-\sqrt{\delta}})$;
- $\int_{B_1} |e(\tilde{u})|^p dx \leq (1 + c\delta^s) \int_{B_1} |e(u)|^p dx$.

(For some $s > 0$, and $c > 0$.)

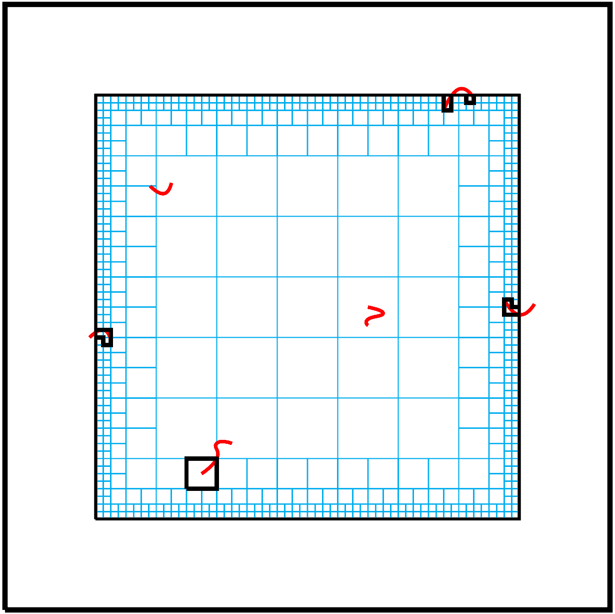
Cleaning lemma: proof

- Pick $R \in (1-\sqrt{\delta}, 1)$ such that there is not too much jump in $B_R \setminus B_{R-2\delta}$;
- Cover most of $B_{R-\delta}$ with cubes of size δ, then build a Whitney covering of B_R by cubes of size $\delta 2^{-k}$ at distance of same order from ∂B_R;
- In "good cubes" with little jump, apply the previous theorem to find ω, a and smooth $u\chi_{Q'\setminus\omega} + a\chi_\omega$. In "bad cubes" with too much jump, do nothing;
- Glue the smoothed functions from neighbouring cubes;
- By construction, all the cubes in $B_{R-2\delta}$ are good: hence one builds a smooth function in most of the ball.
- Some jump (=boundaries of bad cubes) is added only near "big pieces" of jump (at least not infinitesimal).

Cleaning lemma: proof



Cleaning lemma: proof



Main result: wiping out the jump

Consider $\eta > 0$, $\eta \leq \bar{\delta}^d$ (from the previous lemma), and $s > 0$ a small parameter. Assume $w \in GSBD^p(B_\rho)$ with $\mathcal{H}^{d-1}(J_w) \leq \eta(s\rho)^{d-1}$.
For each x point of rectifiability in $J_w \cap B_{(1-s)\rho}$ one defines $r_x \in [0, s\rho]$ such that

$$\begin{cases} \mathcal{H}^{d-1}(J_w \cap B_{r_x}(x)) = \eta r_x^{d-1} \\ \mathcal{H}^{d-1}(J_w \cap B_r(x)) \geq \eta r^{d-1} \qquad \text{for } r \leq r_x \end{cases}$$

Main result: wiping out the jump

Using Besicovitch's theorem, one finds $\mathcal{N}(d)$ families of disjoint balls $B_{r_x}(x)$ which cover $J_w \cap B_{(1-s)\rho}$.

Hence, choosing the family $(B_i)_i$ which covers the most, one can ensure that $\sum_i \mathcal{H}^{d-1}(J_w \cap B_i) \geq \mathcal{H}^{d-1}(J_w \cap B_{(1-s)\rho})/\mathcal{N}(d)$.

In the next step we apply the previous cleaning Lemma to wipe of most of the jump in each B_i: we replace w with $\tilde{w}$ in B_i such that $\tilde{w}$ is smooth in a large part of B_i, and has little additional jump. In particular the choice of r_x ensures that a certain proportion of the jump is erased.

This can be done iteratively in such a way that starting from a $u \in GSBD^p(B)$ we can find a $w \in GSBD^p(B)$ with less jump, no jump at all in a smaller ball, and which differs from u only in a union of small balls with controlled perimeters.

Thank you for your attention