ON THE BLOW-UP OF GSBV FUNCTIONS UNDER SUITABLE GEOMETRIC PROPERTIES OF THE JUMP SET

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Abstract. In this paper we investigate the fine properties of functions whose jump sets have suitable geometric properties. Precisely, given \( p > 1 \) we study the blow-up of functions \( u \in \text{GSBV} \), whose jump sets belongs to an appropriate class \( \mathcal{J}_p \) and whose approximate gradient is \( p \)-th power summable. In analogy with the theory of \( p \)-capacity in the context of Sobolev spaces, we prove that the blow-up of \( u \) converges up to a set of Hausdorff dimension less than or equal to \( n - p \). Moreover, we are able to prove the following result which in the case of \( W^{1,p}(\Omega) \) functions can be stated as follows: whenever \( u_k \) strongly converges to \( u \), then up to subsequences, \( u_k \) pointwise converges to \( u \) except on a set whose Hausdorff dimension is at most \( n - p \).

Keywords: blow-up, special bounded variation, indecomposable set, jump set, perimeter, rectifiable set, capacity, Cheeger’s constant, isoperimetric profile, Poincaré’s inequality.

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Introduction

The following result concerning the Lebesgue points of a Sobolev function is well known (see [9, 7, 18, 20, 11]): given \( 1 < p < n \), if \( u \in L^1_{\text{loc}}(\mathbb{R}^n) \) and its distributional derivatives are \( p \)-th power locally summable, then there exists a set \( A \) with \( \text{dim}_H(A) \leq n - p \), namely with Hausdorff dimension at most \( n - p \), such that every \( x \in \mathbb{R}^n \setminus A \) is a Lebesgue point for \( u \). More precisely, for every \( x \in \mathbb{R}^n \setminus A \) there exists a real number \( a \) such that:

\[
\lim_{r \to 0^+} \frac{1}{r^n} \int_{B_r(x)} |u(y) - a| \, dy = 0. \quad (0.1)
\]

By a change of variables, if we call \( u_x \) the function constantly equal to \( a \), the convergence in (0.1) can be rephrased by saying that \( u_{r,x}(y) := u(x + ry) \), namely the blow-up of \( u \) at \( x \), converges in \( L^1(B_1(0)) \) to \( u_x \), i.e.

\[
\lim_{r \to 0^+} \int_{B_1(0)} |u(x + ry) - u_x(y)| \, dy = 0. \quad (0.2)
\]

Roughly speaking, (0.2) says in a precise way that the values of \( u \) near \( x \) are close to a single constant.

The aim of this paper is to investigate the same issue in presence of a jump discontinuity set. Given \( \Omega \subset \mathbb{R}^n \) an open set, for every function \( u \) which belongs to the space \( \text{GSBV}(\Omega) \) introduced in [1], it is possible to consider an \((n - 1)\)-dimensional jump set \( J_u \) (see Definition 1.23). By using the general theory developed in [1] we know that at every point \( x \) it holds

\[ u_{r,x} \to u_x \text{ in measure in } B_1(0), \text{ as } r \to 0^+, \]

except on a set \( A \) with zero \((n - 1)\)-dimensional Hausdorff measure \( \mathcal{H}^{n-1}(A) = 0 \). Furthermore, if \( x \) is a Lebesgue point then \( u_x \) is a constant function, while if \( x \in J_u \) then \( u_x \) assumes two different values on two disjoint subsets of \( B_1(0) \) separated by an \((n - 1)\)-dimensional hyperplane passing through the origin. In this situation \( u_x \) may assume from one or two values.

In this work we focus our attention on the space \( \text{GSBV}^p(\Omega) \) when \( 1 < p < n \). It is composed by those functions \( u \in \text{GSBV}(\Omega) \) whose approximate gradient \( \nabla u \) is \( p \)-th power summable.
(see Definition 1.3). Precisely, we investigate under which hypothesis on the jump set, the $p$-th power summability of the approximate gradient guarantees $\dim_H(A) \leq n - p$.

To illustrate the result we are going to prove, let us consider the following example. Consider $\Gamma_0 \subset \mathbb{R}^3$ the union of three half lines starting from the origin. Let $\Gamma \subset \mathbb{R}^3$ be defined by $\Gamma_0 \times \mathbb{R}$ and let $l$ be the straight line $\{(0, 0, t) \mid t \in \mathbb{R}\}$. The set $\Gamma$ disconnects $\mathbb{R}^3 \setminus \Gamma$ into three connected components $\Omega_1, \Omega_2, \Omega_3$. Thanks to a well known properties of $GSBV$-functions, every $u \in GSBV^p(\Omega)$ with $J_u \subset \Gamma$, satisfies $u, \Omega_i \in W^{1,p}_{loc}(\Omega_i)$ for $i = 1, 2, 3$. Using a reflection argument, through an obvious modification of the result in [9], there exists a set $A$ with $\dim_H(A) = 3 - p$ such that if $x \in \mathbb{R}^3 \setminus A$ then the blow-up of $u$ at $x$ converges. In addition, on the points $x \in l \setminus A$ the limit $u_x$ can assume three different values $\alpha_i$ each on the set $\Omega_i \cap B_1(0)$, $i = 1, 2, 3$. Therefore, the family of all possible limits $u_x$ is richer than the previous cases.

Nevertheless, the $p$-th power summability of the approximate gradient is in general not enough to guarantee the convergence of the blow-up at every point except on a set of Hausdorff dimension $(n - p)$. Consider for example $u := 1_E$ the characteristic function of a set with finite perimeter. Clearly $\nabla u$ is $p$-summable for every $p \geq 1$, but from the theory of sets of finite perimeter, we know that the blow-up of $u$ in general converges only up to an $H^{n-1}$-negligible set. Precisely, it is possible to construct a set $E \subset \mathbb{R}^2$ with finite perimeter and such that, by setting $u = 1_E$, the set of points $x$ where $u_{x,e}$ does not converge have Hausdorff dimension exactly equal to 1 (see Appendix B). Therefore, it is reasonable to think that the geometry of the jump set affects the local behavior of the functions.

0.1. Main results of the paper. In order to state the main results we need to recall the following space: given $\Gamma \subset \Omega$ a countably $(H^{n-1}, n - 1)$-rectifiable set (see [8, Subsection 3.2.14]) with $H^{n-1}(\Gamma) < \infty$, let $GSBV^p(\Omega; \Gamma)$ be the space of all $u \in GSBV^p(\Omega)$ such that $J_u \subset \Gamma$.

In this paper for every $1 < p < n$ we introduce the class of all admissible jump sets $J_p$ (see Definition 3.3), for which the following two main results hold true.

**Theorem 0.1.** Let $\Omega \subset \mathbb{R}^n$ be open, and let $\Gamma \in J_p$ $(1 < p < n)$. If $u \in GSBV^p(\Omega; \Gamma)$, then there exists a set $A_u$ with Hausdorff dimension at most $n - p$, such that for every $x \in \Omega \setminus A_u$ there exists a function $u_x(\cdot) : B_1(0) \to \mathbb{R}$
\[ u_{r,x} \to u_x, \text{ in measure in } B_1(0), \tag{0.3} \]
as $r \to 0^+$.

**Theorem 0.2.** Let $\Omega \subset \mathbb{R}^n$ be open and let $\Gamma \in J_p$ with $(1 < p < n)$. Suppose $(u_k)_{k=1}^\infty \subset GSBV^p(\Omega; \Gamma) \cap L^p(\Omega)$ is such that
\[ \|u_k - u\|_{L^p(\Omega)} + \|\nabla u_k - \nabla u\|_{L^p(\Omega)} \to 0, \text{ as } k \to \infty. \]
Then there exists a subsequence $(k_j)$, such that for every $x \in \Omega$ except a on set with Hausdorff dimension at most $n - p$ we have
\[ (u_{k_j})_x \to u_x \text{ in measure in } B_1(0), \tag{0.4} \]
as $j \to \infty$.

In (0.4) $(u_k)_x$ is the one given by (0.3) where $u$ is replaced by $u_k$.

Theorem 0.1 can be seen as the analogous of the result (0.2) mentioned above. In the context of Sobolev spaces this is obtained through the theory of capacity, by exploiting the well known fact that smooth functions are dense in $W^{1,p}(\Omega)$. However, it is not known whether there exist dense subspaces of $GSBV^p(\Omega; \Gamma)$ made of regular functions $u$ with the additional constraint $J_u \subset \Gamma$ (see Remark 5.13). For this reason, we decide to perform a different analysis based on geometric measure theory techniques. In particular we prove a weak version of Poincaré’s inequality on balls, which guarantees that the $L^2$-distance of $u$ from a particular piecewise constant function can be controlled in terms of the $L^p$-norm of its approximate gradient plus a small volume error (see Theorem 4.3). This tool, together with a fine analysis of the blow-up of $u$ permits us to obtain the conclusion of Theorem 0.1. Notice that we deduce this result directly working with Hausdorff dimension without passing through the notion of capacity.
Theorem 0.2 is reminiscent of the following result in the context of Sobolev space. If a sequence $u_k$ in $W^{1,p}(\Omega)$ strongly converges to $u$, then up to subsequences the precise value of $u_k(x)$ defined by (0.2) converges to the precise value of $u(x)$, except on a $p$-capacitary negligible set (see for example [11, Lemma 4.8]). In order to prove Theorem 0.2, we pass through a suitable notion of capacity (see (5.2)), which allows us to deduce the convergence (0.4) for every $x$ except on a $p$-capacitary negligible set. The relation between this novel notion of capacity and the Hausdorff measure (see Theorem 5.7) enables us to deduce Theorem 0.2. The dimension $n - p$ in both Theorems 0.1 and 0.2 is optimal, since in the $W^{1,p}(\Omega)$ setting, i.e. $\Gamma = \emptyset$, we already know that it is sharp.

The class $J_p$ is composed of all countably $(H^{n-1}, n - 1)$-rectifiable sets with finite $H^{n-1}$-measure, which satisfy a suitable geometric condition on every points except on a set with Hausdorff dimension $n - p$ (see Definition 3.9). In order to define such properties, we heavily make use of the theory of indecomposable sets, for which we introduce a geometric quantity called upper isoperimetric profile. This quantity plays a similar role to that of the Cheeger’s constant for Riemannian manifolds (see (2.17)). For example, finite union of $(n-1)$-dimensional manifolds of class $C^1$ belong to $J_p$ for every $1 < p < n$. More in general, finite unions of graphs of Sobolev functions in $W^{2,p}$ belong to $J_p$ (see Example A.3). As pointed out in Remark A.2, whenever $n > 2p + 1$, the graph of a $W^{2,p}$-function might have the topological closure with arbitrarily large $n$-dimensional Lebesgue measure. This shows that a generic set in $J_p$ does not need to be essentially closed. In addition, in example A.4 we are able to construct a set in $\mathbb{R}^2$ which cannot be written as a finite union of graphs, but nevertheless it belongs to $J_p$ for every $1 < p < 2$.

0.2. Organization of the paper. The paper is organized as follows. In Section 1 we give some definitions and we prove some preliminary results: we recall the notion of convergence in measure and some related function spaces, we present some technical facts concerning indecomposable sets and finally we remind some properties of $GSBV(\Omega)$-functions.

Section 2 is devoted to the proof of a weak version of Poincaré’s inequality (see Theorem 2.12). We present the upper isoperimetric profile $h_F$ (see (2.17)). Precisely, it is a strictly positive and increasing function $h_F: (0,1/2] \to (0,\infty)$ defined for every indecomposable set $F$. The property $h_F > 0$, which somehow encodes that the concept of indecomposability is the counterpart in measure theory of connectedness in topology, allows us to prove the weak Poincaré’s inequality (2.22).

In Section 3 we introduce the concept of non vanishing upper isoperimetric profile through which we can define the class $J_p$. Some propositions concerning this concept will follow.

In Section 4 we focus on the proof of our first main Theorem 0.1. Here the hardest part is the study of suitably defined medians on balls $B_r(x)$ of functions $u \in GSBV^p(\Omega; \Gamma)$ (see Definition 4.1). Being able to capture the behavior of such medians as $r \to 0^+$, then Theorem 0.1 easily follows by an application of the weak Poincaré’s inequality on balls (see Theorem 4.3).

In Section 5 we define a suitable notion of $p$-capacity in the presence of a prescribed jump set. It turns out to satisfy the axioms of outer measures. We derive the relations between this capacity and the Hausdorff measure (see Subsection 5.2). Finally we prove Theorem 0.2.

We conclude with Appendix A and Appendix B. The first one is dedicated to the construction of sets living in $J_p$. The second one is dedicated to the construction of a set with finite perimeter $E \subset \mathbb{R}^2$ whose blow-up does not converge on a set having Hausdorff dimension exactly equal to one.

1. Notation and preliminary results

1.1. Measures and convergence. We denote by $\mathcal{L}^n$ the $n$-dimensional Lebesgue measure of $\mathbb{R}^n$. In order to simplify the notation, given a set $A \subset \mathbb{R}^n$, we write $|A| := \mathcal{L}^n(A)$. The symbol $B_r(x)$ denotes the $n$-dimensional open ball $\{y \in \mathbb{R}^n \mid |y-x| < r\}$ and $\omega_n := |B_1(0)|$. To simplify the notation we will often write $B_1$ to denote the unitary ball centered at 0. Moreover, given $\Pi^k \subset \mathbb{R}^n$ a $k$-dimensional plane and $x \in \Pi^k$, we denote as $B^k_r(x)$ the $k$-dimensional open
ball \( \{ y \in \Omega^k \mid |y - x| < r \} \) and we denote \( \omega_k := \mathcal{H}^k(B^k_1(0)) \), where \( \mathcal{H}^k \) is the \( k \)-dimensional Hausdorff measure.

Let \( \Omega \subset \mathbb{R}^n \) be an open set. Given a set \( A \subset \Omega \) and \( B_r(x) \subset \Omega \) we define

\[
A_{r,x} := \frac{A - x}{r} \cap B_1(0),
\]

and we will always make use of the following identity

\[
A_{\lambda r,x} = \frac{A_{r,x} \cap B_{\lambda}(0)}{\lambda}, \quad \lambda \in (0, 1].
\]

For a given function \( u: \Omega \rightarrow \mathbb{R} \), we define \( u_{r,x}: B_1(0) \rightarrow \mathbb{R} \) as

\[
u_{r,x}(y) := u(x + ry), \quad y \in B_1(0).
\]

For convenience of the reader we recall the definition of outer measure.

**Definition 1.1** (Outer measure). An outer measure on \( \Omega \) is any set function \( \mu: \mathcal{P}(\Omega) \rightarrow [0, +\infty] \) satisfying the following properties

(a) \( \mu(\emptyset) = 0 \);

(b) \( \mu(A_1) \leq \lambda(A_2) \), whenever \( A_1 \subset A_2 \) (monotonicity);

(c) \( \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \lambda(A_i) \) (countably sub-additivity).

Whenever \( \mu \) is an outer measure on \( \Omega \) and \( 1 \leq k \leq n \), following the notation in \([8, 2.10.19]\) we denote the \( k \)-dimensional upper and lower densities of \( \mu \) at \( x \in \Omega \), respectively as

\[
\Theta^k(\mu, x) := \limsup_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_k r^k}, \quad (1.1)
\]

\[
\Theta^k(\mu, x) := \liminf_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_k r^k}. \quad (1.2)
\]

In the case the upper and lower density coincide, the \( k \)-dimensional density of \( \mu \) at \( x \in \Omega \) is simply defined as

\[
\Theta^k(\mu, x) := \lim_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_k r^k}. \quad (1.3)
\]

Given \( 0 \leq \delta \leq 1 \), we denote the point of density \( \delta \) of \( A \) as

\[
A^{(\delta)} := \{ x \in \Omega \mid \Theta^k(\mathcal{L}^n \setminus A, x) = \delta \}. \quad (1.4)
\]

**Definition 1.2.** Let \( \mu: \mathcal{P}(\Omega) \rightarrow [0, +\infty] \) be an outer measure. Given \( A \subset \Omega \), we say that a property \( \mathcal{P}(x) \), defined for \( x \in A \), holds \( \mu \)-quasi everywhere, and we use the abbreviation \( \mu \)-q.e., if there exists a set \( N \subset A \), with \( \mu(N) = 0 \), such that \( \mathcal{P}(x) \) holds for every \( x \in A \setminus N \).

**Definition 1.3.** We denote by \( L^0(B_1) \) (see [12]) the Fréchet space of all (equivalence classes of) Lebesgue measurable real-functions on \( B_1 \) equipped with the topology of convergence in measure. This topology can be defined for example by the Lévy-metric

\[
\| u - v \|_{L^0(B_1)} := \int_{B_1} |u - v| \wedge 1 \, dx, \quad u, v \in L^0(B_1).
\]

**Definition 1.4.** Let \( \mu \) be an outer measure on \( \Omega \). Let \( X \) be the real vector space of all functions \( u: \Omega \rightarrow L^0(B_1) \), and consider the equivalence relation

\[
u \sim \nu \text{ iff } \mu(\{x \in \Omega \mid u(x) \neq v(x)\}) = 0. \quad (1.5)
\]

We define

\[
U_\mu(\Omega; L^0(B_1)) := X/ \sim
\]

i.e. the space consisting of all equivalence classes obtained by quotient \( X \) with respect to \( \sim \).

**Remark 1.5.** Notice that, since \( \mu \) is an outer measure, (1.5) make sense even without any measurability conditions on the functions \( u \) and \( v \).

With the next definition we want to consider an analogous of the convergence in measure, in the space \( U_\mu(\Omega; L^0(B_1)) \).
Definition 1.6. Let $\mu$ be an outer measure on $\Omega$, and let $(u_k)_{k=1}^\infty$ and $u$ be functions in $U_\mu(\Omega; L^0(B_1))$. We say that $(u_k)$ converges to $u$ in $\mu$-measure (in $\Omega$) if

$$\lim_{k \to \infty} \mu(\{x \in \Omega \mid \|u_k - u\|_{L^0(B_1)} > \varepsilon\}) = 0,$$

for every $\varepsilon > 0$.

Convergence in $\mu$-measure implies up to subsequence point-wise convergence $\mu$-q.e.. This is the content of the next proposition.

Proposition 1.7. Let $\mu$ be an outer measure on $\Omega$, and let $(u_k)_{k=1}^\infty$ and $u$ be functions in $U_\mu(\Omega; L^0(B_1))$. Suppose $u_k \to f$ in $\mu$-measure (in $\Omega$), then there exists a subsequence $(k_j)$ such that

$$\lim_{j \to \infty} \|u_{k_j}(x) - u(x)\|_{L^0(B_1)} = 0, \text{ } \mu \text{-q.e.}$$

Proof. For every $j \in \mathbb{N}$ choose $k_j \in \mathbb{N}$ such that

$$\mu\left( \left\{ x \in \Omega \mid \|u_{k_j} - u\|_{L^0(B_1)} > \frac{1}{j} \right\} \right) \leq \frac{1}{2^j}.$$

Set $A_j := \left\{ x \in \Omega \mid \|u_{k_j} - u\|_{L^0(B_1)} \leq \frac{1}{j} \right\}$, define $B_i := \bigcap_{j \geq i} A_j$ and finally $B := \bigcup_{i=1}^\infty B_i$.

Suppose $x \in B$, then $x \in B_i$ for some $i$ and hence $x \in A_j$ for every $j \geq i$. Therefore

$$\|u_{k_j}(x) - u(x)\|_{L^0(B_1)} \leq \frac{1}{j}, \text{ for } j \geq i,$$

which means

$$\lim_{j \to \infty} \|u_{k_j}(x) - u(x)\|_{L^0(B_1)} = 0.$$

Finally we can use the monotonicity and the countable sub-additivity of $\mu$ to estimate

$$\mu(\Omega \setminus B) \leq \mu(\Omega \setminus B_i) \leq \sum_{j \geq i} \mu(A_j) \leq \frac{1}{2^{i-1}},$$

and by the arbitrariness of $i$ we deduce $\mu(\Omega \setminus B) = 0$. \hfill $\square$

The convergence in measure can be metrized in the following way.

Proposition 1.8. Let $\mu$ be an outer measure on $\Omega$ and let $u, v \in U_\mu(\Omega; L^0(B_1))$. The metric $d(u, v)$ defined by

$$d(u, v) := \inf_{\delta > 0} \mu(\{|u - v| > \delta\}) + \delta,$$

induces the convergence in measure (1.6), and it gives to $U_\mu(\Omega; L^0(B_1))$ the structure of a complete metric space.

Proof. We start by proving that $d(\cdot, \cdot)$ is a metric.

First of all suppose that $d(u, v) = 0$ then we want to prove that $\mu(\{|u - v| > 0\}) = 0$. Indeed, if $d(u, v) = 0$, then for every $\delta > 0$, $\mu(\{|u - v| > \delta\}) = 0$. Since $\{|u - v| > 0\} = \bigcup_{k=1}^\infty \{|u - v| > 1/k\}$, by using the monotonicity and countable sub-additivity of $\mu$ we can write

$$0 \leq \mu(\{|u - v|_{L^0(B_1)} > 0\}) = \mu\left( \bigcup_{k=1}^\infty \{|u - v|_{L^0(B_1)} > 1/k\} \right) \leq \mu\left( \bigcup_{k=N+1}^\infty \{|u - v|_{L^0(B_1)} > 1/k\} \right) \leq \sum_{k=N+1}^\infty \mu(\{|u - v|_{L^0(B_1)} > 1/k\}) \leq 0,$$

which immediately implies $\mu(\{|u - v|_{L^0(B_1)} > 0\}) = 0$.

The equality $d(u, v) = d(v, u)$ is obvious.

Finally we need to prove the triangular inequality. For this purpose notice that for every triple of functions $u, v, g: \Omega \to L^0(B_1)$ it holds

$$\{|u - v|_{L^0(B_1)} > \delta_1 + \delta_2\} \subset \{|u - g|_{L^0(B_1)} > \delta_1\} \cup \{|g - v|_{L^0(B_1)} > \delta_2\}.$$
Given $\epsilon > 0$, let $\delta_1$ and $\delta_2$ be positive real numbers such that

$$
  d(u, g) + \epsilon \geq \mu(\{ \|u - g\|_{L^0(B_1)} > \delta_1 \}) + \delta_1, \quad d(g, v) + \epsilon \geq \mu(\{ \|g - v\|_{L^0(B_1)} > \delta_2 \}) + \delta_2.
$$

Then

$$
  d(u, v) = \inf_{\delta > 0} \mu(\{ \|u - v\|_{L^0(B_1)} > \delta \}) + \delta
$$

$$
  \leq \mu(\{ \|u - v\|_{L^0(B_1)} > \delta_1 + \delta_2 \}) + \delta_1 + \delta_2
$$

$$
  \leq [\mu(\{ \|u - g\|_{L^0(B_1)} > \delta_1 \}) + \delta_1] + [\mu(\{ \|g - v\|_{L^0(B_1)} > \delta_2 \}) + \delta_2]
$$

$$
  \leq d(u, g) + d(g, v) + 2\epsilon,
$$

and letting $\epsilon \to 0^+$ this implies the triangular inequality.

Given $(u_k)_{k=1}^{\infty} \subset U_\mu(\Omega; L^0(B_1))$ and $u \in U_\mu(\Omega; L^0(B_1))$, we claim that $\lim_{k \to \infty} d(u_k, u) = 0$ if and only if $u_k$ converge to $u$ in $\mu$-measure. Let us first suppose $\lim_{k \to \infty} d(u_k, u) = 0$. Then by definition of $d(\cdot, \cdot)$, it turns out that for every $k$ there exist $\delta_k > 0$ such that

$$
  \lim_{k \to \infty} \mu(\{ \|u_k - u\|_{L^0(B_1)} > \delta_k \}) = 0.
$$

Hence, given $\epsilon > 0$ we can find $K$ big enough such that for every $k \geq K$ \( \{ \|u_k - u\|_{L^0(B_1)} > \epsilon \} \subset \{ \|u_k - u\|_{L^0(B_1)} > \delta_k \} \), which implies

$$
  \lim_{k \to \infty} \mu(\{ \|u_k - u\|_{L^0(B_1)} > \epsilon \}) \leq \lim_{k \to \infty} \mu(\{ \|u_k - u\|_{L^0(B_1)} > \delta_k \}) = 0.
$$

This gives the convergence in $\mu$-measure.

Now suppose that $u_k$ converge to $u$ in $\mu$-measure. Then we can write for every $\epsilon > 0$

$$
  \lim_{k \to \infty} \inf_{\delta > 0} \mu(\{ \|u_k - u\|_{L^0(B_1)} > \delta \}) + \delta \leq \lim_{k \to \infty} \mu(\{ \|u_k - u\|_{L^0(B_1)} > \epsilon \}) + \epsilon = \epsilon,
$$

which immediately implies $\lim_{k \to \infty} d(u_k, u) = 0$.

Finally, we have to prove that $U_\mu(\Omega; L^0(B_1))$ endowed with the metric $d(\cdot, \cdot)$ is complete. For this purpose, suppose that the sequence $(u_k)_{k=1}^{\infty}$ is Cauchy. Given a sequence $(\delta_j)_j$ of positive real numbers such that $\sum_{j=1}^{\infty} \lambda_j < \infty$, there exists a subsequence $(k_j)_j$ such that

$$
  d(u_{k_1}, u_{k_2}) \leq \lambda_j, \text{ for every } j_1, j_2 \geq j.
$$

which means that there exists $\delta_j > 0$ such that

$$
  \mu(\{ \|u_{k_1} - u_{k_2}\|_{L^0(B_1)} > \delta_j \}) + \delta_j \leq \lambda_j, \text{ for every } j_1, j_2 \geq j. \quad (1.7)
$$

Define $A_j := \{ \|u_{k_j} - u_{k_{j+1}}\|_{L^0(B_1)} > \delta_j \}$ and set $B_j := \bigcup_{m \geq j+1} A_m$. We claim that $u_{k_j}$ converges point wise for every $x \in \Omega \setminus \bigcap_{j=1}^{\infty} B_j$. Indeed, if $x \in \Omega \setminus \bigcap_{j=1}^{\infty} B_j$ means that there exists $\overline{j}$ such that $x \neq B_{\overline{j}}$, which, by definition of $B_{\overline{j}}$, implies $x \neq A_j$ for every $j \geq \overline{j} + 1$. For this reason we have

$$
  \|u_{k_j}(x) - u_{k_{j+1}}(x)\|_{L^0(B_1)} \leq \delta_j, \text{ for every } j \geq \overline{j} + 1,
$$

and this immediately implies that $(u_{k_j}(x))_j$ is a Cauchy sequence in $L^0(B_1)$. By the completeness of $L^0(B_1)$ we deduce that there exists a function $u: \Omega \setminus \bigcap_{j=1}^{\infty} B_j \to L^0(B_1)$ such that

$$
  \lim_{j \to \infty} \|u_{k_j}(x) - u(x)\|_{L^0(B_1)} = 0.
$$

Since by the monotonicity $\mu$ we have

$$
  \mu\left( \bigcap_{j=1}^{\infty} B_j \right) = 0,
$$

we deduce that the function $u$ is a well defined element of $U_\mu(\Omega; L^0(B_1))$.

Finally, given $k \in \mathbb{N}$ there exists $j_k$ such that $k_{j_k-1} \leq k \leq k_{j_k}$, therefore

$$
  \|u_k(x) - u(x)\|_{L^0(B_1)} \leq \|u_k(x) - u_{k_{j_k}}(x)\|_{L^0(B_1)} + \sum_{j \geq j_k+1} \|u_{k_j}(x) - u_{k_{j+1}}(x)\|_{L^0(B_1)}.
$$
hence if $x \in \Omega \setminus B_r$ we deduce by (1.7)
\[
\|u_k(x) - u(x)\|_{L^p(B_1)} \leq \sum_{j=1}^{\infty} \delta_j.
\]
This means
\[
\left\{\|u_k - u\|_{L^p(B_1)} > \sum_{j=1}^{\infty} \delta_j\right\} \subset B_{bj}.
\] (1.8)

Now given $\epsilon > 0$, consider $j_k$ big enough such that $\sum_{j=j_k}^{\infty} \delta_j \leq \epsilon$. By (1.8), we have that for every $k \geq j_k$
\[
\mu\left(\|u_k - u\|_{L^p(B_1)} > \epsilon\right) \leq \mu(B_{j_k}).
\]
Taking the limit as $k \to \infty$ in both sides of the previous inequality we obtain
\[
\lim_{k \to \infty} \mu\left(\|u_k - u\|_{L^p(B_1)} > \epsilon\right) \leq \lim_{k \to \infty} \mu(B_{j_k}) = 0
\]
We deduce that $u_k$ converge in $\mu$-measure to $u$ which implies $\lim_{k \to \infty} d(u_k, u) = 0$ and we are done.

**Remark 1.9.** The space $U_{\mu}(\Omega; L^0(B_1))$ equipped with the distance defined in the previous proposition is actually a Fréchet space.

### 1.2. Caccioppoli’s partitions and indecomposable sets. Given a $\mathcal{L}^n$-measurable set $E \subset \Omega$ we denote the perimeter of $E$ in $\Omega$ as $P(E; \Omega)$, where
\[
P(E; \Omega) := \sup_{\varphi \in C^1_c(\Omega, \mathbb{R}^n)} \int_E \text{div} \varphi \, dx.
\]
When $\Omega = \mathbb{R}^n$ we simply write $P(E)$ to denote $P(E; \mathbb{R}^n)$.

Whenever $P(E; \Omega) < \infty$, we denote as $\partial^* E$ the reduced boundary of $E$, defined as those $x \in \Omega$ such that there exists $\nu_E(x) \in S^{n-1}$
\[
\lim_{r \to 0^+} \frac{D1_E(B_r(x))}{|D1_E(B_r(x))|} = \nu_E(x).
\] (1.9)
The unitary vector $\nu_E(x)$ is the *theoretical inner normal* of $E$, $D1_E$ denotes the distributional gradient of the characteristic function of $E$ and $|D1_E|$ denotes its total variation. In particular
\[
P(E; \Omega) = |D1_E|^{\#}(\Omega).
\]
Moreover, De Giorgi’s structure Theorem holds (see for example [1, Theorem 3.59]):

**Theorem 1.10.** Let $E \subset \Omega$ be an $\mathcal{L}^n$-measurable set. Then $\partial^* E$ is countably $(\mathcal{H}^{n-1}, n - 1)$-rectifiable and
\[
|D1_E| = \mathcal{H}^{n-1} \cup \partial^* E.
\]
In addition, for every $x \in \partial^* E$ the following properties hold
(a) the sets $(E - x)/r$ locally converge in measure in $\mathbb{R}^n$ as $r \searrow 0$ to the halfspace $H$ orthogonal to $\nu_E(x)$ and containing $\nu_E(x)$;
(b) $\mathcal{G}^{n-1}(\mathcal{H}^{n-1} \cup \partial^* E, x) = 1$.

We recall the definition of Caccioppoli’s partitions (see [1]).

**Definition 1.11** (Caccioppoli’s partition). Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a $\mathcal{L}^n$-measurable partition $(E_i)_{i=1}^{\infty}$ of $\Omega$ is a *Caccioppoli’s partition* if
\[
\sum_{i=1}^{\infty} P(E_i; \Omega) < \infty.
\]
Moreover we say that a Caccioppoli’s partition $(E_i)_{i=1}^{\infty}$ is ordered if $|E_i| \geq |E_j|$ whenever $i \leq j$. 
Definition 1.12 (Indecomposability). Let $F \subset \Omega$ be a set of finite perimeter. We say that $F$ is indecomposable if for every set $E$ satisfying

$$E \subset F, \quad P(F; \Omega) = P(E; \Omega) + P(E \setminus E; \Omega),$$

(1.10)

then $E = \emptyset$ or $E = F$.

Remark 1.13. The notion of indecomposability can be found in for example in [13] and it is in perfect agreement with the following fact (see [6, Proposition 2.12]): the set $F$ is indecomposable if and only if any $u \in BV(\Omega)$ with $|Du|(F) = 0$ is necessarily constant on $F$.

In particular this shows that every connected open set $U \subset \Omega$ with finite perimeter is indecomposable.

Remark 1.14. For every set $E \subset F$ it holds $P(F; \Omega) \leq P(E; \Omega) + P(F \setminus E; \Omega)$. This means that condition (1.10) is equivalent to

$$E \subset F, \quad P(F; \Omega) \geq P(E; \Omega) + P(F \setminus E; \Omega).$$

Moreover, condition (1.10) can be equivalently stated for countable family $(E_i)_{i=1}^\infty$. This means that if $F$ is indecomposable and

$$\bigcup_{i=1}^\infty E_i = F, \quad |E_i \cap E_j| = 0 \quad (i \neq j), \quad \sum_{i=1}^\infty P(E_i; \Omega) = P(F; \Omega),$$

(1.11)

then there exists $i_0$ such that $E_{i_0} = F$ and $E_i = \emptyset$ for $i \neq i_0$.

Indeed condition (1.11) clearly implies (1.10). While if $F$ is indecomposable, by setting $E := E_1$, (1.11) tells us

$$P(E; \Omega) + P(F \setminus E; \Omega) \leq P(F; \Omega),$$

which implies

$$P(E; \Omega) + P(F \setminus E; \Omega) = P(F; \Omega).$$

By the indecomposability of $F$ we deduce that one between $E$ or $F \setminus E$ is the empty set. If $F \setminus E = \emptyset$ we are done. Otherwise $E = \emptyset$ and we can proceed as before by defining $E := E_2$. Clearly, if this procedure do not stop, then $F = \emptyset$ and we are done. Otherwise if it stops at $i_0 \in \mathbb{N}$ this means that $F = E_{i_0}$ and we are done.

We will often make use of the following two results. The first is due to Federer and concerns the structure of sets having finite perimeter. The second asserts that every set of finite perimeter in a sufficiently regular domain, admits a Caccioppoli’s partition into indecomposable sets.

Theorem 1.15. Let $E$ be a set of finite perimeter in $\Omega$. Then

- $\mathcal{H}^{n-1}(E^{(1/2)} \Delta E^*) = 0$;
- $\mathcal{H}^{n-1}(\Omega \setminus \{E^{(1)} \cup E^{(1/2)} \cup E^{(0)}\}) = 0$.

Proof. See for example [1, Theorem, 3.61].

Proposition 1.16. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz-regular domain, and let $E \subset \Omega$ be a set of finite perimeter. Then there exists a Caccioppoli’s indecomposable partition of $E$, which means a family $(F_i)_{i \in I}$ of indecomposable sets such that

1. $|E \setminus \bigcup_{i=1}^\infty F_i| = 0$;
2. $|F_i \cap F_j| = 0$ for $i \neq j$;
3. $\sum_{i=1}^\infty P(F_i; \Omega) = P(E; \Omega)$;
4. $\mathcal{H}^{n-1}(\partial^* E \setminus \bigcup_{i=1}^\infty \partial^* F_i) = 0$;
5. $\mathcal{H}^{n-1}(\partial^* F_i \cap \partial^* F_j) = 0$ for every $i \in \mathbb{N}$;
6. $\mathcal{H}^{n-1}(\partial^* F_i \cap \partial^* E) = 0$ for $i \neq j$.

Moreover the family $(F_i)_{i=1}^\infty$ is unique up to permutation of indices in the sense that given any family of indecomposable sets $(F_i')_{i=1}^\infty$ satisfying (1),(2) and (3), then there exists a bijection $\pi: \mathbb{N} \to \mathbb{N}$ such that

$$|F_i \Delta F_{\pi(i)}^*| = 0 \quad \text{for every } i \in \mathbb{N}.$$
ON THE BLOW-UP OF $\text{GSBV}$ FUNCTIONS

Proof. We can combine [13, Proposition 3.3] with [13, Corollary 3.2] to deduce that there exists a countable family $(\tilde{F}_i)_{i \in I}$ satisfying (1)-(6) with the additional property $\tilde{F}_i \neq \emptyset$ for every $i \in I$. Moreover this family is unique in the sense that if $(\tilde{F}'_i)_{i \in I'}$ is another family satisfying (1)-(3) with the additional property $\tilde{F}'_i \neq \emptyset$ for every $i \in I$, then there exists a bijection $\pi$ of $I'$ onto $I$ such that

$$\tilde{F}_{\pi(i)} = \tilde{F}'_i, \, i \in I'.$$

Then, if the cardinality of $I$ is a natural number $N$, we define $F_i := \tilde{F}_{\pi(i)}$ where $\pi$ is any bijection of $\{1, \ldots, N\}$ onto $I$, and $F_i = \emptyset$ for every $i > N$. While if the cardinality of $I$ is the same as the cardinality of $\mathbb{N}$, we define $F_i := \tilde{F}_{\pi(i)}$ where $\pi$ is any bijection of $\mathbb{N}$ onto $I$. □

Definition 1.17 (Indecomposable components). Let $E \subset \Omega$ with $P(E;\Omega) < \infty$, and let $(F_i)_{i=1}^\infty$ be the unique (up to permutation of indices) indecomposable partition of $E$. For every $i \in \mathbb{N}$ we say that $F_i$ is an \textit{indecomposable components} of $E$.

Proposition 1.18 (Leibniz’s formula). Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E, F \subset \Omega$ be two sets having finite perimeter in $\Omega$. Then $E \cap F$ is a set of finite perimeter in $\Omega$ and moreover

$$\mathcal{H}^{n-1}(\partial^*(E \cap F)) = \mathcal{H}^{n-1}(\partial^*E \cap E^{(1)}) + \mathcal{H}^{n-1}(\partial^*F \cap E^{(1)})$$

$$+ \mathcal{H}^{n-1}([\nu_E = \nu_F]).$$

(1.12)

Proof. See [16, Theorem 16.3] □

We end up this subsection with three technical propositions which will be useful later on.

Proposition 1.19. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F \subset \Omega$ be indecomposable. Suppose $E \subset \Omega$ is a set having finite perimeter in $\Omega$ and such that

$$0 < |F \setminus E| < |F|.$$ 

Then it holds $\mathcal{H}^{n-1}(\partial^*E \cap E^{(1)}) > 0$.

Proof. We can consider the measurable partition of $F$ given by $F = (E \cap F) \cup (F \setminus E)$. By hypothesis $|E \cap F|, |F \setminus E| > 0$. Using Leibniz’s formula (1.12) we can write

$$\partial^*(E \cap F) = [\partial^*E \cap E^{(1)}] \cup [\partial^*F \cap E^{(1)}] \cup [\{\nu_E = \nu_F\}],$$

and

$$\partial^*(F \setminus E) = [\partial^*E \cap E^{(1)}] \cup [\partial^*F \cap E^{(0)}] \cup [\{\nu_E = -\nu_F\}].$$

Since $\partial^*F \cap E^{(1)}$, $\nu_E = \nu_F$, $\partial^*F \cap E^{(0)}$ and $\nu_E = -\nu_F$ are pairwise disjoint subsets of $\partial^*F$, if $\mathcal{H}^{n-1}(\partial^*E \cap E^{(1)}) = 0$ then

$$P(E \cap F;\Omega) + P(F \setminus E;\Omega) = \mathcal{H}^{n-1}(\partial^*F \cap E^{(1)}) + \mathcal{H}^{n-1}(\partial^*F \cap E^{(0)})$$

$$+ \mathcal{H}^{n-1}([\nu_E = \nu_F]) + \mathcal{H}^{n-1}([\nu_E = -\nu_F])$$

$$\leq P(F;\Omega),$$

which by Remark 1.14 implies (1.10) and is in contradiction with the indecomposability of $F$. □

Proposition 1.20. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz-regular domain, and let $E, E' \subset \Omega$ be two sets of finite perimeter in $\Omega$ such that $\partial^*E' \subset \partial^*E$. Let $F \subset E$ be an indecomposable set. Then one and only one of the following holds

1. $F \subseteq E'$
2. $F \subseteq E \setminus E'$.

Proof. It is enough to show that $|F \cap E'| \neq 0$ implies $F \subseteq E'$.

Suppose not. Then $|F \cap E'| > 0$ and also $|F \setminus E'| > 0$. By Leibniz’s formula both $F \cap E'$ and $F \setminus E'$ are sets having finite perimeter in $\Omega$. Moreover, by Proposition 1.19 we would have also $\mathcal{H}^{n-1}(\partial^*E' \cap E^{(1)}) > 0$. But since $F \subset E$ then $F^{(1)} \subset E^{(1)}$, and this implies $\mathcal{H}^{n-1}(\partial^*E' \cap E^{(1)}) > 0$ which is a contradiction with the hypothesis $\partial^*E' \subset \partial^*E$. This proves the proposition. □
Proposition 1.21. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz-regular domain, and let $E' \subset E \subset \Omega$ be two sets of finite perimeter in $\Omega$ such that $\partial^* E' \subseteq \partial^* E$. If $F$ is an indecomposable component of $E'$ then it is also an indecomposable component of $E$.

Proof. Call $(F_i)_{i=1}^{\infty}$ the Caccioppoli’s indecomposable partition of $E$. Since $E' \subset E$ there exists $F_0$ such that $|F_0 \cap F| \neq 0$. We claim that $F_0 = F$.

Indeed $\partial^* F' \subseteq \partial^* E' \subseteq \partial^* E$ and by applying Proposition 1.20, the fact that $F_0$ is an indecomposable component of $E$ leads to one and only one of the followings

$$F_0 \subset F \text{ or } F_0 \subset E \setminus F.$$  

Since $|F_0 \cap F| \neq 0$ we deduce $F_0 \subset F$.

To prove the opposite inclusion, we show that $|F \setminus F_0| = 0$. By point (5) of 1.16 $\partial^* F_0 \subset \partial^* E$ and this together with $F^{(1)} \subset E'^{(1)} \subset E^{(1)}$, implies

$$\partial^* F_0 \cap F^{(1)} = \emptyset.$$  

(1.13)

Moreover, we can consider the measurable partition of $F$ made by

$$F = F_0 \cup (F \setminus F_0).$$

If $|F \setminus F_0| > 0$, by applying Proposition 1.19 we would have $H^{-1}(\partial^* F_0 \cap F^{(1)}) > 0$ which is in contradiction with (1.13). This proves the proposition. \hfill \square

1.3. The space $SBV^p(\Omega; \Gamma)$.

Definition 1.22 (Upper and lower approximate limit). Following [1, Section 4.5] we recall some basic definitions.

Given an $L^n$-measurable function $u: \Omega \rightarrow \mathbb{R}$ we recall that the upper approximate limit of $u$ at $x$ defined as

$$u^+(x) := \inf \{ t \in \mathbb{R} \mid \Theta^n(L^n \setminus \{u > t\}, x) = 0 \}$$

and analogously the lower approximate limit of $u$ at $x$ is defined as

$$u^-(x) := \sup \{ t \in \mathbb{R} \mid \Theta^n(L^n \setminus \{u < t\}, x) = 0 \}.$$  

We say that $u$ admits an approximate limit $a$ at $x$ if $u^+(x) = u^-(x)$.

The singular set of $u$ is defined as

$$S_u := \{ x \in \Omega \mid u^-(x) < u^+(x) \}.$$  

When $x \in \Omega \setminus S_u$ we denote the approximate limit of $u$ at $x$ as $\tilde{u}(x)$.

Definition 1.23 (Jump set). Let $u: \Omega \rightarrow \mathbb{R}$ be an $L^n$-measurable function. A point $x \in \Omega$ is an approximate jump point, and we write $x \in J_u$, if there exists $a, b \in \mathbb{R}$ with $a < b$ and $\nu \in \mathbb{S}^{n-1}$ such that, setting

$$H^+ := \{ y \in \Omega \mid (y - x) \cdot \nu > 0 \}, \quad H^- := \{ y \in \Omega \mid (y - x) \cdot \nu < 0 \}$$

the approximate limit of the restriction of $u$ to $H^+$ is $a$ and the weak limit of the restriction of $u$ to $H^-$ is $b$.

If $x \in J_u$ then $a = u^+(x)$ and $b = u^-(x)$. The vector $\nu$, uniquely determined by this condition, will be denoted by $\nu_u(x)$.

Definition 1.24 (Weak approximate differentiability). Let $u: \Omega \rightarrow \mathbb{R}$ be a $L^n$-measurable function and $x \in \Omega \setminus S_u$. Then $u$ is approximately differentiable at $x$ if $\tilde{u}(x) \in \mathbb{R}$ and there exists a linear map $L: \mathbb{R}^n \rightarrow \mathbb{R}$ such that for any $\epsilon > 0$ the set

$$\left\{ y \in \Omega \setminus \{x\} \mid \frac{|u(y) - \tilde{u}(x) - L(y - x)|}{|y - x|} > \epsilon \right\}$$

has density 0 at $x$; in this case the weak approximate gradient of $u$ is defined as $\nabla u(x) := L$.

Definition 1.25 ($GSBV$ functions). Let $\Omega$ be an open set in $\mathbb{R}^n$. We say that $u: \Omega \rightarrow \mathbb{R}$ is a generalized function of special bounded variation, and we write $u \in GSBV(\Omega)$, if for every $M \in \mathbb{N}$ the truncated function $u^M := (u \vee M) \wedge -M$ belongs to $SBV_{loc}(\Omega)$.
Now we recall the main result about the fine properties of GSBV functions (see [1, Theorem 4.34]).

**Theorem 1.26 (Fine properties).** Let \( u \in GSBV(\Omega) \), let \( M \in \mathbb{N} \). Then

1. \( S_u = \bigcup_{M \in \mathbb{N}} S_{u^M} \) and
   \[
   u^+(x) = \lim_{M \to +\infty} (u^M)^+(x), \quad u^-(x) = \lim_{M \to +\infty} (u^M)^-(x);
   \]
2. \( S_u \) is countably \( \mathcal{H}^{n-1} \)-rectifiable, \( \mathcal{H}^{n-1}(S_u \setminus J_u) = 0 \) and
   \[
   \text{Tan}(S_u, x) = (\nu_u(x))^\perp, \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in S_u;
   \]
3. \( u \) is weakly approximate differentiable \( L^p\text{-a.e.} \) in \( \Omega \) and
   \[
   \nabla u(x) = \nabla u^M(x), \text{ for } L^p\text{-a.e. } x \in \{|u| \leq M\};
   \]
4. \( \{u > t\} \) has finite perimeter in \( \Omega \) for \( L^1\text{-a.e.} \), \( t \in \mathbb{R} \) and
   \[
   \int_{\mathbb{R}} P(\{u > t\}, B) \, dt = \int_B |\nabla u| \, dx + \int_{J_u \cap B} (u^+(x) - u^-(x)) \, d\mathcal{H}^{n-1}(x),
   \]
   for any Borel set \( B \subset \Omega \).

Finally, we introduce suitable subspaces of \( GSBV(\Omega) \).

**Definition 1.27.** Given \( \Gamma \subset \Omega \) a countable \((\mathcal{H}^{n-1}, n-1)\)-rectifiable set with \( \mathcal{H}^{n-1}(\Gamma) < \infty \). We define for every \( p \geq 1 \)

- \( GSBV^p(\Omega) := \{u \in GSBV(\Omega) \mid \nabla u \in L^p(\Omega)\} \);
- \( GSBV^p_p(\Omega) := \{u \in GSBV(\Omega) \mid u \in L^p(\Omega), \nabla u \in L^p(\Omega)\} \);
- \( GSBV^p(\Omega; \Gamma) := \{u \in GSBV^p(\Omega) \mid J_u \subset \Gamma\} \);
- \( GSBV^p_p(\Omega; \Gamma) := \{u \in GSBV^p_p(\Omega) \mid J_u \subset \Gamma\} \).

**Remark 1.28.** The spaces appearing in the previous definition are actually vector spaces (see [4, Proposition 2.3] for the case \( GSBV^p(\Omega) \)). Moreover, by putting together [1, Theorem 4.36] with [4, Remark 2.9], it can be proved that \( GSBV^p_p(\Omega; \Gamma) \) with the norm \( \|u\|_{L^p} + \|\nabla u\|_{L^p} \) is also a Banach space.

## 2. Weak Poincaré’s inequality for indecomposable sets

### 2.1. The upper isoperimetric profile

**Definition 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a Lipschitz regular domain. For every \( k \in \mathbb{N} \), let \( (F_{k,i})_{i=1}^{\infty} \) be a Caccioppoli’s partition of \( \Omega \). We say that \( (F_{k,i})_{i=1}^{\infty} \) converges to a Caccioppoli’s partition \( (F_i)_{i=1}^{\infty} \) as \( k \to \infty \), if and only if

\[
\limsup_{i \to \infty} |F_{k,i} \Delta F_i| = 0, \quad i \in \mathbb{N}.
\]

**Proposition 2.2.** Let \( (E_r)_{r \in (0,1)} \) be a family of sets contained in \( B_1(0) \subset \mathbb{R}^n \) with \( P(E_r; B_1(0)) < \infty \). For each \( r \in (0,1) \) let \( (F_{r,i})_{i=1}^{\infty} \) be the Caccioppoli’s indecomposable partition of \( E_r \). Let \( E_0 \subset B_1(0) \) be an indecomposable set. Suppose that

1. \( \lim_{r \to 0^+} |E_r \Delta E_0| = 0 \)
2. \( \lim_{r \to 0^+} P(E_r; B_1(0)) = P(E_0; B_1(0)) \).

Then, for each \( r \in (0,1) \) there exists \( \sigma_r \in \mathbb{N} \) such that

\[
\lim_{r \to 0^+} |F_{r,\sigma_r} \Delta E_0| = 0, \quad (2.1)
\]

and

\[
\lim_{r \to 0^+} P(F_{r,\sigma_r}; B_1(0)) = P(E_0; B_1(0)). \quad (2.2)
\]
Proof. Suppose that our proposition does not hold. Then there exists a \( \delta > 0 \) such that

\[
\limsup_{r \to 0^+} \left( \inf_{i \in \mathbb{N}} |F_{r,i} \Delta E_0| \right) \geq \delta.
\]

This implies the existence of a subsequence \( (r_m)_{m=1}^{\infty} \) such that

\[
|F_{r_m,i} \Delta E_0| > \delta, \quad (2.3)
\]

for every \( m \in \mathbb{N} \) and for every \( i \in \mathbb{N} \).

Consider the Caccioppoli’s partitions of \( B_1(0) \) made of \( (F_{r_m,i})_{m=1}^{\infty} \). Since \( B_1(0) \) has finite Lebesgue measure, these partitions can be ordered. Thus we can apply the compactness theorem for Caccioppoli’s ordered partition (see [1, Theorem, 4.19] and [1, Remark, 4.20]), to find a Caccioppoli’s (ordered) partition of \( B_1(0) \), say \( (F_{0,i})_{i=1}^{\infty} \) where one of the \( F_{0,i} \) must be equal to \( B_1(0) \setminus E_0 \), such that up to subsequence we have

\[
\lim_{m \to \infty} |F_{r_m,i} \Delta F_{0,i}| = 0 \quad \text{for every} \quad i \in \mathbb{N}. \quad (2.4)
\]

By removing the set \( (B_1(0) \setminus E_0) \) from the partition, we obtain a ordered measurable partition of \( E_0 \), which we still call \( (F_{0,i})_{i=1}^{\infty} \). By (2.3), there exists a family \( I \subset \mathbb{N} \) with cardinality strictly greater than 1, such that \( F_{0,i} \neq \emptyset \) for every \( i \in I \).

Using the lower semi-continuity of the perimeter and (3) of Proposition 1.16, we can write

\[
\sum_{i \in I} P(F_{0,i}; B_1(0)) \leq \liminf_{m \to \infty} \sum_{i=0}^{\infty} P(F_{k_m,i}; B_1(0)) \leq \liminf_{m \to \infty} \sum_{i=1}^{\infty} P(F_{k_m,i}; B_1(0)) \leq \liminf_{m \to \infty} P(E_{k_m}; B_1(0)) = P(E_0; B_1(0)), \quad (2.5)
\]

since \( (F_{0,i})_{i \in I} \) is a (measurable) partition of \( E_0 \), (2.5) implies

\[
\sum_{i \in I} P(F_{0,i}; B_1(0)) = P(E_0; B_1(0)), \quad (2.6)
\]

and by Remark 1.14 this is a contradiction with the indecomposability of \( E_0 \), hence this proves (2.1).

Finally we notice that

\[
P(E_0; B_1(0)) \leq \liminf_{r \to 0^+} P(F_{r,\sigma_r}; B_1(0)) \leq \limsup_{r \to 0^+} P(F_{r,\sigma_r}; B_1(0)) \leq \limsup_{r \to 0^+} \sum_{i=0}^{\infty} P(F_{r,i}; B_1(0)) \leq \limsup_{r \to 0^+} P(E_r; B_1(0)) = P(E_0; B_1(0)),
\]

and this gives (2.2). \( \square \)

The next proposition can be seen as a generalization of the well known result of lower semi-continuity of the perimeter; namely, given a sequence of sets \( (E_k) \) such that \( \lim_k |E_k \Delta E| = 0 \), then for every open set \( \Omega \)

\[
\liminf_{k \to \infty} P(E_k; \Omega) \geq P(E; \Omega).
\]

We are interested in letting also the set \( \Omega \) varies along the sequence.

**Proposition 2.3 (Lower semi-continuity).** Let \( (E_k)_{k=1}^{\infty}, (E'_k)_{k=1}^{\infty}, E, E' \subset \Omega \) be sets of finite perimeter such that \( E'_k \subset E_k \) and

1. \( \lim_{k \to \infty} |E_k \Delta E| = 0 \);
2. \( \lim_{k \to \infty} P(E_k; \Omega) = P(E; \Omega) \);
3. \( \lim_{k \to \infty} |E'_k \Delta E'| = 0 \).

Then it holds the following lower semi-continuity property

\[
\liminf_{k \to \infty} \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) \geq \mathcal{H}^{n-1}(\partial^* E' \cap E^{(1)}). \quad (2.7)
\]
Proof. Using the Leibniz’s formula (1.12) we can write
\[ P(E'_k; \Omega) = P(E'_k \cap E_k; \Omega) = H^{n-1}(\partial^* E'_k \cap E^1_k) + H^{n-1}(\partial^* E_k \cap E^1_k) + H^{n-1}(\nu_{E'_k} = \nu_{E_k}). \] (2.8)

Since \( E'_k \subset E_k \) then \( E^1_k \subset E^1_k \), hence \( E^1_k \cap E^{1(2)}_k = \emptyset \). This implies \( H^{n-1}(\partial^* E_k \cap E^1_k) = 0 \). Moreover, since \( E'_k \subset E_k \) then \( H^{n-1}(\nu_{E'_k} \neq \nu_{E_k}) = 0 \). Therefore (2.8) can be rewritten as
\[ P(E'_k; \Omega) = H^{n-1}(\partial^* E'_k \cap E^1_k) + H^{n-1}(\partial^* E'_k \cap \partial^* E_k). \] (2.9)

Analogously we have
\[ P(E_k \setminus E'_k; \Omega) = H^{n-1}(\partial^* (E_k \setminus E'_k) \cap E^1_k) + H^{n-1}(\partial^* (E_k \setminus E'_k) \cap \partial^* E_k). \] (2.10)

Since \( H^{n-1}(\partial^* (E_k \setminus E'_k) \cap E^1_k) = H^{n-1}(\partial^* E'_k \cap E^1_k) \), then we can rewrite the previous equality as
\[ P(E_k \setminus E'_k; \Omega) = H^{n-1}(\partial^* E'_k \cap E^1_k) + H^{n-1}(\partial^* (E_k \setminus E'_k) \cap \partial^* E_k). \] (2.11)

We claim that
\[ H^{n-1}(\partial^* E_k \setminus (\partial^* E'_k \cup \partial^* (E_k \setminus E'_k))) = 0. \] (2.12)

To show this, notice that by Theorem 1.15, for \( H^{n-1}\text{-a.e. } x \in \Omega \), if \( x \in E^1_k \) then
\[ \{ x \in (E'_k)^{(0)} \text{ or } x \in (E^1_k)^{(1/2)} \} \text{ and } \{ x \in (E_k \setminus E'_k)^{(0)} \text{ or } x \in (E_k \setminus E^1_k)^{(1/2)} \}. \]

But if \( x \in E^1_k \) it cannot happen \( x \in (E'_k)^{(0)} \) and \( x \in (E_k \setminus E'_k)^{(0)} \), otherwise \( x \in E^1_k \) which is a contradiction. This proves (2.12). Also if \( x \in E^1_k \) then it cannot happen \( x \in (E'_k)^{(1/2)} \) and \( x \in (E_k \setminus E'_k)^{(1/2)} \), otherwise \( x \in E^1_k \) which is again a contradiction. This proves (2.13).

By (2.12) and (2.13), summing (2.9) with (2.11) we obtain for every \( k \in \mathbb{N} \)
\[ P(E'_k; \Omega) + P(E_k \setminus E'_k; \Omega) = 2H^{n-1}(\partial^* E'_k \cap E^1_k) + P(E_k; \Omega). \] (2.14)

Since \( E' \subset E \), repeating the same argument we have also in this case
\[ P(E'; \Omega) + P(E \setminus E'; \Omega) = 2H^{n-1}(\partial^* E' \cap E^1_k) + P(E; \Omega). \] (2.15)

Finally if we call \( l := \lim \inf_{k \to \infty} H^{n-1}(\partial^* E'_k \cap E^1_k) \) (without loss of generality we can assume \( l \in \mathbb{R} \)), using (2.14) and the lower semi-continuity of the perimeter on \( \Omega \), we can write
\[ 2H^{n-1}(\partial^* E' \cap E^1_k) + P(E; \Omega) = P(E'; \Omega) + P(E \setminus E'; \Omega) \leq \lim \inf_{k \to \infty} P(E'_k; \Omega) + P(E_k \setminus E'_k; \Omega) = \lim \inf_{k \to \infty} 2H^{n-1}(\partial^* E'_k \cap E^1_k) + P(E_k; \Omega) = 2l + \lim \inf_{k \to \infty} P(E_k; \Omega) = 2l + P(E; \Omega), \]

which is our desired result.

\[ \square \]

Remark 2.4. If the \((E_k)_k\) of the previous proposition are open sets, say for example \((U_k)_k\), such that \( H^{n-1}(U^1_k \Delta U_k) = 0 \) for every \( k \), then we have
\[ \lim \inf_{k \to \infty} P(E'_k; U_k) \geq P(E'; U). \]

Given \( F \subset \Omega \) an indecomposable set, we want to introduce an isoperimetric quantity \( h_F \), which is a function \( h_F : (0, \frac{1}{2}) \to \mathbb{R}^+ \), and which plays a similar role to the so called Cheeger’s constant. We recall that when \( \Omega \) is an open set of \( \mathbb{R}^n \), \((n \geq 2)\) the Cheeger’s constant is defined as (see [14],[15])
\[ h(\Omega) := \left\{ \frac{P(E)}{|E|} \mid E \subset \Omega, |E| > 0 \right\}. \] (2.16)
Let us remind that the Cheeger’s constant was introduced in [3] to study lower bounds for the smallest eigenvalue of the Laplace operator on compact Riemannian manifold without boundary. As a consequence, one obtains a validity of a Poincaré’s inequality with optimal constant uniformly bounded from below by a geometric constant. The analogous problem for the \( p \)-Laplacian \( (1 \leq p < \infty) \) with Dirichlet boundary conditions with \( M \) replaced by a bounded open set \( \Omega \subset \mathbb{R}^n \) such that \( \Omega^{(1)} = \Omega \), can be found for example in [14].

In our case, since we are interested in a weaker version of Poincaré’s inequality for indecomposable sets and without the assumption of Dirichlet boundary conditions, we need a different definition.

**Definition 2.5** (Upper isoperimetric profile). Let \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) be open and let \( F \subset \Omega \) be an indecomposable set with \( |F| < \infty \). For every \( \lambda \in (0,1/2] \) we define the relative Cheeger’s constant of \( F \) at level \( \lambda \) as

\[
h_F(\lambda) := \inf \left\{ \frac{\mathcal{H}^{n-1}(\partial^* E \cap F^{(1)})}{|E|} \mid \lambda |F| \leq |E| \leq |F|/2, \, P(E; \Omega) < \infty \right\}. \tag{2.17}
\]

We call the function \( h_F : (0,1/2] \rightarrow \mathbb{R}^+ \) the upper isoperimetric profile of \( F \).

**Remark 2.6.** The upper isoperimetric profile is a non decreasing function. Moreover, if we take an indecomposable open set \( U \subset \Omega \) such that \( \mathcal{H}^{n-1}(U^{(1)} \Delta U) = 0 \), then (2.17) reduces to

\[
h_F(\lambda) := \inf \left\{ \frac{P(U; E)}{|E|} \mid \lambda |U| \leq |E| \leq |U|/2, \, P(E; \Omega) < \infty \right\}.
\]

Notice that \( \inf_{\lambda > 0} h_U(\lambda) \) is not the Cheeger’s constant in (2.16), since we look only at the relative perimeter of \( E \) inside \( U \), while in (2.16) one is interested in the whole perimeter of \( E \).

Notice also that in literature (in particular in the context of Riemannian manifolds) the isoperimetric profile at \( \lambda \) is defined by considering the infimum among all sets \( E \) with fixed volume \( |E| = \lambda |F| \). Since we ask for \( |F| \leq |E| \) we decide to call it upper isoperimetric profile.

**Proposition 2.7.** Let \( F \subset \Omega \) be an indecomposable set with \( |F| < \infty \). Then \( h_F(\lambda) > 0 \) for every \( \lambda \in (0,1/2] \). In particular, it holds the following relative isoperimetric inequality

\[
|E| \leq \frac{1}{h_F(\lambda)} \mathcal{H}^{n-1}(\partial^* E \cap F^{(1)}), \tag{2.18}
\]

for every \( E \subset F \) with \( \lambda |F| \leq |E| \leq |F|/2 \) and \( P(E; \Omega) < \infty \).

**Proof.** Let \( \lambda \in (0,1/2] \) and consider

\[
h_F(\lambda) = \inf_{E \subset F} \frac{\mathcal{H}^{n-1}(\partial^* E \cap F^{(1)})}{|E|}. \tag{2.19}
\]

Clearly \( h_F(\lambda) \) is finite. Then it is enough so show that it is strictly positive. Consider a minimizing sequence \( (E_k)_{k \in \mathbb{N}} \) i.e.

\[
h_F(\lambda) = \lim_{k \to \infty} \frac{\mathcal{H}^{n-1}(\partial^* E_k \cap F^{(1)})}{|E_k|};
\]

since

\[
P(E_k; \Omega) = \mathcal{H}^{n-1}(\partial^* E_k \cap F^{(1)}) + \mathcal{H}^{n-1}(\{\nu_F = \nu_{E_k}\})
\]

\[
\leq P(F; \Omega) + (h_F(\lambda) + \epsilon)|E_k|
\]

\[
\leq P(F; \Omega) + (h_F(\lambda) + \epsilon)(|F|/2),
\]

then by using [1, Theorem 3.39], up to subsequence there exists a set \( E_\infty \subset F \) having finite perimeter with \( \lambda |F| \leq |E_\infty| \leq |F|/2 \) and such that \( \lim_{k \to \infty} |E_k \Delta E_\infty| = 0 \). Moreover thanks to Proposition 2.7 we have

\[
h_F(\lambda) = \lim_{k \to \infty} \frac{\mathcal{H}^{n-1}(\partial^* E_k \cap F^{(1)})}{|E_k|} \geq \frac{\mathcal{H}^{n-1}(\partial^* E_\infty \cap F^{(1)})}{|E_\infty|},
\]
which means
\[ h_F(\lambda) = \frac{\mathcal{H}^{n-1}(\partial^* E_\infty \cap F^{(1)})}{|E_\infty|}. \]

Finally, since \( \lambda|F| \leq |E_\infty| \leq |F|/2 \) and \( F = E_\infty \cup (F \setminus E_\infty) \), by Proposition (1.19), the indecomposability of \( F \) forces \( \mathcal{H}^{n-1}(\partial^* E_\infty \cap F^{(1)}) > 0 \). This concludes the proof. \( \square \)

**Remark 2.8.** Notice that \( \inf_{\lambda > 0} h_F(\lambda) \) might be equal to zero. Indeed consider two sequences of positive real numbers \( (l_n)_{n=1}^\infty \) and \( (\delta_n)_{n=1}^\infty \) such that \( \sum_{n=1}^\infty l_n^2 < \infty \) and \( \lim_{n \to \infty} \delta_n/l_n^2 = 0 \). Define an open set \( U \subset \mathbb{R}^2 \) made of an union of disjoint open squares \( Q_n \) of side \( l_n \), each connected to an open big rectangle through small bridges of size \( \delta_n \) as in figure (1).

![Figure 1. Indecomposable set with \( \inf_{\lambda > 0} h_U(\lambda) = 0 \).](image)

By our choice of \( l_n \), \( U \) is an open set with finite perimeter, and it can be easily shown that it is indecomposable.

For every \( n \in \mathbb{N} \) we define \( E_n \subset U \) to be the square of side \( l_n \) union half of the \( n \)-th bridge as in figure (1). By our choice of \( l_n \) and \( \delta_n \) we have
\[
\inf_{n \in \mathbb{N}} \frac{\mathcal{H}^1(\partial^* E_n \cap U^{(1)})}{|E_n|} = 0.
\]
However, Proposition 2.7 tells us that this could happen only for sequence of \( E_n \) such that \( |E_n| \to 0 \).

Moreover, by using the Coarea Formula, it can be proved that \( \inf_{\lambda > 0} h_F(\lambda) > 0 \) if and only if for every \( u \in BV(\Omega) \) the following Poincaré’s inequality holds true
\[
\int_F |u - m| \, dx \leq c |Du|(F^{(1)}),
\]
where \( m \) is the median of \( u \) on \( F \) (see Definition 2.10). In this case the best constant \( c \) which satisfies the previous inequality is exactly \( \inf_{\lambda > 0} h_F(\lambda) \).

**Remark 2.9.** Given \( F \subset B_r(x) \subset \Omega \) indecomposable, then simply by definition, we have the following scaling property of the relative upper isoperimetric profile:
\[
h_F(\cdot) = r h_{\frac{F}{r}}(\cdot),
\]
for every \( r > 0 \).

### 2.2. Weak Poincaré’s inequalities

Now we are in position to prove a weak version of Poincaré’s inequality on indecomposable sets. Before we need the following definition.

**Definition 2.10.** Let \( u : \Omega \to \mathbb{R} \) be a measurable function. Given a measurable set \( F \subset \Omega \) we define the **median** of \( u \) in \( F \) as
\[
m(u, F) := \inf \left\{ t \in \mathbb{R} \mid |\{u > t\} \cap F| \leq \frac{|F|}{2} \right\}.
\]

**Remark 2.11.** It holds
\[
|\{u > t\} \cap F| \leq \frac{|F|}{2} \quad \text{for} \quad t \geq m(u, F), \quad |\{u > t\} \cap F| > \frac{|F|}{2} \quad \text{for} \quad t < m(u, F).
\]

\[ (2.20) \]
Theorem 2.12. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\Gamma \subset \Omega$ be a countable $(\mathcal{H}^{n-1}, n - 1)$-rectifiable set. Let $F \subset \Omega$ be an indecomposable set such that $|F| < \infty$ and $\mathcal{H}^{n-1}(\Gamma \cap F^{(1)}) = 0$. Then for every $u \in GSBV^p(\Omega; \Gamma)$ $(p \geq 1)$ and for every $\lambda \in (0, 1/2)$ there exists a measurable set $F^\lambda \subset F$ such that

$$ |F \setminus F^\lambda| \leq 2\lambda|F|, \quad (2.21) $$
and the following inequality holds

$$ \left( \int_{F^\lambda} |u - m|^p \, dx \right)^{\frac{1}{p}} \leq \frac{2p}{h_F(\lambda)} \left( \int_{F^\lambda} |\nabla u|^p \, dx \right)^{\frac{1}{p}}, \quad (2.22) $$

where $m := m(u, F)$.

Proof. Let $v \in GSBV(\Omega; \Gamma)$ be a positive function such that

$$ |\{v > t\} \cap F| \leq \frac{|F|}{2} \quad \text{for } t > 0. \quad (2.23) $$

Define

$$ s := \inf \{t : |\{v > t\} \cap F| \leq \lambda|F|\}, \quad (2.24) $$
and notice that

$$ |\{v > s\} \cap F| \leq \lambda|F| \quad \text{for } s \geq 0, \quad |\{v > t\} \cap F| > \lambda|F| \quad \text{for } t < s. \quad (2.25) $$

If we set $s^e := v \wedge s$ we can write

$$ \int_{F \cap \{v \leq s\}} v \, dx \leq \int_F v^e \, dx = \int_0^s |F \cap \{v > t\}| \, dt. \quad (2.26) $$

Since $\{v > t\} = \{v^e > t\}$ for every $t \in (0, s)$, then by (2.13), (2.25) and the definition of $h_F(\cdot)$ we have

$$ \mathcal{H}^{n-1}(\partial^* \{v > t\} \cap F^{(1)}) \geq h_F(\lambda)|F \cap \{v > t\}| = h_F(\lambda)|F \cap \{v^e > t\}|. $$

Then by (2.26) we can use the Coarea Formula for $BV$ functions (see [1, Theorem 3.40]) to obtain

$$ \int_{F^{(1)} \cap \{v \leq s\}} v \, dx \leq \frac{1}{h_F(\lambda)} \int_0^s \mathcal{H}^{n-1}(\partial^* \{v > t\} \cap F^{(1)}) \, dt $$

$$ = \frac{1}{h_F(\lambda)} |Dv^e|(F^{(1)}) $$

$$ = \frac{1}{h_F(\lambda)} \int_{F^{(1)} \cap \{v \leq s\}} |\nabla v| \, dx, \quad (2.27) $$

where for the last equality we have used $\mathcal{H}^{n-1}(\Gamma \cap F^{(1)}) = 0$ together with the chain rule in $BV$ (see [1, Theorem 3.99]).

Now define $(u - m)_t^p := [(u - m) \vee t]^p$. Since by (2.20)

$$ |\{(u - m)_t^p > t\} \cap F| \leq \frac{|F|}{2} \quad \text{for } t > 0, $$
we can apply (2.27) to the function $(u - m)_t^p$ instead of $v$ to deduce the existence of $s^+ \geq 0$ such that, thanks to the chain rule in $BV$ we can write

$$ \int_{F \cap \{0 < (u - m)_t^p \leq s^+\}} (u - m)_t^p \, dx \leq \frac{p}{h_F(\lambda)} \int_{F \cap \{0 < (u - m)_t^p \leq s^+\}} (u - m)^{p-1}_t |\nabla u| \, dx \quad (2.28) $$

where we used that both integrals vanish on the set $\{(u - m)_t^p = 0\}$ and that $|F \Delta F^{(1)}| = 0$. Analogously, if we set $(u - m)_t^p := [(u - m) \wedge t]^p$ by (2.20)

$$ |\{(u - m)_t^p > t\} \cap F| \leq \frac{|F|}{2}, \quad \text{for } t > 0. $$

Arguing as before there exists $s^- > 0$ such that

$$ \int_{F \cap \{0 < (u - m)_t^p \leq s^-\}} (u - m)_t^p \, dx \leq \frac{p}{h_F(\lambda)} \int_{F \cap \{0 < (u - m)_t^p \leq s^-\}} (u - m)^{p-1}_t |\nabla u| \, dx \quad (2.29) $$
If we set $F^\lambda := \{ t_0 - (s^{-1/p}) \leq u \leq t_0 + (s^{-1/p}) \cap F \}$ by (2.25) we have $|F \setminus F^\lambda| \leq 2\lambda|F|$. By summing the previous two inequalities and by using Hölder inequality we deduce

$$
\left( \int_{F^\lambda} |u - m|^p \, dx \right)^{\frac{1}{p}} \leq \frac{p}{h_F(\lambda)} \left( \int_{F^\lambda} |\nabla u|^p \, dx \right)^{\frac{1}{p}},
$$

(2.30)

which immediately implies (2.22).

\[ \square \]

**Corollary 2.13** (Weak Poincaré’s inequality). Under the same hypothesis of Theorem 2.12 we have for every $\lambda \in (0, 1/2]$ and for every $u \in GSBV^p(\Omega; \Gamma)$

$$
\left( \int_{F} |u - m|^p \wedge 1 \, dx \right)^{\frac{1}{p}} \leq \frac{2p}{h_F(\lambda)} \left( \int_{F} |\nabla u|^p \, dx \right)^{\frac{1}{p}} + (2\lambda|F|)^{\frac{1}{p}},
$$

(2.31)

where $m := m(u, F)$.

**Proof.** Given $u \in GSBV^p(\Omega; \Gamma)$, we can consider $F^\lambda$ and $t_0$ as in Theorem 2.12. Then we can write

$$
\left( \int_{F} |u - m|^p \wedge 1 \, dx \right)^{\frac{1}{p}} \leq \left( \int_{F^\lambda} |u - m|^p \, dx \right)^{\frac{1}{p}} + |F \setminus F^\lambda|^{\frac{1}{p}}
$$

(2.32)

$$
\leq \frac{2p}{h_F(\lambda)} \left( \int_{F} |\nabla u|^p \, dx \right)^{\frac{1}{p}} + (2\lambda|F|)^{\frac{1}{p}},
$$

which is exactly (2.31).

\[ \square \]

3. SETS WITH NON VANISHING UPPER ISOPERIMETRIC PROFILE

**Definition 3.1.** We say that a set $A \subset B_1(0)$ is conical, if

$$
|(A \cap \lambda A) \Delta \lambda A| = 0 \text{ for every } \lambda \in (0, 1).
$$

**Definition 3.2.** Given $(A_r)_{r > 0}$ a family of subsets of $\Omega$, we say that it is left or right continuous, if for every $r_0 > 0$ we have

$$
\lim_{r \to r_0^-} |A_r \Delta A_r| = 0 \quad \text{or} \quad \lim_{r \to r_0^+} |A_r \Delta A_r| = 0,
$$

respectively.

**Definition 3.3** (Non vanishing upper isoperimetric profile). Let $\Gamma \subset \Omega$. Given $x \in \Omega$ we say that $\Gamma$ has a non vanishing upper isoperimetric profile at $x$ if there exists $N_x \in \mathbb{N}$ such that

1. for every $1 \leq j \leq N_x$ there exists $(F_{r,j})_{0 < r \leq r_x}$ a left continuous family of indecomposable subsets of $B_1(0)$, with the following properties

\[ (1.1) \quad \mathcal{H}^{n-1}(T_{r,x} \cap F_{r,j}^{(1)}) = 0, \quad r \in (0, r_x); \]

\[ (1.2) \quad \liminf_{r \to 0^+} h_{F_{r,j}}(\lambda) > 0, \quad \lambda \in (0, 1/2]; \]

2. there exists a measurable partition of $B_1(0)$ made of (nonempty) conical sets $(E_{0,j})_{j=1}^{N_x}$ with the following property

\[ (2.1) \quad \lim_{r \to 0^+} |F_{r,j} \Delta E_{0,j}| = 0. \]

**Remark 3.4.** In order to prevent misunderstandings, since $F_{r,j}$ are subsets of the unitary ball, in the definition of $h_{F_{r,j}}$ (Definition (2.17)), the infimum has to be taken among all sets contained in $F_{r,j}$ satisfying the constraint on the volume and such that their perimeter is finite in $B_1$.

**Remark 3.5.** The property of non vanishing upper isoperimetric profile is stable under inclusion, in the sense that whenever $\Gamma' \subset \Gamma$ and $\Gamma$ has a non vanishing upper isoperimetric profile at $x$, then also $\Gamma'$ has the same property at $x$.

We give a basic example which clarifies the concept of non vanishing upper isoperimetric profile.
Example 3.6. Let $\Sigma \subset \Omega$ be an $(n - 1)$-dimensional manifold of class $\mathcal{C}^1$. Then $\Sigma$ has a non-vanishing upper isoperimetric profile for every $x \in \Omega$.

Let us first suppose $x \in \Sigma$. Then if we call $\nu(x)$ a unit normal to $\Sigma$ at $x$, we know that there exists a sufficiently small value $r_x > 0$ and a $\mathcal{C}^1$ function $f : \nu(x)^\perp \to \mathbb{R}$ such that

$$B_r(x) \cap \Sigma = B_r(x) \cap \text{graph}(f), \ r \in (0, r_x).$$

By writing the generic $y \in \mathbb{R}^n$ as $y = (z, t)$ where $y \in \nu(x)^\perp$ and $t \in \mathbb{R}$, we define

$$F_1 := \{y \in B_{r_x}(x) \mid t < f(z)\} \quad F_2 := \{y \in B_{r_x}(x) \mid t > f(z)\},$$

and

$$N_x = 2, \quad F_{r,1} := (F_1)_{r,x}, \quad F_{r,2} := (F_2)_{r,x}, \quad r \in (0, r_x);$$

$$E_{0,1} := \{y \in B_1(0) \mid \nu(x) \cdot y < 0\}, \quad E_{0,2} := \{y \in B_1(0) \mid \nu(x) \cdot y > 0\}.$$

To prove that the families $(F_{r,j})_{0 < r < r_x}$ are left continuous ($j = 1, 2$), it is equivalent to prove that for every $r \in (0, r_x)$ it holds

$$\lim_{\lambda \to 1-} \|\mathbb{I}_{F_{r,j}} - \mathbb{I}_{F_{r,j}}\|_{L^1(B_1)} = 0. \quad (3.1)$$

Since by definition

$$F_{r,j} = F_{j} - \frac{x}{r} \cap B_1(0) = 1 \left[ F_j - \frac{x}{r} \cap B_1(0) \right] \mathbb{I}_{F_{r,j}}(\mathbb{I}_{F_{r,j}}), \quad j = 1, 2,$$

then we have

$$\mathbb{I}_{F_{r,j}}(x) = \mathbb{I}_{F_{r,j}}(\lambda x), \quad x \in B_1(0), \quad j = 1, 2.$$

This means that the convergence (3.1) can be rewritten as

$$\lim_{\lambda \to 1-} \int_{B_1} |\mathbb{I}_{F_{r,j}}(\lambda x) - \mathbb{I}_{F_{r,j}}(x)| \, dx = 0, \quad j = 1, 2,$$

and this last convergence follows by the continuity of the dilations in $L^1$.

Conditions (1.1),(2) and (2.1) follow easily by construction.

To prove condition (1.2), one can use the $\mathcal{C}^1$ regularity of $f$ and an argument similar to the one in Example A.4, to deduce that the open sets $F_{r,j}$ ($j = 1, 2$) admit a Poincaré’s inequality of the form

$$\int_{F_{r,j}} |u - \int_{F_{r,j}} u| \, dx \leq c |Du|(F_{r,j}), \ u \in BV(B_1) \quad (3.2)$$

where $c > 0$ is a constant independent on $r \in (0, r_x)$. So given $E \subset F_{r,j}$ a set of finite perimeter in $B_1$, we can use $\|E\|_E$ instead of $u$ in (3.2) to deduce that

$$\min\{|E|, |F_{r,j} \setminus E|\} \leq c |Du|(F_{r,j}) = c \mathcal{H}^{n-1}(\partial^* E \cap F_{r,j}) = c \mathcal{H}^{n-1}(\partial^* E \cap F_{r,j}^{(1)}),$$

where the right-most equality follows from the fact $F_{r,j} = F_{r,j}^{(1)}$.

Another possibility to prove that $\Sigma$ has a non-vanishing upper isoperimetric profile at $x \in \Sigma$, is to notice that, since $\Sigma$ is an $(n - 1)$-manifold of class $\mathcal{C}^1$, we can always find a set of finite perimeter $E \subset \Omega$ such that $\Sigma \subset \partial^* E$. In this case we can make use of the next Proposition 3.7, which says that $\partial^* E$ admits a non-vanishing upper isoperimetric profile at every point $x \in \partial^* E$. Since the property of non-vanishing upper isoperimetric profile is stable under inclusion (Remark 3.5), this means that also $\Sigma$ has this property for every $x \in \Sigma$.

Finally, the case $x \in \Omega \setminus \Sigma$ is much easier. Indeed, by the closeness of $\Sigma$ there exists $r_x > 0$ small enough such that $B_{r_x}(x) \cap \Sigma = \emptyset$ for every $r \in (0, r_x)$. Then it is enough to set

$$N_x = 1, \quad F_{r_1} := B_1(0), \quad r \in (0, r_x);$$

$$E_{0,1} := B_1(0).$$

**Proposition 3.7.** Let $E \subset \Omega$ be a set of finite perimeter. Then the reduced boundary $\partial^* E$ has a non vanishing upper isoperimetric profile for every point $x$ which belongs to the following set

$$\{x \in \Omega \mid \Theta^{n-1}(\mathcal{H}^{n-1} \setminus \partial^* E, x) = 0\} \cup \partial^* E.$$

Proof. First we deal with the case \( x \in \partial^*E \).

We denote as \( H \) the half space given by Theorem 1.10 such that
\[
\lim_{r \to 0^+} |E_{r,x} \Delta H \cap B_1(0)| = 0, \quad \lim_{r \to 0^+} P(E_{r,x}; B_1(0)) = P(H; B_1(0)).
\] (3.3)

Clearly \( E_{0,1}, E_{0,2} \) are conical and indecomposable sets. Thus we can apply Proposition 2.2, to find two families \( F_{r,1} \) and \( F_{r,2} \) made of indecomposable components of \( E_{r,x} \) and \( B_1(0) \setminus E_{r,x} \), respectively, such that
\[
\lim_{r \to 0^+} |F_{r,1} \Delta H \cap B_1(0)| = 0, \quad \lim_{r \to 0^+} P(F_{r,1}; B_1(0)) = P(E_{0,1}; B_1(0)),
\] (3.4)
\[
\lim_{r \to 0^+} |F_{r,2} \Delta B_1(0) \setminus H| = 0, \quad \lim_{r \to 0^+} P(F_{r,2}; B_1(0)) = P(E_{0,2}; B_1(0)).
\] (3.5)

Given \( r_x > 0 \) such that \( B_{r_x}(x) \subset \Omega \), we set
\[
E_1 := E \cap B_{r_x}(x), \quad E_2 := B_{r_x}(x) \setminus E,
\]
and
\[
E_{0,1} = H \cap B_1(0), \quad E_{0,2} = B_1(0) \setminus H.
\]

Clearly \( E_{0,1}, E_{0,2} \) are conical and indecomposable sets. This choice guarantees also (1.1) and (2.1) of Definition 3.3. Moreover, using that for every \( r \) the sets \( F_{r,j} \) are indecomposable components of \( (E_j)_{r,x} \) (\( j = 1, 2 \)), respectively, together with (3.3), we can apply Proposition 3.8 to deduce that the family \( r \to F_{r,j} \) are left continuous.

Finally, in order to show (1.2) of Definition 3.3, we claim that
\[
\liminf_{r \to 0^+} h_{F_{r,1}}(\lambda) \geq h_H(\lambda), \quad \lambda \in (0, 1/2],
\] (3.6)
and
\[
\liminf_{r \to 0^+} h_{F_{r,2}}(\lambda) \geq h_{B_1 \setminus H}(\lambda), \quad \lambda \in (0, 1/2].
\] (3.7)

We prove for example (3.6). To this purpose fix \( \lambda \in \left(0, \frac{1}{2}\right] \) and for every \( r \in (0, r_x) \) consider \( E_r \subset F_{r,1} \) with \( P(E_r; B_1) < \infty \), such that
\[
\frac{\mathcal{H}^{n-1}(\partial^* E_r \cap F_r^{(1)})}{|E_r|} \leq h_{F_{r,1}}(\lambda) + r, \quad \lambda|F_{r,1}| \leq |E_r| \leq |F_{r,1}|/2.
\] (3.8)

We show that for every subsequence \( (r_m) \) such that \( r_m \to 0^+ \) as \( m \to \infty \) then
\[
\liminf_{m \to \infty} h_{F_{r_m,1}}(\lambda) \geq h_H(\lambda).
\]

Without loss of generality we assume
\[
\lim_{m \to \infty} h_{F_{r_m,1}}(\lambda) = \lim_{m \to \infty} h_{F_{r_m,2}}(\lambda) = l < \infty.
\]

Since \( E_{r_m} \subset F_{r_m,1} \), by using Leibniz’s formula 1.18 the inequalities (3.8) say to us
\[
\sup_m P(E_{r_m}; B_1) \leq \sup_m \left( |E_{r_m}| h_{F_{r_m,1}}(\lambda) + P(F_{r_m,1}; B_1) \right) < \infty.
\]

This means that, thanks to the compactness result [1, Theorem 3.39], eventually passing through another subsequence we have \( \lim_{m \to \infty} |E_{r_m} \Delta E_0| = 0 \), for some sets \( E_0 \subset H \) with finite perimeter in \( B_1(0) \) and with \( \lambda|H| \leq |E_0| \leq |H|/2 \). Hence thanks to (3.4), we are in position to apply the lower semi-continuity result of Proposition 2.3 to obtain
\[
\liminf_{m \to \infty} h_{F_{r_m,1}}(\lambda) \geq \liminf_{m \to \infty} \frac{\mathcal{H}^{n-1}(E_{r_m} \cap F_{r_m,1}^{(1)})}{|E_{r_m}|} - r_m \geq \frac{\mathcal{H}^{n-1}(E_0 \cap H^{(1)})}{|E_0|} \geq h_H(\lambda).
\]

The same argument shows the validity of (3.7). Since \( h_H(\lambda) > 0 \), this says that \( \partial^* E \) admits a non-vanishing upper isoperimetric profile at \( x \) with \( N_x = 2 \).

In the case \( x \in \Omega \) is such that \( \Theta^{(n-1)}(\mathcal{H}^{n-1} \perp \partial^* E, x) = 0 \), we claim that we have two different sub-cases:
\[
\lim_{r \to 0^+} |B_1(0) \setminus E_{r,x}| = 0,
\] (3.9)
or
\[
\lim_{r \to 0^+} |E_{r,x}| = 0.
\] (3.10)
Indeed by a simple application of the relative isoperimetric inequality in the unitary ball we can write
\[
\min\{|E_{x,r},|,|B_1(0)\setminus E_{x,r}\|\} \leq C(n)\frac{P(E; B_1(0))}{r^{n-1}} = C(n)\frac{\mathcal{H}^{n-1}(\partial^* E \cap B_1(0))}{r^{n-1}},
\]
and by the fact that \( r \mapsto |E_{x,r}| \) is a continuous map on \( (0, r_x) \) we deduce that one between (3.9) and (3.10) must occur. Suppose for example (3.9) holds. Given \( r_x > 0 \) such that \( B_{r_x}(x) \subset \Omega \), we set
\[
E_1 := E \cap B_{r_x}(x), \quad E_{0,1} = B_1(0).
\]
Arguing in the very same way as before, we can make use of Proposition 2.2 to find for every \( r \in (0, r_x) \) an indecomposable component of \( (E_1)_{r,x} \), say \( F_{r,1} \), which form a left continuous family and such that
\[
\lim_{r \to 0^+} |F_{r,1}\Delta B_1(0)| = 0, \quad \lim_{r \to 0^+} P(F_{r,1}; B_1) = 0.
\]
Finally, by using again Proposition 2.3 we can prove in the very same way as before that
\[
\liminf_{r \to 0^+} h_{F_{r,1}}(\lambda) \geq h_{B_1}(\lambda), \quad \lambda \in (0,1/2].
\]
Case (3.10) can be treated in the same way. \( \square \)

The following proposition gives a sufficient condition for which a family of indecomposable sets \( (F_r)_{r>0} \) is left continuous in the sense of Definition 3.2.

**Proposition 3.8.** Let \( E \subset \Omega \) be a set of finite perimeter and let \( r_x > 0 \) be such that \( B_{r_x}(x) \subset \Omega \). Suppose that there exists \( E_0 \subset B_1(0), |E_0| > 0 \) and such that
\[
\lim_{r \to 0^+} |E_{r,x}\Delta E_0| = 0.
\]
If \( F_r \) is an indecomposable component of \( E_{r,x} \) \( (0 < r < r_x) \) such that
\[
\lim_{r \to 0^+} |F_r\Delta E_0| = 0,
\]
then there exists \( 0 < r'_x \leq r_x \), such that the family \( (F_r)_{r \in (0,r'_x)} \) is continuous from the left.

**Proof.** Fix \( r \in (0, r_x) \) and define
\[
\mathcal{F}_{\lambda r} := \frac{F_r \cap B_1(0)}{\lambda}, \quad \lambda \in (0,1]. \tag{3.11}
\]
Notice that arguing as in Example 3.6 it holds \( \lim_{\lambda \to 1^-} |\mathcal{F}_{\lambda r}\Delta F_r| = 0 \). In order to simplify the notation we will write \( E_r := E_{r,x} \).

By applying Proposition 1.20 to the triple \( E_{\lambda r}, \mathcal{F}_{\lambda r}, F_r \), since \( \partial^* \mathcal{F}_{\lambda r} \subset \partial^* E_{\lambda r} \), we deduce that for every \( \lambda \) one only one of the following can hold
\[
F_{\lambda r} \subset \mathcal{F}_{\lambda r} \quad \text{or} \quad \mathcal{F}_{\lambda r} \subset F_{\lambda r} \setminus \mathcal{F}_{\lambda r}. \tag{3.12}
\]
We claim that there exists \( 0 < r''_x < r_x \) such that for every \( r < r''_x \) there exists \( 0 < \delta''_r < 1 \) such that
\[
F_{\lambda r} \subset \mathcal{F}_{\lambda r} \quad \text{or} \quad \mathcal{F}_{\lambda r} \subset F_{\lambda r} \setminus \mathcal{F}_{\lambda r}. \tag{3.13}
\]
Indeed let \( r''_x \) be small enough in such a way that \( |F_r\Delta E_0| \leq |E_0|/8 \) and \( |E_r \setminus E_0| \leq |E_0|/8 \). We define for every \( 0 < r < r''_x \) a real number \( \delta''_r \in (0,1) \) such that \( |\mathcal{F}_{\lambda r}\Delta F_r| \leq |E_0|/8 \) for \( \lambda \in (1-\delta''_r,1] \). If with this choice of \( r''_x \) and \( \delta''_r \) inclusion (3.13) does not hold, then there exists a positive \( r \) with \( r < r''_x \) and a \( \lambda \in (1-\delta''_r,1] \) such that \( F_{\lambda r} \subset E_{\lambda r} \setminus \mathcal{F}_{\lambda r} \), therefore
\[
\frac{7|E_0|}{8} \leq |E_0| - |F_{\lambda r}\Delta E_0| \leq |E_{\lambda r}\setminus \mathcal{F}_{\lambda r}| \leq |E_{\lambda r}\Delta E_0| \setminus \mathcal{F}_{\lambda r}| + |E_0 \setminus \mathcal{F}_{\lambda r}| \leq |E_{\lambda r}\Delta E_0| + |E_0 \setminus F_r| + |F_r \setminus \mathcal{F}_{\lambda r}| \leq \frac{3|E_0|}{8},
\]
which is a contradiction and proves our claim.
Moreover, notice that
\[
\lim_{\lambda \to 1} P(F_{\lambda r}, B_1(0)) = \lim_{\lambda \to 1} \frac{1}{\lambda^{n-1}} P(F_r, B_1(0)) = P(F_r, B_1(0)),
\]
where for the last equality we used the continuity of the Radon measure \( P(F_r, \cdot) \) with respect to monotone sequences. Therefore, since

(1) \( \lim_{\lambda \to 1^-} |F_{\lambda r} \Delta F_r| = 0; \)
(2) \( \lim_{\lambda \to 1^-} P(F_{\lambda r}, B_1(0)) = P(F_r, B_1(0)); \)
(3) \( F_r \) is indecomposable,

we can apply Theorem 2.2 to deduce the existence of a sequence \((\tilde{F}_{\lambda r})_{\lambda \in (0, 1)}\) such that for each \( \lambda \) \( \tilde{F}_{\lambda r} \) is an element of the Caccioppoli’s indecomposable partition of \( F_{\lambda r} \) and
\[
\lim_{\lambda \to 1^-} |\tilde{F}_{\lambda r} \Delta F_r| = 0.
\]
We claim that there exists \( 0 < r'_x < r_x \) such that for every \( r < r'_x \) there exists \( 0 < \delta'_r < 1 \) such that
\[
\tilde{F}_{\lambda r} = F_{\lambda r} \quad \lambda \in (1 - \delta'_r, 1].
\]
Indeed, since \( F_{\lambda r} \subset F_{\lambda r} \) for every \( r < r''_x \) and every \( \lambda \in (1 - \delta''_r, 1], \) since \( F_{\lambda r} \) is indecomposable, and since \( \partial^* F_{\lambda r} \subset \partial^* F_{\lambda r} \) (here we use property (5) for indecomposable components Proposition 1.16), we can apply Proposition 1.20 to the triple \( \tilde{F}_{\lambda r}, F_{\lambda r}, \tilde{F}_{\lambda r}, \) to obtain that one and only one of the followings holds
\[
F_{\lambda r} \subset \tilde{F}_{\lambda r} \text{ or } F_{\lambda r} \subset \tilde{F}_{\lambda r} \setminus F_{\lambda r}.
\]
Since for every \( r < r_x \) \( |F_{\lambda r} \Delta F_{\lambda r}| \to 0 \) as \( \lambda \to 1^- \) and \( |F_r \Delta E_0| \to 0 \) as \( r \to 0^+ \), arguing in the very same way as before, we can find \( 0 < r'_x < r''_x \) such that for every \( r < r'_x \) there exists \( 0 < \delta'_r < \delta''_r \) such that
\[
F_{\lambda r} \subset \tilde{F}_{\lambda r} \quad \lambda \in (1 - \delta'_r, 1].
\]
We want to prove that \( F_{\lambda r} = \tilde{F}_{\lambda r} \) for every \( r < r'_x \) and every \( \lambda \in (1 - \delta'_r, 1] \). It is enough to show
\[
|\tilde{F}_{\lambda r} \setminus F_{\lambda r}| = 0. \tag{3.14}
\]
But since
\[
F_{\lambda r}^{(1)} = \frac{F_r^{(1)} \cap B_\lambda(0)}{\lambda} = \frac{E_r^{(1)} \cap B_\lambda(0)}{\lambda} = E_{\lambda r}^{(1)},
\]
and since \( F_{\lambda r} \) is an indecomposable component of \( E_{\lambda r} \), by property (5) of Proposition 1.16 we must have \( \mathcal{H}^{n-1}(\partial^* F_{\lambda r} \cap F_{\lambda r}^{(1)}) = 0 \). As a consequence, since \( F_{\lambda r}^{(1)} \subset F_{\lambda r}^{(1)} \), then also
\[
\mathcal{H}^{n-1}(\partial^* F_{\lambda r} \cap F_{\lambda r}^{(1)}) = 0. \tag{3.15}
\]
If (3.14) does not hold, since \( \tilde{F}_{\lambda r} = F_{\lambda r} \cup \tilde{F}_{\lambda r} \setminus F_{\lambda r} \), by using Proposition 1.19 and the indecomposability of \( \tilde{F}_{\lambda r} \), immediately follows
\[
\mathcal{H}^{n-1}(\partial^* F_{\lambda r} \cap F_{\lambda r}^{(1)}) > 0,
\]
which is in contradiction with (3.15) and proves the theorem. \( \square \)

Now we are in position to introduce the space of all the admissible jump sets \( \Gamma \).

**Definition 3.9 (Admissible jump sets).** Let \( \Gamma \subset \Omega \) be a countable \( (\mathcal{H}^{n-1}, n-1)-\)rectifiable set with \( \mathcal{H}^{n-1}(\Gamma) < \infty \) and let \( 1 < p < n \). We say that \( \Gamma \) belongs to \( \mathcal{J}_p \) if for every \( x \in \Omega \setminus S_\Gamma \), where \( S_\Gamma \) is a set of Hausdorff dimension \( n - p \), \( \Gamma \) has a non vanishing upper isoperimetric profile at \( x \).

We postpone to appendix A examples of sets living in \( \mathcal{J}_p \).
4. Properties of the blow-up in $\text{GSBV}^p$

4.1. Weak Poincaré’s inequality on balls. We start this section by proving a weak version of Poincaré’s inequality on balls. First, we need the following definition.

**Definition 4.1.** Let $\Gamma \in \mathcal{J}_p$ ($1 < p < n$) and let $x \in \Omega \setminus S_\Gamma$. Let $r_x > 0$ and $N_x \in \mathbb{N}$ be given by Definition 3.3. We define for every $r \in (0, r_x)$, $\pi_{r,x} : B_{r,x}(x) \to \mathbb{R}$ as

$$\pi_{r,x}(y) := \begin{cases} m_j(u, r, x) & \text{on } x + rF_{r,j} \\ 0 & \text{otherwise.} \end{cases}$$

where $m_j(u, r, x) := m(u, F_{r,j})$ (see Definition 2.10), and $(F_{r,j})_{j=1}^{N_x}$ are the indecomposable sets given by Definition 3.3.

**Remark 4.2.** The median of $u$ in $F$ is invariant under rescaling and translation in the sense that

$$m(u, F) = m(u_{r,x}, (F - x)/r), \ x \in \Omega, \ r > 0.$$ 

Therefore, one has the following relation

$$m(u, r, x) = m(u_{r,x}, 1, 0), \ x \in \Omega, \ r > 0. \ (4.1)$$

**Theorem 4.3** (Weak Poincaré’s inequality on balls). Let $\Omega \subset \mathbb{R}^n$ be an open set and suppose that $\Gamma \subset \Omega$ has a non vanishing upper isoperimetric profile at $x$. Then for every $\lambda \in (0, 1/2]$ there exists $r_\lambda > 0$ (depending also on $x$) such that

$$\left( \int_{B_r(x)} |u - \pi_{r,x}|^p \wedge 1 \ dy \right)^{\frac{1}{p}} \leq C(p, n) \left[ \mathcal{H}_x(\lambda) \left( \int_{B_r(x)} |\nabla u|^p \ dy \right)^{\frac{1}{p}} + (r^p \lambda)^{\frac{1}{p}} \right], \ (4.2)$$

for every $r \leq r_\lambda$ and for every $u \in \text{GSBV}^p(\Omega; \Gamma)$, where

$$\mathcal{H}_x(\lambda) := \limsup_{r \to 0+} \left[ \max_{j=1, \ldots, N_x} \left\{ \frac{2}{h_{F_{r,j}}(\lambda)} \right\} \right] < \infty. \ (4.3)$$

**Proof.** Fix $\lambda > 0$ and let $x \in \Omega \setminus S_\Gamma$. By property (2.1) of Definition 3.3 we know that there exists $0 < r_{\lambda}^1 < r_x$ such that for every $r < r_{\lambda}^1$

$$\sup_{r<r_{\lambda}^1} |B_1(0) \setminus \bigcup_{j=1}^{N_x} F_{r,j}| \leq \lambda,$$

which means

$$\sup_{r<r_{\lambda}^1} |B_r(x) \setminus \bigcup_{j=1}^{N_x} x + rF_{r,j}| \leq r^p \lambda. \ (4.4)$$

Moreover by the definition of lim sup we can consider $r_{\lambda}^N$ small enough such that

$$\sup_{r<r_{\lambda}^N} \left[ \max_{j=1, \ldots, N_x} \left\{ \frac{1}{h_{F_{r,j}}(\lambda)} \right\} \right] \leq \mathcal{H}_x(\lambda) < \infty.$$

Since $u_{r,x} \in \text{GSBV}^p(B_1(0); \Gamma_{r,x})$ ($r < r_x$) and thanks to the fact $\mathcal{H}^{n-1}(\Gamma_{r,x} \cap F^{(1)}_{r,j}) = 0$ for every $1 \leq j \leq N_x$, by applying Theorem 2.12 we know that there exist $F_{r,j}^\lambda \subset F_{r,j}$ with

$$|F_{r,j} \setminus F_{r,j}^\lambda| \leq \lambda |F_{r,j}|, \ (4.5)$$

such that

$$\int_{F_{r,j}^\lambda} |u(x + ry) - m_{r,j}|^p \ dy \leq \left( \frac{2pr}{h_{F_{r,j}}(\lambda)} \right)^p \int_{F_{r,j}^\lambda} |\nabla u(x + ry)|^p \ dy,$$

where $m_{r,j} := m_j(u_{r,x}, 1, 0) = m_j(u, r, x)$.

If we define $F_{r}^\lambda := \bigcup_{j=1}^{N_x} F_{r,j}^\lambda$, then by summing on $j = 1, \ldots, N_x$ both sides of the previous inequality if $r \leq \min\{r_{\lambda}^1, r_{\lambda}^N\}$ we obtain

$$\int_{F_{r}^\lambda} |u(x + ry) - \pi_{r,x}(x + ry)|^p \ dy \leq (2p \mathcal{H}_x(\lambda)r)^p \int_{F_{r}^\lambda} |\nabla u(x + ry)|^p \ dy,$$
or equivalently
\[ \int_{x+rF} |u(y) - \overline{u}_{r,x}|^p \, dy \leq (2p\mathcal{H}_x(\lambda)r)^p \int_{x+rF} |\nabla u(y)|^p \, dy. \]

Finally using also (4.4) and (4.5) we can write
\[
\int_{B_r(x)} |u(y) - \overline{u}_{r,x}|^p \, dy + 1 \leq \int_{x+rF} |u(y) - \overline{u}_{r,x}|^p \, dy + \left(1 + \sum_{j=1}^{N_x} |F_{r,j}|\right)r^n \lambda
\]
\[
\leq (2p\mathcal{H}_x(\lambda)r)^p \int_{x+rF} |\nabla u(y)|^p \, dy + (1 + \omega_n)r^n \lambda
\]
\[
\leq (2p\mathcal{H}_x(\lambda)r)^p \int_{B_r(x)} |\nabla u(y)|^p \, dy + (1 + \omega_n)r^n \lambda
\]
\[
\leq C(u,p) \left((\mathcal{H}_x(\lambda)r)^p \int_{B_r(x)} |\nabla u(y)|^p \, dy + r^n \lambda\right).
\]

which is exactly (4.2). \qed

4.2. Convergence of the medians. The following proposition shows that for points \( x \) outside of the singular set \( S_\Gamma \), the medians \( m_j(u, r, x) \) are left continuous as functions of \( r \). This property will play a crucial role in the proof of Theorem 0.1.

Proposition 4.4. Let \( \Gamma \in \mathcal{F}_p (1 < p < n) \), \( x \in \Omega \setminus S_\Gamma \) and \( u \in GSBV^p(\Omega; \Gamma) \). Then the maps
\[ r \mapsto m_j(u, r, x), \quad j = 1, \ldots, N_z, \]
are left continuous for \( r \in (0, r_x) \).

Proof. Let us consider for example \( j = 1 \). Fix \( r \in (0, r_x) \), then we want to show
\[ \lim_{\lambda \to 1^-} m_1(u, \lambda r, x) = m_1(u, r, x). \]

First of all thanks to the invariance property under translation and rescaling of the median (4.1) we can reduce ourselves to prove
\[ \lim_{\lambda \to 1^-} m_1(u_{\lambda r,x}, 1, 0) = m_1(u_{r,x}, 1, 0). \] (4.6)

Hence by definition of median, we have to prove that
\[ \inf\{t \in \mathbb{R} \mid |\{u_{\lambda r,x} > t\} \cap F_{\lambda r,1}| \leq |F_{\lambda r,1}|/2\}, \]
converge to
\[ \inf\{t \in \mathbb{R} \mid |\{u_{r,x} > t\} \cap F_{r,1}| \leq |F_{r,1}|/2\}, \]
as \( \lambda \to 1^- \). First of all notice that
\[ u_{\lambda r,x} \rightharpoonup u_{r,x} \text{ in } L^p \setminus B(0). \] - measure.

This implies that for every \( t \) except a countable set \( A \), we have
\[ |\{u_{\lambda r,x} > t\} \Delta [u_{r,x} > t] \cap B(0)| \to 0, \quad (\lambda \to 1^-). \]

Since \( (F_{r,1})_r \) is a left continuous family, then \( |F_{\lambda r,1} \Delta F_{r,1}| \to 0 \) as \( \lambda \to 1^- \), we have also that for \( t \in \mathbb{R} \setminus A \)
\[ |\{u_{\lambda r,x} > t\} \cap F_{\lambda r,1}| \to |\{u_{r,x} > t\} \cap F_{r,1}|, \quad (\lambda \to 1^-). \] (4.7)
The convergence (4.7) implies that if \( t \in \mathbb{R} \setminus A \) then
\[ |\{u_{r,x} > t\} \cap F_{r,1}| < \frac{|F_{r,1}|}{2}, \]
implies
\[ |\{u_{\lambda r,x} > t\} \cap F_{\lambda r,1}| < \frac{|F_{\lambda r,1}|}{2}, \]
for every $\lambda$ close enough to $1^-$. Analogously if $t \in \mathbb{R} \setminus A$ then

$$\left|\{u_{r,x} > t\} \cap F_{r,1}\right| > \frac{|F_{r,1}|}{2},$$

implies

$$\left|\{u_{\lambda r,x} > t\} \cap F_{\lambda r,1}\right| > \frac{|F_{\lambda r,1}|}{2},$$

for every $\lambda$ close enough to $1^-$. Therefore, the convergence (4.6) is established once we prove that for every $t > m_1(u_{r,x}, 1,0)$ it holds

$$\left|\{u_{r,x} > t\} \cap F_{r,1}\right| < \frac{|F_{r,1}|}{2} \text{ for } t > m_1(u_{r,x}, 1,0),$$

and

$$\left|\{u_{r,x} > t\} \cap F_{r,1}\right| > \frac{|F_{r,1}|}{2} \text{ for } t < m_1(u_{r,x}, 1,0).$$

The last condition follows by the definition of median.

To prove the first condition we argue by contradiction. If it does not hold, then there exist $t_1 < t_2$ such that

1. $\{u_{r,x} > t\} \cap F_{r,1} = \{u_{r,x} > s\} \cap F_{r,1}$ for a.e. every $t, s \in (t_1, t_2)$,

2. $\left|\{u_{r,x} > t\} \cap F_{r,1}\right| = \frac{|F_{r,1}|}{2}$ for a.e. every $t \in (t_1, t_2)$.

By Coarea Formula there exists $t \in (t_1, t_2)$ such that $\{u_{r,x} > t\}$ is a set of finite perimeter in $B_1(0)$ such that $\left|\{u_{r,x} > t\} \cap F_{r,1}\right| = \frac{|F_{r,1}|}{2}$. Since $F_{r,1}$ is indecomposable, Proposition 1.19 says to us that $\mathcal{H}^{n-1}(\partial^*\{u_{r,x} > t\} \cap F_{r,1}^{(1)}) > 0$. We claim that

$$\partial^*\{u_{r,x} > t\} \cap F_{r,1}^{(1)} \subset S_{u_{r,x}}.$$  \hspace{1cm} (4.8)

Indeed, by using Theorem 1.15, we know that $\mathcal{H}^{n-1}$-a.e. point $y \in \partial^*\{u_{r,x} > t\} \cap F_{r,1}^{(1)}$ is a point of density $1/2$ for $\{u_{r,x} > t\}$. By (1) we deduce that every point of density $1/2$ for $\{u_{r,x} > t\}$ is also a point of density $1/2$ for $\{u_{r,x} > s\}$ for every $s \in (t_1, t_2)$. By definition of upper and lower approximate limit (Definition 1.22) we deduce that for every $\mathcal{H}^{n-1}$-a.e. $y \in \partial^*\{u_{r,x} > t\} \cap F_{r,1}^{(1)}$ we have

$$u_-(y) \leq t_1 < t_2 \leq u_+(y),$$

which proves (4.8).

Finally, since by Theorem 1.26 $S_{u_{r,x}} \subset J_{u_{r,x}} \subset \Gamma_{r,x}$ ($\mathcal{H}^{n-1}$-a.e.), condition (4.8) is in contradiction with property (1.1) of Definition 3.3 and this proves our desired result.

The next theorem, which is the core result of this section, tells us that the medians $m_j(u, r, x)$ are convergent for suitable subsequences of radii $r_i \to 0^+$. In the proof we will use the following inequality which is true for each quadruple of measurable sets $A, B, C, D \subset \Omega$

$$|A \Delta B| \leq |C \Delta D| + |A \Delta C \setminus B| + |B \Delta D \setminus A| + |C \setminus A| + |D \setminus B|.$$  \hspace{1cm} (4.9)

In particular, we can write

$$|A \Delta B| \leq |C \Delta D| + 2|A \Delta C| + 2|B \Delta D|.$$  \hspace{1cm} (4.10)

The previous inequality tells us that whenever $|A \Delta C|, |B \Delta D| \leq \epsilon$, then the measure of the symmetric difference between $A$ and $B$ can be estimated through the measure of the symmetric difference between $C$ and $D$ plus an reminder which is of order of $4\epsilon$.

**Theorem 4.5.** Let $\Omega \subset \mathbb{R}^n$ be an open set, let $\Gamma \in \mathcal{F}_p$ $(1 < p < n)$ and let $x \in \partial \Omega \setminus \mathcal{S}_r$. Suppose that there exists some $\delta > 0$ with the following property

$$\limsup_{r \to 0^+} \frac{1}{r^{n-p+\delta}} \int_{B_r(x)} |\nabla u|^p \ dx = 0.$$  \hspace{1cm} (4.10)

Then for every sequence of radii $(r_i)_{i=1}^{\infty}$ such that

1. $(\frac{1}{2})^{\frac{1}{p+\delta}} < \frac{r_{i+1}}{r_i} \leq 1, \ i \in \mathbb{N};$

2. $\sum_{i=1}^{\infty} (r_i)^{\frac{1}{p}} < \infty;$
the sequence of medians \((m_j(u, r_i, x))_{i=1}^{\infty}\) is Cauchy for every \(j = 1, \ldots, N_x\).

**Proof.** Choose \(1 \leq j \leq N_x\). In order to simplify the notation we write
\[
t_{r_i} := m_j(u, r_i, x) \quad F_r := F_{r,j} \quad E_0 := E_{0,j} \quad a_i := \frac{r_i}{r_{i-1}}.
\]
Fix \(0 < \epsilon \leq \frac{2a_i^{n-1}}{40a_i^{n+1} + 1}\), which by condition (1) implies \(0 < \epsilon \leq \frac{2\sqrt{2} - 1}{4\sqrt{2} + 1}\) and consider \(\tilde{t} \in \mathbb{N}\) so big that for every \(i \geq \tilde{t}\)
\[
h_{F_{r_i}}(\epsilon) \geq \frac{1}{c(\epsilon)} := \frac{1}{2} \lim \inf_{i \to \infty} h_{F_{r_i}}(\epsilon) > 0.
\]
This is possible by the definition of \(\lim \inf\).

By using Theorem 2.12 with the function \(u(x + r(\cdot)) \in GSBV^p(B_1(0); \Gamma_{r,x})\) and the indecomposable set \(F_{r,i}\), and by using the invariance under translation and rescaling of the median, i.e.
\[
m_j(u, r, x) = m_j(u_{r,x}, 1, 0),
\]
we deduce that for every \(i \geq \tilde{t}\) and for every \(\epsilon > 0\), there exists \(F_{r_i}' \subset F_{r_i} \subset B_1(0)\) such that
\[
|F_{r_i}'| \geq (1 - 2\epsilon)|F_{r_i}|,
\]
and
\[
\int_{F_{r_i}'} |u_{r_i,x} - t_{r_i}|^p \, dy \leq 2 \epsilon|\epsilon|^p \int_{F_{r_i}} |\nabla u_{r_i,x}|^p \, dy \leq 2 \epsilon|\epsilon|^p \int_{F_{r_i}} |\nabla u_{r_i,x}|^p \, dy.
\]

Now for each \(i \geq \tilde{t}\) define
\[
F_i := a_iF_{r_i}' \cap F_{r_i-1}' \subset B_{a_i}(0).
\]
Since \(F_i \subset a_iF_{r_i}'\), we can give the following estimate
\[
|F_i| = |a_iF_{r_i}' \cap F_{r_i-1}'| = |a_iF_{r_i} \setminus (a_iF_{r_i} \cup a_iF_{r_i} \setminus F_{r_i-1}')| \\
= |a_iF_{r_i} - |a_iF_{r_i} \setminus a_iF_{r_i}'| - |a_iF_{r_i} \setminus F_{r_i-1}'|.
\]
By (4.11) and the fact \(|F_{r_i}, \Delta E_0| \to 0\) we can write
\[
|a_iF_{r_i} \setminus a_iF_{r_i}'| = a_i^\epsilon|F_{r_i} \setminus F_{r_i}'| \leq a_i^\epsilon2\epsilon|F_{r_i}| = a_i^\epsilon2\epsilon|E_0| + o(1),
\]
and by using also inequality (4.9) with \(A = F_{r_i-1} \cap a_iF_{r_i}, B = a_iF_{r_i}, C = E_0 \cap a_iE_0\) and \(D = a_iE_0\), we can write
\[
|a_iF_{r_i} \setminus F_{r_i-1}'| \leq |(F_{r_i-1} \cap a_iF_{r_i}) \setminus F_{r_i-1}'| + |(F_{r_i-1} \cap a_iF_{r_i}) \Delta a_iF_{r_i} \setminus F_{r_i-1}'| \\
\leq |F_{r_i-1} \setminus F_{r_i}'| + |(F_{r_i-1} \cap a_iF_{r_i}) \Delta a_iF_{r_i}| \\
= 2\epsilon|E_0| + |E_0 \cap a_iE_0|\Delta a_iE_0| + o(1),
\]
and since \(E_0\) is conical, then \(|(E_0 \cap a_iE_0)\Delta a_iE_0| = 0\) for every \(i \in \mathbb{N}\); as a consequence we can write
\[
|a_iF_{r_i} \setminus F_{r_i-1}'| \leq 2\epsilon|E_0| + o(1).
\]
Putting together our previous estimates we obtain
\[
|F_i| \geq a_i|E_0| - a_i^\epsilon|E_0| - 2\epsilon|E_0| + o(1) \\
= |E_0|(a_i^\epsilon - a_i^\epsilon - 2\epsilon + o(1).
\]
By our choice of \(\epsilon\), we have \(a_i^\epsilon - \epsilon(2a_i^\epsilon + 2) \geq \frac{1}{2}\), hence
\[
|F_i| \geq \frac{1}{2}|E_0| + o(1), \quad i \in \mathbb{N}.
\]
Therefore, for every \(i \geq \tilde{t}\), we can make use of (4.12), (4.13), and \(a_i \leq 1\), to deduce the following estimates
\[
|t_{r_i} - t_{r_i-1}|^p = \int_{F_{r_i}} |t_{r_i} - t_{r_i-1}|^p \, dy \leq 2^{p-1} \int_{F_{r_i}} |u_{r_i,x} - t_{r_i}|^p \, dy + 2^{p-1} \int_{F_{r_i}} |u_{r_i-1,x} - t_{r_i-1}|^p \, dy \\
= 2^{p-1}a_i^p \int_{F_{r_i}} |u_{r_i,x} - t_{r_i}|^p \, dy + 2^{p-1} \int_{F_{r_i}} |u_{r_i-1,x} - t_{r_i-1}|^p \, dy.
\]
hence by using (4.12) and $a_i \leq 1$ there exists $C = C(p, n, \epsilon) > 0$ such that
\[
|t_{r_i} - t_{r_{i-1}}| \leq C \left[ \int_{F_{r_i}} |\nabla u_{r_i,x}|^p \, dy + \int_{F_{r_{i-1}}} |\nabla u_{r_{i-1},x}|^p \, dy \right]
\]
\[
= C \left[ r_i^p \int_{F_{r_i}} |\nabla u(x + r_i y)|^p \, dy + r_{i-1}^p \int_{F_{r_{i-1}}} |\nabla u(x + r_{i-1} y)|^p \, dy \right],
\]
and finally by using (4.13) we have
\[
|t_{r_i} - t_{r_{i-1}}|^p \leq C \frac{1}{1/2|E_0| + o(1)} \left[ \frac{1}{r_i^{n-p}} \int_{B_{r_i}(z)} |\nabla u|^p \, dx + \frac{1}{r_{i-1}^{n-p}} \int_{B_{r_{i-1}}(z)} |\nabla u|^p \, dx \right]
\]
\[
\leq C' r_i^δ \left[ \frac{1}{r_i^{n-p+\frac{1}{2}p+\frac{1}{2}}} \int_{B_{r_i}(z)} |\nabla u|^p \, dx + \frac{1}{r_{i-1}^{n-p+\frac{1}{2}p+\frac{1}{2}}} \int_{B_{r_{i-1}}(z)} |\nabla u|^p \, dx \right]
\]
where, thanks also to (4.10), $C' > 0$ is a constant which depends only on $x, j, p, n, \epsilon$.

These last inequality means
\[
\sum_{i \geq 1} \frac{|t_{r_i} - t_{r_{i-1}}|}{r_i^\frac{1}{2}} \leq C' \sum_{i = 1}^\infty (r_i)^\frac{1}{p},
\]
and this last series is convergent thanks to our choice of $r_i$. This implies that the sequence $(t_{r_i})_{i=1}^\infty$ is Cauchy. Since $1 \leq j \leq N_x$ was arbitrary, we prove the theorem.

The next proposition shows that for every sequence of radii $r_i$ satisfying (1) and (2) of Theorem 4.5, then $\lim_{i \to \infty} m_j(u, r_i, x)$ is actually unique.

**Proposition 4.6.** Under the hypothesis of Theorem 4.5 we have that there exists $l \in \mathbb{R}$ such that
\[
\lim_{i \to \infty} m_j(u, 1/2^{\alpha_i}, x) = l,
\]
for every $\alpha \in (0, \frac{1}{2n})$.

**Proof.** The fact that it exists for every $\alpha \in (0, 1/2n)$ is simply a consequence of the fact that the sequences $(1/2^{\alpha_i})_{i=1}^\infty$ satisfy (1) and (2) of Proposition 4.5, hence $(m_j(u, 1/2^{\alpha_i}, x))_{i=1}^\infty$ is a Cauchy sequence.

To show that the limits do not depend on $\alpha$, pick $0 < \alpha_1 < \alpha_2 < \frac{1}{2n}$, and consider for every $i \in \mathbb{N}$
\[
r_i' := \frac{1}{2^{\alpha_1}} \text{ and } r_i'' := \frac{1}{2^{\alpha_2}}.
\]
Define a new sequence $(r_i')_{i=1}^\infty$ by reordering the $(r_i')$, $(r_i'')$: set $r_1 := \max\left\{ \frac{1}{2^{\alpha_1}}, \frac{1}{2^{\alpha_2}} \right\}$, and then inductively
\[
r_i := \max\{ r | r \in (r_k')_{k=1}^\infty \cup (r_k'')_{k=1}^\infty \setminus \{r_{k-1}\} \}.
\]
We want to prove that $(r_i')_{i=1}^\infty$ satisfies conditions (1) and (2) of Proposition 4.5. Once we prove this, we can apply Proposition 4.5 to deduce that $(m_j(u, r_i', x))_{i=1}^\infty$ is a Cauchy sequence for every $j = 1, \ldots, N_x$ which implies
\[
\lim_{i \to \infty} m_j(u, r_i', x) = \lim_{i \to \infty} m_j(u, r_i'', x).
\]
To prove (1), notice that for every $i \in \mathbb{N}$
\[
\left( \frac{1}{2} \right)^\frac{1}{p} \leq \min\left\{ \frac{1}{2^{\alpha_1}}, \frac{1}{2^{\alpha_2}} \right\} \leq r_i/r_{i-1} \leq 1.
\]
Indeed $r_i/r_{i-1} \leq 1$ simply follows because by construction $r_i \to 0^+$ as $i \to \infty$. To prove the other inequality, suppose for example $r_{i-1} = r_j'$ and $r_i = r_k''$ for some $j$ and $k$, then this means $r_k'' \geq r_{j+1}'$, and therefore

$$\min\left\{ \frac{1}{2^{\alpha_i}}, \frac{1}{2^{\alpha_j}} \right\} \leq \frac{r_{j+1}'}{r_j'} \leq \frac{r_k''}{r_{j-1}'}.$$ 

The same argument works in the case $r_{i-1} = r_k''$ and $r_i = r_j'$ for some $j$ and $k$.

To prove (2) it is enough to estimate

$$\lim_{N \to \infty} \sum_{i=1}^N (r_i)_{\hat{j}}^2 \leq \lim_{N \to \infty} \left( \sum_{i=1}^N \frac{1}{2^{\alpha_i}/p} + \sum_{i=1}^N \frac{1}{2^{\beta_j}/p} \right) = \frac{2^{\delta_1}/p}{2^{\delta_1}/p - 1} + \frac{2^{\delta_2}/p}{2^{\delta_2}/p - 1} - 2,$$

and this concludes the proof. \qed

The next proposition says that in order to compute the limit at infinity of a right/(or left) continuous function of one variable, it is enough to compute it on arithmetic progressions.

**Proposition 4.7.** Let $f: [0, +\infty) \to \mathbb{R}$ be a right (or left) continuous function. Suppose that there exists $l \in \mathbb{R}$ and $0 < a < b < \infty$ such that for every $\alpha \in [a, b]$ it holds

$$\lim_{i \to \infty} f(i\alpha) = l.$$ 

Then

$$\lim_{t \to +\infty} f(t) = l.$$ 

**Proof.** Suppose not. Then there exists a sequence $t_k \to +\infty$ as $k \to \infty$ and an $\epsilon > 0$ such that

$$|f(t_k) - l| \geq \epsilon, \quad k \in \mathbb{N}.$$ 

By the right continuity of $f$ for every $k$ there exists $\delta_k > 0$ with

$$|f(t) - f(t_k)| \leq \frac{1}{k}, \quad t \in (t_k, t_k + \delta_k).$$ 

In order to simplify the notation write $I_k := (t_k, t_k + \delta_k)$.

We claim that there exists an $\alpha \in [a, b]$ such that for infinitely many $k$ there exists an element of the sequence $(i\alpha)_{i=1}^\infty$ which belongs to $I_k$.

To prove the claim, for each $j \in \mathbb{N}$ define $A_j := \{ \alpha \in [a, b] \mid \exists i \in \mathbb{N}, i\alpha \in I_j \}$, and then set $B_k := \bigcup_{j \geq k} A_j$. Notice that for every $k \in \mathbb{N}$, $B_k$ is an open and dense subset of $[a, b]$. Indeed for every couple $\alpha_1, \alpha_2 \in [a, b]$ with $\alpha_1 < \alpha_2$, there exists $\bar{\alpha} \in \mathbb{N}$ such that

$$\{i + 1\} \alpha_1 < i \alpha_2, \quad i \geq \bar{\alpha}.$$ 

This means that the intervals $[i\alpha_1, i\alpha_2]$ overlap each others for $i \geq \bar{\alpha}$, hence

$$(i\alpha_1, \infty) \subset \bigcup_{i \geq \bar{\alpha}} [i\alpha_1, i\alpha_2].$$ 

(4.17)

To see that $B_k$ are dense in $[a, b]$, it is enough to notice that for every open interval $(\alpha_1, \alpha_2) \subset [a, b]$, by (4.17), there must exists a $\alpha \in (\alpha_1, \alpha_2)$, a $j \geq k$ and a $i \geq \bar{\alpha}$ such that $i\alpha \in I_j$ which implies $\alpha \in B_k$.

The fact that $B_k$ are open is even easier. If $i\alpha_0 \in I_k$ for some $i, k \in \mathbb{N}$ and some $\alpha_0 \in [a, b]$, since $I_k$ is open then by continuity of the map $t \mapsto \alpha t$ there exists a relative open neighborhood $U$ of $\alpha_0$ in $[a, b]$, such that for every $\alpha \in U \cap i\alpha \in I_k$.

Using Baire’s Lemma on the complete metric space $[a, b]$ we deduce that

$$\bigcap_{k \in \mathbb{N}} B_k \neq \emptyset.$$ 

So pick $\alpha \in \bigcap_{k \in \mathbb{N}} B_k$, and notice that by definition of $B_k$, for every $k$ there exists $i_k \in \mathbb{N}$ and $j_k \geq k$ such that $i_k \alpha \in I_{j_k}$. By (4.15) and (4.16) we can write

$$|f(i_k\alpha) - l| \geq |f(t_{j_k}) - l| - |f(t_{j_k}) - f(i_k\alpha)| \geq \epsilon - \frac{1}{j_k}, \quad (k \in \mathbb{N}).$$
Since by construction \( j_k \to \infty \) as \( k \to \infty \), the previous inequality contradicts (4.14) and proves the proposition.

**Theorem 4.8.** Let \( \Omega \subset \mathbb{R}^n \) be open, let \( \Gamma \in \mathcal{F}_p \) (\( 1 < p < n \)) and let \( u \in \text{GSBV}^p(\Omega; \Gamma) \). Then for every \( x \in \Omega \), except a set of Hausdorff dimension \( n - p \), \( \lim_{r \to 0^+} m_j(u, r, x) \) exists finite for every \( 1 \leq j \leq N_x \).

**Proof.** For every \( \delta > 0 \), consider \( A_\delta \subset \Omega \setminus S_\Gamma \) the set of point \( x \) such that

\[
\limsup_{r \to 0^+} \frac{1}{r^{n-p+\delta}} \int_{B_r(x)} |\nabla u|^p \, dx > 0.
\]

(4.18)

By applying for example [7, Theorem 3, Section 2.4.3] we have \( H^{n-p+\delta}(A_\delta) = 0 \). Moreover since \( A_\delta \subset A_{\delta'} \), for \( \delta_1 \leq \delta_2 \), we have that if we fix \( \delta_0 > 0 \) then

\[
\lim_{\delta \to 0^+} \left( \bigcap_{\delta > 0} A_\delta \right) = 0.
\]

Since \( \delta_0 > 0 \) is arbitrary we deduce

\[
\dim_h \left( \bigcap_{\delta > 0} A_\delta \right) = n - p.
\]

(4.19)

Now pick \( x \in \Omega \setminus S_\Gamma \) and \( x \notin \bigcap_{\delta > 0} A_\delta \). If we define for every \( 1 \leq j \leq N_x \)

\[
f(t) := m_j(u, 1/2^t, x), \quad \lambda \geq 0,
\]

then by Proposition 4.4, \( f(t) \) is continuous from the right, and thanks to Proposition 4.6, there exists \( l \in \mathbb{R} \) such that for every \( 0 < \alpha \leq \frac{1}{2^n} \), we have

\[
\lim_{t \to +\infty} f(\alpha t) = l.
\]

Therefore we are in position to apply Proposition 4.7 and to deduce that

\[
\lim_{t \to +\infty} f(t) = l.
\]

As a consequence, since \( \lim_{r \to 0^+} \log \mathcal{H}^j(r) = +\infty \) we finally deduce that

\[
\lim_{r \to 0^+} m_j(u, r, x) = \lim_{r \to 0^+} f(\log \mathcal{H}^j(r)) = \lim_{t \to +\infty} f(t) = l.
\]

\[\square\]

**4.3. Proof of Theorem 0.1.** Finally we can prove the main result of this section.

**Theorem 4.9.** Let \( \Omega \subset \mathbb{R}^n \) be open, let \( \Gamma \in \mathcal{F}_p \) (\( 1 < p < n \)) and let \( u \in \text{GSBV}^p(\Omega; \Gamma) \). Then for every \( x \in \Omega \) except a set of Hausdorff dimension \( n - p \), there exists a measurable function \( u_x(\cdot): B_1(0) \to \mathbb{R} \) such that

\[
\lim_{r \to 0^+} \|u_{r,x} - u_x\|_{L^p(B_1)} = 0.
\]

(4.20)

Moreover using the notation of Definitions 3.3 and 2.10 we also have that

\[
u(x) = m_j(u, x) \text{ if } y \in E_{0,j},
\]

(4.21)

where \( m_j(u, x) := \lim_{r \to 0^+} m_j(u, r, x) \) for \( 1 \leq j \leq N_x \).

**Proof.** Let \( A_\delta (\delta > 0) \) be the sets defined in the proof of Theorem 4.8 and define \( A := \bigcap_{\delta > 0} A_\delta \). By (4.19) \( \dim_H(A) = n - p \), hence also \( \dim_H(A \cup S_\Gamma) = n - p \).

We claim that every \( x \in \Omega \setminus (S_\Gamma \cup A) \) satisfies (4.20) and (4.21). By Theorem 4.8 we know that \( \lim_{r \to 0^+} m_j(u, r, x) \) exists for every \( 1 \leq j \leq N_x \). Therefore, by defining

\[
m_j(u, x) = \lim_{r \to 0^+} m_j(u, r, x), \quad 1 \leq j \leq N_x,
\]

and recalling condition (2.1) of Definition 3.3 we immediately deduce

\[
\lim_{r \to 0^+} \int_{B_1(0)} |\nabla \mathcal{H}^j(x + ry) - u_x(y)| \wedge 1 \, dy = 0.
\]

(4.22)
5. Capacity in $GSBV^p$

5.1. The outer measure $C'_p$. This section is devoted to the proof of Theorem 0.2. For this purpose we need to introduce a suitable notion of capacity for functions in $GSBV^p(\Omega; \Gamma)$.

First of all we want to recall that given $A \subset \mathbb{R}^n$, the classical $p$-capacity in the context of Sobolev functions is defined as (see for example [9] or [7])

$$ C_p(A) := \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^p \, dx \mid u \in K^p, u \geq 1 \text{ a.e. in an open neighborhood of } A \right\}, \quad (5.1) $$

where $K^p := \{ u: \mathbb{R}^n \to \mathbb{R} \mid u \geq 0, u \in L^p(\mathbb{R}^n), \nabla u \in L^p(\mathbb{R}^n) \}$.

The following result can be interpreted as a capacitary version of Chebyshev’s inequality (see for example [9, Paragraph 7] or [7, Lemma 1, Section 4.8]).

**Proposition 5.1.** Assume $u \in K^p$ and $\epsilon > 0$. Let

$$ A := \left\{ x \in \mathbb{R}^n \mid \int_{B_r(x)} u > \epsilon \text{ for some } r > 0 \right\}. $$

Then

$$ C_p(A) \leq \frac{c}{\epsilon^p} \int_{\mathbb{R}^n} |\nabla u|^p \, dx. $$

The previous proposition suggest us that given $A \subset \mathbb{R}^n$, if we define a new capacity $C'_p(A)$ as

$$ C'_p(A) := \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^p \, dx \mid u \in K^p, \limsup_{r \to 0^+} \int_{B_r(x)} u > 1 \text{ for every } x \in A \right\}, $$

then

$$ C'_p(A) \leq C_p(A) \leq cC'_p(A), \text{ for every } A \subset \mathbb{R}^n. $$

Indeed, if $u \in K^p$ is such that $u \geq 1$ a.e. in an open neighborhood of $A$ clearly $u$ satisfies $\limsup_{r \to 0^+} \int_{B_r(x)} u > 1$ for every $x \in A$ and we obtain

$$ C'_p(A) \leq C_p(A). $$

On the other hand, if $\limsup_{r \to 0^+} \int_{B_r(x)} u > 1$ for every $x \in A$, then by definition of $\limsup$ for every $x \in A$ there exists $r_x$ such that $\int_{B_{r_x}(x)} u > 1$. Therefore

$$ A \subset \left\{ x \in \mathbb{R}^n \mid \int_{B_r(x)} u > 1 \text{ for some } r > 0 \right\}. $$

By the capacitary Chebyshev’s inequality the previous inclusion immediately implies

$$ C_p(A) \leq c \int_{\mathbb{R}^n} |\nabla u|^p \, dx. $$

Taking the infimum on the right hand side of the previous inequality one immediately get

$$ C_p(A) \leq cC'_p(A). $$
Hence it is possible to define an equivalent notion of capacity by looking at the averages at small scales. This is the starting point for our definition of capacity in the context of $\text{GSBV}^p(\Omega; \Gamma)$.

**Definition 5.2 (p-Capacity).** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. We define the p-Capacity of $A$ ($p \geq 1$) as

$$C_p(A) := \inf \left\{ \int_{\Omega} (|\nabla u|^p + |u|^p) \, dx \mid u \in \text{GSBV}(\Omega; \Gamma), u^+(x) \geq 1 \text{ on } A \right\}. \quad (5.2)$$

**Remark 5.3.** In (5.2) we consider also the $L^p$-norm of the function, while in (5.1) only the $L^p$-norm of the gradient is present. This is simply because we want to avoid that functions $u$ belonging to the kernel of $\nabla$ could trivialize the infimum in (5.2). We remember that the kernel of the approximate gradient of $\text{GSBV}(\Omega; \Gamma)$ functions is made up of piecewise constant functions whose jump sets are contained in a Caccioppoli’s partition subordinated to $\Gamma$. This result can be found for example in [2] for $\text{SBV}$ functions; the case $\text{GSBV}$ can be easily recovered by a truncation argument. Clearly with this choice, the scaling property $C_p(\lambda A) = \lambda^{n-p}C_p(A)$ (see [7, Section 4.7.1]) is lost. Anyway, we do not need this property to develop our theory.

**Proposition 5.4.** For every set $A \subset \Omega$ we have

$$C_p(A) = \inf \left\{ \int_{\Omega} (|\nabla u|^p + |u|^p) \, dx \mid u \in \text{GSBV}(\Omega; \Gamma), u^+(x) \geq 1 \text{ on } A, 0 \leq u \leq 1 \right\}. \quad (5.3)$$

**Proof.** Let $u^+_1 := (u \wedge 1) \vee 0$. Since $u^+(x) \geq 1$ if and only $u^+_0(x) \geq 1$, it is enough to notice that if $u \in \text{GSBV}^p(\Omega; \Gamma)$ then

$$\int_{\Omega} |\nabla u^+_0|^p + |u^+_0|^p \, dx \leq \int_{\Omega} |\nabla u|^p + |u|^p \, dx,$$

and this concludes the proof. \qed

**Proposition 5.5.** $C_p(\cdot)$ is an outer measure on $\Omega$.

**Proof.** Clearly $C_p(\cdot)$ is monotone and $C_p(\emptyset) = 0$. Hence we need only to prove the countable sub-additivity.

Let $(A_k)_{k=1}^{\infty}$ be a countable family of subsets of $\Omega$ and define $A := \bigcup_{k=1}^{\infty} A_k$. Without loss of generality we can assume $\sum_k C_p(A_k) < \infty$. For each $k$ we can find $u_k \in \text{GSBV}^p(\Omega; \Gamma)$, $0 \leq u_k \leq 1$, and $u^+_k(x) \geq 1$ on $A_k$ such that

$$\int_{\Omega} |\nabla u_k|^p + |u_k|^p \, dx \leq C_p(A_k) + \frac{\epsilon}{2^k}.$$ 

We define $u := \sup_{k \in \mathbb{N}} u_k$, and we claim that $u \in \text{GSBV}^p(\Omega; \Gamma)$ and $u^+(x) \geq 1$ on $A$. Indeed, since the $u_k$ are bounded functions in $\text{GSBV}^p(\Omega; \Gamma)$, we have $u_k \in \text{SBV}(\Omega)$. Therefore by using the chain rule in $\text{BV}$ [1, Theorem 3.99], if we set $u_m := \sup_{1 \leq k \leq m} u_k$, we have

$$\int_{\Omega} |\nabla u_m|^p \, dx \leq \sum_{k=1}^{m} \int_{\Omega} |\nabla u_k|^p \, dx,$$

hence

$$\sup_m \int_{\Omega} |\nabla u_m|^p + |u_m|^p \, dx \leq \sum_{k=1}^{\infty} C_p(A_k) + \frac{\epsilon}{2^k}. \quad (5.3)$$

Thanks to (5.3) we can use the compactness result [1, Theorem 4.36] for $\text{GSBV}(\Omega)$ together with [4, Remark 2.9] to deduce that $u \in \text{GSBV}^p(\Omega; \Gamma)$ and moreover

$$u_m \to u \text{ strongly in } L^1(\Omega) \quad \nabla u_m \rightharpoonup \nabla u \text{ weakly in } L^1(\Omega). \quad (5.4)$$

Moreover if $x \in A$ then $x \in A_k$ for some $k$, therefore $u^+_k(x) \geq 1$, and since $u \geq u_k$ for every $k$ we deduce $u^+(x) \geq u^+_k(x)$, hence

$$A \subset \{ x \in \Omega \mid u^+(x) \geq 1 \}.$$
Therefore by using the lower semicontinuity of the $L^p$-norm with respect to the convergence \((5.4)\), we have

$$C_p(A) \leq \int_\Omega |\nabla u|^p + |u|^p \, dx \leq \liminf_{m \to \infty} \int_\Omega |\nabla u_m|^p + |u_m|^p \, dx \leq \sum_{k=1}^\infty C_p(A_k) + \epsilon,$$

which implies the countably sub-additivity of $C_p(\cdot)$ thanks to the arbitrariness of $\epsilon$. \qed

### 5.2. Relations between $C_p$ and $\mathcal{H}^{n-p}$.

**Proposition 5.6.** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. For every $1 \leq p < n$ there exists a constant $c = c(n, p)$ such that for every $A \subset \Omega$

$$C_p(A) \leq c \mathcal{H}^{n-p}(A).$$

**Proof.** First of all if $B_r(x) \subset \Omega$, then $C_p(B_r(x))$ can be rewritten as

$$\inf \left\{ \int_\Omega (|\nabla v|^p + |v|^p) \, dy \mid v(y) = u \left( \frac{x - y}{r} \right), \quad u \in GSBV(\Omega'; \Gamma'), \quad u^+(x) \geq 1 \text{ on } B_1(0) \right\},$$

where \(\Omega' = (\Omega - x)/r\) and $\Gamma' = (\Gamma - x)/r$.

Notice that for $r \leq 1$ we have

$$\int_\Omega (|\nabla v|^p + |v|^p) \, dy \leq r^p \int_{\Omega'} (r^{-p} |\nabla u|^p + |u|^p) \, dy \leq r^{-n} \int_{\Omega'} (|\nabla u|^p + |u|^p) \, dy.$$

Hence, by choosing $u(x) := \text{dist}(x, \mathbb{R}^n \setminus B_2(0)) \wedge 1$ whenever $x \in \Omega'$, it follows

$$C_p(B_r(x)) \leq 2^{n+1} \omega_n r^{n-p} \quad (r \leq 1).$$

Let $(C_i)_{i=1}^\infty$ be a family of sets contained in $\Omega$ which is a cover of $A$ and $\text{diam}C_i \leq 1$. For each $i$ there exists a ball $B_{r_i}(x_i)$ such that $C_i \subset B_{r_i}(x_i)$ and $r_i = \text{diam}(C_i)$. Therefore

$$C_p(A) \leq \sum_{i=1}^\infty C_p(C_i) \leq \sum_{i=1}^\infty C_p(B_{r_i}(x_i)) \leq 2^{n+1} \omega_n \sum_{i=1}^\infty r_i^{n-p} \leq 2^{2n+1-p} \omega_n \sum_{i=1}^\infty \left( \frac{\text{diam} C_i}{2} \right)^{n-p}.$$

Hence if we set $c := 2^{2n+1-p} \omega_n$, then

$$C_p(A) \leq c \mathcal{H}^{n-p}(A).$$

\qed

Whenever $u : \Omega \to \mathbb{R}$ is such that $u_x$ is a piecewise constant function of the form of Theorem 4.9, then by definition of upper approximate limit (Definition 1.22), it is easy to see that

$$u^+(x) = \max_{1 \leq j \leq N_x} m_j(u, x). \quad (5.5)$$

We shall use this simple observation to deduce more precise relations between the $p$-capacity and the Hausdorff measure.

**Theorem 5.7.** Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Then for every $A \subset \Omega$ and for every $1 < p < n$ we have

1. $C_p(A) = 0$ and $\Gamma \subset \mathcal{J}_p$ imply $\dim_H(A) = n - p$;
2. $\mathcal{H}^{n-p}(A) < \infty$ implies $C_p(A) = 0$.

**Proof.** Suppose $C_p(A) = 0$ and $\Gamma \subset \mathcal{J}_p$. By hypothesis we can find a sequence $(u_k)_{k=1}^\infty \subset GSBV^p(\Omega; \Gamma)$, $0 \leq u_k \leq 1$, such that

(i) $u_k^+(x) \geq 1$, for every $x \in A$;
(ii) $\int_\Omega (|\nabla u_k|^p + |u_k|^p) \, dx \leq \frac{1}{k}$, for every $k \in \mathbb{N}$.

Define $u := \sum_{k=1}^\infty u_k$. Since $GSBV^p(\Omega; \Gamma)$ is a Banach space, by (ii) we deduce that $u \in GSBV^p(\Omega; \Gamma)$. Thanks to Theorem 4.9, if we call $S_k$ the set of $x \in \Omega$ where the blow-up of $u_k$ does not exist, then $\dim_H(S_k) = n - p$. By setting $S := \bigcup_{k=1}^\infty S_k$ clearly

$$\dim_H(S) = n - p. \quad (5.6)$$
For this reason, by using the linearity of the blow-up and again (5.5), we have

\[(u_k)_x = \sum_{j=1}^{N_x} m_j(u_k, x) \| E_{0,j} \|, \quad \text{and} \quad \max_{1 \leq j < N_x} m_j(u_k, x) \geq 1. \tag{5.7}\]

Since \(u_k \geq 0\) for every \(k\), we have \(u(y) \geq \sum_{k=1}^{M} u_k(y)\) for every \(M \in \mathbb{N}\) and every \(y \in B_1(0)\). For this reason, by using the linearity of the blow-up and again (5.5), we have

\[u^+(x) \geq \left( \sum_{k=1}^{M} u_k(x) \right)^+ = \max_{1 \leq j < N_x} \left[ \sum_{k=1}^{M} m_j(u_k, x) \right].\]

By letting \(M \to \infty\), thanks to (5.7) we deduce that \(A \setminus S\) is contained in the set of point \(x\) where \(u^+(x) = +\infty\). By Theorem 4.9 we deduce that

\[\dim_H(A \setminus S) = n - p,\]

which together with (5.6) is exactly (1).

Now suppose \(H^{n-p}(A) \leq 2^{p-n} \gamma < \infty\) for some \(\gamma > 0\). We follow the proof given in [9, Section 3].

By denoting as \(S^{n-p}\) the \((n-p)\)-dimensional spherical measure (see [8, Paragraph 2.10.2]), we have

\[S^{n-p}(A) \leq 2^{n-p} H^{n-p}(A) \leq \gamma.\]

We claim that for every \(m \in \mathbb{N}\) we can find an open set \(V_m\) and a function \(u_m \in W^{1,p}(\Omega)\) such that

\[(a) \ A \subset V_m = \bigcup_{i=1}^{\infty} B_{r_i}(x), \quad \sup_i r_i^p \leq (m + 1)^{-p} \left( \sum_{k=1}^{m+1} k^{-1} \right)^{-p};\]
\[(b) \ B_{2r_i}(x) \subset V_{m-1};\]
\[(c) \ u^+(x) = 1 \quad \text{on} \ V_m, \quad \text{spt} (Du_m) \subset V_{m-1} \setminus V_m, \quad \int_{\Omega} |Du_m|^p \ dx \leq c \gamma,\]

where \(c := c(n, p) > 0\).

We start by setting \(V_0 := \Omega\) and \(u_0 := 1\). Set \(\delta_m := m^{-1} \left( \sum_{k=1}^m k^{-1} \right)^{-1}\). For general \(m\), by using

\[\sum_{i=1}^{\infty} S^{n-p}(A \cap \{ x \mid 2^i < \text{dist}(x, V_{m-1}) \leq 2^{i-1} \}) \leq \gamma,\]

we can find a sequence of balls \((B_{r_i}(x_i))_{i=1}^{\infty}\) such that \(B_{2r_i}(x_i) \subset V_{m-1}\), sup, \(r_i \leq \delta_m\), and

\[A \subset V_m := \bigcup_{i=1}^{\infty} B_{r_i}(x) \quad \text{and} \quad \sum_{i=1}^{\infty} \omega_{n-p} r_i^{n-p} \leq \gamma.\]

Define \(h_i \in W^{1,p}(\Omega)\) as

\[h_i(x) = 1 \quad \text{if} \ |x - x_i| \leq r_i, \quad h_i(x) = 0, \quad \text{if} \ |x - x_i| \geq 2r_i,\]

\[h_i(x) = 2 - |x - x_i|/r_i \quad \text{if} \ r_i < |x - x_i| < 2r_i.\]

Since \(\int_{\Omega} |Dh_i|^p \ dx = r_i^p \omega_n [(2r_i)^n - r_i^n] = c \omega_{n-p} r_i^{n-p}\), if we define \(u_m := \sup_{i=1}^{\infty} h_i\), then

\[\int_{\Omega} |Du_m|^p \ dx \leq \int_{\Omega} \sum_{i=1}^{\infty} |Dh_i|^p \ dx \leq c \gamma.\]

In this way (a), (b) and (c) are satisfied.

Define \(u := \sum_{k=1}^{\infty} k^{-1} u_k\). Since by construction \(\text{spt} (Du_m) \subset V_{m-1} \setminus V_m\) we have \(\text{spt} (Du_m) \cap \text{spt} (Du_{m+1}) = 0\) for every \(m \in \mathbb{N}\). Therefore we can write

\[\int_{\Omega} |Du|^p \ dx = \int_{\Omega} \sum_{k=1}^{\infty} k^{-p} |Du_k|^p \ dx \leq c \gamma \lambda,\]

where \(\lambda := \sum_{k=1}^{\infty} k^{-p}\). By using

\[u(x) \leq \sum_{k=1}^{m} k^{-1} \quad \text{if} \ x \in V_{m-1} \setminus V_m,\]

we have
we can estimate
\[
\int_{\Omega} |u|^p \, dx = \int_{\Omega} \left| \sum_{k=1}^{\infty} k^{-1} u_k \right|^p \, dx \leq \sum_{m=1}^{\infty} \int_{V_{m-1} \setminus V_m} \left( \sum_{k=1}^{m} k^{-1} \right)^p \, dx
\]
\[
\leq \sum_{m=1}^{\infty} \left( \sum_{k=1}^{m} k^{-1} \right)^p |V_{m-1}|
\]
\[
\leq \sum_{m,i=1}^{\infty} \left( \sum_{k=1}^{m} k^{-1} \right)^p \omega_n r_i^n
\]
\[
\leq \sum_{m,i=1}^{\infty} m^{-p} \omega_n r_i^{n-p}
\]
\[
\leq \sum_{m=1}^{\infty} m^{-p} \gamma
\]
\[
= \gamma \lambda,
\]
where the radii \( (r_i)_{i=1}^{\infty} \) are those who satisfy property (a) with \( m - 1 \).

We claim that
\[
u^+(x) \geq \sum_{k=1}^{m} k^{-1}, \quad x \in V_m.
\] (5.8)

To prove (5.8) it is sufficient to show that for every \( t < \sum_{k=1}^{m} k^{-1} \) the superlevel \( \{ u > t \} \) has strictly positive density at every \( x \in V_m \).

Using the fact that \( V_m \subset V_{m+1} \) together with property (c), we have that
\[
u^+(x) \geq 1, \quad 1 \leq k \leq m, \quad x \in V_m.
\] (5.9)

Hence, by choosing any \( t < \sum_{k=1}^{m} k^{-1} \), since
\[
\bigcap_{k=1}^{m} \{ u_k > (1 - \delta) \} \subset \{ u > t \},
\]
for any \( 0 < \delta < 1 \) such that \( \sum_{k=1}^{m} (1 - \delta) k^{-1} = t \), and since by (5.9) each set \( \{ u_k > (1 - \delta) \} \) has density greater or equal than one at \( x \in V_m \), we deduce that \( \{ u > t \} \) has strictly positive density at every \( x \in V_m \).

For this reason by definition of \( p \)-capacity it immediately follows
\[
C_p(V_m) \leq \left( \sum_{k=1}^{m} k^{-1} \right)^{-p} \int_{\Omega} (|\nabla u|^p + |u|^p) \, dx \leq \left( \sum_{k=1}^{m} k^{-1} \right)^{-p} \epsilon' (\gamma \lambda).
\]

Sending \( m \to \infty \) in the previous inequality we deduce \( C_p(A) = 0 \). \( \Box \)

5.3. Convergence in measure. Let \( 1 < p < n \) and \( \Gamma \in \mathcal{J}_p \). Given \( u \in GSBV^p(\Omega; \Gamma) \), by Theorem 4.9 we know that \( u_x \in L^p(B_1) \) is well defined for every \( x \in \Omega \) up to a singular set with Hausdorff dimension \( n - p \). If we call \( S \) such a set, this means that for every \( 1 < q < p \) we have \( H^{n-q}(S) = 0 \), and by Theorem 5.7 also \( C_q(S) = 0 \). Therefore, for every \( 1 < q < p \), \( u_x \) is a well defined element of the Fréchet space \( U_{C_q}(\Omega; L^p(B_1)) \) (see Definition 1.4). Unfortunately, we cannot conclude the same for \( q = p \). For this reason we need to introduce a further outer measure.

**Definition 5.8** (Lower \( p \)-capacity). Let \( \Omega \subset \mathbb{R}^n \) be open, and let \( \Gamma \in \mathcal{J}_p \) \((1 < p < n)\). Given any set \( A \subset \Omega \) we define the **lower \( p \)-capacity** as
\[
C_p^-(A) := \sup_{1 < q < p} C_q(A).
\] (5.10)

**Proposition 5.9.** \( C_p^-(\cdot) \) is an outer measure. In addition,
\[
C_p^-(A) = 0 \iff C_q(A) = 0 \text{ for every } 1 < q < p.
\] (5.11)
Finally, by the definition of

\[ \text{Proof.} \]

By Theorem 4.9 we know that \( u_x \) exists everywhere except on a singular set \( S \) of Hausdorff dimension \( n - p \). This means that for every \( \delta > 0 \) \( \mathcal{H}^{n-p}(S) = 0 \), and as a consequence, by Theorem 5.6 this means \( C_{p-\delta}(S) = 0 \). Finally, relation (5.11) gives the conclusion of the theorem.

**Proposition 5.11** (Capacitary Chebyshev’s inequality). Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and let \( \Gamma \in \mathcal{F}_p \) with \( (1 < p < n) \). Then, for every \( \epsilon > 0 \) and for every \( 1 < q < p \) it holds

\[
C_q(\{ x \in \Omega \mid \| u_x \|_{L^p(B_1)} > \omega_n \epsilon \}) \leq \frac{1}{\epsilon^q} \int_{\Omega} (|\nabla u|^q + |u|^q) \, dx, \tag{5.12}
\]

for every \( u \in GSBV^p(\Omega; \Gamma) \).

**Proof.** Renormalizing by \( \epsilon \), in order to prove (5.12), it is enough to show the following inclusion

\[
\{ x \in \Omega \mid \| u_x \|_{L^p(B_1)} > \omega_n \} \subset \{ x \in \Omega : u^+(x) \geq 1 \}. \tag{5.13}
\]

By Theorem 4.9 together with Theorem 5.6 we know that except a \( C_q \)-negligible set, we have

\[
u_x(y) = \sum_{j=1}^{N_x} m_j(x) \mathbb{I}_{E_{n,j}}(y), \quad y \in B_1(0).
\]

Using (5.5) we know that \( u^+(x) \geq 1 \) if and only if at least one of the \( (m_j(x))_{j=1}^{N_x} \) is greater or equal than one.

Now suppose by contradiction that \( \max_{1 \leq j \leq N_x} m_j(x) < 1 \) and \( \| u_x \|_{L^p(B_1)} > \omega_n \). Then

\[
\| u_{0,x} \|_{L^p(B_1)} = \left\| \sum_{j=1}^{N_x} m_j(x) \mathbb{I}_{F_{r,j}} \right\|_{L^p(B_1)} \leq \sum_{j=1}^{N_x} \int_{F_{r,j}} |m_j(x)| \wedge 1 \, dy \leq \omega_n,
\]

which immediately implies (5.13) and the proposition.

**Theorem 5.12.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and let \( \Gamma \in \mathcal{F}_p \) with \( (1 < p < n) \). Suppose \( (u_k)_{k=1}^{\infty} \subset GSBV^p(\Omega; \Gamma) \) is such that

\[
\| u_k - u \|_{L^p(\Omega)} + \| \nabla u_k - \nabla u \|_{L^p(\Omega)} \to 0, \quad \text{as } k \to \infty.
\]

Then \( (u_k)_x \) converge to \( u_x \) in the Fréchet space \( U_{C_p^p}(\Omega; L^0(\Gamma)) \).

**Proof.** We shall prove that given \( \epsilon, \delta > 0 \), then there exists \( \overline{\epsilon} \) such that for every \( k \geq \overline{\epsilon} \)

\[
C_p(\{ x \in \Omega : (u_k)_x - u_x \|_{L^p(B_1)} > \omega_n \epsilon \}) \leq \delta.
\]

By linearity we have that \( (u_k)_x - u_x = (u_k - u)_x \). Therefore, by using the capacitary Chebyshev’s inequality, for every \( 1 < q < p \) we get

\[
C_q(\{ x \in \Omega : \| (u_k)_x - u_x \|_{L^p(B_1)} > \omega_n \epsilon \}) \leq \frac{1}{\epsilon^q} \int_{\Omega} (|\nabla u_k - \nabla u|^q + |u_k - u|^q) \, dx.
\]

Finally, by the definition of \( C_p \) it is enough to choose \( \overline{\epsilon} \) big enough such that for every \( k \geq \overline{\epsilon} \)

\[
\sup_{1 < q < p} \frac{1}{\epsilon^q} \int_{\Omega} (|\nabla u_k - \nabla u|^q + |u_k - u|^q) \, dx \leq \delta,
\]

which is possible since \( \Omega \) is bounded.

Putting together Theorems 5.7, 5.12 and (5.11) we are able to prove our second main result.
Proof of Theorem 0.2. Let us first suppose $\Omega$ bounded. Putting together the previous result with Theorem 1.7 we have that there exists a subsequence $k_j$ such that
\[
\lim_{j \to \infty} \| (u_{k_j})_x - u_x \|_{L^p(B_1)} = 0,
\]
for every $x \in \Omega$ except a $C^0_\infty$-negligible set $S$. Putting together Theorem 5.7 with (5.11) it easily follows that
\[
\dim_H(S) = n - p.
\]
For general $\Omega$, we set $\Omega_i := \Omega \cap B_i(0)$, $i \in \mathbb{N}$. For every $i$ we can apply the previous result on the bounded open set $\Omega_i$ to obtain a sequence $(k_j^i)_{i=1}^\infty$ and a set $S_i \subset \Omega_i$ with $\dim_H(S_i) = n - p$, such that
\[
\lim_{j \to \infty} \| (u_{k_j^i})_x - u_x \|_{L^p(B_1)} = 0, \quad \text{for every } x \in \Omega_i \setminus S_i.
\]
We can also suppose that $(k_j^i)_{i=1}^\infty$ is a subsequence of $(k_j^i)_{i=1}^\infty$ for every $i$. By a diagonal argument we define for every $j \in \mathbb{N}$ $k_j := k_j^i$, and we obtain that
\[
\lim_{j \to \infty} \| (u_{k_j})_x - u_x \|_{L^p(B_1)} = 0, \quad \text{for every } x \in \Omega \setminus \bigcup_{i=1}^\infty S_i.
\]
Finally, since every $S_i$ has Hausdorff dimension equal to $n - p$, then also $\bigcup_{i=1}^\infty S_i$ has Hausdorff dimension equal to $n - p$. This proves the theorem.

Remark 5.13. In [5] the authors are able to prove a density result for the space $SBV^p(\Omega)$. More precisely, let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary, and let $u \in SBV^p(\Omega)$. Then, there exists a sequence of functions $u_j \in SBV^p(\Omega)$ and of compact $\mathcal{C}^1$ manifolds with $\mathcal{C}^1$ boundary $M_j \subset \subset \Omega$ such that $J_{u_j} \subseteq M_j$ but $\mathcal{H}^{n-1}(M_j \setminus J_{u_j}) = 0$ and
\[
u_j \in C^\infty(\Omega \setminus J_{u_j}), \quad \| u_j - u \|_{L^1} \to 0, \quad \| \nabla u_j - \nabla u \|_{L^p} \to 0, \quad \mathcal{H}^{n-1}(J_{u_j} \Delta J_u) \to 0.
\]
It is natural to ask whether the hypothesis $\mathcal{H}^{n-1}(J_{u_j} \Delta J_u) \to 0$ can be improved in the sense
\[
J_{u_j} \subset J_u \quad \text{for every } j \in \mathbb{N}.
\]
In other words we can rephrase this question in the following way: given $\Gamma \subset \Omega$ a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set, then is it true that the closure in $SBV^p$ with respect to the norm given by $\| \nabla u \|_{L^p} + \| u \|_{L^1}$ of all functions $v$ such that
\[
v \in C^\infty(\Omega \setminus J_{u_j}), \quad J_{u_j} \subset M \subset \Gamma, \quad M \text{ is any } \mathcal{C}^1 \text{ manifolds with } \mathcal{C}^1 \text{ boundary},
\]
is the whole of $SBV^p(\Omega; \Gamma)$?

The answer is in general no. Indeed consider for example in $\mathbb{R}^3$ the set $\Gamma$ given by the union of three half spaces intersecting on a line $l$. $\Gamma$ disconnects $\mathbb{R}^3$ into three connected components, and consider $v: \mathbb{R}^3 \to \mathbb{R}$ the function which assumes three different constant values on each of the connected components, say $\alpha_1 \neq \alpha_2 \neq \alpha_3$. Clearly $v \in SBV^p(\Omega; \Gamma)$ for every $p \in [1, 3)$. We claim that for $p \in (2, 3)$, the function $v$ cannot be approximated in $SBV^p$ by functions satisfying (5.14). Indeed, any function $u \in SBV^p(\Omega)$ satisfying (5.14) has the property that $v_x$ is defined everywhere, except a $(3 - p)$-dimensional Hausdorff set, and is a function taking at most two values. By using a slightly modified version of Theorem 0.2 (where we have to substitute the $L^p$ convergence of the functions with the $L^1$ convergence), but that still holds, we deduce that any limit $u \in SBV^p(\Omega; \Gamma)$ of functions satisfying (5.14), inherits the property that its blow-up converge to a function $u_x$ which takes at most two values for every $x$ except a set of Hausdorff dimension $3 - p$. However for every point $x \in l$ our function $v$ has a blow-up which converge to a function $v_x$ which takes three different values $\alpha_1, \alpha_2, \alpha_3$. Since $\dim_{H}(l) = 1$, this implies that for every $p \in (2, 3)$, $v$ cannot be approximated by functions satisfying (5.14).

Appendix A. Example of sets living in $J_p$

Let $n \geq 3$ and $1 < p \leq n - 1$. We write the generic point $x \in \mathbb{R}^n$ as $x = (y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

We define $W^{1,p}(\mathbb{R}^{n-1})$ as the space of all Sobolev functions $f \in W^{1,p}(\mathbb{R}^{n-1})$ such that for every $y \in \mathbb{R}^{n-1}$ except a set of Hausdorff dimension $n - 1 - p$, $y$ is a Lebesgue point for the distributional gradient $Df$.

1By using the theory of capacity it is easy to see that $W^{2,p}(\mathbb{R}^{n-1}) \subset W^{1,p}(\mathbb{R}^{n-1})$. 

1
Now let \( f \in W^{1,p}(\mathbb{R}^{n-1}) \) and consider its sub-graph
\[
S_f^- := \{ x \in \mathbb{R}^n \mid t < f(y), \ y \in \mathbb{R}^{n-1} \}.
\]
It is well known that \( S_f^- \) is a set having locally finite perimeter in \( \mathbb{R}^n \).

Consider the following sets
\[
A := \left\{ y \in \mathbb{R}^{n-1} \mid \exists \hat{f}(y) \in \mathbb{R}, \ \lim_{r \to 0} \int_{B_r^{n-1}(y)} |f(z) - \hat{f}(y)| \ dz \to 0 \right\},
\]
and
\[
B := \left\{ y \in \mathbb{R}^{n-1} \mid \exists \tilde{D}f(y) \in \mathbb{R}^n, \ \lim_{r \to 0} \int_{B_r^{n-1}(y)} |Df(z) - \tilde{D}f(y)| \ dz \to 0 \right\},
\]
where \( B_r^{n-1}(y) \) is the \((n-1)\)-dimensional ball of radius \( r \) centered at \( y \). To be precise we will call the graph of \( u \) the set of point of the form
\[
\text{graph}(u) := \{ (y,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y \in A \cap B, \ t = \hat{f}(y) \}.
\]

**Proposition A.1** (Example of sets living in \( J_\rho \)). Let \( u \in W^{1,p}(\mathbb{R}^{n-1}) \) with \( 1 < p \leq n-1 \) and \( n \geq 3 \). Then graph(\( u \)) belongs to \( J_\rho \).

**Proof.** Using the theory of capacity developed in [9] (see also [7, Section 4.7]), and the definition of \( W^{1,p} \), we know that
\[
\dim_H(\mathbb{R}^{n-1} \setminus A \cap B) \leq n - 1 - p.
\]
Therefore, it follows for example by [17, Corollary 8.11] that
\[
\dim_H(\mathbb{R}^{n-1} \setminus (A \cap B) \times \mathbb{R}) \leq n - p.
\]

We claim that for every \( x = (y,t) \in \mathbb{R}^n \) such that \( y \in A \cap B \) one and only one of the following conditions occur
- \( x \in \partial^* S_f^- \);
- \( \Theta^{*(n-1)}(\mathcal{H}^{n-1} \cap \partial^* S_f^-, x) = 0 \).

By Proposition 3.7, this would imply that \( \partial^* S_f^- \) has a non vanishing upper isoperimetric profile at \( x \).

We first prove that for every \( x = (y,t) \in (A \cup B) \times \mathbb{R} \) such that \( t < \hat{f}(y) \), it holds
\[
\Theta^{*(n-1)}(\mathcal{H}^{n-1} \cap \partial^* S_f^-, x) = 0,
\]
(A.1)
or equivalently
\[
\lim_{r \to 0^+} \mathcal{H}^{n-1}_r((\partial^* S_f^-)_{r,x}) = 0.
\]
(A.2)

Now, since \( \lim_{r \to 0} \int_{B_r^{n-1}(y)} |f(z) - \hat{f}(y)| \ dz = 0 \), then by a change of variable in the integral we have
\[
\lim_{r \to 0^+} \|f(y + r(\cdot)) - \hat{f}(y)\|_{L^1(B_r^{n-1}(0))} = 0.
\]
In particular, this means that for every \( \epsilon > 0 \)
\[
\lim_{r \to 0^+} |\{ |f(y + r(\cdot)) - \hat{f}(y)| > \epsilon \} \cap B_r^{n-1}(0)| = 0.
\]
(A.3)

For every \( z \in B_1^{n-1}(0) \) such that \( |f(y + rz) - \hat{f}(y)| \leq \epsilon \) we have
\[
|f(y + rz) - t| \geq \frac{|f(y) - t|}{r} - \frac{|f(y + rz) - \hat{f}(y)|}{r} \geq \frac{|\hat{f}(y) - t|}{r} - \frac{\epsilon}{r},
\]
and if \( \epsilon < \frac{|\hat{f}(y) - t|}{2r} \), by the previous inequalities we deduce
\[
|f(y + rz) - t| > \frac{\hat{f}(y) - t}{2r}.
\]
Hence, for sufficiently small value of \( r \), we have
\[
|f(y + rz) - t| > 1.
\]
Therefore, for sufficiently small value of \( r \) we have
\[
\{ |f(y + r(\cdot)) - t|/r \leq 1 \} \cap B_{1}^{n-1}(0) \subset \{ |f(y + r(\cdot)) - \tilde{f}(y)| > \epsilon \} \cap B_{1}^{n-1}(0).
\]
Notice that
\[
(\partial^* S_f)_{r,x} \subset \left\{ (z, s) \in B_{1}^{n-1}(0) \times (-1, 1) \mid s = \frac{f(y + rz) - t}{r} \right\}.
\]
As a consequence, for sufficiently small value of \( r \) we have the following inequalities
\[
\mathcal{H}^{n-1}(\partial^* S_f)_{r,x} \leq \int_{\{ |f(y + r(\cdot)) - t|/r \leq 1 \} \cap B_{1}^{n-1}(0)} \sqrt{1 + |Df(z)|^2} \, dz \tag{A.4}
\]
\[
\leq \int_{\{ |f(y + r(\cdot)) - \tilde{f}(y)| > \epsilon \} \cap B_{1}^{n-1}(0)} \sqrt{1 + |Df(z)|^2} \, dz. \tag{A.5}
\]
Therefore, by using (A.3) and the definition of \( B \) we deduce
\[
\lim_{r \to 0^+} \frac{1}{\int_{B_{1}^{n-1}(0)}} \frac{|f(y + rz) - f(y) - Du(y) \cdot rz|}{r} \, dz = 0,
\]
which proves the claim (A.1).

Analogously one can prove that if \( x = (y, t) \in (A \cup B) \times \mathbb{R} \) is such that \( \tilde{f}(y) < t \) then (A.1) holds.

Finally it remains to prove that if \( x \in \text{graph}(u) \) then \( x \in \partial^* S_f \). First of all, since \( y \) is a Lebesgue point for \( u \) and a Lebesgue point for \( Du \), by using [1, Theorem 3.83], \( u \) is approximately differentiable at \( y \)
\[
\lim_{r \to 0^+} \frac{1}{\int_{B_{1}^{n-1}(0)}} \frac{|f(y + rz) - f(y) - \tilde{D}u(y) \cdot rz|}{r} \, dz.
\]
This means that if we set \( L_y(z) := \tilde{D}u(y) \cdot z \) then
\[
\frac{f(y + r(\cdot)) - \tilde{f}(y)}{r} \to L_y(\cdot), \text{ in } L^1(B_{1}^{n-1}(0)) \tag{A.6}
\]
as \( r \to 0^+ \). Moreover, if we call \( H^+_x \) the lower half space relative to the unit vector \( \frac{(-\tilde{D}u(y), 1)}{\sqrt{1 + |\tilde{D}u(y)|^2}} \) and \( C_1(0) \) the cylinder given by \( B_{1}^{n-1}(0) \times (-1/2, 1/2) \), we have
\[
\lim_{r \to 0^+} P((S_f^* - x)/r; C_1(0)) = \lim_{r \to 0^+} \int_{B_{1}^{n-1}(0)} \sqrt{1 + |Df(y + rz)|^2} \, dz
\]
\[
= \mathcal{H}^{n-1}(B_{1}^{n-1}(0)) \sqrt{1 + |\tilde{D}f(y)|^2}
\]
\[
= P(H^+_x; C_1(0)), \tag{A.7}
\]
where we used that \( y \) is a Lebesgue point for \( Du \).

Putting together (A.6) with (A.7) we deduce
(i) \( (S_f^* - x)/r \to H^+_x \) in measure in \( C_1(0) \) as \( r \to 0^+ \);
(ii) \( \lim_{r \to 0^+} P((S_f^* - x)/r; C_1(0)) = P(H^+_x; C_1(0)) \).

Since \( B_1(0) \subset C_1(0) \), condition (i) implies in particular
\[
(S_f^* - x)/r \to H^+_x \text{ in measure in } B_1(0) \text{ as } r \to 0^+ \tag{A.8}
\]
the convergence in measure in $B_1(0)$. Moreover, since $P(H_x^-; \partial B_1(0)) = 0$ we have

$$P(H_x^-; C_1(0)) = \lim_{r \to 0^+} P((S_f^- - x)/r; C_1(0))$$

$$\geq \limsup_{r \to 0^+} [P((S_f^- - x)/r; B_1(0)) + P((S_f^- - x)/r; C_1(0) \setminus B_1(0))]$$

$$\geq \limsup_{r \to 0^+} P((S_f^- - x)/r; B_1(0)) + \inf_{r \to 0^+} P((S_f^- - x)/r; C_1(0) \setminus B_1(0))$$

$$\geq \inf_{r \to 0^+} P((S_f^- - x)/r; B_1(0)) + \inf_{r \to 0^+} P((S_f^- - x)/r; C_1(0) \setminus B_1(0))$$

$$\geq P(H_x^-; B_1(0)) + P(H_x^-; C_1(0) \setminus B_1(0))$$

$$= P(H_x^-; C_1(0)),$$

which implies

$$\limsup_{r \to 0^+} P((S_f^- - x)/r; B_1(0)) = \inf_{r \to 0^+} P((S_f^- - x)/r; B_1(0)) = P(H_x^-; B_1(0)). \quad (A.9)$$

Putting together (A.8) and (A.9) we can use [1, Proposition 1.62] for the measures $\mu_r := D_{\mu}(S_f^- - x)/r \ (0 < r < 1)$, to deduce that

$$\mu_r(B_1(0)) \to \omega_{n-1} \nu(x) \text{ as } r \to 0^+, \quad (A.10)$$

where $\nu_{H_x^-}$ is the unit vector relative to $H_x^-$. Finally, by (A.9) we deduce

$$\lim_{r \to 0^+} P(S_f^-; B_r(x)) = \omega_{n-1},$$

which together with (A.10) implies

$$\lim_{r \to 0^+} \frac{D_{\mu_r}(B_r(x))}{D_{\mu_r}(S_f^-)(B_r(x))} = \lim_{r \to 0^+} \frac{r^{n-1} \mu_r(B_1(0))}{P(S_f^-; B_r(x))} = \lim_{r \to 0^+} \frac{\mu_r(B_1(0))}{\omega_{n-1}} = \nu(x),$$

and this is exactly (1.9), hence we can conclude $x \in \partial^* S_f^-$. \hfill $\Box$

**Remark A.2.** Since for $n - 1 - 2p > 0$ it is possible to construct functions $u \in W^{2,p}(\mathbb{R}^{n-1})$ such that the topological closure of their graphs have arbitrarily large $n$-dimensional Lebesgue measure, with the previous example we have shown that a generic set in $\mathcal{F}_p$ is not essentially closed.

**Proposition A.3.** Let $\Omega \subset \mathbb{R}^n$ be an open set ($n \geq 3$), and let $(\Gamma_i)_{i=1}^M \ (M \in \mathbb{N})$, be sets such that for every $i$ there exists $\xi_i \in S^{n-1}$ and $f \in W^{1,p}(\xi_i^+)$ ($1 < p \leq n - 1$) with $\Gamma_i := \text{graph}(f_i) \cap \Omega$. Then $\Gamma := \bigcup_{i=1}^M \Gamma_i$ belongs to $\mathcal{F}_p$.

**Proof.** Proposition A.1 shows that for every $x \in \Omega$ and for every $1 \leq i \leq M$, except an $(n-p)$-dimensional Hausdorff set, one and only one of the following conditions occurs

- $x \in \partial^* S^+_{f_i}$;
- $\Theta^{(n-1)}(H^{n-1} \cap \partial^* S^+_{f_i}, x) = 0$.

Now fix such an $x \in \Omega$. By reordering the index $i$ we may suppose for example that there exists $k \in \mathbb{N}$ such that for every $1 \leq i \leq k$ $x \in \partial^* S^+_{f_i}$ and for every $k < i \leq M$ $\Theta^{(n-1)}(S_{f_i}, x) = 0$. Without loss of generality we may also suppose that if $1 \leq i_1 < i_2 \leq k$ and $\Gamma_{i_1}, \Gamma_{i_2}$ have the same tangent space at $x$, then the theoretic normals of $S^+_{f_{i_1}}$ and $S^+_{f_{i_2}}$ are the same at $x$. For the same reason, without loss of generality, we may suppose that for every $k < i \leq M$ than $x$ is a point of density 1 for $S^+_{f_i}$.

Given $r > 0$ such that $B_r(x) \subset \Omega$, we set for $1 \leq i \leq k$

$$E^-_{i} := S^{-}_{f_i} \cap B_r(x) \quad \text{and} \quad E^+_{i} := S^{+}_{f_i} \cap B_r(x).$$

and

$$E^{-}_{0,i} := \{y \in B_1(0) \mid \nu_{\Gamma_i}(x) \cdot y < 0\} \quad \text{and} \quad E^{+}_{0,i} := \{y \in B_1(0) \mid \nu_{\Gamma_i}(x) \cdot y > 0\},$$

For $k < i \leq M$ we set

$$E_{i,1} := S^{-}_{f_i} \cap B_r(x),$$
and  

\[ E_{0,i} := B_1(0). \]

By eventually reordering again the first \( k \) index, we may assume that there exist \( k_1, k_2, \ldots, k_m \) (\( m \leq k \)) such that  

\[ \nu_{r_{k_1}} = \nu_{r_{k_2}} \text{ if and only if } k_j \leq i_1, i_2 < k_{j+1}. \]

Now we want to define the sets \( F_{r,j} \) and \( E_{0,j} \) of Definition 3.3. For this purpose let us denote as \( \Sigma^M_2 \) the family of maps from \( \{1, \ldots, M\} \) into \( \{-, +\} \). We define  

\[ E_\sigma = \bigcap_{\sigma \in \Sigma^M_2} E^\sigma_i(i), \]

and  

\[ E_{0,\sigma} = \bigcap_{\sigma \in \Sigma^M_2} E^0_{0,i}(i), \]

whenever \( E_{0,\sigma} \neq \emptyset \).

We have \( 1 \leq N_x \leq 2^M \). Instead of the index \( j = 1, \ldots, N_x \) appearing in Definition 3.3, we have indexed our sets by \( \sigma \in \Sigma^M_2 \). Notice that  

\[ \lim_{r \to 0^+} |(E_\sigma)_{r,x} \Delta E_{0,\sigma}| = 0. \]

Moreover \( E_{0,\sigma} \) are conical and indecomposable sets, since they are intersection of half spaces.

Notice that by our choice of \( x \in \Omega \) we have that  

\[ \lim_{r \to 0^+} P((E^\pm_i)_{r,x}; B_1(0)) = P(E^\pm_i; B_1(0)), \quad i = 1, \ldots, M. \]

By construction we have also that, since \( E_{0,\sigma} \neq \emptyset \), then \( \sigma(i_1) = \sigma(i_2) \) for every \( k_j \leq i_1, i_2 < k_{j+1} \) and for every \( j = 1, \ldots, m \). This means that the family \( (E^\sigma_{0,i})_{i=1}^M \) satisfies also point (3) of Lemma A.5. Therefore we can deduce that  

\[ \lim_{r \to 0^+} P((E_\sigma)_{r,x}; B_1(0)) = P(E_{0,\sigma}; B_1(0)). \]

Hence, we are in position to apply Proposition 2.2 and to deduce that for every \( \sigma \in \Sigma^M_2 \) such that \( E_{0,\sigma} \neq \emptyset \), there exist indecomposable components of \((E_\sigma)_{r,x}\), say \( F_{r,\sigma} \) such that  

\[ \lim_{r \to 0^+} |F_{r,\sigma} \Delta E_{0,\sigma}| = 0, \]  

(A.11)

and  

\[ \lim_{r \to 0^+} P(F_{r,\sigma}; B_1(0)) = P(E_{0,\sigma}; B_1(0)). \]  

(A.12)

This gives immediately condition (1.1) and (2.1) of Definition 3.3, and using Proposition 3.8 we deduce also that for every \( 1 \leq j \leq M \) the families \((F_{r,j})_{0 < r < r_j}\) are left continuous.

Finally, by (A.12) we can use the same argument as in the proof of Proposition 3.7 to deduce  

\[ \liminf_{r \to 0^+} h_{F_{r,\sigma}}(\lambda) \geq h_{E_{0,\sigma}}(\lambda), \quad \lambda \in (0, 1/2], \]

which implies condition (1.2) of Definition 3.3 since \( h_{E_{0,\sigma}}(\lambda) > 0 \) for every \( \lambda \in (0, 1/2] \). \( \square \)

The purpose of the previous propositions is to show that the class \( J_0 \) is much richer than the class of \( \mathcal{C}^1 \)-manifolds. Nevertheless, we were able to cover condition (1.2) of Definition 3.3, by using the convergence of the perimeter  

\[ \lim_{r \to 0^+} P(F_{r,j}; B_1(0)) = P(E_{0,j}; B_1(0)), \quad 1 \leq j \leq N_x. \]  

(A.13)

However, we want to show that (A.13) is not necessary in order to have a non-vanishing upper isoperimetric profile at \( x \). In the next example we exhibit a rectifiable set \( \Gamma \) in \( \mathbb{R}^2 \) such that there exists a set of Hausdorff dimension \( \alpha \) (\( 0 < \alpha < 1 \)) on which \( \Gamma \) admits an asymptotic upper isoperimetric profile but the limit (A.13) diverges to \( +\infty \).
Example A.4 (Cantor’s home). We work in $\mathbb{R}^2$. We define a sequence of closed set, say $(J_n)_{n=1}^{\infty}$, following the usual way to construct the Cantor’s middle third set (see [17, Subsection 4.10]): let $J_1 := [0, 1]$ and define $J_n := J_{n-1} \cup \left( \frac{2}{3} + J_{n-1} \right)$.

Now fix $2 < s < 3$ and consider by induction the following sets:

$$C_1 := J_1 \times \left[0, \frac{1}{s-1}\right],$$

and

$$C_n := C_{n-1} \setminus \left( J_{n-1} \times \left(-\infty, \frac{s^{1-n}}{s-1}\right) \right),$$

where

$$\frac{s^{1-n}}{s-1} = \sum_{i=n}^{\infty} \frac{1}{s^i} \quad \text{and} \quad \frac{1}{s-1} = \sum_{i=1}^{\infty} \frac{1}{s^i}.$$

We define the Cantor’s home $C \subset \mathbb{R}^2$ as

$$C := \bigcap_{n=1}^{\infty} C_n.$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{cantor-home.png}
\caption{Cantor's home}
\end{figure}

By construction $C$ is a closed set and $P(C) < \infty$. Indeed it can be easily verified that

$$P(C_{n+1}) := P(C_n) + \frac{2^n}{(s-1)s^n}, \quad n = 1, 2, \ldots,$$

which means

$$P(C) \leq \liminf_{n \to \infty} P(C_n) = P(C_1) + \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2^n}{(s-1)s^n} < \infty,$$

where in the last inequality we have used $s > 2$.

We claim that $\partial^* C$ admits a non vanishing upper isoperimetric profile at every $x \in \mathbb{R}^2$. As a consequence $\Gamma \in J_p$ for every $p > 1$.

To prove our claim, notice that if we call $C \subset [0, 1]$ the Cantor’s set, i.e.

$$C = \bigcap_{n=1}^{\infty} J_n,$$

then it is easy to see that for every $x \in \mathbb{R}^2 \setminus (C \times \{0\})$ our claim is satisfied. Therefore, we need only to prove that for $x \in C \times \{0\}$ our claim holds.

If $x \in C \times \{0\}$, by using the fact that the number of connected components of $J_n \cap (x_1 - \frac{2}{3}, x_1 + \frac{2}{3})$ can be asymptotically estimated by $2^{n-\log_{3/2} r}$, together with the fact that $s < 3$, it is possible to check that

$$\lim_{r \to 0^+} \frac{|C \cap B_r(x)|}{r^2} = 0,$$  \hfill (A.14)
and
\[
\lim_{r \to 0^+} \frac{P(C, B_r(x))}{r} = +\infty. \tag{A.15}
\]

Now we denote the generic point \(x \in \mathbb{R}^2\) as \(x = (x_1, x_2)\) where \(x_1, x_2 \in \mathbb{R}\) and we prove that conditions (1) and (2) of Definition 3.3 are satisfied with \(N_x = 1\). Instead of the balls \(B_r(x)\) we prefer to work with the squares \(Q_r(x)\). It is clear that everything will be true also for balls.

Pick \(x \in C \times \{0\}\) and consider \(r_x > 0\) such that \(Q_{r_x}(x) \subset \Omega\). Set
- \(E_{0,1} := Q_1(0)\);
- \(F_{r,1} := Q_1(0) \setminus \frac{c_r}{r} \) for every \(r \leq r_x\).

First of all, since for each \(r\) the sets \(F_{r,1}\) are connected open sets with finite perimeter, then they are indecomposable (see Remark 1.13). Thanks to (A.14), we can apply Proposition 3.8 to deduce that there exists an \(0 < r' < r_x\) such that for \(r \in (0, r')\) the family \((F_{r,1})_r\) is left continuous. Moreover, conditions (1.1) and (2.1) immediately follow from construction and from (A.14), respectively.

In order to show that also condition (1.2) is satisfied, first of all notice that for each \(r < r_x\) the sets \(F_{r,1}\) are open connected and of finite perimeter, hence in particular they are indecomposable. We claim that for every \(r \in (0, r_x)\) and every \(\lambda \in (0, 1/2)\) we have
\[
h_{F_{r,1}}(\lambda) \geq \frac{1}{3}. \tag{A.16}
\]

In order to show (A.16) we shall prove that for every \(r \in (0, 1)\)
\[
\min\{|E|, |F_{r,1} \setminus E|\} \leq 3\mathcal{H}^1(\partial^* E \cap F_{r,1}^{(1)}), \quad E \subset F_{r,1}. \tag{A.17}
\]

This can be achieved by proving that for every \(r \in (0, 1)\) it holds the following Poincaré's inequality
\[
\int_{F_{r,1}} |u - \bar{u}| \, dx \leq 3|Du|(F_{r,1}), \quad u \in BV(Q_1(0)), \tag{A.18}
\]
where \(\bar{u} := \int_{F_{r,1}} u\). Then (A.17) follows by choosing \(u = 1_E\) in (A.18), since in this case
\[
\int_{F_{r,1}} |u - \bar{u}| \, dx \geq \min\{|E|, |F_{r,1} \setminus E|\},
\]
and
\[
\mathcal{H}^{n-1}(\partial^* E \cap F_{r,1}^{(1)}) \geq \mathcal{H}^{n-1}(\partial^* E \cap F_{r,1}) = |Du|(F_{r,1}).
\]

Given \(t \in \mathbb{R}\) we write
\[
F_t := \{x_2 \in \mathbb{R} \mid (t, x_2) \in F\}, \quad \text{and} \quad F^t := \{x_1 \in \mathbb{R} \mid (x_1, t) \in F\}.
\]

Notice that for each \(r \in (0, r_x)\) the sets \(F_{r,1}\) have the following two properties
- (1) \((x_1, x_2) \in F_{r,1}\) and \((x_1, y_2) \in F_{r,1}\) implies \((x_1, \lambda x_2 + (1-\lambda)y_2) \in F_{r,1}\) for every \(\lambda \in [0, 1]\);
- (2) \((x_1, x_2) \in F_{r,1}\), \((y_1, x_2) \in F_{r,1}\) and \(x_2 \in (-1/2, 0)\) implies \((\lambda x_1 + (1-\lambda)y_1, x_2) \in F_{r,1}\) for every \(\lambda \in [0, 1]\).

We show that any set \(F \subset Q_1(0)\) satisfying (1) and (2) admits a Poincaré's inequality like (A.18).

Indeed we have
\[
\int_F |u - \bar{u}| \, dx = 2 \int_{-1/2}^0 \left[ \int_F u(x_1, x_2) - \left( \int_F u(y_1, y_2) \, dy_2 \right) \, dx_1 \, dx_2 \right] \, dt
\leq 2 \int_{-1/2}^0 \left[ \int_F \left( u(x_1, x_2) - u(y_1, y_2) \right) \, dy_2 \, dx_1 \, dx_2 \right] \, dt.
\]

If \(t \in (-1/2, 0)\), using the triangle inequality we can write
\[
|u(x_1, x_2) - u(y_1, y_2)| \leq |u(x_1, x_2) - u(x_1, t)| + |u(x_1, t) - u(y_1, t)| + |u(y_1, t) - u(y_1, y_2)|,
\]
hence by the Fundamental Theorem of Calculus we have
\[
\int_F |u - \bar{u}| \, dx \leq 2 \int_{-1/2}^0 \left[ \int_F \left| D_2 u(F_{x_1}) \right| \, dy_1 \, dy_2 \right] \, dx_1 \, dx_2 \, dt
\]
Since \( \Omega \subset \mathbb{R}^n \) be an open set. Let \((E_{r,i})_{i=1}^M\) (\(M \in \mathbb{N}\)) be sets having finite perimeter in \(\Omega\). Suppose that there exist sets \((E_{0,i})_{i=1}^M\) having finite perimeter in \(\Omega\) such that

1. \(\lim_{r \to 0^+} |E_{r,i} \Delta E_{0,i}| = 0\), \(1 \leq i \leq M\);
2. \(\lim_{r \to 0^+} P(E_{r,i}; \Omega) = P(E_{0,i}; \Omega), 1 \leq i \leq M\);
3. \(H^{n-1}(\partial^* E_{0,i_1} \cap \partial^* E_{0,i_2} \cap \{\nu_{E_{0,i_1}} \neq \nu_{E_{0,i_2}}\}) = 0, 1 \leq i_1 < i_2 \leq M\).

Then we have

\[
\lim_{r \to 0^+} P\left( \bigcap_{i=1}^M E_{r,i}; B_1(0) \right) = P\left( \bigcap_{i=1}^M E_{0,i}; B_1(0) \right).
\]

Proof. We proceed by induction on \(M\). For \(M = 1\) there is nothing to prove. By induction suppose that our statement holds for \(M - 1\), then we want to show that it still holds for \(M\). For this purpose, suppose to have \((E_{r,i})_{i=1}^M\) satsifying (1)-(3). If we consider the first \(M - 1\) sets \((E_{r,i})_{i=1}^{M-1}\), then they clearly still satisfy (1)-(3), hence by inductive hypothesis we have

\[
\lim_{r \to 0^+} P\left( \bigcap_{i=1}^{M-1} E_{r,i}; B_1(0) \right) = P\left( \bigcap_{i=1}^{M-1} E_{0,i}; B_1(0) \right).
\]

If we define \(E'_r := \bigcap_{i=1}^{M-1} E_{r,i}, E'_0 := \bigcap_{i=1}^{M-1} E_{0,i}\) and \(E_r := E_{r,M}, E_0 := E_{0,M}\), then we have that the couple \(E_r, E'_r\) still satisfies (1)-(3): the first is clearly satisfied; the second follows from (A.19); for the third just notice that if \(x \in \partial^* E'_0 \cap \partial^* E_0\) then there must exist \(1 \leq i \leq M - 1\) such that \(x \in \partial^* E_{0,i} \cap \partial^* E_{0,M}\), therefore if \(\nu_{E'_0}(x) = -\nu_{E_0}(x)\) also \(\nu_{E_{0,i}}(x) = -\nu_{E_{0,M}}(x)\). This immediately implies \(H^{n-1}(\partial^* E'_0 \cap \partial^* E_0 \cap \{\nu_{E'_0} \neq \nu_{E_0}\}) = 0\). Hence we are reduced to prove our statement for \(M = 2\).

In order to do that, we notice that by Theorem 1.15 the following identities hold

\[
P(E'_r; B_1(0)) = H^{n-1}(\partial^* E'_r \cap E^{(1)}_r) + H^{n-1}(\partial^* E'_r \cap E^{(0)}_r) + \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} = \nu_{E_r}\}) + H^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}),
\]

and analogously

\[
P(E_r; B_1(0)) = H^{n-1}(\partial^* E_r \cap E^{(1)}_r) + H^{n-1}(\partial^* E_r \cap E^{(0)}_r) + \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} = \nu_{E_r}\}) + H^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}).
\]

Summing both sides of (A.20) and (A.21), and using Leibniz formulas (1.12) for the reduced boundary of an intersection of sets with finite perimeter we get

\[
P(E'_r; B_1(0)) + P(E_r; B_1(0)) = P(E'_r \cap E_r; B_1(0)) + P(E'_r \cap E_r; B_1(0)) + 2\mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\}).
\]
Taking the lim sup on both sides we get

$$P(E'_0; B_1(0)) + P(E_0; B_1(0)) = \limsup_{r \to 0^+} P(E'_r \cap E_r; B_1(0)) + P(E'_r \cap E_r; B_1(0))$$

\[= \limsup_{r \to 0^+} \left[ 2H^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\} ) \right] \]

\[ \geq 2H^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\} ) \]

\[ \geq P(E'_0 \cap E_0; B_1(0)) + P(E'_0 \cap E_0; B_1(0)) \]

\[ + 2 \liminf_{r \to 0^+} H^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\} ) \]

\[= P(E'_0; B_1(0)) + P(E_0; B_1(0)) \]

\[ + 2 \liminf_{r \to 0^+} H^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\} ) \]

where in the last equality we use again identity (A.22) for $E'_0, E_0$ and the fact $H^{n-1}(\partial^* E'_0 \cap \partial^* E_0 \cap \{\nu_{E'_0} \neq \nu_{E_0}\} ) = 0$.

By (A.23) we immediately deduce

$$\liminf_{r \to 0^+} H^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\} ) = 0.$$ 

Moreover since (A.23) is true for every subsequence $r_j \to 0^+$ we can choose $r_j$ such that

$$\limsup_{r \to 0^+} H^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\} ) = \lim_{j \to \infty} H^{n-1}(\partial^* E'_{r_j} \cap \partial^* E_{r_j} \cap \{\nu_{E'_{r_j}} \neq \nu_{E_{r_j}}\} ),$$

and we immediately deduce that

$$\lim_{r \to 0^+} H^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \neq \nu_{E_r}\} ) = 0.$$ 

Using this last information again in (A.23), we obtain

$$\lim_{r \to 0^+} [P(E'_r \cap E_r; B_1(0)) + P(E'_r \cap E_r; B_1(0))] = P(E'_0 \cap E_0; B_1(0)) + P(E'_0 \cap E_0; B_1(0)),$$

which by the lower semi-continuity of the perimeter implies separately

$$\lim_{r \to 0^+} P(E'_r \cap E_r; B_1(0)) = P(E'_0 \cap E_0; B_1(0)),$$

and

$$\lim_{r \to 0^+} P(E'_r \cap E_r; B_1(0)) = P(E'_0 \cap E_0; B_1(0)).$$

This is exactly our desired result.

**Appendix B. Not convergence of the blow-up**

In this last part we construct a set in $E \subset \mathbb{R}^2$ with the property that its blow-up $(E - x)/r$ does not converge in measure on every point of a set having Hausdorff dimension equal to 1. To show this, we need the following theorem which can be deduced from the results obtained in [19] (see also [10] for a simpler proof). Anyway we decide to present this result with an alternative proof which is more convenient for our purpose.

**Theorem B.1.** Let $N \subset (0, 1)$. Then $N$ has zero Lebesgue measure if and only if there exists a Lipschitz function $u : (0, 1) \to \mathbb{R}$ such that $u$ is not differentiable from the right at every point of $N$.

**Proof.** If $f$ is lipschitz then the set of point where it is not right differentiable has Lebesgue measure zero from Rademacher’s Theorem.

If $N \subset \mathbb{R}$ is such that $|N| = 0$. We claim that there exists a Borel set $F \subset \mathbb{R}$ such that for every $x \in N$ we have

$$0 = \liminf_{r \to 0^+} \frac{|F \cap (x, x + r)|}{r} \leq \sup_{r \to 0^+} \frac{|F \cap (x, x + r)|}{r} = 1. \quad (B.1)$$

To prove this, notice that for every $0 < \epsilon \leq 1/6$, since $|N| = 0$, we can find a cover of $N$ made of open and disjoint subintervals of $(0, 1)$, say $(I_i)_{i=1}^\infty$, such that
Indeed (1) simply follow by the fact that $|N| = 0$. Moreover, notice that it is always possible to enlarge the interval $I_i$ by moving its end points of a quantity small than $\epsilon/2^{j+1}$ (this preserve (1) with $2\epsilon$) in order to satisfy (2). To see this, call $a < b$ the end points of $I_i$, then since $|N| = 0$ also
\[
|N - 1/2^j \cap (a - \epsilon/2^{j+1}, a)| = 0, \ j = 1, 2, 3, \ldots,
\]
hence,
\[
\left| \bigcup_{j=1}^{\infty} N - 1/2^j \cap (a - \epsilon/2^{j+1}, a) \right| = 0.
\]
This means that it is enough to choose the left end point $a'$ in the set $(a - \epsilon/2^{j+1}, a) \setminus \bigcup_{j=1}^{\infty} N - 1/2^j$ to guarantee that
\[
a' + 1/2^j \notin N, \ j = 1, 2, 3, \ldots.
\]
By repeating the same argument for the right hand point we obtain (2).

Define $\mathcal{I} \subset (0,1)$ to be a cover of $N$ made of open intervals satisfying (1) and (2) with $\epsilon = 1/6$. By induction we define $\mathcal{I}$ in the following way. For every $I \in \mathcal{I}^{i-1}$ we consider the set
\[
N_i := N \cap \{ x \in I \mid 1/2^{j+1} < \text{dist}(x, R \setminus I) < 1/2^j \},
\]
and since $|N_i| = 0$ we can use the claim to find a cover of $N_i$ made of open and disjoint subintervals of $\{ x \in I \mid 1/2^{j+1} < \text{dist}(x, R \setminus I) < 1/2^j \}$, say $(I_{i,j})_{j=1}^{\infty}$, satisfying (1) and (2) with $\epsilon = 1/6^j$. Finally, we call $\mathcal{I}^i$ the family made of all open intervals obtained as in the previous procedure, by letting $I$ varies in $\mathcal{I}^{i-1}$ and $j$ varies in $\mathbb{N}$.

We set
\[
F := \bigcup_{i=1}^{\infty} \left( \bigcup_{I \in \mathcal{I}^{i-1}} I \setminus \bigcup_{I \in \mathcal{I}^i} I \right),
\]
and we claim that $F$ does the job. Clearly $F$ is Borel since every $\mathcal{I}^i$ is a family made of open intervals. Moreover, whenever $x \in N$ then for every $i$ there exists $I \in \mathcal{I}^i$ such that $x \in I$. Moreover, by property (2) there exists $j$ such that $x \in \{ x \in I_i \mid 1/2^{j+1} < \text{dist}(x, R \setminus I_i) < 1/2^j \}$.

Therefore
\[
\left( x, x + \frac{1}{2^{j+2}} \right) \subseteq \bigcup_{k=j-1}^{j+1} \{ x \in I_i \mid 1/2^{k+1} < \text{dist}(x, R \setminus I_i) < 1/2^k \},
\]
which means by property (1) with $\epsilon = 1/6^j$ that
\[
\left| \bigcup_{I \in \mathcal{I}^{i+1}} I \cap \left( x, x + \frac{1}{2^{j+2}} \right) \right| \leq \sum_{k=j-1}^{j+1} \frac{1}{6^k}.
\]
If $i$ is odd, by (B.2), then
\[
\frac{|F \cap (x, x + 1/2^{j+2})|}{1/2^{j+2}} \geq 2^{j+2} \left( \frac{1}{2^{j+2}} - \sum_{k=j-1}^{j+1} \frac{1}{6^k} \right) = 1 - \frac{2^3}{3^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k}.
\]
(B.3)

Therefore for each $i$ odds there exists a corresponding $j_i$, with $j_i \to \infty$ as $i \to \infty$, satisfying (B.3), hence by letting $i \to \infty$ among all odds numbers, this proves
\[
\limsup_{r \to 0^+} \frac{|F \cap (x, x + r)|}{r} = 1.
\]
If $i$ is even, by (B.2), then
\[
\frac{|F \cap (x, x + 1/2^{j+2})|}{1/2^{j+2}} \leq 2^{j+2} \sum_{k=j-1}^{j+1} \frac{1}{6^k} = \frac{2^3}{3^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k},
\]
(B.4)
Arguing as before, this proves
\[
\liminf_{r \to 0^+} \frac{|F \cap (x, x + r)|}{r} = 0.
\]

Finally, define
\[
u(t) := |F \cap (0, t)|, \quad t \in (0, 1).
\]
Clearly \(u\) is 1-Lipschitz and moreover \((u(x + r) - u(x))/r = |(F \cap (x, x + r))|/r\) for every \(0 < r < 1 - x\). By (B.1) we immediately deduce that \(u\) is not right differentiable at every \(x \in N\). \(\square\)

**Theorem B.2.** There exists a set \(E \subseteq \mathbb{R}^2\) of finite perimeter, such that the set of points where its blow-up \((E - x)/r\) does not converge locally in measure has Hausdorff dimension 1.

**Proof.** Let \(N \subseteq (0, 1)\) be a set of Hausdorff dimension equal to 1. It can be easily constructed as a countable union of sets \(N_k\) with positive \(H^{1-1/k}\)-measure. Clearly \(N\) has zero Lebesgue measure.

Let \(u : (0, 1) \to \mathbb{R}\) be the 1-lipschitz function given by the previous proposition, which is not right differentiable at every point of \(N\). Define \(E := \{x \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < u(x_1)\}\). We claim that at every point \(x\) of the form \(x_1 \in N\) and \(x_2 = u(x_1)\) the blow-up of \(E\) at \(x\) does not converge in measure.

Indeed, since \(u\) is 1-Lipschitz then in the cylinder \(C_1(0)\) the set \((E - x)/r\) can be described as the subgraph of the function \(y_1 \to (u_{r,x_1}(y_1) - u(x_1))/r\) for \(y_1 \in (-1, 1)\) and for every \(r < \min x_1, 1 - x_1\). Hence the the convergence of \((E - x)/r\) locally in measure implies in particular the convergence in \(L^{1}_{loc}(0, 1)\) of the sequence \((u_{r,x_1}(y_1) - u(x_1))/r\) to some function \(v(y_1)\). Moreover, since for every \(\lambda > 0\) we have
\[
v(\lambda y_1) = \lim_{r \to 0^+} (u_{r,x_1}(\lambda y_1) - u(x_1))/r = \lambda \lim_{r \to 0^+} (u_{r,x_1}(y_1) - u(x_1))/r = \lambda v(y_1),
\]
we have that \(v\) is positively one-homogeneous. But since \(u\) is 1-Lipschitz, then the \(L^{1}_{loc}\) convergence can be improved to a uniform convergence on the closed interval \([0, 1]\), i.e.
\[
\lim_{r \to 0^+} \sup_{y_1 \in [0, 1]} \left| \frac{u(x_1 + r y_1) - u(x_1)}{r} - v(y_1) \right| = 0,
\]
and thanks to the positively one homogeneity of \(v\) this immediately implies the right differentiability of \(u\) at \(x_1\) with \(u'(x_1) = v(1)\) which is a contradiction. This proves the theorem. \(\square\)

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**References**


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