# AN ASYMPTOTIC DESCRIPTION OF THE NOETHER-LEFSCHETZ COMPONENTS IN TORIC VARIETIES


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**Abstract**

We extend the definition of Noether-Leschetz components to quasi-smooth hypersurfaces in a projective toric variety $\mathbb{P}^{2k+1}_\Sigma$ having orbifold singularities, and prove that asymptoticaly the components whose codimension is bounded from above are made of hypersurfaces containing a small degree $k$-dimensional subvariety. As a corollary we get an asymptotic characterization of the components with small codimension, generalizing the work of Otwinowska for $\mathbb{P}^{2k+1}_\Sigma = \mathbb{P}^{2k+1}$ and Green and Voisin for $\mathbb{P}^{2k+1}_\Sigma = \mathbb{P}^3$. Some tools that are developed in the paper are a generalization of Macaulay's theorem for Fano, irreducible normal varieties with rational singularieties, satisfying a suitable additional condition, and an extension of the notion of Gorenstein ideal for normal varieties with finetely generated Cox ring.



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## 1 Introduction

The classical Noether-Lefschetz theorem states that a very general surface $X$ in $\mathbb{P}^3$ of degree $d \geq 4$ has Picard number 1. In recent years generalizations have been proved for simplicial projective toric threefolds using Hodge theory [4], and more generally by Ravindra and Srinivas for normal projective threefolds using a purely algebraic approach [15].

The Noether-Lefschetz locus is the subscheme of the (hyper)surface parameter space where the Picard number is greater than the Picard number of the ambient variety. Green and Voisin proved that if $N_d$ is the Noether-Lefschetz locus for degree $d$ surfaces in $\mathbb{P}^3$, with $d \geq 4$, the codimension of every component of $N_d$ is bounded from below by $d-3$, with equality exactly for the components formed by surfaces containing a line. Otwinowska gave an asymptotic generalization of Green and Voisin's results to hypersurfaces in $\mathbb{P}^n$ [18].

In [5] (see also [14]) it was proved that for simplicial projective toric varieties the codimension of the Noether-Lefschetz components are also bounded from below, but leaving open the question if the only components with smallest codimension are those corresponding to surfaces containing a "line," defined as a curve which is minimal in a suitable sense. The purpose of the present paper is to extend and generalized Otwinowska's ideas to odd dimensional simplicial projective toric varieties. In section 2 we present a generalization of the restriction theorem due to Green [9] and we obtain a generalization of the classical Macaulay's theorem, while in section 3 we introduce a generalization of the notion of Gorenstein ideal, which we call a Cox-Gorestein ideal; these will be the key tools in the proof of our main result. Section 4 is a more technical; there we prove some application of Macaulay's theorem to Cox-Gorenstein ideals. In section 5 using Hodge theory we explicitly construct the tangent space at a point in the Noether Lefschetz loci, which turns out to be a graded part of a Cox-Gorenstein ideal. In section 6 using all the machinery so far developed we prove our main result.

We shall consider a a projective toric variety $\mathbb{P}^{2k+1}_{\Sigma}$ with orbifold singularities, whose fan is $\Sigma$, and an ample line bundle $L$ on $\mathbb{P}^{2k+1}_{\Sigma}$, with $\deg L = \beta \in \mathrm{Pic}(\mathbb{P}^{2k+1}_{\Sigma})$ satisfying for some $n \geq 0$ and $k \geq 1$ the condition

$$k\beta - \beta_0 = n\eta$$

where $\beta_0$ is the class of the anticanonical bundle and $\eta$ is class of a primitive ample Cartier divisor. $f \in \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_{\Sigma}}(\beta)))$ will be a section such that $X_f = \{f = 0\}$ is quasi-smooth hypersurface.[1] Let $\mathcal{U}_\beta \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_{\Sigma}}(\beta)))$ be the open subset parameterizing the quasi-smooth hypersurfaces and let $\pi : \chi_\beta \to \mathcal{U}_\beta$ be its tautological family. Let $H^{2k}_{\mathbb{Q}}$ be the local system $R^{2k}\pi_\star\mathbb{Q}$ and let $\mathcal{H}^{2k}$ be the locally free sheaf $H^{2k}_{\mathbb{Q}} \otimes \mathcal{O}_{\mathcal{U}_\beta}$ over $\mathcal{U}_\beta$. Let $0 \neq \lambda_f \in H^{k,k}(X_f, \mathbb{Q})/i^*(H^{k,k}(\mathbb{P}^{2k+1}_{\Sigma}))$ and let $U$ be a simply connected open subset around $f$, so that $\mathcal{H}^{2k}(U)$ is constant Finally, Let $\lambda \in \mathcal{H}^{2k}(U)$ be the section defined by $\lambda_f$ and let $\bar{\lambda}$ its image

[1] Heuristically this means that $X_f$ has only singularities inherited from the ambient space, or more precisely, regarding $\mathbb{P}^{2k+1}_{\Sigma}$ as a smooth orbifold, that $X_f$ is a smooth sub-orbifold, see e.g. [2].

in $(\mathcal{H}^{2k}/F^k\mathcal{H}^{2k})(U)$, where $F^k\mathcal{H}^{2k} = \mathcal{H}^{2k,0} \oplus \mathcal{H}^{2k-1,1} \oplus \cdots \oplus \mathcal{H}^{k,k}$.

**Definition 1.1** (Local Noether-Lefschetz Locus)**.** $N^{k,\beta}_{\lambda,U} := \{G \in U \mid \bar{\lambda}_G = 0\}$.

The following is of our main result.

**Theorem 1.2.** *For every positive $\epsilon$ there is positive $\delta$ such that for every $m \geq \frac{1}{\delta}$ and $d \in [1, m\delta]$, if $\operatorname{codim} N^{k,\beta}_{\lambda,U} \leq d\frac{m^k}{k!}$ where $m = \max\{i \mid i\eta \leq \beta\}$, then every element of $N^{k,\beta}_{\lambda,U}$ contains a $k$-dimensional subvariety whose degree is less than or equal to $(1+\epsilon)d$.*

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## 2 A restriction theorem

Every positive integer $c$ can be written in the form

$$\binom{k_n}{n} + \cdots + \binom{k_\delta}{\delta},$$

with $k_n > k_{n-1} > \ldots k_\delta \geq \delta > 0$. This is called the $n$-th Macaulay's decomposition of $c$. Let $c$ be the codimension of a sublinear system $W \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d))$ and let $W_H \subset H^0(\mathcal{O}_H(d))$ be the restriction of $W$ to a general hyperplane $H$ of codimension $c_H$. Then the classical restriction theorem says that

$$c_H \leq c_{<n>},$$

where

$$c_{<n>} := \binom{k_n - 1}{n} + \cdots + \binom{k_\delta - 1}{\delta}.$$

We generalize this result for a Fano, irreducible, projective normal variety $Y$ with rational singularities, satisfying a suitable additional condition. We note two elementary properties of the function $\phi : c \mapsto c_{<n>}$:

(i) If $c' \leq c$, then $c'_{<n>} \leq c_{<n>}$, i.e , the map $\phi$ is non-decreasing;

(ii) If $k_\delta > \delta$ then $(c-1)_{<n>} < c_{<n>}$ i.e the map $\phi$ is increasing.

**Lemma 2.1.** *Let $Y$ be an irreducible, normal projective variety with $H^1(\mathcal{O}_Y) = 0$. Let $W \subset H^0(Y, \mathcal{O}_Y(D))$ be a sublinear system, $D$ a generic ample Cartier divisor and let $W_D \subseteq H^0(D, \mathcal{O}_Y(D))$ be its restriction. Then*

$$c_D = \operatorname{codim}(W_D, H^0(\mathcal{O}_D(D))) \leq c_{<1>} = \operatorname{codim}(W, H^0(\mathcal{O}_Y(D))) - 1$$

*Proof.* Taking cohomology in the fundamental short exact sequence of the divisor $D$ we obtain

$$0 \to H^0(\mathcal{O}_Y) \to H^0(\mathcal{O}_Y(D)) \to H^0(\mathcal{O}_D(D)) \to 0 \to \cdots$$

so that

$$h^0(\mathcal{O}_Y(D)) = h^0(\mathcal{O}_Y) + h^0(\mathcal{O}_D(D)) = 1 + h^0(\mathcal{O}_D(D)).$$

Let $W_D = \{w_{|D} \mid w \in W\}$. Denoting by $r$ the projection $W \to W_D$ one has

$$\dim W = \dim \ker r + \dim W_D.$$

so (2) minus (2) we have

$$\operatorname{codim} W = \operatorname{codim} W_D + 1 - \dim \ker r.$$

If $s_D$ a section in $H^0(\mathcal{O}_Y(D))$ such that $D = \operatorname{div}_0(s_D)$, hence

$$\ker r = \{w \in W \mid w = \lambda s_D \in W, \lambda \in \mathbb{C}\}$$

and since $D$ is general so that $s_D \notin W$, then $\ker r = \{0\}$. □

**Lemma 2.2.** *Let $W \subset H^0(\mathcal{O}_{\mathbb{P}^1}(n))$ $(n > 1)$ be a subsystem, $D$ be a generic point and let $W_D \subset H^0(\mathcal{O}_D(n))$ be its restriction . Then*

$$c_D \leq c_{<n>}$$

*Proof.* Clearly $H^0(\mathcal{O}_D(n)) = \mathbb{C}$ and since $D$ is generic $c_D = 0$. On the other hand because $n > 1$ we have that $k_n > 1$ so that $c_{<n>} > 0$ i.e., $c_D \leq c_{<n>}$ □

**Definition 2.3.** *A strongly Fano variety is a pair $(Y, D)$, where $Y$ is an irreducible normal projective variety with rational singularities, and $D$ is an ample ample Cartier divisor such that $-K_Y - (k-1)D$ is ample, where $k = \dim Y$.*

**Theorem 2.4** (Restriction Theorem). *Let $(Y, D)$ be a strongly Fano variety, and let $W \subset H^0(X, \mathcal{O}_X(nD))$, with $n \geq 1$, be a subsystem, and let $W_D \subseteq H^0(D, \mathcal{O}_D(nD))$ be its restriction. Then*

$$c_D \leq c_{<n>}.$$

*Proof.* Let $l_n, \dots l_\delta$ be the coefficients of the $n$-th Macaulay decomposition of $c_D$. The inequality of the statement is equivalent to

$$\binom{l_n+1}{n} + \binom{l_{n-1}+1}{n-1} + \dots + \binom{l_\delta+1}{\delta} < c$$

By contradiction, and recalling that $\binom{l+1}{n} = \binom{l}{n} + \binom{l}{n-1}$, we have

$$c \leq \binom{l_n}{n} + \binom{l_n}{n-1} + \dots + \binom{l_\delta}{\delta} + \binom{l_\delta}{\delta-1}$$

or equivalently

$$c - c_D \leq \binom{l_n}{n-1} + \dots + \binom{l_\delta}{\delta-1}.$$

From the exact sequence

$$0 \to W(-D) \to W \to W_D \to 0$$

one has

$$\dim W = \dim W_D + \dim W(-D).$$

By a generalized Kodaira vanishing theorem [17] applied to the divisor $(n-1)D - K_Y$ $(n \geq 1)$, we have $H^1(Y, K_Y + (n-1)D - K_Y)) = 0$ so that

$$0 \to H^0(\mathcal{O}_Y(n-1)D) \to H^0(\mathcal{O}_Y(nD)) \to H^0(\mathcal{O}_D(nD)) \to 0$$

and thus

$$h^0(\mathcal{O}_Y(nD)) = h^0(\mathcal{O}_Y(n-1)D) + h^0(\mathcal{O}_D(nD)).$$

Then (2) minus (2) yield

$$c = c_D + \operatorname{codim} W(-D).$$

Taking $D' \in |D|$ generic we are within the same assumptions of the theorem on $D$ i.e.,

- $D \cap D'$ is a generic Cartier divisor in $D$;
- Moreover $D$ is irreducible, normal with rational singularities [3];
- $-K_D - (k-2)D_{|D}$, where $k = \dim Y$, is ample because $Y$ has rational singularities so it is Cohen-Macaulay [11] so we can apply the adjunction formula [12] to get

  $$-K_D - (k-2)D_{|D} = -K_{Y|D} - D_{|D} - (k-2)D_{|D} = (-K_Y - (k-1)D)_{|D}, \quad (k-1 = \dim D)$$

  by assumption the last divisor is ample.

Now we have the short exact sequence

$$0 \to W_D(-(D \cap D')) \to W_D \to W_{D|D'} \to 0$$

which gives

$$c_D = \operatorname{codim} W_{D|D'} + \operatorname{codim} W_D(-(D \cap D'))$$

Note that $W(-D')_D \subset W_D(-(D \cap D'))$, hence

$$c_D \leq \operatorname{codim} W_{D|D'} + \operatorname{codim} W(-D')_D$$

Note that strongly Fano implies Fano, so by generalized Kodaira vanishing theorem $H^1(\mathcal{O}_Y) = 0$; moreover since at each step of taking a successive generic divisor, the divisor is Fano, we have that $h^1(\mathcal{O}_D) = 0 = h^1(\mathcal{O}_{D \cap D'})$ and so on. Now by induction on $n$ and the dimension $k$ the theorem is true for $W_D$ and $W(-D)$, note that Lemmas 2.1 and 2.2 provide the induction basis.)

Now applying the theorem to $W_D$ and $W(-D)$ we get

- $(c_D)_{|D'} \leq (c_D)_{<n>} = \binom{l_n - 1}{n} + \cdots + \binom{l_\delta - 1}{\delta}$
- $(c - c_D)_{|D'} \leq (c - c_D)_{<n-1>}$

Adding the two inequalities and keeping in mind that $D' \sim D$

$$c_{D'} = c_D \leq (c_D)_{<n>} + (c - c_D)_{<n-1>}$$

by (3)

$$(c - c_D)_{<n-1>} < \binom{l_n - 1}{n-1} + \cdots + \binom{l_\delta - 1}{\delta - 1}$$

thus

$$c_D < \binom{l_n - 1}{n} + \cdots + \binom{l_\delta - 1}{\delta} + \binom{l_n - 1}{n-1} + \cdots + \binom{l_\delta - 1}{\delta - 1} = c_D$$

which is a contradiction. □

*Example* 2.5. Taking $Y = \mathbb{P}^k$ and $D = H$ a generic hyperplane, we recover the classical restriction theorem [9]. Clearly

$$-K_{\mathbb{P}^{k+1}} - (k-1)H = (k+1)H - (k-1)H = 2H$$

which is ample △

More generally:

*Example* 2.6. Let $Y = \mathbb{P}[q_0, q_1, \ldots, q_k]$ be a weighted projective space with $gcd(q_0, \ldots, q_k) = 1$ and $\delta = lcm(q_0, \ldots, q_k)$. Then for each $0 \leq j \leq k$ by [16] we have that $\frac{\delta}{q_j} D_j$ is a generator of $Pic(Y)$ and $-K_Y = \frac{\sum_i q_i}{\delta}\left(\frac{\delta}{q_j} D_j\right)$. So taking $D = \frac{\delta}{q_j} D_j$ we get that

$$K_Y - (k-1)D \ \ is \ \ ample \iff \frac{\sum_i q_i}{\delta} \geq k$$

△

**Lemma 2.7.** *Let $\mathbb{P}_\Sigma$ be a Fano projective simplicial toric $3-fold$ . Then every general nef $D$ Cartier divisor with $\rho(D) \leq 4$ is toric.*

*Proof.* By the adjunction formula $D$ is Fano and being nef is smooth by Bertini's theorem. The smooth Fano surfaces are either $\mathbb{P}^1 \times \mathbb{P}^1$ which is toric or the projective plane blown up in at most 8 points. Since $\rho(D) < 4$, $D$ is the blow up of $\mathbb{P}^2$ in at most 3 points. Applying an appropriate automorphism we can take these at most 3 points to the 3 toric points of $\mathbb{P}^2$, making $D$ isomorphic to a toric variety. □

**Macaulay's theorem.** A generalization of the classical Macaulay theorem can be obtained from the restriction Theorem 2.4. Let $W \subset H^0(\mathcal{O}_Y(nD))$ be a subsystem and let $k_n, k_{n-1}, \dots k_\delta$ be the Macaulay coefficients of its codimension $c$; let $W_1$ be the image of the multiplication map $W \otimes H^0(\mathcal{O}_Y(D)) \to H^0(\mathcal{O}_Y(n+1)D)))$, and $c_1$ be the codimension of its image. Let us denote

$$c^{<n>} := \binom{k_n+1}{n+1} + \dots + \binom{k_\delta+1}{\delta+1}.$$

which has the next elementary properties

- if $c' \leq c$ then $c'^{<n>} \leq c^{<n>}$,i.e, the map $c \mapsto c^{<n>}$ is non-decreasing
- $(c+1)^{<n>} = \begin{cases} c^{<n>} + k_1 + 1 & if\ \delta = 1 \\ c^{<n>} + 1 & if\ \delta > 1 \end{cases}$

**Theorem 2.8** (Generalized Macaulay's Theorem)**.** $c_1 \leq c^{<n>}$

*Proof.* Let $l_{n+1}, l_n, \dots l_\delta$ be the $(n+1)$-th Macaulay coefficients od $c_1$; then

$$(c_1)_D \leq c_{<n>} = \binom{l_{n+1}-1}{n+1} + \dots + \binom{l_\delta - 1}{\delta}$$

and by the sequence obtained by restriction it follows that

$$c_1 \leq c + (c_1)_D$$

so that

$$\binom{l_{n+1}-1}{n} + \dots + \binom{l_\delta - 1}{\delta - 1} \leq c$$

and then

$$\binom{l_{n+1}}{n+1} + \dots + \binom{l_\delta}{\delta} = c_1 \leq c^{<n>}.$$

□

## 3 Cox-Gorenstein Ideals

To any normal complete variety $Y$ with free finitely generated class group $\mathrm{Cl}(Y)$ one can associate a Cox ring (see [1], Construction 4.1.1):

**Definition 3.1.**

$$S(Y) := \bigoplus_{D \in \mathrm{Cl}(Y)} H^0(Y, \mathcal{O}_Y(D))$$

*Example* 3.2. Let $Y$ be a smooth projective variety with $\mathrm{Pic}(Y)_{\mathbb{R}} = N^1(Y)$. Then, $Y$ is a toric variety if and only if its Cox ring is a polynomial ring (see [10]). △

*Example* 3.3. The Cox ring need not be finitely generated; a counterexample is provided by a K3 surface with Picard number 20 [13]. △

It what follows we will assume that the Cox ring of $Y$ is finitely generated.

**Definition-Proposition 3.4** (Irrelevant Ideal)**.** *Let $D$ be an ample Cartier divisor on $Y$ and let $R_D = \bigoplus_{m=0}^{\infty} S(Y)_{mD}$. The* irrelevant ideal *is defined as*

$$B(Y,D) := \sqrt{J_{Y,D}} \text{ where } J_{Y,D} =< R_D >$$

*Actually $B(Y,D)$ it is independent of the choice of the ample Cartier divisor $D$ so we denote it $B(Y)$ (see [1]).*

**Definition 3.5** (Cox-Gorentein Ideals)**.** *An ideal $I \subset B(Y) = \mathbb{C}[x_1, \dots, x_r]$ is a* Cox-Gorenstein ideal *of socle degree $N \in Cl(Y)$ if $I$ is Artinian and there exists a nonzero linear map $\Lambda \in (S^N)^\vee$ such that for every ample class $\beta \in Cl(Y)$ one has*

$$I^\beta = \{P \in B(Y)^\beta \mid \Lambda(PQ) = 0 \textit{ for all } Q \in S(Y)^{N-\beta}\}$$

Note that the linear map $\Lambda$ induces a dual isomorphism

$$B^\beta(Y)/I^\beta \cong \left(B(Y)^{N-\beta}/I^{N-\beta}\right)^\vee$$

for every $\beta$ such that $N-\beta$ is ample. In particular $\operatorname{codim} I^\beta = \operatorname{codim} I^{N-\beta}$.

**Proposition 3.6.** *If $I$ and $I'$ are two Cox-Gorentein ideals with socle degree $N$ and $N'$ with $I \subset I'$, there exists $F \in B(Y)^{N-N'} \setminus I^{N-N'}$ such that $I' = (I : F)$.*

*Proof.* Note that $N'$ is less than or equal to $N$, and $\Lambda$ induces the isomorphism

$$B^{N-N'}(Y)/I^{N-N'} \cong \left(B(Y)^{N'}/I^{N'}\right)^\vee,$$

so that, as $\Lambda'$ (the linear map defining the ideal $I'$) yields a nonzero element in $(B(Y)^{N'}/I^{N'})^\vee$, if $[F]$is the unique element in $B^{N-N'}(Y)/I^{N-N'}$, taking a representative $F \in B^{N-N'}(Y)$, we get $\Lambda'(Q) = \Lambda(QF)$ for every $Q \in B(Y)^{N'}$. In particular

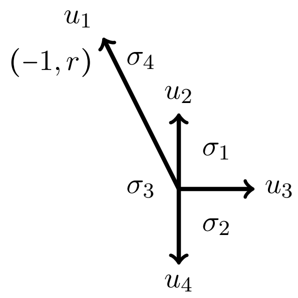
$$I' = \{Q \in B(Y) \mid QF \in I\}.$$

□

*Remark* 3.7. Artinian monomial ideals can be characterized as those whose minimal generators have the form $x_i^{a_i}$ with $a_i > 0$ for all $i\{1,\dots r\}$ ([21], Def. 2.2.13). △

*Example* 3.8. If $Y = \mathbb{P}^k$ one recovers the classical Gorenstein ideals. Other natural examples are the Artinian base point free ideals. △

*Example* 3.9. The Hirzebruch surface $\mathcal{H}_r$ $(r \geq 1)$ has fan




Denoting by $D_i$ the toric divisor corresponding to $u_i$ the are the equivalences $D_1 \sim D_3$ $D_4 \sim rD_1 + D_2$, so that $\mathrm{Pic}(\mathcal{H}_r) = \langle D_1, D_2\rangle$. The generators of its irrelevant ideal are

$$x^{\hat{\sigma_1}} = x_1x_4,\ \ x^{\hat{\sigma_2}} = x_1x_2,\ \ x^{\hat{\sigma_3}} = x_2x_3,\ \ x^{\hat{\sigma_4}} = x_3x_4.$$

Introducing variables

- $w := x^{\hat{\sigma_1}} = x_1x_4$ with $\deg w = (r+1, 1)$
- $x := x^{\hat{\sigma_2}} = x_1x_2$ with $\deg x = (1, 1)$
- $y := x^{\hat{\sigma_3}} = x_2x_3$ with $\deg y = (1, 1)$
- $z := x^{\hat{\sigma_4}} = x_3x_4$ with $\deg z = (r+1, 1)$

one can write

$$B(\Sigma) = \mathbb{C}[w, x, y, z].$$

Let us consider a monomial ideal $I$ with minimal generator elements of the form $w^{d_1}, x^{d_2}.y^{d_3}, z^{d_4}$ with $d_i > 0$, i.e,

$$I = \left\langle w^{d_1}, x^{d_2}, y^{d_3}, z^{d_4}\right\rangle \ \mathit{with}\ d_i > 0.$$

Let us check that $I$ is Cox-Gorenstein with socle degree

$$N = \deg\left(\frac{w^{d_1}x^{d_2}y^{d_3}z^{d_4}}{wxyz}\right) = (d_1 - 1)\deg w + (d_2 - 1)\deg x + (d_3 - 1)\deg y + (d_4 - 1)\deg z.$$

Let $F = \frac{w^{d_1}x^{d_2}y^{d_3}z^{d_4}}{wxyz} = w^{d_1-1}x^{d_2-1}y^{d_3-1}z^{d_4-1}$, which can be seen as one of the generators of $S^N$, and denote by $G_1,\dots,G_s$ the other generators, i.e, $P \in S^N$ is $\sum_i a_i G_i + aF$. We define $\Lambda : P \mapsto a$. Note that, if $R \in B(\Sigma)^\beta$,

$$\begin{aligned}&\Lambda(RQ) \neq 0 \ \forall Q \in S^{N-\beta} \Leftrightarrow \\ &\quad R = \sum_{k_1,k_2,k_3,k_4} a_{k_1k_2k_3k_4} w^{k_1}x^{k_2}y^{k_3}z^{k_4} \text{ such that there exists } k_1,k_2,k_3,k_4 \text{ with } 0 < k_i < d_i,\end{aligned}$$

or equivalently,

$$\Lambda(RQ) = 0 \Leftrightarrow R = \sum_{k_1,k_2,k_3,k_4} a_{k_1k_2k_3k_4} w^{k_1}x^{k_2}y^{k_3}z^{k_4} \text{ such that } k_i \geq d_i \ \forall k_1,k_2,k_3,k_4,$$

i.e, $R \in I$. △

*Example* 3.10. For $\beta$ ample, let $f \in B(\Sigma)^\beta = \mathbb{C}[x_1,\dots,x_r]$ be a quasi-smooth hypersurface in a normal toric variety. Then the Jacobian ideal $J(f) =< \frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_r} >$ is a Cox-Gorentein ideal with socle degree $N = \deg(\mathrm{Hess}(f))$ where $\mathrm{Hess}(f)$ is the Hessian polynomial of $f$, that is, the determinant of the Hessian matrix of $f$. △

# 4 Applications of Macaulay's theorem

In this section we prove some applications of Macaulay's theorem to Cox-Gorenstein ideals. We assume that $(Y,D)$ is a strongly Fano variety and we denote $\deg D = \eta \in \mathrm{Pic}(Y)$.

**Lemma 4.1.** *Let $W \subset H^0(\mathcal{O}_Y(n\eta))$ be a linear subspace whose base locus has dimension $k$ and degree $d$. Then*

$$\mathrm{codim}(W) \geq \binom{n+k+1}{k+1} - \binom{n-d+k+1}{k+1}$$

*Proof.* Let $Z$ be the base-locus of $W$ and $I_Z$ its ideal. Since $W \subset I_Z$ and $\mathrm{codim}\, W \geq \mathrm{codim}\, I_Z^n$ we can just prove that the result holds true for $\mathrm{codim}\, I_Z^n$. We shall prove that by induction over $n$ and $k$. For $n = 0$ it is clear. For $k = 0$ and $n > 0$ we need to show that $\mathrm{codim}\, I_Z^n \geq d$. Taking cohomology in the exact sequence

$$0 \to \mathcal{I}_Z(rD) \to \mathcal{O}_Y(rD) \to \mathcal{O}_Z(rD) \to 0$$

we have

$$0 \to H^0(\mathcal{I}_Z(rD)) \to H^0(\mathcal{O}_Y(rD)) \to H^0(\mathcal{O}_Z(rD)) \to H^1(\mathcal{I}_Z(r)) \to \cdots$$

where by Serre's vanishing theorem $H^1(\mathcal{I}(rD)) = 0$ for $r >> 0$. Thus

$$c := \mathrm{codim}\, I_Z^{rD} = h^0(\mathcal{O}_Y(rD)) - h^0(rD) = h^0(\mathcal{O}_Z(rD)) = d$$

as $Z$ has degree $d$. Taking $n > d$ and reasoning by contradiction we have $c < d < n$, so that

$$d = \binom{n}{n} + \cdots + \binom{n-(d-1)}{n-(d-1)} = \underbrace{1 + \cdots + 1}_{d\text{–}times}.$$

By applying the generalized Macaulay theorem and using the fact that the map $^{<n>} : c \mapsto c^{<n>}$ is increasing, we have

$$c_1 \le c^{<n>} < d \text{ where } c_1 = \operatorname{codim} I_Z^{(n+1)D};$$

repeating the same argument replacing $c$ with $c_1$ we have

$$c_2 \le c_1^{<n+1>} \le (c^{<n>})^{<n+1>} < d \text{ where } c_2 = \operatorname{codim} I_Z^{(n+2)D},$$

so that

$$c_r \le (c^{<n>})^{<n+1>\cdots<n+r-1>} < d$$

which implies $c_r \le d-1$. This is a contradiction as $c_r = d$.

Now let us assume that the result is true for $n-1$ and $k-1$. To easy the notation we write $I_Z^n$ instead of $I_Z^{nD}$.

**Claim:** Since $D$ is general, the multiplication for $x_D$

$$\mu_D : B^{(n-1)}/I_Z^{(n-1)} \to B^n/I_Z^n,$$

where $D = \operatorname{div}_0(x_D)$, is injective.

In principle the base locus $Z$ may contain $D$ but since $D$ is general we may assume by Bertini's theorem that $Z \cap D = \varnothing$, i.e., $\mu_D \neq 0$. Now, if $\mu(f) = 0$ then $f.x_D = 0$ and since $x_D \neq 0$ then $f = 0$.

We have a well defined surjective restriction map ($D$ is general), $B^n/I_Z^n \xrightarrow{r} B^n/I_{Z\cap D}^n$. There is a short exact sequence

$$0 \to \ker r \xrightarrow{\mu_D} B^n/I_Z^n \xrightarrow{r} B^n/I_{Z\cap D}^n \to 0.$$

It is clear that $\ker r$ contains $B^{n-1}/I_Z^{n-1}$. By induction we have

$$\operatorname{codim} I_Z^{n-1} \ge \binom{n+k}{k+1} - \binom{n-d+k}{k+1}$$

and

$$\operatorname{codim} I_{Z\cap D}^n \ge \binom{n+k}{k} - \binom{n-d+k}{k};$$

thus adding (4) and (4), and keeping in mind that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, we get the result. □

**Corollary 4.2.** *Let $W \subset H^0(\mathcal{O}_Y(n\eta))$ a subsystem whose base locus has dimension and degree greater than or equal to $k$ and $d$, respectively. Then for every $x \le min(k,n)$ one has*

$$\operatorname{codim} W \ge x\frac{(n-x)^k}{k!}$$

*Proof.* Since $\binom{n+k+1}{b+1} - \binom{n-d+k+1}{k+1} = \sum_{j=1}^{d} \binom{k+1+n-j}{n-j+1}$ applying the above lemma we get

$$\sum_{j=1}^{d} \binom{k+1+n-j}{n-j+1} \geq \sum_{j=1}^{d} \frac{(k+1+n-j)\dots(k-(k-1)+1+n-j)}{k!}$$
$$\geq \sum_{j=1}^{d} \frac{(n-j)^k}{k!} \geq d\frac{(n-d)^k}{k!} \geq x\frac{(n-x)^k}{k!}$$

□

From now we assume that $Y$ is $\mathbb{Q}$-factorial, i.e., for every Weil divisor $D$ there is an integer number $m$ such that $mD$ is Cartier. We establish a preorder in $N^1(Y) = \operatorname{Pic}(Y)\otimes\mathbb{Q}$ by letting $N < N'$ when $N' - N$ is numerically effective.

**Proposition 4.3.** *For every $\epsilon_1 > 0$ there exists $\delta_1 > 0$ such that for every $m \geq \frac{1}{\delta_1}$ and every real $d \in [1, \delta_1 m]$, if a Cox-Gorenstein ideal $I$ with socle degree $N$ satisfies*

- $\beta - \beta_0 \leq N - \beta = n\eta$ *with* $n \geq 1$
- $\operatorname{codim} I^\beta \leq d\frac{m^k}{k!}$ *where* $m = \max\{i \in \mathbb{N}^+ \mid i\eta \leq \beta\}$

*then*

1. *For every integer $i \in \{0, \dots, \lfloor \delta_1 m \rfloor\}$ one has*

$$\operatorname{codim} I^{\beta - i\eta} \leq (1+\epsilon_1)d\frac{m^k}{k!}$$

2. *For every $i \in \{0, \dots m\}$ one has*

$$\operatorname{codim} I^{\beta - i\eta} \leq 4^k d\frac{m^k}{k!}$$

*Proof.* First note that since $I$ is Gorenstein of socle degree $N$,

$$\operatorname{codim} I^{\beta - i\eta} = \operatorname{codim} I^{N-(\beta - i\eta)} = \operatorname{codim} I^{(n+i)\eta}.$$

So by the generalized Macaulay theorem (2.8)

$$\operatorname{codim} I^{\beta - i\eta} \leq (\operatorname{codim} I^{n\eta})^{<n>\cdots<n+i-1>}$$

and since for a fixed $c$ the map $c^{<->}$ is decreasing, and for a fixed $n$ the map $c \mapsto c^{<n>}$ is increasing, for every natural number $x \leq n$

$$\operatorname{codim} I^{\beta - i\eta} \leq (\operatorname{codim} I^{n})^{<x>\cdots<x+i-1>}$$

Also note that if

$$\operatorname{codim} I^n \leq \binom{\tau + x}{x} + \cdots + \binom{\alpha + x - \upsilon}{x - \upsilon} \ where\ \tau, \upsilon \in \mathbb{N}$$

as the map $c \mapsto c^{<n>}$ is increasing, (4) and (4) imply

$$\operatorname{codim} I^{\beta - i\eta} \leq \binom{\alpha + x + i}{x + i} + \cdots + \binom{\alpha + x - \beta + i}{x - \beta + i}$$

Suppose that $\delta_1$ is small enough that $d \leq \frac{m-2r}{2^{k+1}}$ for $r = \min\{i \mid \beta \leq i\eta\}$. By assumption $\beta - \beta_0 \leq n\eta$ i.e, $(m - r)\eta \leq n\eta$, so that

$$\lfloor \frac{m}{2} \rfloor + 2^k d \leq \lfloor \frac{m}{2} \rfloor + \frac{m - 2r}{2} \leq m - r \leq n.$$

Let $\gamma$ be the smallest positive real number such that $(2 + \gamma)^k d$ is an integer and

$$\lfloor \frac{m}{2 + \gamma} \rfloor + (2 + \gamma)^k d \leq n;$$

then the inequality (4) is true for $x = \lfloor \frac{m}{2+\gamma} \rfloor + (2 + \gamma)^k d$. On the other hand,

$$m^k \leq (\gamma + 2 + m)^k = (1 + \frac{m}{2 + \gamma})^k \leq (1 + \lceil \frac{m}{2 + \gamma} \rceil)^k =$$
$$(2 + \lfloor \frac{m}{2 + \gamma} \rfloor)^k \leq (k + \lfloor \frac{m}{2 + \gamma} \rfloor) \dots (2 + \lfloor \frac{m}{2 + \gamma} \rfloor) = \frac{(k + \frac{m}{2+\gamma})!}{(\frac{m}{2+\gamma} + 1)!}$$

so that

$$\frac{m^k}{k!} \leq \binom{k + \lfloor \frac{m}{2+\gamma} \rfloor}{\lfloor \frac{m}{2+\gamma} \rfloor + 1}$$

and

$$d \frac{m^k}{k!} \leq \underbrace{\binom{k + \lfloor \frac{m}{2+\gamma} \rfloor + (2 + \gamma)^k d - 1}{\lfloor \frac{m}{2+\gamma} \rfloor + (2 + \gamma)^k d} + \cdots + \binom{k + \lfloor \frac{m}{2+\gamma} \rfloor}{1 + \lfloor \frac{m}{2+\gamma} \rfloor}}_{(2+\gamma)^k d - terms}$$

Then by the second assumption we have that the inequality (4) is true for

- $x = \lfloor \frac{m}{2+\gamma} \rfloor + (2 + \gamma)^k d$,
- $\tau = k - 1$,
- $\upsilon = (2 + \delta)^b t - 1$;

thus inequality (4) holds, i.e.,

$$\begin{aligned}
\operatorname{codim} I^{\beta - i\eta} &\leq \binom{\lfloor \frac{m}{2+\gamma} \rfloor + (2+\gamma)^k d + k - 1 + i}{\lfloor \frac{m}{2+\gamma} \rfloor + (2+\gamma)^k d + i} + \cdots + \binom{\lfloor \frac{m}{2+\gamma} \rfloor + k + i}{\lfloor \frac{m}{2+\gamma} \rfloor - 1 + i} \\
&\leq (2 + \gamma)^k d \frac{(\frac{m}{2+\gamma} + (2+\gamma)^k d + k + i)^k}{k!} \\
&\leq (m + (2 + \gamma)^{k+1} d + (2 + \gamma) k + (2 + \gamma) i)^k \frac{d}{k!} \\
&\leq (1 + \frac{(2+\gamma)^{k+1} d + (2+\gamma) k + (2+\gamma) i}{m})^k d \frac{m^k}{k!}
\end{aligned}$$

Now if $0 \le i \le \lfloor m\delta_1 \rfloor$ we have

$$\operatorname{codim} I^{\beta - i\eta} \le (1 + \big((2+\gamma)^{k+1} + (2+\gamma)k + (2+\gamma)\big)\delta_1)^k d, \frac{m^k}{k!}$$

so that, given $\epsilon_1 > 0$, we take $\delta_1 > 0$ smaller enough such that

$$\big((2+\gamma)^{k+1} + (2+\gamma)k + (2+\gamma)\big)\delta_1 < \epsilon_1,$$

i.e., one gets claim 1 and taking $0 \le i \le m$ one gets claim 2. □

**Definition 4.4.** *Let $I \subset B(Y)$ be an ideal. For $i \in \{0, \dots, 2k\}$ and a fixed $n \in \mathbb{N}^+$ we define*

$$l_i^n(I) := \min\{l \in \mathbb{N} \cup \infty \mid dim V(I^{(n+l)\eta}) \le 2k - i\},$$

*or, equivalently,*

$$l_i^n(I) := \max\{l \in \mathbb{N} \cup \infty \mid \dim V(I^{(n+l-1)\eta}) > 2k - i\}.$$

We let $\dim \varnothing = -1$, and $l_i = \infty$ when this number does not exist.

*Remark* 4.5.
- We shall write $l_i(I)$ instead of $l_i^n(I)$.
- Note that $l_0(I) \le \cdots \le l_{2k}(I)$.
- If $I$ is base point free, then $l_{2k}(I) \in \mathbb{N}$.

△

**Lemma 4.6.** *For every $\epsilon_2 > 0$ there exists $\delta_2 > 0$ such that for every $m \ge \frac{1}{\delta_2}$ and $d \in [1, \delta_2 m]$, if a Cox-Gorenstein ideal $I \subset B(Y)$ with socle degree $N$ satisfies*

- $N - \beta = n\eta$
- $\operatorname{codim} I^\beta \le d\frac{m^k}{k!}$, *where $m = \max\{i \in \mathbb{N}^+ \mid i\eta \le \beta\}$,*

*then*

$$l_i(I) - 1 \le \epsilon_2(m-2) \ \ \forall i \in \{k, \dots, 2k\}.$$

*Proof.* Note that it is enough to prove the Lemma for $i = k$, so we apply the previous Proposition for $\epsilon_1 = 1$, and the Corollary for $x = 1$. Then for $l = \min(l_k(I) - 1, m)$ we have

$$\frac{(l-1)^{k+1}}{(k+1)!} \le \operatorname{codim} I^{l\eta} \le 4^k d \frac{m^k}{k!}$$

so that

$$l \le 1 + \big(4^k d m^k (k+1)\big)^{\frac{1}{k+1}} \le \big(\frac{1}{m} + (4^k(k+1)\frac{d}{m})^{\frac{1}{k+1}}\big)m \le (\delta_2 + (4^k(k+1)\delta_2)^{\frac{1}{k+1}})m$$

and since $2 \le 2m\delta_2$,

$$l \le (3\delta_2 + (4^k(k+1)\delta_2)^{\frac{1}{k+1}})m - 2.$$

So, given $\epsilon_2 > 0$, we take $\delta_2$ small enough to have $3\delta_2 + (4^k(k+1)\delta_2)^{\frac{1}{k+1}} < \min\{1, \epsilon_2\}$; then $l < m$ i.e $l = l_k(I) - 1$ or, in other words, $l_k(I) - 1 < \epsilon_2 m - 2$, and taking $\epsilon_2 \le 1$ we get that $l_k(I) - 1 < \epsilon_2(m-2)$ as desired. □

The following Proposition will be the technical core of what follows.

**Proposition 4.7.** *For every $\epsilon > 0$ there exists $\eta > 0$ such that for every integer $m > \frac{1}{\delta}$ and for every $d \in [1, \delta m]$, if a Cox-Gorenstein ideal $I \subset B(Y) = \mathbb{C}[x_1, \dots, x_r]$ with socle degree $N$ satisfies*

*i) $N = (k+1)\beta - \beta_0$ and $N - \beta = n\eta$;*

*ii) $I$ contains $r$ polynomials $\{F_i\}_{i=1}^r$ with $degF_i = \beta - degx_i$ and whose associated ideal is base point free;*

*iii) $\operatorname{codim} I^\beta \le d\frac{m^k}{k!}$ where $m = \max\{i \in \mathbb{N}^+ \mid i\eta \le \beta\}$,*

*then $I$ contains the ideal $I_V$ of a closed scheme $V \subset Y$ of pure dimension $k$ and degree less than or equal to $(1+\epsilon)d$. Moreover, $I$ and $I_V$ coincide in degree less than or equal to $\big(m - 2 - (r-j)\deg V\big)\eta$.*

*Proof.* By definition $\dim V(I^{l_k(I)}) \le k$, so that there exist $j \in \mathbb{N}^+$ and $f_1, f_2, \dots f_{r-j} \in I^{l_k(I)}$ such that $\dim V(< f_1, \dots, f_{r-j} >) = k$; more precisely, note that $j = k+1$. Moreover, as $I$ satisfies the assumptions of the previous Lemma, $f_1, f_2, \dots f_{r-j} \in I^{\le \frac{\epsilon_2}{2}(m-2)+1}$, and by the second assumption it is possible to find $r-j$ polynomials $f_{r-j+1}, \dots, f_r$ ,where $\deg(f_i) = \beta - deg(x_i)$ ( $i > j$), so that the ideal $< f_1, \dots f_r >$ is base point free and it is a Cox-Gorentein ideal of socle degree

$$\sum_{i=1}^{r-j} \deg(f_i) - \deg(x_i) + \sum_{i=r-j+1}^{r} \deg(f_i) - \deg(x_i) \le (r-j)\Big((m-2)\frac{\epsilon_2}{2} + 1\Big)\eta + j\beta - \beta_0$$

Now, by Proposition 3.6 there exists a polynomial $P$ with

$$\deg P \le (r-j)\Big((m-2)\frac{\epsilon_2}{2} + 1\Big)\eta + j\beta - \beta_0 - N = (r-j)\Big((m-2)\frac{\epsilon_2}{2} + 1\Big)\eta$$

and $I = \big((f_1, \dots f_r) : P\big)$. Moreover $I$ and $J = \big((f_1, \dots f_{r-j}) : P\big)$ coincide in degree less than or equal to

$$\beta - 2\eta - degP \ge (m-2)\eta - (r-j)\Big((m-2)\frac{\epsilon_2}{2} + 1\Big)\eta \le (m-2)\eta - (r-j)\big((m-2)\epsilon_2\big)\eta;$$

the last inequality is true when for $\delta_2 < \frac{\epsilon_2}{2}$ and $\frac{1}{\delta_2} + 2 \le m$. Now let us consider $l = \lfloor\big(1 - (r-j)\epsilon_2\big)(m-2)\rfloor$ and let us apply the previous results to $I^{l\eta}$. Then for every $x \le \min(degV, (r-j)\epsilon_2 m) \le \min(k, l)$

$$x\frac{(l-x)^k}{k!} \le \operatorname{codim} I^l \le (1+\epsilon_1)d\frac{m^k}{k!}$$

and

$$x\Big(1 - \frac{\lfloor\epsilon_2(r-t)m\rfloor + x}{m}\Big)^k \le (1+\epsilon_1)d$$

so that

$$x \leq \frac{(1+\epsilon_1)}{\big(1-2\epsilon_2(r-j)\big)^k} d;$$

then, given $0<\epsilon<1$ and taking $\epsilon_1$ and $\epsilon_2$ so that

$$\frac{(1+\epsilon_1)}{\big(1-2\epsilon_2(r-j)\big)^k} d \leq (1+\epsilon)d,$$

one has $x \leq (1+\epsilon)d < 2d < 2\delta m$. Thus taking $\eta < \frac{\epsilon_2}{2}$ we have $x < \epsilon_2 m \leq (r-j)\epsilon_2 m$, i.e., $x = \deg V$ and $\deg V \leq (1+\epsilon)d$. Moreover, $I$ and $I_V$ coincide in degree less than or equal to

$$\big(m-2-(r-j)\deg V\big)\eta$$

□

## 5 The tangent space to the Noether Lefschetz Locus

Since $\mathbb{P}_\Sigma^{2k+1}$ has a pure Hodge structure [20, 23], there is a well defined residue map for it, and we can use it to construct the tangent space at a point of the Noether-Lefschetz locus. Let $X = \{f=0\}$ be a quasi-smooth hypersurface in $\mathbb{P}_\Sigma$, with $\deg f = \beta$. Denote by $i : X \to \mathbb{P}_\Sigma$ the inclusion, and by $i^* : H^\bullet(\mathbb{P}_\Sigma^{2k+1}, \mathbb{Q}) \to H^\bullet(X, \mathbb{Q})$ the associated morphism in cohomology; $i^* : H^{2k}(\mathbb{P}_\Sigma^{2k+1}, \mathbb{Q}) \to H^{2k}(X, \mathbb{Q})$ is injective by Proposition 10.8 in [2].

**Definition 5.1.** *The primitive cohomology group $H^{2k}_{\mathrm{prim}}(X)$ is the quotient*

$$H^{2k}(X,\mathbb{Q})/i^*(H^{2k}(\mathbb{P}_\Sigma^{2k+1},\mathbb{Q})$$

Both $H^{2k}(\mathbb{P}_\Sigma^{2k+1},\mathbb{Q})$ and $H^{2k}(X,\mathbb{Q})$ have pure Hodge structures, and the morphism $i^*$ is compatible with them, so that $H^{2k}_{\mathrm{prim}}$ inherits a pure Hodge struture.

Also, we shall denote by $M$ the dual lattice of the lattice $N$ which contains the fan $\Sigma$, i.e., $\Sigma \subset N \otimes \mathbb{R}$.

**Definition 5.2.** *Fix an intger basis $m_1, \ldots m_{2k+1}$ for the lattice $M$. Then given a subset $\iota = \{i_1, \ldots, i_{2k+1}\} \subset \{1, \ldots, \#\rho(1)\}$, where $\#\rho(1)$ is the number of rays, we define*

$$\det(e_\iota) := \det\big( < m_j, e_{i_h} >_{1\leq j,h \leq 2k+1} \big);$$

*moreover, $dx_\iota = dx_{i_1} \wedge \cdots \wedge dx_{i_{2k+1}}$ and $\hat{x}_\iota = \Pi_{i \notin \iota} x_\iota$.*

**Definition 5.3.** *The $(2k+1)$–form $\Omega_0 \in \Omega_S^{2k+1}$ is defined as*

$$\Omega_0 := \sum_{|\iota|=2k+1} \det(e_\iota)\hat{x}_\iota dx_\iota$$

*where the sum is over all subsets $\iota \subset \{1, \ldots, 2k+1\}$ with $2k+1$ elements.*

For more details about these definitions see [2].

**Theorem 5.4.** *$T_{[f]}\big(NL^{k,\beta}_{\lambda,U}\big)\cong E^{\beta}$, where*

$$E=\{K\in B(\Sigma)^{\bullet}\mid \sum_{i=1}^{b}\lambda_i\int_{\mathrm{Tub}\,\gamma_i}\frac{KR\Omega_0}{f^{k+1}}=0\ for\ all\ R\in S^{N-\bullet}\},$$

*and* Tub$(-)$ *is the adjoint of the residue map.*

*Proof.* By [4, Prop. 2.10] the $p$-th residue map

$$r_p:H^0(\mathbb{P}_\Sigma,\Omega^{2k+1}_{\mathbb{P}_\Sigma}(2k+1-p)X)\to H^{p,2k-p}_{\mathrm{prim}}(X)\ \ for\ \ 0\le p\le 2k$$

exists; it is surjective and has kernel $H^0(\mathbb{P}_\Sigma,\Omega^{2k+1}_{\mathbb{P}_\Sigma}(2k-p)X)+dH^0(\mathbb{P}_\Sigma,\Omega^{2k}_{\mathbb{P}_\Sigma}(2k-p)X)$. So

$$resH^0(\Omega^{2k+1}(2k+1)X)=r_{2k}H^0(\Omega^{2k+1}(X))\oplus\cdots\oplus r_0H^0(\Omega^{2k+1}(2k+1)X))$$

by definition of $H^0(\Omega^{2k+1}(2k+1)X)$. Or, equivalently,

$$\mathrm{res}\,H^0(\Omega^{2k+1}(2k+1)X)=H^{2k,0}_{\mathrm{prim}}(X)\oplus\cdots\oplus H^{0,2k}_{\mathrm{prim}}(X)=H^{2k}_{\mathrm{prim}}(X).$$

Similarly

$$\mathrm{res}\,H^0(\Omega^{2k+1}(kX)=F^{k+1}H^{2k}_{\mathrm{prim}}(X).$$

On the other hand by [2, Thm 9.7] we have

$$H^0(\Omega^{2k+1}_{\mathbb{P}_\Sigma}(kX)=\{\frac{K\Omega_0}{f^k}\mid K\in S^{k\beta-\beta_0}\}=\{\frac{K\Omega_0}{f^k}\mid K\in B^{k\beta-\beta_0}_{\Sigma}\};$$

the last equality holds true because we are assuming that $k\beta-\beta_0$ is ample and hence $B^{k\beta-\beta_0}_{\Sigma}=S^{k\beta-\beta_0}$ by Lemma 9.15 in [2].

Now fixing a basis $\{\gamma_i\}_{i=1}^{b}$ for $H_{2k}(X,\mathbb{Q})$ we have that the components of any element in $F^{k+1}H^{2k}_{\mathrm{prim}}(X)$ are

$$(\int_{\gamma_1}\mathrm{res}\,\frac{K\Omega_0}{f^k},\dots,\int_{\gamma_b}\mathrm{res}\,\frac{K\Omega_0}{f^k}),$$

or, equivalently,

$$(\int_{\mathrm{Tub}(\gamma_1)}\frac{K\Omega_0}{f^k},\dots,\int_{\mathrm{Tub}(\gamma_b)}\frac{K\Omega_0}{f^k})$$

where Tub$(\gamma_j)$ is the adjoint to the residue map. Now taking $0\neq\lambda_f\in H^{k,k}(X,\mathbb{Q})$ one has $\lambda_f\perp F^{k+1}H^{2k}_{\mathrm{prim}}(X)$ (see [22]) and since the sheaf $\mathcal{H}^{2k}$ is constant on $U$ we have

$$NL^{k,\beta}_{\lambda,U}=\{G\in U\mid \lambda_G\in F^kH^{2k}_{\mathrm{prim}}(X_G)\}=\{G\in U\mid \lambda_f\perp F^{k+1}H^{2k}_{\mathrm{prim}}(X_G)\}.$$

More explicitly, if $(\lambda_1,\dots\lambda_b)$ are the components of $\lambda_f$, one gets

$$\lambda_f\perp F^{k+1}H^{2k}_{\mathrm{prim}}(X_G)\quad\Leftrightarrow\quad \textstyle\sum_{i=1}^{b}\lambda_i\int_{\mathrm{Tub}\,\gamma_i}\frac{K\Omega_0}{G^k}=0\ \forall K\in S^{N-\beta}$$

where $N$ is equal to $(k+1)\beta - \beta_0$.Thus we can characterize the local Noether-Lefschetz locus in the following way:

Let us consider the differentiable map $\psi$ which assigns to every homogeneous polynomial $G \in B_\Sigma^\beta$ a linear map $\psi_G \in (B_\Sigma^{N-\beta})^\vee$ ,i.e.,

$$\begin{array}{rclcl} \psi : B_\Sigma^\beta & \longrightarrow & (B_\Sigma^{N-\beta})^\vee & & \\ G & \mapsto & \psi_G : & B^{N-\beta} & \to \mathbb{C} \\ & & & K & \mapsto \sum_i \lambda_i \int_{\mathrm{Tub}(\gamma_i)} \frac{K\Omega_0}{G^k} \end{array}$$

then $NL_{\lambda,U}^{k,\beta} = \psi_{|U}^{-1}(0)$ , hence the tangent space at $f$ is the kernel of $d\psi_f$. Now $T_{[f]}U \simeq S_\beta$ and since $\beta$ is ample, $S^\beta = B^\beta$. Thus we can identified canonically $T_{[f]}\big(NL_{\lambda,U}^{k,\beta}\big)$ with the subspace $E^\beta \subset B_\Sigma^\beta$, which is the $\beta$-summand of the Cox-Gorenstein ideal

$$E = \{K \in B_\Sigma^\bullet \mid \forall R \in S^{N-\bullet}, \sum_{i=1}^b \lambda_i \int_{\mathrm{Tub}\,\gamma_i} \frac{KR\Omega_0}{f^{k+1}} = 0\}$$

whose socle degree is $N = (k+1)\beta - \beta_0$. $\square$

*Remark* 5.5. Note that $E$ contains the Jacobian ideal $J = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_r})$, where $< x_1, \dots x_r >= B_\Sigma$, and since $X$ is quasi-smooth, there exists $l \in \mathbb{N}$ such that

$$B(\Sigma)^l \subset J \subset B(\Sigma);$$

by the Toric Weak Nullstellensatz $V(J) = \varnothing$ [6]. $\triangle$

We also consider the Cox-Gorenstein ideals

$$E_s := \{K \in B_\Sigma^\bullet \mid \forall R \in S^{N+r\beta-\bullet}, \sum_{i=1}^b \lambda_i \int_{\mathrm{Tub}\,\gamma_i} \frac{KR\Omega_0}{f^{k+r+1}} = 0\}$$

with $s \in \mathbb{N}^+$, which have socle degree $N + r\beta$. For a fixed $s$, the ideal $E_s$ describes the deformation of order $s+1$ of $NL_{\lambda,U}^{k,\beta}$ in a neighborhood of $f$.

**Proposition 5.6.** *The Cox-Gorenstein ideals $E_s$ have the following properties:*

*i.* $E_s = (E_{s+1} : f)$*;*

*ii. If $f$ is a generic point of $NL_{\lambda,U}^{red}$ then $(E_r)^2\Theta \subset E_{s+1}$, where $\Theta \subset S_\beta = B_\Sigma^\beta$ is the image of the tangent space $T_f(N_{\lambda,U})^{red}$*

*iii.* $\forall K \in E_s$ *and* $\forall j \in \{1, \dots, r\}$, $\frac{\partial K}{\partial x_j} f - (k+s+1)K; \frac{\partial f}{\partial x_j} \in E_{s+1}$.

*Proof.* 1. Clear.

2. For every $G \in NL_{\lambda,U}^{k,\beta}$ and for every $i \in \mathbb{N}^+$ such that $N + r\beta - i\eta$ is ample, consider the bilinear map

$$\begin{array}{rcl} \mathcal{Q}_i(G): \ B_\Sigma^{i\eta} \times B_\Sigma^{N+r\beta-i\eta} & \to & \mathbb{C} \\ (K,R) & \mapsto & \sum_{i=1}^b \lambda_i \int_{\mathrm{Tub}\,\gamma_i} \frac{KR\Omega_0}{G^{k+r+1}} \end{array}$$

For a fixed $R$ we have $\ker \mathcal{Q}_i(G) = E_s^{i\eta}(G)$, and for a fixed $K$ we have $\ker \mathcal{Q}_i(G) = E_s(G)^{N+rL-iD}$, where $E_s(G)$ is the Cox-Gorenstein ideal associated to the class $\lambda_G$. Since $f$ is a quasi-smooth point of $(NL^{k,\beta}_{\lambda,U})^{red}$, the map $G \mapsto \mathcal{Q}_i(G)$ has constant rank for every $G$ close to $f$. So for each $\vec{v} \in T_f(N_{\lambda,U})^{red}$ associated to $M \in \Theta$ the differential of the bilinear map

$$\begin{array}{rccl} d\mathcal{Q}_i(f)(\vec{v}): & B_\Sigma^{i\eta} \times B_\Sigma^{N+r\beta-i\eta} & \to & \mathbb{C} \\ & (K,R) & \mapsto & -(k+s+2)\sum_{i=1}^t \lambda_i \int_{\mathrm{Tub}\,\gamma_i} \frac{KRM\Omega_0}{f^{k+s+2}} \end{array}$$

is zero on $E_s^{i\eta} \times E_s^{\eta+r\beta-i\eta}$, or, in other words, $E_s^{i\eta} E_s^{N+r\beta-i\eta}\Theta \subset E_{s+1}^{N+(s+1)\beta}$.

3. Given $K \in E_s$, for every $R \in B_\Sigma^{N+s\beta+\eta-deg(K)}$ we have

$$R\left(\frac{\partial K}{\partial x_i} f - (k+s+1)K\frac{\partial f}{\partial x_i}\right) = \underbrace{\frac{\partial (KR)}{\partial x_i} f - (k+r+1)KR\frac{\partial f}{\partial x_i}}_{A} - \underbrace{KF\frac{\partial R}{\partial x_i}}_{B}.$$

Note that $\frac{A\Omega_0}{f^{n+r+2}}$ is an exact form in the kernel of the residue map, so that $A \in E_{s+1}$. By assumption $K\frac{\partial R}{\partial x_j} \in E_s$ so $B \in E_{s+1}$ by the first property. Thus $R(\frac{\partial K}{\partial x_i} f - (k+r+1)K\frac{\partial f}{\partial x_i}) \in E_{s+1}$ and since $R$ is arbitrary we get the result. □

# 6 Proof of the Main Theorem

Now we have all the machinery necessary to prove our main result.

**Theorem 6.1.** *For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $m \geq \frac{1}{\delta}$ and for all $d \in [1, m\delta]$, if $\operatorname{codim} N^{k,\beta}_{\lambda,U} \leq d\frac{m^k}{k!}$ where $m = \max\{i \mid i\eta \leq \beta\}$ and if $G \in N^{k,\beta}_{\lambda,U}$, then there exists a $k-$dimensional subvariety $V \subset X_G$ with degree less than or equal to $(1+\epsilon)d$.*

*Proof.* If $f$ is a generic point in $(NL^{k,\beta}_{\lambda,U})^{red}$, by proposition 4.7 there exists a subscheme $V \subset \mathbb{P}_\Sigma$ of pure dimension $k$ and degree $d' \leq (1+\epsilon)d \leq 2\delta m$ such that $I_V \subset E$; the two ideals in degree less or equal to $\big(m-2-(r-j)d'\big)\eta$, so it is enough to prove that $f \in \sqrt{I_V}$. Moreover

**Claim 1.** $\left(I_V^{\leq d'\eta}\right)^2 \subset E_1$.

Let $R \in (I_V^{\leq d'\eta})^2$, then the partial derivatives of $R$ belong to $E$, and by items (i) and (iii) of Proposition 5.6, the partial derivatives of $f$ belong to $(E_1 : R)$. Since $f$ is quasi-smooth, its Jacobian is base point free, and $(E_1 : R)$ contains a base point free ideal whose socle degree is less than or equal to

$$(r-(k+1))(\epsilon_2(m-2))\eta + (k+1)\beta - \beta_0.$$

Byy contradiction $R \notin E_1$ then $(E_1 : R)$ has socle degree greater than or equal to

$$N + \beta - \deg R \geq N + \beta - 2d'\eta \geq N + \big((1-4\delta)m\big)\eta.$$

Now by (ii) in Proposition 5.6 we have $\Theta \subset (E_1 : R)$, and by assumption $\operatorname{codim}(\Theta) \le d\frac{m^k}{k!}$, so then $\operatorname{codim}(E_1 : R)^\beta \le d\frac{m^k}{k!}$, i.e, $(E_1 : R)$ satisfies the assumptions of lemma 4.6. Then taking $\epsilon_2 = \frac{1}{2(r-(k+1))}$ and $\delta_2 = \delta < \frac{1}{4(r-(k+1))}$ we get

$$\frac{m-2}{2}\eta + N \ge N + \big((1-4\delta)m\big)\eta,$$

which imples $\delta > \frac{1}{8}$. Since

$$r - (k+1) \ge k+1 \Leftrightarrow \frac{1}{4(k+1)} \ge \frac{1}{4(r-(k+1))}$$

so that $\delta < \frac{1}{8}$, which is a contradiction. So one has $R \in E_1$ as desired.

**Claim 2**: $f \in \sqrt{I_V}$.

Since $V$ is of pure dimension $k$, it is enough to show that $f \in \sqrt{I_W}$ for every irreducible subscheme $W$ of $V$ associated to the primary ideal decomposition of $I_V$. Let $W'$ be the smallest subscheme of $V$ such that $I_V = I_W \cap I_{W'}$, and let $P \subset \mathbb{P}_\Sigma$ be a projective linear space of dimension $k-1$, for which we can suppose without loss of generality that it has equations $x_1 =, \dots, x_{r-k} = 0$ and we set $B_P = \mathbb{C}[x_1, \dots, x_{r-k}]$. Since $W$ and $W'$ are of pure dimension $k$, the homogeneous ideals $I_W \cap B_P \subset B_P$ and $I_{W'} \cap B_P \subset B_P$ are of pure codimension 1 for $P$ generic; therefore they are principal. Let $K_{P,W}$ and $K_{P,W'}$ be the images of the generators in $B_\Sigma$. Let $\kappa = \deg K_{P,W}$ and $\kappa' = degK'_{P,W}$; by construction we have that $\kappa \le degW$ and $\kappa' \le degW'$. Considering $K_P = K_{P,W}K^2_{P,W'}$, we have $K_P \in E$, $K_P \notin E_1$, so that the ideal $(E_1 : K_P)$ has socle degree $N + \beta - (\kappa + 2\kappa')$ and moreover contains the ideal

$$J_P = \Big\langle f, I_W^{degW}, \frac{\partial f}{\partial x_{r-k+1}}, \dots, \frac{\partial f}{\partial x_r}\Big\rangle.$$

More precisely, the following facts hold true:

- $K_P \in E$ as $\kappa + 2\kappa' \le m - 2 - (r-j)d'$;
- $K_P \notin E_1$. Otherwise, $(k+r+1)K_P\frac{\partial f}{\partial X_i} \in E_1$ and then, using property (iii) of Proposition 5.6, $\frac{\partial K_P}{\partial X_i} f \in E_1$ and by property (i), $\frac{\partial K_P}{\partial x_i} \in E$ for all $i$; however, by construction not all partial derivatives of $K_P$ are in $E$, so this is a contradiction.
- $J_P \subset (E_1 : K_P)$; indeed, as $\frac{\partial K_P}{\partial x_{r-k+1}} = 0, \dots, \frac{\partial K_P}{\partial x_r} = 0$ then $(E_1 : K_P)$ contains $\frac{\partial f}{\partial x_{r-k+1}}, \dots, \frac{\partial f}{\partial x_r}$ by property 3 of proposition 2. On the other hand by lemma 2 we have $\big((I_V)^{\le d'}\big)^2 \subset E_1^{\le 2d'}$ and since $I_W^{degW} K_P \subset \big((I_V)^{\le d'}\big)^2$, we have $I_W^{degW} \subset (E_1 : K_P)^{degW}$.

Now by contradiction, if $f \notin I_W$, then $\dim V(f, I_W^{\deg W}) \le k-1$, and moreover $J_P$ contains a Cox-Gorentein ideal with socle degree less than or equal to $N + (k+1)d'\eta$. On the other hand, $(E_1 : K_P)$ has socle degree greater than or equal to $N + \beta - 2d'\eta$, so that

$$N + (r-(k+1))2\delta m\eta \ge N + (r-(k+1))d'\eta \ge N + \beta - 2d'\eta \ge N + (1-4\delta)m\eta$$

which implies that $\delta \ge \frac{1}{2(r-(k+1)+2)} \ge \frac{1}{2(k+3)}$, contradicting our choice of $\delta$. Thus $f \in I_W$. $\square$

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